MATHEMATICAL TASKS AND THE COLLECTIVE MEMORY: HOW DO TEACHERS MANAGE STUDENTS' PRIOR KNOWLEDGE WHEN TEACHING GEOMETRY WITH PROBLEMS?

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Education)
in The University of Michigan
2009

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To the González-Rivera family.

ACKNOWLEDGEMENTS

I visited the University of Michigan for the first time in 1993. At that moment I started thinking about doing a doctorate in mathematics education. I had been to the NCTM annual conference in Seattle and my professor at the University of Puerto Rico, Milagros Martínez, invited me to visit her daughter in Ann Arbor. I will always be grateful to her for encouraging me to pursue graduate studies.

I want to thank Professor Patricio G. Herbst for his work as my advisor and for chairing my dissertation committee. I have learned much from him about mathematics education. He has mentored me and has always expected my best work. I appreciate the intellectual challenge of working with him and his deep care for mathematics.

I also want to thank the other dissertation committee members who enriched my work with their expertise. Professor Magdalene Lampert, Professor Jay Lemke, Professor Mary J. Schleppegrell, and Professor Priti R. Shah. Their insights and their questions enabled me to think deeper about my work and to make connections with educational research. I appreciate their commitment to work with me.

The materials developed and the data collected were part of a research project supported by NSF grant REC 0133619 (PI: Patricio Herbst). My doctoral work was supported by a Rackham Merit Fellowship and a School of Education Scholars Award. I also received the support from the Center for Proficiency in Teaching Mathematics. I appreciate the opportunity to pursue doctoral work because of this support.

I have been fortunate to be a member of the mathematics education research group GRIP (Geometry Reasoning and Instructional Practices) directed by Dr. Herbst. I want to thank Dr. Herbst, Wendy Aaron, Chia-Ling Chen, Angela Hsu, Manu Mehrotra, Takeshi Miyakawa, Chieu VuMinh, and Michael Weiss for the opportunity to discuss my ideas with them. I appreciate the valuable feedback from Wendy Aaron to parts of my writing. Chia-Ling Chen, Wendy Aaron, and Angela Hsu have shared many important moments with me and I am grateful for their friendship.

My family in Puerto Rico has been very supportive. My mother, Gloria Esther González, and my father, Carlos Emilio González, helped me to move to Ann Arbor. The love I have received from them has shaped my life. My brother, Carlos González, and his wife, María Aponte, have also made it possible for me to pursue my doctoral work. The laughter of my nieces Isabel and Natalia has motivated me. I am grateful to my aunt, María Antonia Rivera, for her letters and her support. I also want to thank Virginia Arzuaga for encouraging me to start a doctorate when I was an undergraduate and for listening to me. My new family in Denmark has also been very important, especially the encouragement of Morten's parents, Jonna and Henrik Lundsgaard.

Finally, my husband, Morten Lundsgaard, has given me strength in difficult moments and has pushed me to finish this work. Every day he teaches me about the important things in life. I am grateful for his patience, his help, and his love. Our daughter, Claudia Esther, has spent more hours doing this work with me than anybody else. One day, I hope, she will have her own computer. Claudia's curiosity, her sense of wonder, and her adventurous spirit have motivated me. I am grateful for the joy she brings to our lives.

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ABSTRACT

This is a study of how teachers manage students' prior knowledge in problem-based mathematics teaching. I propose that geometry teachers manufacture a collective memory which they use to hold students accountable for what they should remember and what they should forget when they work on problems. This hypothesis is put to work in analyzing two corpuses of data. I inspect a corpus of video records of a problem-based unit on quadrilaterals, where a teacher made changes to usual practices in two ways, by asking students to call forth knowledge from prior mathematics classes and by having students anticipate a theorem that the teacher had not stated yet. The second corpus consists of proceeds of five focus group sessions in which experienced geometry teachers viewed and discussed records of problem-based teaching in geometry and where they designed tasks in which they would engage their students. The analysis uncovered teachers' assumptions and normative stances on how to manage students' prior knowledge. In addition, from the analysis I describe a catalogue of teaching actions that teachers accept they might avail themselves for shaping the collective memory of a class. Methodologically, the study shows how to investigate teachers' management of prior knowledge by applying tools from Systemic Functional Linguistics to transcripts of mathematics classroom talk and to transcripts of conversations among practitioners. This work is a contribution to the study of teaching by describing the kinds of resources that

teachers could use in teaching with problems, and the underlying rationality for teaching actions to manage students' prior knowledge.

CHAPTER 1

INTRODUCTION

This study of geometry instruction focuses on teachers' management of students' prior knowledge in problem-based instruction. I study how teachers make students accountable for remembering what they ought to know to do a novel problem. I also study what teachers hold themselves accountable for doing about students' prior knowledge when teaching with a problem. The high school geometry course gives an opportunity to investigate the tensions that problem-based instruction causes on the work of the teacher regarding students' prior knowledge. Students take geometry in the 9th or 10th grade, but they have had the opportunity to learn about geometric ideas in previous mathematics courses. However, the geometry curriculum usually makes use of these properties only after they have formally been stated in the geometry class. The geometry course is rarely problem-based because it has been traditionally conceived of as an opportunity for introducing students to an axiomatic system in mathematics. As a consequence, the geometry curriculum ignores students' prior knowledge from other mathematics courses. Besides the curricular constraints, particular activities of teaching may involve different ways to evoke and to use prior knowledge.

Teachers' work to manage students' prior knowledge can be particularly challenging when teaching geometry with a problem. Recent reform documents in mathematics education convey the expectation for teachers to use problems in teaching mathematics (National Council of Teachers of Mathematics [NCTM], 2000). Teachers could make use of problems for having students experience authentic mathematics and in finding out new ideas (Lampert, 2001). However, one of the difficulties teachers may have in using problems to teach is that of managing students' prior knowledge. A teacher could possibly ponder: "Will students think about using the Pythagorean theorem?"; "Will students remember that we solved a similar problem about parallelograms?"; "Will students know the definition of an angle bisector?"; or "Will students remember what we did yesterday?" These questions illustrate the kinds of issues that teachers consider regarding students' prior knowledge when teaching with a problem.

How teachers make use of students' prior knowledge is a fundamental question to be asked about the geometry course. Students possess prior knowledge that they could apply when solving problems. However, the geometry curriculum does not make use of students' prior knowledge about geometric concepts, with the expectation that students will come to build their geometric knowledge anew. In teaching, students' actual memories could hinder a teacher's work of sorting through what knowledge students can call forth when solving a problem. If students were to use their prior knowledge to solve a problem, this might jeopardize a teacher's effort to bring about new knowledge as a result of students' work on the problem. For example, the knowledge that a teacher might like to bring forth with a problem may be something that students already possess in their actual memories. Students' prior knowledge may allow them to work on a problem in ways that would make the new knowledge that is supposed to come forth with their work on the problem irrelevant. In addition, not all students in a class may share the same prior knowledge, despite the acknowledgement that individual students do possess prior knowledge. These examples illustrate some of the difficulties that a teacher may confront if he or she were to draw upon students' prior knowledge to teach something new. In the next section I elaborate on the instructional problem of managing students' prior knowledge in the case when a teacher uses a problem to teach geometry.

The Problem of Managing Students' Prior Knowledge

A central notion in this study is that of a *didactical contract* that binds teachers and students together through implicit sets of norms in a class. This notion, as described by Brousseau (1997) in his theory of didactical situations, postulates that teachers and students have distinct responsibilities that tie them in different ways to the knowledge at stake. Different contracts specify those different responsibilities further. The notion of structuring a course of studies by teaching with problems (Lampert, 2001) involves a particular didactical contract. I study the demands that the expectation to teach with a problem poses on the geometry teacher from the observer and the participants' point of view. At the same time, I study the responsibilities that teachers think students should take on about memories of prior knowledge when teaching with a problem. These responsibilities that teachers and students share about remembering prior knowledge when working on a problem could be different when teaching with a problem than in other activities of teaching. In particular, I compare how the way a teacher manages students' prior knowledge when teaching with a problem and when doing a proof may be different.

Prior work has demonstrated that it is difficult for a teacher of high school geometry to sustain the work involved in using a problem to teach new geometric content

(Herbst, 2006). The activity of teaching geometry with a problem is not customary. That means that if and when a teacher chooses to initiate the teaching of a new idea with a problem, they would have to negotiate the contract for what it would mean to teach and learn with that problem, increasing the cost for the teacher. So, one could conceive that the kind of prior knowledge that students would need to bring to bear for working on a problem is one of the elements of the contract that a teacher would have to negotiate with the students. Here, the constructs of *task* and *instructional situation* are useful to study the work on a problem in a class. In particular, these constructs are useful when examining how teachers and students negotiate the memories of prior knowledge needed to work on a problem.

According to Herbst (2006), a mathematical *task* is the deployment over time of students' work on a problem. For a task to unfold in a class, it has to be viable within the contract. In particular, the teacher and the students must deem the task as appropriate for the teacher to teach content and for the student to learn content. The kind of work classes do on a mathematical problem, the task, depends upon the *instructional situation* in which the problem is posed. Those instructional situations frame how teachers and students trade work on mathematical tasks for knowledge claims. The construct of *situation* helps to account for the exchange between a teacher and the students. An instructional situation enables the teacher to get some work done without having to negotiate how the contract applies to a particular task.

The same mathematical problem can prompt different tasks, and as a consequence, different kinds of work, depending on the *resources* and the *operations* one would bring to bear in order to achieve a *product* (Doyle, 1988; Herbst, 2006). Different tasks may require students to use different memories from prior knowledge. So a teacher may want to activate specific pieces of prior knowledge for students to work on a task. At the same time, students' work on different tasks may yield products that a teacher may want students to remember as evidence of some knowledge. Some prior knowledge may be more or less valuable according to the situation framing the exchange between students' work on a problem and claims about learning. Also, some knowledge may be more or less memorable in the future according to the situation framing that exchange. For example, in the situation of *doing proofs* the proven statement is not meant to be memorable (Herbst & Brach, 2006). In contrast, when a teacher *installs* a theorem by means of providing a proof, the proven statement ought to be remembered (Herbst & Nachlieli, 2007). Therefore, an instructional situation implies contractual obligations that suggest what from students' work is valuable, and thus, memorable.

The focus of this dissertation is to investigate the possibility that a teacher could make use of students' prior knowledge in the activity of teaching with a problem. The study contributes to understanding what it may take to teach geometry with problems. Thus, if routinely teachers were to develop new ideas through problems, what sort of mutual understandings, and in particular, what sorts of tacit norms might help them carry through the work on mathematical tasks for students to make use of prior knowledge.

A Hypothesis and the Research Questions

I hypothesize that as teachers and students interact around problems, they create a public representation of past events, conceivably different from those past events. Teachers in particular push a privileged representation of the past—a discourse of the past—as the orthodox collective memory of the class. With this purpose, a teacher makes moves to make students remember (or forget) things and events from the class.

In studying how geometry teachers create a discourse of the past as the collective memory of the class, I ask: *How can the hypothesis that the teacher creates a collective memory of the class be demonstrated in the case of a teacher who attempts to teach geometry with problems?* I investigate this question with the following research questions:

1. How does a teacher hold students accountable for remembering what they ought to know in order to do a problem?

2. How does student participation feature in a teacher's management of the collective memory?

3. What do teachers hold themselves accountable for doing to make students remember or forget past things and events when using a problem to teach something new?

4. How do teachers perceive and appreciate alternative ways of managing students' prior knowledge when using a problem to teach new geometric content?

While the first and second research questions involve the analysis of teaching actions, the third and fourth questions involve the analysis of teachers' commentary on action. By answering these four research questions, I address the main question guiding this work regarding the hypothesis of the collective memory of the class. I gather examples of teaching actions geared toward making students remember prior knowledge and also, teaching actions with the purpose of making something memorable in the future. In doing these actions, I expect that teachers would set boundaries between those things and events that students should remember and those things and events that students should forget.

My use of "forgetting" is metaphorical. I do not mean to imply that students are unable to recall prior knowledge. The question of whether students have the neurological or psychological capabilities to remember prior knowledge is beyond the scope of this dissertation. My focus is on what memories of prior knowledge are unavailable for students' use when they work on problems in the geometry class. In particular, I study what the teacher does to deauthorize students' use of some prior knowledge that they might otherwise remember, thus filtering out students' memories.

My choice of studying teaching actions for managing students' prior knowledge when teaching geometry with a problem intends to confront current teaching practices with the new expectations of the reform. According to the Standards (NCTM, 2000), problem-based instruction is key for students to have meaningful understandings of mathematics. The document reads,

> Problem solving means engaging in a task for which the solution method is not known in advance. In order to find a solution students must draw on their knowledge, and through this process, they will often develop new mathematical understandings. Solving problems is not only a goal of learning mathematics but also a major means of doing so. Students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking. (NCTM, 2000, p. 52)

This new demand for students to learn through problems may require teachers to teach differently than how they currently do. In particular, teachers would need to match their expectations about students' prior knowledge with their anticipations about students' work on a problem. While learning mathematics through problem solving is something desirable, there is still much to learn about teaching actions to manage students' prior knowledge in problem-based instruction. In this dissertation I present two studies with the intention of identifying teaching actions for organizing old and new knowledge as students work on a problem. In the following section I give a rationale for the two studies.

Rationale for Two Studies on Teachers' Use of Students' Prior Knowledge

This dissertation includes two complementary studies that address teachers' management of students' prior knowledge from two different perspectives. Each study involves a different kind of empirical data.[1] Classroom episodes of two geometry classes allow me to do a descriptive study of how prior knowledge management plays out in classroom interaction when a teacher uses a problem to teach, and focus group sessions with geometry teachers allow me to study what teachers consider themselves accountable to do when teaching new content with a problem. While the first study provides an observer's account of classroom data, the second study analyzes teachers' perspectives on teaching actions. In the two studies I examine how teachers manage students' prior knowledge when teaching geometry with problems.

[1] Data for this dissertation is part of the archive of GRIP (Geometry Reasoning and Instructional Practices), a research group at the University of Michigan directed by Dr. Herbst. The data was collected with the support of NSF grant REC 0133619 to Dr. Herbst; all opinions expressed are the sole responsibility of the author and do not necessarily represent the views of the Foundation.

A Teacher's Attempt to Shape the Collective Memory of Her Class when Teaching with Problems: An Observer's Account of Classroom Data

The main question guiding this study is: *How does a teacher hold students accountable for remembering what they ought to know in order to do a problem?* A specific subsidiary question is: *How does student participation feature in a teacher's management of the collective memory?*

This study focuses on teaching actions as seen from an observer's perspective. I present the case of a teacher who attempted to shape the collective memory of her class when teaching with problems. The study compares and contrasts the activity of teaching with a problem with other, more established activities in high school geometry such as *doing proofs* (Herbst, 2002; Herbst & Brach, 2006) in relation to how teachers make use of students' prior knowledge for managing students' work on mathematical tasks. The data for this study comes from a problem-based instructional unit to teach quadrilaterals, taught by a teacher in two geometry classes. This unit was conceived of as an "instructional experiment" (Herbst, 2006). Similar to "breaching experiments" (Garfinkel, 1967), the unit introduced the activity of teaching with a problem as a change to the didactical contract in the class. As a result, we had the opportunity to observe how the teacher and the students responded to those changes by *repairing* (Mehan & Wood, 1975) the task or the situation framing the activity, thus negotiating the contract. I analyze video-records of episodes where students' reliance on past memories (or lack thereof) brought about tensions for the teacher in teaching with a problem. I examine how that teacher managed those tensions by performing different actions that required students to remember prior knowledge.

The Rationality Involved in Managing Students' Prior Knowledge when Teaching with a Problem: The Perspective of Teachers

The main question guiding this study is: *What do teachers hold themselves accountable to do regarding students' memories as students work on a problem?* A more specific subsidiary question is: *How do teachers perceive and appreciate alternative ways of managing students' prior knowledge when using a problem to teach new geometric content?*

This study focuses on teachers' commentary on action, using records of teaching to provoke their thinking. I examine records of focus group sessions where the teachers who participated (hereafter, "participants") talked about the possibility of teaching with a problem. Participants in the focus group sessions watched and commented on video records. These video records showed examples of a teacher using a problem to teach properties of quadrilaterals in different geometry classes. I analyze participants' reactions to those videos. In their reactions to the videos, participants brought up the issues they would have to consider if and when they were to use the same problem to teach. I concentrate my analysis on participants' comments about what students should or should not remember if they were to use a problem for teaching properties of quadrilaterals. Moreover, I identify what teachers consider to be normative teaching practices toward managing students' prior knowledge when using a problem to teach new geometric content.

Altogether, the two studies showcase teaching actions and teachers' commentary on teaching actions. The two studies thus inspect teaching practices and the attitude that the profession has towards teaching practices. Methodologically, the two studies show alternative ways to address questions on how teachers make use of students' prior knowledge when teaching with a problem.

Dissertation Overview

The rest of the dissertation is organized as follows. Chapter two provides an overview of related literature, including references on the study of practices that involve collective remembering in other areas besides education. I also present a framework for the study of teaching that makes it possible to study collective remembering in relation to mathematical tasks and how issues regarding remembering and forgetting surface in activities of teaching within the geometry course. In chapter three I lay out the methodological aspects of the study including the selection of data sources and the tools for the analysis of data. In chapter four I present the analysis of classroom data and show how the teacher manipulated the memory of the class by means of discursive actions. In chapter five I present the analysis of the focus group data and identify how teachers activate memories of prior knowledge and control what students will remember with different kinds of actions. In chapter six I integrate the two studies into an answer of the question of how teachers manage students' prior knowledge, providing the conclusions of the study. This chapter also includes a reflection on the methodological contributions of this work. In this final chapter I also elaborate on implications of this dissertation for the study of teaching, teacher education, and policy.

Summary

This chapter describes the purpose of the dissertation to examine how geometry teachers build a representation of the past by attempting to impact students' collective memory while at the same time utilizing feedback from students' actual memories. I examine records of practice from a problem-based teaching unit implemented in two geometry classes and records from focus group sessions where geometry teachers discussed records like those. In each of these studies, I use the hypothesis that geometry teachers manipulate the past to explain how they manage students' prior knowledge.

CHAPTER 2

REVIEW OF LITERATURE AND THEORETICAL FRAMEWORK

In this chapter I review related literature and elaborate on how I use different sources in setting the theoretical framework that guides the two studies. First, I present a framework for the study of teaching. Within this framework, I study how prior knowledge is connected to mathematical tasks. Then, I discuss how collective memory and collective remembering are useful constructs in studying how classes deal with the past. From this discussion I elaborate on studies on remembering in discourse. I suggest how these studies could be helpful in analyzing remembering and forgetting in teaching geometry. I also review a set of psychological studies on factors associated with how children and adolescents encode and retrieve prior memories, because these suggest other ways in which teachers could influence students' remembrances in addition to conversations. In addition, I review implications for teaching from studies on students' prior knowledge. Then, I review studies on remembering and forgetting in relation to teaching, including specific examples about mathematics teaching. Finally, I discuss issues pertaining to remembering and forgetting in activities of teaching within the geometry course. With this discussion, I frame the study of remembering in the activity of teaching geometry with a problem.

A Framework for the Study of Teaching

Task and Situation

I take the notion of *mathematical task* to be useful in understanding how teachers make use of students' prior knowledge in problem-based instruction. Prior research in mathematics education has centered on studying the place of mathematical tasks in teaching (Christiansen, 1997; Henningsen & Stein, 1997; Herbst, 2003, 2006; Stein, Grover, & Henningsen, 1996). This work has shown that even when teachers pose meaningful problems to their students, they need to work to sustain the cognitive demands of those problems as they unfold over time. I draw on Herbst's notion of task as a "specific *unit of meaning* (i.e., the actions and interactions with the symbolic environment) that constitutes the intellectual context in which individuals think about the mathematical ideas at stake in a problem" (Herbst, 2006, p. 315). A mathematical task is the space of possible encounters of a student with a mathematical problem—the deployment of a problem as a cognitive activity over time. Thus, the same problem may elicit different tasks, depending on the resources used, the operations involved, and the products sought and achieved.

Herbst (2006) has used the expression "instructional situation" to name each of the various systems of norms that frame how chunks of work (one or more tasks) are exchanged for claims on knowledge in a class. Students and teachers engage in interactive work over time; that work can be symbolically traded for claims on the knowledge at stake. Teachers, in particular, use students' work to attest that learning has happened or that knowledge of something exists in the class. Different kinds of work trade for different kinds of claims on knowledge; thus what counts as valuable mathematical work varies. Instructional situations are frames in the sense that Goffman (1974/1986) gave to the word "frame." Frames assist participants of a social event to interpret that event and to know what they have to do.

One such instructional situation in geometry classrooms is the situation of *doing proofs* (Herbst, 2002; Herbst & Brach, 2006; Herbst, Chen, Weiss, & González, in press), in which students produce a proof in response to a problem posed by the teacher. Another situation is *installing a theorem* (Herbst & Nachlieli, 2007) that includes activities by which a new theorem becomes part of the public knowledge in a geometry class. These situations frame recurrent interactions in which similar kinds of work are exchanged for similar claims on knowledge. They save the teacher and student from having to explicitly negotiate the contract for the particulars that they are handling.

In contrast with these instructional situations that are usual in the geometry class, the activity of teaching geometry with a problem is unusual and requires the teacher to negotiate the contract for a task (Herbst, 2006). By negotiating the contract it means that the teacher and the students attempt to find some value in the work that they do by labeling their work according to customary patterns of interactions of an instructional situation. The patterns of interactions in an instructional situation are implicit as part of the didactical contract. The *didactical contract* (Brousseau, 1997) binds teachers and students together around the content by specifying (in a general way) the kinds of obligations teachers and students have to each other and to content in a class.

In this dissertation, I inspect a case of a teacher's management of prior knowledge when teaching with a problem. My conjecture is that teachers might evoke memories from different places in a timeline regarding students' immediate or remote past and teachers consider those immediate events within the geometry class to be more significant (Lemke, 2000). In setting expectations for what students should remember, teachers constrain students' prior knowledge to that knowledge which is part of the collective memory of the geometry class, making students disregard prior knowledge from other mathematics courses or individual knowledge that they may posses. Geometry teachers could employ different objects to link long-term and short-term events. These objects, which could be artifacts (e.g. notes in students' notebook, textbooks, and diagrams) or conceptual objects (e.g. definitions and theorems), enable students to remember things or events from the geometry class. Thus, in conceiving classrooms as places where there are symbolic exchanges, I hypothesize that when students use knowledge from the remote past, students' work on mathematical tasks is not perceived as valuable by teachers.

I am interested in how teachers use the collective memory in launching, negotiating, sustaining, and closing mathematical tasks. In particular, I examine three elements to model a task from the observer's perspective: *resources, products*, and *operations* (Doyle, 1988; Herbst, 2003, 2006). Students who work on a mathematical task are expected to achieve certain products by using some resources and by performing certain operations. Shared knowledge from the past—such as available mathematical language or proven propositions—can be part of the resources. I focus then on how teachers manage prior knowledge to hold students accountable for what they should remember and what they should forget in setting up resources and operations, and in achieving certain products when teaching with a problem.

Practical Rationality of Mathematics Teaching

Through this work I intend to make a contribution to our understanding of teaching as a practice that happens over time. Cohen, Raudenbush, and Ball (2003) argue that teachers and students use the resources they have available through their interactions around content. One of those resources is the curriculum. The geometry curriculum sets constraints and possibilities for teachers to engage students in solving problems differently than other mathematics classes. One of the expectations of the geometry course is that teachers teach students to appreciate Euclidean geometry as an example of an axiomatic system in mathematics. In encountering new mathematical knowledge, students take into account only those theorems, postulates, and definitions that had been already introduced in class (González & Herbst, 2006). The way teachers manage students' prior knowledge illustrates how teachers deploy tactical and strategic moves to fulfill the demands of their work.

Prior work in the study of teaching has centered on describing teachers' actions as a result of their beliefs and judgments (Cooney & Shealy, 1997; Cooney, Shealy, & Arvold, 1998; Ernest, 1989; Isenberg, 1990; King, 2000; Pajares, 1992; Shavelson, 1981). This work has been helpful in illustrating personal characteristics of teachers that could shape their work such as their views about mathematics or their perspectives about learning. There has been work on teacher thinking with the goal of describing individual teaching actions at the moment of making decisions (Borko, Atwood, & Shavelson, 1979; Clark, & Peterson, 1986). Recent work on the study of teaching describes teaching as a cultural activity where teachers transmit and transform their practices over time according to shared values (Jacobs & Morita, 2002; Santagata & Stigler, 2000;

Stigler & Hiebert, 1999). These researchers' novel perspective focuses on teaching as an activity rather than as the expression of personal traits of individual teachers. In my work I embrace the perspective that teaching is an activity that needs to be understood independently of whether it achieves learning, even when it depends upon the existence of a learner and the expectation that the learner should learn (Fenstermacher, 1985).

Herbst and Chazan (2003) have proposed the construct of *practical rationality of mathematics teaching* to denote the "wisdom of practice" of teachers who teach the same course of studies. Because teachers who teach the same course of studies are accountable to similar curricular and institutional obligations in their work, it is conceivable that they would perceive and appreciate teaching actions in a similar way. Teachers of the same course of studies can utilize the same set of dispositions in responding to the norms of particular instructional situations. In particular, they hypothesize that teachers of the same course of studies draw from the same set of resources to construct attitudes toward what a teacher should do or should not do in an instructional situation.

Within this framework, I place my work in studying how teachers manage students' prior knowledge when teaching with a problem. I inquire on the norms and dispositions that regulate how teachers manage students' prior knowledge within the activity of teaching geometry with a problem. As geometry teachers manage students' prior knowledge, they could deploy tactical or strategic moves that in effect shape the collective memory of the class. The distinction between *tactical* and *strategic* moves (Erickson, 2004) is important, because it points to actions that a teacher can control beforehand, or that a teacher would perform during the enactment of a task. By tactical moves, I mean teaching actions that take into account the moment-by-moment interactions with the students in the class. By strategic moves, I mean teaching actions that respond to a plan. So, in looking for elements of the practical rationality when teaching with a problem, I study possible teaching actions to manage students' prior knowledge during the enactment of a task and the attitude that the profession has towards those actions. This goal follows the path of prior studies on teachers' decision-making, in looking at choices that teachers make or the choices that teachers perceive when commenting on action.

Different reform movements in mathematics education have attempted to change teaching practices disregarding the role played by existing practices. A problem with these initiatives is that clashes between what the reform asks teachers to do and what teachers are able to do prevent teachers from fulfilling the expectations of the reform (Wilson, 2003). An agenda for reform in mathematics education could start by understanding those practices that teachers currently do in order to distinguish what is viable from what is desirable. In order to improve mathematics education, research on teaching could provide a language to talk about teaching practices and a stance that would give more insights into what teaching is like (Chazan & Ball, 1999). My goal is to unveil elements that characterize the practical rationality of geometry teachers with regards to how teachers manage students' prior knowledge. For example, I intend to describe what actions teachers do to activate or to deauthorize students to use prior knowledge as students work on a problem. In addition, I intend to study whether it matters *when* teachers do those actions. If so, teaching actions would point to decision moments for a teacher—at particular times in the enactment of a mathematical task—that could change students' work towards solving a problem according to their memories of past things or events.

Summary

My study of how teachers manage students' prior knowledge when teaching with a problem is situated in a larger theoretical perspective that looks at classrooms as places where there are symbolic exchanges and at the work of teaching as one of managing exchanges between work and knowledge claims. I make the hypothesis that the kinds of resources that students have available when working on a problem vary according to temporal constraints. At the same time, teachers assign value to students' work, trading students' work for claims on knowledge. I conjecture that when students rely on prior memories that are different from those memories that teachers utilize to represent the past, teachers do not take the work that results from that work as valuable. I use my hypothesis to explain why teachers might limit students' memory to a privileged representation of the past, setting boundaries to the kinds of resources and operations students have available in solving a problem.

I take the notion of *practical rationality of mathematics teaching* to be a useful construct in understanding those practices that teachers do as part of their work. I describe teaching as adaptation to specific instructional situations. Teachers, despite their individual differences, share similar constraints and possibilities that shape the resources available for those instructional situations. I explore how temporal demands in the work of teaching impact how they manage students' prior knowledge.

A Framework to Study Memory in Connection to the Work of Teaching

Collective Memory and Collective Remembering

In setting the theoretical framework guiding this dissertation, I draw from literature on memory from different sources, but with the common assumptions that individual memories are shaped by social experiences and that there are practices which sustain those memories shared by a group. Mary Douglas' (1986) work characterizing social institutions demonstrates that institutions keep their stability by setting boundaries to the memory of individuals and also by making use of systems of classification, which highlight important elements of what is known. This work suggests that memories transcend the individual because of social processes connected to what individuals remember.

I use Halbwachs' (1992) and Wertsch's (2002) work in my theoretical framework because they give credence to the notion of collective memory by illustrating how social institutions can shape the memories of individuals. These perspectives set the tone for studying how geometry teachers play a role in creating a representation of the past. In teaching, teachers face tensions when attending individual students, while at the same time attending the whole class (Lampert, 2001). Therefore, one could expect that a representation of the past could be helpful for a teacher to manage the complexities of dealing with students' individual memories of prior knowledge needed to work on a problem. In what follows, I describe how Halbwachs' and Wertsch's work have been useful in my work to study how teachers manage students' prior knowledge.

Halbwachs' Notion of Collective Memory

Halbwachs uses the term "collective memory" to denote those frameworks that a group uses for reading past things or events. He argues that personal memories are intertwined with those memories shared by a group. Yet, even when individuals within the same group could possibly have different memories than those memories avowedly shared, society strives to bring about coherence through frameworks of collective memory.

> The individual calls recollections to mind by relying on the frameworks of social memory. In other words, the various groups that compose society are capable at every moment of reconstructing their past. But, as we have seen, they most frequently distort that past in the act of reconstructing it. There are surely many facts, and many details of certain facts, that the individual would forget if others did not keep their memory alive for him. But, on the other hand, society can live only if there is a sufficient unity of outlooks among the individuals and groups comprising it. (Halbwachs, 1992, p. 182)

Halbwachs' examples of families, religious groups, and social classes illustrate how frameworks of collective memory provide continuity of experience by making individuals within each group accountable for remembering and forgetting the past. Different social groups select and manipulate past remembrances according to collective frameworks. Thus, Halbwachs argues that even though memories appear to be individual, these are a property of the collective because individuals within the same group share the same values towards their representation of the past.

Halbwachs elaborates on the idea of frameworks of collective memory by examining differences between two social classes: the nobility and the bourgeoisie. According to Halbwachs, the noble keeps its stability through symbols that represent his rank such as titles, portraits, and coats of arms. Those symbols sustain the memories that bring together the noble class. In contrast, the bourgeois attempts to keep its newly acquired status in society by erasing its remote past. Hence, the bourgeois limits past memories to those memories within the immediate past. In the absence of symbols from a remote past that would point to a belonging to the nobility class, the bourgeois places more importance than the noble in valuing its members according to other standards, such as the morality of recent actions. Thus, what is remembered and what is forgotten is related to the values that societal groups assign to those memories.

Although the composition of a geometry class is of a different sort than the groups under Hallbwachs' study—families, religious groups, and members of a social class—it is possible that teachers would manufacture a collective memory for their geometry class and that this would be different than the memories of individual students. By imposing a collective memory a teacher could avoid dealing with possible differences between individual students' memories. For example, in a geometry class a teacher could have students share the same geometric terms, symbols, and operations. In doing so, teachers could be communicating to their students which memories are valuable.

Wertsch's Notion of Collective Remembering

Wertsch (1998, 2002) has elaborated Halbwachs' notion of collective memory by examining how individuals make use of cultural tools that mediate the act of remembering. By mediated action, he refers to possible interactions between an agent and those material means that provide constraints and affordances in achieving some goals, such as the goal of representing the past (Wertsch, 1998). Wertsch argues that because cultural tools pertain to a group, remembering is a collective activity more so than an individual activity. He proposes that the availability of cultural tools, such as language, shapes what individuals could remember. Wertsch uses the expression "collective remembering" to refer to actions involved in making a group remember the past. He argues that collective remembering is dynamic rather than static, because the act of remembering transforms collective memory.

In his book *Voices of Collective Remembering,* Wertsch (2002) examines how narratives from Soviet Russia and post-Soviet Russia served as cultural tools for transmitting an ideology of the past. These narratives gave some continuity to the reading of past events by keeping the same structure in light of changes to the content of the narratives. During the Soviet era, the state intended to influence the reading of the past by controlling collective remembering through official narratives. Individuals took these narratives produced by the state as true, without disagreeing publicly about their content. To remember was a dangerous endeavor, particularly, if it was to contradict unequivocal official narratives. However, the state had to frequently rewrite their recount of historical events according to changes in the political landscape. By keeping the same narrative structure the state could impose a reading of the past in spite of changes in the content of the narratives. Towards the end of the Soviet era, individuals inserted their own voices into the narratives, blurring differences between private and public opinion. This change in how individuals positioned themselves before narratives of the past continued to depend upon memories shared by a collective that would give an alternate reading of past events than the one proposed and controlled by the state.

Wertsch's perspective on remembering seems relevant in studying how geometry teachers make students accountable for remembering past things or events. Geometry teachers could make use of material means to mediate students' remembrances. For example, geometry teachers could make use of diagrams to filter those things that students should remember and those things that students should forget. In doing so, geometry teachers could bring about regularity to those memories that individual students possess, shaping what they collectively should remember. In addition, because the geometry class shares the same resources to retrieve prior memories—such as geometric terms, procedures, and diagrams—remembering in the geometry class is bound to be a collective rather than an individual activity.

Although there are conceivably many differences between a teacher and a state apparatus in the purposes and in the resources for creating and for imposing a representation of the past, the notion of mediation tools could be useful in understanding how geometry teachers make students remember something as they work on a problem. In posing a problem, geometry teachers could call forth prior knowledge that is relevant for understanding that problem. Geometry teachers could create a representation of the past by making use of different mediation tools such as common terms and symbols. Students then could deploy those mediation tools to remember what they need to know in solving a problem.

In sum, the notions of collective memory and collective remembering could help in understanding how teachers manage students' prior knowledge when teaching new geometric content with a problem. In particular, these notions could be helpful in understanding whether a shared representation of the past is useful for teachers to handle different senses of time. Herbst asserts that,

> There is one temporality or sense of time which deals with the articulation of old and new in instruction, namely, what the child had the chance to learn yesterday or last year comes before what he had the chance to learn today or this year. This carries with it one sense of memory. Another sense of memory is tied to the temporality associated with the implicative structure of mathematics, where knowing that something is true now enables you to deduce later that something else is true. (P. Herbst, personal communication, November 26, 2006)

These two different forms of temporality could become evident when teaching geometry with a problem, because even when students possess prior knowledge that they could apply in solving a problem, teachers might expect students to disregard that prior knowledge because it is not shared by the whole class.

Remembering in Discourse

In this section I review studies that analyze the phenomenon of remembering in discourse. In particular, I review studies on remembering in conversations, which identify the resources speakers make use of in engaging others with the past during social interactions. These studies suggest ways to study how geometry teachers make students accountable for remembering in talking to their students.

Prior studies suggest that remembering is a direct or an implicit resource in discourse. I have reviewed studies that apply methodologies from two different discourse analysis traditions. One set of studies draws from Conversation Analysis to examine how interlocutors engage in acts of remembering in informal conversations among friends, within parent-and-child interactions, and in the workplace (Goodwin, 1987; Middleton, 1997; Middleton & Brown, 2005). The other set of studies can be described as applications of Systemic Functional Linguistics (Halliday & Matthiessen, 2004). Among them, one study examines how newspaper editorials make a representation of past events. Another study investigates literacy practices between mothers and children, revealing what resources mothers use when recalling past events in connection to stories that they read to their children.

While reviewing prior studies on remembering in discourse I consider how geometry teachers make students remember and lay out important considerations in my study. Particularly, I examine what resources interlocutors make use of in reference to prior memories and what methods could be applicable in the analysis of classroom discourse. I admit that there are differences between the studies I review and studies of remembering in classrooms because of special characteristics of the work of teaching. Teachers interact with the whole class as much as with individual students. In contrast, studies on parent-child relationships involve one-on-one interactions and studies in the workplace use data from conversations in small teams. Also, teachers meet with students on a regular basis over an extended period of time (one semester or one school year), which contrasts with other relationships in the workplace that could be more or less sporadic and with parental relationships, which last a lifetime. Moreover, teachers' position is different than that of parents because teachers have the main goal of teaching students a particular subject matter, geometry. That is, teachers are accountable to the discipline they teach. So the specialization of the geometry teacher contrasts with the lack of specialization of the parents, who deal with many more issues in their relationship with their children than geometry teachers with their students. In spite of these differences, I find points of contact between the studies I reviewed and my work: discourse is the medium where the remembering happens. So, language operates as a resource for remembering. I take the analysis of discourse as central in understanding how remembering could engage interlocutors in creating (and utilizing) a representation of the past in the present.

Studies on Collective Remembering using Conversation Analysis

Typical of research in the tradition of Conversation Analysis, Goodwin (1987) analyzes an informal discussion where a speaker's failure to recall someone's name while relating a story involves others in the conversation in the goal of remembering. That speaker's forgetfulness has consequences in other speakers' reactions in successive turns. Goodwin shows how speakers cope with patterns of social interaction through emergent tactical moves in a conversation. Other speakers engaged with the goal of remembering because of that speaker's difficulty in recalling the past. The speaker who failed to remember turned to his wife, inviting her to help him remember. With that invitation, the speaker made a distinction between those who know the story and those who do not. In the case of the speaker, him and his wife shared a story that others did not necessarily know. But, with the invitation for his wife to speak, he opened the possibility for others to contribute to the story, thus engaging them into the conversation. In the conversation, the speaker recalled the name, and then asked for verification of that name. With that move, he kept control of his position as the narrator of the story. This example shows how interactions in conversations are affected by the shared knowledge that those participants of the conversation may possess.

Middleton and Brown inquire into practices that involve collective remembering through the analysis of discourse from the perspective of social psychology (Middleton, 1997; Middleton & Brown, 2005). They apply Conversation Analysis to examine how memories of past events influence interlocutors' understanding of current events in social interactions. Their analysis shows how language becomes a trigger for past memories: At times past memories become the explicit focus of a conversation, with the intention of co-opting others into remembering. A speaker may impose a particular reading of past events by presenting evidence to sustain their avowedly shared remembrances. According to Middleton, "Conversational remembering involves contesting versions and details in the creation of accounts and the interpretation we place on them through argument and persuasion" (1997, p. 75). This perspective is similar to Halbwachs' notion of collective memory in that there could be different versions of past events, but one version of the past prevails against other versions, as speakers deploy rhetorical devices for imposing a privileged version of the past.

A significant feature of using remembering in conversations is that a speaker could commit others to the consequences of embracing a particular account of the past. For example, Middleton (1997) presents an excerpt of a conversation within a medical team where one doctor recalls agreed upon decisions about medical practices in a prior meeting. A second doctor concurs with the first doctor, giving more evidence about their agreement on implementing a procedure. Then, the first doctor reveals that another doctor (who is not part of the conversation) neglected that procedure. The first doctor's remembrance of a collective agreement contrasts with that doctor's personal recount of an exchange with another doctor who disregarded their prior agreement. At the point where the first doctor brings about personal memories, others in the conversations were already co-opted into remembering their prior agreement. This example reveals that speakers could draw upon personal memories or shared memories in order to seek agreement. When speakers make use of avowedly shared memories, they take these as resources for building a persuasive argument, co-opting other speakers into the consequences of sharing the same representation of the past in the present.

The notion that interlocutors could establish mutual commitments according to memories of past events through rhetorical means has relevance in understanding teachers' work managing mathematical tasks. In classroom discussions, teachers could bring about memories with the purpose of holding students accountable for using prior knowledge when solving a problem. The question of what rhetorical resources geometry teachers call forth to co-opt students into accepting a representation of the past is central in this dissertation, regardless of how accurate that representation of the past is.

Besides rhetorical resources, interlocutors could employ artifacts to trigger remembrances. This is relevant to the teaching of geometry because in making students recall prior knowledge, teachers could make use of diagrams to highlight properties of geometric figures that students ought to know. Middleton (1997) shows an example of a conversation between a mother and a child where they look at a photograph to surface memories of a family daytrip. In this conversation, the mother makes the child recall some of his feelings when riding a horse. Through questions, the mother contrasts her son's feelings with feelings that she attributes to other members of the family. The mother orients her son's memories, giving an interpretation to those past events captured in the photograph. One could expect a similar kind of interaction between teachers and students by means of a diagram, where a teacher manipulates prior knowledge that students should posses through questions about the diagram. In this case, the teacher might expect students to make statements about the diagram based on theorems and postulates already studied in class. Teachers could also make use of contested information that different students provide about a diagram to provoke students to recall prior knowledge relevant in solving a problem.

In sum, interlocutors could trigger remembrances of past things and events through conversation. In doing so, interlocutors show expectations about what can be remembered and what can be forgotten. At the same time, interlocutors could impose a privileged version of the past according to the evidence they deploy in conversations. Interlocutors could also make use of artifacts to trigger remembrances during a conversation. These elements, which emerge from applying Conversation Analysis to examine collective remembering, could be useful in studying the kinds of resources that geometry teachers call forth in making students accountable for remembering what they ought to know when doing a problem. Thus, through class discussions, teachers could co-opt students into the goal of remembering prior knowledge as part of those emergent tactical moves in teaching, where the timing of teachers' actions affect students' work.

Applying Systemic Functional Linguistics to Study Remembering in Discourse

Systemic Functional Linguistics (SFL) is the theory of language developed by M.A.K. Halliday (Halliday & Matthiessen, 2004). Proponents of this theory postulate that speakers realize meanings through language choices. So, the analysis of discourse from the perspective of SFL involves studying what resources speakers utilize to create meaning. In particular, from the perspective of SFL, speakers' linguistic choices allow them to construct different kinds of meanings according to different functions of language: (1) **interpersonal** metafunction "to enact relationships," (2) **ideational** metafunction "to represent experience," and (3) **textual** metafunction "to organize text" (Martin & Rose, 2007, p. 7).

I find in SFL useful resources for a comprehensive analysis of discourse where speakers make use of remembrances. Even though there are not many studies that analyze remembering and forgetting using SFL, I discuss the findings of some studies which draw upon SFL to show what resources speakers call forth to construct a representation of the past.

In her analysis on how newspapers editorials from Uruguay describe the events of September 11, Achugar (2004) shows textual resources that writers utilize to characterize those actors involved in the events. She examines how speakers take different evaluative stances towards the past through discourse according to different ideologies. The study of evaluative stances focuses on speakers' ways of appraising people, things, or events and it is related to the **interpersonal** metafunction of language. By evoking judgments and feelings, different editorials construct a view of "them" against "us," revealing that their reading of past events was not neutral and reflected a privileged view of the past.

From Achugar's study, I take that it is possible to analyze teachers' evaluative stances towards prior knowledge and the way by which those evaluative stances influence how teachers make students accountable for remembering and for using prior knowledge. In my analysis of classroom discourse, I find it useful to examine whether teachers value students' prior knowledge differently according to what basis students use to legitimize knowledge claims. For example, teachers could demonstrate different evaluative stances about knowledge from the geometry class and other mathematics courses, or about individual students' memories and those memories purportedly shared by the class. Thus, the attitudes that teachers show towards different knowledge claims could shape what prior knowledge students decide to call forth in solving a problem.

In a study of literacy practices in early childhood, Williams (2001, 2005) reports differences between literary practices of mothers from a Lower-Autonomy Professional (LAP) group and mothers from a Higher-Autonomy Professional (HAP) group.[2] This division into two groups has to do with "the relative autonomy of the family breadwinners to exercise power in the workplace" (Williams, 2005, p. 18). For example, the LAP group included mothers with professions such as factory assistant, waitress, and clerk. In contrast, the HAP included mothers with professions such as teacher, secretary, and medical specialist. For the HAP group, mothers explicitly engaged their children into using remembrances during joint book-reading activities. Through questions, mothers drew children's attention to prior events in the story. HAP mothers also asked children to remember things and events that were outside of the scope of the story. HAP mothers tended to help their children in understanding a story by using children's remembrances as resources. In doing so, mothers from the HAP group socialized their children into reading by connecting children's past experiences with fictional events. In contrast, mothers from the LAP group made fewer references to objects that were not part of the text and their references were very brief. It is possible that, similar to the mothers

[2] This study draws from a prior study by Hasan, reported in Hasan and Cloran (1990).

from the HAP group, teachers could use prefacing to make explicit connections to prior knowledge.

I draw from Williams' study methodological tools for considering whether teachers point to remembrances from the geometry class or from prior mathematics classes. For example, Williams (2005) identified resources within the **textual** metafunction in order to make connections between past and new events in the mothers' reading of a story. In particular, he studied the logical connections that mothers would establish in their talk by *prefacing* (e.g. "Remember when we looked at the ferry at Balmain?"). Then, he classified whether the objects referred to were internal to the text (what he called metaphorically an "anaphoric reference") or external to the text (what he called metaphorically an "exophoric" reference). He also studied the relationships between references by using *lexical cohesion analysis*, a resource related to the **ideational** metafunction, to trace semantic links in a text. In addition, he studied who was the agent of the interaction. That is, who—the mother or the child—would bring about remembrances in the discussion. In my analysis I study how teachers draw connections with prior knowledge and if teachers expect students to make use of knowledge from the geometry class more so than knowledge from other sources.

In sum, studies that draw from Systemic Functional Linguistics identify particular language resources that speakers could utilize in order to make a representation of past events. For example, speakers could make use of the appraisal system of language to encode their evaluative stances toward remembrances. Speakers could also organize discourse by pointing to past things or events previously mentioned in a text or outside of a text. In doing so, speakers could use identification to keep track people or things previously said or known. Similarly, teachers could make use of these resources for making students remember, revealing what expectations they hold about how students should use prior knowledge in solving a problem. In the following section I consider actions that may induce children to remember or forget past things or events besides discursive resources.

Studies on Factors that Shape Children's Remembrances

In this section I review prior psychological studies geared towards understanding factors that shape remembrances, in particular with children and adolescents (Greenhoot, 2000; Johnson, Greenhoot, Glisky, & McCloskey, 2005; Saylor, Baird, & Gallerani, 2006). I find these studies relevant because they suggest resources that teachers could employ in calling forth students' remembrances. In addition, while other studies on collective remembering examine how institutions could shape the memory of individuals, these studies investigate specific strategies in one-on-one relationships with children.

In a study by Greenhoot (2000) children listened to stories where a main character was involved in a series of events. The stories did not include enough evidence to support whether the main character in the story provoked those events or not. Interviewers made a deliberate attempt to manipulate children's memories by controlling their prior knowledge. Each child received positive, negative, or neutral information about the main character, before they listened to the story. Afterwards, children shared their perspectives about the main characters in an interview. In following sessions, children listened to another story, which gave more evidence with either positive or negative descriptions about the main character's actions. Children usually attributed intentionality to the actions of the characters when the initial description of the characters was either positive or negative. Children with neutral information about the main character did not have positive or negative impressions in a coherent way afterwards. This study suggests that children filter new information according to their prior knowledge and that researchers were able to manipulate what children remembered.

Another study by Saylor and colleagues (2006) suggests that the organization and classification of new knowledge could affect what children remember. Researchers showed a video to young children and then asked them to recall events in the video. Prompts, such as voice-over narrations preceding the actions in the video and actors' lines of dialogue, helped children remember. When researchers presented videos of the same events without those prompts but with wordless music, there was more variability in children's answers to a video-retelling task. Saylor and colleagues conclude, "children primarily relied on the video script in structuring their retellings" (p. 358). This finding suggests that the prompts used in the video to organize information affected what children remembered.

The study of how individuals cope with prior memories involve not just those things and events that people could remember, but also what factors are related to forgetting. A study with adolescents who were victims of depression or abuse reveals that they tend to suppress negative remembrances (Johnson, Greenhoot, Glisky, & McCloskey, 2005). The interviewers had to use various prompts to elicit adolescents' recollection of the past, suggesting that they make use of forgetting as a resource to control unpleasant memories. So, even though this study involves a special population, different from a more general population of high school geometry students, I am interested in this study as an example of how individuals may forget events from the past.

In conclusion, prior studies on how children and adolescent remember and forget reveal that there are factors that shape memories such as the classification of new information and the prior knowledge that individuals possess. Children's ability to incorporate new information increases with age and is supported by the prompts used to recall remembrances. Adolescents could manipulate not just what they remember but also what they forget, such as unpleasant remembrances. I understand that these studies report on situations with different goals and expectations for remembering and forgetting than those of classroom situations. However, I find their methodology useful because researchers attempted to manipulate prior memories.

These studies suggest that even when geometry students could bring about prior knowledge from other mathematics classes, teachers could manipulate those memories by using different prompts to recall prior knowledge or to prevent students from using prior knowledge. A big part of the work of teaching involves organizing new knowledge by setting systems of classification of knowledge, by labeling new knowledge as such, by highlighting important things to be remembered, and by making explicit those connections between prior knowledge and new knowledge (Brach, 2004). In these actions, teachers make new knowledge memorable to students so that they could hold students accountable later for applying that knowledge in solving a problem.

Summary

In order to study how teachers make use of a representation of the past to teach with a problem, I review literature on social practices that involve collective remembering. I elaborate on the notion that individual memories are connected to those memories shared by a group because of common resources to encode and to retrieve those remembrances. Prior work that looks into collective remembering examines how individuals make use of those common resources, such as narratives, to remember. Prior studies on remembering in discourse tell us that speakers could make use of conversational resources to co-opt others into a version of the past through emergent tactical moves. Also, speakers could impose an ideology of the past by making use of evaluative stances towards remembrances. There are possibly other means to manipulate prior memories besides discourse such as the classification of new knowledge.

Implications for Teaching in Studies about Students' Prior Knowledge

Studies on students' prior knowledge focus on assessing how prior knowledge affects students' experiences learning something new. In a literature review of 183 studies on prior knowledge by Dochy, Segers, and Behl (1999), they found that, in general, students' prior knowledge affected positively their performance in assessment measures. However, performance was related to the kind of instrument used to measure students' prior knowledge. In the implications for teaching, the authors suggest that prior knowledge should be activated and that this activation should happen at a particular time. The authors say,

> From this review, we can conclude that prior knowledge is indeed an effective aid for learning new knowledge. This result supports the current practice of activating prior knowledge at the beginning of a learning process…. In problem based learning and the problem method, for example, activating prior knowledge is an explicit phase. These students' reflection on their prior knowledge is facilitating learning. Likewise, students' reflection on what knowledge is

important for the learning process probably enhances learning. (Dochy, Segers, & Behl, 1999, p. 173)

So, the suggestions specify that the activation of prior knowledge should happen before starting an instructional activity. In addition, the activation of prior knowledge in activities that involve problem-based instruction is one example of how instruction should accommodate to the learner.

The recommendation of drawing upon students' prior knowledge also appears in the report *How people learn* by the National Research Council (2000). An important conclusion of the report is that students' misconceptions could make it difficult for them to learn something new. The authors say, "Teacher can help students change their original conceptions by helping students make their thinking visible so that misconceptions can be corrected and so that students can be encouraged to think beyond the specific problem or to think about variations on the problem" (p. 78). Therefore, the prior knowledge that students possess includes misconceptions that a teacher should try to surface in order to correct them.

From these studies one could conclude that it is important for teachers to activate prior knowledge. On the one hand, teachers could call forth knowledge that students will make use of when working on a problem. Teachers, on the other hand, could identify misconceptions that students would need to correct when learning something new. According to these studies, activating prior knowledge should precede problem-based instruction. In addition, problems can provide opportunities for students to correct their misconceptions.

Remembering and Forgetting in Relation to Teaching

In this section I start by reviewing examples from the teaching of science and social science, which suggest that a teacher could assume an active role in shaping students' memories in a class. Even though these studies focus on students' memories, their implications involve a study of activities of teaching that influence students' memories. Finally, I review studies that are specific about how teachers manage students' prior knowledge in mathematics classrooms. These studies take the perspective of the work of a teacher by describing how teachers handle the tension between individual memories that students' possess and the memories of the class.

Students' Memories about Things and Events in a Classroom

Even though my work focuses on the study of mathematics teaching, prior work in other subject areas is relevant to this work because it suggests that teachers create a representation of the past in helping students to connect old knowledge and new knowledge. In a study of a social studies unit in an elementary classroom, Nuthall (2000a) notes that the teacher of that unit held students accountable for remembering what they had discussed in class. At the same time, the teacher posed questions to students that were leading them to elaborate on their answers, in remembering what they had studied in the past. Nuthall concludes,

> The way the teacher managed the class discussion and structured the subsequent report developed the children's expectations that their knowledge should be elaborated with reasons and implications, and that they should be able to recall selectively what they knew in order to carry out the required tasks. Developing

> these ways of organizing memory and recall is part of becoming an "expert" in classroom routines. (Nuthall, 2000a, p. 48)

Nuthall shows a case for which a teacher's expectations about what students should remember from prior class discussions had leverage in what students brought up in subsequent discussions. The teacher evaluated students' responses and asked students to elaborate their answers so as to shape the memory of the class to conform her expectations.

Nuthall's (2000b) study on how students remember things and events in science and social studies is of interest here because of the methodology involved in the analysis of classroom episodes and his findings. Nuthall applies *genre* theory to the study of classroom interactions. By genre he means particular patterns of interaction that could be evident in discourse. Nuthall found that students tended to remember or to forget things and events in their class according to students' participation in activities within a genre. Moreover, students' memories matched teachers' expectations for what students should remember and forget in activities pertaining to a particular genre. Nuthall argues that there are genre-like structures that organize students' experiences. As a consequence, students develop their expertise dealing with the expectations about what to remember and what to forget in different classroom activities.

In addition, Nuthall found that at times students blurred differences between their individual memories and others' memories. This finding gives evidence to the building of a collective memory through classroom interactions. However, at other times, students remembered things that did not happen in class. Also, students remembered things in the long-term interview, which they had not mentioned in a test taken immediately after the unit or during the unit. According to Nuthall these findings make it more complex to characterize students' memories as stemming solely from classroom interactions because students may not always give evidence for what they remember about classroom events and because sometimes what students remember did not really happen.

In sum, the studies by Nuthall consider students' perspectives about things and events that they remember or forget. In spite of the focus on learning instead of on teaching, Nuthall's studies are useful in my work because they suggest teaching actions geared towards engaging students in remembering and forgetting. According to Nuthall, one way in which a teacher can shape students' memories is through classroom dialogue. Another way is by setting up expectations about students' memories in relation to the activities involved in a particular genre. These findings suggest that there are deliberate actions of a teacher that could affect what students remember and also what students forget. Also, students' memories could vary in different classroom activities according to the expectations set by the teacher. Therefore, one could expect that different classroom activities would have different norms about the memory.

Individual and Collective Memories in a Mathematics Class

In studying teaching practice, Lampert (2001) proposes that the memory of past events impacts current relationships between students and teachers. She argues that the work of teaching involves constantly reshaping individual or collective memories to ensure continuity across time.

> The memory of the teacher and the memories of the students, as well as their anticipations of where they are going, separately and together, in the near and longer-term future, contribute to the knowledge that informs how the teacher and

> students manage relationships with content in any particular moment. Again, the teacher can make use of individual and collective memory and anticipation, or not, but how memories and anticipations are managed affects what can be learned. (Lampert, 2001, p. 428)

Some traditional practices in teaching that rely on the collective memory include teachers' explicit or implicit moves to encourage students to remember what they had learned in the past such as highlighting technical terms or asking students to recall past events. For example, in a study of geometry teachers, Brach (2004) identifies particular teaching moves geared towards making students remember things studied in the past or towards making something memorable for the future. Also, Marie-Jeanne Perrin-Glorian (1993) has identified what she calls situations of *recall* (*Fr. situations de rappel*) where teachers retrieve past knowledge of immediate and remote memories.

Within the French paradigm of Didactics of Mathematics, prior work has focused on issues of memory in relation to teaching. Brousseau and Centeno (1991) argue that a teacher can control students' memories about that prior knowledge which is relevant in solving a problem. They talk about the *teacher's didactic memory* as an element in the work of the teacher that could allow students to organize new knowledge. They found that the absence of the teacher's memory about prior knowledge affected how the teacher organized students' connections between prior knowledge and new knowledge, requiring of the teacher to make those connections explicit.

Flückiger (2005) has used the notion of *pupil's didactic memory* to denote those moments where a student recalls and makes use of prior knowledge when working on a problem. She identified three kinds of recall, prompted by students: recalling a previous event, recalling a previous result, and recognizing a new class of problems because of students' connections with prior knowledge. This work shows that part of the teachers' work is to connect students' prior knowledge with new knowledge, and that at times, teachers can provide opportunities for students to make those connections on their own.

Summary

In sum, teachers can play an active role making students remember or forget prior knowledge. Past memories could influence students' new work on a problem. At the same time, these memories could help a teacher anticipate the kind of work that students would be doing in the future. In these studies, I examine how teachers manage students' prior memories in a particular activity of teaching within the geometry class: the teaching of new geometric content with a problem. In order to do this, I start reviewing issues that pertain to remembering and forgetting in one of the instructional situations found in the geometry class: doing proofs.

Remembering and Forgetting in the Geometry Class

In this section I review results of a previous study on the situation of *doing proofs* that are related to students' prior knowledge. These results are relevant to establish a comparison between a stable situation within the geometry course—the situation of doing proofs—and a novel activity in geometry classrooms— teaching geometry with a problem. It is possible that the two activities of teaching would involve different demands on memory. With this in mind, I end this section with a discussion that anticipates issues of memory that may surface in the activity of teaching geometry with a problem.

Remembering within the Situation of "Doing Proofs"

Prior work investigating the situation of doing proofs has revealed that there are usual practices regulating the division of labor—who has to do what—when doing a proof (Herbst, 2002; Herbst & Brach, 2006). With regards to remembering, teachers are usually accountable for providing all the objects that need to be used in the proof. This includes laying out triggers for students' activation of the prior knowledge relevant for doing a proof. Students, on the other hand, need to remember the definitions and theorems so triggered.

Based upon a set of interviews with geometry students, Herbst and Brach (2006) reported that students expect teachers to cue them to use those ideas that they will need in a proof. Moreover, students stated that they do not expect to make use of prior knowledge that does not belong to the geometry class in a proof.

> With regards to concepts, students recognized, within their share of labor, the expectation that they be able to translate diagrammed objects into concepts and search their memory for definitions and statements of properties. However, they did not expect to have to bring in concepts not identified by the problem or pointed to by the diagram, make conceptual connections that had not been featured in the course, or bring in very old concepts from memory. (Herbst & Brach, 2006, p. 95)

This finding suggests that to consider whether a concept is relevant in doing a proof, students take into account the temporal proximity between the proof problem and the moment when these concepts featured in the course. In the interviews, students revealed that concepts studied much earlier in the geometry course are usually not relevant when doing a proof, unless students are explicitly prompted to use those concepts. Also, students said that they tend to disregard geometric concepts that they have studied earlier in past mathematics courses when working on a proof. The demands on memory seems to depend on the timescales of the events to be remembered (Lemke, 2000), one timescale governing events within the geometry course and another timescale which locates all mathematics courses, including the geometry course as events in time. However, students tend to take as more significant those concepts that they have studied recently in the geometry course.

In addition, there are temporality issues regarding differences between using prior knowledge and foreshadowing new knowledge within the situation of doing proofs. Herbst and Brach (2006) reported that students were more amenable to the idea that proofs required them to apply prior knowledge sanctioned within the geometry class than to the idea that proofs would require them to use knowledge they anticipated to be true.

> The data allow us to see that problems, which required students to anticipate a future result and use it as a justification for a step of an argument, might appear to them as illegitimate for reasons that have to do with the norms of the situation (not to use anything they do not yet know.) (Herbst & Brach, 2006, p. 102)

Students in the interviews revealed that the boundaries for the kinds of resources they could make use of in a proof are not only related to prior knowledge, but also related to possible anticipations of knowledge.

These findings suggest that there are norms that regulate what kind of knowledge students could draw upon as a resource within the situation of doing proofs. One could expect that in teaching, teachers have an important role in shaping the collective memory, actually creating a preferred representation of the past of the class to conform the expectations for proving.

Remembering when Teaching Geometry with a Problem

In contrast with the situation of doing proofs, teaching geometry with a problem is uncommon. Whereas the idea that one could teach with problems is appealing and some examples exist of courses that developed norms and routines to study mathematical concepts from problems (Lampert, 2001), teaching new ideas with problems is not customary. Problems are used to practice old ideas, but not to develop new ones. Prior work has demonstrated that the expectation that a problem be used as a context for students to make a *reasoned conjecture* created difficulties regarding the division of labor between teacher and students (Herbst, 2006). Customary practices in that geometry class did not include an instructional situation where students could be held accountable for making a conjecture and building the grounds for its reasonableness, and as a result the task changed into one that could be contained within situations that existed in that class—students were first encouraged to make a conjecture without attention to reasons, then after the conjecture was sanctioned they were asked to prove it deductively. This finding is consistent with prior work that suggests that in geometry classrooms the work of conjecturing and the work of proving are conceived as separate from each other (Chazan, 1995). The larger point is that one cannot speak of teaching new material with a problem in the same way one speaks of "doing proofs" in geometry. Doing proofs is a stable, sustainable instructional situation, but teaching with problems is not. So if in one class a teacher wants to use a problem to teach a new idea, he or she would have to negotiate how the didactical contract applies for that particular problem.

One of the challenges of teaching with problems seems to be that of altering the sequence by which the textbook organizes the content of the geometry course. Lampert (1993) reports that geometry teachers, "rely on textbooks for guidance about the order in which to introduce topics, because they assume that the books present material in the order in that geometrical knowledge logically develops" (p. 154). Thus, teaching with a problem would require teachers to cope with possible changes in the sequence of topics established in the geometry curriculum. The way students work on a problem may require them to draw upon knowledge that has not been covered in class yet; and teachers would need to alter the sequence of topics in the course to support students' ideas in solving a problem.

One could expect that the problem of managing students' prior knowledge would be more salient in teaching with problems than in doing proofs, because there seems to be more flexibility in deciding what kind of knowledge students can draw upon as a resource in solving problems than in doing proofs. In these studies, I examine whether there are differences in the way teachers value students' prior knowledge when teaching with problems and when engaging students in proving. If so, I would characterize those differences according to the temporal markers of that prior knowledge that could be used (or not) as a resource in doing a mathematical task.

Summary

In my work, I compare and contrast two activities of teaching within the geometry class in terms of how teachers manage students' prior knowledge: *doing proofs* and *teaching with a problem.* Doing proofs is a usual activity in geometry classroom, with a set of norms regulating what teachers and students should do. However, teaching geometry with a problem is an unusual activity in geometry classrooms. Prior studies reveal that it is difficult for a teacher who attempts to use a problem to teach new content to sustain this activity over time, and make claims that students learned something new as a result of their work on a problem. The contrast between the two activities of teaching provides an opportunity to examine what demands on memory would be useful in order for teachers to draw from students' prior knowledge to teach new content with a problem. If the reform wants teachers to engage in problem-based instruction, then it is important that research examines what would make this activity viable in mathematics classrooms.

CHAPTER 3

METHODOLOGY

I use empirical data of two kinds to ground discussions about teaching actions to manage students' prior knowledge when teaching geometry with a problem. I examine two kinds of video-records: classroom episodes of two geometry classes and focus group sessions with geometry teachers.

The classroom episodes are an example of problem-based instruction. By studying classroom episodes I analyze how a teacher holds students accountable for remembering what they should know to do a problem from an observer's perspective. The focus group data, on the other hand, provides practitioners' perspectives on action as they considered the possibility of using a problem to teach properties of quadrilaterals. Participants commented on videos of a teacher using a problem to teach geometry and considered various mathematical tasks to teach with that problem. I distill normative statements on what teachers hold themselves accountable for doing to make students remember. My analysis of teachers' statements that involve how they draw upon the collective memory in envisioning mathematical tasks aggregates teachers' practices and teachers' dispositions towards those practices.

Data Collection and Analysis of Study with Classroom Data

Data Sources

Classroom Episodes

The classroom episodes are video records of an “instructional experiment” (Herbst, 2006) in two sections of honors geometry (2nd period and 7th period) taught by the same teacher, Megan Keating.[3] This instructional experiment replaced the textbook unit on quadrilaterals with a collection of 12 lessons in consecutive days.[4] The unit is described as an instructional experiment because during the unit the class experienced changes to their usual routines—it was thus an experiment in a sociological sense, building on the notion of “breaching experiment” of ethnomethodologists (Garfinkel, 1967; Mehan & Wood, 1975), not an experiment in the sense usually given to the word in psychological research. Students did not use their textbook during the unit as they usually do, neither in the presentation of content nor in the assignment of homework.

The unit started with students working on a problem for several days in small groups. The solution to this problem required a theorem that students had not studied in class yet but that they would encounter during the unit. Students had access to dynamic geometry software in graphing calculators throughout the unit; they could take the calculators home to explore further some of the questions posed in class. The homework assignments included some open-ended questions mixed with the usual questions from the textbook. Since the homework problems were distributed to students in separate worksheets, students were not able to tell apart the problems that came from the textbook

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[3] All names are pseudonyms.
[4] The design of the unit included problems and activities authored in collaboration by Dr. Herbst, the teacher, and myself.

and non-routine problems. While some problems required students to apply knowledge they had just studied, other problems required them to anticipate knowledge about properties of quadrilaterals that they had not studied in class yet.

Prior to the replacement unit, the class had studied parallelograms and their properties, using their textbook (Boyd, Burrill, Cummins, Kanold, & Malloy, 1998). We gave an assessment instrument to students before starting the unit. This assessment showed that some individual students possessed prior knowledge about properties of special quadrilaterals. For example, all students answered correctly that a square and that a rectangle have two pairs of parallel sides, and 84% of students answered correctly that a rhombus has two pairs of parallel sides. To the question of whether the diagonals of a special quadrilateral bisect each other, 96% answered correctly that this is true in a square; 84% answered correctly that this is true in a rectangle; and 88% answered correctly that this is true in a rhombus. However, when asked to give a definition of a rectangle, only 10% of the answers included sufficient information about what a rectangle "has" or what a rectangle "is." Most definitions (45%) were insufficient. So even though students possessed some prior knowledge about quadrilaterals, they did not necessarily know the sufficient conditions that define a special quadrilateral.

Our group had previously collected video-records of intact geometry lessons taught by Megan Keating— lessons where we had not done anything to alter the instruction. In examining those records our research group had seen that she usually follows the sequence of topics set by the textbook and that, in spite of the fact that they might have studied figures in middle or elementary school, students would not be expected to make use of prior knowledge about geometric figures before these figures are introduced in class. The textbook officially introduced special quadrilaterals and their properties in the quadrilaterals chapter.

In conversations when planning the unit, Megan was aware that the replacement unit would be different than the regular unit on quadrilaterals in that students would be able to draw upon their prior knowledge about properties of quadrilaterals. Megan was also aware that the unit would provide opportunities for students to investigate properties of quadrilaterals that are traditionally taught in the geometry course, as they were useful in solving problems, but not necessarily in the same order set by their textbook.

Description of the Unit

The replacement unit started with defining a midpoint quadrilateral, namely an "m-quad," as a quadrilateral made by connecting successive midpoints of a quadrilateral, and posing the question, "What quadrilateral would you need to start from in order to get an interesting m-quad?" This kind of question, asking students whether they could get an "interesting" figure, was unusual in Megan's class. Students speculated about would make the midpoint quadrilateral an interesting one. Megan had regarded this problem as an opportunity to introduce the medial-line theorem, which is usually taught later in the year. The medial-line theorem states that the segment connecting the midpoints of two sides of a triangle is half of the length of the third side and parallel to the third side. This theorem helps prove that the midpoint quadrilateral is always a parallelogram. By posing the question at the beginning of the unit students could conjecture that the midpoint quadrilateral would be a parallelogram. To prove that conjecture, they might need the medial-line theorem. But the medial-line theorem was not going to be discussed until the end of the unit. Throughout the unit, students worked on this problem that was conceived as an opportunity to teach properties of quadrilaterals. Students started to make conjectures about the midpoint quadrilaterals since the first day of the unit. However, they were not able to prove this conjecture because they lacked sufficient resources to do so. The unit gave the opportunity for students to want to prove the medial-line theorem in order to solve other problems. But we expected conflicts if students were to take for granted some prior knowledge that had not been officially introduced by the teacher yet. My analysis focuses on those moments of conflict where the teacher and the students differed about what constituted prior knowledge.

Methodology for the Analysis of Data

For the analysis of data I select episodes where there was an apparent conflict between what the teacher and the students took as prior knowledge. With that purpose, I perform task analysis. I model a task by studying what *resources* and *operations* could be deployed to achieve some *products* (Doyle, 1988; Herbst, 2003, 2006). Task analysis a priori (i.e., without use of information on the actual enactment of a task) allows me to anticipate what resources and operations students could have made use of to solve a problem. Moreover, I can identify tasks that may require negotiations of the didactical contract because students might not possess the resources needed to solve the problem at hand. Then, task analysis of the enacted task allows me to identify negotiations of the task. These negotiations may result when there are disputes about the status of a statement. For example, students may take a statement for granted, but for the teacher that statement needs to be proven before the class can use it. I parse the selected episodes into segments according to changes in the task. I focus on conflicts between what the teacher and the students take as resources for a task: Students want to use a resource for a task that the teacher does not approve because it is not part of the collective memory of the class.

In my analysis of the classroom episodes I find ways in which remembering becomes relevant in conversations, particularly in how Megan stated her expectations for students to remember prior knowledge and in how Megan co-opted students in remembering. I use discourse analysis to identify remembering and forgetting in successive interactions between speakers. I draw resources from Systemic Functional Linguistics (Halliday & Matthiessen, 2004), to analyze how collective remembering makes use of resources from language to create a representation of the past. This theory of language postulates that there are three meta-functions of language: ideational, interpersonal, and textual. To fulfill these three meta-functions of language, there are different discourse systems such as conjunctions and appraisals. I look into particular discourse systems to get information about how memories of prior knowledge are realised in talk.

The content of memories is related to the ideational metafunction. The ideational metafunction pertains to how discourse conveys speakers' experiences in the world. A *figure* is the way in which the clause represents the world. As a template of the clause, a figure contains three elements: a Process,[5] Participants (of the Process), and Circumstances (Halliday & Matthiessen, 2004, p. 169). So, to examine what the memories are, I parse the text into clauses. Then, I identify Participants (people, things, or events) referred to in discourse. I identify Participants that refer to concepts and propositions studied in the geometry class. In addition, I identify clauses that involve

[5] I capitalize these terms to represent elements of discourse analysis.

mathematical statements about geometric properties. I also perform conjunction analysis to study the logical connections speakers use to organize experience (or to organize discourse). These experiences are related to prior knowledge of things and events from the past. In particular, when students are working on a task, they use resources for that task. Conjunctions enable them to connect the resources for a task with the purpose of making an argument.

The attitudes towards memories are related to the interpersonal metafunction. The linguistic system of appraisal realizes attitudinal meanings. So, I perform appraisal analysis to understand the evaluative stances that speakers use in their talk. In particular, I study what attitudes speakers convey towards claims that use prior knowledge. Possible conflicts between the teacher and the students regarding their attitudes for using prior knowledge are important, because they demonstrate differences in the resources from memory they are entitled to use.

The organization of the memories is related to textual metafunction. The system of conjunctions, in addition to the purpose of conveying meanings, also organizes talk to fulfill the textual metafunction. So, I look at the system of conjunctions to see how these help speakers to give logical "waves of information" (Martin & Rose, 2007, p. 116). In addition, I examine whether participants in discourse achieve cohesion by pointing to past things or events that belong to the immediate past or the remote past. Cohesion is related to the textual metafunction of language. According to Halliday and Hasan (1976), cohesion is a system that includes lexicogrammatical resources beyond the meanings in a clause and within larger textual units. At the same time cohesion includes the study of "semantic and contextual resource for creating and interpreting text"

(Halliday & Matthiessen, 2004, p. 532). Cohesion analysis includes identifying five elements in discourse: *conjunctions,*[6] *references, substitution, ellipsis* and *lexical organization*.[7] Speakers' use of reference, substitution, and ellipsis is important in my analysis to trace the appearance of the same Participant throughout the text, beyond a particular figure.

From this analysis I gather how the teacher and the students used language to manage changes in the temporal organization of knowledge—by relying on knowledge from the remote past or by relying on future memories—to support students' achievement of products. So, in my analysis, I start by identifying conjunctions. Since speakers may use conjunctions to connect different clauses, the conjunction analysis allows me to parse turns of speech in the transcript into clauses, using as a template for those clauses the notion of figure. Because speakers may refer to some of the elements of a figure in many ways, I draw from cohesion analysis tools to track how speakers maintain continuity within larger textual units, across clauses. I study the resources that the teacher and the

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[6] Halliday and Hasan (1976) include conjunctions as part of the resources for achieving lexical cohesion. However, conjunctions are also a system within the ideational metafunction of language to achieve meanings. This points to the dual role of conjunctions: conjunctions are used to convey logical meanings (a resource within the ideational metafunction) and also to organize text (a resource within the textual metafunction). In relation to conjunctions, Martin and Rose (2007) say, "CONJUNCTION in other words has two faces. One side of the system interacts with IDEATION, construing experience as logically organized sequence of activities. The other side of the system interacts with PERIODICITY, presenting discourse as logically organized waves of information" (p. 116). In my work, I use Martin and Rose's perspective on conjunctions as resources to convey logical meanings and also to organize text.

[7] Instead of "lexical organization," Halliday and Hasan (1976) call this element "lexical cohesion." I have decided to use the term by Halliday and Matthiessen (2004), "lexical organization," to denote that all these elements in a text help achieve lexical cohesion. Also, in contrast with Halliday and Hasan (1976), Halliday and Matthiesen group substitution and ellipsis as one cohesive type. In this work, I keep them separate.

students used in order to achieve cohesion in the transcripts of the episodes. With this analysis, I examine possible connections with prior knowledge in discourse through grammatical choices and through the choice of lexical terms. In particular, I focus on Megan's references to the shared knowledge of geometric properties and her use of a common vocabulary of terms associated with special quadrilaterals as they signal what the teacher expected students to hold as past knowledge of the class. By identifying *references*, I examine possible changes in ways of naming the same entity in order to solve a problem. The same geometric object could be conceived of as a different sort configuration, which could point (or not) to resources and operations helpful in solving a problem.

Finally, I apply appraisal analysis to examine Megan's evaluative stances towards students' knowledge claims. The system of appraisal is part of the interpersonal metafunction of language, and includes resources for evaluation. In my analysis of transcripts of classroom interactions, I study how the teacher and the students negotiate and value claims made when doing a proof. As students worked on a problem, they made different sorts of claims and Megan accepted, rejected, or modified those claims assigning value to students' mathematical work. By studying Megan's evaluative stances, I contrast some of the underlying norms regarding prior knowledge in the situation of doing proofs and when teaching geometry with problems.

Data Collection and Analysis of Study with Focus Group Data

Data Sources

The focus group data includes a collection of five sessions with geometry teachers from different schools. At the beginning of the three-hour session, named *Teaching with*

Problems, participants had the opportunity to work on the problem, "what can be said about the angle bisectors of a quadrilateral?" Then, teachers watched a collection of video clips where a teacher had worked on this problem with different geometry classes. These videos provided a common context for teachers to discuss different ways of using this problem in their own class.

The focus group sessions were intended to provoke conversations among geometry teachers in reaction to the videos of a teacher teaching with a problem (see Herbst & Chazan, 2003). In contrast with one-on-one interviews, the focus groups gave each participant the opportunity to calibrate their comments in relation to comments by other participants. A moderator, who capitalized on his or her experience teaching geometry, led the sessions and a researcher asked questions to probe the acceptability of participants' comments about the video (Nachlieli & Herbst, 2006). In the sessions, participants also reviewed other artifacts, such as samples of students' work and worksheets to teach with the angle bisectors problem. These artifacts were useful for teachers to talk about what they would need to do to teach with the angle bisectors problem.

The videos of a teacher using the angle bisectors problem in several geometry classes made participants comment on how students should be accountable for remembering and for using prior knowledge sanctioned by the class. Moreover, in looking at a video where a teacher had asked students to prove that the angle bisectors of a rectangle make a square, participants discussed how students should make use of their prior knowledge. Participants debated how to transform the mathematical task to cue students about the resources needed to solve such a problem. So, even when participants were not designing tasks for a particular class, they made statements about task design that disclose their ideology of the past.

The discussions in the focus group sessions with geometry teachers constitute a valuable data source to examine normative statements about teaching with problems. In my analysis I show actions associated with how teachers manage students' prior knowledge when teaching with a problem. By studying teachers' arguments about to how use of students' prior knowledge, I describe elements in the *practical rationality of mathematics teaching* in the actions that teachers describe when teaching geometry with a problem.

Criteria for Data Selection

The sessions are parsed into intervals. Intervals are changes in the activity structure in the session (see Herbst, Chazan, & Nachlieli, 2007). Intervals are emic units of analysis; they signal when participants changed the patterns of interaction during the session. It is conceivable that in each one of these changes of activity, participants could also change the topic of the conversation, but this is not necessarily the case. I select intervals from the transcripts of the focus group sessions where participants made statements about students' prior knowledge. In particular, I identify intervals where teachers described mathematical tasks with explicit references to the knowledge that students should or should not remember. For example, I code intervals where participants spoke about how to use the statement of a task to trigger the resources students need to work on a problem, or whether students should remember to draw diagrams when working on a problem. So, even though the tasks that participants designed were not for a particular class, in their reactions to the videos, participants demonstrated how they would draw upon students' prior knowledge if they were to use the angle bisectors problem in their class.

I code intervals according to teaching actions at different moments during the enactment of a task—before, during, and after students work on a problem. I select actions that a teacher could do within the time span of a lesson such as handling a diagram or managing a discussion, in contrast with actions outside of the time span of a lesson such as planning or grading students' work. As an example of an action that a teacher could do before students start working on a problem, I select the action of giving the statement of a task. I select two actions during students' work on the problem: making discursive moves and handling a diagram on the board. As an example of an action after students have concluded their work on the problem, I select naming a theorem and handling an incorrect diagram. These actions are not meant to be comprehensive, but examples of how teachers manage students' prior knowledge when teaching with a problem.

Methodology for the Analysis of Data

I focus on resources related to two metafunctions of language: the ideational metafunction and the interpersonal metafunction. In my analysis I look for ways in which participants reconstruct the stories presented in the video episodes, and also for ways in which they propose alternative stories. I distill these stories and examine the different attitudes they have towards teaching actions in the stories (see for example González & Herbst, 2008 or Nachlieli & Herbst, in press). By applying Analysis of Participation I identify, from participants' comments, their perceptions about what is the division of labor regarding who is responsible for activating memories, whether it is the teacher or the students. So I look at clauses where the agent is the teacher or where the agent is the student. In particular, I look at clauses where the teacher prompts students' actions. In addition, I look at temporal markers in participants' talk to see whether there are actions that teacher should perform at a particular moment in his or her interaction with students in a class. These temporal markers are important in order to identify possible tactical moves of a teacher in order to manage students' prior knowledge when teaching with a problem.

I analyze participants' evaluative stances towards statements regarding actions to manage students' prior knowledge by applying appraisal analysis (Martin & Rose, 2003; Martin & White, 2005). Further elaborations by Lemke (1998) with his notion of *attitudinal meanings* have been useful for me to identify how participants evaluated teaching actions for managing students' prior knowledge. There are seven possible evaluative orientations: (a) desirability, (b) probability, (c) normativity, (d) usuality, (e) importance, (f) comprehensibility, and (g) humorousness. I translate statements made by participants into statements that represent participants' evaluative orientation regarding teaching actions.

My choice of SFL as opposed to Conversation Analysis (CA) is related to the research questions guiding the two studies. Studies that use CA give evidence to the phenomenon that speakers co-opt others into remembering or forgetting through talk. In these studies, researchers identified linguistic resources that speakers used with the purpose of making others share a representation of the past through talk. In the first study I focus on how teachers make a shared representation of the past with the content of the talk. CA would be useful in the study of classroom interactions if one were to study resources that teachers could use for making students remember by performing specific speech acts such as stressing a word or gazing at interlocutors at a particular moment in the interaction. Brach's (2004) study of teaching actions to make students remember could be placed in this line of research because she identified linguistic resources that teachers deploy. In the second study, CA could be useful to study participants' speech acts in their agreements or disagreements about how to manage students' prior knowledge in their conversations about the classroom videos. However, my focus is not on participants' speech acts. The research questions required the analysis of participants' comments about prior knowledge as a theme of the conversation and the attitude about those comments. Therefore, I found in SFL the tools for the analysis of data to answer the research questions.

Overall, these analytical tools from SFL have enabled me to examine themes in conversations with participants regarding students' prior knowledge (or their lack of). In addition, I have applied SFL to study participants' evaluative stances towards teaching actions to make students remember prior knowledge when working on a problem.

Summary

In order to study how geometry teachers manage students' prior knowledge in relation to mathematical tasks, I investigate two kinds of data for the activity of teaching with a problem. One data set involves videos of an instructional unit where a teacher, Megan, made changes to her usual practices to make students accountable for remembering. I focus on episodes where there was a conflict between Megan and the students, because students brought about prior knowledge from other mathematics classes or because students anticipated knowledge. I use analytical tools from Systemic

Functional Linguistics to point to the linguistic resources that the teacher used in managing students' prior knowledge.

Another data set involves focus group data from five sessions where participants commented on videos of a teacher who used a problem to teach new geometric content about quadrilaterals. In their discussion, participants shared what expectations they have for students' use of prior knowledge. Participants also made statements about what they need to do in order to make students remember what they should know. I draw from Systemic Functional Linguistics tools for analyzing the transcripts of these sessions, looking for participants' references to the use of students' prior knowledge in relation to mathematics tasks and for their evaluative stances towards those statements about teaching actions that draw upon students' memories.

CHAPTER 4

A TEACHER'S ATTEMPT TO SHAPE THE COLLECTIVE MEMORY OF HER CLASS WHEN TEACHING WITH PROBLEMS: AN OBSERVER'S ACCOUNT OF CLASSROOM DATA

In order to answer the question of *how teachers manage students' prior knowledge when teaching geometry with a problem,* I analyze teaching practices from an observer's perspective. I have selected episodes from video records of classroom work with the purpose of describing a teacher's handling of students' prior knowledge. How does a teacher make use of students' prior knowledge when engaging them with a problem? What does a teacher expect students to remember in the future from their work on a problem? I use the collective memory construct to propose explanations for a teacher's actions when deciding what students should remember or what students should forget.

From a 12-day unit on quadrilaterals taught in two geometry classes by Megan Keating, I have chosen episodes where there was a conflict between what the teacher and the students considered to be prior knowledge that could be used as they worked on a mathematical task. The unit replaced the usual chapter on quadrilaterals in the textbook from Megan's class. During the unit, students did not use their textbooks and engaged in different classroom activities to learn the properties of quadrilaterals. We expected that a teacher's work of managing students' prior knowledge would become apparent when teaching the unit. The unit, designed as a problem-based inquiry, triggered students' prior knowledge from previous mathematics classes and from earlier experiences in the same unit. The conflicts between the teacher and the students followed activities in the unit where students were working on a problem. On the one hand, as they worked on mathematical tasks students made claims using prior knowledge. Megan, on the other hand, did not let students take these claims for granted, even though students were relying upon prior knowledge. From Megan's perspective, students did not know those claims yet.

The unit included some alterations to usual practices in Megan's class. One of these alterations involved how new topics were introduced. In our observations prior to the unit, Megan usually introduced concepts to students and then asked students to work on problems. However, during the unit, students encountered concepts as they worked through problems in their class. In some of the unit's activities, students might find it helpful to make use of prior knowledge from earlier mathematics courses in order to work on a problem. In addition, students might be able to anticipate knowledge that had not been officially introduced in class yet. We expected that these two circumstances—relying upon prior knowledge beyond the geometry course and anticipating new knowledge—would bring about conflicts between the teacher and the students. In particular, Megan would need to make some decisions about how to handle students' memories of prior knowledge and students' anticipations of new knowledge while engaging the class to work on problems. In this sense the unit was an experiment.

In the following three sections I discuss characteristics of the replacement unit that frame my analysis of the episodes in this chapter. I start by presenting what topics from the geometry class preceded the teaching of the unit. These topics suggest the prior knowledge from the geometry class that the teacher could expect students to possess. Then, I give an overview of the 12-day unit on quadrilaterals and explain how the unit was conceived as an opportunity to teach properties of quadrilaterals with a problem. In particular, I discuss how the replacement unit altered usual practices in Megan's geometry class. Finally, I present particular events in the unit that preceded the selected episodes. These events provide the background for understanding how activities around teaching with a problem provoked the conflicts in the selected episodes. After these first three sections, I provide a detailed analysis of the selected episodes. The results from this analysis give evidence for how Megan organized students' memories of past events when teaching with a problem.

Prior to the Quadrilaterals Replacement Unit in Megan's Geometry Class

A discussion of what topics had been officially introduced in Megan's geometry class before starting the unit is of interest here for understanding what prior knowledge the teacher and the students could take for granted. Prior to the quadrilaterals unit, Megan had covered chapters 1 to 5 by Boyd, Burrill, Cummins, Kanold, and Malloy (1998). Chapter 1 gives the basic postulates about points, lines, and planes. Chapter 2 introduces inductive and deductive reasoning, and also asks students to do their first proofs, using a two-column proof format. Chapter 3 includes theorems about perpendicular lines and parallel lines. Chapters 4 and 5 include theorems about triangle congruence.[8]

Immediately before the replacement unit on quadrilaterals, Megan's class had studied the first two sections of chapter 6, the chapter on quadrilaterals. In the first

[8] Appendix A includes a list of the titles of the chapters in the geometry textbook from Megan's class. This list suggests the topics included in each chapter.

section of chapter 6, the authors give a definition of parallelograms: "A parallelogram is a quadrilateral with *both opposite sides parallel"* (Boyd et al., 1998, p. 291). In the same section, the authors introduce four theorems about parallelograms. These theorems state that in a parallelogram opposite sides are congruent, opposite angles are congruent, consecutive angles are supplementary, and diagonals bisect each other. Students could apply these properties of parallelograms when solving problems that involve a parallelogram. So, if students were to know that a geometric figure is a parallelogram, they could deduce that that figure would possess those four properties.

The second section of chapter 6 mirrors the first section in that students study tests for parallelograms. That is, what are sufficient conditions to assert that that quadrilateral is a parallelogram? The authors present five possible conditions that would warrant asserting that a quadrilateral is a parallelogram: two pairs of parallel sides, two pairs of congruent sides, two pairs of congruent opposite angles, diagonals that bisect each other, or one pair of parallel and congruent sides. So, properties of parallelograms and tests for parallelograms were part of the official knowledge about quadrilaterals that had been introduced in Megan's class.

At the moment of starting the replacement unit on quadrilaterals, students were about to study special quadrilaterals and their properties. This is a standard topic in the geometry curriculum. In the textbook from Megan's class, the authors introduce special quadrilaterals after the two sections on parallelograms. The quadrilaterals chapter has sections for rectangles, squares, rhombi, and trapezoids.[9] The replacement unit also

[9] Some special quadrilaterals such as kites and "darts" are not included in the textbook from Megan's class in a separate section. For example, there is not a section for kites, but there is a construction activity about kites.

included the study the properties of these special quadrilaterals, but, different from the textbook, these properties were introduced within the context of problem-based instruction. In the following section I explain how the replacement unit altered usual practices in Megan's class by means of teaching properties of quadrilaterals with a problem.

Overview of the Quadrilaterals Replacement Unit: A Case of Teaching with Problems

One of the distinguishable features of the replacement unit on quadrilaterals is that the unit started with a problem and that this problem makes reference to a concept that is not usually emphasized in the geometry curriculum. Megan started the replacement unit by defining a *midpoint quadrilateral*, abbreviated "m-quad," as the quadrilateral made by connecting midpoints of consecutive sides in a given quadrilateral. The given quadrilateral was called the "outer quadrilateral," or the "o-quad." After introducing the definition of midpoint quadrilateral, Megan posed the question, "What quadrilateral would you need to start from in order to get an interesting m-quad?" In different activities through the unit, students continued to work on answering that question.

In contrast with special quadrilaterals (e.g., rectangles, squares, trapezoids, and rhombi), midpoint quadrilaterals do not usually feature as an object of study in the US geometry curriculum. For example, in the textbook used in Megan's class (Boyd et al., 1998), there is no definition of midpoint quadrilaterals; midpoint quadrilaterals do not have any other name that students are expected to remember. Moreover, the properties of midpoint quadrilaterals are not featured in the main text of any of the sections of the chapter on quadrilaterals of the textbook. Only one exercise makes references to midpoint quadrilaterals. However, the exercise does not give them a name. The exercise says,

> Draw any quadrilateral *ABCD* and connect the midpoints, *E, F, G, H* of the sides in order.
> a. Determine what kind of figure *EFGH* will be. Use the information from this lesson to prove your claim.
> b. Will the same reasoning work with five-sided polygons? Explain why or why not.
>
> (Boyd et al., 1998, p. 368, exercise 36)

The fact that the only reference to midpoint quadrilaterals appears in an exercise (as one of the last exercises), and not in the main text of a chapter, suggests that midpoint quadrilaterals are marginal; they are not among the objects of study around which the didactical contract for the geometry class is established. Since midpoint quadrilaterals are not objects of study in the didactical contract of the geometry class, a teacher would need to do something else to turn the work around the problem on midpoint quadrilaterals into valuable work. Otherwise, the work on the midpoint quadrilaterals problem would not count as having learned something from the geometry curriculum.

One difference between the problem posed by Megan to introduce the replacement unit on quadrilaterals and the exercise regarding midpoint quadrilaterals in the textbook has to do with the kinds of resources that Megan's students had available to answer the question. The exercise in the textbook asks for a proof and hints to use the theorems included in the section where the problem comes from. The exercise is in the chapter on similarity, and, in particular, in the section where the authors present the medial-line theorem. The medial-line theorem (a.k.a., the midpoint connector theorem) states that the segment connecting the midpoints of two sides of a triangle is parallel and half of the third side. At the time of the replacement unit on quadrilaterals, Megan's class had not studied the medial-line theorem yet. So, students did not have the medial-line theorem as a resource to work on the problem. The answer to the problem is another theorem stating that the midpoint quadrilateral of any quadrilateral is a parallelogram. As the reader probably knows, the medial-line theorem applied to the triangles determined by each diagonal of the original quadrilateral allows a straightforward proof of this theorem, also known as Varignon's theorem.[10]

A different solution to the midpoint quadrilateral problem would involve similarity. One could prove that there is a pair of similar triangles using the Side-Angle-Side similarity theorem. As a consequence of having similar triangles, corresponding angles of those triangles would be congruent. One could deduce that a side of the midpoint quadrilateral is parallel to one diagonal of the outer quadrilateral (see Figure 1 and Figure 2). Then, one could conclude that the midpoint quadrilateral of any quadrilateral is a parallelogram. While students had studied similarity in middle school, the concept of similarity had not been introduced in Megan's class yet. Also, students would need to take the initiative to draw in the diagonal of the outer quadrilateral, and then establish a relationship between the outer quadrilateral and the midpoint quadrilateral.

It is conceivable that some students would remember the concept of similarity from earlier mathematics classes. Students who were to remember similarity would bring prior knowledge from outside of Megan's geometry class. These students could observe that the ratio of two pairs of sides is 1:2, but they would need to establish a relationship

[10] Professional journals such as the *Mathematics Teacher* by the National Council of Teachers of Mathematics have featured articles discussing Varignon's theorem (see Oliver, 2001, for example).

between the third pair of sides. They probably had not studied the Side-Angle-Side Similarity theorem before. However, they would probably know that if corresponding sides of two figures are in the same ratio, then the figures are similar. So they would need to rely on visual perception or on measurements to notice that the segment connecting the midpoints of two consecutive sides of the outer quadrilateral (*EF* in Figure 1) and one diagonal connecting the vertices of those sides of the outer quadrilateral (*DB* in Figure 1) are in a 1:2 ratio as well. Then, they would conclude that there is a pair of similar triangles, in a 1:2 ratio. From that similarity, they could find pairs of congruent angles leading them toward concluding that opposite sides of the midpoint quadrilateral are parallel.

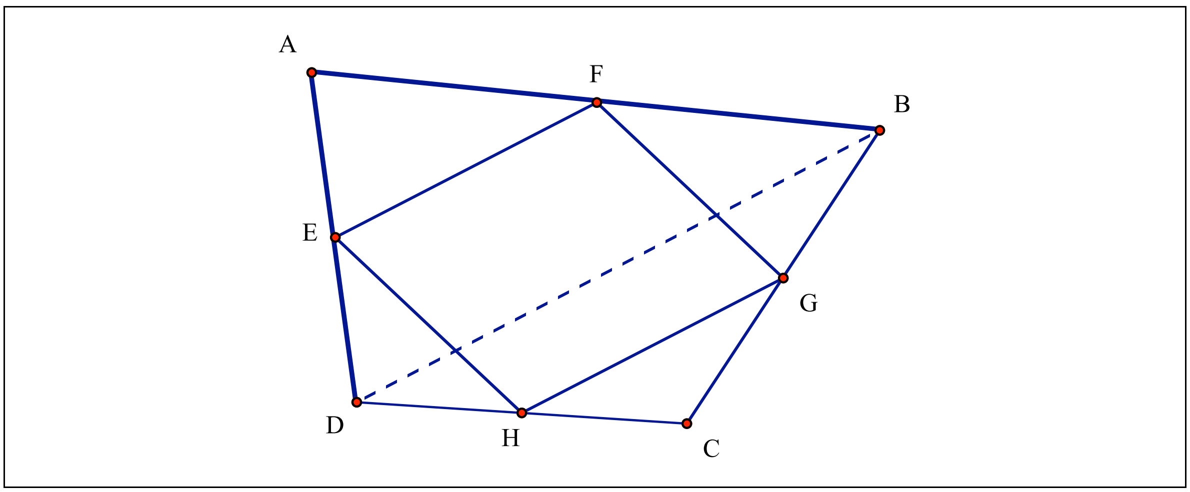

Figure 1. *ABCD* with a diagonal *DB* and its midpoint quadrilateral, *EFGH*.

Alternatively, students could use visual perception to notice that the sides of the midpoint quadrilateral are parallel. This observation does not rely upon drawing the diagonal of the original quadrilateral. So, in this case, students' reading of the diagram would allow them to see that the midpoint quadrilateral is a parallelogram.

In Figure 1, *E*, *F*, *G*, and *H* are the midpoints of the sides of quadrilateral *ABCD*. Triangles *AEF* and *ADB* are similar because two pairs of sides (*AE* and *AD*; *AF* and *AB*) are in the ratio of 1:2 and the included angle, Angle A, is the same in both triangles. So, *EF* and *DB* are in a ratio of 1:2. Also, since triangles *AEF* and *ADB* are similar, there are pairs of congruent angles, such as *AEF* and *ADB*. From this result, one could deduce that *EF* and *DB* are parallel. Because a similar result holds for triangles *CHG* and *CDB*, then *EF* and *HG* are both parallel to *DB*. In addition, *GH* and *DB* are in a ratio of 1:2. One could conclude that *EFGH* is a parallelogram because it has one pair of opposite sides congruent and parallel: *EF* and *HG.*

Figure 2. A proof for the claim that midpoint quadrilaterals are always parallelograms.

There are other reasons why the problem opening the unit was unusual. Students' initial work was intended to lead them to do a proof of the medial-line theorem in the last day of the unit. The proof at the end of the unit would be a proof by contradiction using triangle congruency and parallel lines. Proofs by contradiction were unusual in Megan's class.

In addition, the statement of the problem—*What quadrilateral would you need to start from in order to get an interesting M-Quad?*—was unusual. It asked to find something "interesting," without cueing students about what is means to have an interesting finding. To the extent that no official sense of "interesting" existed, it is conceivable that students might find any finding "interesting" even if these might not be mathematically interesting. For example, students could find something interesting about one diagram that could not be generalized for other figures, or students could use visual perception to point to specific parts of the midpoint quadrilateral. However, these findings may not lead to the formulation of a theorem applicable to many cases.

In sum, the unit altered usual practices in Megan's geometry class by drawing attention to a geometric figure that is not specially featured in the geometry curriculum, by presenting a problem for which the usual solution relies on a theorem that students had not studied yet, and by asking a strange question. Moreover, the unit started by posing a problem instead of asking students to apply concepts that they had already learned to solve a problem. So, the question did not appear to conform to the norms of the usual instructional situations of the geometry class such as doing a construction, an exploration, or a calculation.

We had predicted that those changes that the replacement unit brought about to the usual practices in Megan's geometry class, would perturb the way prior knowledge would be used in students' work. In some activities, students would be expected to draw upon their prior knowledge of special quadrilaterals and their properties, even though special quadrilaterals had not been officially introduced in Megan's class. Students had studied special quadrilaterals in other mathematics classes, and they showed some knowledge about properties of special quadrilaterals in an assessment we gave before starting the unit.

Another perturbation has to do with how students would start to conjecture that the midpoint quadrilateral of any quadrilateral is always a parallelogram, even though they only had perceptual evidence to sustain this conjecture. During the development of the unit, we expected students to remember the conjecture that had not been proven in class yet. The prolonged time between the moment when students started to gather evidence for the conjecture that all midpoint quadrilaterals (day 1) are parallelograms and the moment when they could prove that conjecture (day 12) was unusual. Usually, if students in Megan's class were to propose conjectures about geometric figures, then they would have the resources to prove (or reject) a conjecture immediately after the conjecture had been proposed. During the unit, we expected that students would remember the conjecture that all midpoint quadrilaterals are parallelogram for several days, and use it to build new ideas, without having produced a proof for that conjecture.

We had anticipated that the changes in the unit would provoke students to make two unusual actions: rely on prior knowledge from outside of the geometry class and anticipate knowledge that required a theorem that had not been proven yet. These two actions differ from how students in Megan's class usually relied on memories to solve a problem. As a result, we had expected conflicts between the teacher and the students. I selected two episodes from the replacement unit that showcase how these two actions provoked conflicts. In the following section, I present a chronology of events leading towards the two selected episodes.

Chronology of Events in the Quadrilaterals Replacement Unit Leading to the Selected Episodes

From the 12-day replacement unit on quadrilaterals taught by Megan in two geometry classes, I selected two episodes. One of the episodes, which I call "the rectangle episode," happened in the seventh period class, on the sixth day of the unit. The other episode, which I call "the kite episode," happened in the second period class, on the eleventh day of the unit. The rectangle episode showcases how a teacher manages students' prior knowledge about concepts that had not been officially introduced in the geometry class. Students remembered properties of special quadrilaterals from prior mathematics classes and from their work on a problem the previous day. The kite episode showcases how a teacher manages students' anticipation of new knowledge. Students remembered a conjecture for which they had gathered empirical evidence during the unit, but for which they had not yet supplied a proof. The analysis of these two episodes is the core of this chapter. Even though the two classes covered the same lessons in the same days, I present a separate chronology, with the purpose of connecting relevant events that preceded the selected episodes in each class.

Chronology of Events Leading to the Rectangle Episode: Conflicts when students rely on prior knowledge that had not been officially introduced

In the sixth day of the unit during the seventh period class, there was a conflict between the teacher and the students. The teacher used a homework problem as an opportunity to prove a property of diagonals of a rectangle. While doing the proof, students took that property for granted. In my analysis I demonstrate that students' lack of appreciation for the proof of that property of diagonals is related to what students remembered from an activity on the previous day. I use the episode to showcase the difficulties for a teacher when relying upon students' prior knowledge that has not been officially introduced in class yet. Prior to this conflict, some of the activities of the unit had activated students' prior knowledge about properties of special quadrilaterals as students worked on a problem. In particular, the day before the rectangle episode, the class played a game called "Guess My Quadrilateral." In this game students could make use of their prior knowledge about special quadrilaterals. The teacher's acceptance of students' prior knowledge during the play of the Guess My Quadrilateral game contrasts with the teacher's reluctance to let students take for granted properties of a rectangle in the rectangle episode.

I present brief summaries of events prior to the rectangle episode to illustrate how students' prior knowledge about properties of special quadrilaterals, and in particular, properties about the diagonals of special quadrilaterals, were part of the discussions in the seventh period class. In those discussions, the teacher did not show objections to students who brought about prior knowledge about properties of special quadrilaterals. To facilitate the understanding of the events of the Guess My Quadrilateral Game (day 5), I briefly narrate what had happened in the first and the second day of the unit

Day 1

In the first day of the unit, students in the seventh period class made diagrams of different quadrilaterals and their midpoint quadrilaterals (m-quads). Some students used names for special quadrilaterals to refer to their drawings and also to propose their conjectures. Others referred to properties of the quadrilaterals they had drawn, without naming specific quadrilaterals. For example, at the end of the class, Bart and Hu-yen went to the board to present the findings of their respective groups. Bart made references to parallelograms, which they had studied prior to the unit. Bart made a diagram of a convex quadrilateral with no specific features (no congruent sides and no congruent angles) and its midpoint quadrilateral (see Figure 3). Bart presented his conjecture to class: "If you drew a quadrilateral and no matter what–which quadrilateral you draw, as long as it doesn't have an angle above the measure of 180, you'll come out with a parallelogram inside."[11]

In contrast with Bart, Hu-yen referred to special quadrilaterals in her presentation. She had made a dart-looking quadrilateral and its midpoint quadrilateral. The dart-looking quadrilateral was drawn as a concave quadrilateral with exactly two distinct pairs

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[11] Bart's statement before the class was consistent with what he had written on his worksheet, which he submitted at the end of the class, "We believe that no matter what quadrilateral you draw, you will get a m-quad out of it that is a parallelogram." His worksheet included a drawing of a square and another quadrilateral like the one he presented at the board, with their respective midpoint quadrilaterals.

of adjacent congruent sides. Hu-yen said, “it looks like a trapezoid.” Bart replied, “It looks like a rectangle” (implicitly rejecting his own conjecture about concave quadrilaterals). When looking at Hu-yen’s diagram, Bart and Hu-yen identified them with names for special quadrilaterals. However, at this point, they did not talk about properties of special quadrilaterals. They seemed to rely upon a holistic, visual perception of the diagrams to recognize these figures. In that sense, students related to the diagrams as *icons* representing a geometric figure (Herbst, Hsu, Chen, González, & Jeppsen, 2007).

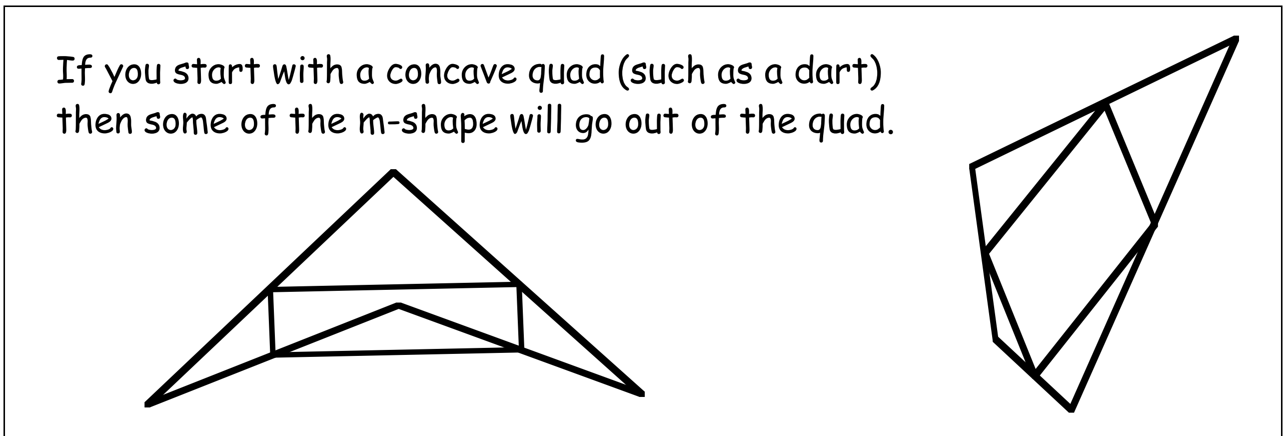


Figure 3. Hu-yen’s and Bart’s diagrams in the 7th period class.

Day 2

In the second day of the unit, students had to complete a table relating properties of the outside quadrilateral and properties of its midpoint quadrilateral (see Table 1). First, students completed the table in their own groups. At the end of the class, Megan led a discussion and compiled entries for the table from different groups. Megan called this list, “the master list,” and asked students to copy this list in their notebooks for them to work on homework.[12] With this request, Megan held students accountable for the results recorded by the whole class and not necessarily for the results obtained in each group.

Students in the seventh period included properties of diagonals in the sixth entry of the table (see Table 1). Earlier, Megan had led a discussion about properties of kites and "darts." Even though the textbook used in Megan's class does not include a definition of "darts," Megan treated these as special quadrilaterals. The class came up with a way to describe kites and "darts" in terms of a common property: a quadrilateral with two pairs of adjacent congruent sides. They recorded this property in the fifth entry of the table. Then, Megan pointed out that no group had said anything about the diagonals of a quadrilateral. Megan asked, "What seems to be true about the diagonals of kites?" A student, Anil, said that diagonals of a kite are perpendicular. Megan omitted any references to the diagonals of a rectangle. Instead, Megan illustrated with separate diagrams that the diagonals of a rhombus and the diagonals of a square are perpendicular. Then, she drew the midpoint quadrilateral of a square and the midpoint quadrilateral of a rhombus, concluding that in both quadrilaterals the midpoint quadrilateral is a rectangle. In the table, Megan recorded that if the outer quadrilateral has perpendicular diagonals, then the midpoint quadrilateral is a rectangle.

In the discussion, Megan did not consider other quadrilaterals that have perpendicular diagonals, but that do not have a special name. So, even though the table was meant to record the properties of an outer quadrilateral, class discussion centered on special quadrilaterals. The names of special quadrilaterals were connected with their

[12] The following day, Megan distributed copies of the table produced by the group to each student.

properties. Once the class had agreed upon what properties a special quadrilaterals possesses, then Megan would write on the table the properties of that outer quadrilateral instead of writing its name. This suggests that it unusual for the class to have a discussion without relating possible properties of quadrilaterals to quadrilaterals for which there is not a name. An alternative would be for the teacher to guide a discussion about properties of quadrilaterals without making references to names of special quadrilaterals. Then, the class would consider quadrilaterals for which there is not a name.

Table 1
Table of properties made by the seventh period class

Properties of O-Quad	**M-Quad**
1. Two pairs of congruent opposite sides	Parallelogram
2. Sides and angles are not congruent	Parallelogram
3. Four 90-degree angles	Rhombus
4. Four 90-degree angles and four congruent sides	Square
5. Two pairs of adjacent congruent sides	Rectangle
6. Perpendicular diagonals	Rectangle
7. Two pairs of congruent opposite angles	Parallelogram
8. No parallel sides	Parallelogram
9. Four congruent sides	Rectangle

Day 5-Beginning of class

In the fifth day of the unit, students played a game called "Guess my Quadrilateral." In that game, students had to ask the minimum number of yes/no questions about properties of a special quadrilateral that the teacher would draw from a bag and hide from their sight until they figure out what the hidden quadrilateral was. In the design of the unit, we intended that the game would provide an opportunity for students to discuss possible properties of quadrilaterals that could be used to define the special quadrilaterals. The game would also impress upon the students the value of finding conditions that were necessary and sufficient when writing a mathematical definition.

The game prompted students' memories about the names and the properties of special quadrilaterals. At the beginning of the seventh period class, Megan modeled the kind of questions students could ask when playing the game. Megan used the table of properties they had written in the second day of the unit to exemplify a possible question. In particular, she referred to the sixth entry of the table, with a question about the diagonals of a quadrilateral. This sample question by Megan is important because the question she chose to ask when modeling a play of the game is a property that some special quadrilaterals possess. With this question, Megan implicitly allowed students to rely on their memories from earlier mathematics courses or to develop new ideas about special quadrilaterals. At the moment, the class had not studied special quadrilaterals. However, Megan enabled students to use properties of special quadrilaterals publicly.

> Megan: I'll pull out a shape for [each] group […] And then each group is gonna get to ask me questions, yes or no questions, about their shape. And they can't, your question can't be, "Is it a square?" No. It has to be some of the properties. I want at least one person from each group to get out that list of properties we made before—the o-quad and the m-quad. You probably only need one out per group. Okay, I'm gonna print more so that you have one later […] Like, if you look at number… 6, "Does it have perpendicular diagonals?" And I'll say yes or no about the shape …

Megan illustrated the possible questions that students could pose by asking if the hidden quadrilateral has perpendicular diagonals. However, students in Megan's class had not officially studied which quadrilaterals have perpendicular diagonals or any other property besides those of parallelograms. Even though there are quadrilaterals with no specific names that may possess perpendicular diagonals, some special quadrilaterals do: a rhombus, a kite, and a square. In the two sections on parallelograms prior to the beginning of the unit, Megan's class had studied that in a parallelogram diagonals bisect each other. Also, they had studied that if diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram. Other properties that involve diagonals of a quadrilateral (for example, whether the diagonals are angle bisectors, or whether they are perpendicular) are characteristic of some special quadrilaterals. So, Megan, in her example, referred to a property that some special quadrilaterals possess and that they had mentioned when doing the table the day before. However, Megan had not covered in class quadrilaterals with perpendicular diagonals in the sections prior to the replacement unit.

Since Megan had discussed properties of parallelograms in class, it could be expected that students would make use of their prior knowledge about those properties when playing the game. Also, the exploration during the first two days of the unit, where students had made a table of properties relating an outer quadrilateral and its midpoint quadrilateral, allowed students to create other memories about properties of quadrilaterals. In particular, when completing the table, students relied upon their prior knowledge of names of special quadrilaterals and upon their visual perception of possible properties the figures they drew. The game tacitly prompted students to confront memories of possible properties of special quadrilaterals with the actual properties that those special quadrilaterals possess.

Before starting the play of the Guess My Quadrilateral Game, Megan reviewed the names of special quadrilaterals. Upon Megan's request, a student, Lily, went to the board to sketch special quadrilaterals. These sketches were iconic representations of special quadrilaterals (see Figure 4). In particular, the sketches did not have any markings to indicate properties. In a question and answer session with Megan, students recognized the figures by their names and added new figures that Lily had omitted.

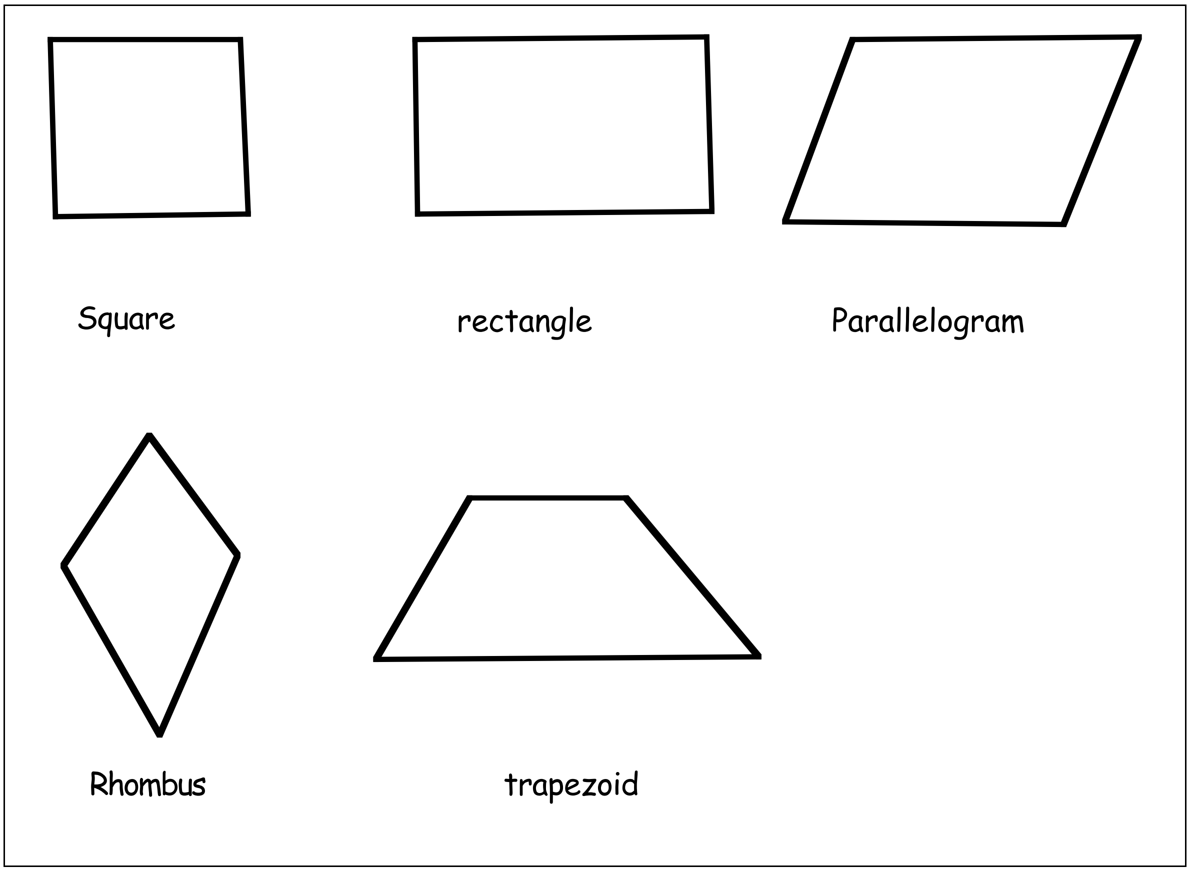


Figure 4. Diagrams by Lily prior to the play of the Guess My Quadrilateral game.

Megan corrected some possible misconceptions about special quadrilaterals. For example, Megan said that Lily's drawing of a rhombus looked like a kite and emphasized that trapezoids do not necessarily have a pair of congruent sides (see Figure 5). Megan also asked students what would a third-grader call a rhombus, and students replied, "a diamond." Then, Megan asked students about other possible special quadrilaterals. Students suggested a kite; they also proposed a "dart." Megan drew sketches for a kite and for a dart on the board. With this action of adding these two figures on the board, Megan accepted students' memories of a kite and of a dart, even though the dart is not one of the special quadrilaterals in their geometry textbook.

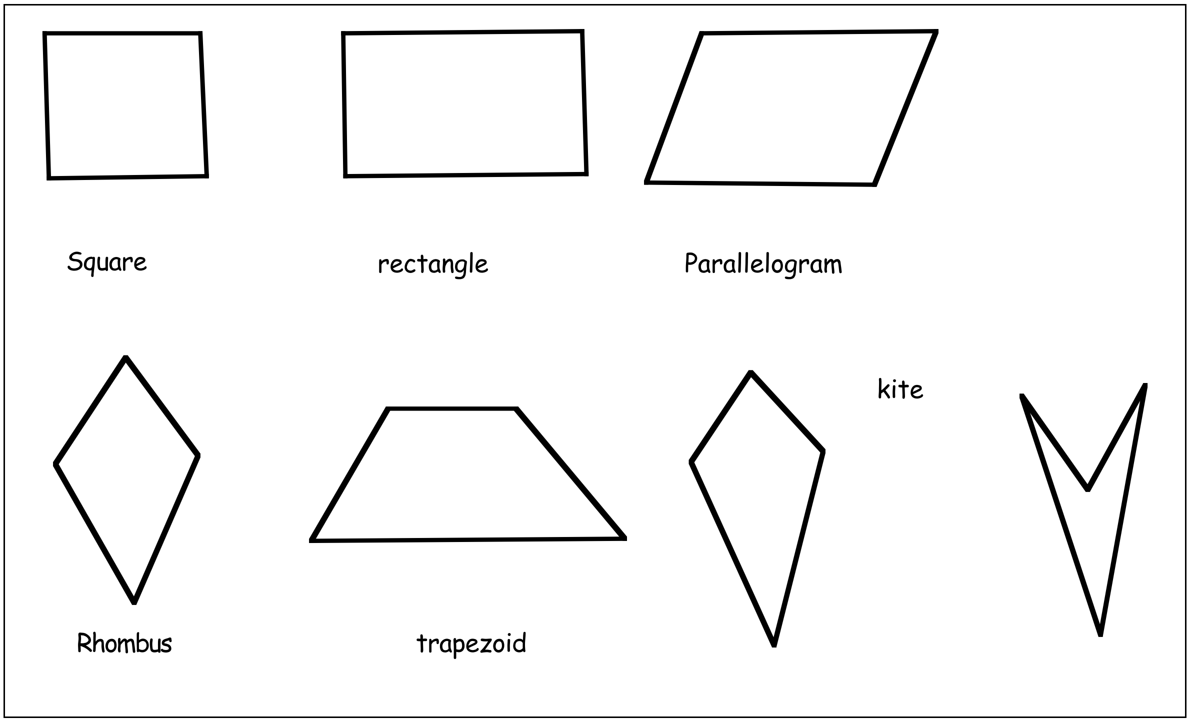


Figure 5. Modified diagrams before the play of the Guess My Quadrilateral game.

So, in preparation for the play of the Guess My Quadrilateral game, Megan made students remember their prior knowledge about special quadrilaterals. They used iconic representations to match each figure with the name of a special quadrilateral. Megan made references to informal terms to name the figures when she asked students to say how would a third grader would call a rhombus. She also corrected possible misconceptions by redrawing a rhombus so that it would appear different than a kite and by redrawing a trapezoid so it would not appear to be isosceles. Even though Megan did not provide a definition of special quadrilaterals, she made students remember what they knew about special quadrilaterals.

Day 5-End of class

At the end of the class all groups put on the board a poster with the questions they had asked. Students in the seventh period class asked a total of 67 questions in all small groups about properties of quadrilaterals. Table 2 summarizes the kinds of questions students asked. The most frequently asked questions were if the quadrilateral had four congruent angles and if the quadrilateral had four congruent sides. Only one group, formed by Chase, Hu-yen, Lawrence, and Tabitha, asked a question about the diagonals of a quadrilateral. They asked twice, "Are diagonals perpendicular bisectors of each other?" This question is different from the question Megan had posed earlier, when exemplifying how to play the game.[13] No other groups asked about the diagonals of the hidden quadrilateral. The property of having congruent diagonals would become important the next day, in the rectangle episode. However, in students' records of questions asked during the play of the game, there is no evidence that students considered whether diagonals are congruent as a plausible property of quadrilaterals.

[13] The question posed by Chase's group included more conditions than Megan's question: the question asked not only whether diagonals are perpendicular, but also whether they bisect each other.

Table 2

Questions asked in the Guess-my-Quadrilateral game in 7th period

Parts of the quadrilateral alluded in the question	**Totals**	**Property of quadrilateral asked for**	**Frequency**
Sides	35	Two pairs of opposite congruent sides	1
		Two pairs of congruent sides	3
		Four congruent sides	15
		Any parallel sides	3
		No parallel sides	1
		Two parallel sides	3
		Two pairs of parallel sides	9
Angles	30	Four congruent angles	20
		Two pairs of congruent angles	3
		Two pairs of congruent opposite angles	4
		Concave	3
Diagonals	2	Diagonals are perpendicular bisectors of each other	2
Total	**67**		**67**

In the discussion at the end of the class, the class discussed the question about the diagonals. Megan went over the questions posed by the group, naming the possible quadrilaterals that they could have considered. Megan analyzed the play of the game, by suggesting what questions were unnecessary because students could have inferred the answers to those using the answers to previous questions. Students identified Chase as the member of the group who posed the question of whether the diagonals of the quadrilaterals were perpendicular bisectors or not. Chase named quadrilaterals that could possess diagonals that are perpendicular bisectors: a square and a rhombus. This question led them to guess correctly the hidden quadrilateral as a rhombus. At the end of this exchange, Megan asked other groups whether they had posed a question regarding the diagonals of the hidden quadrilateral. Students replied that they had not.

The Guess my Quadrilateral game was conceived as an activity of teaching with a problem. The problem (to win the game) was supposed to teach them to value having necessary and sufficient conditions to identify a figure. In order to find what was the hidden quadrilateral with the least number of questions, students had to relate to quadrilaterals by their properties. Megan allowed students to refer to special quadrilaterals by their names. Megan also allowed students to make use of the properties of special quadrilaterals to pose questions during the play of the game. The class discussed a question about diagonals at the end of the class. Megan did not show objections about referring to the diagonals of special quadrilaterals. On the contrary, Megan asked what quadrilaterals have diagonals that are perpendicular bisectors. This property of diagonals is more specific than what they had studied about diagonals of parallelogram prior to the replacement unit. Megan's openness to talk about a property of special quadrilaterals regarding its diagonals, contrasts with her reluctance to take other properties of diagonals for granted in the rectangle episode the following day, as we will see.

This concludes the chronology leading to the rectangle episode that I will present later. The other chosen episode, the kite episode, had its own chronology. In the following section, I present the chronology of events leading to the kite episode. While the rectangle episode showcases conflicts when students rely on prior knowledge that had not been officially introduced in class, the kite episode illustrates conflicts when students remember a conjecture that had not been proven.

Chronology of Events Leading to the Kite Episode:
Conflicts when students remember a conjecture that has not been proven

In the eleventh day of the unit, there was a conflict between the teacher and the students. A student used a conjecture in order to prove that the midpoint quadrilateral of a kite is a rectangle. However, the teacher did not allow students to use that conjecture because it had not been proven yet. The kite episode illustrates the case when students recall a proposition that they have not proven, even though they had gathered empirical evidence to support that conjecture. To prove the conjecture students might have to make use of a theorem that they had not discussed in class: The usual proof for this conjecture in geometry textbooks requires the medial-line theorem (or midpoint connector theorem). Megan's class had not studied this theorem yet, which is usually covered later in the year.[14] As I mentioned earlier, one of the goals of the replacement unit was to create a context where students would come up with the conjecture of the medial-line theorem. Thus, the student who was trying to prove that the midpoint quadrilateral of a kite is a rectangle anticipated knowledge that had not been officially introduced in class. The kite episode shows a case where students insisted on using this conjecture to justify a statement in a proof, anticipating that it would be true. In doing that, they were breaking a norm of the situation of *doing proofs* according to which students are only allowed to assume to be true those theorems they have proved earlier, in addition to the definitions,

[14] The textbook used in Megan's class introduces the medial-line theorem as an application of similar triangles, in the chapter that follows quadrilaterals (Boyd et al., 1998, p. 363). The textbook shows the main ideas involved in the proof and asks students to do a two-column proof of this theorem in one of the exercises in the same section where the theorem is presented (Boyd et al., 1998, p. 368, #33). It can, however, be proved by contradiction using triangle congruence and Megan knew this proof.

postulates and what is specified in the initial conditions or “givens” (Herbst & Brach, 2006).

I report events prior to the kite episode where students in the second period class referred to the conjecture that all midpoint quadrilaterals are parallelograms. Megan allowed students to talk about this conjecture in class in different problem-based activities during the unit. Some of these activities enabled students to gather empirical evidence to support this conjecture. However, Megan did not allow students to take this conjecture for granted at the moment of doing a proof in the eleventh day of the unit. This contrast is of interest here because even though students remembered the conjecture, Megan did not allow them to use the conjecture as a resource when doing a proof.

Day 1

At the end of first day of the unit, students in the second period class had started to gather perceptual evidence to conjecture that the midpoint quadrilateral is always a parallelogram. Ada, a student in the second period class, shared an answer with the class to the question *what quadrilateral do you need to start with in order to get an interesting m-quad?* Megan had asked Sally, another student in the second period class, about findings in her group. Ada, who was in Sally’s group, said that when drawing the midpoint quadrilateral of different quadrilaterals they always got a parallelogram. Megan understood that the group had inferred all midpoint quadrilaterals are parallelograms. However, Ada hesitated to call a parallelogram the midpoint quadrilateral of a square, which she correctly identified as a square. Later, Ada elaborated that in some cases the midpoint quadrilateral could be classified as something more than a parallelogram. For example, the midpoint quadrilateral of a square is a square, which is a special kind of parallelogram.

> Megan: Sally, what about your group? What kinds of interesting shapes did you get?
> Ada: Well, we didn't really do interesting shapes. We came up with an interesting um, idea.
> Megan: Okay.
> Ada: Like, we did a lot of kites and um, we even did a dart, and um, and a lot of other funky shapes and it always came back to a parallelograms and um rectangles and it would alternate.
> Megan: Okay, I think that, that's an interesting conjecture. So your group thinks that no matter what—
> Ada: No, no, no, no matter what, it's just like, I mean like squares don't do this, but like—
> Megan: Squares you don't get a parallelogram? (Sammy just did one.)
> Ada: (Well, you get a parallelogram) but like, you get like a square, not just like a random parallelogram. But like, they always come back to like parallelograms and stuff.
> Megan: Okay. I want to know what group didn't get a parallelogram—started with a shape and didn't get a parallelogram.

Here, Ada answered Megan's question and spoke on behalf of her group. She talked about finding an interesting idea, which could be understood as a generalization of results for different kinds of quadrilaterals. Students' worksheets for that group included a "dart," a kite, a parallelogram, and other quadrilaterals with no specific features. Some students had made drawings of successive midpoint quadrilaterals, which was a common exploration in both classes (see Figure 6). Ada could have been talking about comparisons between every other figure made by successive midpoint quadrilaterals when she mentioned that parallelograms would alternate. This episode is relevant because Ada's comments provoked a public discussion in this class around the question of whether the midpoint quadrilateral is always a parallelogram or not.

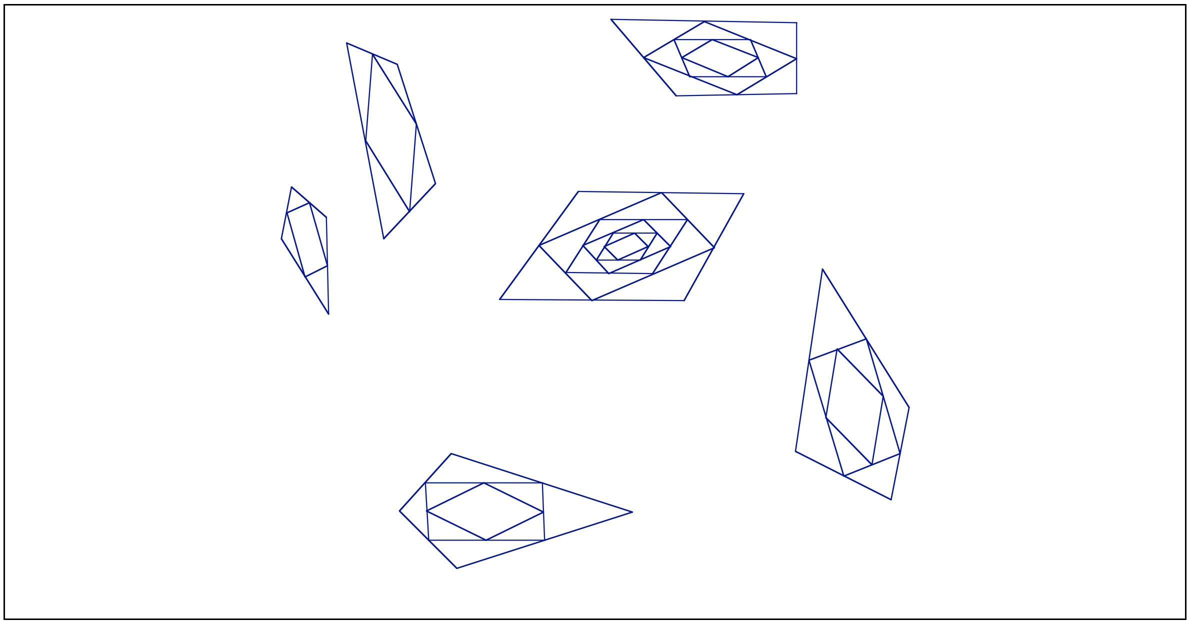


Figure 6. Representation of one worksheet submitted at end of the first day.

After this episode, in trying to answer Megan's question, students looked at their diagrams to check whether they always got a parallelogram or not as a midpoint quadrilateral. Mitchell was the only student who volunteered to go to the board to show that he had not gotten a parallelogram. However, Mitchell's midpoint quadrilateral also appeared to be a parallelogram[15] and he went back to his seat (see Figure 7).

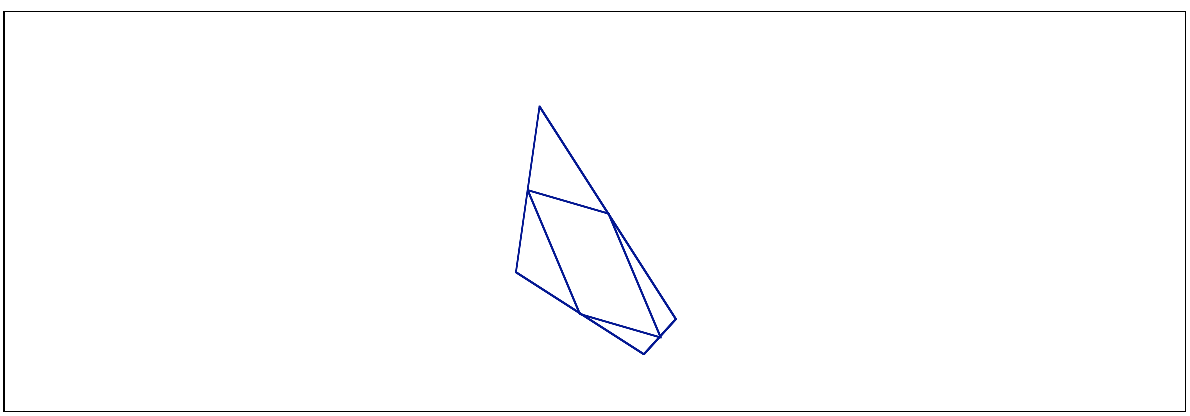


Figure 7. Mitchell's diagram of a quadrilateral and its midpoint quadrilateral.

[15] Mitchell's worksheet also included the diagram of a quadrilateral with one side much shorter than the other three. Its midpoint quadrilateral appeared to be a parallelogram with the same shape and in the same orientation as the one he drew on the board.

Therefore, at the end of the first day of the unit, students in the 2nd period class had perceptual evidence to support the notion that regardless of which quadrilateral they started with, the midpoint quadrilateral appeared to be a parallelogram. They had started to formulate the conjecture that all midpoint quadrilaterals are parallelograms. During the following days, students continued to gather empirical evidence to support this conjecture in different activities, some in which they used dynamic geometry software.

Day 2

At the beginning of the class, some students went to the board and showed diagrams for a solution of a homework problem. The homework problem asked students to draw a quadrilateral and then to draw successive midpoints quadrilaterals, starting with the original one. At the end of this discussion, students started to make conjectures about relationships between the perimeter and the area of successive midpoint quadrilaterals. At this point, Megan asked students if they remembered something they had studied in middle school about relationships between the perimeter and the area. A student said, "is it scale factor?" Megan replied, "something to do with scale factor." Then, Megan added, "some of that old seventh grade stuff might come in handy this week." Megan continued with the discussion of other homework problems, but in class they did not make reference to scale factor again. This provides evidence for how Megan activated students' memories from prior mathematics classes and connected them to topics in the unit.

After homework discussion, Megan asked students to complete the table relating properties of the outer quadrilateral and the midpoint quadrilateral in their groups. In the last 15 minutes of class, Megan led a discussion about the table with the whole group.

She asked each group to give one entry of the table. Megan wrote the entries of the table in a transparency that everyone could see projected on the board (see Table 3). After writing the first seven entries of the table, Megan requested for other possible entries. Megan recalled that the previous day someone had made a diagram of a dart, "a pushed-in kite." Then, Megan asked, "What about kites in general?" She asked, "What shape do I get for just a kite?" A student said, "a rhombus." Megan asked students to see if they had made a diagram of a kite. A student said, "a rectangle." Megan asked students to describe the properties of a kite without saying it is a kite. They came up with a description of a kite that would be different from a description of a parallelogram: "two pairs of congruent sides not opposite each other." Megan asked, "What do I get? What shape do I get?" Some students replied, "a rectangle."

Table 3

Table of properties produced by students in the seventh period class

Properties of O-Quad	**M-Quad**
1. All angles are 90°	Rhombus
2. No sides or angles congruent	Parallelogram
3. Two pairs of congruent opposite sides	Parallelogram
4. Congruent sides and angles	Square
5. All congruent sides	Rectangle
6. Two pairs of congruent opposite angles	Parallelogram
7. One pair of parallel sides, another pair of congruent sides	Rhombus
8. Two pairs of congruent sides (not opposite each other)	Rectangle
9. Diagonals bisect each other	Parallelogram
10. Diagonals are perpendicular	Rectangle

For the last entry of the table, Megan drew some diagrams of quadrilaterals with perpendicular diagonals, including a diagram of a kite, its diagonals, and its midpoint quadrilateral (see Figure 8). Students said that the midpoint quadrilateral of a quadrilateral with perpendicular diagonals would be a rectangle. Megan said, "So, maybe we should write that one. If the diagonals are perpendicular, then it looks that I get a rectangle. Now, I may get, I may get something more special, like a square." Then, Megan moved on to read the questions for homework and to distribute the homework for the next day.

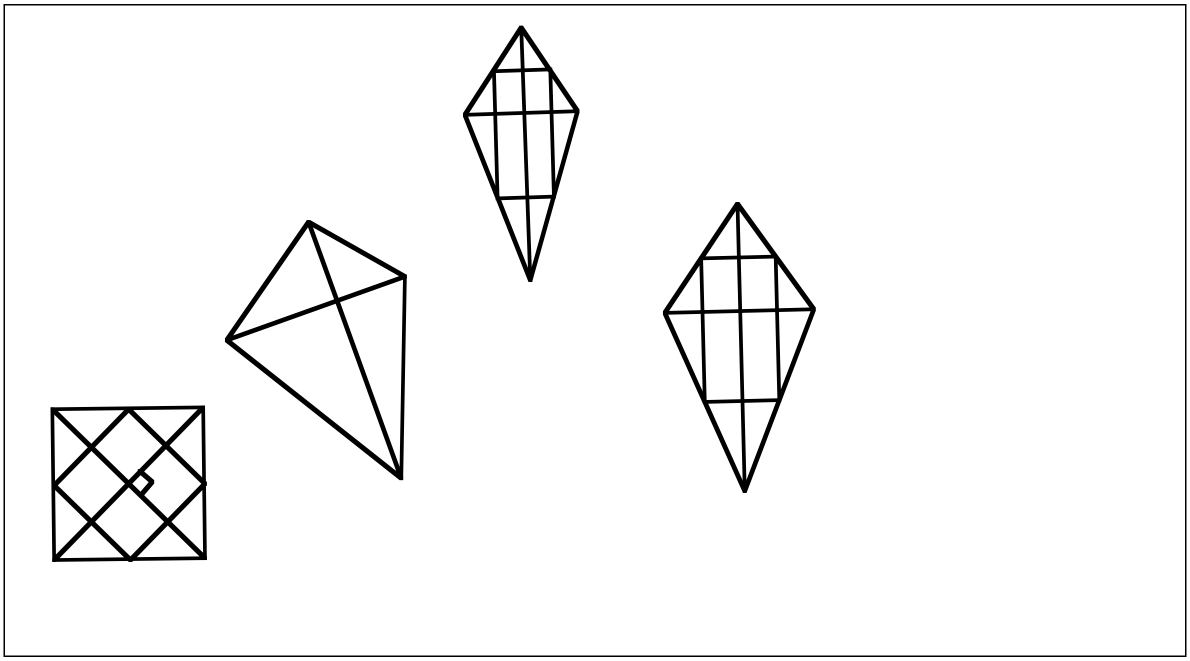


Figure 8. Quadrilaterals with perpendicular diagonals drawn by Megan on the board.

By the end of the class, the table showed that in four out of ten entries, the midpoint quadrilateral is a parallelogram, even though they started with a quadrilateral with different properties. Other entries included the names of special quadrilaterals for the midpoint quadrilateral. All of these special quadrilaterals are parallelograms, but these were not identified as such in class.

In sum, the table of properties of the second period class included parallelograms in all the entries for the midpoint quadrilateral. Megan called students' attention to the midpoint quadrilateral of a quadrilateral with perpendicular diagonals. This is a characteristic of the kite. In the kite episode, which happened in the eleventh day of the unit, a student comes to the board to prove that the midpoint quadrilateral of a kite is a rectangle. This was a public result in the second period class on the second day of the unit.

Considerations Regarding Both Episodes

Both episodes illustrate a phenomenon in the geometry class where a teacher asks students to disregard properties of geometric figures that are not part of what the class has studied though they are "known" by students in the class. The replacement unit, which was conceived as an opportunity for teaching geometry with problems, brought with it changes in the usual sequence in which new knowledge was introduced to class. In the Guess my Quadrilateral game, the teacher allowed students to draw upon knowledge from the remote past when trying to identify the hidden quadrilateral by posing questions about properties of quadrilaterals. In the problem of finding something interesting about midpoint quadrilaterals, the teacher allowed students to formulate a conjecture. The proof of this conjecture requires using the medial-line theorem. So, if students were to do the proof, they might get to anticipate needing that theorem. These activities created opportunities for students to remember or come to know some geometric properties, though the manner in which they knew them was not as deductive consequences of accepted hypotheses. The two selected episodes illustrate how Megan dealt with difficulties as students worked on mathematical tasks, after introducing changes in how temporal boundaries sustain the organization of knowledge in the geometry class. In the following sections I present the two episodes and my analysis.

The Rectangle Episode

The rectangle episode happened at the beginning of the 6th day into the quadrilaterals unit in the seventh period class. At the beginning of the class, Megan spent some time talking about students' improvements in a recent quiz and about her expectations for students to elaborate on their answers for open-ended problems. Then she started a discussion about the homework, which had been assigned at the end of the previous lesson (when students had played the Guess my Quadrilateral game). Megan selected an exercise that she called typical of the quadrilaterals chapter in the book, disregarding some of the open-ended problems in the homework worksheets from the replacement unit. This exercise required knowledge about properties of rectangles as a context for doing algebra (Figure 9). However, Megan used this exercise as an opportunity to prove a property of rectangles, transforming a calculation exercise into an opportunity for installing a new property (and giving the proof of it).

3. Quadrilateral *MNOP* is a rectangle. Find the value of x.

a. $MO = 2x - 8$; $NP = 23$
b. $CN = x^2 + 1$; $CO = 3x + 11$
c. $MO = 4x - 13$; $PC = x + 7$

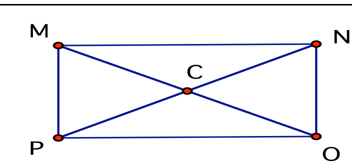


Figure 9. Homework problem (copied from Boyd et al., 1998, p. 309, exercises 5-7).

The three questions in the exercise are independent even though they refer to the same diagram. Thus, the value of x may differ from answer to answer. All questions require knowing that the diagonals in a rectangle are congruent. My analysis focuses on the class discussion of part (a) of the homework exercise. The exercise is an example of an “algebraic calculation in geometry,” characteristic of a *situation of calculation* in the geometry class (Herbst, Hsu, Weiss, Chen, & González, 2008).[16] Students’ work on the exercise would require them to apply properties of geometric figures and algebraic skills to find an unknown quantity. In the following section, I present an analysis of the mathematical task in this homework exercise.

Analysis of the Initial Task in the Rectangle Episode

In order to study the initial task in the rectangle episode, I refer to the model of a task as a set of resources and operations that one could make use of in order to achieve a particular goal (Doyle, 1988; Herbst, 2003, 2006). By applying the model of a task to the homework exercise discussed during the rectangle episode, one could see that the goal of finding the value of *x* would require a theorem about rectangles (Table 4). This theorem states that the diagonals of a rectangle are congruent. Students could apply the theorem to make an equation, solve the equation, and thus, find the value of the variable that would make true the statement *MNOP* is a rectangle. Alternatively, it is conceivable that students would not rely on the theorem to set up the equation. Students could use visual perception to study the diagram and conclude that *MO* and *NP* appear to be congruent. A third option to solve the problem would be for students to use the only two algebraic expressions in the statement of the problem in order to make an equation. A fourth option would be for students to confirm that the diagonals of a rectangle are congruent by doing a proof, after having set up the equation. So, students could achieve the same goal of finding the value of the variable, even though the resources for solving the problem could vary substantively. Moreover, students do not need to rely on the theorem, and

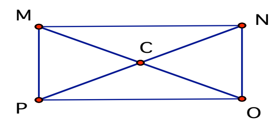
[16] In a *situation of calculation,* the norm is to set up and solve operations between quantities or algebraic expressions using properties of the figure.

could rely on other resources—the diagram and the norms of the situation of calculation—in order to set up an equation correctly.

The intersection point of the diagonals is more of a distraction than a resource in part “a” of the homework exercise. Even though the intersection of the diagonals has a label, point C, there are not algebraic expressions with that point in part “a” of the homework exercise. Consequently, students could ignore the intersection of the diagonals to get an answer to question “a,” different from other parts of the exercise that make references to the intersection of the diagonals. So, it is unlikely that students would propose other solutions taking into account the intersection point of the diagonals for part “a” of the homework exercise.

Table 4

Analysis of the initial task in the rectangle episode

	Solution with the theorem about rectangles	**Solution with visual perception**	**Solution by using the given algebraic expressions**	**Solution with proof of the property of rectangles**
Problem	Quadrilateral *MNOP* is a rectangle. Find the value of x. $MO = 2x - 8$; $NP = 23$.			
Goal	Find the value of x.			Prove that the diagonals of a rectangle are congruent.
Resources	• Diagram of rectangle *MNOP* and its diagonals. • Algebraic expressions for the diagonals of *MNOP*. • **Theorem: The diagonals of a rectangle are congruent.**	• Diagram of rectangle *MNOP* and its diagonals. • Algebraic expressions for the diagonals of *MNOP*.	• Diagram of rectangle *MNOP* and its diagonals. • Algebraic expressions for the diagonals of *MNOP*.	• Diagram of rectangle *MNOP* and its diagonals. • Definition of a rectangle as a quadrilateral with four right angles. • Property of a rectangle stating that opposite sides are congruent. • Reflexive property. • Side-Angle-Side theorem of triangle congruency.
Operations	• Set up an equation to represent that the diagonals of the rectangle are equal; *MO*	• Use visual perception to study the diagram and notice that the diagonals of the rectangle, *MO* and	• Notice that there are <u>only two</u> algebraic expressions in the statement of the	• Notice triangles *MPO* and *NOP*. • Apply reflexive property to show that

	Solution with the theorem about rectangles	**Solution with visual perception**	**Solution by using the given algebraic expressions**	**Solution with proof of the property of rectangles**
	= *NP* ⇒ 2*x* – 8 = 23. • Solve an algebraic (linear) equation in one variable.	*NP* are congruent. • Set up an equation to represent that the diagonals of the rectangle are equal; *MO* = *NP* ⇒ 2*x* – 8 = 23. • Solve an algebraic (linear) equation in one variable.	problem. • Set up an equation to make the given algebraic expressions equal to each other; *MO* = *NP* ⇒ 2*x* – 8 = 23. • Solve an algebraic (linear) equation in one variable.	*PO* is congruent to *OP*. • Apply definition of a rectangle to show that angles *MPO* and *NOP* are congruent. • Apply property of rectangles to show that *MP* and *NO* are congruent. • Apply Side-Angle-Side theorem of triangle congruency to prove triangles *MPO* and *NOP* congruent. • Apply definition of congruency to conclude that diagonals of a rectangle are congruent.

In the rectangle episode, Megan made students aware that a theorem was needed to set up the equation in part (a) of the problem, $2x - 8 = 23$. However, some students showed difficulties in understanding what properties of a rectangle they could take for granted and what properties needed to be proven. This conflict is of interest in this study because, as I have shown in my review of events prior to the rectangle episode, the theorem that diagonals of a rectangle are congruent had not been mentioned in class discussions. In contrast, Megan had *installed* properties of diagonals of parallelograms prior to the unit. In addition, there had been public discussions in the unit about quadrilaterals with perpendicular diagonals and quadrilaterals with diagonals that are perpendicular bisectors of each other. So, students were expected to do a homework problem that required a theorem: The diagonals of a rectangle are congruent. This theorem had not been *installed* in class. By assigning the problem without having installed the theorem, Megan changed usual practices in her class. Students could have possibly taken for granted that the theorem had already been installed. They could also follow the norms of the situation of calculation and solve the problem of finding the value of *x*. However, Megan wanted to use the discussion of the homework to install the theorem. With that purpose, Megan put together elements characteristic of a *situation of calculation* and of a situation of *doing proofs.*

Description of the Rectangle Episode

The rectangle episode lasted approximately 4 minutes and 15 seconds. During the discussion of the homework problem, Megan asked students to prove properties of rectangles that they could deduce from the definition of a rectangle as a quadrilateral with four congruent angles. In doing the proof, students kept mentioning properties of rectangles. They said that in a rectangle opposite sides are congruent, opposite sides are parallel, and diagonals are congruent. However, Megan did not allow students to take these properties for granted. She drew upon their prior knowledge about parallelograms, about parallel lines, and about congruent triangles to do a proof for those properties of a rectangle that students wanted to use.

Transcription of the Rectangle Episode Parsed by Clause[17]

A transcription of the rectangle episode parsed by clause[18] follows. This parsing allows me to point to particular moments and to study the transcript in depth. Each clause includes a process, either expressed or elided. I do not count as clauses interjections or false starts (for example, turn 4). I identify adjuncts to clauses, such as vocatives, with the same number as its related clause and a letter (see 1.22a).

In order to make my decisions about parsing the text into clauses explicit, I connect interrupted clauses and add elided text, whenever possible. I note interrupted clauses with conventional notation, "<< >>." For example, in the first turn, Megan said, "Okay, I want to go over homework, I wanna skip problems 1 and 2, even though I think

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[17] In the parsed transcript, I take out notes regarding overlapping speech, interruptions, pauses, and interpersonal gestures, in order to simplify the reading of the parsed transcript. I also take out fillers and interjections.
[18] Eggins and Slade (1996) give a definition of a clause as follows, "A clause can now be identified as a sequence of some of the constituents identified above: Subject + Finite, plus a Predicator, and combinations of Complements and Adjuncts, with some elements possibly ellipsed but recoverable from prior clauses. However, in the dynamics of casual talk speakers do not always finish clauses that they start, either because they run out of steam, or because they are interrupted" (p. 106). According to Eggins and Slade, "The Subject is the pivotal participant in the clause, the person of thing that the proposition is concerned with and without whose presence there could be no argument or negotiation" (p. 45). Also, "The Finite expresses the process part of the clause that makes it possible to argue about the Subject participant" (p. 77).

those are some of the more interesting questions, for a few minutes." In my reconfiguration of clauses in this text, I consider three clauses:

(1) I want to go over homework << >> for a few minutes,

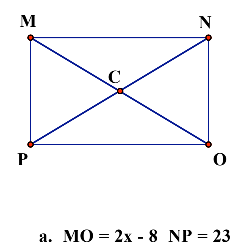
(2) <<I wanna skip problems 1 and 2,

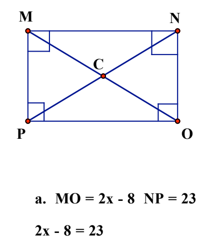
(3) even though I think

(4) those are some of the more interesting questions.>>.

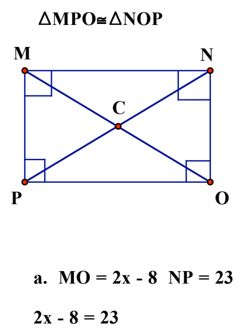
The text in between marks is the one that should be interpolated in clause 1.1 in order to have the original text.

I identify in the transcript actions that refer to the diagram on the board. I describe these actions in squared brackets. These actions are important to identify what has been elided or referred to in spoken text. Actions follow the nearest clause, with a letter added to the number of the clause, in the order that the action took place. That is, actions are not interpolated within a clause. For example, Megan performed two actions associated with clause 1.14: 1.14a-draws diagonals of rectangle and 1.14b-labels points. She drew the diagonals first and then labeled the points. I expand the text, whenever possible, to include ellipsis, substitutions, and references. When there is elided text, I note it with an empty set notation, "{ }" and recover what has been elided between the curly brackets (see clause 28.89). I also add the reference to the clause or to the action that gives evidence for how I recovered the elided material. For example, I expand clause 11.42 as follows, "Which triangles {are congruent (10.41)}?" In turn 11, Megan said "Which triangles?" I point the reader to the previous clause, 10.41, where Alana said, "'cause the triangles are congruent." The expanded clause 11.42 includes the same process as in clause 10.41. The same applies for actions that support my decision about how I expand a clause (see clause 30.93). Expanded text could refer to clauses or actions that either precede or follow a particular clause. That is, speakers could achieve lexical cohesion by connecting a clause with something that was previously said (or done), or with something that I note later in the text as being said or done (even when simultaneous, as when pointing to the diagram while speaking). Altogether, the expanded text illustrates how speakers used different resources to attain cohesion in this excerpt of classroom talk.

Turn #	**Speaker**	**Turn**
1.	Megan	1. Okay, I want to go over the homework << >> for a few minutes. 2. <<I wanna skip problems 1 and 2, 3. even though I think 4. those are some of the more interesting questions,>> 5. and look at some of the more traditional ones, 6. so we can flesh out some of the properties of rectangles. 7. Okay, where is my homework? 8. Here we go. 9. Okay, {look at (1.5)} problems like number three. 10. Why didn't I write this on the board in between? 11. They told you 12. that you had a rectangle. 12a. [Draws rectangle.] 13. They said 14. that this outer thing was a rectangle, *MNOP* 14a. [Draws diagonals of rectangle.] 14b. [Labels points.] 15. And then they give you a little bit of information.  16. Okay, part "a" they say 17. that *MO* is equal to 2*x* minus 8, 18. and *NP* is equal to 23. 19. Okay, someone talk about 20. how I would solve for *x* on this particular problem. 21. *MO* is equal to 2*x* minus 8, 22. *NP* is equal to that, 22a. Alana?

Turn #	Speaker	Turn
2.	Alana	23. Two *x* minus 8 equals 23.
3.	Megan	24. Okay, what << >> is true? 25. <<are you assuming>>
4.	Alana	That—
5.	Anil	26. *MO* is congruent to…
6.	Alana	27. That's a rectangle?
7.	Megan	28. They told you 29. it was a rectangle. 30. I'm not assuming that (7.29). 30a. [Marks right angles.] 31. They told you that (7.29). 31a.Anil 32. What is it {the assumption (3.25)}? 
8.	Anil	33. *MO* is congruent to *NP*.
9.	Megan	Yeah. 34. We haven't had a theorem about that (9.35). 35. That the diagonals are congruent. 36. But she's right. 37. They {the diagonals (9.35)} are congruent. 38. Can someone say 39. how we could prove 40. that the diagonals of a rectangle are congruent, 40a. Alana?
10.	Alana	41. 'cause the triangles are congruent.
11.	Megan	42. Which triangles {are congruent (10.41)}? 43. The big triangles {*MPO* and *NOP* (11.43a) are congruent (10.41)}? 43a.[Traces triangles *MPO* and *NOP*.]
12.	Alana	44. {Triangles (11.42)} *MCN* and *PCO* {are congruent (10.41)}.
13.	Megan	45. Okay, how would I prove 46. those {triangles *MCN* and *PCO* (12.44)} were congruent?
14.	Alana	47. {I would prove triangles *MCN* and *PCO* congruent (13.46)} By side...[19]

[19] Alana started turn 14 by naming a possible theorem of triangle congruency, when she said "By side." However, Alana elided the reason that could justify why triangles *MCN* and *PCO* would be congruent. Alana could have used two theorems to prove triangle congruency that have a statement starting with "side": side-angle-side or side-side-side. It could be the case that in making the statement she realized that the proof she had intended was not possible, and thus retracted from giving a reason. I take "by side" as a clause with elided text, in response to the teacher's question. Alana's use of the

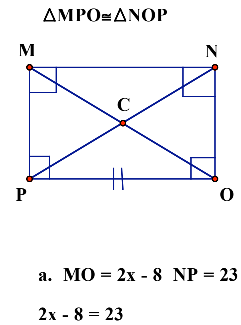
Turn #	Speaker	Turn
		48. I don’t know, 49. but I know 50. that angle *PCO* and angle *MCN* are equal,[20] 51. ‘cause they’re {*PCO* and *MCN* (14.50)} vertical angles. 51a. [Megan marks vertical angles congruent.] 52. And then *MN* and *PO* are a side[21] 53. that’s {*MN* and *PO* (14.52)} are equal.[22]
15.	Megan	54. Okay, I’m gonna give you a hint. 55. Quit looking at the little triangles. 55a. [Erases marks for vertical angles.] 56. Let’s look at the big triangles. 57. Look at triangle *MPO*. 57a. [Writes Δ*MPO*.] 58. {Look at (15.57)} This big triangle {*MPO* (15.58a)}. 58a. [Traces Δ*MPO*.] 59. And then {look at (15.57)} this big triangle {*NPO* (15.59a)}. 59a. [Traces Δ*NPO*.]  60. What’d I have? 61. {I would have triangle (15.60)} *MPO*. 62. So it would be *NO*, *P*. 62a. [Writes that Δ*MPO* is congruent to Δ*NOP*.] 63. How can I prove this? 64. Okay, that this big triangle {Δ*MPO* (15.64a)} is equal to this one {Δ*NOP* (15.64b)}, 64a. [Traces Δ*MPO*.] 64b. [Traces Δ*NOP*.] 64c. Anil?
16.	Anil	65. {I can prove Δ*MPO* is congruent to Δ*NOP* by (15.64)} SSS 66. or like {I can prove Δ*MPO* is congruent to Δ*NOP* by (15.64)} SAS.

conjunction “by” suggests that the expected answer to the teacher’s question in turn 13 is a reason by which the two triangles are congruent.

[20] In mathematics, there is a difference between congruency and equality. However, in GRIP’s archive of classroom videos from geometry classes, it is usual for speakers to use the term “equal” for parts that are “congruent.” In this episode, I take that when speakers talk about equal parts of triangles or equal triangles they mean that these are congruent.

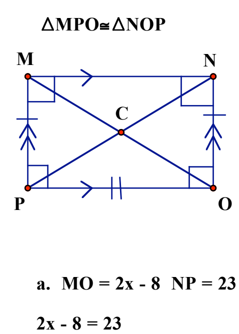
[21] I take Alana’s use of “a” and “side” as a mistake in speech, when she could have said, “And then *MN* and *PO* are sides.”

[22] I take Alana’s use of two processes “is” and “are” as a self-correction. The clause could read as “that {*MN* and *PO*} are equal.”

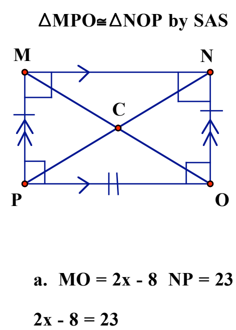
Turn #	Speaker	Turn
17.	Megan	67. Okay, I know 68. these {*PO* (17.68a)} sides are equal, the bottom, 68a. [Adds hash marks to *PO*.] 69. because it's part of both triangles. 70. It's part of this one {Δ*MPO* (17.70a)}, 70a. [Traces Δ*MPO*.] 71. and it's part of this one {Δ*NPO* (17.71a)}. 71a. [Traces Δ*NPO*.] 72. Okay, what else << >> is true, 73. <<do I know>> 73a. Anil? 
18.	Anil	74. The opposite sides are equal Because[23]
19.	Megan	75. Why do I know that (18.74)?
20.	Anil	76. 'cause in a rectangle the opposite sides {are equal (18.74)}.
21.	Megan	77. I don't know that (20.76) in a rectangle.
22.	Anil	78. {in a (21.77)} parallelogram {the opposite sides are equal (18.74)}.
23.	Megan	79. It's {*MNOP* (1.14b)} a parallelogram. 80. How could I prove 81. it's {*MNOP* (1.14b)} a parallelogram?
24.	Anil	82. Opposite sides are parallel.
25.	Megan	83. How do I know 84. the opposite sides are parallel?
26.	Anil	'cause[24]
27.	Megan	85. Maybe we need to talk about 86. what do you need to know 87. for it {*MNOP* (1.14b)} to be a rectangle. 88. What's definitely true about a rectangle?
28.	Alana	89. {A rectangle (27.88) has (30.92)} Four right angles.

[23] Anil repeated "because" in two turns, 18 and 20. I do not take "because" in turn 18 as a clause because it is a false start. I take turn 20 as a clause because he included more information that he elided in turn 18, when interrupted by Megan. This information included a statement with an implicit process: in a rectangle the opposite sides are equal. In clause 18.74, I take that Anil referred to the diagram of quadrilateral *MNOP*. However, in clause 20.76, I take that Anil referred to a general property of rectangles.

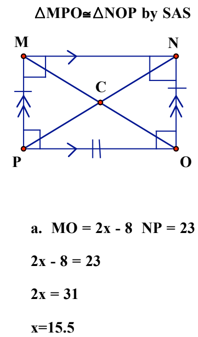
[24] In turn 26, I cannot reconstruct the elided text. Therefore, I do not take turn 26 as a clause, even though Anil's use of the conjunction "because" suggests that Anil intended to give a reason to the question posed by the teacher in turn 25. Anil looks down and flips a paper he has on his desk. He tenses his lips and raises his eyebrows, while looking at a classmate across his desk.

Turn #	Speaker	Turn
29.	Anil	90. {A rectangle (27.88) has (30.92)} Four right angles.
30.	Megan	91. {A rectangle (27.88) has (30.92)} Four right angles. 92. Okay, if I have four right ang—{angles (28.89)} 93. If these two {angles *MPO* and *NOP* (30.93a)} are right angles, 93a. [Points to angles *MPO* and *NOP*] 94. why are these sides {*MP* and *NO* [30.94a]} parallel, 94a. [Waves along *MP* and *NO* up and down.] 94b.Anil?
31.	Anil	'cause, 95. 'cause of the, the same side interior angles are…
32.	Megan	96. {The same side interior angles are (31.95)} supplementary.
33.	Anil	Yeah.
34.	Megan	97. I've got two {angles *MPO* and *NOP* (34.97a)} there. 97a. [Points to angles *MPO* and *NOP*.] 98. I could do the same thing here {angles *NMP* and *MPO* (34.98a)}. 98a. [Points to angles *NMP* and *MPO*.] 99. So you're {Anil (31.95)} right. 100. I could prove 101. these {*MP* and *NO*; *MN* and *PO* (34.101a)} were parallel. 101a. [Marks opposite sides parallel.]  102. So, it's {*MNOP* (1.14b)} a parallelogram. 103. So then that (34.102) tells me 104. the opposite sides {*MP* and *NO* (34.104a)} are equal 104a. [Marks sides *MP* and *NO* congruent.] 105. 'cause it's {*MNOP* (34.102)} a parallelogram. 106. So I have side {*MP* (34.106a)}, side {*PO* (34.106a)}, equal to side {*NO* (34.106b)}, side {*PO* (34.106b)}. 106a. [Points to *MP* and *PO*.] 106b. [Points to *NO* and *PO*.] 107. What else do I get?[25] 108. It's {*MNOP* (1.14b)} a rectangle. 108a. Anil?
35.	Anil	109. The two diagonals {are congruent (34.107)}.
36.	Megan	No,

[25] I take Megan's question in 34.107, "What else do I get?," as a probe for Anil to identify other congruent parts.

Turn #	Speaker	Turn
		110. I don't know 111. the two diagonals {are congruent (34.107)}. 112. I'm trying to prove that (9.39, 9.40).
37.	Anil	Oh.
38.	Megan	Ebony?
39.	Ebony	113. The angles {*PMN, MNO*, *NOP*, and *OPM* (28.89)} are equal? 114. They're {*PMN*, *MNO*, *NOP*, and *OPM* (28.89)} right angles.
40.	Megan	Yeah, 115. they're {*PMN*, *MNO*, *NOP*, and *OPM* (28.89)} all 90. 116. Like Alana said, 117. they {*PMN*, *MNO*, *NOP*, and *OPM* (28.89)} are all 90. 118. So I have side {*MP* and *NO* (40.118a, 34.104)}, angle {*MPO* and *NOP* (40.118b, 40.117)}, side {*PO* (40.118c, 17.68)}. 118a. [Points at side *MP*.] 118b. [Points at angle *MPO*.] 118c. [Points at side *PO.]* 118d. [Writes on the board, "by SAS."] 119. I could prove this {Δ*MPO* congruent to Δ*NOP* (15.62a)} by side-angle-side. 120. Then, why are the diagonals {*MO* and *NP* (40.120a)} equal? 120a. [Traces *MO* and *NP*.] 120b. Anil? 
41	Anil	<< >>
42.	Megan	121. If this big triangle {*MPO* is congruent to Δ*NOP* (15.62a, 42. 121a)}. 121a. [Traces Δ*MPO*.]
43.	Anil	122. {*MO* and *NP* are equal (40.120)} <<Because[26]>> CPCTC.
44.	Megan	123. {*MO* and *NP* are equal because (43.122)} CPCTC.
45.	Anil	Yeah

[26] Since there is not a process, I do not take turn 41 as a clause. However, Anil gave a reason for Megan's question about the diagonals (40.120) in 43.122. I take turns 41 and 43 as an interrupted clause, with elided text about the diagonals *MO* and *NP*. Later, in my analysis of conjunctions of this episode, I only count the conjunction "because" in turn 41 once, since it was expanded in turn 43. I do not count the repetition of "because" by Megan in clause 44.123, which I added to make the elided text part of the clause.

Turn #	Speaker	Turn
46.	Megan	124. Okay, so I could prove that. 125. She {Alana (2.23)} is right. 126. Okay, I am going to move the 8 over. 127. Two x equals << >> 31 128. <<what is that,>> 129. So *x* is 15.5. 130. Is that what you got, 130a. Alana? 
47.	Alana	Yeah.

Summary of the Parsed Transcript of the Rectangle Episode

The parsed transcript of the rectangle episode includes a total of 130 clauses. The teacher was the speaker in 104 clauses, whereas the students were speakers in 26 clauses. This suggests that the teacher was in control of classroom discourse, leading the discussion of the homework exercise and managing the production of the proof of the claim that the diagonals of a rectangle are congruent. This fact is important because the focus of my analysis is on teaching actions in which a teacher manages students' work on a mathematical task, and in particular, what kinds of actions have to do with how a teacher deals with students' prior knowledge. Since there are so many turns by the teacher, it is very likely that there will be evidence for how a teacher manages students' prior knowledge.

Highlights of the Rectangle Episode

In my retrospective analysis of the rectangle episode, I observe that Megan had to compensate for changes when teaching this problem-centered unit. Megan had not proved the theorem that the diagonals of a rectangle are congruent in any of the preceding lessons. The homework exercise she had assigned for the night before did not require students to do a proof. Considering where in the textbook that problem was located, it is fair to say that the problem assumed that students knew that diagonals of a rectangle are congruent. In the rectangle episode, Megan intended to use the homework problem to introduce officially properties of rectangles that they had not proven in class yet. In particular, Megan said that she wanted to focus on that homework problem, disregarding other problems, to " flesh out some of the properties of rectangles" (clause 1.6). She transformed the homework exercise into a proof. Megan put to play some of the norms characteristic of the situation of *doing proofs* (Herbst & Brach, 2006) and some other features of the situation of *installing a theorem* (Herbst & Nachlieli, 2007). She asked students to state the assumption needed to solve the problem and guided students to prove that that assumption is true in a rectangle. However, students faced several difficulties when doing the proof in identifying what properties of a rectangle they could make use of. Students did not make clear distinctions between what was given and what they were trying to prove. Moreover, students had difficulties in deducing properties of a rectangle from its definition, because they took these properties for granted. In what follows, I make an analysis of the mathematical task in the rectangle episode.

Analysis of the Task in the Rectangle Episode

At the beginning of the rectangle episode, Megan transformed the homework exercise into a new mathematical task. The *goal* of the homework exercise was to find the value of a variable. The theorem stating that diagonals of a rectangle are congruent was a *resource* needed to achieve that goal but covertly given. That the figure was a rectangle was a given resource as well as expressions for its diagonals. The fact that the problem included a rectangle and its diagonals was meant to trigger a property of rectangles: the diagonals of rectangles are congruent. Students would need to perform the *operations* of setting the expressions equal to each other and solving an algebraic equation in one variable. The task so modeled fits squarely within an instructional situation that is common in geometry—*situation of calculation.* In contrast, the discussion of the homework theorem had the *goal* of producing a proof of that theorem.[27] By changing the goal, Megan also changed the customary situation that would frame the homework exercise as a *situation of calculation*. In my analysis I will provide evidence for how Megan dealt with the unusual circumstance that students were supposed to do a homework problem using a theorem that had not been installed, by negotiating a new situation. This new situation has some characteristics of *doing a proof* and of *installing a theorem.*

In Table 6, I present an analysis of the new mathematical task including what resources (Table 5) and operations it would take to produce a proof for the theorem that the diagonals of a rectangle are congruent.[28] I use this analysis to divide the rectangle episode into different segments according to the resources and operations used to achieve the goal of producing a proof.

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[27] In the textbook from Megan's class, the proof for this theorem is assigned as one of the "Classroom Exercises" (Boyd et al., 1998, p. 310, exercise 14). The authors give a two-column proof with some statements and some reasons missing. Students need to fill in the blanks. The proof uses triangle congruency to prove that the two overlapping triangles made by two sides of a rectangle and one of the diagonals are congruent.
[28] Even though it is conceivable to have other definitions of a rectangle, I take the definition of a rectangle that Megan used in class: a rectangle has all angles congruent.

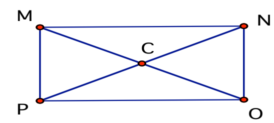
Table 5

Possible resources for doing a proof for the new mathematical task in the rectangle episode

#	**Resource**
1.	Definition of a rectangle: A rectangle is a quadrilateral with four congruent angles.
2.	Theorem: If a quadrilateral has two pairs of opposite angles congruent, then the quadrilateral is a parallelogram.
3.	Theorem: If two lines are intersected by another line, and same-side interior angles are congruent, then the two lines are parallel.
4.	Definition of a parallelogram: A parallelogram is a quadrilateral with opposite sides parallel to each other.
5.	Theorem: A parallelogram has opposite sides congruent to each other.
6.	Reflexive property of congruency of angles and segments.
7.	Side-Angle-Side theorem of congruency of triangles (SAS): Two triangles are congruent if two sides and the included angle of one triangle are congruent to two sides and the included angle of another triangle.
8.	Definition of congruency of triangles: Two triangles are congruent if and only if corresponding parts (sides and angles) are congruent.
9.	Theorem: In a right triangle, the midpoint of the hypotenuse is equidistant to the three vertices.
10.	Substitution property of equality.
11.	Segment addition postulate.
12.	Postulate that one can draw a perpendicular to a line passing from a point not on the line.
13.	Angle-Side-Angle theorem of congruency (ASA): Two triangles are congruent if two angles and the included side of one triangle are congruent to two angles and the included side of another triangle.
14.	Angle-Angle-Side theorem of congruency (AAS): Two triangles are congruent if two angles and a side (not the included side) are congruent to two angles and a side (not the included side) of another triangle.
15.	Property: Opposite sides of a rectangle are congruent.
16.	Definition of perpendicular bisector: A line, ray, or plane that bisects a segment and is perpendicular to the segment.
17.	Definition of isosceles triangle: A triangle with exactly two sides congruent.

Table 6

A priori task analysis of the new mathematical task in the rectangle episode

	A. Solution proving triangles *MPO* and *NOP* congruent	**B. Solution using the midpoint of the hypotenuse of Δ*MPO***	**C. Solution drawing an auxiliary line**
Problem	Quadrilateral *MNOP* is a rectangle. Find the value of x. $MO = 2x - 8$; $NP = 23$.		
Goal	A proof of the statement the diagonals of a rectangle are congruent.		
Resources (Table 5)	1-8	1-4, 9-11	1-4, 8, 10, 11, 13 or 14, 15, 16, 17
Operations	Prove that a rectangle is a parallelogram (by using resources 1 and 2, or by using resources 1, 3, and 4). Use properties of parallelograms to prove opposite sides (*MP* and *NO*) are congruent. Prove that triangles *MPO* and *NOP* are congruent by Side-Angle-Side. Apply the definition of congruency to prove that *MO* and *NP* are congruent.	Prove that a rectangle is a parallelogram (by using resources 1 and 2, or by using resources 1, 3, and 4). Use properties of parallelograms to prove that *C* is the midpoint of the diagonals. Show that *MPO* is a right triangle. Apply theorem about the midpoint of the hypotenuse to prove *CP*, *MC* and *OC* congruent. Use substitution to prove *NC* congruent to *CP*, *MC*, and *OC*. Apply segment addition postulate to prove diagonals congruent.	Prove that a rectangle is a parallelogram (by using resources 1 and 2, or by using resources 1, 3, and 4). Draw an auxiliary line, perpendicular to a side of the rectangle and passing through *C* (see Figure 10). Prove that triangles *QMC* and *ROC* are congruent by Angle-Side-Angle or by Angle-Angle-Side. Apply the definition of congruency to prove that *MQ* is congruent to *OR*. Prove that *MQ* and *PR* are congruent, applying the definition of a rectangle and the property of a rectangle that opposite sides are congruent. Apply substitution to prove that *MQ*, *PR* and *RO* are congruent. Show that *QR* is a perpendicular bisector of triangle *PCO*. Prove that triangle *PCO* is isosceles. Use definition of congruency and substitution to prove that *PC*, *OC*, *NC* and *MC* are congruent. Apply the segment addition postulate to prove that diagonals of a rectangle are congruent.

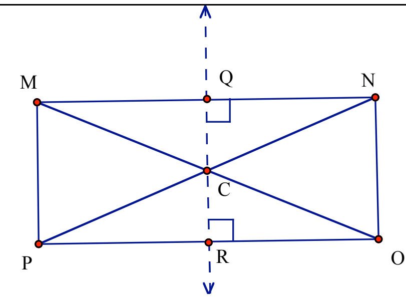


Figure 10. Diagram for a solution of the proof with an auxiliary line.

In the analysis a priori of the task I consider three possible solutions of the problem in the rectangle episode. These solutions draw upon tests for parallelograms, properties of parallelograms, the definition of a parallelogram, the definition of a rectangle, and theorems to prove triangles congruent. Solution A involves proving that triangles *MPO* and *NOP* are congruent. Then, as a consequence of proving those triangles congruent, the diagonals of the rectangle are congruent. Solution B uses a theorem that given a right triangle, the midpoint of the hypotenuse is equidistant to all vertices. If one were to apply this theorem to triangle *MPO*, then one could prove that the diagonals of the rectangle are congruent. Solution C would require drawing an auxiliary line to prove that the triangles made by the intersection of the diagonals are isosceles.[29] Consequently, the segments determined by the intersection of the diagonals are congruent, leading to prove that the diagonals are congruent.

In terms of the resources involved in these solutions, solutions A and C rely on triangle congruency and properties of parallelograms. However, solution B differs from solutions A and C in that it uses a theorem about right triangles. This theorem does not

[29] Alternatively, one could prove that an auxiliary line perpendicular to a side of the rectangle passing through the intersection of the diagonals is a line of symmetry (Figure 10). However, proofs by symmetry are unusual in the traditional geometry curriculum.

appear in the textbook from Megan's class. So, it would be unlikely that students in Megan's class would use this theorem, and thus apply solution B.[30] I include this solution in order to examine what options students could have to solve this problem by using possible resources within the geometry curriculum. Solution C is different from the other solutions in that it requires applying properties of isosceles triangles. Isosceles triangles had been covered in Megan's class prior to the quadrilaterals unit, specifically in chapter four. Megan could expect students to make use of properties of isosceles triangles, if needed to solve a problem.

In terms of the operations involved in solving the problem, solution C would require students to draw in an auxiliary line. Prior studies on the situation of doing proofs show that it is unlikely for students to add an auxiliary line unless prompted by the teacher or asked for in the statement of the problem (Herbst & Brach, 2006, p.104-105). Therefore, even though students in Megan's class possessed all the resources needed to achieve solution C, I conjecture that there is a low probability that students would have the initiative to alter a diagram on their own.

Other possible solutions using triangle congruency would lead to dead ends, because one would need to assume other properties of rectangles, including the goal of the problem: that the diagonals of a rectangle are congruent. For example, one could prove that a pair of triangles such as triangles *MCP* and *OCN,* or triangles *MCN* and *OCP*, are congruent. First, one would need to prove that a rectangle is a parallelogram.

[30] This theorem does not appear in the standard geometry curriculum. In the geometry textbook by Jurgensen, Brown, and Jurgensen (1994), this theorem appears in the quadrilaterals chapter, in the section on special quadrilaterals, where theorems about rectangles are included (p. 185). In the textbook by Clemens, O'Daffer, Cooney, & Dossey (1992) this theorem does not appear.

Then, these triangles have two pairs of congruent sides because in a parallelogram diagonals bisect each other. So, $\overline{PC}$ and $\overline{NC}$ are congruent; $\overline{MC}$ and $\overline{OC}$ are congruent. Finally, the pairs of triangles would be congruent by side-angle-side (because the included angles are congruent by the vertical angle theorem) or by side-side-side (because opposite sides of a rectangle are congruent). The difficulty with this proof is that one would need a way to state that $\overline{MC}$, $\overline{NC}$, $\overline{OC}$, and $\overline{PC}$ are all congruent. For this purpose one would need to assume that triangles *MCN*, *OCN*, *OCP*, and *MCP* are isosceles; or that if C is the vertex, of any of the triangles (*MCN*, *OCN*, *OCP*, and *MCP*) then the base angles of those triangles are congruent.[31] Alternatively, one would need to assume that the diagonals of a rectangle are congruent, which is what we are trying to prove. So, it is most likely that students in Megan's class would work on solution A, which focuses on proving triangles *MPO* and *NOP* congruent.

Segments in the Rectangle Episode

To do an analysis a posteriori of the task, I divide this episode into nine segments, according to changes in the goals, the resources, and the operations used. Some of these segments overlap because there is some transition in the restatement of the goals of the task. Table 7 shows the segments and a brief description of each.

Table 7

Segments in the rectangle episode

Segment number	**Title**	**Turns**	**Description**
I	*Megan's presentation*	1	Megan presents an algebraic calculation problem. The goal of the problem is to find the value of an unknown.

[31] One could prove that the base angles are congruent or that the triangles are isosceles by making an argument using symmetry. However, proofs relying on symmetry are unusual in the traditional geometry course.

	of the problem		
II	*Alana's partial solution of the algebraic calculation*	2	Alana sets up an equation, using that diagonals of a rectangle are congruent.
III	*Megan's transformation of an algebraic problem into a proof*	3-9	Megan changes the goal of the problem from finding the value of a variable to proving a property. Megan asks students to prove that the diagonals of a rectangle are congruent.
IV	*Alana's failed attempt to prove triangles MCN and PCO congruent*	10-14	Alana tries to prove that the diagonals of a rectangle are congruent by proving a pair of triangles, *MCN* and *PCO*, congruent. Alana uses the vertical angles theorem as a resource to show that there is a pair of congruent angles, $\angle MCN$ and $\angle PCO$. Alana also uses that opposite sides of a rectangle are congruent as a resource to show that $\overline{MN}$ is congruent to $\overline{PO}$. However, Alana does not have sufficient evidence to claim that triangles *MCN* and *PCO* are congruent.
V	*Megan's lead to prove triangle MPO and NOP congruent*	15-26	Megan suggests proving that the diagonals of a rectangle are congruent by proving triangles *MPO* and *NOP* congruent.
VI	*Proving Anil's claim that a rectangle is a parallelogram*	23-34	Anil states that a rectangle is a parallelogram as a rationale for why opposite sides of a rectangle are congruent. Megan changes the goal of the task, asking to prove that a rectangle is a parallelogram. They rely on the definition of a rectangle as a quadrilateral with four right angles. Then they apply tests for parallel lines and the definition of a parallelogram. The product of this task—*MNOP* is a parallelogram—will be a resource for proving triangles *MPO* and *NOP* congruent.
VII	*Continuation of the proof that triangles MPO and NOP are congruent*	34-40	The goal is to prove that triangles *MPO* and *NOP* are congruent. The resources are the definition of a rectangle, the property that opposite sides of a rectangle are congruent, the reflexive property, and the Side-Angle-Side theorem. This proof will be a resource to prove that the diagonals of a rectangle are congruent. Then, in trying to prove triangles congruent, Anil says that the diagonals of a rectangle are congruent. However, Megan has guided the class to prove that the triangles are congruent with the purpose of proving that the diagonals of a rectangle are congruent. Megan reacts to Anil.
VIII	*Conclusion of the proof that the diagonals of a rectangle*	40-46	By applying the definition of congruency and the earlier product that *MPO* and *NOP* are congruent triangles, then the diagonals of the rectangle are congruent.

	are congruent		
IX	*Solution of algebraic calculation*	46-47	The goal is to find a value of x that would make the statement that diagonals are congruent true. The resources are the proven result that the diagonals of a rectangle are congruent. The operations involve setting up an equation and applying algebra to solve for x.

I now summarize the episode by using the segmentation in Table 7. Alana chose to prove two triangles congruent for which not enough information was available to prove congruent.[32] Then, Anil stated that opposite sides in a rectangle are equal. However, Megan did not accept Anil's claim. Anil retracted from his claim and proposed that a parallelogram has two sides that are equal, which prompted Megan's request to prove that a rectangle is a parallelogram. By proving that a rectangle is a parallelogram, a rectangle would inherit those properties of parallelograms that they had studied in class before. Megan recalled the definition of a rectangle and expected students to use that definition to prove that a rectangle is a parallelogram. Then, Anil argued that diagonals of a rectangle are congruent, taking this property as one of the known properties of rectangles, even though this claim is what Megan wanted students to prove. Megan insisted on proving triangles congruent in order to conclude that parts of the triangles would also be congruent as a consequence. Once they finished the proof for the claim that the diagonals of a rectangle are congruent, then, they could use the theorem to solve the original homework exercise.

In different segments, there is evidence for how the teacher and the students negotiated a situation for the new task proposed by Megan. In segment IV, Alana did not

[32] Alana's choice of triangles was not useful to prove the property that diagonals of a rectangle are congruent. If students were to do solution C in Table 6, then they would find useful those triangles that Alana identified. However, as I said earlier, this solution is unlikely.

choose what triangles to prove congruent according to the statement to be proved. Instead, Alana selected a pair of triangles that she thought she could prove congruent. As a result, she was unable to do the proof. In contrast, within the norms of a situation of doing proofs there would not be ambiguities about what triangles to select in relation to the statement to be proved. In segment V, Megan suggests an alternative reading of the diagram than that proposed by Alana. Megan's suggestion could be interpreted as a repair of what would be a usual way to interpret the diagram when *doing proofs*. The diagram suggests a "descriptive mode of interaction," (Herbst, 2004) where one could take the signs given in the diagram, without changing the diagram to identify properties of the diagram. However, Megan had to tell students to focus on overlapping triangles, which involve a different way to visualize the diagram by taking pieces of the diagram apart.

The segmentation of the episode shows that the proof of the claim that the diagonals of a rectangle are congruent included a series of shorter proofs. Megan took the product of a shorter proof as a resource for a subsequent proof, until they proved that the diagonals of a rectangle are congruent. This action is uncharacteristic of the situation of *doing proofs*, where the statement to be proven is unimportant (Herbst & Brach, 2006). At the end of the episode, Megan used the proven result—the diagonals of a rectangle are congruent—as a resource to do the algebraic calculation problem. Thus, Megan took the property as installed. In the following sections I apply discourse analysis to examine what they are talking about and the nature of the conflict between the teacher and the students in the rectangle episode. I start by presenting a conjunction analysis of the rectangle episode. With this analysis, I describe what the teacher and the students were doing, from an observer's perspective. In addition, I identify the avowed purpose of the teacher's work in the rectangle episode.

Conjunction Analysis of the Rectangle Episode

My analysis of conjunctions in the rectangle episode concentrates on identifying which discursive resources speakers made use of in order to represent experience and to establish logical connections. As noted in the methodology chapter, conjunctions are one of the resources speakers use to convey meanings. I examine what speakers are talking about in the rectangle episode. In particular, the study of conjunctions shows the logical links in the text. Thus, the analysis of conjunctions in the rectangle episode allows me to identify the logical links in the proof produced.

I use Martin and Rose's (2003, 2007) tools[33] for the analysis of conjunctions as a discourse system. According to Martin and Rose, conjunctions are "a set of meanings that organize activity sequences on the one hand, and text on the other" (2007, p. 116). They draw upon Halliday and Hasan's (1976) categorization of conjunctions into two major groups: *external conjunctions* and *internal conjunctions.* External conjunctions are those used to organize "events in an activity sequence" (Martin & Rose, 2007, p. 117). Internal conjunctions are those used to organize discourse itself. This distinction is helpful here because it allows establishing differences between those conjunctions that had the purpose of connecting specific statements about geometric figures and those conjunctions intended to connect stages in the argument.

[33] Halliday and Matthiessen (2004, p. 545) refer to internal and external conjunctions in their analysis of enhancing conjunctions, but do not take these functions as central in their organization. For a discussion on how external and internal conjunctions derive from functional uses of language, I refer the reader to Halliday and Hasan (1976, p. 240-241).

Table 8 shows the conjunctions in each clause of the parsed transcript. It will show the reader how speakers established logical relations in the rectangle episode. First, I classify each conjunction according to its type, whether it is an internal conjunction or an external conjunction. Second, I classify conjunctions according to four possible logical relations: *addition*, *comparison*, *time*, and *sequence.* Martin and Rose's (2003) have identified these four categories as the possible logical relations when using conjunctions in the English language. Third, I specify the role of each conjunction following Martin and Rose's (2003, 2007) taxonomy of options of subtypes of conjunctions. Finally, I describe the purpose of the conjunction in the rectangle episode by specifying how a particular conjunction is useful to establish logical connections and to convey meanings in the rectangle episode.

Table 8

Analysis of conjunctions used in the rectangle episode

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
1.	Megan	1. Okay, I want to go over the homework << >> for a few minutes.	Okay	internal	addition	staging, framing	Adding a new stage by framing the homework discussion.
		2. <<I wanna skip problems 1 and 2,					
		3. even though I think	even though	external	consequence	cause, concessive	Conceding that there could be a reason for discussing problems 1 and 2 (as in "although").
		4. those are some of the more interesting questions,>>					
		5. and look at some of the more traditional ones,	and	external	addition	additive, adding	Adding a new action to the action of skipping problems 1 and 2.
		6. so we can flesh out some of the properties of rectangles.	so	external	consequence	purpose, desire, expectant	Explaining the purpose of selecting a traditional problem (as in "so that" or "in order to").

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		7. Okay, where is my homework?	okay	internal	addition	staging, framing	Adding a new stage by framing her search for the handout.
		8. Here we go.					
		9. Okay, {look at} problems like number three.	okay	internal	addition	staging, framing	Adding a new stage by framing the selection of the problem.
		10. Why didn't I write this on the board in between?					
		11. They told you					
		12. that you had a rectangle.					
		13. They said					
		14. that this outer thing was a rectangle, *MNOP*					
		15. And then they give you a little bit of information.	and then	external external	addition time	additive, adding successive, immediate	Adding new information about the problem. Providing successive information about the problem.
		16. Okay, part "a" they say	okay	internal	addition	staging, framing	Adding a new stage by framing the statement of the problem.
		17. that *MO* is					

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		equal to 2*x* minus 8, 18. and *NP* is equal to 23. 19. Okay, someone talk about 20. how I would solve for *x* on this particular problem. 21. *MO* is equal to 2*x* minus 8, 22. *NP* is equal to that. Alana?	and okay	external internal	addition addition	additive, adding staging, framing	Adding more information about *MNOP*. Adding a new stage by framing her request for a volunteer to solve the problem.
2.	Alana	23. Two *x* minus 8 equals 23.					
3.	Megan	24. Okay, what << >> is true? 25. << are you assuming>>	okay	internal	addition	staging, framing	Adding a new stage by framing the request for new information about the assumption made.
4.	Alana	That—					
5.	Anil	26. *MO* is congruent to...					

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
6.	Alana	27. That's a rectangle?					
7.	Megan	28. They told you 29. it was a rectangle. 30. I'm not assuming that. 31. They told you that. Anil, 32. what is it?					
8.	Anil	33. *MO* is congruent to *NP*.					
9.	Megan	Yeah. 34. We haven't had a theorem about that. 35. That the diagonals are congruent. 36. But she's right.	but	external	consequence	cause, concessive	Conceding the reason why the statement proposed by Alana (8.33) is correct even though the class had not discussed the theorem that states that diagonals of a rectangle are congruent

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		37. They are congruent. 38. Can someone say 39. how we could prove 40. that the diagonals of a rectangle are congruent? Alana?					(9.34, 9.35).
10.	Alana	41. ‘cause the triangles are congruent.	‘cause	external	consequence	cause, expectant	Stating the reason why diagonals of a rectangle are congruent.
11.	Megan	42. Which triangles {are congruent}? 43. The big triangles {are congruent}?					
12.	Alana	44. *MCN* and *PCO* {are congruent}.					
13.	Megan	45. Okay, how would I prove 46. those were	okay	internal	addition	staging, framing	Adding a new stage by framing her request for a proof.

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		congruent?					
14.	Alana	47. {I would prove triangles *MCN* and *PCO* congruent} By side…	by	external	consequence	means, expectant	Explaining by what means one would prove that that the two triangles are congruent.
		48. I don't know,					
		49. but I know	but	external	comparison	different, opposite	Contrasting what she knows and what she does not know.
		50. That angle *PCO* and angle *MCN* are equal,					
		51. 'cause they're vertical angles.	'cause	external	consequence	cause, expectant	Stating the reason why two angles are congruent.
		52. And then *MN* and *PO* are a side,	and then	external external	addition time	additive, adding successive, immediate	Adding new information. Adding successive information.
		53. that's, are equal.					
15.	Megan	54. Okay, I'm gonna give you a hint.	okay	internal	addition	staging, framing	Adding a new stage by framing new information.
		55. Quit looking at the little triangles.					
		56. Let's look at					

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		the big triangles.					
		57. Look at triangle *MPO*.					
		58. {Look at} This big triangle.					
		59. And then {look at} this big triangle.	and	external	addition	additive, adding	Adding new information.
			then	external	time	successive, immediate	Adding a new action to be performed.
		60. What'd I have?					
		61. {I would have triangle} *MPO*.					
		62. So it would be *NO*, *P*.	so	external	consequence	cause, expectant	Stating the consequence of the correspondence between pairs of congruent triangles caused by the order established with triangle *MPO*.
		63. How can I prove this?					
		64. Okay, that this big triangle is equal to this one, Anil?	okay	internal	addition	staging, framing	Adding a new stage by framing her request for reasons in the proof.
16.	Anil	65. {I can prove					

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		triangle *MPO* is congruent to triangle *NOP* by} SSS					
		66. or like {I can prove triangle *MPO* is congruent to triangle *NOP* by} SAS	or	external	addition	alternative	Showing an alternative to SSS.
			like	external	comparison	similar	Showing similarity between two ways to prove congruent triangles.
17.	Megan	67. Okay, I know	okay	internal	addition	staging, framing	Adding a new stage by framing the focus on parts of the triangles.
		68. these sides are equal, the bottom					
		69. because it's part of both triangles.	because	external	consequence	cause, expectant	Stating the reason why there is a pair of congruent sides in two triangles.
		70. It's part of this one,					
		71. and it's part of this one.	and	external	addition		Adding statements about two triangles that share the same side.
		72. Okay, what else << >> is true?	okay	internal	addition	staging, framing	Adding a new stage by framing the request of new information.
		73. << do I					

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		know>>					
		Anil?					
18.	Anil	74. The opposite sides are equal Because					
19.	Megan	75. Why do I know that?					
20.	Anil	76. ‘cause in a rectangle the opposite sides {are equal}.	‘cause	external	consequence	cause, expectant	Giving the reason why two opposite sides are equal (19.75).
21.	Megan	77. I don’t know that in a rectangle.					
22.	Anil	78. {in a} parallelogram {the opposite sides are equal}.					
23.	Megan	79. It’s a parallelogram. 80. How could I prove 81. it’s a parallelogram?					
24.	Anil	82. Opposite sides are parallel.					
25.	Megan	83. How do I know					

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		84. the opposite sides are parallel?					
26.	Anil	‘cause					
27.	Megan	85. Maybe we need to talk about 86. what do you need to know 87. for it to be a rectangle. 88. What’s definitely true about a rectangle?	for	external	consequence	purpose, desire, expectant	Establishing what conditions should be met for the purpose of having a rectangle (as in “in order to.”)
28.	Alana	89. {A rectangle has} Four right angles.					
29.	Anil	90. {A rectangle has} Four right angles.					
30.	Megan	91. {A rectangle has} Four right angles. 92. Okay, if I have four right	okay	internal	addition	staging, framing	Adding a new stage by framing a new question.

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		ang {angles} 93. If these two are right angles, 94. why are these sides parallel, Anil?	If	external	consequence	condition, open, expectant	Establishing the condition of a statement.
31.	Anil	‘cause, 95. ‘cause of the, the same side interior angles are…	‘cause	external	consequence	cause, expectant	Giving the reason of a statement.
32.	Megan	96. {The same side interior angles are} supplementary.					
33.	Anil	Yeah.					
34.	Megan	97. I’ve got two there. 98. I could do the same thing here. 99. So you’re right.	So	internal[34]	consequence	concluding, conclude	Showing the conclusion of an argument (as in “thus,” or “in conclusion”).

[34] According to Martin and Rose (2007, p. 139), speakers use “so” to show a conclusion in spoken language. In clause 34.99, I take “so” as an internal conjunction with the purpose of showing the conclusions of an argument. Here there is a connection to something that had been said before, beyond connecting immediate steps.

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		100. I could prove					
		101. these were parallel					
		102. So, it’s a parallelogram.	So	external	consequence	cause, expectant	Showing the mathematical implication (as in “therefore”).
		103. So then that tells me	So	external	consequence	cause, expectant	Showing the consequence of the statement (34.102) that *MNOP* is a parallelogram (as in “therefore”).
			then	external	consequence[35]	condition, open, expectant	Showing what would follow from the condition that two triangles are congruent.
		104. the opposite sides are equal					
		105. ‘cause it’s a parallelogram.	‘cause	external	consequence	cause, expectant	Giving the reason why opposite sides are equal (34.104).
		106. So I have side, side, equal to side, side.	So	external	consequence	cause, expectant	Showing the consequence of previous statements about pairs of congruent sides in two triangles.
		107. What else					

[35] The conjunction “then” could also be coded as an external conjunction to denote time when there is an immediate succession of events. Here, I take “then” as a marker of condition. In mathematical discourse “if…then” statements are usual (O’Halloran, 2005, p. 124) to show the conditions in a mathematical statement.

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
		do I get? 108. It's a rectangle. Anil?					
35.	Anil	109. The two diagonals {are congruent}.					
36.	Megan	No. 110. I don't know 111. the two diagonals {are congruent}. 112. I'm trying to prove that.					
37.	Anil	Oh.					
38.	Megan	Ebony?					
39.	Ebony	113. The angles are equal? 114. They're right angles.					
40.	Megan	Yeah, 115. they're all 90. 116. Like Alana said,	Like	internal	comparison	similarity	Making references to something said before to order arguments, equivalent to "as." Confirming and

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
							affirming what Alana had previously said.
		117. they are all 90.					
		118. So I have side, angle, side.	So	external	consequence	cause, expectant	Showing the mathematical implication of pairs of congruent parts of two triangles.
		119. I could prove this by side-angle-side.					
		120. Then, why are the diagonals equal? Anil?	Then	internal	addition	staging, framing	Showing a new stage of the argument, equivalent to "now."
41.	Anil	<< >>					
42.	Megan	121. If this big triangle {*MPO* is congruent to triangle *NOP*}.	if	external	consequence	condition, open, expectant	Establishing the condition of a statement.
43.	Anil	122. {*MO* and *NP* are equal} <<because>> CPCTC.	because	external	consequence	cause, expectant	Giving the reason why diagonals are congruent.
44.	Megan	123. {*MO* and *NP* are equal because} CPCTC.					

Turn #	Speaker	Clause	Conjunction	Type of conjunction	Logical relation	Role	Purpose
45.	Anil	Yeah.					
46.	Megan	124. Okay, so I could prove that.	okay	internal	addition	staging, framing	Adding a new stage by framing the end of the proof.
			So	internal	consequence	concluding, conclude	Showing the conclusion of an argument.
		125. She is right.					
		126. Okay, I am going to move the 8 over.	okay	internal	addition	staging, framing	Adding a new stage by framing algebraic steps.
		127. Two *x* equals << >> 31					
		128. <<what is that>>					
		129. So *x* is 15.5.	So	external	consequence	cause, expectant	Showing the mathematical implication (as in "therefore").
		130. Is that what you got, Alana?					
47.	Alana	Yeah.					

Logical Links in the Rectangle Episode:
How are statements about rectangle MNOP connected?

In the following sections I discuss how statements about rectangle *MNOP* were connected with logical links using conjunctions. The distinction between external and internal conjunctions is helpful to identify possible differences between the kinds of logical links in the transcript. One could expect that some conjunctions would be used to organize a sequence of activities whereas other conjunctions would be used to organize discourse itself. In addition, different kinds of logical relationships may imply different kinds of meanings. So the logical relations and the roles of the conjunctions in the text are important markers to interpret what speakers are talking about.

External Conjunctions in the Rectangle Episode

After identifying all the conjunctions in the rectangle episode, I found that external conjunctions predominated over internal conjunction. There were a total of 33 external conjunctions and a total of 18 internal conjunctions. Most of the external conjunctions, used by Megan and by those students participating in the discussion, indicate consequence (21 out of 33). The external conjunctions with a logical relation of addition follow the external conjunctions for consequence. There is a total of 7 conjunctions for addition; six of the conjunctions for addition use "and." However, the predominance of the conjunction *and* is not surprising because this conjunction is frequently used in English (Martin & Rose, 2003, p. 113). The finding that most external conjunctions denote consequence is important because it suggests that the rectangle episode is a case where the teacher and the students were making an oral proof. I will argue that, even though neither the teacher nor the students wrote a proof on the board, the logical links in the rectangle episode maintain the same structure than a proof.

My analysis of external conjunctions used to denote consequence shows that, for the most part, the logical links helped speakers to connect components of a mathematical argument. External conjunctions for consequence could be classified into four different types: cause, means, purpose, and condition. Conjunctions for causation—*because, but, even though,* and *so*—were the most common in the rectangle episode (see Table 9). All the uses of "because" (7 in total) preceded a mathematical reason to justify a particular claim. All the uses of "so" to denote consequence (6 in total) connected a previous clause with a mathematical consequence. The conjunctions "but" and "even though" are an exception from the set of external conjunctions to denote causation in that they do not link mathematical statements. Therefore, speakers used a total of 13 out of 33 external conjunctions to denote the cause or the reason of a mathematical statement.

Table 9

External conjunctions in the rectangle episode

Logical relation	Totals	Role	Totals	Conjunction	Frequency
Addition	7	Additive	6	*and*	6
		Alternative	1	*or*	1
Comparison	2	Different	1	*but*	1
		Similar	1	*like*	1
Time	3	Successive	3	*then*	3
Consequence	21	Cause	15	*because, 'cause*	7
				but	1
				even though	1
				so	6
		Condition	3	*if*	2
				then	1
		Means	1	*by*	1
		Purpose	2	*for*	1
				so	1
Total	33				

Other external conjunctions used to denote consequence also have the purpose of connecting mathematical arguments. Speakers used the conjunctions "if" and "then" to draw attention to particular conditions why something may hold true. In her study of mathematical discourse, O'Halloran (2005, p. 124) has pointed out that "if…then" statements are usual conjunctions in mathematics. In fact, many geometry textbooks start the year with a section on how to write the so-called conditional statements in "if…then" form. The conjunction "by" preceded a mathematical justification for a statement (clause 14.47). It is a usual practice in the geometry class to write the theorems and postulate of triangle congruency in the "reasons" column using the "by." For example, one could write, "by SAS" to explain that two triangles are congruent "by means of" or "by using" the side-angle-side (SAS) theorem. Thus, "by" usually precedes the name of the theorem that would back up an assertion. Similarly, the conjunction "for" denoted consequence when Megan said, "what do we need to know for it to be a rectangle" (27.86 and 27.87). I take "for" to be equivalent to "in order to," which Martin and Rose (2003, p. 119) classify as a conjunction to denote purpose. In this turn, "to be a rectangle" is a process. Megan asked for the conditions needed to achieve a purpose: to have a rectangle.

The conjunction "so" in clause 1.6 does not have the role of connecting mathematical arguments. However, this conjunction is important because it shows the purpose for Megan's selection of the homework exercise. Megan said that she wanted to focus on some of "the more traditional" problems "so we can flesh out some of the properties of rectangles." The use of "so" denotes her purpose for deliberately selecting a homework problem to teach students properties of rectangles. Therefore, the consequence of going over homework is that she would expand the discussion of properties of rectangles. Megan's use of "so" stresses her avowed purpose for choosing a particular homework problem in the rectangle episode.

In sum, from a total of 21 external conjunctions to denote consequence, only 3 of those (in clauses 1.3, 1.6, and 9.36) do not link mathematical arguments. Therefore, 18 out of 33 external conjunctions show the consequences of a mathematical argument. This finding suggests that the rectangle episode is a case where the class, led by the teacher, produced an oral proof. Speakers used conjunctions to connect different statements by following the same logical structure than a mathematical proof.

Internal Conjunctions in the Rectangle Episode

In contrast with the variety of external conjunctions, there were only four kinds of internal conjunctions in the rectangle episode: *okay*, *so*, *like*, and *then* (see Table 10). The total of internal conjunctions in the rectangle episode (18) was less than the total of external conjunctions (33). Notably, the teacher was the only one who used internal conjunctions in the rectangle episode; students did not. The internal conjunctions were a resource for the teacher to organize the ideas that went into making an oral proof. Because internal conjunctions have the main function of organizing discourse, the finding that the teacher was the only one who used these conjunctions suggests that the teacher was in control of leading and organizing the argument made through the discussion of the homework problem.

Table 10

Internal conjunctions in the rectangle episode

Logical relation	Totals	Role	Conjunction	Frequency
Addition	15	Staging	*okay*	14
			then	1
Consequence	2	Concluding	*so*	2
Comparison	1	Similar	*like*	1
	18			18

The conjunction *okay* was the most frequent internal conjunction in the rectangle episode (see Table 10).[36] According to Martin and Rose (2007, p. 134), *okay* denotes a logical relation of addition. In particular, *okay* has the function of adding a new stage in an argument. There are two options for adding a new stage using conjunctions. One option is to add stages by framing and another option is to add stages by sidetracking. The conjunction *okay* is used for framing. Other conjunctions that are analog to *okay* are *now, well,* and *alright.* By framing one could focus the presentation of new information. In contrast, by sidetracking one could add a new argument by deviating from the information that has been already presented. Examples of conjunctions used for sidetracking are *anyway, anyhow, incidentally,* and *by the way.* If we were to substitute

[36] Halliday and Hasan (1976, p. 268-269) have proposed that "now" and "well" belong to a third category of conjunctions, which they called "continuatives." These conjunctions share characteristics of internal and of external conjunctions. They note that some of these continuatives, such as "now" and "well" could have the function of connecting responses in a dialogue and are used to make transitions between speakers. Halliday and Hasan have noted that a particular feature of continuatives in a dialogue is that speakers use them at the beginning of an utterance, sometimes in order to stage the response to a question, and by making use of a particular pattern of intonation. Halliday and Hasan's discussion does not list "okay" as a possible conjunction. In some sense, Megan's uses of "okay" could be considered continuatives because most of them were initial responses to previous turns. However, I want to be consistent with the tools provided by Martin and Rose (2003, 2007). They include "okay" as a marker of internal conjunction. Also, they suggest that most continuatives are used inside of a clause, which is not the case for Megan's uses of "okay" in the rectangle episode. Megan always used "okay" at the beginning of a clause.

any of Megan's uses of "okay" for "anyway," one would expect to have a statement that is not directly related to what was said before. So, Megan's uses of "okay" helped students to focus their attention, prior to adding a new stage of the argument, by following the same flow of ideas.

A study by Christie has explained that teachers use conjunctions for framing in early literacy teaching. According to Christie's (2002, p. 66-67) analysis, the teacher's use of conjunctions guided students' work when engaging them in a task. During the task orientation, the teacher "takes a dominant role, introducing what is to be done, and generally marshalling the energies of students towards the achievement of a common purpose" (2002, p. 65). One way in which the teacher oriented students was to say a monologue. Another way in which the teacher oriented students was by marking some of the themes with conjunctions[37] such as "now" and "well." Similarly, in the rectangle episode, Megan used the conjunction *okay* five times in the first turn. The first turn includes 22 clauses by Megan and it is the longest turn in the episode, which has a total of 130 clauses. This suggests that Megan's long monologue and her use of internal conjunctions had the function of setting up the mathematical task in the rectangle episode.

In other turns, besides the first turn, the teacher's use of *okay* seems to serve a dual purpose. On the one hand, Megan framed the introduction of a new idea. In a previous section, I showed a segmentation of the rectangle episode according to changes in goals, the resources, and the operations used. Here, I will argue that for turns other than the first turn, Megan's use of "okay" coincides with the introduction of a new

[37] Christie (2002) follows Halliday and Hasan's (1976) work and calls "now" and "well" continuatives. Continuatives are part of the system of conjunctions.

segment, where there is a change in the mathematical task. At the beginning of segment III, when Megan transforms the algebraic problem into a proof, Megan says, "Okay, what are you assuming is true?" Here, the internal conjunction stages a change of the mathematical task (clauses 3.24 and 3.25). For example, at the beginning of segment V, when Megan asks students to prove triangles *MPO* and *NOP* congruent, Megan says, "Okay, I'm gonna give you a hint" (clause 15.54). Here, the use of "okay" allows her to change the strategy for the proof by focusing on proving congruent another set of triangles, different from the ones suggested earlier. Also, in segment IX, when Megan starts solving the algebraic equation, she says, "Okay, I am going to move the 8 over" (clause 46.126). In this segment, she transitions from having concluded a proof towards finding the value of a variable. These examples show how Megan used an internal conjunction to stage a new mathematical task.

On the other hand, Megan, at times, evaluated previous ideas by saying "okay." This use of "okay" is more than a resource to achieve the ideational metafunction, but a resource within the interpersonal metafunction. For example, when Alana gave the equation to solve the algebraic problem, Megan answered, "Okay, what are you assuming is true?" (clauses 3.24 and 3.25). Megan use of "okay" could be interpreted as a way to give feedback to Alana by stating that her answer is correct. Also, at the end of the proof for the claim that the diagonals of a rectangle are congruent, Megan said, "Okay, so I could prove that" (clause 46.124). This use of "okay" could be interpreted as Megan's approval of the proof. In contrast, Megan was selective about those statements that students could make use of in the proof. For those statements she disapproved, she did not say *okay*. For example, when Anil suggested that they could use that diagonals of a rectangle are congruent (turn 35), Megan replied, “No, I don’t know the two diagonals. I’m trying to prove that” (turn 36). So, Megan’s uses of *okay* allowed students to move forward with the proof once Megan granted permission to do so.

Megan used another internal conjunction for addition besides “okay.” Her use of “then” when she said, “Then, why are the diagonals equal? (clause 40.120), has the purpose of adding a new stage of the argument. In this clause, “then” is analogous to using “now.” One could rewrite the clause as, “Now, why are the diagonals equal?” With this interpretation, I take “then” to be an internal conjunction to develop the argument. So, even though “then” could be used to denote a sequence of events over time or to show a condition, in clause 40.120 Megan frames a new stage of the argument by starting her question with “then.”

Another internal conjunction that Megan used twice was “so.” Megan’s use of “so” allowed her to denote consequence when stating partial conclusions in the development of the proof. Megan used “so” in reference to Anil’s justification for why opposite sides of the figure were parallel. Megan said, “So, you’re right” (clause 34.99) and summarized what they had concluded so far: opposite sides of *MNOP* are parallel. Megan also used “so” to conclude the proof that the diagonals of a rectangle and congruent when she said, “Okay, so I could prove that” (clause 46.124). In both cases, the use of the conjunction “so” coincides with the end of a segment. Turn 34 is the end of segment VI and turn 46 is the end of segment VIII.

In addition to “okay,” “then,” and “so,” Megan used the conjunction “like” to connect something that Ebony said (turn 39) with something that Alana had said before (turn 28). The conjunction “like” could be used as an external conjunction to denote similarity. However, in this case I interpret the use of "like" as an internal conjunction for comparing with a statement previously made. With "like," Megan organized the discourse by making references to what Alana had said earlier. The logical connection is one of comparison, similarity. Megan said, "Like Alana said, they are all 90" (clauses 40.116 and 40.117). Megan connected with what Alana had said previously. This connection is important because Ebony answered a question posed by Megan about conclusions that they could make from the fact that the figure was a rectangle when she asked, "What else do I get? It's a rectangle" (clauses 34.107 and 34.108). Megan related the conclusion that a rectangle has right angles with what Alana had said earlier in the discussion. Megan affirmed what Alana had said.

In conclusion, Megan used the internal conjunction *okay* to link changes in the task in most turns. At the same time, with *okay* Megan showed her approval of what had been said. Megan used *then* to frame a new stage in the argument. Megan used *so* to organize discourse, showing partial conclusions. Also, Megan used *like* as *as,* to relate a statement with what had been said earlier. Some internal conjunctions coincide with the beginning turn or with the end turn of a segment, thus denoting changes in the stages of an argument. So, the internal conjunctions in this episode gave some continuity to a proof that was produced with the input of several students and was never written in full on the board. In the following sections I examine specific turns in depth to show how conjunctions, external and internal, enabled speakers to do a proof.

Clauses and Conjunctions per Turn

In the rectangle episode, there are a total of 130 clauses and 51 conjunctions. This is a ratio of 2.5 clauses per conjunction, which means that for every two to three clauses one could expect a conjunction. The total number of turns in the episode is 47, which is roughly a 1:1 ratio between conjunctions and turns. However, turns vary in the number of clauses. There are turns with many more clauses than others. Also, there are turns with more conjunctions than others. The average of clauses per turn is 2.8, which characterizes the episode as a series of many brief exchanges between speakers. By finding the number of conjunctions per turn, I identify those turns that have a high concentration of conjunctions (see Table 11).

The four turns with the highest number of conjunctions are: turn 1 with 11 conjunctions, turn 34 with 6 conjunctions, turn 14 with 5 conjunctions, and turn 15 with 5 conjunctions. Table 11 shows that these turns, especially turn 1, have more conjunctions than the other turns. These turns have also more clauses than the average number of clauses per turn. Turn 1 has 22 clauses, turn 34 has 12 clauses, turn 15 has 11 clauses, and turn 14 has 7 clauses. The teacher was the speaker in three of these four turns. A student was the speaker in turn 14. I conduct further analysis of these four turns to study how speakers use conjunctions to make logical connections. Through my analysis, I show the content of their talk.

Table 11

Number of clauses and conjunctions in the rectangle episode

Cluster	Turn #	Speaker (Teacher or Student)	Number of clauses	Total number of conjunctions	External conjunctions	Internal conjunctions
	1*	T	22	11	6	5
	2	S	1	0	0	0
	3	T	2	1	0	1
	4	S	0	0	0	0
	5	S	1	0	0	0
	6	S	1	0	0	0
	7	T	5	0	0	0
	8	S	1	0	0	0
[9, 10]	9	T	7	1	1	0
[9. 10]	10	S	1	1	1	0
	11	T	2	0	0	0
	12	S	1	0	0	0
[13, 17]	13	T	2	1	0	1
[13, 17]	14*	S	7	5	5	0
[13, 17]	15*	T	11	5	3	2
[13, 17]	16	S	2	2	2	0
[13, 17]	17	T	7	4	2	2
	18	S	1	0	0	0
	19	T	1	0	0	0
	20	S	1	1	1	0
	21	T	1	0	0	0
	22	S	1	0	0	0
	23	T	3	0	0	0
	24	S	1	0	0	0
	25	T	2	0	0	0
	26	S	0	0	0	0
	27	T	4	1	1	0
	28	S	1	0	0	0
	29	S	1	0	0	0
[30, 31]	30	T	4	2	1	1
[30, 31]	31	S	1	1	1	0
	32	T	1	0	0	0
	33	S	0	0	0	0
	34*	T	12	6	5	1
	35	S	1	0	0	0
	36	T	3	0	0	0
	37	S	0	0	0	0
	38	T	0	0	0	0
	39	S	2	0	0	0

Cluster	Turn #	Speaker (Teacher or Student)	Number of clauses	Total number of conjunctions	External conjunctions	Internal conjunctions
	40	T	6	3	1	2
	41	S	0	0	0	0
[42, 43]	42	T	1	1	1	0
[42, 43]	43	S	1	1	1	0
	44	T	1	0	0	0
	45	S	0	0	0	0
	46	T	7	4	1	3
	47	S	0	0	0	0
	Totals		**130**	**51**	**33**	**18**

Since the rectangle episode has sequences of brief exchanges between speakers, as noted by how speakers alternate clauses, it is possible that conjunctions are grouped in successive turns, and not just in one turn. I call a *cluster* a set of successive turns, where all turns have at least one conjunction. There are four clusters: (1) turns 9 and 10, (2) turns 13 through 17, (3) turns 30 and 31, and (4) turns 42 and 43. I define the *density* of a cluster as the ratio of total number of conjunctions in the cluster to the total number of turns in the cluster (conjunctions per turn). The cluster with the highest density is the cluster [13, 17] (see Table 12). The cluster [13, 17] involves more than two successive turns. Also, the cluster [13, 17] has a total of 17 conjunctions, which is a third of the total number of conjunctions in the episode. As I noted earlier, turns 14 and 15 have a high number of conjunctions per turn. So, it seems appropriate to zoom into the analysis of the cluster that includes turns 13 through 17 in order to examine how speakers used conjunctions to convey meanings.

Table 12

Density of clusters of conjunctions in the rectangle episode

Cluster	**Turns in the cluster**	**# of conjunctions in the cluster**	**# of turns in the cluster**	**Density of a cluster (conjunctions per turn)**
[9, 10]	9, 10	2	2	1
[13, 17]	13, 14, 15, 16, 17	17	5	3.4
[30, 31]	30, 31	3	2	1.5
[42, 43]	42, 43	2	2	1

In conclusion, I used two criteria to select particular turns for further analysis. One criterion is the number of conjunctions per turn. I selected the four turns with the highest number of conjunctions: turns 1, 14, 15, and 34. Another criterion is the cluster with the highest density of conjunctions per turn. This cluster includes turns 13 through 17. In the following section I present the analysis of specific turns.

Analysis of Specific Turns

Turn 1.

In turn 1, Megan presents the homework problem to be discussed in class (see Table 13). She begins the turn with an internal conjunction, "okay," to stage the presentation of the homework problem. Megan said, "Okay, I want to go over the homework." By staging, she prepares students to present new information. Megan uses "okay" to frame her introduction of the activity of homework discussion. Instead of *framing*, another option that she could have to stage her presentation of a homework problem would be *sidetracking* (Martin & Rose, 2007, p. 134). By sidetracking she could have said, "Anyway, I want to go over the homework." Here, the conjunction "anyway" would mean that she is taking off from whatever they were doing before. However, she uses "okay" to frame a new stage, focusing students' work.

Megan makes the deliberate choice of skipping the first two problems. She uses the conjunction "even though" to concede that these could be interesting problems. Then, she uses "and" to add the action of looking at "more traditional" problems to the action of skipping some problems. The conjunction "so" has the role of giving the purpose of Megan's choice of a problem: to "flesh out some of the properties of rectangles." This is important because Megan makes explicit her reasons for selecting a specific homework problem as an opportunity to teach some of the properties of a special quadrilateral, a rectangle.

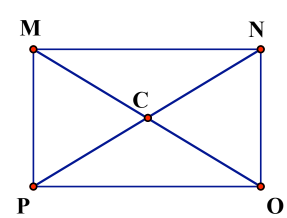
In clause 1.7, Megan changes the content of her talk momentarily, with her search of the handout with the homework problems on her desk. She uses "okay" to frame this diversion from the discussion of the homework problem. Then, she uses "okay" in clause 1.9 to introduce problem number three, preceding her reading of the problem and her drawing on the board. Clause 1.10 is another brief diversion where she is apparently stating her regret for not writing the problem on the board before the start of this class, in between different class periods. This clause gives more evidence for her deliberate choice to discuss problem 3, even before the beginning of the class. Megan wished that she had prepared earlier to discuss this homework problem. The conjunctions "and" and "then" in clause 1.15 provide continuity to the process of adding new information given in the problem. Megan uses "okay" to read the first part of the problem, part "a," which is the focus of the discussion of the homework problem. Finally, Megan uses "okay" in clause 1.19 with the purpose of framing her request for a volunteer to solve the problem.

In sum, Megan uses "okay" five times during the first turn. Her frequent use of this conjunction suggests that in this turn she is framing the discussion of the homework problem. Megan also uses "okay" when performing managerial tasks such as finding the handout with the homework problems, choosing the problem that should be discussed, and deciding who should present the problem to class. Megan uses "and" three times with the purpose of adding more information. The frequent use of "and" is characteristic of the English language, and it is not surprising in this text.

A salient feature of this turn is her use of "so" to denote purpose (clause 1.6). There are only two times in the rectangle episode when a speaker uses a conjunction to denote purpose (clauses 1.6 and 27.87). This time, Megan uses a conjunction to explicate the purpose of her choice of a problem that would lead them to work on properties of rectangles.

Table 13

Megan's turn 1 in the rectangle episode parsed by clause

Clause	Conjunction	Board content
1. Okay, I want to go over the homework << >> for a few minutes.	Okay	 a. MO = 2x - 8 NP = 23
2. <<I wanna skip problems 1 and 2,		
3. even though I think	even though	
4. those are some of the more interesting questions,>>		
5. and look at some of the more traditional ones,	and	
6. so we can flesh out some of the properties of rectangles.	so	
7. Okay, where is my homework?	Okay	
8. Here we go.		
9. Okay, {look at} problems like number three.	Okay	
10. Why didn't I write this on the board in between?		
11. They told you		
12. that you had a rectangle.		

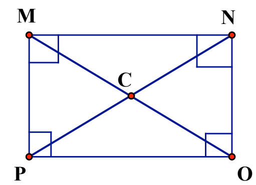
12a. [Draws rectangle.] 13. They said 14. that this outer thing was a rectangle, *MNOP*. 14a. [Draws diagonals of rectangle.] 14b. [Labels points.] 15. And then they give you a little bit of information. 16. Okay, part “a” they say 17. that MO is equal to 2*x* minus 8, 18. and *NP* is equal to 23. 19. Okay, someone talk about 20. how I would solve for *x* on this particular problem. 21. *MO* is equal to 2*x* minus 8, 22. *NP* is equal to that. 22a. Alana?	 And then Okay and Okay	

Turn 13.

Prior to turn 13, Alana had suggested a reason why diagonals of a rectangle should be congruent. Alana identified a pair of congruent triangles made by the diagonals: triangles *MCN* and *PCO*. In turn 13, Megan requested Alana for an idea of a proof for those triangles she had said would be congruent (Table 14). Megan used “okay” to frame a new stage in the argument. One could expect that in response, Alana would provide a justification for proving that triangles *MCN* and *PCO* are congruent.

Table 14

Megan's turn 13 in the rectangle episode parsed by clause

Clause	Conjunction	Board content
45. Okay, how would I prove 46. those {triangles *MCN* and *PCO*} were congruent?	okay	 a. MO = 2x - 8 NP = 23 2x - 8 = 23

Turn 14.

Alana starts turn 14 by giving a reason to prove triangles *MCN* and *PCO* congruent (Table 15). She uses the conjunction "by" to name a theorem for triangle congruency. It is a usual practice in a geometry class to precede the statement of the theorem with the conjunction "by." For example, one would say, "by definition" or "by subtraction property of equality."[38] This use of the conjunction "by" could be read as "this claim could be made by means of theorem X."

Alana elides the reason why the triangles are congruent. She said, "By side," and then starts a new clause (clause 14.47). This is an example of an "abandoned clause" (Eggins & Slade, 1996, p. 106). However, there are two possible theorems that could be used to prove triangle congruency that start with "side." One theorem is Side-Angle-Side and the other theorem is Side-Side-Side. She followed her attempt to give a reason to

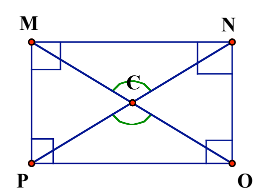
[38] In the congruent triangles chapter of the textbook used in Megan's class, paragraph proofs include this way of presenting reasons preceded by the conjunction "by" (see the paragraph proof in Boyd et al., 1998, p. 215, example 1).

prove that the triangles are congruent with, “I don’t know” (clause 14.48). At this point of the class, Alana did not have enough evidence to use any of those two theorems. This may explain why she did not give the full statement of the theorem.

In clause 14.49, Alana uses “but” to contrast what she knows and what she doesn’t know. Then, Alana presents new information about a pair of congruent angles: angles *PCO* and *MCN* (clause 14.50). Alana uses “*‘cause”* to state the reason why the two angles are congruent. So, one could expect that the two angles are congruent because they are vertical angles. Finally, Alana uses “and” and “then” to provide new successive information to prove that the two triangles are congruent (clause 14.52). In particular, Alana says that there is a pair of congruent sides. However, she is missing information about the triangles to prove that the triangles are congruent by Side-Angle-Side (SAS) or by Side-Side-Side (SSS).

Table 15

Alana’s turn 14 in the rectangle episode parsed by clause

Clause	Conjunction	Board content
47. {I would prove triangles *MCN* and *PCO* congruent} By side{ }.	By	
48. I don’t know,		
49. but I know	but	
50. that angle *PCO* and angle *MCN* are equal,		
51. ‘cause they’re vertical angles.	‘cause	
51a. [Megan marks vertical angles congruent.]		a. MO = 2x - 8 NP = 23
52. And then *MN* and *PO* are a side,	And then	2x - 8 = 23
53. that’s, are equal.		

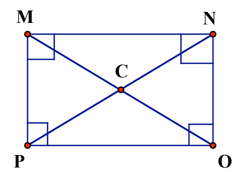
Turn 15.

Megan starts turn 15 by using "okay" to frame the presentation of new information (see Table 16). She calls this information "a hint" for Alana, who is having difficulties with the proof. The hint suggests Alana to look for another pair of triangles. Then, Megan focuses students' attention to triangles *MPO* and *NOP*. In order to add to this new information, Megan uses the conjunctions "and" and "then" (clause 15.59). Megan uses "so" to denote the consequence of having chosen a particular order for the vertices of the first triangle. Since the first triangle is triangle *MPO*, then the second triangle should be triangle *NOP*. In geometry, the order in which one names the vertices of pairs of congruent triangles is important because the statement of a congruence between triangles implies a correspondence between the vertices of the triangles that are purportedly congruent. Megan pauses to get the correct correspondence, so that the vertices of triangle *MPO* correspond to the vertices of triangle *NOP*, in that order. That is, vertex *M* in the first triangle corresponds to *N* in the second triangle, and so forth.

Megan uses "okay" to frame her request for new information in clause 15.64. In the preceding clause, she had already asked how to prove that triangles *MPO* and *NOP* are congruent. Her use of "okay" is a way to rephrase the question while tracing the triangles. The answer to this question would constitute a new stage in the proof.

Table 16

Megan's turn 15 in the rectangle episode parsed by clause

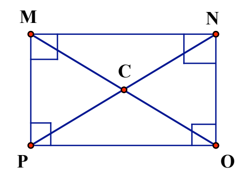
Clause	Conjunction	Board content
54. Okay, I'm gonna give you a hint.	Okay	$\Delta MPO \cong \Delta NOP$  a. MO = 2x - 8 NP = 23 2x - 8 = 23
55. Quit looking at the little triangles.		
55a. [Erases marks for vertical angles.]		
56. Let's look at the big triangles.		
57. Look at triangle *MPO*.		
57a. [Writes Δ*MPO*.]		
58. {Look at} This big triangle.		
59. And then {look at} this big triangle.	and then	
59a. [Traces Δ*NPO*.]		
60. What'd I have?		
61. {I would have triangle} *MPO.*		
62. So it would be *NO*, *P*.	so	
62a. [Writes that Δ*MPO* is congruent to Δ*NOP*.]		
63. How can I prove this?		
64. Okay, that this big triangle is equal to this one,	Okay	
64a. [Traces Δ*MPO*.]		
64b. [Traces Δ*NOP*.]		
64c. Anil?		

Turn 16.

Anil names two possible theorems to justify that the triangles are congruent: Side-Side-Side (SSS) or Side-Angle-Side (SAS). These two alternatives are of the same kind, so they pertain to the same class of theorems for proving triangles congruent. With the conjunction "or," Anil adds a new alternative (Table 17). With the conjunction, "like," Anil denotes that SAS is a co-class when compared to SSS.

Table 17

Anil's turn16 in the rectangle episode parsed by clause

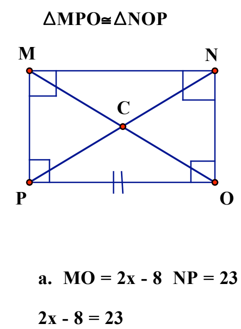
Clause	Conjunction	Board content
65. {I can prove triangle *MPO* is congruent to triangle *NOP* by} SSS 66. or like {I can prove triangle *MPO* is congruent to triangle *NOP* by} SAS	 or like	ΔMPO≅ΔNOP  a. MO = 2x - 8 NP = 23 2x - 8 = 23

Turn 17.

In clause 17.67, Megan uses "okay" to add a new stage to the proof and in particular, to frame the focus on congruent parts of the two triangles (see Table 18). In clause 17.68, Megan introduces a pair of congruent sides: $\overline{PO}$ is congruent to $\overline{PO}$. Megan uses "because" to preface the reason by which this side is of interest, "because it's part of both triangles" (clause 17.69). Then she elaborates on what triangles she is referring to, using "and" to connect statements about the triangles (clause 17.71). Megan concludes her turn by requesting more information (clauses 17.72 and 17.73). Megan uses "okay" to frame her request of more information that would lead to prove that triangles *MPO* and *NOP* are congruent.

Table 18

Megan’s turn 17 in the rectangle episode parsed by clause

Clause	Conjunction	Board content
67. Okay, I know	okay	
68. these sides are equal, the bottom the bottom,		
68a. [Adds hash marks to *PO*.]		
69. because it’s part of both triangles.	because	
70. It’s part of this one,		
70a. [Traces triangle *MPO*.]		
71. and it’s part of this one.	and	
71a. [Traces triangle *NPO*.]		
72. Okay, what else << >> is true?	okay	
73. << do I know>>		
73a. Anil?		

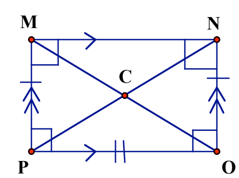
Turn 34.

Megan starts the turn by elaborating on an idea proposed earlier by Anil (turn 31). Anil had said how to prove that opposite sides of the figure are parallel. Megan uses the conjunction “so” four times in turn 34 (see Table 19). The first *so* (clause 34.99) denotes her agreement with Anil’s claim that there is a pair of supplementary angles. Anil’s claim was the basis for concluding that opposite sides are parallel. Megan uses “so” as one would use “ “thus” to denote that a conclusion of a part of the argument. The second *so* (clause 34.102) is a marker for a mathematical implication based upon the preceding statement: *MNOP* is a parallelogram. The conjunctions *so* (clause 34.103), *then* (clause 34.103) and *‘cause* (clause 34.105) denote that, they can prove opposite sides congruent as a consequence of proving that *MNOP* is a parallelogram. Finally, Megan uses *so* (clause 34.106) to point those sides that are congruent as a consequence of having a parallelogram.

Turn 34 illustrates how Megan uses conjunctions as a resource for going over the steps of a proof. Megan was going over the steps for proving that *MNOP* is a parallelogram. The six conjunctions in this turn (*'cause*, *so*, and *then*) denote consequence. In the rectangle episode, there are 23 conjunctions (external and internal) to show a logical relationship of consequence. So, the conjunctions to denote consequence in this turn are a little bit more than one fourth of the conjunctions to denote consequence in the rectangle episode.

Table 19

Megan's turn 34 in the rectangle episode parsed by clause

Clause	Conjunction	Board content
97. I've got two there.		ΔMPO≅ΔNOP
97a. [Points to angles *MPO* and *NOP*.]		
98. I could do the same thing here.		
98a. [Points to angles *NMP* and *MPO*.]		
99. So you're right.	so	
100. I could prove		
101. these were parallel		
101a. [Marks opposite sides parallel.]		a. MO = 2x - 8 NP = 23
102. So, it's a parallelogram.	so	2x - 8 = 23
103. So then that tells me	so then	
104. the opposite sides are equal		
104a. [Marks sides *MP* and *NO* congruent.]		
105. 'cause it's a parallelogram.	'cause	
106. So I have side, side, equal to side, side.	so	
106a. [Points to *MP* and *PO*.]		
106b. [Points to *NO* and *PO*.]		
107. What else do I get?		
108. It's a rectangle.		
108a. Anil?		

The proof about the claim that a rectangle is a parallelogram had started prior to turn 34, when Megan asked students to recall the definition of a rectangle in turn 27.

Megan held students accountable for remembering this definition when she asked, “What’s definitely true about a rectangle?” (clause 27.88). Megan had made Anil aware of the “given” condition in the proof (clause 30.92). She used the conjunction *if* to say that *MNOP* has four right angles. In stating the condition of the proof, Megan was making use of the definition of a rectangle stated earlier in the class. Then she asked a “why” question, probing Anil to state the reason why $\overline{MP}$ and $\overline{NO}$ are parallel (clause 30.94). Anil answered using a conjunction that denoted consequence, “’cause of the, the same side interior angles are…” (clause 31.95). Anil’s answer is an abandoned clause, which Megan completed when she said, “supplementary” (clause 32.96).

Summary of Analysis of Specific Turns

From my analysis of specific turn in the rectangle episode, I found that the external conjunctions were mostly used to denote causal relationships between different statements about properties of geometric figures. The external conjunctions carried the function of giving some structure to the oral proof that resulted from class discussion. Even though Megan used the board to keep a record of essential steps in the proof, Megan did not write each step in a two-column proof. Megan recorded in the diagram those statements that they could make use of, as they justified each statement. The final diagram did not illustrate the temporal nature of how these logical statements were connected. However, by looking at the sequence of diagrams in the transcript, I could match added features to the diagram (such as hash marks and markings for angles) with those new statements about the diagram. The external conjunctions helped in separating proven statements and their consequences, as one would list statements and reasons in a two-column proof, from left to right (see Figure 11).

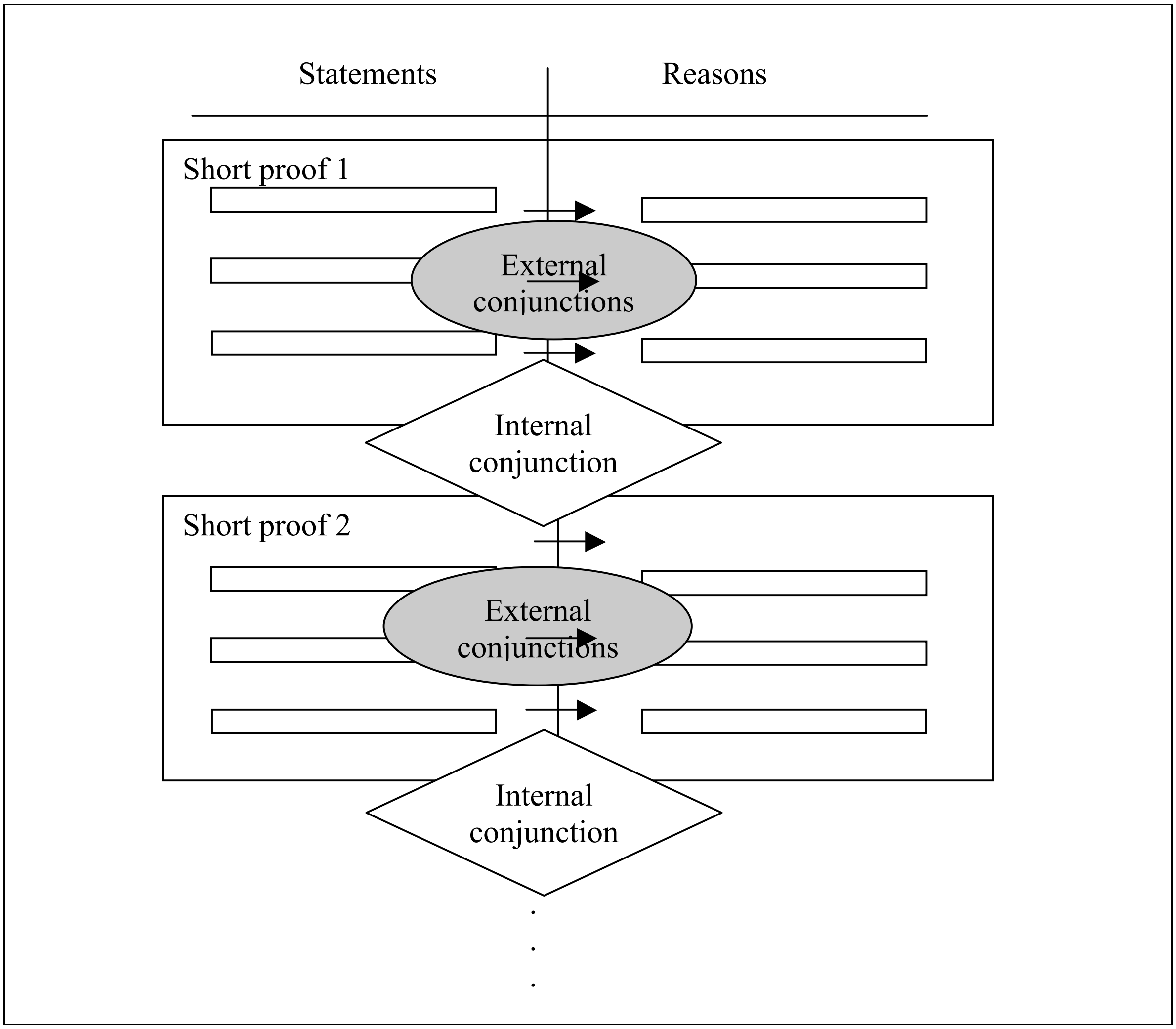


Figure 11. How conjunctions structured the argument as a two-column proof.

The internal conjunctions helped Megan to explicate the reasons for choosing the homework problem to discuss in the first turn. In other turns, internal conjunctions were useful for adding new stages in the argument or by bringing to closure what had been said so far, thus concluding a part of the argument in the proof. So, internal conjunctions could have the same function as the spatial organization horizontal rows in a two-column proof (see Figure 11). In particular, internal conjunctions may be helpful to group a sequence of statements and reasons. That group of sequence of statements and reasons could make up a short proof. By connecting all those short proofs, one gets the final proof of the statement that the diagonals of a rectangle are congruent.

Summary of Conjunction Analysis of the Rectangle Episode

I have presented two ways of analyzing conjunctions in the rectangle episode. After coding individual clauses, looking for external or internal conjunctions, I made a summary of the conjunctions used throughout the episode. From this analysis, I showed that conjunctions performed the function of organizing an oral proof in the rectangle episode for the claim that diagonals of a rectangle are congruent. The evidence for this result is the predominance of conjunctions to denote consequence. The teacher was the only speaker using internal conjunctions, which suggests that she was in charge of organizing the instructional dialogue. In addition, I demonstrated that the teacher had a deliberate purpose when she chose the homework problem to be discussed in class. Her choice was geared towards explicating some of the properties of rectangles.

The second approach to the analysis of conjunctions in the rectangle episode has to do with the selection of particular turns, with the purpose of analyzing how speakers used conjunctions to convey meanings and to organize the oral proof. I used two criteria to choose the specific turns: selecting the four turns with most conjunctions and selecting the cluster with the highest density of conjunctions per turn. I selected seven turns (1, 13, 14, 15, 16, 17, and 34) for an in-depth analysis of the conjunctions speakers used. From this analysis, I concluded that the external and the internal conjunctions enabled speakers to organize differently the ideas of the oral proof in the rectangle episode. On the one hand, the external conjunctions were useful to link particular statements with the reasons supporting that statement. The internal conjunctions, on the other hand, were a resource for dividing different parts of a long proof into shorter proofs. Internal conjunctions were useful to reach to a partial conclusion or to start a new stage in the argument. In particular, the teacher brought about structure to the proof by giving students some benchmarks whenever they needed to start or to finish a piece of the argument in the proof. In the case of her choice of an internal conjunction, "okay," the teacher gave feedback to students' ideas, approving those that would go into the proof. So, the internal conjunction "okay" was also convenient for the teacher to evaluate statements proposed by students.

In sum, in the rectangle episode students were given a diagram of a rectangle and its diagonals. Students took for granted a property of a rectangle that had not been officially introduced in class yet: rectangles have congruent diagonals. The teacher asked students to prove this property. In doing the proof, the teacher and the students used conjunctions to create a discourse that offered a different interpretation of the diagram. In this new interpretation of the diagram, students could only take for granted some properties. For example, students could assume that the given figure was a rectangle. Through class discussion, students got to know other properties of the figure. As a result, they proved that the diagonals of a rectangle are congruent. In the following section I study other discursive resources that speakers made use of in the rectangle episode in order to bring about continuity to the proof. In particular, I examine how speakers visualized the rectangle as other geometric figures in order to produce the proof. This question is important because it points to the kinds of resources that students had in order to do a mathematical task. Some of these resources include students' prior knowledge.

What Kind of Geometrical Object is MNOP?

In this section, I examine different ways in which speakers, the teacher and the students, talked about the quadrilateral *MNOP* during the rectangle episode. Megan wanted to prove that the diagonals of a rectangle are congruent. It was given that *MNOP* was a rectangle. Yet, in the process of doing the proof, speakers visualized *MNOP* (and its diagonals) as other kinds of configurations different from a rectangle such as a set of congruent triangles, pairs of parallel lines, or a parallelogram. By visualizing *MNOP* as something else rather than a rectangle, speakers used known properties of those other configurations in the proof. I focus on identifying what properties speakers ascribed to *MNOP* in doing the proof for the claim that diagonals in a rectangle are congruent.

Table 21 lists all clauses and actions in the rectangle episode that include a "being" process (Martin & Rose, 2003, p. 76-83).[39] In this text, "being" processes state what a geometric object *is*, or what a geometric object *has*, as speakers talk about definitions and attributes of geometric figures.[40] I have written each clause as a statement with a "being" process, with the purpose of examining how speakers used these statements to establish relationships between geometric objects. Statements with "being" processes are important because they show what kind of geometric figure *MNOP* is taken to be, and what kind of properties speakers ascribe to it.

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[39] For the purpose of this analysis, the simplified version of processes by Martin and Rose (2003) suffices because I am mostly interested in "being" processes, which are equivalent to the relational and existential processes identified by Halliday and Matthiessen (2004, p. 210).

[40] Halliday and Matthiessen (2004, p. 214) state, "the verbs that occur most frequently as the Process of a 'relational' clause are *be* and *have;* and they are typically both unaccented and phonologically reduced (e.g. */z/* in *she's happy*)—the 'copula' or 'copular verb' of traditional grammar."

In the statements, I recover elements that may have been omitted or implied by speakers. Speakers could have pointed to or construed these elements in pronouns. Speakers could have also elided elements in their talk. So, with the aim of writing a statement, I refer to the expanded clause. I use the expanded clause to show what speakers were alluding to in their talk. For example, the teacher said, "It's part of this one" in clause 17.70. A statement equivalent to this clause is "*PO* is part of triangle *MPO*." The pronoun "it" corresponds to side "*PO*," and "this one" corresponds to "triangle *MPO*." The expanded clause in the transcript includes references to gestures that match speakers' use of pronouns. In this example, the teacher had added hash marks to side *PO* (17.68a) prior to saying the clause. The teacher had also traced triangle *MPO* (17.70a) immediately after saying the clause. Therefore, in the statements I reconstruct and I specify elements that may be missing in the corresponding clause, based upon other evidence in the text.

Statements include spoken clauses, inscriptions on the board, and alterations to the diagram on the board. Sometimes the teacher made changes to the diagram, such as adding markings for congruent parts or for parallel sides. With these markings the teacher made statements about the diagram, even when these were not articulated in words. For example, when the teacher wrote on the board "$\Delta MPO \cong \Delta NOP$ *by SAS*," I render this as: *Triangle MPO is congruent to triangle NOP by side-angle-side* (statement 65 in Table 21). Also, when the teacher marked all angles of *MNOP* right, I represent this action as a "being" statement: *A rectangle has all angles right* (statement 11 in Table 21). So, I give as much importance to changes to the diagram on the board as to verbal statements by speakers, because the diagram is part of the multi-semiotic meaning-making system in the geometry class and is a focal point for discussing and evolving understanding about quadrilateral *MNOP* during the rectangle episode.

Table 21 also shows "being" processes classified into five different categories according to their purpose: to describe a *quality*, to *classify* something (or someone), to show a relationship between *parts* and a whole, to give the *identity* of an entity, or to present an entity as simply *existing* (Martin & Rose, 2003, p. 76-82). The classification of "being" processes was geared toward pointing to taxonomic relationships in the statements about *MNOP*. In particular, I wanted to extract from the "being" statements information about how speakers visualized different configurations to describe *MNOP*. Also, I expected that the classification of these statements would point to mathematical relationships or mathematical properties of the geometric figures. For example, a statement like "*MNOP* is a parallelogram" has the purpose of classifying *MNOP* as a member of a particular *class* of quadrilaterals: parallelograms. Yet, a statement such as "*MNOP* has four right angles," has the purpose of identifying *parts* of *MNOP*: the angles. Thus, by classifying "being" statements according to their purpose I identify what kind of geometrical object *MNOP* is visualized as throughout the rectangle episode.

Finally, Table 21 shows what the proclaimed status of the statement is from the teacher's or the student's perspective. Sometimes speakers labeled the statement with their status. I use a speaker's label of a statement to code each statement according to the possible status of a statement in a proof or in an algebraic problem. Table 20 shows a description of the seven codes I used: (1) *give*n, (2) *to prove,* (3) *installed proposition,* (4) *not installed proposition,* (5) *claim,* (6) *proven,* and (7) *solution.* These codes are not meant to be exhaustive, but they are sufficient for coding statements in the rectangle episode. Usually, the status of a statement would change according to the moment when speakers state it during the episode. That is, a statement does not have a fixed status; the status of a statement may change as the proof is developed over time. "Given" statements are exceptions in that they state the conditions of a problem, and these do not change through the proof. Similarly, speakers could have already known the propositions because these could have been installed in class in the past. However, within the rectangle episode, speakers had the opportunity to change the status of the statement "the diagonals of a rectangle are congruent" from a "not installed proposition" to an "installed proposition." Once a statement has been "proven" or otherwise "installed," then its status remains the same thereafter.

Table 20

Codes for classifying the proclaimed status of a statement

Code for the proclaimed status of a statement	**Description of the code**	**Examples from the rectangle episode**
1. *Given*	Conditions taken for granted in the problem, including elements of the diagram that could be assumed (such as incidence properties) or that are stated with markings.	*MNOP* has diagonals MO and *NP*.
2. *To prove*	Unknowns that must be found out by applying deductive reasoning.	*MO* and *NP* are congruent.
3. *Installed proposition*	Theorems, definitions, and postulates already accepted as true statements in the class. Sometimes propositions are not proved (e.g., postulates and definitions are not proved).	A parallelogram has two pairs of parallel sides.
4. *Not installed proposition*	Theorems, definitions, and postulates that had not been accepted as true statements in	The diagonals of a rectangle are congruent.

	the class yet.	
5. Claim	Reasons that connect inferences from a given statement or a previously proven statement with an installed proposition.	*MNOP* is a parallelogram.
6. Proven	Conclusions about the geometric diagram that had been accepted as true through deductive reasoning. Some proven statements are preceded by the conjunctions "so" or "then."	Triangle *MPO* is congruent to triangle *NOP* (once proven to be true).
7. Solution	Algebraic statements that give the answer to a problem.	Two *x* minus 8 equals 2.

In Table 21 I also identify changes to the status of a statement. These changes point to knowledge that the teacher did not allow students to use when doing a proof.

Table 21

Statements with "being" processes, the classification according to their purpose, and speakers' proclaimed status of the statement

Number of statement	**Speaker**	**Clause #**	**Clause**	**Statements with "being" processes**	**Purpose (*quality, class, part, identity, existence*)**	**Teacher's proclaimed status**	**Students' proclaimed status**	**Evidence for status**	**Are there changes to the status of a statement?**
1.	Teacher	1.12	that you had a rectangle	There is a rectangle.	existence	given		"They told you" (1.11)	
2.	Teacher	1.14	that this outer thing was a rectangle, *MNOP*	*MNOP* is a rectangle.	class	given		"They said" (1.13)	
3.	Teacher	1.14a	[Draws diagonals of rectangle.]	*MNOP* has diagonals *MO* and *NP*.	part	given		Draws diagonals (1.14a)	
4.	Teacher	1.17	that *MO* is equal to 2*x* minus 8	*MO* is equal to 2*x* minus 8.	identity	given		"they say" (1.16)	
5.	Teacher	1.18	and *NP* is equal to 23.	*NP* is equal to 23.	identity	given		"they say" (1.16)	
6.	Teacher	1.21	*MO* is equal to 2*x* minus 8	*MO* is equal to 2*x* minus 8.	identity	given		"this particular problem" (1.20)	
7.	Teacher	1.22	*NP* is equal to that	*NP* is equal to 23.	identity	given		"this particular problem" (1.20)	
8.	Student	2.23	two *x* minus 8 equals 23	Two *x* minus 8 equals 23.	identity		Solution	"how would I solve for *x*" (1.20)	

Number of statement	**Speaker**	**Clause #**	**Clause**	**Statements with "being" processes**	**Purpose (*quality, class, part, identity, existence*)**	**Teacher's proclaimed status**	**Students' proclaimed status**	**Evidence for status**	**Are there changes to the status of a statement?**
9.	Student	6.27	That's a rectangle	*MNOP* is a rectangle.	class		to prove	"what are you assuming is true?" (3.24 and 3.25)	Yes
10.	Teacher	7.29	it was a rectangle	*MNOP* is a rectangle.	class	given		"they told you" (7.28); "I'm not assuming that" (7.30)	Yes
11.	Teacher	7.30a	[Marks right angles.]	A rectangle has four right angles.	part	installed proposition		Megan marks right angles. (7.30a)	
12.	Student	8.33	*MO* is congruent to *NP*.	*MO* is congruent to *MP*.	identity		to prove	"what is it {the assumption}" (7.32)	
13.	Teacher	9.35	That the diagonals are congruent.	The diagonals of a rectangle are congruent.	identity	to prove		"what is it {the assumption}" (7.32); "we haven't had a theorem about that" (9.34)	Yes
14.	Teacher	9.37	They {the diagonals} are congruent.	The diagonals of *MNOP* are congruent.	identity	claim		"she's right" (9.36)	Yes
15.	Teacher	9.40	that the diagonals of a rectangle are	The diagonals of a rectangle are congruent.	identity	to prove		"how we could prove" (9.39)	Yes

Number of statement	**Speaker**	**Clause #**	**Clause**	**Statements with "being" processes**	**Purpose (*quality, class, part, identity, existence*)**	**Teacher's proclaimed status**	**Students' proclaimed status**	**Evidence for status**	**Are there changes to the status of a statement?**
			congruent						
16.	Student	9.41	'cause the triangles are congruent	The triangles [non-specified] are congruent.	identity		to prove	"how we could prove" (9.39)	
17.	Teacher	11.43	The big triangles {*MPO* and *NOP*} are congruent.	Triangles *MPO* and *NOP* are congruent.	identity		to prove	"which triangles?" (11.42)	
18.	Student	12.44	{Triangles} *MCN* and *PCO* {are congruent}.	Triangles *MCN* and *PCO* are congruent.	identity		claim	"'cause" (10.41)	
19.	Teacher	13.46	those {triangles *MCN* and *PCO*} were congruent	Triangles *MCN* and *PCO* are congruent.	identity	to prove		"how would I prove" (13.45)	
20.	Student	14.50	that angle *PCO* and angle *MCN* are equal	Angle *PCO* and angle *MCN* are equal.	identity		to prove	"how would I prove" (13.45)	
21.	Student	14.51	'cause they're {*PCO* and *MCN*} vertical angles.	Angles *PCO* and *MCN* are vertical angles.	class		claim	"'cause" (14.51)	
22.	Teacher	14.51a	[Megan marks vertical angles congruent.]	Angles *PCO* and *MCN* are congruent.	identity	proven		Megan marks vertical angles congruent	

Number of statement	**Speaker**	**Clause #**	**Clause**	**Statements with "being" processes**	**Purpose (*quality, class, part, identity, existence*)**	**Teacher's proclaimed status**	**Students' proclaimed status**	**Evidence for status**	**Are there changes to the status of a statement?**
								(14.51a)	
23.	Student	14.52	And then *MN* and *PO* are a side.	*MN* and *PO* are sides of triangles *MCN* and *PCO* respectively.	part		claim	"how would I prove" (13.45)	
24.	Student	14.53	that's {*MN* and *PO*" are equal	*MN* and *PO* are equal.	identity		claim	"how would I prove" (13.45)	
25.	Teacher	15.62a	[Writes that Δ*MPO* is congruent to Δ*NOP*.]	Triangle *MPO* is congruent to triangle *NOP*.	identity	to prove		"How can I prove this?" (15.63)	
26.	Teacher	15.64	Okay, that this big triangle {Δ*MPO*} is equal to this	Triangle *MPO* is congruent to triangle *NOP*.	identity	to prove		"How can I prove this?" (15.63)	
27.	Student	16.65	{I can prove Δ*MPO* is congruent to Δ*NOP* by} SSS.	Triangle *MPO* is congruent to triangle *NOP* by SSS.	identity		claim	"How can I prove this?" (15.63)	
28.	Student	16.66	or like {I can prove Δ*MPO* is congruent to Δ*NOP* by} SAS.	Triangle *MPO* is congruent to triangle *NOP* by SAS.	identity		claim	"How can I prove this?" (15.63)	

Number of statement	Speaker	Clause #	Clause	Statements with "being" processes	Purpose (***quality, class, part, identity, existence***)	Teacher's proclaimed status	Students' proclaimed status	Evidence for status	Are there changes to the status of a statement?
29.	Teacher	17.68	these {*PO*} sides are equal, the bottom	Side *PO* is equal to side *PO.*	identity	claim		"How can I prove this?" (15.63)	
30.	Teacher	17.68a	[Adds hash marks to *PO*.]	Side *PO* is equal to side *PO*.	identity	claim		"How can I prove this?" (15.63)	
31.	Teacher	17.69	because it's part of both triangles	*PO* is part of both triangles.	part	claim		"How can I prove this?" (15.63)	
32.	Teacher	17.70	It's part of this one {Δ*MPO*}	*PO* is part of *MPO.*	part	claim		"How can I prove this?" (15.63)	
33.	Teacher	17.71	and it's part of this one {Δ*NPO*}	*PO* is part of triangle *NPO*.	part	claim		"How can I prove this?" (15.63)	
34.	Student	18.74	The opposite sides are equal.	Opposite sides [non-specified figure] are equal.	identity		no evidence	"what else do I know it's true" (17.72 and 17.73)	
35.	Student	20.76	'cause in a rectangle the opposite sides {are equal}	In a rectangle, the opposite sides are equal.	identity		installed proposition	"why do I know that?" (19.75)	Yes
36.	Teacher	21.77	I don't know that in a rectangle	In a rectangle the opposite sides are equal.	identity	not installed proposition		"I don't know that" (21.77)	Yes
37.	Student	22.78	{in a}	In a	identity		installed	"why do I	

Number of statement	**Speaker**	**Clause #**	**Clause**	**Statements with "being" processes**	**Purpose (*quality, class, part, identity, existence*)**	**Teacher's proclaimed status**	**Students' proclaimed status**	**Evidence for status**	**Are there changes to the status of a statement?**
			parallelogram {the opposite sides are equal}	parallelogram the opposite sides are equal.			proposition	know that?" (19.75)	
38.	Teacher	23.79	It's {*MNOP*} a parallelogram.	*MNOP* is a parallelogram.	class	claim		"It's a parallelogram" (23.79)	Yes
39.	Teacher	23.81	It's {*MNOP*} a parallelogram?	*MNOP* is a parallelogram.	class	to prove		"How could I prove" (23.80)	Yes
40.	Student	24.82	Opposite sides are parallel	Opposite sides of *MNOP* are parallel.	class[41]		claim	"Opposite sides are parallel" (24.82)	Yes
41.	Teacher	25.84	the opposite sides are parallel?	Opposite sides of *MNOP* are parallel.	class	to prove		"How do I know" (25.83)	Yes
42.	Teacher	27.87	for it {*MNOP*} to be a rectangle	*MNOP* is a rectangle.	class	given		"what do you need to know" (27.86)	
43.	Student	28.89	{A rectangle has} Four right angles	A rectangle has four right angles.	part		installed proposition	"What's definitely true about a rectangle?" (27.88)	
44.	Student	29.90	{A rectangle has} Four right	A rectangle has four right	part		installed proposition	"What's definitely true	

[41] Non-gradable because being parallel is a property that cannot be intensified.

Number of statement	Speaker	Clause #	Clause	Statements with "being" processes	Purpose (*quality, class, part, identity, existence*)	Teacher's proclaimed status	Students' proclaimed status	Evidence for status	Are there changes to the status of a statement?
			angles	angles.				about a rectangle?" (27.88)	
45.	Teacher	30.91	{A rectangle has} Four right angles	A rectangle has four right angles.	part	installed proposition		"Four right angles" (30.91)	
46.	Teacher	30.92	Okay, if I have four right an—{angles}	There are four right angles in *MNOP*.	existence	given		"if" (30.92)	
47.	Teacher	30.93	If these two {angles *MPO* and *NOP*} are right angles.	Angles *MPO* and *NOP* are right angles.	class	given		"if" (30.93)	
48.	Teacher	30.94	why are these sides {*MP* and *NO*} parallel?	*MP* and *NO* are parallel.	class	to prove		"why are these sides parallel" (30.94)	
49.	Teacher	32.96	{The same side interior angles are} supplementary	The same side interior angles of *MNOP* are supplementary.	class	claim		"why are these sides parallel" (30.94)	
50.	Teacher	34.97	I've got two {angles *MPO* and *NOP*} there.	There are two supplementary same-side-interior angles, *MPO* and *NOP*.	existence	proven		"I've got" (34.97)	
51.	Teacher	34.98	I could do the same thing	There are two supplementary	existence	proven		"I could do" (34.98)	

Number of statement	**Speaker**	**Clause #**	**Clause**	**Statements with "being" processes**	**Purpose (*quality, class, part, identity, existence*)**	**Teacher's proclaimed status**	**Students' proclaimed status**	**Evidence for status**	**Are there changes to the status of a statement?**
			here {angles *NMP* and *MPO*}	same-side interior angles, *NMP* and *MPO*.					
52.	Teacher	34.101	these {*MP* and *NO*; *MN* and *PO*} were parallel	*MP* and *NO*, and *MN* and *PO*, are parallel.	class	proven		"I could prove" (34.100)	
53.	Teacher	34.102	So, it's {*MNOP*} a parallelogram.	*MNOP* is a parallelogram.	class	proven		"it's a parallelogram" (34.102)	
54.	Teacher	34.104	The opposite sides {*MP* and *NO*} are equal.	Opposite sides of *MNOP*, *MP* and *NO*, are congruent.	identity	claim		"So then that tells me" (34.103)	
55.	Teacher	34.105	'cause it's {*MNOP*} a parallelogram	*MNOP* is a parallelogram	class	proven		"'cause" (34.105)	
56.	Teacher	34.106	So I have side {*MP*}, side {*PO*), equal to side {*NO*}, side {*PO*}	*MP* is congruent to *NO* and *PO* is congruent to *PO*.	identity	proven		"So I have" (34.106); (34.104); (17.68)	
57.	Teacher	34.108	It's {MNOP} a rectangle.	*MNOP* is a rectangle.	class	given		"that this outer thing was a rectangle" (1.14)	

Number of statement	Speaker	Clause #	Clause	Statements with “being” processes	Purpose (*quality, class, part, identity, existence*)	Teacher’s proclaimed status	Students’ proclaimed status	Evidence for status	Are there changes to the status of a statement?
58.	Student	35.109	The two diagonals {are congruent}	The two diagonals (of *MNOP* or of a rectangle) are congruent.	identity		proven or installed proposition	“What else do I get?” (34.107)	Yes
59.	Teacher	36.110 and 36.111	I don’t know the two diagonals {are congruent}	The two diagonals (of *MNOP* or of a rectangle) are congruent.	identity	to prove or not installed proposition		“I don’t know” (36.110); “I’m trying to prove that” (36.112)	Yes
60.	Student	39.113	The angles {*PMN*, *MNO*, *NOP*, and *OPM*} are equal.	All angles of *MNOP* are congruent	identity		proven	“What else do I get?” (34.107)	
61.	Student	39.114	They’re {*PMN*, *MNO*, *NOP*, and *OPM*} right angles.	All angles of MNOP are right angles.	class		proven	“What else do I get?” (34.107)	
62.	Teacher	40.115	They’re {*PMN*, *MNO*, *NOP*, and *OPM*} all 90.	All angles of *MNOP* measure 90 degrees.	class	proven		“Yeah” (turn 40)	
63.	Teacher	40.117	They {*PMN*, *MNO*, *NOP*, and *OPM*} are all 90	All angles of *MNOP* measure 90 degrees.	class	proven		“Like Alana said” (40.116)	

Number of statement	**Speaker**	**Clause #**	**Clause**	**Statements with "being" processes**	**Purpose (*quality, class, part, identity, existence*)**	**Teacher's proclaimed status**	**Students' proclaimed status**	**Evidence for status**	**Are there changes to the status of a statement?**
64.	Teacher	40.118	So I have side {*MP* and *NO*}, angle {*MPO* and *NOP*}, side {*PO* and *PO*}	Sides *MP* and *NO* are congruent; Angles *MPO* and *NOP* are congruent; Sides *PO* and *PO* are congruent.	identity	claim		"So I have" (40.118); (34.106)	
65.	Teacher	40.118d	[Writes on the board "by SAS."]	Triangle *MPO* is congruent to triangle *NOP* by side-angle-side.	identity	proven		"So I have" (40.118)	
66.	Teacher	40.119	I could prove this {Δ*MPO* congruent to Δ*NOP*} by side-angle-side.	Triangle *MPO* is congruent to triangle *NOP* by side-angle-side.	identity	proven		"I could prove" (40.119)	
67.	Teacher	40.120	Then, why are the diagonals {*MO* and *NP*} equal?	The diagonals, *MO* and *NP*, are congruent.	identity	to prove		"why" (40.120)	
68.	Teacher	42.121	If this big triangle {*MPO* is congruent to triangle *NOP*}	Triangle *MPO* is congruent to triangle *NOP*.	identity	proven		"I could prove" (40.119); "why" (40.120)	

Number of statement	**Speaker**	**Clause #**	**Clause**	**Statements with "being" processes**	**Purpose (*quality, class, part, identity, existence*)**	**Teacher's proclaimed status**	**Students' proclaimed status**	**Evidence for status**	**Are there changes to the status of a statement?**
69.	Student	43.122	{*MO* and *NP* are equal because} CPCTC	*MO* and *NP* are congruent because CPCTC.	identity		claim	"why" (40.120)	
70.	Teacher	44.123	{*MO* and *NP* are equal because} CPCTC	*MO* and *NP* are congruent because CPCTC.	identity	proven		"I could prove that" (46.124)	
71.	Teacher	46.127	Two x equals 31	Two x equals 31.	identity	solution		"she is right" (46.125)	
72.	Teacher	46.129	So x is 15.5	x is 15.5.	identity	solution		"she is right" (46.125)	

Most “being” processes in the rectangle episode, in 39 out of 71 statements, were used to establish *identity* (see Table 22). Statements regarding an identity say something about geometric figures (or parts of geometric figures) that are equal or congruent. Speakers used 19 out of 71 “being” processes to say that a geometric figure is part of a particular *class*. There are 8 out of 71 statements where speakers used a “being” process to specify characteristics that *parts* of a geometric figure possesses. There are 4 out of 71 “being” processes of *existence* in the episode. Existence statements have to do with conditions either given in the homework problem or taken as the steps towards finding a conclusion.

These four types of “being” processes—*identity, class, parts*, and *existence*—could be used for making statements in a proof. For example, a statement such as “angle *PCO* and angle *MCN* are equal,” is an identity, because it identifies two triangles as being the same. In contrast, a statement such as “angle *PCO* and *MCN* are vertical angles” implies that the pair of angles pertains to a class of pairs of angles, that of vertical angles. A statement such as “A rectangle has four right angles” points to a characteristic that parts of a geometric figure possess. Also, a statement such as “There are four right angles in *MNOP*,” emphasizes the existence of four right angles.

In contrast with the four types of “being” processes aforementioned, there were no “being” processes regarding *quality* in the rectangle episode. A key characteristic of “being” processes describing a quality is that qualities are gradable in some sense (Martin & Rose, 2003, p. 76-77). By gradable, it means that one could assign different degrees to evaluate something as having more or less of a quality. For example, the statement “opposite sides of *MNOP* are parallel” (statement 40 in Table 21) is not gradable, because the quality of being parallel is not intensified. This is a statement regarding *class,* because it refers to a class of pairs of lines: the class of parallel lines. In contrast, one could say that two lines are somewhat parallel or that an angle is almost ninety-degrees. So, in the rectangle episode, the statements about geometric figures were not gradable; geometric figures were said to be something or not.

Table 22

Summary of the purpose of "being" processes in the rectangle episode

Purpose	**Percent of total**	**Total**	**Statements**
Identity	55	39	4, 5, 6, 7, 8, 12, 13, 14, 15, 16, 17, 18, 19, 20, 22, 24, 25, 26, 27, 28, 29, 30, 34, 35, 36, 37, 54, 56, 58, 59, 60, 64, 65, 66, 67, 68, 69, 70, 71
Class	27	19	2, 9, 10, 21, 38, 39, 40, 41, 42, 47, 48, 49, 52, 53, 55, 57, 61, 62, 63,
Part	13	9	3, 11, 23, 31, 32, 33, 43, 44, 45
Existence	5	4	1, 46, 50, 51
Quality	0	0	
		71	

Table 21 also shows the proclaimed status of a statement by the teacher and by the students. In the rectangle episode, there are six changes in the status of a statement. In four of those changes, the teacher and the students dispute the status of a statement. In the other two, the teacher changes the status of a statement. Table 23 summarizes changes in the status of a statement. Half of the changes in status have to do with identifying whether a geometric figure belongs or not to a particular class. The other half of the changes have to do with whether two geometric objects are congruent or not.

The first change in the status of a statement results from a disagreement between the teacher and a student (statements 9 and 10). The student says that the assumption that sustains her solution is that *MNOP* is a rectangle. For the teacher, this is not the underlying assumption in the proposed solution of the problem. The teacher states that the statement, "*MNOP* is a rectangle," is one of the given statements in the problem.

The second change happens when a student makes the statement "*MO* is congruent to *NP*" (statement 12). The teacher translates this statement, which is expressed in the *diagrammatic register* of *MNOP*, into a general statement about a property of rectangles, in the *conceptual register* (Weiss & Herbst, 2007). That is, the teacher changed a statement made in terms of parts of the diagram into a statement about concepts related to those parts of the diagram. The teacher says, "that the diagonals are congruent" in reference to the diagonals of a rectangle (statement 13). The teacher also changes the status of this statement from something stated by the student as a fact of the figure to something that had to be proved because it was not a theorem yet, and eventually to a statement to prove. The teacher's change in the status of this statement is important, because it illustrates how she uses the homework problem to "flesh out" properties of quadrilaterals (clause 1.6). In this case, she uses the homework problem, which made references to a specific geometric figure, *MNOP*, and hence elicits a property of the figure, to establish its general statement and its conjectural status, to later prove a general property that the class will be able to use for all rectangles: the diagonals of a rectangle are congruent.

The third change corresponds to a disputed statement, where a student and the teacher disagree about its status. The student says that in a rectangle the opposite sides are equal (statement 35). The student takes this statement as an installed proposition to provide a reason for why opposite sides of *MNOP* are congruent. However, the teacher says that this is a statement that they do not know to be true yet. For the teacher, the statement "in a rectangle the opposite sides are equal" is a proposition that has not been installed yet. This disagreement gets partially resolved when the student changes the class of geometric figures being talked about. Instead of making a statement about rectangles, the student makes a statement about parallelograms (statement 37). The student says that in a parallelogram opposite sides are congruent. So, even though statements 35 and 37 are identities stating that diagonals are equal, in statement 37 the student changed the class of the geometric figures, from a rectangle to a parallelogram. The teacher does not react negatively to the statement "in a parallelogram the opposite sides are equal." This is one of the theorems that the class had studied prior to the replacement units on quadrilaterals. Then, the teacher asks students to prove that rectangles are a member of the class of parallelograms.

The fourth change in the status of a statement involves statements 38 and 39. The teacher changes the status of the statement "*MNOP* is a parallelogram" from a statement of fact to a statement to be proved. By proving that *MNOP* is a parallelogram, they intend to show that rectangle *MNOP* belongs to the class of parallelograms. If proven true, rectangle *MNOP* would inherit properties of parallelograms that they had already proven in class. So, installed propositions about parallelograms would also apply to rectangles.

In statements 40 and 41 there is another change in the status of a statement. A student claims that the opposite sides of *MNOP* are parallel. However, the teacher takes this statement as something that is unknown. The teacher asks, "How do I know the opposite sides are parallel?" (clauses 25.83 and 25.84). So, they would need to prove that *MNOP* is a parallelogram in order to claim that the opposite sides are parallel.

Finally, the teacher and a student disagree about the status of statements 58 and 59: the two diagonals are congruent. According to the student, the statement that the two diagonals (of *MNOP* or of a rectangle) are congruent is an installed proposition. However, the teacher considers this statement to be what needs to be proven or a proposition that has not been installed yet. The teacher had said earlier that this is what they should prove earlier in the episode (statement 15). Despite of the teacher's comment, the student takes what needs to be proven as something already known.

Table 23

Changes in the status of a "being" statement

Statement number	**"Being" statement**	**Purpose of the "being" statement**	**Teacher's proclaimed status**	**Students' proclaimed status**
9 and 10	*MNOP* is a rectangle.	classify	Given	to prove
13, 14, and 15	The diagonals of a rectangle (*MNOP*) are congruent.	identify	to prove; claim; to prove	
35 and 36	In a rectangle, the opposite sides are equal.	identify	not installed proposition	installed proposition
38 and 39	*MNOP* is a parallelogram.	classify	claim; to prove	
40 and 41	Opposite sides of *MNOP* are parallel.	classify	to prove	claim
58 and 59	The two diagonals (of *MNOP* or of a rectangle) are congruent.	identify	to prove or not installed proposition	proven or installed proposition

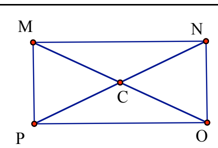
In sum, the teacher utilized disputes regarding the status of a statement as an opportunity to install the theorem that the diagonals of a rectangle are congruent. She changed the status of a statement as a step toward doing the proof of that theorem. The teacher's intention to do a proof brings about the question of what resources students could deploy in order to do a proof. The question of resources is important because it points to the prior knowledge students could make use of. In the rectangle episode, the class visualized *MNOP* as different kinds of objects. Some statement about *MNOP* relied upon prior knowledge that the teacher approved. However, other statements about *MNOP* relied upon knowledge that the teacher did not accept. The teacher's approval or rejection of a statement demonstrates what kind of prior knowledge was acceptable for doing a proof. In the following section I examine the different ways in which the class visualized *MNOP* and the prior knowledge that they used in that visualization.

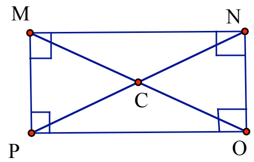
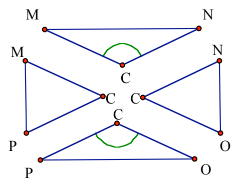
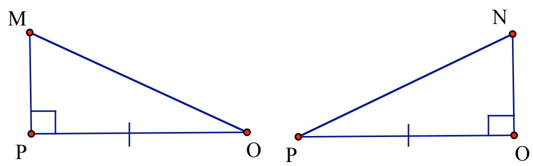
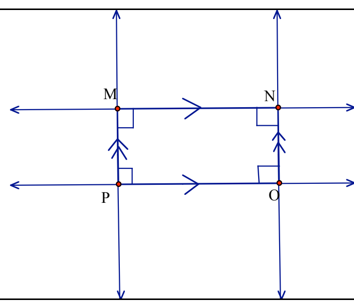
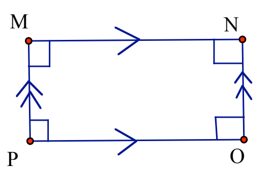
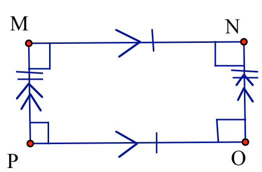
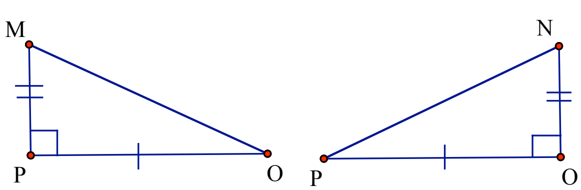
Ways of Visualizing Quadrilateral *MNOP* and the Prior Knowledge Needed for that Visualization

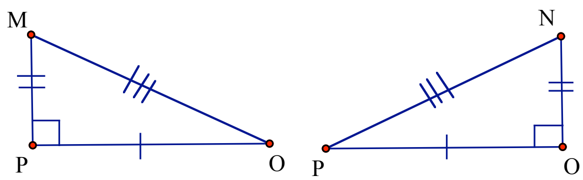
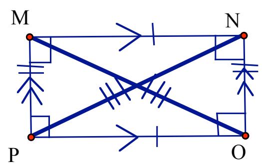
The different statements about quadrilateral *MNOP* (and its diagonals) imply visualizing that geometric figure as other kinds of configurations. This visualization of *MNOP* as other configurations was useful in order to use prior knowledge about properties of geometric figures when proving that the diagonals of a rectangle are congruent. So, speakers looked at the diagram of *MNOP* in different ways to get a configuration for which there were enough resources to do the proof. This is a standard mathematical practice in that one would designate a geometric object as another one with known properties, in order to do a proof. For example, it may be convenient to look at a rectangle as if it was a parallelogram, if one knows properties of parallelogram that could be applicable to a rectangle as well. Therefore, by visualizing quadrilateral *MNOP* as other configurations, students could conceivably apply prior knowledge about properties of those other configurations to do a proof. By virtue of changing the configuration, students also change what is to be proved.

Table 24 shows a sequence of diagrams with different ways in which speakers visualized quadrilateral *MNOP* (and its diagonals) as other configurations. Speakers' visualization of the diagram support statements made about the geometric figure in the rectangle episode. I focus on statements that were accepted by the teacher. In the rectangle episode, the diagram was an affordance, enabling the teacher and the students to have different visualizations of the same geometric figure. In Table 24, the diagrams are my interpretation from the statements made about the diagram and do not correspond directly to what the teacher wrote on the board. For example, the teacher did not make a diagram with separate triangles, the teacher did not extend the lines to show that they are parallel, and the teacher did not mark diagonals congruent. However, in my interpretation of ways of visualizing *MNOP*, I represent the configurations that I hypothesize speakers visualized when they looked at the diagram.

Table 24
Ways of visualizing *MNOP* in the rectangle episode

Diagrams representing different configurations of *MNOP*	**Ways of visualizing *MNOP***	**Reference of statements (from Table 21)**
 MNOP is a rectangle	• *MNOP* as a rectangle with diagonals *MO* and *NP*.	2, 3

Diagrams representing different configurations of *MNOP*	**Ways of visualizing *MNOP***	**Reference of statements (from Table 21)**
	• *MNOP* as a quadrilateral with four right angles.	11
	• *MNOP* includes a pair of triangles, *MCN* and *PCO*, with a pair of congruent angles, *MCN* and *PCO*.	19, 20, 21
	• *MNOP* as two overlapping triangles, *MPO* and *NOP*, with a pair of congruent angles and a pair of congruent sides.	25, 26, 29, 30, 31, 32, 43, 44, 45, 46
	• *MNOP* as two sets of parallel lines, *MN* || *PO* and *MP* || *NO*.	52
	• *MNOP* as a parallelogram.	53
	• *MNOP* as a parallelogram with opposite sides congruent.	54
	• *MNOP* as two pairs of congruent overlapping triangles, *MPO* and *NOP*, by SAS.	65

Diagrams representing different configurations of *MNOP*	**Ways of visualizing *MNOP***	**Reference of statements (from Table 21)**
	• *MNOP* as two pairs of congruent overlapping triangles with congruent parts, *MO* congruent to *NP*.	69, 70
	• *MNOP* as a rectangle with congruent diagonals.	67, 69, 70

In the rectangle episode, the teacher accepted statements that involved visualizing *MNOP* (and its diagonals) into the following sequence of configurations: a rectangle, a quadrilateral with four right angles, a pair of triangles (*MCN* and *PCO*), a pair of triangles (*MPO* and *NOP*), two sets of parallel lines, a parallelogram, a pair of congruent triangles (*MPO* and *NOP*) with all corresponding parts congruent, and a rectangle with congruent diagonals. Of all these configurations, having a pair of triangles *MCN* and *PCO* was not useful to prove the theorem that the diagonals of a rectangle are congruent. However, all the other configurations were useful to prove that theorem.

By visualizing *MNOP* and its diagonals as different configurations, the class could draw upon their prior knowledge of geometric figures. This prior knowledge involved theorems stating sufficient conditions for triangle congruence, the definition of congruency, the vertical angles theorem, the definition of a rectangle, theorems stating sufficient conditions for proving lines parallel, the definition of a parallelogram, and some properties of parallelograms. Table 25 shows a summary of lexical terms and propositions used in the rectangle episode that could be associated with each one of the concepts involved in the proof. All these concepts are part of the usual geometry curriculum. In particular, all these concepts had been covered in Megan's class prior to the replacement unit on quadrilaterals.

The teacher did not need to use all the ideas mentioned in the final proof. For example, the teacher did not use the vertical angles theorem nor did she use side-side-side theorem for triangle congruency. However, the teacher did not say to students that they should not use these statements. So, I take the teacher's lack of reaction to disapprove references to these propositions as her acknowledgement of the prior knowledge that students possessed and could make use of in a proof.

Table 25

References to concepts and propositions in the geometry curriculum

Concepts and Propositions	**Lexical choices** (in order of appearance in the transcript)
the vertical angle theorem	• vertical angles (14.51)
Theorems stating sufficient conditions for triangle congruence	• SSS (16.65) • SAS (16.66, 40.118d) • side-side, equal to side-side (34.106) • side, angle, and side (40.118, 40.119)
the definition of a rectangle	• four right angles (28.89, 29.90, 30.91)
Theorems stating sufficient conditions for proving lines parallel	• same side interior angles (31.95)
the definition of a parallelogram	• these {opposite sides} were parallel (34.101)
properties of parallelograms	• the opposite sides are equal (34.104)
the definition of congruency of triangles	• CPCTC (43.122, 44.123)

Megan held students accountable for using prior knowledge of concepts and propositions studied in class prior to the quadrilateral replacement unit. In contrast, Megan did not allow students to use their knowledge about properties of rectangles for doing the proof of the claim that the diagonals of a rectangle are congruent. This happened in two instances (see the second and the sixth entries in Table 23). First, when Anil asserted that in a rectangle opposite sides are equal (20.76), Megan replied "I don't know that in a rectangle" (21.77). In the second instance, Anil said that the diagonals of a rectangle are congruent (35.109). In response, Megan replied that this is something they did not know and they were trying to prove (turn 36). She said, "No. I don't know the two diagonals. I'm trying to prove that." Therefore, when students brought about properties of rectangles— different from the avowedly shared definition of a rectangle as a quadrilateral with four right angles—the teacher did not allow students to make use of that knowledge. That is, students could not take for granted properties of rectangles stating that opposite sides are congruent and that diagonals are congruent.

Figure 12 shows a representation of the steps in the proof produced in class. Megan guided students through different steps in the proof, by making them aware of the status of their claims. In order to prove that diagonals of a rectangle are congruent, the prior knowledge students could draw upon was restricted to that knowledge previously studied in their geometry class. Students could make use of theorems about parallel lines, properties of parallelograms, and theorems about triangle congruence. In contrast, students could not draw upon knowledge outside of what had been officially studied in the geometry class. Therefore, they needed to visualize *MNOP* as different kinds of mathematical objects whose properties they had studied before. In that way, Megan disciplined students to read the diagram so that they could do a proof using deductive reasoning. Moreover, Megan shaped students' memories so that those concepts and propositions studied in class prior to the replacement unit on quadrilaterals would be the ones students had to remember.

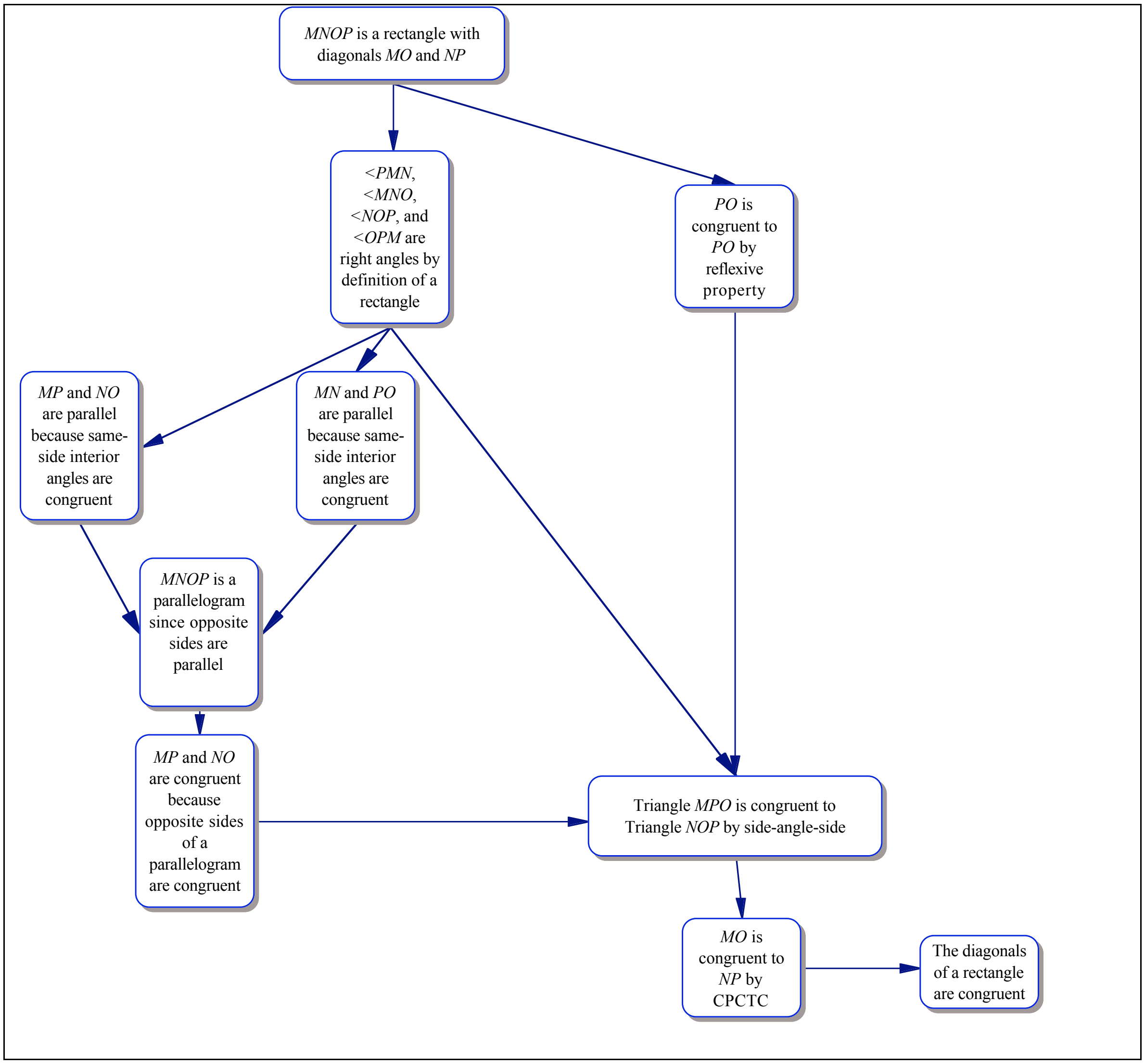


Figure 12. A representation of the proof produced in the rectangle episode.

Megan dismissed some of the claims made on *MNOP*, which took for granted properties of rectangles or that could not get the class to prove what they needed to prove. For example, Megan did not follow through Alana's idea of translating *MNOP* as a configuration of two pairs of congruent triangles (Δ*MCN* congruent to Δ*PCO* and Δ*MCP* congruent to Δ*NCO*). Megan's focus on a pair of overlapping triangles (Δ*MPO* and Δ*NOP*) was useful because she could guide students in finding a pair of congruent sides (*PO* congruent to *PO*) and a pair of congruent angles (∠*MPO* and ∠*NOP*), just by using the definition of a rectangle. Still, proving those overlapping triangles congruent required finding another pair of congruent parts. Anil wanted to take for granted that a rectangle has opposite sides congruent. Anil said that in a rectangle opposite sides are congruent (20.76). However, Megan did not want to use properties of rectangles they had not proven yet (21.77). At that moment, Anil retracted his previous claim and called *MNOP* a parallelogram, using prior knowledge about properties of parallelograms (22.78). The class had already studied a theorem stating that opposite sides of a parallelogram are congruent prior to the replacement unit on quadrilaterals. Megan insisted on proving first that a rectangle is a parallelogram, in order to use that property which they had already proven for parallelograms. So, they visualized *MNOP* as a parallelogram, to get opposite sides congruent (*MP* congruent to *NO*).

Other kinds of operations using visual perception could have led to similar results. By using visual perception, one could say that *MP* and *NO* appear to be congruent and finish the proof. Moreover, visual perception could have led students to take the diagonals, *MO* and *NP* as congruent in the first place, without the need of doing a proof. Students could have arrived at the same result of making an equation if they had used visual perception to conclude that the diagonals were congruent, and then finding the value of x. However, Megan had changed the mathematical task into one of doing a proof to install a theorem. With that purpose, Megan had to set boundaries to the prior knowledge that students could make use of as a resource in the proof: Students could only rely upon prior knowledge officially introduced in class before the replacement unit on quadrilaterals.

Megan's action to limit students' prior knowledge to memories about concepts and propositions studied prior to the replacement unit on quadrilateral contrasts her actions during the play of the Guess my Quadrilateral game. The play of the game required students to make use of properties of special quadrilaterals even when these had not been officially introduced in class. Thus, the day before the rectangle episode, Megan had to allow students to draw upon their prior knowledge of special quadrilaterals, even when special quadrilaterals had not been officially introduced in class. This prior knowledge included properties of the diagonals. But, in order to install the theorem that the diagonals of a rectangle are congruent, she had to make students remember only those concepts and propositions that had been officially introduced in class.

Conclusions about the Rectangle Episode

In leading the class through proving that the diagonals of a rectangle are congruent, the teacher took as useful those configurations that drew upon what the class had studied before the quadrilaterals unit. The teacher held students accountable for remembering prior knowledge that had been officially introduced in class, excluding what the class had discussed during that unit. On the one hand, the teacher did not allow students to assume that a rectangle is a parallelogram. In contrast, she asked students to produce a proof to assert that a rectangle is a parallelogram, making them remember properties of parallel lines and tests for parallelograms. These two topics were covered during the days prior to the replacement unit on quadrilaterals. Later, the teacher had students apply theorems related to triangle congruency in order to prove that the diagonals of a rectangle are congruent. Triangle congruency was also another topic studied in that class earlier in the semester. The teacher, on the other hand, relied on students' individual memories about special quadrilaterals and its properties to play the Guess my Quadrilateral game. Students reviewed the names and the diagrams of special quadrilaterals before starting the play of the game. During the play of the game, students used the properties to make the least number of questions and guess the hidden special quadrilateral correctly. In particular, there were public discussions about properties of diagonals of some quadrilaterals. So, there is a contrast between the teacher's willingness to accept students' individual prior knowledge when teaching with a problem during the play of the Guess my Quadrilateral game and the teacher's insistence to set boundaries to the prior knowledge students could make use of in the rectangle episode.

In the rectangle episode, the teacher wanted to prove the theorem stating that diagonals of a rectangle are congruent, even when students had taken this theorem for granted when solving the homework exercise. The teacher did not take as valid students' claims about properties of rectangles, besides the definition of a rectangle as a quadrilateral with four congruent angles. The teacher made a deliberate effort to disregard properties of a rectangle and to make students remember concepts and propositions prior to the quadrilaterals unit, in order to install the theorem. However, in the previous day, the teacher had enabled students to draw upon properties of special quadrilaterals, including properties of rectangles, in order to play the Guess my Quadrilateral game.

Students' actions to bring about properties of rectangles during the proof could be explained because of changes in what could be taken as prior knowledge during the play of the game and in the problems assigned for homework which took properties of rectangles as known properties. The Guess my Quadrilateral game had changed the way new knowledge was introduced in this class. During the play of the game students were able to use as prior knowledge things that had not been discussed in class, but that they individually possessed or came to realize. I did not find evidence that the property about the diagonals of a rectangle had surfaced the day before, even when it could have been fair for students to make use of this property in guessing the hidden figure. It could also be the case that students had taken this property for granted while playing the game, but did not use it because it was not strategic to guess a quadrilateral. So, even though there are no records of students' prior knowledge of this property of diagonals of a rectangle, it is possible that students already knew it from prior mathematics classes or that students were cued in by the statement of the homework exercise, which included just the sufficient information to do the equation assuming that diagonals of a rectangle are congruent. In any case, the Guess my Quadrilateral game allowed students to know properties of quadrilaterals, including knowledge of properties of rectangles, even when these properties had not been officially introduced in class.

From my examination of copies of homework worksheets that seventh period students submitted at the end of the class, I found that 18 students had answered correctly homework problem #3a (which is the problem featured in the rectangle episode), one student had set up the equation correctly but made mistakes solving the equation, two students had provided a wrong solution, and one student had skipped that question. Even though the teacher did not have this information at the moment when she discussed the homework in class, the records from the class show that most students had been able to do the problem by applying the property that diagonals of a rectangle are congruent. The fact that students were held accountable for doing the homework shows that the teacher expected students to do something to solve it.

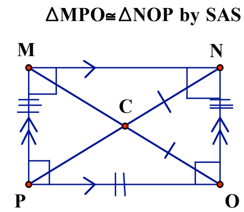
The teacher's actions could be explained in terms of the collective memory hypothesis. The teacher shapes the collective memory of the class to make it serve the students' work on a task. While doing the proof about the diagonals of a rectangle, the teacher acted as if she had to set strong boundaries to the knowledge that students could make use of. She did not let students take properties of rectangles from granted, even if they possessed individual knowledge about those properties. The contrast between how the teacher let students make use of prior knowledge when playing the Guess my Quadrilateral game and when doing the proof demonstrates that the teacher manufactures the memory of the class by playing an active role legitimizing what can be remembered, when it is useful to solve a problem.

The play of the Guess my Quadrilateral game was conceived of as an activity of teaching with problems. The game activated students' prior knowledge about special quadrilaterals and their properties. However, the game was not enough for the teacher to install theorems about properties of special quadrilaterals. So, while the game provoked class discussions among students about properties of quadrilaterals, the teacher did not have the opportunity to correct misconceptions that could have surfaced, or even to list those properties that she needed to teach. Therefore, the next day, the teacher repaired the play of the game by choosing a problem that would allow her to install a theorem.

The teacher’s decision to put boundaries between what students should remember and what students should forget while doing the proof of the theorem could be explained because this decision helped her identify the shared knowledge of the class. If the teacher were to allow any memories from prior knowledge to be part of the shared knowledge of the class, then it would be difficult for her to hold students accountable for using that knowledge in the future; students’ memories could be different from what she wants students to remember and not all students may possess that knowledge. The collective memory hypothesis helps explains the teacher’s actions because in shaping students’ memories the teacher can control the content of the memories and can hold students accountable to bring about those memories later.

Following the Rectangle Episode

Once they had solved part “a” of the homework problem, Megan continued with the discussion of the other parts. Megan restated that the diagonals of the rectangle are congruent, because they had proven it. Megan said that they could assume that diagonals bisect each other because *MNOP* is a parallelogram. Students could draw upon properties of parallelograms in part “b” of the problem, because they had proven that *MNOP* is a parallelogram when doing part “a.” So, Megan used this new problem to remind students of a property of parallelograms that they had studied before: The diagonals of a parallelogram bisect each other. Then they could deduce that all segments made by the intersection of the diagonals of a rectangle are congruent, and solve the algebraic problem.

Turn #	Speaker	Turn
48.	Megan	Okay, "b." "b" is the one with the quadratic, right? [Grabs handout from desk and reads the handout.] Yeah. Okay, let's look at "b." They told you that *CN* was equal to *x* squared plus one and *CO* is equal to three *x* plus 11 [writes equations on the board]. Okay *CN* [points to *CN*] and *CO* [points to *CO*]. Well, the diagonals are congruent. We just established that. Why are these two equal? [Adds hash marks to *CN* and *CO*. Adds more hash marks to *NO* and *MP*.] Why are these two [points to *CN* and *CO*] equal?  a. MO = 2x - 8 NP = 23 2x - 8 = 23 2x = 31 x=15.5 b. CN = x^2+ 1 CO = 3x + 11 x^2+ 1= 3x + 11
49.	Alana	Because the diagonals bisect each other? [Megan starts writing the quadratic equation on the board.]
50.	Megan	Right, because it's a parallelogram [while writing equation on the board], so the diagonals bisect each other. So I can say this is equal. Who got this far? Okay, Sharon, do you know what to do next?
51.		[Keeps solving quadratic equation with Sharon's input.]

Following the rectangle episode, Megan reminded students about the theorem they had installed: A rectangle has congruent diagonals. Because the class had just proven that a rectangle is a parallelogram, students could also apply properties of a parallelogram—the diagonals bisect each other—to a rectangle. Megan made it official when those properties of a rectangle could be taken for granted and when students could say those properties—namely, *after* they had produced the proof.

The exchange between Megan and her students after the rectangle episode gives more evidence that Megan granted permission for students to remember theorems only after they had been officially introduced in class. The homework problem assumed that students knew two theorems: The diagonals of a rectangle are congruent and the diagonals of a rectangle bisect each other. The second theorem is a property of parallelograms, and not just of rectangles. Even though students could have individually remembered these theorems before, the teacher expected students to remember them after installing them in class by means of doing the proof. I argue that the collective memory hypothesis helps explain the teacher's actions because even though students possibly knew these theorems from before, the teacher controlled when they had the right to remember what they knew.

This concludes my analysis of the rectangle episode. The rectangle episode is a case where the teacher did not allow students to use prior knowledge that had not been officially introduced to class, even though students remembered that prior knowledge. In the next section I present another episode, the kite episode, with the intention of showing a case where the teacher prevented students from using a conjecture that had not been proven, even though students remembered that conjecture.

The Kite Episode

The kite episode happened at the end of the 11th day into the quadrilaterals unit in the 2nd period class. Students had been working in groups, using a worksheet that had two questions. The first question was to name which midpoint quadrilateral corresponds to each special quadrilateral listed in a table—squares, rectangles, rhombi, trapezoids, kites, and "darts"—and to explain how they reached their conclusions. The second question asked to consider which quadrilaterals have a rectangle as their midpoint quadrilateral: "Last year, when I asked students what quadrilateral I should start from to get an M-Quad that is a rectangle, Bubba said that to get a rectangle M-Quad, one must start from a rhombus. Is Bubba right or wrong?" In the following section I present an analysis of this task.

Analysis of the Task in the Kite Episode

In the analysis a priori of the task in the kite episode, I have considered four possible solutions (see Table 26). A resource that could be important in this task, the medial-line theorem, had not been discussed in class yet. Therefore, it was not likely that students would make use of that theorem to solve the problem. However, students had conjectured since the first day of the unit that the midpoint quadrilateral of all quadrilaterals is a parallelogram. As I have explained earlier, this conjecture relies on the medial-line theorem. If they were to use this conjecture, students could apply properties of parallelogram to the solution of the problem (solution A). Because of the medial-line theorem, each pair of opposite sides of the midpoint quadrilateral is parallel to one of the diagonals of the original quadrilateral. One would need to know the proof of Varignon's theorem. In particular, one could use that the diagonals of the original quadrilateral are parallel to the sides of the midpoint quadrilateral to prove that the midpoint quadrilateral of a quadrilateral with perpendicular diagonals is a rectangle.

Figure 13 shows a quadrilateral with perpendicular diagonals, *ABCD,* and its midpoint quadrilateral, *MNOP.* If one were to know that the diagonals of the original quadrilateral are parallel to the sides of the midpoint quadrilateral, then $\overline{BD}$ would be parallel to $\overline{NO}$. Given that diagonals, $\overline{AC}$ and $\overline{BD}$ are perpendicular, then one could conclude that $\overline{AC}$ is perpendicular to $\overline{NO}$. Similarly, if one were to know that $\overline{AC}$ is parallel to $\overline{MN}$, then one could conclude that $\overline{MN}$ and $\overline{BD}$ are perpendicular. Since $\overline{NO}$

is parallel to $\overline{BD}$, and $\overline{BD}$ is perpendicular to $\overline{MN}$, then one could conclude that $\overline{NO}$ and $\overline{MN}$ are perpendicular. As a result *MNOP* would be a rectangle. So, students' solution would contradict Bubba's conclusion: All quadrilaterals with perpendicular diagonals have a rectangle as its midpoint quadrilateral. But this result would depend upon students' memory of a conjecture that had not been proven in class yet.

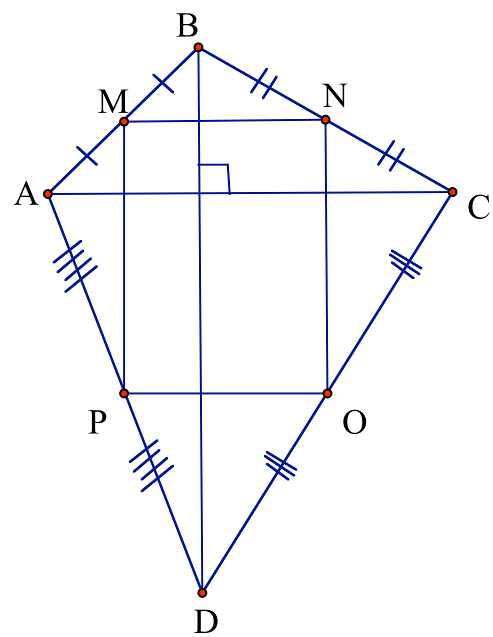


Figure 13. A quadrilateral with perpendicular diagonals and its midpoint quadrilateral.

Solution B relies on properties of special quadrilaterals. Students would make the mistake to conclude the same result as Bubba: A rhombus is the only quadrilateral that has a rectangle as a midpoint quadrilateral. In order to achieve this solution, students would need to apply prior knowledge about triangle congruency, isosceles triangles, and the definitions of special quadrilaterals. Solutions C and D involve measurements or visual perception to find the answer to the problem. However, these two operations are usually not allowed in Megan's class.

From the analysis of the task, I conclude that if students were to use their prior knowledge of geometric properties that had been officially introduced in Megan's class before the unit on quadrilaterals, they might not be able to prove that other figures different than the rhombus (and consequently the square) have a rectangle as its midpoint quadrilateral. If students were to use visual perception or measurement tools, they may be able to find out other special quadrilaterals that satisfy the conditions of the problem, namely, the kite and the "dart." If students were to use the conjecture that all midpoint quadrilaterals are parallelograms, they would be able to find that Bubba's conclusion was incorrect. Since students did not possess prior knowledge about the medial-line theorem, it was less likely that they would conclude that all quadrilaterals with perpendicular diagonals would have a rectangle as its midpoint quadrilateral. Students would have had to consider drawing a quadrilateral that could be identified by its properties, instead of a quadrilateral that could be identified by its name. Since quadrilaterals with perpendicular diagonals do not have a name, then it was not very likely that students would draw these quadrilaterals on their own. So, with the prior knowledge that students already possessed in Megan's class they would have had to rely on measurements, on visual perception, or on a conjecture that had not been proven yet in order to reject Bubba's conclusion. All of these operations were not the standard way of working with diagrams in Megan's class. Usually, these claims would require a proof, but the problem did not explicitly ask for a proof.

From the perspective of the design of the unit, the homework problem was an opportunity for students to investigate common characteristics of quadrilaterals with a rectangle as its midpoint quadrilateral. A kite, a rhombus, a square, and a "dart" are special quadrilaterals with perpendicular diagonals. Some students who may ponder about common properties of these figures may come up with the idea that quadrilaterals with perpendicular diagonals have a rectangle as its midpoint quadrilateral. This would depend upon students' initiative to draw the diagonals of the outer quadrilaterals. Ideally, some students would start sketching other quadrilaterals with perpendicular diagonals, and without other special characteristics, to find out a general case. In doing so, students would see that the diagonals of a quadrilateral are parallel to the sides of its midpoint quadrilateral. This could have led to an introduction of the medial-line theorem, which would be discussed by the end of the replacement unit on quadrilaterals.

While Eva was at the board making a diagram, Megan asked the class the answer to the second question in the day's worksheet. Students replied that Bubba was wrong because in addition to the rhombus, the midpoint quadrilaterals of a kite and a dart are also rectangles. Megan stated that her main interest was to focus on a rhombus and a kite. Then Megan asked, "My question is, why do they work? Does anyone have a reason?" Brett volunteered to present his diagram. The discussion surrounding Brett's solution is the core of the kite episode. In the following section I present an exchange between Eva and Megan that preceded the kite episode. This exchanges is important because it gave an opportunity for Megan to clarify what kind of prior knowledge was officially permitted to answer the homework problem.

Table 26

Task analysis of the homework problem at the beginning of the kite episode

	A. Solution using properties of diagonals	**B. Solution using properties of special quadrilaterals**	**C. Solution using measurements**	**D. Solution using visual perception**
Problem	What quadrilateral has a midpoint quadrilateral that is a rectangle?			
Goal	Name quadrilaterals that have a rectangle as a midpoint quadrilateral.			
Key Resources	The diagonals of a quadrilateral are parallel to the sides of the midpoint quadrilateral. (The midpoint quadrilateral is always a parallelogram.)	Definitions of special quadrilaterals and theorem stating that in an isosceles triangle base angles are congruent.	Definitions of special quadrilaterals. Diagrams of special quadrilaterals and measuring tools.	Names and diagrams of special quadrilaterals. Drawing tools.
Operations	Consider a quadrilateral with perpendicular diagonals. Since the diagonals of any quadrilateral are parallel to the sides of its midpoint quadrilateral, then these sides are perpendicular to each other. As a result, the midpoint quadrilateral is a rectangle.	Sketch diagrams of special quadrilaterals and their midpoint quadrilaterals. Identify congruent angles made by isosceles triangles and by congruent triangles. Use deductive reasoning to prove all angles of the midpoint quadrilateral congruent, or make an equation relating an internal angle of the midpoint quadrilateral and its two adjacent angles. Identify midpoint quadrilaterals for which all angles are congruent.	Make accurate diagrams of special quadrilaterals by relying on their definition and by using measuring tools. Measure the sides of the special quadrilaterals to find the midpoints. Draw the midpoint quadrilateral of each special quadrilateral. Measure the angles of the midpoint quadrilaterals. Find out for which special quadrilaterals the midpoint quadrilateral has all right angles.	Sketch diagrams of special quadrilaterals. Estimate the midpoint of the quadrilaterals and draw their respective midpoint quadrilateral. Look at the drawing and see whether the midpoint quadrilateral appears to be a rhombus.
Possible solutions	all quadrilaterals with perpendicular diagonals	a rhombus and a square	a rhombus, a kite, a "dart," and a square (all quadrilaterals with perpendicular diagonals)	a rhombus, a kite, a "dart," and a square (all quadrilaterals with perpendicular diagonals)

A Special Request Prior to the Kite Episode

Prior to the kite episode, while Eva and Brett were at the board making their respective diagrams, Eva had asked Megan whether she could assume that the midpoint quadrilateral is always a parallelogram (see Figure 14). Megan recognized that students had noticed this result since the beginning of the unit. However, when Eva insisted in asking whether she could use that assumption, Megan refused. The transcript of this exchange follows.

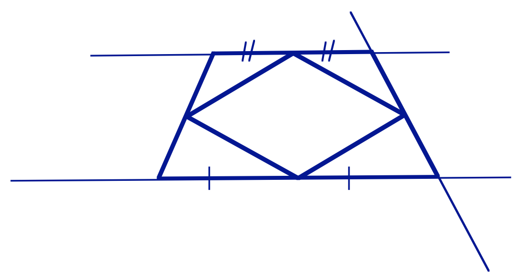


Figure 14. Diagram drawn by Eva prior to the kite episode.

Turn #	Speaker	Turn
1.	Eva	1. Can you assume 2. that you already know 3. there's always gonna be a parallelogram in the center? 4. When we are talking about a quadrilateral?
2.	Megan	5. Okay, you know what, 6. while we are waiting for Brett for a minute, 7. Eva's got an interesting question. 8. She said, 9. "Can we assume 10. that the inside is always going to be a parallelogram?" 11. Okay, we, we sort of have been, you know, 12. right from the first day people noticed, 13. "God, no matter what quadrilateral I draw 14. I get a parallelogram. 15. It could be um [3 s] 16. sometimes I get a special parallelogram like the square, or

Turn #	Speaker	Turn
		rhombus, or something. 17. But I always get a parallelogram on the inside. 18. Well, why is that? 19. You know, we all agree 20. that we do. 21. Why do we always get a parallelogram, 22. even though you can see… 23. what is that thing you drew, a trapezoid?
3.	Eva	Yes.
4.	Megan	24. She started with the trapezoid 25. and got a parallelogram. 26. If I just start with, you know, Joe Quadrilateral, 27. I'll get a parallelogram too. 28. We really need to ah [2 s] prove that before the end of the chapter. Okay, here--
5.	Eva	29. Can I assume that 30. when I do my thing?
6.	Megan	31. Well, I need you to sit down 32. because Brett is gonna talk 33. and you're gonna listen.
7.	Eva	34. Okay. 35. But like can I assume it anyway?
8.	Megan	36. Not right now.
9.	Eva	Okay. [Shrugs and goes back to her seat.]

This exchange between Eva and Megan before the kite episode is significant because it shows Megan's stance regarding students' finding that all midpoint quadrilaterals are parallelograms. Megan enacted the voice of students who have said, "I always get a parallelogram on the inside" (clause 2.17). According to Megan, that was a result the class had agreed upon (clause 2.19 and 2.20). So, on the one hand, Megan recognized that result as true, even for "Joe Quadrilateral," a non-descript quadrilateral (clause 4.26). Megan, on the other hand, reminded students that the result was something they had not proven yet (clause 4.28). Megan asked Eva to sit down (clause 6.31) and did not let Eva make this assumption in her proof (clause 8.27). However, later in the kite episode, Brett had difficulties because Megan noticed that he had used this assumption as the basis of his argument.

Description of the Kite Episode

Brett's presentation followed Eva's request. He intended to show that the midpoint quadrilateral of a kite is a rectangle. His diagram on the board of a kite and its midpoint quadrilateral had many markings (Figure 15). Some of these markings were incorrect because he did not use that *J*, *K*, *L*, and *M* are midpoints of congruent sides to mark *AM*, *DM*, *CL*, and *DL* as congruent segments. Also, different from what he had written on his worksheet, he used the same markings for *AJ*, *DM*, *CK*, and DL, even though there are only two pairs of congruent segments: *AJ* and *CK*, and *AM* and *CL*. Nobody in class noticed those mistakes. Brett did not label the points in the diagram, and referred to parts of the diagram using indexicals and gestures. I have added labels to the diagram for the reader to follow Brett's ideas and my analysis.

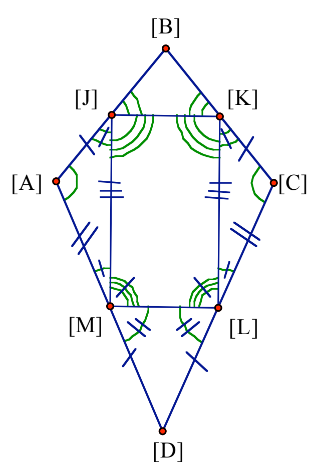
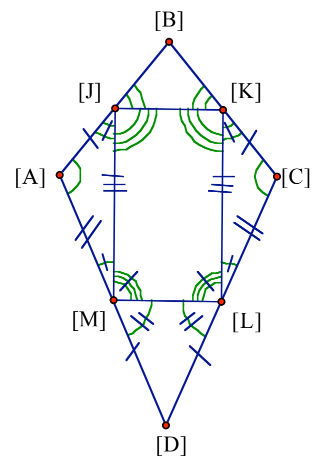


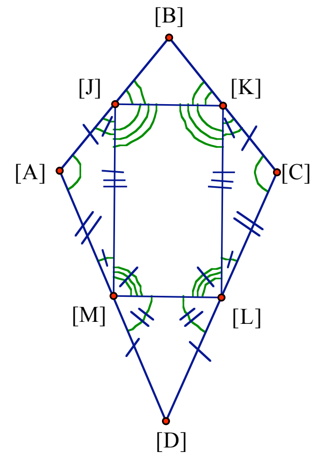
Figure 15. Brett's kite and its midpoint quadrilateral.

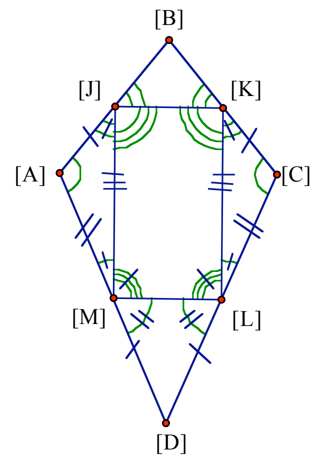
In the kite episode, Brett started by using triangle congruency to prove that the midpoint quadrilateral has a pair of congruent sides. Then he proved that the midpoint quadrilateral of a kite has two pairs of adjacent congruent angles. Brett had difficulties when he wanted to say that the midpoint quadrilateral had opposite sides congruent and therefore, it is a parallelogram. Megan pointed out that they only knew one pair of opposite sides congruent. Megan and Brett debated about what kind of quadrilateral would have one pair of congruent sides, agreeing that it could be a trapezoid. Finally, Megan asked Brett whether he had assumed the midpoint quadrilateral was a parallelogram. Brett said that he had. In the following section I present the transcription of the kite episode.

Transcription of the Kite Episode

The kite episode lasted 5:45 minutes. Brett was the only student at the board when he presented his findings. Megan was standing by the side of the room. The rest of the students were at their desks. I added labels to the diagram, in brackets, to mark those elements in the drawing that Brett pointed to as he talked, even though Brett did not use any labels.

Turn #	Speaker	Turn
10.	Brett	Okay, 1. well if you start with a kite, 2. then we get… 2a. [Turns around.]
11.	Megan	3. Is that a rectangle?
12.	Brett	Oh yeah.
13.	Megan	Okay. [Laughs.]
14.	Brett	Okay, 4. we get these two congruent triangles right here {ΔJAM and ΔKCL} due to uh, side angle side 5. because the midpoints {J and K} of these two sides {AB and CB}, 6. so these {JA and KC} are gonna be equal, 7. and these two {AM and CL} are going to be equal, 8. so we've got congruent triangles. 
		9. And then we have two congruent sides of the m-quad {JM and KL} due to CPCTC.
		And then, because the, then this triangle {ΔJBK}…
15.	Megan	10. Hey, you guys see? 11. Brett, scoot that way, some. 12. They can't even see you. Okay, that triangle…
16.	Eva	13. You gotta use the other hand. 14. Then people can see.
17.	Brett	[Switches hands and moves to the side.]
18.	Megan	15. Okay, you proved 16. that the two outer ones are congruent by side angle side. 16a. [Gesturing to ΔJAM and ΔKCL.] And then what?
19.	Brett	17. Okay, then these two triangles {ΔBJK and ΔMDL} are going to be isosceles.
20.	Megan	18. Okay, we agree with that, 19. a lot of groups thought 20. they were isosceles

Turn #	Speaker	Turn
21.	Brett	Okay. 21. So, that means 22. that these two angles {*DML* and *DLM*} are going to be congruent 23. and, these two angles {*AMJ* and *CLK*} are going to be congruent << >> because of CPCTC. 23a. <<but not to these two {*DML* and *DLM*}>> 24. And then because it has to add up to 180, 24a. [Traces lines *AD* and *CD*.] 
		25. then these two, angles {*JML* and *KLM*} are gonna also be congruent to each other 'cause (angle subtraction).
22.	Megan	(Oh!) 26. Okay, this is pretty nice.
23.	Brett	27. And then, you do the same thing on the other side. 27a. [Points to angles *MJK* and *LKJ*.] 28. And, you can prove that this is a parallelogram {*JKLM*} 29. ‘cause the opposite sides are congruent. 29a. [Pointing to *JM* and *KL*.]
24.	Megan	30. Both pairs of opposite sides?
25.	Brett	31. Well one pair (of opposite sides). 31a. [Pointing to *JM* and *KL*.]
26.	Megan	32. (I've only got) one pair.
27.	Brett	33. Then [4 s] but isn't… 33a. [Mumbles "a paral.”]
28.	Megan	34. Okay, stop for a minute. Uh…Yeah. 35. I only have one pair. Right?
29.	Brett	Yeah. 36. Don't you only need one pair? 37. If you had… [5 s] one… 38. If you had one pair of congruent sides in a quadrilateral 38a. [Positions hands as parallel sides.] 39. there is no way 40. it's not going to be a parallelogram.

Turn #	Speaker	Turn
30.	Megan	Um. 41. No, I can have a trapezoid. 42. Let me think about that, 43. one pair of congruent sides in a quadrilateral. 44. No, I don't know anything about those, those other sides. 45. Okay, let's think about this. 
31.	Brett	46. Oh, I know, 47. you could like… 47a. [Moves hand tilting it.]
32.	Megan	48. I can have, 49. I can have 50. things that aren't parallelograms, 51. but you already know 52. that it has two angles congruent. Right?
33.	Brett	Yeah.
34.	Megan	53. Does that help you?
35.	Brett	Umm… [7 s]
36.	Megan	54. (I've got two angles congruent)
37.	Brett	55. (It could still be a trapezoid.)
38.	Megan	56. and two sides congruent. 57. Is that enough?
39.	Brett	58. No, it could still be a trapezoid.
40.	Megan	59. It could be a trapezoid, 60. it could be isosceles trapezoid. So we… 61. you used that it's a parallelogram, right?
41.	Brett	Yeah.
42.	Megan	62. To get that 63. those other angles were congruent too?
43.	Brett	[Moves head assenting.]

Highlights of The Kite Episode

Brett went to the board to prove that the midpoint quadrilateral of a kite is a rectangle. However, in doing the proof, he had assumed that the midpoint quadrilateral was a parallelogram. Brett said that the midpoint quadrilateral is a parallelogram, because opposite sides were congruent (clauses 23.28 and 23.29). However, he did not have enough evidence to show that both pairs of opposite sides were congruent. Brett said that he had assumed that the midpoint quadrilateral was a parallelogram (turn 41). Earlier, Megan had not allowed Eva to make this assumption in doing a proof about the midpoint quadrilateral of a trapezoid. However, making this assumption would have enabled Brett to prove that the midpoint quadrilateral of a kite is a rectangle.

Brett was able to prove that midpoint quadrilateral of a kite has two pairs of congruent consecutive angles (clauses 21.25, 23.27, and 23.27a). By assuming that all midpoint quadrilaterals are parallelograms, he could deduce opposite angles congruent. He could connect these two statements to claim that all angles are congruent—which would imply that the midpoint quadrilateral is a rectangle. Brett made use of a result for which they had perceptual evidence since the first day of the quadrilaterals unit, but that they had not proven yet. The question is, why would Megan prevent students from using that result to solve the problem? The answer to this question is what motivates the analysis in this chapter.

Brett tried to see whether he had enough information to assert that the midpoint quadrilateral of a kite is a rectangle. When Megan noticed that so far he had proven only one pair of opposite sides congruent, Brett asked, “Don’t you only need one pair?” (clause 29.36). Then, he added, “If you had one pair of congruent sides in a quadrilateral, there is no way it’s not going to be a parallelogram” (29.38, 29.39, and 29.40). Here, Brett said that quadrilaterals with one pair of opposite sides are parallelograms. In response, Megan gave him a counterexample: a trapezoid (clause 30.41). Further discussion led them to consider other properties that Brett had already found about the midpoint quadrilateral of a kite. The midpoint quadrilateral of a kite has a pair of opposite sides congruent and also two pairs of congruent angles. So, they concluded the midpoint quadrilateral could be an isosceles trapezoid (clause 40.63).

By the end of the kite episode there was a conflict about what could be taken as prior knowledge to work on a problem. The class lacked a valid reason to prove that the midpoint quadrilateral of a kite is a rectangle. By using visual perception, students could see that the midpoint quadrilateral appeared to be a rectangle. Moreover, since the conjecture that all midpoint quadrilaterals are parallelograms had been suspected true since the first day of the unit, Brett had used this conjecture as the reason needed to prove that the midpoint quadrilateral of a kite is a rectangle. Yet, Megan did not grant permission to use this conjecture in the proof.

An argument that uses the conjecture that midpoint quadrilaterals are always parallelograms has more mathematical value than concluding that the midpoint quadrilateral is a rectangle by visual perception. Students had actual memories of the conjecture because of their work in the unit. However, Megan did not allow students to rely upon these memories. Usually, Megan did not allow students to use a result that they anticipated would be true but for which they had not yet produced a proof. If Megan were to allow students use that result, they would violate customary practices in a geometry class. In order to prove that the midpoint quadrilateral is always a parallelogram, they needed to make use of the medial-line theorem, and this was a theorem Megan intended to discuss at the end of the unit.

In one of the interviews with Megan after the quadrilaterals unit had concluded, she commented that during the unit, students were using a conjecture, even though they had not proven that conjecture yet. Megan said,

> Megan: And, I mean, there were some things like that the m-quad was a parallelogram that after a couple of days they just all were taking that for granted. We hadn't even proved it, but they all were like, "Well, it's always a parallelogram." You know? They just saw that enough times that they just believed it.
>
> Researcher: So, sometimes when it became something that they would use as a given--
>
> Megan: Yeah. I think there's some comfort in realizing, "Well, everyone thinks that, so we're all just using it," do you know what I mean? There's like this comfort level all of a sudden where they just accepted that, "Okay I don't know why it's true, but everybody else seems to think it's true too. So, this is something I can just use."

Megan made this comment in reaction to the question, "how did students know when they had learned something new?" For her, students' reliance on the conjecture posed at the end of the first day of the unit—that the midpoint quadrilateral is always a parallelogram—could be taken as a token for something that students learned in the unit. However, when Megan said that students, "just believed" a conjecture that had not been proven yet, she was hinting that this practice of taking a result for granted without proving it was unusual in her class. Megan's use of the word "just" adds graduation as a resource to soften a veiled critique (Martin & Rose, 2003, p. 43) to students' decision to take an assumption for granted.[42]

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[42] Wagner and Herbel-Eisenmann (2008) reported different meanings of "just" in teacher talk, including the case of using "just" to denote frustration. Megan's use of "just" when she said "just believed" could be interpreted as a resource to show her frustration with students who took an assumption for granted.

In mathematics, there are times where statements are taken for granted, as it is the case of postulates and conjectures. Moreover, some conjectures that had not been proven yet could yield some novel (and even true) results. So, in mathematics, the value of a conclusion that has been deduced logically from a statement is independent of the statement itself. However, Megan prevented students from taking this conjecture as true at the moment when Brett was doing the proof.

Table 27 shows the steps that Brett took in doing the proof. He visualized the kite and its midpoint quadrilateral as a configuration of four triangles and a quadrilateral. This decomposition into separate figures allowed Brett to apply theorems and properties of geometric figures previously studied in class. In his proof he referred to the definition of a kite as a quadrilateral with two pairs of adjacent congruent sides, the property of the kite that it has a pair of congruent opposite angles, the various criteria for proving triangles congruent, the definition of triangle congruence (CPCTC),[43] the definition of isosceles triangles, the theorem about the base angles of an isosceles triangle, and the angle addition postulate. Megan accepted statements that relied on knowledge discussed prior to the quadrilaterals unit. At this point in the unit, the definition of a kite and its properties had been discussed in this class during the unit. One could possibly explain Megan's ease with how students used prior knowledge about properties of a kite because this knowledge was already accepted as installed and it belonged to the collective memory of the class.

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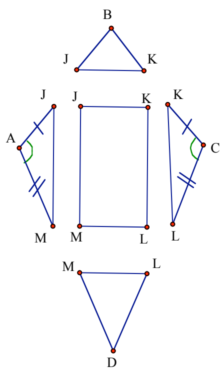
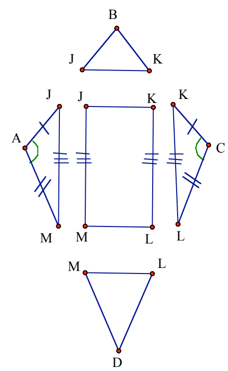
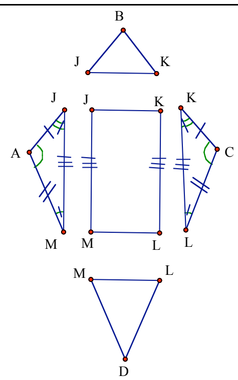
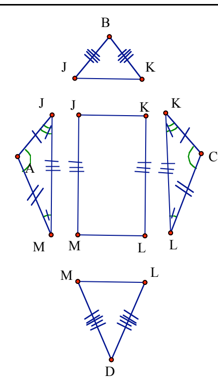
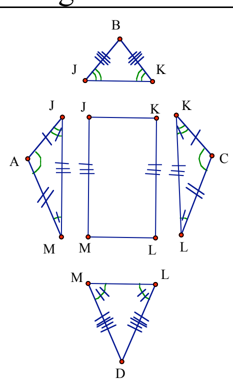
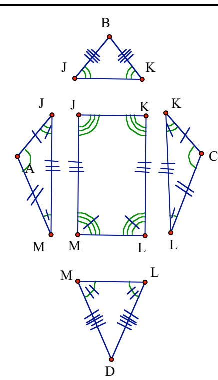
[43] "CPCTC" stands for "corresponding parts of congruent triangles are congruent," which is a statement of what it means to say that two triangles are congruent.

Table 27

Ways of visualizing the kite and its midpoint quadrilateral in Brett’s proof

 1. There is a pair of congruent triangles, *JAM* and *KCL*.	 2. *JM* and *KL* are congruent by CPCTC.
 3. There are pairs of congruent angles in triangles *JAM* and *KCL*, by CPCTC.	 4. There are two isosceles triangles, *BJK* and *DML*.
 5. The base angles are congruent for each isosceles triangle.	 6. *JKLM* has two pairs of congruent adjacent angles by angle addition.

Brett was able to prove that the midpoint quadrilateral of a kite has two pairs of consecutive angles congruent and a pair of congruent sides. He made use of the definition of a kite, properties of a kite, triangle congruency, and angle addition (Figure 16). However, if he were to assume that the midpoint quadrilateral is always a parallelogram, he could have made a stronger statement: The midpoint quadrilateral of a kite is a rectangle. Brett still had some more work to do to complete the proof, even when using the assumption.

By assuming that all midpoint quadrilaterals are parallelograms one could prove that the midpoint quadrilateral of a kite is a rectangle. The class had previously studied before that parallelograms have opposite angles congruent. They had proven that the midpoint quadrilateral of a kite has two pairs of consecutive angles congruent. A quadrilateral with two pairs of consecutive angles congruent and with opposite angles congruent would have all angles congruent. Therefore, the midpoint quadrilateral of a kite would have all angles congruent, and it would be a rectangle. One interesting feature about this proof is that it uses an interesting strategy to prove that all angles of the midpoint quadrilateral are congruent. Any angle of the midpoint quadrilateral of the kite is congruent to its adjacent angle and to its opposite angle. As a result, all angles are congruent. This strategy could be applied to other proofs.

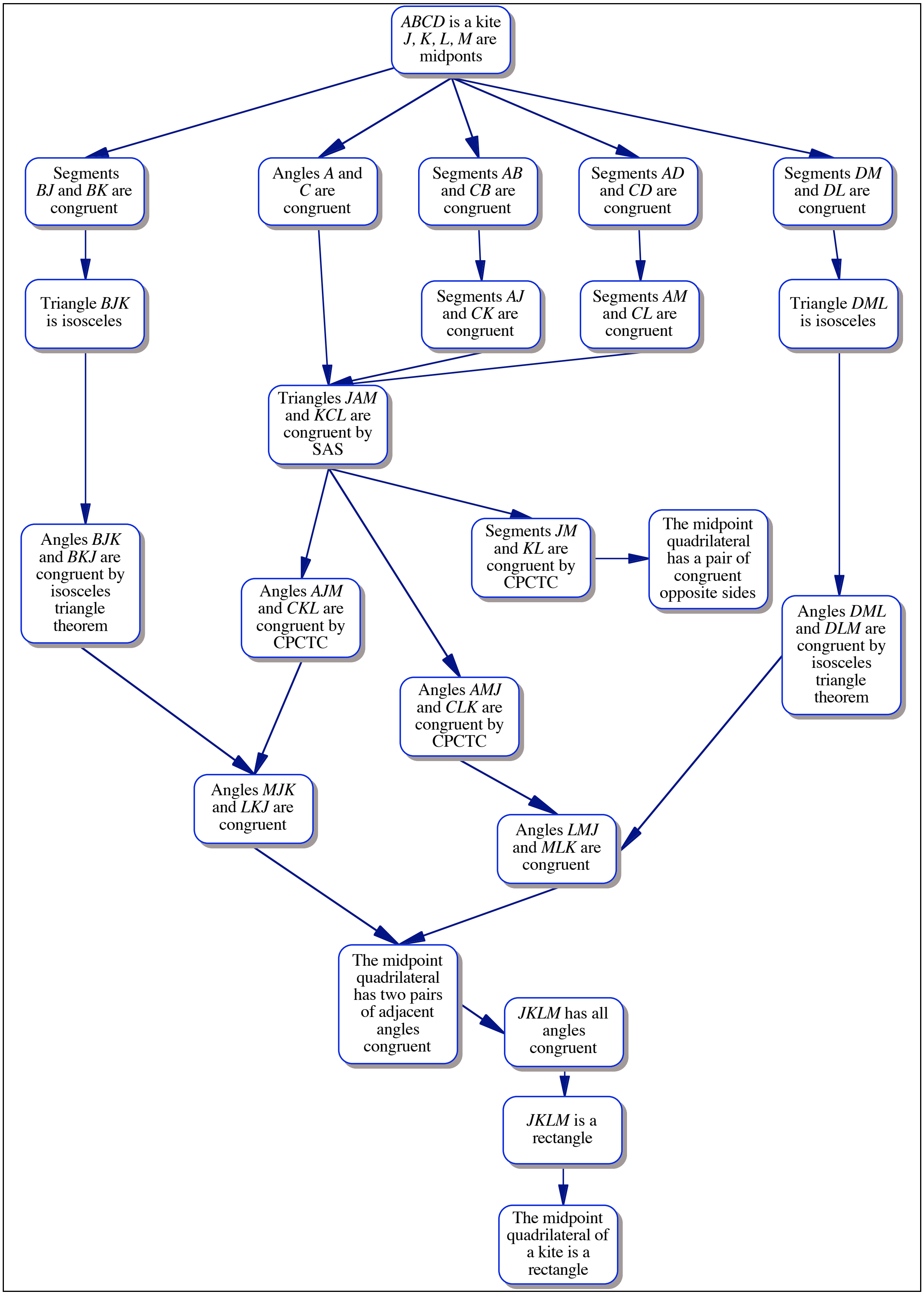


Figure 16. A representation of Brett's proof.

Figure 17 illustrates what steps were needed to complete the proof. Brett had done a lot of work. He had remembered concepts previously studied in the geometry class, and he had applied these concepts to solve the problem. Brett was very close to solving the problem by concluding that the midpoint quadrilateral of a kite is a rectangle. Yet, Megan settled for a minor result: The midpoint quadrilateral of a kite is a quadrilateral with two pairs of adjacent congruent angles and with a pair of congruent sides connecting non-congruent angles.

From an observer's perspective one could ask why the teacher would accept a minor result instead of solving the problem. One possible explanation using the hypothesis of the collective memory is that if the teacher were to accept a conjecture based upon the medial-line theorem, she would transgress usual practices about what should be remembered when working on a mathematical task that involves doing a proof. If instead, the teacher were to start to prove all midpoint quadrilaterals are parallelograms, she could have had the opportunity to conjecture the medial-line theorem.

Figure 17 shows statements in the proof that depended upon the medial-line theorem. The interest in proving that all midpoint quadrilaterals are parallelograms—to solve the kite problem and to solve other sorts of problems— could have led to the discovery of the medial-line theorem. However, it appears that it was difficult for the teacher to accept the conjecture that all midpoint quadrilaterals are parallelograms. If she were to accept the conjecture, it would have altered the collective memory of the class. Contrary to usual practices, the class would have been allowed to remember a result that had not been proven yet. So, even though students had actual memories about the conjecture, these were not officially part of what the teacher wanted students to remember.

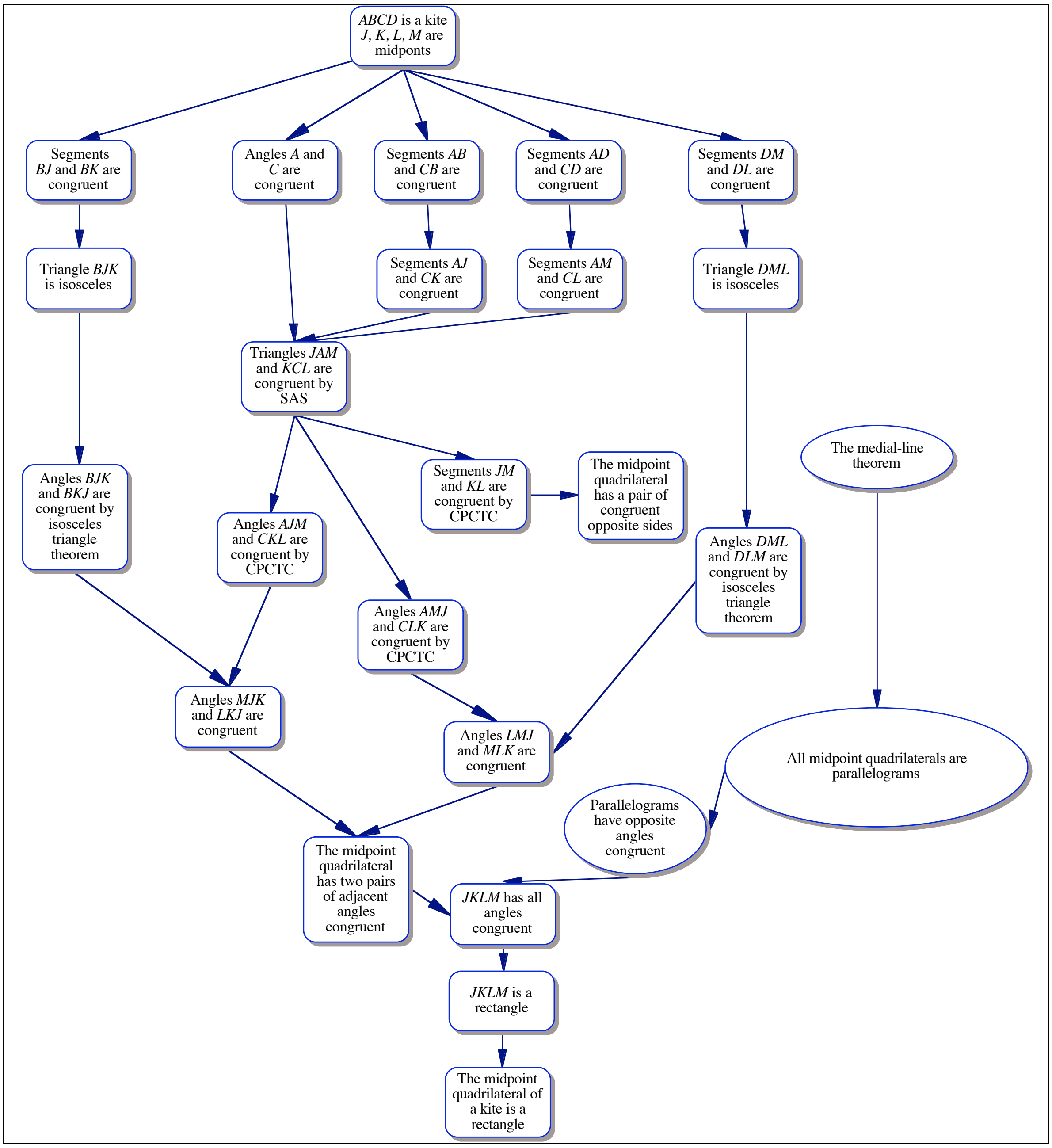
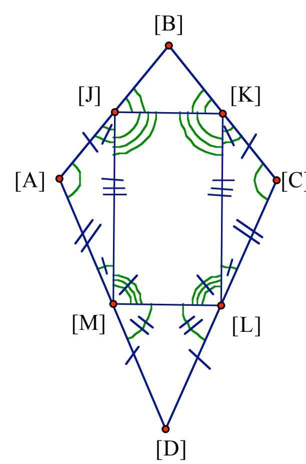


Figure 17. Steps for proving that the midpoint quadrilateral of a kite is a rectangle.

Continuation to the Kite Episode

After Brett's presentation, the class continued to struggle to complete the proof for the next 3 minutes, until the bell rang. Adriana, another student, gave a new idea to complete the proof. Megan did not follow up on Adriana's idea. However, it seems as if Adriana had intended to prove that the midpoint quadrilateral of a kite was a parallelogram, instead of using this statement as a reason to prove that the midpoint quadrilateral of a kite is a rectangle. Adriana spoke from her seat while Brett was still at the board. Megan took this opportunity to walk to the board and restate Brett's proof.

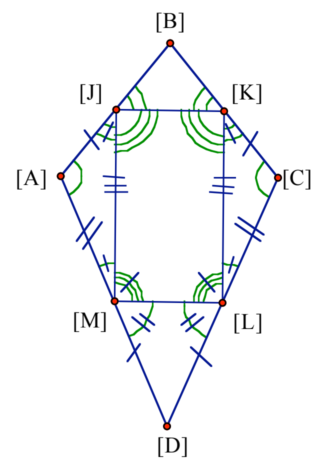
Turn #	**Speaker**	**Turn**
44.	Adriana	[Raises hand.]
45.	Megan	Okay, Adriana?
46.	Adriana	64. Couldn't you just do the exact same thing as that 65. to find out that 66. all the angles are congruent, 67. they all equal 90-degrees 68. so then you would have to say 69. it is a parallelogram? 
47.	Megan	[Megan walks to the board.] 70. Okay, well, here is the problem with that, 71. I know, 72. I know these two {angles *MJK* and *LKJ*} are equal to these two {angles *JML* and *KLM*}. 72a. [Points to angles *MJK* and *LKJ*.] 72b. [Points to angles *JML* and *KLM*.] 73. Er, wait a minute.
48.	Brett	No.
49.	Megan	74. No, I don't know. 75. I know, 76. um, I know 77. these two {angles *JML* and *KLM*} are equal 77a. [Points to angles *JML* and *KLM*.]

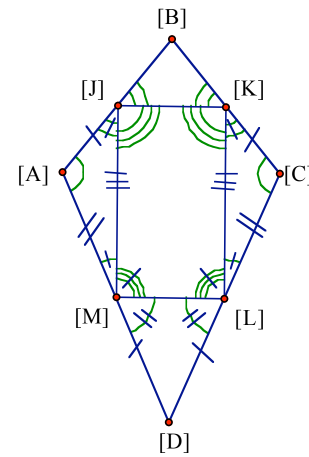
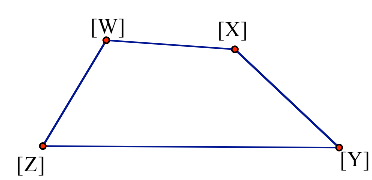
Turn #	Speaker	Turn
		78. and these two are equal 78a. [Points to angles *MJK* and *LKJ*.] 79. but I don't know 80. they are equal to each other. 81. that's the problem. 82. Do you know 83. what I mean?

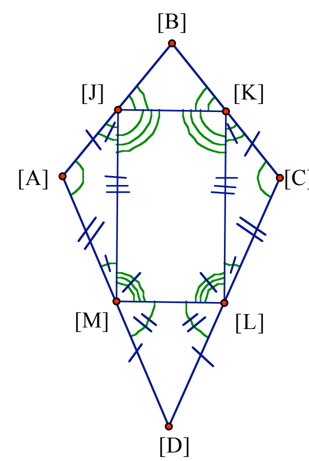
Megan made a mistake because she said that all angles were congruent (clause 47.72), but she corrected herself quickly (clause 49.74). Megan repeated Brett's conclusion—the midpoint quadrilateral of a kite has two distinct pairs of adjacent congruent angles (clauses 49.77 and 49.78). Her restatement of what they had gotten so far from Brett's proof was a way to accept Brett's argument up until that point. Megan, then, stated the problem of the proof—they did not know that the angles were equal to each other (clauses 49.79 and 49.80). With this statement, Megan implied that they could not take for granted as an assumption that all midpoint quadrilaterals are parallelograms.

Ada made a third attempt to complete the proof about the midpoint quadrilateral of a kite, following Megan's restatement of Brett's proof. Ada proposed that the midpoint quadrilateral had pairs of supplementary angles. Since those pairs of supplementary angles were also congruent, then each angle had to be 90-degrees. Brett reminded Ada that they did not yet know that the midpoint quadrilateral was a parallelogram, which incidentally provides evidence about Brett's awareness of the difficulty with the proof because they could not assume that the midpoint quadrilateral is a parallelogram. Ada explained her idea again, but Megan did not seem to understand Ada's suggestion. Ada's idea, which she explained from her seat, is significant, because it gave the class another chance to do a proof about the midpoint quadrilateral of a kite.

Brett was still at the board. By default, his stay at the board suggests that Brett was still accountable for producing the proof. Brett used Ada's idea to do a new proof. In this new proof, Brett assumed that the midpoint quadrilateral was an isosceles trapezoid. The transcription of this exchange follows.

Turn #	**Speaker**	**Turn**
50.	Megan	Okay, 84. go ahead Ada.
51.	Ada	85. But, the, the two pairs have to supplement each other, 86. because they are supplementary, 87. which means that, um, 88. and if they are congruent also 89. it means that 90. they have to be right angles. 
52.	Brett	91. But you don't know 92. it's a parallelogram.
53.	Ada	No. 93. I'm just going by supplement…by supplementary angles 94. because you know that 95. because of the isosceles triangles that the, the two << >> that they are equal.
54.	Brett	<<Yeah, um, yeah.>>
55.	Brett	96. That these two [points to angles *JKL* and *MLK*] have to be (supplementary).
56.	Ada	97. (But like), not that one 97a. [Moves her hand vertically.] 98. but like that one 98a. [Moves her hand horizontally.]

Turn #	Speaker	Turn
57.	Brett	99. Well, if it was an isosceles triangle 100. then these two angles, 100a. [Points to *JKL* and *MLK*.] 101. if this is an isosceles triangle 101a. [Traces Δ*KCL*.] 102. this is the base 102a. [Points to angle *CKL*.] 103. and this is the base, 103a. [Points to angle *CLK*.]  104. and these {angles *JKL* and *MLK*} would be supplementary, 104a. [Points to angles *JKL* and *MLK*.] 105. and these {angles *KJM* and *LMJ*} would be supplementary, 105a. [Points to angles *KJM* and *LMJ*.] 106. because of the way it's made, 106a. [Makes parallel lines with his hands.] 106b. because of alternate, er no same-side-interior angles.
58.	Megan	Okay, 107. wait a minute, 108. Say it again. 109. It's a combination of 110. what Ada is saying.
59.	Ada	No, but…
60.	Megan	111. Do you see 112. what he's saying? 113. No, I'm asking [the Researcher] 114. because we only have a minute. 115. Do you see 116. what they are saying?
61.	Researcher	Um, not yet.
62.	Brett	117. Okay, well, if this is a trapezoid—well… 117a. [Draws a trapezoid without labels but sides *WX* and *ZY* don't look parallel.].  118. Pretend it's a trapezoid. Okay.

Turn #	Speaker	Turn
63.	Megan and students	[Laugh.]
64.	Megan	119. Pretend it's a trapezoid.
65.	Brett	120. This {angle X} is gonna be supplementary to this {angle Y} 120a. [Points to angle X.] 120b. [Points to angle Y.] 121. because they are parallel lines 122. and same side interior angles {angles X and Y}… 122a. [Points to angles X and Y.] 123. So if we use that, um here—{$JKLM$} 123a. [Points to $JKLM$.] 
66.	Megan	124. But I don't have any parallel lines. 125. Where are they?
67.	Brett	Oh. 126. Wait. 127. Well, we're, we're assuming 128. that it might be a trapezoid, right? 129. If it wasn't a trapezoid, 130. it's gotta be—
68.	Megan	131. Okay, if it's not a parallelogram, 132. you are saying—
69.	Brett	133. Well, I'm saying that--
70.	Megan	134. It could be a trapezoid.
71.	Brett	135. No what I'm saying is—
72.	Megan	136. (Is that what you're—) Okay.
73.	Brett	Ah! 137. Well, (if it is—)
74.	Megan	138. (I can go with that) (for a minute.) 139. Okay, wait a minute. 140. I wanna… 141. Wait.
75.	Brett	142. (If it is—) 143. we can prove that 144. it's not a trapezoid, 145. leaving the only option to be a rectangle.
76.	Megan	146. Okay, stop for a minute,

Turn #	Speaker	Turn
		147. because we need to go only in like a minute. 148. I want to say that 149. this is a great strategy. 150. When you are doing a proof 151. you think, 152. "okay, I'm stuck here, 153. what else could it be." 154. And then examine, 155. if it was that, 156. will that help me 157. or will that hurt me. 158. Okay, you know what, 159. I want 160. you to keep thinking about this problem. 161. I want 162. you to keep thinking. 163. Tonight's homework is relatively easy 164. compared to the homework we've had. So if I, [44] 165. this is a problem 166. that you would have time to maybe <<>> think about. 166a. <<[Bell rings.]>>

The exchange between Megan and Brett after Ada's suggestion illustrates differences in the kinds of assumptions that students are allowed to make when doing a proof. Earlier, while Brett was presenting his original proof, Megan had suggested that the midpoint quadrilateral could be a trapezoid as a counterexample to Brett's claim that a quadrilateral with a pair of congruent sides must be a parallelogram (clause 30.41). Then, Brett had accepted Megan's idea that the midpoint quadrilateral of a kite could be a trapezoid (clause 37.55). Ada suggested that the pair of congruent angles in the midpoint quadrilateral is also a pair of supplementary angles. Ada's idea, prompted Brett to conclude that the midpoint quadrilateral of a kite was a trapezoid (clause 62.117). In

[44] I take this as a false start.

contrast to her previous rejection to the assumption that the midpoint quadrilateral is always a parallelogram, Megan let Brett suppose that the midpoint quadrilateral of a kite was a trapezoid, by giving him the chance to explain his ideas on several occasions (clauses 58.108 and 74.138). It could have been the case that they had gathered evidence by deduction for concluding that the midpoint quadrilateral of a kite was a trapezoid. They had applied the theorem that states that a quadrilateral with two pairs of congruent and supplementary angles is an isosceles trapezoid. In contrast, the conclusion that the midpoint quadrilateral of a kite is a rectangle was based upon the assumption that all midpoint quadrilaterals are parallelograms for which they had only gathered empirical evidence, but they had not proven yet. There is no explicit evidence in the transcript to explain why Megan let Brett assume that the midpoint quadrilateral was a trapezoid when she Brett's words and said, "pretend it's a trapezoid" (clause 64.119). Yet, at the end of the class, Megan called Brett's move to make an assumption "a great strategy" when doing proofs (clauses 76.149 and 76.150). So, the assumption that the midpoint quadrilateral is a trapezoid could be taken as a hypothesis when doing a proof. In contrast, the assumption that the midpoint quadrilateral is a parallelogram results from gathering empirical evidence. By allowing students to make assumptions based upon empirical evidence, a teacher could threaten the chances of doing a proof.

Brett wanted to prove that the midpoint quadrilateral of a kite was a rectangle by contradicting the assumption that the midpoint quadrilateral was a trapezoid. He took *JK* and *ML* as a pair of parallel sides. Brett did not get to complete the proof. However, his choice of $\angle KJM$ and $\angle JML$ as one pair of supplementary angles was a dead end because there was not enough information about other relationships between ΔKJM and ΔJML.

Ada had implied that the pair of parallel sides were *KL* and *JM* (see Figure 18), when she referred to the pair of congruent angles (clause 51.88). Later she clarified what she meant when she made a gesture to identify as possible pairs of supplementary angles those angles oriented horizontally, such as ∠*JML* and ∠*KLM* (clauses 56.98 and 56.98a). Ada's conclusion would have been that ∠*JML* and ∠*KLM* are supplementary and congruent, so each angle would measure 90 degrees. However, Ada's assumption that *JKLM* was a trapezoid with *KL* parallel to *JM* could have been contradicted because *KL* and *JM* were already stated to be congruent. And if quadrilateral *JKLM* were to be a trapezoid with bases *KL* and *JM*, these could not be congruent. Ada's idea was different than Brett's, who assumed that *JK* and *ML* were the bases of the trapezoid.

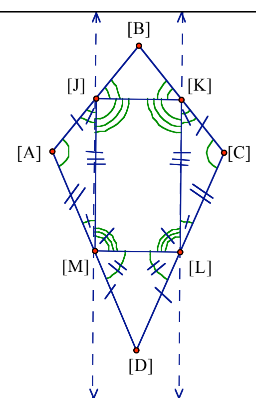


Figure 18. A representation of Ada's assumption that *KL* and *JM* were parallel.

In contrast with Ada, Brett had assumed that the midpoint quadrilateral was a trapezoid with bases *JK* and *ML*. He started by stating that triangles *KCL* and *JAM* were isosceles (clause 57.101) with bases *KL* and *JM*, respectively. However, these triangles were not isosceles, and already had markings to denote that the sides were not necessarily congruent. Then, Brett said that the midpoint quadrilateral had two pairs of supplementary angles. One pair of supplementary angles would be ∠*JKL* and ∠*MLK*.

The other pair of supplementary angles would be ∠*KJM* and ∠*LMJ* (clauses 57.104 and 57.105). This shows that Brett's interpretation of the orientation of the trapezoid was different than Ada's. While Ada considered the vertical sides of the midpoint quadrilateral to be parallel, Brett considered the horizontal sides of the midpoint quadrilateral to be parallel. Brett's interpretation can be proved; Ada's cannot be proved.

Brett elaborated on his answer, using the assumption that the midpoint quadrilateral was a trapezoid, in turns 62 and 65. With this assumption, he concluded that a pair of angles such as *JKL* and *KLM* would be supplementary. Brett used a simplified version of the diagram of the midpoint quadrilateral (see Figure 19), assuming that it was a trapezoid and pointing to angles that were oriented similarly as in quadrilateral *JKLM* (clauses 65.120 and 65.123). In the event that *JKL* and *KLM* were congruent, then the angles would be right angles, and the midpoint quadrilateral would be a rectangle. In the event that *JKL* and *KLM* were not congruent, then the midpoint quadrilateral would be a trapezoid.[45]

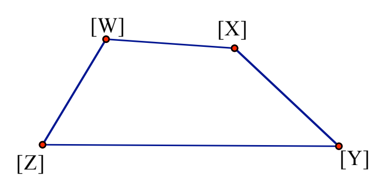


Figure 19. Brett's diagram of a trapezoid.

After having considered different alternatives, the class had not been able to label the midpoint quadrilateral of a kite as one special quadrilateral. They were able to state some properties of that quadrilateral: It had two pairs of congruent adjacent angles and it

[45] It is noteworthy that in this discussion, students only considered special quadrilaterals as possible midpoint quadrilaterals of a kite, disregarding other geometric figures that do not necessarily have a name, but that could be described according to their properties.

had one pair of congruent sides (connecting the vertices of the non-congruent angles). There is not a special quadrilateral named with those characteristics. They could have proved that it was an isosceles trapezoid. However, they needed more information about parallel lines to label the midpoint quadrilateral of a kite as a rectangle.

Before the end of the class, Megan talked about the value of Brett's work by emphasizing how his work showcased a problem solving method when doing proofs (clauses 76.148 through 76.157). Megan's meta-talk about the solution method, not about the problem itself, was a way to salvage Brett's work and to illustrate to the class that making hypotheses was good as a problem solving strategy for anticipating results. By the end of the class, Megan had not allowed students to assume that the midpoint quadrilateral is a parallelogram, even though this was the assumption they needed in order to prove that the midpoint quadrilateral of a kite is a rectangle. So, the prior knowledge that students had about midpoint quadrilaterals was not useful to do the proof.

In contrast with Megan's refusal to take the assumption as prior knowledge to work on the proof, Megan allowed students to draw upon concepts already studied in the geometry class. Table 28 shows a list of concepts and propositions that students referred to in the kite episode with Megan's approval. Students made use of terms and propositions pertaining to concepts studied prior to the quadrilaterals unit such as triangle congruency, midpoints, parallelogram, the sum of angles on a straight line, and parallel lines. Students also made use of concepts introduced in the unit pertaining special quadrilaterals: kites, rectangles, and trapezoids. However, the assumption made about midpoint quadrilaterals was a topic of discussion during the unit that Megan did not allow students to take for granted for doing the proof. Therefore, Megan established a difference between resources within the unit that could be taken as prior knowledge and resources that could not be used in doing the proof. Even though students remembered that they had made a conjecture about midpoint quadrilaterals, Megan did not allow students to rely on that memory to do the proof.

Table 28

References to concepts and propositions in the geometry curriculum

Concepts and Propositions	**Lexical choices** (in order of appearance in the transcript)
kites	• Quadrilateral with a pair of opposite angles congruent (10.1, diagram) • Quadrilateral with two adjacent congruent sides (diagram) (Although the diagram does not show markings to denote *BJ* congruent to *AJ*, it does show *BJ* and *BK* congruent.)
theorems stating sufficient conditions for triangle congruence	• Side-Angle-Side (14.4)
midpoints	• The definition of a midpoint as a point dividing a segment into two congruent segments (14.5)
the definition of congruency of triangles	• CPCTC (14.9, 21.23)
isosceles triangles	• Triangles with two congruent sides (19.17) • Base angles are congruent (53.95, 55.96)
postulates about sum of angles	• Sum of angles from the same vertex on a line (21.25) • Angle subtraction (21.25)
properties of parallelograms	• Opposite sides are congruent (23.29) • All angles congruent (46.69) • Consecutive angles are supplementary (51.85)
trapezoids	• Isosceles trapezoid as a quadrilateral with two congruent sides (37.55) • Quadrilateral with consecutive angles supplementary (65.120)
properties of angles made by parallel lines	• Same side interior angles congruent (65.122)

After the Kite Episode

As students left the classroom, Ada went to the board. Ada explained her idea of taking *JM* and *KL* as the parallel sides of a trapezoid to Megan. However, Megan restated the difficulty of relying on the assumption that the midpoint quadrilateral is always a parallelogram. Megan said that the core of Brett's proof rested on that assumption, which Eva wanted to take for granted at the beginning of the class. If they were to accept the assumption, the proof would have been much easier to do.

> Megan: They are only supplementary if this is a parallelogram. That's what Brett's saying is—
> Ada: (Well that would make it a lot easier.)
> Megan: (Well, if that's a parallelogram,) it would make it a lot easier up here [laughs].
> Ada: Oh!
> Megan: We're getting back to that. That's sort of what Eva was saying, "Can I assume that it's a parallelogram?"
> Ada: It's just like, to prove it from there.

Ada had continued to be interested in showing her proof after the class had ended. The exchange between Ada and Megan demonstrates that Megan was aware of the difficulties in completing the proof because they could not assume what Eva had asked earlier—that the midpoint quadrilateral is always a parallelogram. So, even though students remembered this result from their work earlier in the unit, Megan did not allow them to use this conjecture as a resource to do the proof.

Conclusions about the Kite Episode

The kite episode reveals tensions in teaching provoked by changes in usual practices during the unit. Brett drew upon a result that the class had suspected true since the beginning of the unit. However, this result had not been proven. It required a theorem that had not been officially installed. Usually, these kinds of difficulties do not surface because the textbook sets the sequence of topics in the class and the assigned problems do not require students to anticipate a theorem. In contrast, in mathematics, a conjecture could be the source of valuable work done by mathematicians who would continue to work on a problem assuming that a conjecture is true, until proven so. A recent example in the history of mathematics is the proof of Fermat's Last Theorem, which was proved by Kenneth A. Ribet in 1986 to follow as a consequence of the Taniyama-Shimura conjecture at a time when this conjecture had not yet been proved (Cox, 1994).[46] Students had gathered empirical evidence to support that the midpoint quadrilateral is always a parallelogram. However, they could not make such a statement in class, because they did not have the resources to prove it. Brett had done valuable work, but it was hard for the teacher to trade that work for a proof, unless he could have assumed that the midpoint quadrilateral was a parallelogram. Instead, the teacher traded Brett's work of assuming that the midpoint quadrilateral could be a trapezoid as an illustration of a useful problem-solving strategy.

I want to highlight that my recount of the kite episode is not intended to show a case of the teacher's mathematical knowledge in teaching with problems, even though the teacher's knowledge of the mathematics involved in students' suggestions was an important element in the discussion. Megan was able to cope with the solutions proposed by different students and to identify assumptions in students' arguments that they were not making explicit at the moment when they presented their proofs. Megan was also able to make students aware of the assumptions they were making, holding students

[46] The conjecture was eventually proved by Richard Taylor and Andrew Wiles.

accountable for the prior knowledge they could make use of in solving the problem. Megan did not have difficulties with the mathematical knowledge required for understanding students' solutions. Megan faced difficulties in teaching when students insisted on making an assumption that they had not proven in class yet. This assumption would have allowed them to complete the proof about the midpoint quadrilateral of a kite.

The Day After the Kite Episode

The kite problem continued to be a topic of discussion in the 2nd period class the following day. The day following the kite episode was meant to be the last day of the unit (the 12th day). When planning the unit, that lesson was expected to include a proof of the medial-line theorem. The homework problems from day 11 included problems in which students had to draw upon the medial-line theorem, even though they had not installed this theorem in class yet. While discussing solutions to homework problems, Megan called students' attention to the line connecting two midpoints of two sides of a triangle, which students had started to note is half of the third side. Megan related this finding and the assumption that the midpoint quadrilateral is always a parallelogram. Megan drew a quadrilateral and its midpoint quadrilateral and said,

> Megan: Eva said, "can we assume that?" yesterday. And it seems like—that seems like a very valid request, because we've seen them for two or three weeks, that every time we draw a quadrilateral we get these parallelograms. She said, "we know that that is a parallelogram, can't we assume that?" That would have made [laughs] Brett's problem a lot easier, 'cause he was up there—You know, he had two angles equal. If he could assume that that thing was a parallelogram then he would have been done, 'cause he could say that opposite angles are all congruent. The, so the thing is a rectangle. That was that problem where we were trying to prove that the kite [goes to the board and draws a kite and its midpoint quadrilateral as she speaks] when you connect the m-quads, or the midpoints of a kite, you get a rectangle [completes the drawing.]

Here, Megan takes as a valuable outcome of Brett's work the relationship between a quadrilateral and its midpoint quadrilateral. The 11th day of the unit, Megan ended the class by asking students to use Brett's strategy when doing homework. However, on the 12th day of the unit, Megan revisited Brett's problem, emphasizing the result that the midpoint quadrilateral of a kite is a rectangle. Megan recalled the difficulty with Brett's solution because it was based upon an assumption that had not been proven yet. With the assumption, Brett could have applied a property of parallelograms already studied in class: parallelograms have opposite angles congruent.

Following Megan's previous comments, she called students' attention to possible relationships between the diagonals of a quadrilateral and the sides of its midpoint quadrilateral, noting that this was the focus of the lesson. And in response to a question posed by a student about homework, Megan said, "If we could get this theorem, it would help us with quite a few of the other little wrenches that had come up." This is the second time when Megan called upon the difficulties because of the lack of a theorem to prove the assumption made. For the first time, Megan brought up a possible relationship between the diagonals of a quadrilateral and its midpoint quadrilateral. Megan asked Brett to repeat what he had said the day before about the midpoint quadrilateral of a kite. Brett went to the board and drew a diagram, similar to the one he had done the day before—of a kite and its midpoint quadrilateral—with markings to denote those parts he had proven to be congruent.

Brett had difficulties remembering his argument and other students started to make suggestions. Walter remembered that they had assumed that the midpoint quadrilateral of a kite could be a trapezoid. Chad asked, "How can it be a trapezoid if the m-quad is a parallelogram?" Brett replied, "But we don't know that." And Megan took this as an opportunity to launch the medial-line theorem.

> Megan: We don't know that. That's the whole problem. That's what Eva said yesterday, "can't we just assume is a parallelogram?" Because if we could do that, you are right, we could be done. So maybe that's what we should quit now and start working on. Okay, I'm gonna leave that picture up there, but the thing that maybe we should be working on is—why is it always a parallelogram? [Megan walks to the board.] 'Cause if we can get this thing [points at the midpoint quadrilateral of the kite] is a parallelogram, we would be done, 'cause we'd know that the opposite angles would be congruent then. So, if these two are congruent then they are all congruent. And we would be able to show that this thing is a rectangle.

Megan's recall of Eva's question, "can't we just assume is a parallelogram?," is important because it frames the discussion of what is the important thing to prove at the moment. Megan had moved from considering that the important result is that the midpoint quadrilateral of a kite is a rectangle towards introducing the medial-line theorem. The need to prove this theorem was evidenced by Eva's and Brett's difficulties to complete their proofs. Even though they knew what the resulting figure would be when drawing the midpoint quadrilateral of a trapezoid and of a kite, they could not do the proof without using the assumption.

Brett made a last attempt to get that the angles of the midpoint quadrilateral of a kite is a rectangle by assigning hypothetical values to the angle measures. However, he was not able to use this strategy to show that the kite had right angles. Megan asked Brett to stop working on the problem, and introduced the next activity geared towards proving the medial-line theorem.

The relevance that this problem had in this class during two consecutive days was unusual. The second take on proving that the midpoint quadrilateral of a kite is a rectangle took about 6 minutes during the 12th day of the quadrilaterals unit. It is significant that in none of the attempts, did Megan let students make the assumption that the midpoint quadrilateral is always a parallelogram. Megan kept recalling Eva's plea to make use of that assumption and identified their inability to put to use that assumption as the main difficulty in completing the proof. Students tried to get around that assumption, finding alternative ways to show that the midpoint quadrilateral of a kite has four right angles. The last attempt was Brett's initiative to assign numerical values to the angles in his diagram. However, none of these attempts were fruitful in completing the proof. Megan used this opportunity to launch the discussion of the medial-line theorem, as it had been previously planned when designing the unit.

Discussion

I contend that there exist such as thing as the collective memory of the class, different from students' actual memories. The collective memory of the class includes those things and events that students are entitled to remember, according to the teacher. The two episodes—the rectangle episode and the kite episode—show cases where a teacher's work shaping the collective memory of the class was evident as she tried to cope with the demands of teaching with a problem.

The rectangle episode and the kite episode involve similar mathematical tasks. In the rectangle episode, it was convenient to visualize a rectangle as a parallelogram. By visualizing a rectangle as a parallelogram, the class could prove the theorem that diagonals of a rectangle are congruent. Megan guided students into doing a proof, using as resources prior knowledge that had been officially introduced into the collective memory of the class. Properties of parallelograms were some of the resources for that proof. Instead of doing the proof, Megan could have considered the property of diagonals of a rectangle prior knowledge from the Guess my Quadrilateral game. However, she did not, because the collective memory is different from the actual memories that students may individually possess. In the kite episode, it seemed convenient to prove that a quadrilateral is a rectangle by assuming that that quadrilateral is a parallelogram in order to prove that the quadrilateral had all angles congruent. However, the class could not finish the proof because the resources to prove the needed assumption could not be considered part of the collective memory of the class yet. The theorem needed to prove that assumption had not been stated and proven in class yet. So in the two episodes the teacher set boundaries for the resources that students had available to work on a proof, limiting resources to those that were part of the collective memory of the class.

Both episodes illustrate tensions in teaching. Students insisted upon using resources that the teacher could not bring herself to allow them to use. In the rectangle episode, students wanted to take for granted that the diagonals of a rectangle are congruent. In the kite episode, students wanted to assume that the midpoint quadrilateral of all quadrilaterals is a parallelogram. I explain the teacher's decision to restrain students from using these resources with the hypothesis of the collective memory. The resources that students wanted to use, but that the teacher did not let them use, were not part of the collective memory of the class. The teacher had not officially sanctioned that prior knowledge as shared knowledge of the class. So, when students insisted upon using that prior knowledge to work on a mathematical task, Megan made students aware of what kind of knowledge they could draw upon, setting the boundaries between what students should remember and also what students should forget.

The replacement unit on quadrilaterals introduced changes to usual practices of teaching by structuring the teaching through problems. Some of the phenomena that became visible such as a teacher's attempt to manipulate what prior knowledge students should remember could be explained by the hypothesis that there is a collective memory that is shaped by the teacher in the geometry class. Students showed that they could make deductions from possibilities as they worked making conjectures about midpoint quadrilaterals from the first day of the unit, anticipating the need for a theorem that was not known but that could be used to deduce other things. To make the teaching with a problem work, the teacher had to be open to a range of individual, actual memories. Students relied on their individual prior knowledge about quadrilaterals and their properties; students remembered results that were based upon visual perception; and students requested to use an assumption of a conjecture that they remembered but that they had not been proven yet. To make the problems viable, the teacher encouraged students to enlist individual knowledge about properties of quadrilaterals and invited them to make conjectures about midpoint quadrilaterals. Both of these practices, although useful for working on a problem, were problematic for the teacher.

In the rectangle episode, the teacher had to make students aware that they could not assume properties of rectangles that had not been proven in class, making them prove that a rectangle is a parallelogram and using that result to prove a property of diagonals of rectangles. In the kite episode, the teacher mended the difficulty of not getting a proof by stressing the value of making hypotheses as a problem-solving strategy, coming back to the problem the next day to prove the missing theorem. In both cases, the teacher had difficulties because students were relying on knowledge that was not yet part of the shared knowledge of the class.

The teacher's tensions had to do with differences in how to manage students' memories of prior knowledge when teaching with a problem and when doing a proof. The rectangle episode and the kite episode show two ways in which teaching geometry with a problem challenged usual practices in Megan Keating's class. The rectangle episode highlights how the Guess my Quadrilateral game required Megan to enable students to draw upon their memories, contrary to what she would usually do. Students could rely on prior knowledge that they individually held. However, all students in the class did not necessarily share these memories. Moreover, students could have been using other resources, such as visual perception, to make new memories about properties of quadrilaterals, without relying on deductive reasoning. So by enabling students to draw upon their memories while playing the Guess my Quadrilateral game, Megan lost control of knowing the content of students' memories, the way they had acquired those memories, and whether all students shared those memories or not.

The Guess my Quadrilateral game, as an example of a problem to teach with, was an opportunity to teach students about mathematical definitions. As a result of the play of the game, students talked about characteristics of definitions of special quadrilaterals, such as their conciseness. With the aim of finding the hidden quadrilateral, students discussed the kinds of questions they asked about properties of quadrilaterals and the order of those questions. Students also discussed relationships between the properties of different special quadrilaterals. For example, students classified quadrilaterals according to characteristics of their sides—such as quadrilaterals with two pairs of congruent sides—or characteristics of their angles—such as quadrilaterals with all angles congruent. However, the game was insufficient for the teacher to ask students to do proofs of theorems based upon a definition. Moreover, the game was insufficient for the teacher to establish an official record of what the class had learned.

The kite episode showcases how the midpoint quadrilateral of a kite task challenged the teacher. The teacher could not stop memories of past conjectures from being invoked. Students started to make assumptions that relied upon a theorem. However, the installation of that theorem, including the proof of that theorem, was yet to come. In contrast with the situation of doing proofs, the teacher provoked students to anticipate new knowledge. But the unit, as an activity of teaching with a problem, was insufficient for the teacher to start a proof of the medial-line theorem with the aim of proving the conjecture that all midpoint quadrilaterals are parallelograms.

In both cases, the activity of teaching with a problem implied changes in the usual temporal boundaries that demarcate the kinds of resources students could make use of when doing proofs. In the situation of doing proofs, students can use knowledge from the geometry class, disregarding knowledge from previous mathematics classes. In addition, teachers activate immediate knowledge from the geometry class for students to use this knowledge in the proofs. However, when teaching with a problem, the teacher activated memories from previous mathematics classes and from shared experiences. The teacher also allowed students to anticipate new knowledge. As a result, the teacher had to do a great deal of effort to shape the collective memory of the class.

I have presented evidence for some of the tensions that the teacher needed to manage as a consequence of breaches in usual practices. In particular, the replacement unit on quadrilaterals introduced changes to the usual organization of knowledge over time. I propose that usual practices depend upon the existence of an apparent shared knowledge of the class—a framework of collective memory. The apparently contradictory actions of the teacher could be explained as a struggle between making the resources that students needed to solve problems public, and yet, holding the position that those resources were not really known. So even when those resources were public, because students had talked about their ideas in class, the teacher did not conceive of those resources as part of the collective memory of the class. I propose then that the teacher's usual reliance on practices that set boundaries for what can be remembered—limiting the shared memory of the class to the immediate past—sustain the work of doing proofs.

Conventional wisdom has it that in the geometry course new mathematical propositions build upon proven propositions. I have shown that the teacher can use the collective memory to bring about coherence and continuity to class work. Coherence entails a connection among topics whereas continuity stresses the timely sequence in which these topics are linked. Connections with past problems give some sort of credibility to new problems, and the collective memory of the class prolongs the past to the present.

Conclusions

The work of teachers has been under public scrutiny because of concerns regarding their apparent inability to embrace the ideals of the reform in mathematics education (Jacobs et al., 2006). Teachers are supposed to use problem-based instruction in their teaching. Teachers are also supposed to draw upon students' prior knowledge when teaching new mathematical ideas. In this study I show that the kinds of resources from prior knowledge that students could make use of are different when *teaching geometry with a problem* and when *doing proofs*. The teacher's expectation for students' work on a mathematical task differed substantially according to the activity framing that mathematical task. Consequently, the teacher shaped the collective memory of the class differently to suit those expectations.

Within the situation of engaging students in proving, students can only make use of theorems, definitions, and postulates that have been officially introduced in the geometry class. However, within the activity of teaching with a problem, students seem to need to rely upon knowledge from the remote past (of previous mathematics classes or of experiences outside of school). Students also seem to need to anticipate knowledge by making conjectures that rely on theorems that they have not studied in the geometry class yet. By enabling students to use as resources knowledge from the remote past or knowledge that they suspect to be true, a teacher changes usual practices in the geometry class when teaching with a problem. Students come to know those resources differently through their work on a problem because of a teacher's moves to shape the memory of the class.

This work suggests that geometry teachers do organize new knowledge, limiting what can be remembered to the immediate past—the set of propositions endorsed in the geometry class—and disregarding knowledge from other sources besides the geometry class. The teacher attempted to shape what students remembered and what student forgot through classroom discussions. In this study, classroom discussions conveyed not just statements about mathematical concepts, but evaluative stances towards the kind of knowledge students can make use of when working on a problem. The teacher showed that the knowledge that students could make use of in doing a proof ought to be part of the collective memory of the class.

One of the possible differences when bridging the work of teaching geometry with a problem and engaging students in proving may have to do with the apparent incompatibilities of the paradigms behind those two activities. The history of mathematics includes debates about different perceptions of what is mathematical knowledge and how new mathematical knowledge comes about (Kitcher, 1983). Ways of knowing in the discipline of mathematics are transformed within schooling to suit the demands of the knowledge communication (Chevallard, 1985). In addition, a teacher has the responsibility to introduce students into broader practices of the discipline. How a teacher shapes the collective memory to sustain the work on proving in mathematics classrooms could be in conflict with the demands of teaching with a problem, where memories are less controllable. Reconciling those differences may require more than imposing reform ideals, but a deep understanding of how teachers do their work.

CHAPTER 5

THE RATIONALITY INVOLVED IN MANAGING STUDENTS' PRIOR KNOWLEDGE WHEN TEACHING WITH A PROBLEM: THE PERSPECTIVE OF TEACHERS

In this chapter I study the question of what teachers accept responsibility for doing in order to manage students' prior knowledge when *teaching with a problem*. This question complements the question studied in chapter four in the following way: while in that study I examine a teacher's actions from the point of view of the observer, in this study I examine teaching actions from the practitioners' point of view, using teachers' descriptions of their anticipated actions and teachers' commentary on actions of an observed teacher. I study that question using records of focus group discussions among teachers who participated in sessions centered on teaching geometry with a problem. I identify the practices that teachers hold themselves responsible for doing in order to have students remember (or forget) things and events in their geometry classes. These practices include actions at specific moments in the enactment of a mathematical task—before, during, and after students work on a task—such as prompting students to remember what they need to use when solving a problem or asking students for justifications.

I use the focus group data to investigate, from the perspective of the teacher, how a teacher would manage students' prior knowledge so that students' work on a problem could count as an opportunity to learn. I present examples of teaching practices that involve teachers' actions within a lesson. These actions are not meant to be comprehensive, but they illustrate what a teacher could do to manage students' prior knowledge when teaching with a problem. For example, managing a discussion and using a diagram on the board are actions that could conceivably happen during a lesson. A teacher also does things before or after a lesson. For example, planning a lesson and reading a student's journal are actions that happen before and respectively after a lesson. For this reason, I selected data from the focus group sessions where participants commented on particular actions for activating or for preventing students from using prior knowledge as a teacher interacts with students in the class. These actions, such as talking or altering a diagram, function as levers that a teacher could conceivably manipulate in order to manage students' prior knowledge.

The focus group sessions were designed around the question, "What can one say about the angle bisectors of a quadrilateral?" That question was considered as an example of a problem that could be used to teach new geometry material. The 3-hour sessions started with teachers working on the problem on their own and discussing their findings. Then, the moderator of the session started a discussion by asking participants to identify what topics of the geometry curriculum could be taught with that problem. Following that discussion, the agenda of the focus group sessions included the presentation of videos of classroom instruction where a teacher, Ms. Keating, had used the angle bisectors problem in several of her classes. Approximately one hour after the beginning of the session, participants watched a 10-minute montage of clips of different videos. These videos showed various findings that Ms. Keating's students had made when working on the angle bisectors problem, including conjectures about the figure formed by the intersection of the angle bisectors of a quadrilateral and proofs of some of those conjectures. In the last hour of the session, the moderator presented short video clips where Ms. Keating and her students had worked on a proof for a special case: (a) the angle bisectors of a square make a point, (b) the angle bisectors of a kite make a point, or (c) the angle bisectors of a rectangle make a square. The choice of the special case varied according to the session. In addition, during the session, the moderator presented other artifacts to elicit teachers' responses about how they would use the angle bisectors in their class. These artifacts included quotes from participants of previous sessions, examples of worksheets, and samples of students' work.

The angle bisectors problem and the video clips from Ms. Keating's class were particularly useful because they provided grounds for discussing moment-to-moment actions and decisions of a teacher to manage students' prior knowledge as they worked on the problem. In particular, one of these videos showed that when proving that the angle bisectors of a rectangle make a square, some students had suggested to Ms. Keating to add an auxiliary line to the diagram. Ms. Keating had had to decide whether to allow students to add this auxiliary line or not. In making this decision, I conjecture there are memory issues at play: A solution of the problem using an auxiliary line would require students to make use of a different set of resources (e.g., what concept that line represents, what is known about that concept) that the solution without the auxiliary line might not require. I expected that participants' reactions to the video for the case where a rectangle makes a square would provide opportunities for those teachers watching the videos to comment on actions made by Ms. Keating and to propose alternatives to those actions.

In my analysis of the transcripts of the focus group sessions, I identify moments where participants referred to the work of managing students' prior knowledge as they work on a mathematical task. From participants' reactions to the videos and to other artifacts, I gather examples that attest to the role they attribute to memory and that inform the question of what conditions would make it possible for a teacher to manage students' prior knowledge when using a problem as an opportunity to teach something new. Participants talked about Ms. Keating's actions and at times proposed alternatives to those actions. Participants' perceptions about teaching actions (in the video or in the alternative stories they proposed in response) as well as participants' evaluative stances towards those actions, constitute the basis for understanding the rationality involved in managing students' prior knowledge when teaching with a problem.

My findings include examples of actions that participants said they would need to take in order to manage students' prior knowledge when students work on a problem. Those actions take place at different times in the timeline of a task (students' work on a problem): *before* students work on a task, *while* students are working on a task, and *after* students' work on a task has ended. In my analysis of how participants anticipated a task would unfold, I examine which resources and operations participants expect students to bring to bear as they work on a problem and distill what from students' work on a problem ought to be memorable according to participants. The results show teachers' perceived levers for actions and the possible consequences of those anticipated actions to manage students' prior knowledge.

Overview of the Chapter

I make the hypothesis that a teacher's work while students work on a problem may involve two sorts of actions. On the one hand, teachers can activate and make use of prior memories as students are working on a problem. Teachers, on the other hand, can underscore those memories that students should keep for the future as students are working on a problem. So, the first two sections of the results include teaching actions to manage prior knowledge in the set up and the implementation of a task. Then, the other two sections of the results involve teaching actions in the implementation of a task and after its conclusion, with the purpose of building knowledge for the future. I start by presenting a summary of the video episode from Ms. Keating's class where students prove the claim that the angle bisectors of a rectangle make a square. This video is the basis for many of the discussions in the focus group sessions that I examine in this chapter. Altogether, there are six sections: a section summarizing the video, four main sections about results, and a final section with the conclusions.

A Proof for the Claim that The Angle Bisectors of a Rectangle Make a Square

The episode[47] starts with an exchange between the teacher, Ms. Keating, and a student, Jack. In answering Ms. Keating's questions, Jack says that the angle bisectors of a rectangle make a square. Other students say that they got the same result when working with their graphing calculator. Ms. Keating projects a diagram made by a student,

[47] Appendix B includes the full transcript of the episode. This transcript was available to participants of the focus group sessions.

Dewey, using his calculator.   Dewey's diagram of a rectangle and its angle bisectors is on the board (see Figure 20).

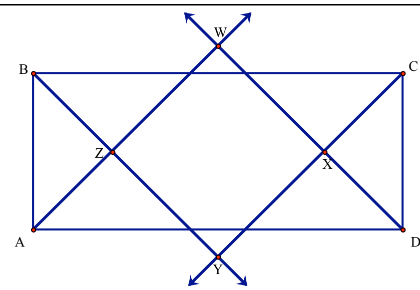


Figure 20. Dewey's diagram of a rectangle and its angle bisectors.

Ms. Keating drags the vertices of the rectangle so as to force the avowed square inside the rectangle, in response to other students who note that Dewey's diagram has two pairs of consecutive angle bisectors intersecting outside of the rectangle (see Figure 21). Then, Ms. Keating asks students how they could show that the angle bisectors of a rectangle make a square. Anthony suggests using the measuring tool, and Ms. Keating reframes the question asking students to recall properties of squares. Anthony gives out a definition of a square: All angles measure 90-degrees and all sides are congruent.

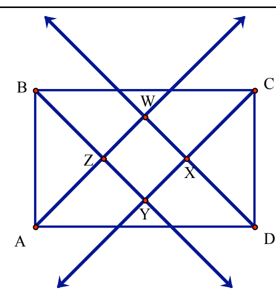


Figure 21. Modified diagram where all angle bisectors intersect inside the rectangle.

A student, Jackie, goes to the board and starts the proof for the claim that the angle bisectors of a rectangle make a square. She uses the definition of angle bisector, the triangle sum theorem, and the definition of a rectangle to deduce that angle bisectors of consecutive angles of a rectangle are perpendicular to each other. Jackie concludes that the figure resulting from the intersection of angle bisectors is either a rectangle or a square. However, the class confronts difficulties in proving that all sides are congruent. Jack, Dewey, and Jackie suggest drawing the diagonal of the avowed square, but Ms. Keating asks them for a rationale before adding any auxiliary lines to their drawing (see Figure 22). Ms. Keating asks for an alternative strategy. Jackie identifies isosceles triangles in the diagram and uses the segment addition postulate to prove that the inner figure has all sides congruent. Ms. Keating asks Jackie to go to her seat and summarizes the most important pieces of the argument to prove that the angle bisectors of a rectangle make a square.

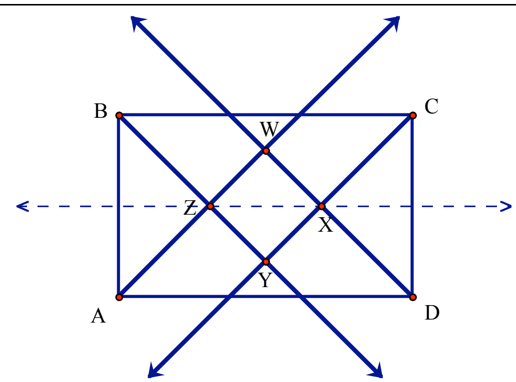


Figure 22. A rectangle, its angle bisectors, and an auxiliary line *ZX*.

The video episode is approximately 10 minutes long. Appendix B includes the full transcript of the video episode, which participants of the session had available. Throughout this chapter, I make references to events in this video as I analyze teachers' reactions to it during the focus group sessions.

Teachers' Actions to Trigger Students' Prior Knowledge Before They Work on a Problem

Before engaging students in a task, a teacher does some work to set up the task. One could expect that issues regarding students' prior knowledge would surface when setting up the task. Participants in the focus group sessions talked about a teacher's actions before students work on a problem. I use the model of a task as a set of *resources* and *operations* to achieve *a goal* (Doyle, 1988; Herbst, 2003, 2006) with the purpose of identifying in the data actions that teachers think they need to do before making students responsible for a task. In particular, I focus on the role of prior knowledge in a teacher's work in relation to each one of those components of a task.

The Stated Goal of a Task: A Possible Lever to Prompt Students' Prior Knowledge

To engage students in a task, a teacher needs to make them responsible for it. This is often done by stating the goal of the task, namely what students are expected to obtain through the work they are invited to do. Often the statement of the task contains a common core of the mathematical problem behind the task. I argue that the statement of the goal of the task is one of the levers that a teacher could use to prompt students' prior knowledge. For example, if the teacher were to say, "Prove that the angle bisectors of a rectangle make a square," the goal of that task is to produce a proof. In this example, there is an assumption that students would remember the meaning of the geometrical terms mentioned: angle bisector, rectangle, and square. It is conceivable that one could modify the statement of the problem so as to remind students about these terms in the following way: "Prove that the angle bisectors of a rectangle make a square. A rectangle is a quadrilateral with all angles congruent and a square is a rectangle with all sides congruent." By including particular definitions of a rectangle and of a square, a teacher could suggest that those definitions are useful in the proof. In addition a teacher could prevent students from remembering (and using) alternative definitions. If a teacher were not to remind students about the definitions of a square and of a rectangle, students would have to remember these on their own and consider whether they are useful for doing the proof. Moreover, students may draw upon alternative definitions, different from the definition the teacher had thought of. So, the statement of a problem could point to characteristics of the product of a task (in this case, how to tell it is a square). If students were to know characteristics of the product of a task, then they would work towards achieving a goal taking into account these characteristics.

My interest focuses on how teachers set up the task considering students' prior knowledge. When coding data from focus group sessions, I looked for intervals where participants talked about how to set up a task to teach with the angle bisectors problem. There were 51 such intervals out of a total of 313 intervals in all sessions. From that subset of intervals, I selected the intervals with specific references to students' prior knowledge and the goals of a task. There were 31 such intervals. TWP052405-11, illustrates how teachers tie their expectations about prior knowledge that students should remember to the goal of a task, when using a problem to teach.[48] In what follows I describe the discussion at that point in the session; then I present the transcript of the interval and my analysis.

At the beginning of the TWP052405 session, teachers worked in pairs on a version of the angle bisectors problem stated as follows: "In a triangle, the angle

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[48] The reference for an interval means that this is the focus group on "Teaching with Problems" followed by the date (month, day, year)-interval number. For example, interval 11 of the session on May 24th, 2005 is denoted as TWP052405-11.

bisectors meet at a point. What about a quadrilateral?" This statement does not identify the goal of the task. For example, the statement does not say explicitly if students have to do a proof or make a statement as a result of their work on the angle bisectors problem. By default it might call for a yes or no answer but that is unlikely a valuable mathematical answer. This omission is relevant because students' awareness of the goal of the task could be helpful for them to figure out the resources and operations needed to solve the problem, when these are not stated explicitly. Moreover, students would be held accountable for setting the goal of a task, which is rarely the case in the instructional situations of *doing proofs* and *installing theorems* in a geometry class (Herbst & Brach, 2006; Herbst & Nachlieli, 2007).

After 15 minutes of work in pairs on the problem, participants reported their findings. They all began the problem by choosing a special quadrilateral to work on. Participants said what special quadrilaterals they chose to investigate and whether the angle bisectors make another special quadrilateral. Some participants reported that for some special quadrilaterals, the angle bisectors meet at a point. Participants mentioned properties of quadrilaterals that they found helpful to pursue their investigation. For example, participants considered quadrilaterals with line symmetry as cases where students could use that symmetry to study the question about angle bisectors. Participants also discussed how they would pursue their inquiries. After participants' reports of their initial experiences, the moderator asked them about the possibility of teaching with the angle bisectors problem. In this discussion, participants talked about the statement of the task.

When the moderator asked about the plausibility of expecting that students produce a proof in response to the same statement of the task given at the beginning of the session, participants said that students would be confused or that students would quit. From these comments I glean that participants worried that students would have difficulties identifying the goal of the task. That is, participants' comments gave idea of what they anticipated might happen if a teacher were to ask students the same question posed to them in the session: "In a triangle, the angle bisectors meet at a point. What about in a quadrilateral?" In my analysis, I identify those possible storylines with which participants responded to the moderator's prompt. From these possible storylines, I explain teachers' underlying rationality for making decisions about the statement of a task related to students' prior knowledge.

I present the transcript of interval 11, TWP052405, divided into two parts. This division results from my analysis of themes within the interval, while the interval itself is defined as a unit based on considerations of patterns of interaction. Throughout the interval, Lynne[49] and Karen alternated turns to answer the moderator's question with some input from Nicola who interjected comments as others spoke. The first part of the interval includes answers to the moderator's question about expecting students to produce a proof. When giving these answers, participants assumed the moderator's scenario that a teacher would expect students to do a proof as the goal of the task. In the second part of the interval, participants changed the task and said what they would do as an alternative to the given task. The transcript of the first part of the interval follows.

[49] All participating teachers are identified by a pseudonym of their first names.

Turn #	Speaker	Turn[50]
90.	Moderator	If you were to put some stakes on it. You know, "I want you to work on this and I expect a proof of something at the end." Do you think that would sort of get them more focused or it would create a panic kind of response? Would that help them, or would it turn them more into the mode of, "what do you want from me?"
91.	Lynne	If you said, "I want a proof, a definitive response [Moderator: Yeah.] at the end of this," I think they would get confused by what you want because this is an open-ended question, asking them for something definitive would send them spiraling towards what you [Nicola] were saying, [Nicola: Yes, yes.] "What do you want?"
92.	Moderator	(Let's)-yeah. Go ahead.
93.	Karen	(What I'm), what I'm concerned about, if I gave this particular one to my class, that they would just answer, "it's not a point." And stop.
94.	Moderator	Oh, just a yes or no.
95.	Karen	Yeah. Like, "well I drew a rectangle and it's not a point," and so that would be it, rather than going - I mean it's like, we push to, well what does it make?
96.	Moderator	Right.
97.	Karen	So, um, the *what about a quadrilateral*, I think they would just take it literally, and being lazy, they would stop instead of them thinking about, what would be there.
98.	Nicola	What other quadrilaterals can you think of?
99.	Karen	Right.

(TWP052405, interval 11)

In this interval, participants reacted negatively to the possibility of expecting students to do a proof if asked, "In a triangle, the angle bisectors meet at a point. What about a quadrilateral?" The negative reactions surfaced in response to a story proposed by the moderator, where a teacher would ask, "I want you to work on this and I expect a proof of something at the end." This is the first story in the interval. According to participants, students would be confused with the open-endedness of the question (turn

[50] Each turn introduces a new speaker. The number of the turns for each transcript corresponds to the order in which participants spoke within each focus group session. Overlapping speech is denoted with parenthesis and explanatory comments are denoted in brackets.

91) or students would quit (turn 93). These anticipated reactions from students are the conclusions of two different emergent stories. In what follows, I present the other emergent stories and participants' underlying rationale (see Table 29).

The second story starts when the teacher asks "In a triangle, the angle bisectors meet at a point. What about a quadrilateral?" There is evidence for the starting point of the story when Lynne said, "If you said, 'I want a proof, a definitive response at the end of this'…." (turn 91). Lynne chose to continue the scenario proposed by the moderator (turn 90). According to Lynne, if at the end of their work students should provide a proof, students would ask the teacher, "What do you want?," confused by the teacher's request.

The third story also starts when the teacher asks "In a triangle, the angle bisectors meet at a point. What about in a quadrilateral?" In response, students answer, "It's not a point." Alternatively, a student says, "Well, I drew a rectangle and it's not a point." Then, students stop their work on the problem. The third story contrasts with the possibility that students would consider what figures result from the intersection of the angle bisectors of a quadrilateral.

Table 29

Emergent stories in interval 11, TWP052405

Moments	**Story 1**	**Story 2**	**Story 3a**	**3b**	**4**
1	The teacher asks "In a triangle, the angle bisectors meet at a point. What about a quadrilateral? I want you to work on this and I expect a	The teacher asks "In a triangle, the angle bisectors meet at a point. What about a quadrilateral? I want a proof."	The teacher asks "In a triangle, the angle bisectors meet at a point. What about in a quadrilateral?"	The teacher asks "In a triangle, the angle bisectors meet at a point. What about in a quadrilateral?"	The teacher asks "In a triangle, the angle bisectors meet at a point. What about in a quadrilateral?"

	proof of something at the end."				
2		Students work on the problem.	A student says, “Well, I drew a rectangle and it’s not a point.”	Students say, “It’s not a point.”	Students think about what would be there.
3		Students are confused.	Students stop working on the problem.	Students stop working on the problem.	
4		Students ask, “What do you want?”			

Participants had negative attitudes towards both stories 2 and 3. In story 2, the choice of the word “confused” to denote students’ state of being shows that students are not clear about the goals of the task. This confusion is more evident when students ask the teacher, “What do you want?” In story 3, by prefacing the story with the words, “I’m concerned,” there is an appraisal of *undesirability* related to the students’ actions. The anticipation that students would “just take it literally” (turn 97) shows the negative consequence of asking the question, “what about a quadrilateral.” The negative consequences of story 3 contrast the alternative in story 4. In story 4, students look for the figure made by the angle bisectors. The use of “just” in reference to story 3 is a resource to devalue students’ interpretation of the problem as asking whether the angle bisectors meet at a point like in a triangle. In Wagner and Herbel-Eisenmann’s study (2008, p. 150) teachers used “just” synonymously with the word “simply” in most cases. Here, one could also take the interpretation that students would “simply take it literally.” So, in story 3, students’ interpretation of the goal of the task is that they should find whether the angle bisectors meet at a point or not, and this interpretation is different from what a teacher expects.

According to participants these are the stories that could happen if they were to pose the angle bisectors problem as stated in the session. The use of the modal verb "would' by Lynne and by Karen when they said, "would get confused," "would send them spiraling towards what you were saying," "that would be it," "they would just take it literally," and "they would stop," shows that these stories are plausible. Thus, the appraisal for stories 2 and 3 is one of *probability*, where it is probable that these stories would follow a teacher's decision to ask "In a triangle, the angle bisectors meet at a point. What about in a quadrilateral?"

From my analysis of the interval it is apparent that the underlying rationale for these stories is related to contradictory nature of the goals of the tasks proposed by the teacher. In relation to story 2, Lynne said, "this is an open-ended question, asking them for something definitive would send them spiraling towards what you [Nicola] were saying, 'What do you want?'" According to Lynne, the teacher, on the one hand, gives a task that she labeled as "open-ended." The teacher, on the other hand, asks for a proof. Lynne created a contrast by opposing the terms "open-ended question" and "definite answer. " In particular, the contrast between a question and the answer is realized by using *converses* (Martin & Rose, 2003, p. 102). So, the statement of the goal of a task—the question—does not match what students are to produce—a definite answer.

Unlike usual statements of a task in the situation of *doing proofs,* the statement of the task given in the session did not specify what was to be proved. In the situation of doing proofs, it is usual for the teacher to state what is to be proved. In particular teachers parse the "givens" and the statement "to prove." One could expect that the lack of a statement "to prove" in the question "What can be said about the angle bisectors of a quadrilateral?" would provoke students' distress, because the kind of product expected had not been specified in the statement of the task. If a teacher were to expect students to produce a proof in this scenario there would be a conflict: While the product would be characteristic of the situation of doing proofs, the statement of the task does not comply with the norms in that situation.

The consequence of story 3 is the worst-case scenario of all stories. Students would stop working on the question, without investigating further with other quadrilaterals, or other options, such as different meanings of what can be said. Karen described students' possible action of stopping when she said, "that would be it," and "they would stop." From my analysis of story 3, I found two possible explanations for why students would quit. One explanation is that students would think that they had answered the question once they find out that angle bisectors of a quadrilateral do not meet at a point (turns 93 and 95). So, students would believe that they have answered the question, because they thought it was a yes or no question. Another explanation is that students' laziness would prevent them from exploring the question further on their own (turn 97). So, students would decide not to do more, even though they are aware that they could do more. Both explanations support the point that students would stop working on the problem.

Karen reacted to the lack of explicit direction in the goals of the task when she said, "We push to, 'well what does it make?'" (turn 95). She enacted the voice of a teacher who would ask further questions, "pushing students" to describe the figure that the intersection of the angle bisectors would make. From this choice of words, I interpret that cueing students about the kinds of products expected contrasts with letting students stop their investigation. It is undesirable that students would stop the investigation with the apparent answer of, "it's not a point." So, the teacher should ask further questions for the students to consider other cases.

The stories delineated by participants suggest that the goal of a task should be explicit in the statement of the task. One way to control for the kinds of products that students will achieve when working on the angle bisectors problem is by including references to resources in the statement of the task. Diagrams are an example of resources for a mathematical task. In the next section I discuss how by including diagrams a teacher could clarify what the goal of the task is, thus controlling the kind of prior knowledge that students use when working on a problem.

Including Diagrams in the Statement of the Task

I contend that a teacher can use the statement of a task to make students aware of the prior knowledge they should call upon for working on that task. In particular, a teacher can use the statement of the task to cue students about the resources and operations they should deploy to solve a problem. In the second part of interval 11, TWP052405, participants modified the statement of the task as given in the session to include two elements. First, participants used the statement of the task to trigger identification of the resources that students should make use of in working on the problem. Secondly, participants were explicit about the kinds of products that students should produce such as a diagram, a list, a conjecture, or a proof. The transcript of the rest of the interval 11, TWP052405 reads:

Turn #	**Speaker**	**Turn**
100.	Lynne	That's what I thought about it. You know, if I was gonna use this I thought about drawing several examples for them. Instead of,

		we request to hand-draw but we know some of the symmetry properties, lots of little rules already, um haven't taught the course a long time, so, I thought, well maybe I would draw a square, a couple of different versions of a parallelogram, isosceles trapezoid, different shapes for them to play with on the sheet at the same time. That, I was thinking that that might limit them a little bit (in the freedom).
101.	Karen	(Well, that would get I mean part of) the interest of the problem, is to start thinking about what other kinds are there? And looking in it, I would want to rephrase the question though, um to say, "what does it make in a quadrilateral?" Rather than, "what about in a quadrilateral."
102.	Moderator	So, it's interesting 'cause both of you are saying you need to make the problem more focused in a way. To get students to work on it productively and you have different ideas about how to do it. So your suggestion is I mean, just you know off the top of your head, you'd [Karen] reword it so that it asks specifically what is made?
103.	Karen	And then what kinds of quadrilaterals. So, encourage them with the question to try out other kinds of quadrilaterals.
104.	Moderator	And your [Lynne's] idea was to provide drawings, specific examples to work with.
105.	Lynne	I guess it would depend on what we just talked about. If they are really familiar with all the types of quadrilaterals at that point in our discussion or if we're just starting to work on properties.

(TWP052405, interval 11)

Lynne suggested including diagrams of quadrilaterals in the statement of the task to cue students about the special quadrilaterals they should focus on (turn 100). However, she hesitated to pursue this idea further, because a teacher would limit students' inquiry by requiring them to study those quadrilaterals already included by the teacher in the worksheet. This hesitation is evident when Lynne said, "I was thinking that that might limit them [the students] a little bit in the freedom." So, the value of being explicit about the quadrilaterals students should work with is counterbalanced by the value of letting students choose how to pursue the investigation. However, one could expect that since there is a possibility that students might get confused or quit, a teacher may need to sacrifice students' freedom in order to have students' investment to work on the problem.

Then, a teacher would use the statement of the task to communicate resources, such as diagrams of special quadrilaterals, and operations, such as a request for students to make diagrams of special quadrilaterals. By doing so, a teacher avoids ambiguities about what resources and operations are needed to work on the problem, prompting students to use their prior knowledge about special quadrilaterals.

Karen embraced Lynne's idea of providing the diagrams of different quadrilaterals as a way to "get them [students] to start thinking" about the problem (turn 101). Karen extended Lynne's idea by modifying the statement of the task. Karen would ask students to find out what results from the intersection of the angle bisectors of a quadrilateral. This question would prevent students from stopping their investigation as soon as they observed whether the angle bisectors met or did not meet at a point. In addition, this question would ask students to achieve a specific product: the name of the special quadrilateral that results from the intersection of the angle bisectors of another quadrilateral.

At the end of the interval, Lynne concluded that students' prior knowledge about quadrilaterals (or their lack of) would influence her decision to present sample diagrams in the statement of the problem. Lynne set a condition for a teacher to include diagrams in the statement of the problem: if students are "familiar with all sorts of quadrilaterals," or if students are starting to study quadrilaterals (turn 105). Students may possess knowledge about quadrilaterals from their experiences in other mathematics classes besides the geometry class. However, Lynne did not hold students accountable for remembering prior knowledge on their own. On the contrary, Lynne set the boundaries of students' prior knowledge to that knowledge of quadrilaterals that is part of the geometry course. Lynne's linguistic choices situated her comments about students' prior knowledge to prior knowledge of topics studied in the geometry course. She used the first person in apparent reference to her geometry class. She said, "I guess it would depend on what **we** just talked about. If they are really familiar with all types of quadrilaterals at that point in **our** discussion or if **we**'re just starting to work on properties" (turn 105).[51] In addition, Lynne used temporal markers to consider the case when the class had started to work on properties of quadrilaterals, when she said, "if we're ***just*** start***ing*** to work on properties." The use of "just" here is synonymous with "recently" (Wagner & Herbel-Eisenmann, 2008, p. 151). Thus, Lynne and Karen seemed to agree that the teacher should tell students what prior knowledge—from the geometry class— they could make use of in order to control the kinds of resources students would use when working on a mathematical task.

In conclusion, participants said that from a teacher's perspective the statement of a task should specify what *products* should result from students' work. By modifying the statement of the task toward more specificity (or by making the statement more specific), teachers expect to hold students accountable for remembering what they need to know in order to solve a problem and for using that prior knowledge in their solution. Also, they would hold students accountable for using those resources to achieve specific products, as required in the modified statement of the task. Students would not get to produce a proof as part of their work on a problem, unless they are explicitly asked to do so.

[51] Throughout this chapter, I add emphasis to the quote to show important elements I considered in the analysis. This emphasis does not intend to portray emphasis by the speakers, for example by changing their intonation. The transcription of the focus group sessions does not include information about emphasis by other means besides the choice of words.

According to participants, the kinds of products that students would achieve with a question like "What can be said about the angle bisectors of a quadrilateral" would not be valuable because students would not necessarily get to use their prior knowledge about quadrilaterals. Moreover, if, as part of the geometry curriculum, the class had not covered quadrilaterals yet, a teacher would need to include resources—such as diagrams of quadrilaterals—in the statement of the task. Otherwise, teachers would not be entitled to hold students responsible for using any prior knowledge about quadrilaterals from other courses. In the following section, I analyze teachers' decisions about making students aware of the resources for a task.

The Resources for a Task: A Teacher's Decision to Remind Students about the Prior Knowledge Needed to Work on a Problem

It is conceivable that a teacher would decide to identify the resources needed to work on a mathematical task before students start working on a problem. It could also be the case that a teacher would let students identify the resources needed for a task on their own. The statement of the task may or may not include references to the resources students need to work on a problem. Some of the resources needed such as a diagram or construction tools, could be material resources. Other resources could be mathematical propositions that a teacher may assume that a student already knows. Here I focus on a teacher's decision to identify the resources in the statement of the task to make students remember prior knowledge. When coding the transcripts of the focus group sessions, I looked for intervals where teachers talked about whether to include explicit references to resources in the statement of the task.

From 51 intervals with references to the statement of the task, there were 24 intervals where participants talked about reminding students about resources in the statement of the task. I selected TWP052405-17. In this interval, which happened 40 minutes into the three-hour session, the moderator presented the scenario of using the angle bisectors problem to start the quadrilaterals unit. In this interval, Lynne and Nicola responded to questions by the moderator. Lynne and Nicola were concerned that students at the beginning of the unit of quadrilaterals would not distinguish between different kinds of quadrilaterals. However, Lynne anticipated that students would ask for the names of the quadrilaterals if students were to have diagrams of different quadrilaterals. The use of diagrams in the statement of the problem appears to be a *resource* to prompt students' memory of prior knowledge that is useful in order to work on the angle bisectors problem. The transcript of the part of the interval where Lynne and Nicola talked about the use of diagrams follows.

Turn #	**Speaker**	**Turn**
203.	Moderator	Let me go forward. I mean, there is no reason why we can't do that. Let me skip forward a little bit to this slide. Okay, so um, so a couple of different things have been suggested already as ways you could use this problem. You could use it at the beginning of the unit, someone said, to start off the quadrilaterals unit. Oh you asked
204.		[Laughs.]
205.	Moderator	[Inaudible] and nobody liked the idea.
206.	Lynne	Well, it's just that they're not um (extremely—)
207.	Nicola	(familiar with the shapes.)
208.	Lynne	Familiar with which quadrilaterals are which. But if you gave it to them with several different drawings they would be asking what's the name of this one because they would have to call it something (so…)
209.	Moderator	So is vocabulary really the only hitch in, in doing it at the beginning of the unit. They just don't know it?
210.	Karen	I think that, that you use properties of them. So, without them you get bogged down in discovering the properties of a quadrilateral. And, I think it would get really messy.
211.	Moderator	Okay, 'cause you need the properties to do the work.
212.	Karen	Yeah. I mean to make it into something that can be done in ten, fifteen minutes.
213.	Moderator	What if you weren't limiting yourself to ten, fifteen minutes?

214.	Lynne	Well, you'd make it to, though [laughs].
215.	Karen	Yes.

(TWP052405-17)

In this interval, participants stated difficulties of starting the unit on quadrilaterals with the angle bisectors problem. In particular, participants mentioned students' lack of familiarity with quadrilaterals as a potential barrier (turns 206-208). Lynne suggested that a teacher could modify the statement of the task given in the focus group session by including diagrams of different quadrilaterals. This alternative was supposed to compensate for students' lack of familiarity with those quadrilaterals. Lynne said, "But if you gave it to them with several different drawings they would be asking what's the name of this one because they would have to call it something." From this comment one could conclude that students would not be responsible for producing diagrams, rather the teacher would be providing the diagrams. This is an important point because each diagram constitutes a different resource for students to work on the task. In addition, each diagram triggers different sorts of resources about propositions associated with a particular geometric figure that students may already know. According to Lynne, diagrams drawn by the teacher, without any sort of label for identification, would prompt students to ask for the names of special quadrilaterals. With this change in the statement of the task, Lynne anticipated that students would feel responsible for knowing the names of special quadrilaterals. In this proposed task, students would have more guidance for relying on prior knowledge about special quadrilaterals than with the question, "What can be said about the angle bisectors of a quadrilateral?"

Prior knowledge of vocabulary about special quadrilaterals is a needed resource that Lynne had identified. In addition, Karen argued that, besides the names for special quadrilaterals, students' knowledge about the properties of quadrilaterals would be crucial for them to work on the angle bisectors problem. According to Karen, the main difficulty in using the angle bisectors problem at the beginning of the quadrilaterals unit is that students would not possess knowledge about the properties of quadrilaterals. In response to the moderator's question about students' lack of vocabulary, Karen said, "I think that, that you use properties of them [quadrilaterals]" (turn 210). Karen could not conceive of the problem as an opportunity to learn properties of quadrilaterals. On the contrary, Karen stated that students would "get bogged down in discovering the properties of a quadrilateral." Then she added a concluding statement on the difficulties of teaching with the angle bisectors problems at the beginning of the unit: "And, I think it would get really messy." Her evaluative stance regarding the use of the angle bisectors problem at the beginning of the unit is that of *undesirability*. So, Karen would expect students to get frustrated with the problem, because of their lack of prior knowledge about properties of quadrilaterals.

Participants' comments suggest that diagrams of special quadrilaterals without their names are an insufficient resource for students to work on a task. One could conceive of a scenario where a teacher would give students only the diagrams, without their names. However, according to participants, if the diagrams do not have a name, students would be confused. A possible explanation for students' confusion could be related to a teacher's anticipation of how students would report their findings. Besides using the names of quadrilaterals to identify the properties of those quadrilaterals needed to work on a task, students could also use the names of quadrilaterals to report their findings. But, without the names of special quadrilaterals, students could have difficulties to report their findings, because they would need to rely upon properties of the quadrilaterals. For example, instead of talking about a square, students could say something about a quadrilateral with four congruent angles, four congruent sides, congruent diagonals, and perpendicular diagonals. By stating all these properties with the purpose of naming a square, students' findings would be "messy." So, in a way, the name of a quadrilateral would allow students to concentrate on a configuration when reporting their findings, and to forget about specific properties.

At the end of this interval, Karen and Lynne joked about limiting for 10 to 15 minutes students' work with the angle bisectors problem, even when the moderator had pushed them to conceive of using the problem for an extended time (turns 213-215). The humorous reaction could be a way to denote the potential difficulties of using this problem in teaching. Karen and Lynne conceived of other possibilities to use the angle bisectors problem throughout a unit on quadrilaterals later on in the session. However, their reaction to questions about the plausibility of using the angle bisectors problems at the beginning of a unit on quadrilaterals suggests that a teacher cannot expect students to remember properties of special quadrilaterals without giving out sufficient prompts.

In sum, according to participants, the statement of the task should cue students about the *resources* needed to work on the problem. Therefore, students would not deploy the resources necessary to work on the problem such as relevant vocabulary or properties of geometric figures, without a teacher's prompt. A teacher could include diagrams to organize students' work on a problem, especially if quadrilaterals have not been covered yet in the geometry course.   Thus, according to participants, whenever a teacher intends to draw upon students' prior knowledge—from the geometry class or from other mathematics classes—a teacher should suggest in the statement of a task those resources that students need to work on the problem. In the next section I present other alternatives for providing resources to students in the statement of a task.

More About The Teacher's Work Providing Resources

In the previous section, there was evidence for teachers' preference to let students know about the resources they should draw upon when working on a problem. In this section I put forward the following tension for a teacher: providing enough cues for students to remember resources needed to work on a problem versus letting students remember the resources needed on their own. A teacher's decisions about how much guidance they should provide to their students is important because in that decision there are some assumptions about students' prior knowledge. From 24 intervals where participants talked about including resources in the statement of the task, I selected an interval where tensions about making that decision surfaced. In response to that tension, participants said how they would make explicit the resources in the statement of a task.

By the end of the session TWP020805, participants had seen videos from two different classes taught by Ms. Keating where they were working on a proof for the claim that the angle bisectors of a square meet at a point. The interval I selected, interval 67, happened in the last 7 minutes of the three-hour session and lasted approximately two minutes. The moderator asked about best and worse expectations from using "the angle bisectors problem." In response, Mitch made a confession that appears to point to a decision. On the one hand, a teacher could prompt the prior knowledge students need to work on a problem. A teacher, on the other hand, could let students figure out what prior knowledge is needed. There is a potential risk in the first scenario that teachers would be giving out too much information before students start working on the problem. The second scenario is also risky because students would have to possess the prior knowledge needed, and also remember it. Mitch stated a preference for making explicit to students the prior knowledge needed to pursue the problem. The transcript of the interval follows.

Turn #	**Speaker**	**Turn**
575.	Moderator	I think we've come to the end of the 10 minute question, I said, you know? So um, let me very, very quickly ask, you've seen, saw the slide before um, we're gonna skip - we have a whole 10 minute thing on rectangle and [inaudible] and squares, triangle and [inaudible], um, you saw this slide before. Uh, now that you've seen some, some uh, footage, does anybody want to, uh add anything to what they said earlier about how this problem could be used? Different ways it could be used, and what kinds of issues you may be conscious of in the use of this problem? Either difficulties you'd be worrying about or things you'd be hoping for?
576.		[Long pause.]
577.	Mitch	[Raises hand.]
578.	Moderator	Yeah.
579.	Mitch	I'm worried about kids putting it all together, and really working with it, but I think it'd be great.
580.	Moderator	When you say putting it all together, do you…
581.	Mitch	Well, being able to actually produce proofs for um, with, with concurrency, and uh, and um, and seeing, one, to, to maybe use that um, well yeah, to to, take to go in a creative direction with it. I would love to see my kids producing proofs of these facts. I guess I'd worry about how I'm gonna structure it, how I'm gonna uh, make sure everybody has the prerequisites, when I assign, you know, "now you're gonna do this one." Say in, in groups everybody's doing the same, the um, say the, the, bisectors of a parallelogram are parallel, the angle bisectors of a parallelogram are parallel. So start with that, and then to get to, to some of the um, proving, proving the square which was being done in a previous video - proving that the bisectors of a rectangle form a square. I guess that I'm just worried that, I'd be there telling them, [laughs] what to do.

(TWP020805-67)

This interval starts when the moderator asked participants how they could use the angle bisectors problem. Participants had watched videos where Ms. Keating and her students were proving that the angle bisectors of a square meet at a point. I interpret Mitch’s comment about, “being able to actually produce proofs for um, with, with concurrency” as a statement about the product of a mathematical task with the angle bisectors problem. So, according to Mitch, a teacher could give a mathematical task for students to produce proofs about concurrency. It is unclear in this interval whether Mitch was referring to the same task of proving that the angle bisectors of a square make a point as in the videos or to another task. In any case, Mitch argued for the *desirability* of students producing proofs as a way to "put together" their work on the problem (turns 579 and 580). Mitch stated, "I would **love** to see my kids producing proofs of these facts.” Mitch's choice of the word "love" displays a positive attitude towards proofs that could result from students' work. Therefore, Mitch's highest expectation for *products* resulting from investigating the "angle bisectors problem" would be for students to do proofs.

Mitch had concerns about what kinds of *resources* students would have available. He said, "I guess I’d **worry** about how I’m gonna structure it, how I’m gonna uh, make sure everybody has the prerequisites, when I assign, you know, 'now you’re gonna do this one.'" Mitch's choice of the word "worry" signals the potential difficulties of teaching with this problem. When Mitch talked about how to "structure" the problem, he seemed to be referring to how to organize students' work on the problem, starting with the goal of the problem task. Then, he posed a rhetorical question about how to make sure that all students possess the "prerequisites" to work on a problem. Mitch's choice of the word "prerequisites" implies a reference to the resources that all students should possess before they start working on the problem. In particular, if different groups of students were to work on a task based on the "angle bisectors problem," then Mitch would need to monitor what resources students possess to work on their assigned task.

Mitch illustrated his concerns about how to structure the task to make sure that students have the prior knowledge they need to solve the problem when he proposed the scenario of assigning students to prove the claim "the angle bisectors of a parallelogram are parallel" (turn 581). The products of students' work on this proof could be the basis for students' work on a new proof: "the angle bisectors of a rectangle form a square" (turn 581). Mitch would be organizing students' work on "the angle bisectors problem," by relying on a hierarchical arrangement of quadrilaterals. Since rectangles are parallelograms, those proven claims about parallelograms would also apply to rectangles. Therefore, Mitch would be providing resources for students to work on a problem by having them prove a claim that could become useful in proving other claims. That is, Mitch would organize students' work as to make sure that he introduces the *resources* needed to continue to work on a variation of the "angle bisectors" problem **prior** to their subsequent work in other variations of the problem.

Mitch pondered whether a teacher would need to organize students' work on the problem so that students would use prior *products* as *resources* for further work on the problem. It is conceivable that if students get to prove that the angle bisectors of a parallelogram are parallel, then they could use this proven result to say that the angle bisectors of a rectangle are parallel, without having to reproduce the earlier proof. As a consequence, the problem of proving that the angle bisectors of a rectangle make a square becomes a simpler problem. Students could draw upon the definition of parallelograms (parallelograms are quadrilaterals with two pairs of parallel sides) or on properties of parallelograms (for example, parallelograms have opposite sides congruent, parallelograms have opposite angles congruent) to make the new proof about the angle bisectors of a rectangle.

Mitch's final statement evidences a teacher's tension about the decision of providing enough resources for students to work on a problem against letting students decide the resources. At the end of his turn, Mitch showed reservations about "telling" students "what to do" (turn 581). Mitch's concern was similar to Lynne's when she had stated that a teacher who decides to give out diagrams of special quadrilaterals might limit students to investigate only those given diagrams (TWP052405-67, turn 100). A teacher could cue students about the *resources* they need to work on a problem with the statement of the problem. Also, a teacher could organize students' work by controlling the order of the products they are to produce. In that way, a proven result, which was the *product* of students' prior work with a problem, could be taken as a *resource* that is already part of the collective memory of the class.

So far I have discussed teachers' actions to trigger students' prior knowledge are related to the *goals* and the *resources* of a task. A teacher could use the statement of a task to remind students about the prior knowledge they need to work on a problem. In addition, a teacher could organize the sequence of problems, optimizing opportunities for students to remember products from a mathematical task immediately after they have produced them, when doing the next task. In the next section I discuss teachers' expectations for students to remember *operations* for a task.

The Operations for a Task: A Teacher's Expectation about What Students Should Remember to Do

In contrast with students' memory of resources, which could entail remembering concepts or propositions useful to solve a problem, students' memory of operations involves remembering what to do. Examples of mathematical operations that students could remember include the steps taken to solve a quadratic equation, to graph a linear function, to construct the bisector of an angle, or to calculate the area of a figure formed of juxtaposed triangles. As the examples suggest, one could expect that operations in geometric tasks involve in one way or another actions with diagrams. Some of these operations include to measure, to draw auxiliary lines, to add markings to parts of the diagram, to perform a geometric transformation to a diagram, and to draw a diagram. When coding intervals from focus group sessions, I selected intervals where participants made references to the operation of drawing diagrams. In the following section I present the analysis of intervals with data that speaks to how teachers expect students to use diagrams. I present that as a case that shows students' memory of operations for a task.

Expecting Students to Remember to Draw Diagrams

In the focus group sessions, participants stated their expectations for students to remember to draw diagrams as they work on a problem in 25 intervals from a total of 313 intervals. Here, I present my analysis of two intervals. The first interval illustrates how a teacher could assign to students the responsibility of making a diagram to start working on a problem. In particular, this interval showcases that a teacher could make students responsible for remembering *resources* needed to create a diagram and also the *operations* involved in drawing diagrams. The second interval illustrates a teacher's expectation for students to remember the operations involved when drawing a diagram with the aid of a technological tool. The data from these two intervals demonstrate that a teacher can expect students to rely upon their prior knowledge of *operations* when working on a problem.

Approximately one hour and 10 minutes into the TWP031506 session, the moderator invited participants to modify a set of sample worksheets that they could use in teaching with the angle bisectors problem. Figure 23 shows one of those worksheets.

Angle Bisectors Worksheet (Draft 0.3)

Answer each of the questions below, and explain why it is true.

1. The intersection points of the angle bisectors of a rectangle form a ____________.
2. The intersection points of the angle bisectors of a parallelogram form a ____________.
3. The intersection points of the angle bisectors of an isosceles trapezoid form a ____________.
4. The intersection points of the angle bisectors of a kite meet at a ____________.
5. The intersection points of the angle bisectors of a square meet at a ____________.
6. The intersection points of the angle bisectors of a rhombus form a ____________.
7. The intersection points of the angle bisectors of a trapezoid (not isosceles) form a ____________.
8. The intersection points of the angle bisectors of a general quadrilateral form a ____________.

Figure 23. A worksheet.[52]

[52] This sample worksheet is an artifact made for the focus group session. The moderator presented the worksheets as drafts and asked teachers to modify them for their class. Items 4 and 5, while grammatically incorrect, present the case where a teacher suggests to students that the angle bisectors of a quadrilateral are concurrent with the article "a"

Tabitha, Stan, and Minnie commented on what elements should be considered in posing the angle bisectors problem to their students. In particular, they discussed whether they would provide diagrams of special quadrilaterals or, instead, ask students to draw those diagrams. The interval I selected, interval 43, immediately follows that discussion. In that interval, Denis talked about teaching students to draw diagrams.

Denis selected a worksheet that did not have any diagrams (see Figure 22). He stated that he would expect students to draw a diagram, even when it was not specified in the instructions of the worksheet. Denis said that he asks students to draw diagrams in the "second day of school" in his geometry class. Reportedly, Denis continues to emphasize the importance of drawing diagrams throughout the year, by requesting students to draw a diagram whenever they are working on a problem. The transcript of TWP031506-43 follows. This interval lasted approximately one minute and the main activity in this interval is Denis' turn of speech regarding the worksheet, which is longer than turns taken by other participants in the previous interval.

Turn #	**Speaker**	**Turn**
397.	Researcher	Which one did you say would be a good warm up?
398.	Stan and Tabitha	Zero.
399.	Denis	See, draft three, to me, I would be okay giving my students because the second day of school the first thing I tell them is the one thing you are always going to do in this class is you are always going to draw a picture and you are always going to use that picture. So, if you are not doing that you're gonna hear me say it fifty million times for the rest of the year and they, they would look at this and say, "oh, I gotta draw a picture because Maple[53] already told us, we have to do this every time." You

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following the phrase, "meet at a." Students are supposed to say that the angle bisectors meet at a point.

[53] Denis Maple is the pseudonym of this teacher.

		know, and some of them would freehand it, some of them would use a ruler or a compass, or a, you know a nice protractor, I mean it just depends on the kid, you know, but they, I think they would, I would stake my money on them that they would at least probably draw the picture.
400.	Jillian	And again that's that you reap what you sow.
401.	Denis	Yeah.

(TWP031506-43)

I find this example relevant in broadening our understanding of how a teacher shapes the collective memory of the class. The collective memory of the class includes operations a teacher might rely on students making use of when working on a problem. Denis gave an example of an operation that students would do in his class to work on a task: to draw a diagram. Denis demonstrated how he makes his expectation explicit. Reportedly, Denis inculcates in his students the habit of drawing a diagram when working on a geometric problem. Denis said that he tells his students, "you are **always** going to draw a picture and you are **always** going to use that picture." Denis repeated use of the word "always" is an appraisal resource of modality for emphasizing that students *should* perform this operation always (Martin & Rose, 2003, p. 50). The appraisal associated with the action of drawing and using a diagram is a *normativity* appraisal, denoted by the choice of words "are to," describing what students are supposed to do. As Denis examined different worksheets, he stated that it is highly *probable* that his students would draw diagrams, even if the worksheet would not ask them to do so. Denis said, "I would stake my money on them [the students] that they would at least probably draw the picture." Therefore, in Denis' class, students should apply their prior knowledge of operations for working on a problem by accepting responsibility for producing diagrams even if not directly asked to.

Jillian's comment, "you reap what you sow," (turn 400) uses the analogy between harvesting and a teacher's work. With this comment there is an emphasis on a teacher's responsibility to set expectations about what kind of work students should produce and seeing those expectations fulfilled in students' work. So, according to Jillian, if a teacher had instilled in students the habit of doing some operations when working on a problem, then the teacher could rely on students performing those operations.

Denis' reaction to the absence of diagrams in the worksheet when he said, "I would be okay," suggests that a norm has been breached in the didactical contract. The implied norm is that teachers should provide a diagram for students to work on a problem. It has been reported that in the situation of doing proofs it is usual for a teacher to include a diagram with the statement of the task (Herbst & Brach, 2006, p. 91-92). In Denis' class, students would know what to do to overcome the absence of a diagram, if they were to remember the operation of drawing a diagram. One could expect that in another class, students would need to remember what to do or ask the teacher what they should do, in the absence of specific instructions about operations in the statement of the task.

From Denis' comments, it appears that students in his class can decide whether to sketch diagrams freehand or with the aid of construction tools. Denis said, "You know, and some of them would freehand it, some of them would use a ruler or a compass, or a, you know a nice protractor, I mean it just depends on the kid." His statement, "it just depends on the kid," points to the circumstance conditioning the choice of a construction tool, without pointing to the teacher as responsible for choosing the tool. So, identifying what kinds of tools students have available to perform a particular operation is a resource that students ought to remember, if a teacher does not specify what resources they could make use of to work on a problem.

Kathy, a teacher in the TWP020805 session, provided another example where students would need to rely upon their memory of *operations* to pursue a problem. One hour into the three-hour session, Kathy answered questions to the moderator about how to use dynamic geometry software in interval 17. This interval lasted approximately 3 minutes and is characterized by the exchange between Kathy and the moderator.

Kathy said that she would expect students to remember how to manipulate tools in a dynamic geometry software. Kathy estimated that the time students would spend working on "the angle bisectors" problem would vary according to the skills they possess when using dynamic geometry software. These skills have to do with the operations that students could execute when creating a diagram with the aid of dynamic geometry software.

Turn #	**Speaker**	**Turn**
226.	Researcher	So Kathy, if you were going to do that then, with the software thing right, what would you, what would you want to get from them at the end of the period? Would you want them to turn something in or?
227.	Kathy	A period? [Laughs.]
228.	Researcher	Well, a week or whatever. How actually, how would you pace them. I mean, would you use um, one class for this, or would you want to…
229.	Kathy	I think it would depend on if the kids knew the software or not. And if they didn't know the software, then two weeks later [laughs, inaudible]. I mean, that's probably a little bit of a stretch, but um, and also depending on whether they know um the symbols and the terminology, and how long it would take them to draw it and um, you know, if you can give them pictures of, "This is a right trapezoid. Recreate this on the screen and now try bisecting the angles, and what shapes do you get, and can you use your measurements on your computer software to begin to get stuff out of it?" And that can be kind of cool and then you can have, single instant proofs that's enough to say, "Oh, I'm thinking from a right

		trapezoid it's something like a kite.” But it would probably take a couple of blocks because you’re getting the measurements and now you have to go back and look and say, "do those measurements mean anything to me? If I have two of them that are equal here, is there a shape that has anything like that? Or do I know that?" So...that’s off the top of my head.

(TWP020805-17)

Here, Kathy mentioned some of the different operations that students could potentially do with a dynamic geometry software to study the "angle bisectors" problem. First, students would need to translate a static diagram into a dynamic diagram. Kathy talked about students' use of the "symbols" and the "terminology" of a dynamic geometry software package to "recreate" a diagram of a right trapezoid "on the screen." Secondly, students would need to draw the angle bisectors of the quadrilateral by using the tools provided by the dynamic geometry software. Thirdly, students would begin formulating some conjectures about the figure that results from drawing the angle bisectors of a quadrilateral. Kathy suggested that students would start thinking about those figures that result from the intersection of the angle bisectors. While it is unclear how students could get to formulate their conjectures from Kathy's comment, one could expect that students would use visual perception to identify the resulting figure as a special quadrilateral. Finally, students could apply other operations, such as measuring parts of the dynamic diagram, to pursue their investigation of the problem or as Kathy said, "to begin to get stuff out of it."

Figure 24 shows a possible diagram by a student who would follow Kathy's description of operations, assuming that students would start measuring angles. It is conceivable that students would use the tools to measure other parts of the diagram, such as the sides of the quadrilateral or the slopes of the lines. Kathy did not specify what measurements students would do. Yet, from Kathy's comments I infer that students would need to decide what to measure according to the tools available in a dynamic geometry software. Moreover, students would need to remember how to perform operations to get those measurements.

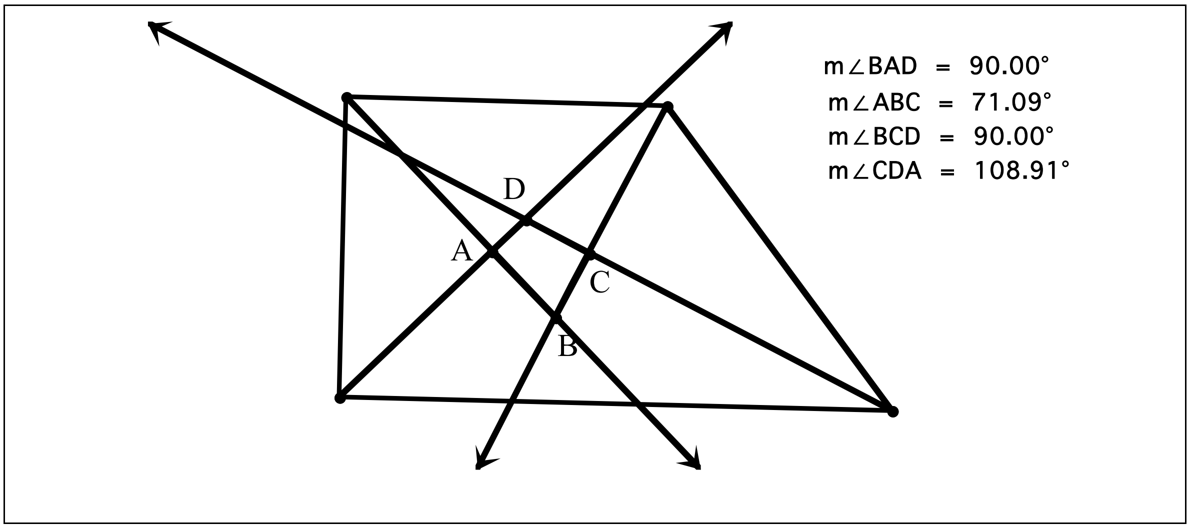


Figure 24. The angle bisectors of a right trapezoid.

Kathy's example illustrates that a teacher could rely on students' memory of *operations* to pursue a problem. There are particular operations (such as constructing and measuring) connected to particular tools (such as dynamic geometry software) that a student would need to remember in order to work on a problem with the aid of technological artifacts. For example, when making diagrams, students would need to remember a sequence of steps involved to draw a geometric figure—what to draw first and what to draw next; when measuring, students would need to remember what geometric object to select and in what order in order, to get the desired measurement. In the case of making measurements, students would also need to remember how to select the unit of measurement. In addition, students would need to remember how to manipulate the tools they have available to get what they want. For example, in some dynamic geometry software packages (such as *The Geometer's Sketchpad*®)[54] one must to select the givens needed to make a construction before selecting the operation for making that construction. So, one option to draw a quadrilateral would be to draw four points, and then draw segments to connect successive points. Or, to draw an angle bisector, one would need to select an angle, and then choose the appropriate option in the construction menu of the program to draw the angle bisector. But in other dynamic geometry software packages (such as *Cabri* ®or *GeoGebra*©)[55] one needs to choose the operation first and then the givens. Thus, students' memory of operations would shape their work on a problem because students will do operations that they remember how to perform.

The two examples by Denis and Kathy illustrate how a teacher could hold students responsible for remembering different sorts of *operations* to work on a problem. Denis, on the one hand, said that he would expect students to remember to draw diagrams whenever they face a new problem. Kathy, on the other hand, presented an example where students would need to remember how to use different tools from dynamic geometry software to make a dynamic diagram. The dynamic diagram could be helpful for students to gather information leading to formulating conjectures based upon that information. In both cases, students would need to remember operations with some set of resources. That is, students would have to remember how to utilize the tools available—

[54] *The Geometer's Sketchpad*® is a registered trademark of Key Curriculum Press.
[55] *Cabri* ® is a registered trademark of Texas Instrument. *GeoGebra*© is a free software developed by Markus Hohenwarter (geogebra.org).

for example, a compass, a protractor, a ruler, or a straightedge—to investigate a problem, using the given conditions.

In conclusion, according to participants, teachers rely on students' memory of operations. An example of an operation is how to draw a diagram when working on a problem. Students use the available resources—such as definitions, postulates, theorems, and construction tools—to perform particular operations to draw diagrams. There are other operations that students could also perform with diagrams such as marking a diagram, drawing an auxiliary line, or making measurements. Their available memory of operations shapes students' work, enabling and constraining what students are able to do. A teacher has to decide whether to make students remember the operations needed to solve a problem or to expect students to remember these operations on their own. In making this decision, a teacher may assume that students would rely on usual operations or, in contrast, a teacher may make the operations needed to solve a problem explicit.

Summary

In this section I have presented possible teaching actions towards making students remember prior knowledge before they start working on a problem. According to participants, teachers make important decisions about what resources and operations students should make use of when working on a problem. When using a problem to teach something new, the resources may be visible in the statement of a task, but the operations are often implied. Moreover, teachers must decide whether to prompt students' prior knowledge about those resources needed to solve a problem, or to let students figure out resources and operations on their own. This could bring about tensions for a teacher in deciding upon who is responsible for triggering the prior knowledge needed to solve a problem.

I contend that in the division of labor between a teacher and the students, the teacher assumes responsibility to trigger students' prior knowledge before students start working on a problem. This is important because there is prior knowledge that a teacher could assume that students already possess. However, according to participants, it seems unlikely that a teacher would leave students to decide on their own what resources are needed to work on a problem. Therefore, the tension for a teacher lies on identifying how much they should tell their students, while still allowing students to come up with ideas by themselves. The decision to tell or not to tell what resources are needed to solve a problem is important because when using a problem to teach a teacher could jeopardize students' opportunities to learn something new if students do not bring about on their own those resources needed to work on a problem.

Overall it seems that letting students decide the resources of a problem could be risky. According to participants, students could forget the resources they need or they could trivialize the goals of a task. Students' apparent forgetfulness about resources or their misunderstanding for the goals of a task would be a barrier for the achievement of the task. As a result, participants' comments suggest that teachers may prefer to be clear about the resources and about the goals of a task, thus fulfilling their responsibility in the didactical contract. Teachers could use the statement of the task to cue students' prior knowledge of resources. Teachers could also organize students' work on a problem so that the products of a mathematical task become the resources of another mathematical task. In that way, teachers control students' memories, by giving some temporal order to the sequence of products achieved.

Teachers' expectations for students to draw and to use diagrams for working on a problem could be explained with the collective memory hypothesis. The collective memory may include operations that a teacher expects students should do despite a lack of reminders about what to do. For example, the operation of drawing diagrams could be one of the elements of the didactical contract in a geometry class that a student must do, regardless of the omission of this operation in the statement of a task. In the case of dynamic geometry software, students would need to remember how to perform different operations with the software, such as drawing diagrams and making measurements, to achieve some products. The memory of operations that students put to play for the achievement of a particular goal is part of the collective memory of the class once a teacher relies upon what students remember, without cueing them about what to do. In conclusion, students' prior knowledge of resources and operations when doing diagrams—either freehand or with the aid of technological tools—is an example of one of the elements that teachers consider as they set up mathematical tasks for their students to teach new geometric content with a problem. With the statement of the task, teachers control students' memories about the resources and operations needed to solve a problem.

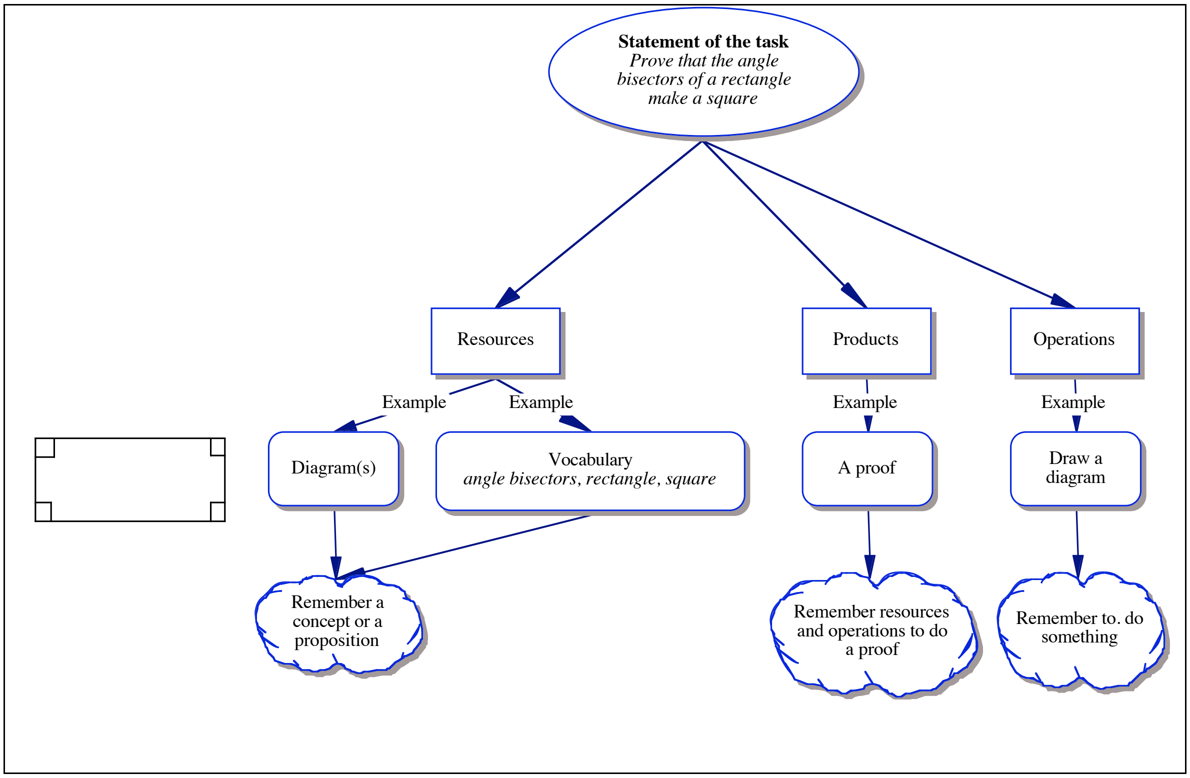


Figure 25. Elements in the statement of the task to make students remember.

In Figure 25 I present what elements a teacher can manipulate to trigger students' prior knowledge using the *statement of a task*. The statement of a task may include specifics about products that would make students remember relevant resources and operations to solve a problem. For example, the statement of the task may ask students to do a proof, cueing students about the resources and operations they would need to remember to do that proof. The statement of a task may also include instructions regarding operations that students should remember to do. For example, a teacher could remind students to do a diagram, or else assume that students would remember to draw a diagram on their own. The resources may be implied in the statement of a task with diagrams or with specifics with the purpose cueing students about they need to remember. For example, there could be vocabulary terms (e.g. angle bisector) that would make students remember the resources and the operations they should use. The statement of the task could include diagrams to remind students about figures and their properties. Overall, the work of the teacher stating the task potentially aims at having students remember something. By including or excluding different elements from the statement of a task, a teacher controls what students are expected to remember. From the participants' perspective, the statement of a task becomes is important element for a teacher to shape students' memories.

Teachers' Actions to Manage Students' Prior Knowledge While Students Are Working on a Task

In this section I discuss teachers' actions towards managing students' prior knowledge at the same time that students are working on a task. The resources and operations for a task could change while students are working on a problem. For example, if a teacher scaffolds the task, the teacher makes available a resource (such as a theorem or a construction tool) that was not mentioned in the statement of a task, or tells students to use an operation (such as drawing a diagram or measuring parts of a diagram) that was implicit in the statement of a task. These changes in the resources and the operations for a task result from negotiations of the didactical contract between a teacher and the students (Herbst 2003, 2006). So, the same problem could entail different tasks at different moments in the lesson according to changes in the availability of resources and operations. While students are working on a problem, a teacher may point to prior knowledge of resources and operations for students to achieve the goal of a task, thus changing the task.

In this section, my focus is on how teachers' control of discursive patterns is an example of a lever that a teacher can use to prompt students' prior knowledge while students are working on a problem. Prior research has reported the importance of classroom discourse in managing students' work on mathematical tasks and in guiding students towards building publicly shared knowledge (Ball & Bass, 2000). One could expect that discourse is a means for making students remember prior knowledge. For example, Brach (2004) found that geometry teachers do specific discursive actions such as giving students auditory cues by emphasizing a statement with their intonation, repeating terms, or telling students explicitly that they should remember something. According to Brach, a common characteristic of these actions is that the teacher assumed responsibility for making students remember something. Similarly, in chapter 4, I presented evidence for how a teacher held students responsible for remembering prior knowledge as they worked on a problem through that teacher's interaction with the students.

In order to study the question of how teachers manage students' prior knowledge while students are working on a task, I selected data that exemplifies actions by which a teacher triggers students' prior knowledge through discursive patterns of interaction. I looked in the focus group data for evidence of how participants reportedly use discourse to make students remember. In particular, I selected intervals where participants commented on the video episode where Ms. Keating and her students worked on proving that the angle bisectors of a rectangle make a square. There were 38 such intervals. From those intervals, I selected intervals with explicit references to discursive moves such as asking students for justifications or restating students' ideas. I found 10 intervals with references on how to make students remember something by performing discursive moves. I present data that gives evidence for four discursive moves that the participants said they would perform as they watched the video episodes. I start with evidence on how, according to the participants, a teacher manages the discursive channel to make students remember prior knowledge needed to solve a problem. Secondly, I elaborate on how participants expected that asking for justifications would be a key move for a teacher to make students rely upon their prior knowledge. Then, I present the participants' comments on a teachers' work in making individual prior knowledge public. Finally, I present the participants' perspectives on teaching actions demanding students to make explicit how they rely on shared knowledge. Altogether, these four actions illustrate how a teacher may shape students' memories about prior knowledge through discourse. I argue that by being attuned to making students remember something in their public interactions with students, teachers may be able to shape the collective memory of the class. Thus, the collective memory of the class contains those resources and operations that a teacher expects students to remember as they work on a problem.

Managing the Discursive Channel to Prompt Students' Prior Knowledge

By prompting students' memories about prior knowledge a teacher may engage in negotiating the task. Here I present examples of participants' statements about how they would prompt students' memories about prior knowledge by means of managing the discursive channel. Management of the discursive channel means here a teacher's guidance of what students say in class. Suppose that students were already working on a problem and their work was framed by a *situation.* That situation may already include some norms regulating discursive actions. The teacher might then perform actions that lead class discussions, conforming to the norms of the situation. For example, if the class was engaged doing a proof, where the norm is that they should produce a two-column proof, a teacher could manage the discursive channel by asking students to identify the statements and reasons. The kinds of questions that the teacher would pose would be of the sort, "what is your next statement?" or, "what is your reason?" So the teacher would orchestrate the writing of a proof, following some sort of protocol and keeping the flow of the discussion. I expect that when managing the discursive channel, a teacher could provoke students to remember prior knowledge.

I chose two intervals for in depth analysis: one from the TWP040505 session and another one from the TWP050306 session. Both intervals illustrate in different ways how a teacher could prompt students' memories about prior knowledge when managing the discursive channel. The selected interval from the TWP040505 session happened in the last half hour of the three-hour session. The interval is 2:25 minutes long and is characterized by a question-and-answer pattern of interaction, with Alice answering questions posed by the moderator and the researcher. In the video, Ms. Keating had stated that the goal of the task was to prove that the angle bisectors of a rectangle make a square. In that statement of the task, it was given that the figure formed by the intersection of the angle bisectors of a rectangle is a square. In contrast, Ms. Keating could have asked students to conjecture and to prove what figure is formed by the intersection of the angle bisectors of a quadrilateral. So, the task, as stated by Ms. Keating, was likely to have prompted students to remember properties of a square that they could apply to solving the problem.

In this interval, Alice pointed out that the task of proving that the angle bisectors of a rectangle make a square started with the assumption that the angle bisectors of a rectangle make a rectangle. Instead, students should have proved that the angle bisectors of a rectangle make a rectangle, before proving that the angle bisectors of a rectangle make a square. According to Alice, Ms. Keating did not guide students enough for them to use properties of a rectangle in their proof. Alice disapproved that Ms. Keating did not require students to use their prior knowledge to prove that the angle bisectors of a rectangle make a rectangle. In her critique of the video episode, Alice described the kind of reasoning that she would expect students to use to explicate what properties of a rectangle and of angle bisectors could be applied to the solution of the problem. Alice disagreed with the possibility of using measurements to confirm that the intersection of the angle bisectors of a rectangle make a right angle. A piece of the conversation in this interval where Alice made a critique of the video episode follows.

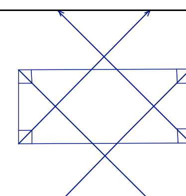
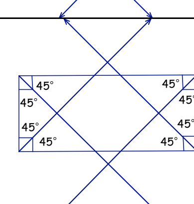
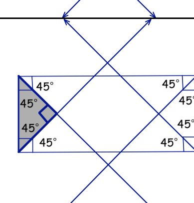
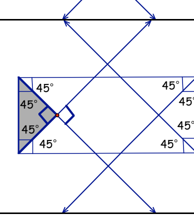
Turn #	**Speaker**	**Turn**
483.	Researcher	So, I'm still kind of confused in what is that makes this episode be say you know, like--
484.	Alice	She, she never used…the properties of a rectangle to establish that it was a rectangle. She stated, "this is a rectangle, now let's prove that this is a square." And the students had to use, "this is a 90-degree angle, I bisected the angle so I've created two 45's. The sum of the interior angles of a triangle are 180, so this one has to be 90, and this is vertical." So they, they reasoned their way through it, without ever establishing that, "yes indeed by the architects, you know, uh, right angle, that this is definitely a right angle." It was assumed that that was the case.
485.	Moderator	By measuring?
486.	Alice	Correct, correct.

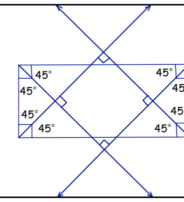
(TWP040505-57)

Here, Alice described an alternative to the video episode of the proof for the claim that the angle bisectors of a rectangle make a square. In her alternative, she would start by showing how to deduce that the angle bisectors of a rectangle make right angles. This alternative could be conceived of as an emergent story with the following moments. First, students would say that the angles of a rectangle measure 90-degrees. Secondly, students would say that the angle bisectors of those angles would make 45-degree angles. Thirdly, students would apply the triangle sum theorem to get that the angle bisectors meet at a right angle. Finally, students would apply the vertical angles theorem to show that the angles of the figure made by the angle bisectors are right angles. All these steps would constitute a proof for the claim that the angle bisectors of a rectangle make another rectangle. Table 30 includes the moments in the emergent story delineated by Alice. The last moment is one that I infer as a conclusion to the story, in order to prove that the resulting figure is a rectangle. There would be more work to do in order to prove that the figure made by the angle bisectors of a rectangle is a square.

Table 30

Emergent stories delineated by Alice (TWP040505-57)

Moments	Story	Diagrams
1	There is a rectangle and its angle bisectors.	
2	Students say that in a rectangle all angles measure 90-degrees.	
3	Students deduce that the angle bisectors of a rectangle make 45-degree angles.	
4	Students apply the triangle sum theorem to deduce that the angle made by the angle bisectors of a rectangle measures 90-degrees.	
5	Students apply the vertical angle theorem to show that the internal angle of the figure made by the angle bisectors of a rectangle is a right angle.	

6	Students apply a similar argument to deduce that all angles made by the intersection of the angle bisectors of a rectangle are right angles, and consequently, the figure made is a rectangle.	

In this emergent story, students are active participants. Alice denoted students' agency with her use of the first person to enact what a student would say as they went over the steps of the proof. Alice said: "And **the students** had to use, 'this is a 90-degree angle, **I** bisected the angle so **I**'ve created two 45's. The sum of the interior angles of a triangle are 180, so this one has to be 90, and this is vertical.'" (TWP040505-57, turn 484). Students in Alice's emergent story would be making logical deductions, connecting one step to the next, and justifying their answers relying on their prior knowledge.

Students' actions in the emergent story differ from the actions in the video, where, according to Alice, Ms. Keating let students assume that the angle bisectors of a rectangle make a rectangle. Alice attributed agency to Ms. Keating when she said, "she [Ms. Keating] never used…the properties of a rectangle to establish that it was a rectangle. She [Ms. Keating] stated, 'this is a rectangle, now let's prove that this is a square.'" (TWP040505-57, turn 484). So, in Alice's narration of the video episode, she identified the moment of the statement of the task as a decision moment for a teacher to possibly ask students to prove that the angle bisectors of a rectangle make a rectangle.

Alice contrasted a teacher's decision to establish that the figure was a rectangle and a teacher's decision to assume that the figure was a rectangle. Alice reacted negatively to Ms. Keating's decision to start proving that the figure resulting from the intersection of the angle bisectors of a rectangle is a square by assuming that the figure was already a rectangle. Alice indicted Ms. Keating for not using the properties of a rectangle and for relying on measurements instead (which she had not).

In the alternative story delineated by Alice, there is a *normativity* stance towards a teacher's decision to make connections with prior knowledge in the proof. In reference to Ms. Keating's actions, Alice said, "she **never** used the properties of a rectangle," which contrasts an expectation that using the properties of a rectangle is something that Ms. Keating should have done. Also, when Alice said, "they [the class] reasoned their way through it, **without ever establishing** that, 'yes indeed by the architects, you know, uh, right angle, that this is definitely a right angle.' It was assumed that that was the case," there is a contrast between making an assumption and establishing a result through logical reasoning, with the expectation that a teacher should use logical reasoning. The word choices of "never" and "ever" stress that it would be appropriate for a teacher to do things differently than in the video, where Ms. Keating did not require students to apply prior knowledge to state that the angle bisectors of a rectangle make a rectangle. So, there is an expectation that students should use prior knowledge to justify each step in the proof, following a teacher's request.

The previous discussion of TWP040505-57 illustrates a case where, from the participant's perspective, a teacher is responsible for managing the discursive channel to prompt students' memories about prior knowledge. According to Alice, the teacher should have asked students to rely upon prior knowledge in order to justify each step of the proof, instead of letting students make assumptions by measurements. Alice perceived this decision as timely. Alice's comments suggest that, according to participants, a teacher should expect students to provide justifications based upon the prior knowledge they have throughout the development of the proof. The absence of such a line of reasoning in the video from Ms. Keating's class shows that there were not enough connections to prior knowledge in order to do the proof. Alice noted that Ms. Keating "used…the properties of a rectangle to establish that it was a rectangle." One could conceive that properties of a rectangle are part of the prior knowledge that students possess. So, from a practitioner's perspective, a teacher is responsible for activating prior knowledge when managing the discursive channel. Moreover, according to a practitioner, a teacher should activate students' prior knowledge at a particular moment—at the moment when justifications are needed for each step of the proof.

In the video episode from Ms. Keating's class, they had been trying to prove that the angle bisectors of a rectangle make a square. In the focus group discussion, Alice flagged a moment when a teacher could expect students to draw upon students' prior knowledge. In the decision to prompt students' memories of prior knowledge, Alice saw the teacher follow the steps of the proof to guide **when** to rely upon prior knowledge to justify each step of the proof. She used the definition of a rectangle, the definition of angle bisectors, the triangle sum theorem, and the vertical angle theorem to develop the first part of the proof—the angle bisectors of a rectangle make another rectangle. According to Alice's expectation, a teacher should hold students responsible for drawing upon prior knowledge in making deductions based upon the givens. Alice worried about the absence of such a line of reasoning in the video episode, when they were making the claim that the angle bisectors of a rectangle meet at right angles. So, from a practitioner's perspective, a teacher is responsible for not letting students make assumptions and for demanding students to justify steps in the proof, thus connecting with the prior knowledge needed to solve a problem.[56] There is a mathematical rationale in this demand for prior knowledge to justify each step in the proof. In this case, the grounds for making decisions about relying upon prior knowledge have to do with a mathematical justification. However, a teacher could have other grounds to prompt students' memories of prior knowledge besides an alignment with practices in the discipline of mathematics as I show with an example from another interval.

Different from the previous interval, TWP050306-43 shows a case where a teacher's actions to make students remember prior knowledge needed to solve a problem is a reaction to students' forgetfulness. In this interval a teacher's decision to remind students about prior knowledge is justified on the grounds of sustaining an activity of teaching, and in particular for moving a discussion forward. Analogue to the interval I presented earlier, according to participants, a teacher is responsible for drawing upon students' prior knowledge. Also, participants say that the decision to draw upon students' prior knowledge is timely because it happens at the moment when a teacher perceives that students are not moving forward with the proof. My discussion of this new interval follows.

The interval starts approximately one minute after participants had finished watching the video of the proof of the angle bisectors of a rectangle make a square from Ms. Keating's class. The interval lasted 1:55 minutes, with the main participation of Jenna who commented on the video. Jenna took a long turn to describe what she would do in case that students were not moving forward in finding the solution of the angle bisectors problem: She would cue students about relevant concepts that are needed to

[56] In Ms. Keating's class, they had already studied that in a parallelogram consecutive angle bisectors are perpendicular.

solve the problem. Jenna described her teaching move as that of "throwing out words," with the intention of making students remember things that they know and that are useful for working on the proof. The relevant turns from the interval where Jenna made statements about students' prior knowledge follow.

Turn #	Speaker	Turn
776.	Researcher	So, was there any moment where you thought that…as a teacher you could have done something else?
777.	Jenna	I wouldn't have no. I mean, I, I wou— if I were direct teaching that, you know, I would say, "hey, can't you see that it's…" y'know, don't—"have you ever—remember anything about equidistant?" Y'know. (and) I, um…if they never did get somewhere, like they did with this teacher, if they, they never were getting somewhere, I might throw out a couple words like that, say, "is…equidistant mean anything to you?" and, and see if somebody comes up with something. Y'know, um…or…y'know, throw out segment addition. Y'know, uh, "see any isosceles triangles," or y'know I'd throw out words like that to, to lead them on if they weren't going anywhere. But, once they're going a direction, you don't want to stop that just to get the direction you want to go, because that's obviously—their direction is what's making sense to them and they bring along a lot of the other students and then if there's some that, that never got…there, the other students to know why they did that, you can say, "okay, what's another way we could look at this?" and then throw out something about equidistant and, something y'know, the way you were thinking about it, and see if that helps the other students that didn't get there.

(TWP050305-43)

Jenna said that her decision to intervene with students' work on the problem would depend on whether students are making progress in achieving a solution or not. This decision is timely because she would consider if the work students have done so far is valuable or not. Jenna stated a condition to cue students about concepts they should remember. She said, "**if they [students] never did get somewhere**, like they did with this teacher [Ms. Keating]**, if they, they [students] never were getting somewhere**, I might throw out a couple of words like that…" Jenna perceived that in the video from

Ms. Keating's class students were moving forward with the solution. However, according to Jenna, if students were not moving forward with the solution, she would remind students about concepts needed in the proof such as the meaning of "equidistant" or the "segment addition postulate." These were explicit references to things that Jenna would expect students to remember in solving the angle bisectors problem. The evaluative stance associated with this teaching move was one of *probability*, signaled by the use of “would” and “might.”

Jenna described the action of cueing students about relevant concepts needed to solve the angle bisectors problem as "direct teaching." She used the metaphor of "throwing out" ideas to denote how in her statements she would be telling students to consider particular concepts relevant for achieving a solution. Jenna said that the purpose of saying relevant concepts would be to "lead them [students] on if they weren't going anywhere." In contrast, according to Jenna, if students were to show commitment to a solution and also engagement with the work entailed in that solution, a teacher should not interrupt students. Jenna said, "But, once they’re going a direction, **you don’t want to stop** that just to get the direction you want to go, because that’s obviously—their direction is what’s making sense to them and they bring along a lot of the other students….” So, from the participant’s perspective, a rationale for letting students pursue their solution is that students should work on something that makes sense to them. Also, according to the participant, those students who are convinced about the possibility of pursuing a solution would "bring along" other students as well.

It seems as if this justification for letting students work on a solution of their choice does not take into account students' prior knowledge. Instead, a teacher’s decision to allow students to work on a particular solution is tied to a student's drive to solve a problem according to his or her intuition, and to other students' understanding of that solution. One could infer from the participant's comments that the teacher is responsible to let students follow their intuition. From the participant's perspective, it appears that students are not responsible for remembering prior knowledge relevant to solve a problem on their own. If students don't have ideas of their own, the teacher is responsible to move forward a discussion.

Besides helping students who are stumped, a teacher could have other goals when cueing students' about prior knowledge. For example, a teacher could activate relevant concepts pertaining to a particular desired solution. That is, by making students remember what they need to know to solve a problem, a teacher can control the kind of solution that students will end up working on. However, in the case illustrated above, a teacher's action to make students remember geometric terms is a particular case of providing the resources students should use to work on a problem. A teacher might use geometric terms with the purpose of reminding students about prior knowledge when class discussions are not moving forward. Thus, the importance of sustaining an activity of teaching—holding a discussion—contrasts with other possible rationales for activating prior knowledge, such as the objective of making a mathematical argument.

In conclusion, according to participants, instead of giving students about the resources needed before starting a problem, a teacher could bring about memories of those resources <u>while</u> students are working on the problem. The teacher is responsible for managing the discursive channel and for drawing upon students' prior knowledge. Although it is conceivable that a teacher would expect students to remember relevant resources to do the proof on their own, it seems as if students might not be able to make connections between prior knowledge and parts of the proof by themselves. Participants perceive that a teacher's suggestion at the right moment could motivate students to continue to work on the problem. Participants identified timely actions by the teacher to make students remember the prior knowledge needed to solve a problem. With these actions, a teacher can sustain students' engagement with the problem. In addition, a teacher could control what students remember so as to make students solve a problem the way the teacher has intended to. The teacher could do that at the moment when students could potentially be stumped with the problem or at the moment when students should provide justifications for their claims.

The hypothesis of the collective memory helps explain teachers' decisions to prompt students' memories of prior knowledge when managing the discursive channel. A teacher may want students to rely upon their memories of particular resources that have been sanctioned in class and that are useful for solving a problem in a particular way. However, it seems that teachers feel the need of prompting students' memories to prevent students from making unwarranted assumptions and to sustain the activity of teaching when students do not remember things on their own.

I have presented two examples of how a teacher can use prior knowledge to manage the discursive channel. In the first example, the discursive channel should follow the structure of a two-column proof. In a two-column proof every statement should be justified by a reason. However, there was a perception that the teacher in the video had not required students to justify a statement. The teacher breached a norm that regulates the discursive channel by allowing students to make a statement without a reason.

Though on her defense, per the participants who watched this episode, in order to manage the discursive channel, the teacher should ask students to rely upon their prior knowledge when establishing the sequence of statements and reasons in the proof. In the second example, a teacher reminds students about the prior knowledge needed to work a problem in order to keep the flow of the discursive channel. If students were to be working on a solution that made sense to them, then the teacher would not need to make students remember their prior knowledge. So, a teacher's action to make students remember prior knowledge has the purpose of moving the discussion along. In both examples, a teacher is responsible for activating students' memories.

By assuming the responsibility of making students remember the resources needed to solve a problem, a teacher controls what students remember: Students should remember what has been said in class. Moreover, a teacher can control the resources that students should remember instead of letting students decide the resources. Instead, a teacher could let individual students remember resources on his or her own. However, it seems that teachers do not allow students to assume responsibility to remember what they need to solve a problem on own. In the next section I present data on how a teacher asks students to use prior knowledge to justify their actions as they work on a problem as a case where a teacher's demand for justification allows the teacher to draw upon specific memories of resources. I will argue that by asking students to give justifications a teacher shapes the collective memory of the class.

Asking Students to Justify Actions Based upon Prior Knowledge

Another action that a teacher could do in order to manage students' prior knowledge as they work on a problem is to ask students for justifications. This action came about in reactions to the videos from Ms. Keating's class. Participants commented on what the teacher and the students were doing, noticing that Ms. Keating required students to justify their answers. In my analysis I focus on the use of metaphors to speak about students' prior knowledge, because they point to different actions that teachers and their students might do while students are working on a task.

The metaphor of the "geometry toolbox" illustrates participants' perspectives regarding the prior knowledge that students could make use of when working on a problem. Lucille used this metaphor in the discussion that followed the viewing of a video-montage in the first part of the interval (TWP040505-23). Even though the moderator attempted to reframe the discussion by presenting a new slide with questions about when it would be appropriate to use the angle bisectors problem, the idea of "the toolbox," originally presented by Lucille, continued to have relevance in Mara's comments. In the interval following interval 23, Mara alluded to this metaphor when elaborating on how, in order to teach with the problem, a teacher could capitalize on what students individually know. My analysis focuses on Lucille's and Mara's turns, where they used the metaphor of "geometry toolbox" in reference to students' prior knowledge.

The interval is approximately 3 minutes long and starts approximately one hour and 8 minutes from the beginning of the session. The interval starts with a question by the researcher regarding why students in the video did not use the graphing calculators they had available when working on the angle bisectors problem. In response, Mara said that she thought students had worked with their calculators prior to the moment when they presented their results. Lucille commented on the questions posed by Ms. Keating to her class. She argued that Ms. Keating intended to have students justify their answers by asking them to remember things that they knew. Then, Lucille introduced the metaphor of the "geometry toolbox," in reference to students' prior knowledge (turn 181 below). The full transcript of TWP040505-23, follows.

Turn #	**Speaker**	**Turn**
179.	Researcher	But an interesting thing about what you saw is that none of them are actually doing that in the video right? Even though they have it at some point in the lesson.
180.	Mara	Right, so, so what I would suggest that they had done at their own table and then they had to get up there. So they had to visually show it. But I still think they were doing it on their calculators, some of 'em.
181.	Lucille	But I think, but I think what you might be trying to um direct our thoughts to is that despite that, they had to justify (Mara: Right) what they saw and that was the, that was the growing process. I mean, seeing it but then relating it to the various things that they knew from, you know, it's kind of like, everything overlaps, every theorem seems to like make more sense when you see where it came from, you see something else. So, they, you know, a couple of 'em got up and they were using similar triangles and, you know, she was saying, "Well, how do you know, how do you know?" You know, trying to get them to go back to their foundational information that they'd already kind of stored in their geometry toolbox and kind of build from there. So, it was more than just, it seemed like it was more than just, "What's the answer?" "What logic can we use to, to—" It, it—You know, sh—she was writing a proof but and she was forcing them to verbalize the proof.
182.	Unidentifiable speaker	Right.
183.	Alice	But I thought—what struck me with that—the problems up here was that none of them took your generic quadrilateral. They all started with—they, they did talk about the rhombus and the square—that they would intersect at a point, um, and then it seemed that they focused on the rectangle. And, I don't think they went beyond that special uh quadrilateral. And so I guess I was curious (Mara: Yes.) as to whether the question started (Mara: Right.) the way our question did, or whether it was started as a more focused question, that you know, "Get—what can you say about the angle bisectors of a rectangle?" Well, that's a much more focused question than a quadrilateral.
184.	Candace	They did have on there the kite and the isosceles trapezoid and a few other things though.

185.	Nathan	Towards the end.
186.	Candace	Yeah.
187.		[4 seconds pause]

(TWP040505-23)

I focus on how participants' metaphors for students' prior knowledge give evidence for teacher's actions while students work on a mathematical task. In particular, how a teacher's demand for students to justify their actions impinges upon students' prior knowledge. The metaphor of "geometry toolbox" came about in reaction to the video montage of Ms. Keating's class using the angle bisectors problem with different classes in different years. Lucille's turn is an example of how a participant in the focus group session perceived and appreciated Ms. Keating's actions in the video. In the following discussion, I marshal evidence from my analysis to support the claim that Lucille established contrasts between different sorts of actions that she saw in the video. According to Lucille, these actions have an effect on a teacher's management of students' prior knowledge.

Lucille established a contrast between two possible teaching moves: asking students to give justifications and asking students to give their answers without any justification. Lucille made references to students' prior knowledge in describing the kinds of resources that students could make use of when giving justifications. In Lucille's statements there is yet another contrast in reference to the video: The teacher is responsible for doing some things and the students are responsible for doing other things.

The contrast between things that the teacher should do and things that the students should do is relevant in identifying the share of labor for how teachers and students are expected to draw upon students' prior knowledge when working on a problem. I applied Analysis of Participation to identify how different linguistic choices point to relationships between people or things (Participants[57]) and also to identify Processes in which those Participants are involved. After parsing turns into clauses and identifying Participants, Processes, and Circumstances, I aggregated figures that pertain to teaching actions and figures that pertain to students' actions. In addition, I searched for lexical cohesion chains that linked Participants and Processes that belong to the same cohesion chain, because the linguistic choices point to the same kinds of meanings. From this analysis I expected to understand whether Lucille saw that Ms. Keating and her students engaged in different kinds of actions, thus, taking different responsibilities in their class. The linguistic choices of participants who watched the video point to their perception about the division of labor in Ms. Keating's class.

Analysis of Participation[58] (see Table 31) shows distinct responsibilities for the teacher and for the students according to what each does when working on a problem. In particular, Lucille identified how a teacher could require students to remember what they ought to know through the process of justification as they work on a problem. But before presenting results pertaining to the division of labor that was apparent in the viewing of the video from Ms. Keating's class, I present other results about contrasts in Processes that show through from the Analysis of Participation. These contrasts in Processes are important because Lucille related prior knowledge to only one of these Processes.

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[57] I capitalize "Participants," "Processes," and "Circumstances" when referring to the components of a figure.

[58] Halliday coined the term "figure" to refer to how a clause fulfills the ideational metafunction of language—or how the clause represents the world, and in contrast with other roles of the clause that concern the interpersonal and textual metafunctions (Halliday & Matthiessen, 2004). Analysis of Participation involves parsing figures into three elements: *Participants, Processes,* and *Circumstances.* Processes are the "goings on" represented in the clause—often embodied in the verb but not always. Participants are "people and things that participate in the process" (Martin & Rose, 2003, p. 70). In a text, some of these elements may be omitted or implied.

The Analysis of Participation of turn 181 shows a contrast between two Processes, “seeing” and “justifying.” There is a clause that includes “seeing” as a Process (clause 9) and another clause that includes “seeing” as a Circumstance (clause 8). When Lucille talked about what students "saw," she used the Process of “seeing” in a Participant to characterize students' discoveries (clause 3). However, “justifying” appears as an action demanded by the teacher (clause 3). Lucille’s use of “to justify” refers to students’ actions, in response to Ms. Keating‘s demands.

The linguistic choices in turn 181 also suggest a contrast between “justifying” and “seeing.” There are semantic links of terms related to the act of justification: "to justify" (clause 3), "how" (clause 13), "logic" (clause 20), "a proof" (clause 21), "to verbalize" (clause 22), and "the proof" (clause 22). The choice of terms emphasizes those things that students could do or produce whenever they justify their answers. In contrast, another constellation of terms relates to the act of seeing: "what they saw" (clause 3), "seeing" (clause 5), and "the answer" (clause 19). These terms refer to students’ claims about the diagram, based upon what they gather from visual perception.

Table 31

Analysis of Participation (TWP040505-23, turn 181)[59]

	Clause	Figures present in Lucille's speech	Participant			Process	Circumstance
				The teacher	The students		
1.	But I think, but I think	I think	• I			think	
2.	what you might be trying to direct our thoughts to is <<>> despite that	what you might be trying to direct our thoughts to is that despite that	• what you might be trying to direct our thoughts to • [that]			is	despite that
3.	<<that>> they had to justify what they saw	they (the students) had to justify what they saw	• what they (the students) saw		they (the students)[60]	had to justify	
4.	and that was the, that was the growing process.	that was the growing process	• that (to justify what they saw) • the growing process			was	
5.	I mean, seeing it	(the students are) seeing it	• it (what they saw)		(the students)	(are) seeing	

[59] There are figures with Processes that have been expanded by the speaker, 3. “had to justify,” 8. “seems to make sense,” and 14. “get to go back.” Analog to Martin and Rose’s decision (2003, p. 74), I decided to keep these as one Process. Also, whenever a Process has other words in between, I noted the ellipsis with a slash "/."

[60] When there is ellipsis or substitution, I note in parenthesis my interpretation of the reference, as it is the case for the use of pronouns.

	Clause	Figures present in Lucille's speech	Participant			Process	Circumstance
				The teacher	The students		
6.	but then relating it to the various things that they knew from, you know,	(the students are) relating it (what they saw) to the various things that they (the students) knew from (before)	• it (what students saw) • the various things that they (the students) knew from (before)		(the students)	(are) relating / to	
7.	it's kind of like, everything overlaps,	everything (what the students saw and what the students knew) overlaps	• everything (what the students saw and what the students knew)			overlaps	
8.	every theorem seems to like make more sense when you see where it came from,	every theorem seems to like make more sense when you (anybody) see where it (the theorem) came from,	• every theorem • [you (anybody)][61]			seems to / make / sense	when you (anybody) see where it (the theorem) came from
9.	you see something else.	you (anybody) see something else.	• you (anybody) • something else			see	
10.	So, they, you know, a couple of 'em got up	they (the students)…, a couple of 'em (the students) got up			• they (the students) • a couple of 'em (the students)	got up	

[61] I note in brackets when a key Participant is part of the Circumstances in a figure.

	Clause	Figures present in Lucille's speech	Participant			Process	Circumstance
				The teacher	**The students**		
11.	and they were using similar triangles	and they (the students) were using similar triangles	• similar triangles		they (students)	were using	
12.	and, you know, she was saying,	she (the teacher) was saying,		she (the teacher)		was saying	
13.	“Well, how do you know, how do you know?”	"Well, how do you (the students) know, how do you (the students) know?"			“you” (the students	know”	"how"
14.	You know, trying to get them to go back to their foundational information	(the teacher was) trying to get them (the students) to go back to their (students') foundational information		(the teacher)	them (the students)	(was) trying to get / to go back	to their (students') foundational information
15.	that they’d already kind of stored in their geometry toolbox	that (foundational information) they’d (the students) already kind of stored in their (students') geometry toolbox[62]	• that (foundational information)		they (the students)	'd / stored	• already • in their (students') geometry toolbox

[62] This figure (#15) is an elaboration, which explicates the preceding figure (#14). I decided to separate the original clause into two figures to distinguish between the current Process where the teacher relies on students' prior knowledge and a past Process where students' acquired new knowledge.

	Clause	Figures present in Lucille's speech	Participant			Process	Circumstance
				The teacher	**The students**		
16.	and kind of build from there.	and (the teacher) kind of build from there (students' foundational information).		(the teacher)		build	from there (students' foundational information)
17.	So, it was more than just,	it (what the teacher was trying to do) was more than just,	• it (what the teacher was trying to do)			was	more than
18.	it seemed like it was more than just,	it (what the teacher was trying to do) seemed like it (what the teacher was trying to do) was more than just,	• it (what the teacher was trying to do)			seemed like / was	more than
19.	"What's the answer?"	"What's the answer?"	• "what" • "the answer"			"(is)"	
20.	"What logic can we use to, to—"	"What logic can we (the class) use to, to—"	• "what logic" • "we" (the class)			"can / use to"	
21.	It, it—You know, sh—she was writing a proof	It, it—You know, sh—she (the teacher) was writing a proof	• a proof	she (the teacher)		was writing	
22.	but and she was forcing them to verbalize the proof.	but and she (the teacher) was forcing them (the students) to verbalize the proof.	• the proof	she(the teacher)	them (the students)	was forcing / to verbalize	

The evaluative stances expressed by Lucille toward the two Processes differ, valuing more "justifying" than "seeing." For example, Lucille animated how a teacher could pose questions to students to establish a difference between seeing and justifying (clauses 19 and 20) and used "just" to pose a critique to those teachers' questions that merely look for an answer (clauses 17 and 18).[63] However, Lucille had stated earlier that the process of justification was "the growing process" (clause 4), which is a desirable characteristic of learning. At the same time, it is understandable that a "growing process" would involve some sorts of pains that the person involved in the process would be able to overcome, as commonly implied in the term "growing pains." Thus, Lucille had made positive remarks regarding the process of justification as opposed to critiques to the process of seeing without relating it to what they already knew (clause 6)

In relation to teachers' and students' responsibilities, Analysis of Participation shows that the teacher prompted students' actions (see Table 32). According to Lucille, Ms. Keating asked questions to students, demanded students to justify their answers, and requested students to go back to their "foundational information." Moreover, Lucille saw that Ms. Keating "forced" students "to verbalize the proof." The choice of "to force" positions the teacher in the video as the one empowered to demand from the students actions that would prompt them to justify their answers (instead of just letting students state their answers without justification or instead of relying on students' willingness to justify answers of their own). In addition, Lucille described Ms. Keating with active Processes—saying, asking, and writing—which shows that, according to Lucille, the teacher was in control of the instructional dialogue.

[63] Lucille's use of "just" could be interpreted as a resource within the system of appraisal for graduation, in particular for softening focus (Martin & Rose, 2003, p. 41-43).

Table 32

Sequence of figures that involve the teacher and the students (TWP040505-23, turn 181)

Clause	Lucille's speech	Participant	Process	Participant	Does the teacher ask students to do something?
3	**they (the students)** had to justify what **they** saw	**they (the students)**[64]	had to justify	what **they (the students)** saw	
5	(**the students** are) seeing it	**(the students)**	(are) seeing		
6	but then (**the students** are) relating it to the various things that **they** knew from,	**(the students)**	(are) relating to	the various things that **they (the students)** knew from	
10	So, **they (the students)**, you know, **a couple of 'em (the students**) got up	**-they (the students) -a couple of 'em (the students)**	got up		
11	and **they** were using similar triangles	**they (the students)**	were using		
12	and, you know, *she (the teacher)* was saying	*she (the teacher)*	was saying		
14	trying to get **them (students)**	*(the teacher)*	(was) trying to get	**them (the students)**	yes
14	[**them**] to go back to **their**	**them**	to go back	**their (students')**	

[64] As a convention, I show references to the students in bold and references to the teacher in italics.

Clause	Lucille’s speech	Participant	Process	Participant	Does the teacher ask students to do something?
	(students') foundational information	**(the students)**	to	foundational information	
15	**that they**’d already kind of stored in their geometry toolbox	**they (the students)**	had stored		
21	You know, sh—*she (the teacher)* was writing a proof.	*she (the teacher)*	was writing		
22	but and *she (the teacher)* was forcing them to verbalize the proof.	*she (the teacher)*	was forcing	**Them (the students)**	yes

According to Lucille, when students are working on a problem, students' past actions continue to have relevance as they justify their answers. In particular, students could relate what they see in solving a problem with what "they knew." Thus, students possess some knowledge, which Lucille called "their foundational information." Students' prior knowledge is composed of memories that they have as resources in their "geometry toolbox." Lucille's reference to students' prior knowledge as "their geometry toolbox" is an example of an **ideational metaphor**, where one lexical item—"toolbox" in "geometry toolbox"—carries on the meanings associated with another one—the "handyman's toolbox." Lucille had achieved cohesion by connecting sets of terms related to students' prior knowledge; "the various things that they (students) knew," "every theorem," and "their foundational information" are references to those resources that students have to work on a problem. In Lucille's comments, a teacher's demand for justification could prompt students to connect what they see in a problem with those resources they already have "in their geometry toolbox." When Lucille said, "it's kind of like everything overlaps," she bridged students' prior knowledge and new knowledge through justifications of statements made in a proof.

Thus, the rationality underlying Lucille's comments could be summarized as follows: Students possess some prior knowledge that could be a resource in solving problems. For example, in the video episode, students volunteered to present their answers and some made specific connections with concepts and theorems that they had studied in the past (such as similar triangles). The teacher is the key agent in helping students to connect prior knowledge and new knowledge through the process of justification. Part of the work of the teacher is to hold students accountable for justifying their answers, beyond just giving out an answer. In doing so, she gets students to deploy prior knowledge as a resource for solving a problem.

The analysis of Lucille's reaction to the video episode from Ms. Keating's class illustrates a case where Lucille saw how a teacher could trigger students' prior knowledge by demanding that they justify their answers. So, if a teacher were to require justifications when holding a discussion with students in class, it would be expected that these justifications would use prior knowledge that students possess. It is conceivable that a teacher would not ask students to justify their answers. As a result, students may not justify their answers. In that case, the possibility that students would rely upon prior knowledge on their own is unwarranted. But one of the avowed purposes of the geometry course is that of teaching students deductive reasoning (González & Herbst, 2006). So, it appears that a teacher's demand for justification is an important step for teaching deductive reasoning. In the next section I analyze how the metaphor of the geometry toolbox was developed further in the following interval, with references to the knowledge that all students, as a collective, should share.

Making Students' Individual Prior Knowledge Public

The ideational metaphor of "students' geometry toolbox" continued to have resonance in the TWP040505 session, after Lucille's comments, when Mara brought up the idea that the prior knowledge that individual students possess could be a resource for all students in the class. Mara extended Lucille's ideas about students' prior knowledge, by proposing how a teacher could profit from having students share what they individually know in solving a problem.

Mara made her comments at the beginning of interval 24. This interval follows interval 23, where Lucille referred to the "geometry toolbox" for the first time in the session. In interval 24, Nathan, Mara, Candace, and Lucille alternated turns, answering questions by the moderator. Throughout the interval, Mara took most turns. The transcript, at the end of interval 24, reads:

Turn #	Speaker	Turn
188.	Moderator	So, so our question uh would be like, what could you use this problem for? And, and, and we've talked about using the problem as introducing the unit. Or you could think about it as you know, well if you use it to develop the unit. Or do you use it at the end, or you could use it instead of the unit. Um what, what would you hope for in, in, in this kind of situation? When you use the problem at the beginning of the unit, or when you use the problem at the end of the unit?
189.		[4 seconds pause]
190.	Nathan	I think when you use it at the beginning you're just hoping to get a lot of ideas on the table to be further analyzed later maybe. But at least you get started and get them thinking about, you know, how a square is different than a rhombus, and then how the rectangle behaves differently and just get a lot started.
191.	Mara	And I think you're trying to have students draw on the other knowledge of other kids in the class, that other kids already have caught some of this knowledge in there and some of them have no knowledge. So by that problem you're starting to give them a bigger toolbox for the—each individual student in there to use as opposed to—It's like putting all of our collective thoughts together at one time as opposed to just starting kind of blindly into some [inaudible].

(TWP040505-24)

In contrast with Lucille, Mara was not focused on the video presented in the session. Her use of the second person, "you," in most of the clauses situated her comments within the discussion in the focus group. In particular, her use of the pronoun "you" could be interpreted as a generalization of teaching practices, something that any teacher would do (Halliday & Hasan, 1976, p. 53). The choice of the words "you're trying" suggests an evaluative stance of *desirability*, as opposed to one of *normativity*, where everyone is obliged to follow a norm.

Mara spoke about differences among students. She contrasted those students who have "caught some of this knowledge" with others who have "no knowledge." Mara described teachers as agents who provide students more resources, "a bigger toolbox." Analysis of Participation of those clauses that involve the teacher and the students shows that the teacher is a Participant in active Processes whereas students are patients or receivers of the teacher's actions. Table 33 illustrates clauses where the teacher prompts students to do some actions (clauses 2, 5, and 6). The teacher asks students to "draw on" other's knowledge (clause 2) and gives students "a bigger toolbox" (clause 5). Mara used "them" to talk about students (clauses 4 and 5), suggesting that students are receivers or beneficiaries of the teacher's actions. Mara said that a teacher puts together "our collective thoughts," in apparent reference to ideas presented in class (clause 6). In addition, Mara described students' actions according to things that they did in the past when she stated that students "have caught" knowledge in the past (clause 3). Hence, students' past actions of acquiring knowledge influence new actions of working on a problem.

Table 33

Figures that involve the teacher and the students (TWP040505-24, turn 182)

Clause	**Mara's speech**	**Participant**		**Process**	**Participant**	**Circumstance**	**Does the teacher prompt students to do some actions?**
		the teacher	**the students**				
1.	And I think						
2.	you're (the teacher) trying to have students draw on the other knowledge of other kids (students) in the class,	*you (the teacher)*		are trying to have **students** draw on	• the knowledge of **other kids (students)**		yes
3.	that other kids (students) already have caught some of this knowledge in there		**other kids (students)**	have caught	• some of this knowledge	in there	no
4.	and some of them (students) have no knowledge.		some of **them (the students)**	have no knowledge			no
5.	So by that problem you're (the teacher) starting to give them (students) a bigger toolbox for the—each individual student in there to use as opposed to—	*you (the teacher)*		are starting to give	• **them (students)** • a bigger toolbox	for each individual **student** to use	yes
6.	It's like (the teacher is)	*(the*		(is) putting	• our (the	at one time	yes

Clause	Mara's speech	Participant		Process	Participant	Circumstance	Does the teacher prompt students to do some actions?
		the teacher	**the students**				
	putting all of our (the class') collective thoughts together at one time as opposed to just starting kind of blindly into some [inaudible].	*teacher)*		together	class') collective thoughts		

Mara started the turn responding to a prior comment by Nathan who had suggested how to use the angle bisectors problem at the beginning of a unit on quadrilaterals. Mara's turn started with the conjunction "and," connecting and adding to one elaboration of the scenario of using the angle bisectors problem at the beginning of the unit on quadrilaterals. Mara categorized students into two groups: those kids who possess knowledge and those kids who do not possess knowledge (clauses 3 and 4). Mara considered those memories that individual students possess at the beginning of the unit on quadrilaterals, but that not all students necessarily have. So, students possess individual memories from other mathematics classes. Nevertheless, teachers cannot assume that all students in a class possess this knowledge yet. Mara suggested that by using the angle bisectors problem and by having students share what they know and what they do not know, a teacher can provide students more resources, "a bigger toolbox," for them to work on the problem (clause 5). This "bigger toolbox" would result from the teacher's actions of gathering students' collective thoughts.

I contend that the hypothesis of the collective memory could explain the decision of gathering students' ideas. Mara described three actions when she said, "you're (the teacher) **trying to have students draw** on the other knowledge of other kids (students) in the class" (clause 2); "you're (the teacher) **starting to give** them (students) a bigger toolbox…" (clause 5); "**putting all of our (the class') collective thoughts together** at one time…" (clause 6). I infer that by drawing on students' knowledge, by giving students new knowledge, and by bringing together individual memories that students possess, a teacher creates the collective memory of the class.

It is conceivable that a teacher would act differently. A teacher could let individual students remember those resources they need to solve a problem on their own and decide not to make students' individual memories public. In this alternative scenario, a student's memories of prior knowledge do not become part of the collective. So, instead of assuming responsibility for activating prior knowledge and for making sure that the class holds the same memories of prior knowledge, in this alternative scenario, a teacher would hold students responsible to bring about their individual memories. Since there is not a collective memory, individual memories could vary (in their content and in the degree of how much each individual actually remembers). Then, the work students would produce to achieve the goal of a mathematical task will vary significantly, as each student would have different access to resources from his or her own memories. These variations may make more difficult the work of teaching with a problem. Or, the variations between alternative solutions of a problem that use different resources could be so wide that it may be hard for a teacher to hold students accountable for one solution that uses something that the teacher expects students to remember.

As I illustrated in chapter 4, the work of the student with a problem requires developing a new idea and using that idea in the problem. This requirement contrasts with usual work on problems in geometry where students are supposed to apply prior knowledge that they remember from the class. A teacher has the challenge of using a problem to raise students' awareness about something that they do not know yet and that the work with the problem would allow them to know. So, by restricting what students remember, a teacher has the opportunity to shape the answers to a problem according to those resources that the teacher thinks that students should know.

Both Lucille and Mara referred to the teacher's responsibility for prompting and demanding students' actions. Lucille talked about the actions of the teacher in the video. Mara, in contrast, did not make specific references to the video and seemed to be making generalizations about teaching practices. Both Lucille and Mara said that, in response to the teacher's demands, students are to remember prior knowledge to accomplish different goals: to share what they know, to see new things in a problem, to make connections between old and new knowledge, and to justify their answers. In addition, Mara stated that students are not homogenous in terms of the knowledge that they possess—some students have knowledge and other students have no knowledge. A teacher can use a problem as an opportunity for students to share what they know and to expand their knowledge, drawing upon each others' knowledge as a resource and thus, contributing to the collective memory of the class. In the next section, I present results pertaining to teaching actions with the purpose of drawing upon shared knowledge.

Drawing upon Shared Knowledge

The possible choice between different solutions to the problem of proving that the angle bisectors of a rectangle make a square brought about discussions about how to draw upon students' shared knowledge. In particular, debates regarding the value of having students draw an auxiliary line exposed the kind of prior knowledge needed to do a proof with an auxiliary line. In an interval where Lucille commented on a solution to the problem with an auxiliary line she said that this solution would require students to make use of properties of rhombi. Students would need to know that the diagonals of a rhombus are perpendicular to prove that the figure is a rhombus. On the other hand, a solution to the problem without the auxiliary line would also require prior knowledge to prove that the figure is a rhombus. In particular, students would need to apply the segment addition postulate. Geometry teachers, following a mainstream geometry textbook, would likely teach the segment addition postulate early in the year.[65] In contrast, geometry teachers often discuss properties of rhombi later in the year when covering the quadrilaterals unit. Lucille conditioned her decision to let students draw an auxiliary line according to students' prior knowledge at the moment of presenting the problem. She situated her decisions within the timeline of the geometry course. A piece of the transcript from the interval where Lucille commented on students' prior knowledge to do the proof of the claim that the angle bisectors of a rectangle make a square follows.

Turn #	**Speaker**	**Turn**
438.	Researcher	On the other hand, somebody who really was interested in going orderly towards the argument, and had seen the, the segment addition argument is quite compelling, actually. So you would say, "Well, this is—this isn't really relevant for segment addition." So, if I go there, then they may get lost, you know, they get, you know, outside of the focus that what I want them to do is to compare this segment with this segment.
439.	Lucille	But again it goes in what did they have previous knowledge of. If they didn't have the previous knowledge, that to be a rhombi that the diagonals would, um, intersect at a perpendicular angle, and they, what- and then what they knew was similar tr-, er, congruent triangles, and segment addition, and all of that, then I could see—I mean that would be a good thing to pull out…later. So, again, I think it's "where you are starting from?"

(TWP040505-54)

From my analysis of this interval, I claim that, from the participants' perspective, a teacher's decision to incorporate an idea for solving a problem depends upon what prior knowledge students would need to apply for working on a solution that uses that idea. Here, Lucille conditioned a preference for a given solution on the prior knowledge that

[65] For example, in the geometry textbook by Boyd, Burrill, Cummins, Kanold, and Malloy (1998) the segment addition postulate is in chapter 1 (p. 29).

students possess, when she said, "But again it goes in what did they [students] have **previous knowledge** of." Lucille identified the property of diagonals of a rhombus to be crucial for students to follow through the idea of drawing an auxiliary line. She said, "If they [students] didn't have the previous knowledge, that to be a rhombi that the diagonals would, um, intersect at a perpendicular angle." With this comment, Lucille emphasized that students' prior knowledge that diagonals of a rhombus are perpendicular would determine whether students could pursue the solution that uses the auxiliary line. Therefore, Lucille proposed that to make it *comprehensible* a teacher might postpone the discussion of a solution that uses the auxiliary line for later in the year. That is, in the event that students did not have prior knowledge about properties of rhombi. Lucille allowed for postponing the discussion when she said, "that would be a good thing to pull out…later." Thus, according to Lucille, the timeline of the course provides constraints and possibilities for teachers to decide whether relevant resources for solving a problem are already part of the knowledge that students possess.

From Lucille's comments, it seems as if the angle bisectors problems was not perceived as a vehicle to bring about properties of a rhombus. On the contrary, a solution of the angle bisectors problem that would make use of properties of rhombi would require students to know these properties beforehand. That is, students would bring about properties of rhombi as resources to solve the angle bisectors problem instead of using the angle bisectors problem with the goal of finding out properties of rhombi. Prior research has shown that in order for teachers to use a problem as an opportunity to teach something new, they need to negotiate with students the situation that frames the exchange between work produced and the knowledge at stake (Herbst, 2006). In particular, in a case where a teacher used a problem of the concurrency of the medians to teach a theorem about equal area, Herbst found that the teacher had to give students an intermediate result as a resource, instead of having students produce this result on their own. So, it seems that in order to manage students' work on a problem, a teacher might have to make sure that the resources students need to solve the problem are part of the collective memory of the class. As an alternative, a teacher could use the problem to attain products that would then become part of the collective memory of the class.

Further discussion of the video episode about the proof of the claim that the angle bisectors of a rectangle make a square led Lucille to say that Jackie (the student in the video) had drawn upon prior knowledge of theorems in doing the proof. Lucille noticed that Ms. Keating had asked Jackie for justifications and that request had intimidated Jackie. Thus, according to Lucille, Ms. Keating's decision to ask for justifications was perceived by Jackie as a challenge.

Lucille's comments came about approximately seven minutes before concluding the three-hour session, when the researcher asked some final comments about the video. In this interval, Lucille and Alice alternated turns of speech to answer the researcher's question. The whole interval is approximately 2:30 minutes long. Here I include the transcript of the moderator's question and Lucille's response.

Turn #	**Speaker**	**Turn**
563.	Researcher	We'll, we'll have to finish soon. But I have two moments during the video that I heard some amusement in here. I was wondering if I could just remind you what those were and you could tell me what made you um, what amused you in case it did. Um, so at some moment, eh, Jackie says, "Geez, you scared me!" [Soft laughter from the group.] That was one. And then at another moment Jackie again says, "let's not look at those triangles." And the teacher looks at her and like rolls her eyes. And I, in both cases I heard some

		laughs. I, I didn't pay attention to who was laughing, but I was wondering what made, what made that amuse—amusing?
564.	Lucille	Jackie was going based on theorems that she knew. And when, when the teacher challenged that it made her now start to doubt, like, the reasoning she was having and just the way she said it like, "don't scare me." Like, "don't change the rules on me now." And I just thought that was funny because she felt pretty confident but when the teacher, you know, gave her the indication that she could be wrong, even though she didn't really say that. She said, "How do you know?" or "Are you sure?" It was like, "Oh!" You know? And that was funny [Alice: Yeah.] because kids do stuff like that, you know.

(TWP040505-64)

Here, Lucille noticed that Jackie applied prior knowledge about theorems studied in the geometry class to the solution of the problem of the angle bisectors of a rectangle. Lucille said, "Jackie was going based on theorems that she [Jackie] knew." However, according to Lucille, Ms. Keating's request for justifications seemed to have threatened Jackie's confidence. Lucille noticed that it was not enough for Jackie to make use of what she knew, because Ms. Keating required Jackie to justify her answers. According to Lucille, Ms. Keating was holding Jackie accountable for making use of prior knowledge about theorems in her proof. Lucille described the effect that Ms. Keating's questions had on Jackie when she said, "when the teacher [Ms. Keating] **challenged** that [the proof] it made her [Jackie] now **start to doubt**." So, Lucille said that even though Ms. Keating had not suggested that Jackie's answers were incorrect, Ms. Keating's questions challenged Jackie, who second-guessed her decisions.

One possible explanation for the interactions in the video could be that Jackie reacted to Ms. Keating's changes in the expectations for the activity they were doing in class. Even though Jackie was giving a deductive argument by presenting different statements, she was not giving the reasons for those statements. Jackie's actions of giving reasons without justifications could be explained on the grounds that Ms. Keating had not sufficiently framed the activity as one of "doing proofs." For example, there was not a two-column proof form on the board, which is often used in the situation of doing proofs. Also, at the beginning of the activity Ms. Keating had asked students to make a conjecture. So, students did not get any hints beforehand to recognize the activity of solving the problem of proving that the angle bisectors of a rectangle make a square as one of doing proofs. However, Ms. Keating asked Jackie to justify the statements in the proof. So, Jackie was surprised by the change of the activity from one in which she was making statements without reasons to doing proofs, where reasons are required. However, Jackie obliged by providing the reasons that Ms. Keating requested.

Lucille found Jackie's reaction both *plausible* and *humorous*. Lucille said that Jackie's reaction would be usual for a student whose answers to a problem are questioned by a teacher. Students who are requested to give out more justifications tend to perceive a teacher's questions as a sign that their answers are incorrect, even when their answers could be correct. According to Lucille, Ms. Keating's questions had the effect to shatter Jackie's confidence, provoking participants' humorous reaction to the video. Lucille noticed that Ms. Keating required Jackie to draw upon prior knowledge of theorems previously discussed in class. Moreover, Ms. Keating required Jackie to make her justifications explicit as they worked through the problem.

Lucille's comments illustrate a participant's perspective on how a teacher could make use of students' prior knowledge from the geometry class to work on the angle bisectors problem. A teacher would consider the kinds of resources that students possess at the moment of posing the problem, with the purpose of evaluating which answers students could work on. Then, while students were working on the problem, the teacher would consider whether students possess or not prior knowledge from the geometry class for pursuing a particular solution. If students were already using prior knowledge from the class, the teacher would hold students accountable for justifying how that prior knowledge is relevant in solving the problem. Thus, it seems that, from a participant's perspective, a teacher would decide to make use of students' prior knowledge when a student suggested a solution and also at the moment when a student justified the solution. At both moments, a teacher could hold students responsible for making explicit how their work on the problem draws from what they already know from the geometry class.

In sum, Lucille's comments illustrate a case where, from a participant's perspective, teachers hesitate to encourage solutions that rely on knowledge that students have not officially covered yet. For example, it is possible to work on a solution for the problem of the angle bisectors of a rectangle by using properties of rhombi. However, teachers usually discuss properties of rhombi in the quadrilaterals unit. Therefore, any discussions that would make use of properties of rhombi prior to the quadrilaterals unit would require teachers to alter the usual order of topics within the geometry course. It seems that teachers perceive these solutions as difficult for students to achieve, because students would need to anticipate knowledge that has not been officially introduced as part of the collective memory of the class. Students could rely upon resources that are within the collective memory of the class to solve a problem, and the teacher would not need to alter the usual order of topics in the geometry class. So, instead of using a problem to teach something new, it seems that teachers prefer to use a problem for students to apply ideas that they have learned before.

It seems that teachers prefer solutions that rely upon students' prior knowledge. In the usual contract of the geometry class, students use prior knowledge to justify their answers. However, when using a problem as an opportunity to teach something new, a teacher might expect students to create new knowledge as they provide justifications (see for example chapter 4, or Herbst, 2005). So, there is a tension associated with a change of expectations about what to remember when teaching with a problem because when learning something new as a result of working with a problem, the instructional goal of the activity is something that ordinarily would be a resource in a task, say in a task framed as “doing proofs.”

Summary

In this section I have presented teachers’ actions to manage students’ prior knowledge as students are working on a task. I have referred to discursive moves as examples of teaching moves to draw upon students’ prior knowledge. These discursive moves include managing the discursive channel, asking student to justify actions, making students’ individual prior knowledge public, and drawing upon students’ prior knowledge. Within the didactical contract of the geometry class, there is a division of labor. In that division of labor, teachers are active agents who are expected to demand students use their prior knowledge when working on a problem.

Overall, teachers’ expectation to control of instructional discourse shows how they might filter the prior knowledge that students could make use of. While students work on a problem, the kind of work that teachers expect to do is more tactical than strategic, because they expect to respond to the immediate feedback that they get from students’ work in order to affect students’ subsequent actions. This is an important finding in that problem-based teaching includes not just the setting up of the problem, but tactical ways to respond to students' work in order to activate prior knowledge.

The action of drawing upon students' prior knowledge to assess the value of a particular solution is an example of a tactical move that teachers might perform while students are working on a problem. Even though an alternative solution that uses different kinds of resources than those in the collective memory of the class could be possible, it is unlikely that a teacher would let students work on that alternative solution. The decision of allowing students to work on an alternative solution has to do with the resources they possess to work on a problem. It seems that, according to participants, it is preferable that students would use the problem as a context to apply resources that they already know than as a context to bring about new products.

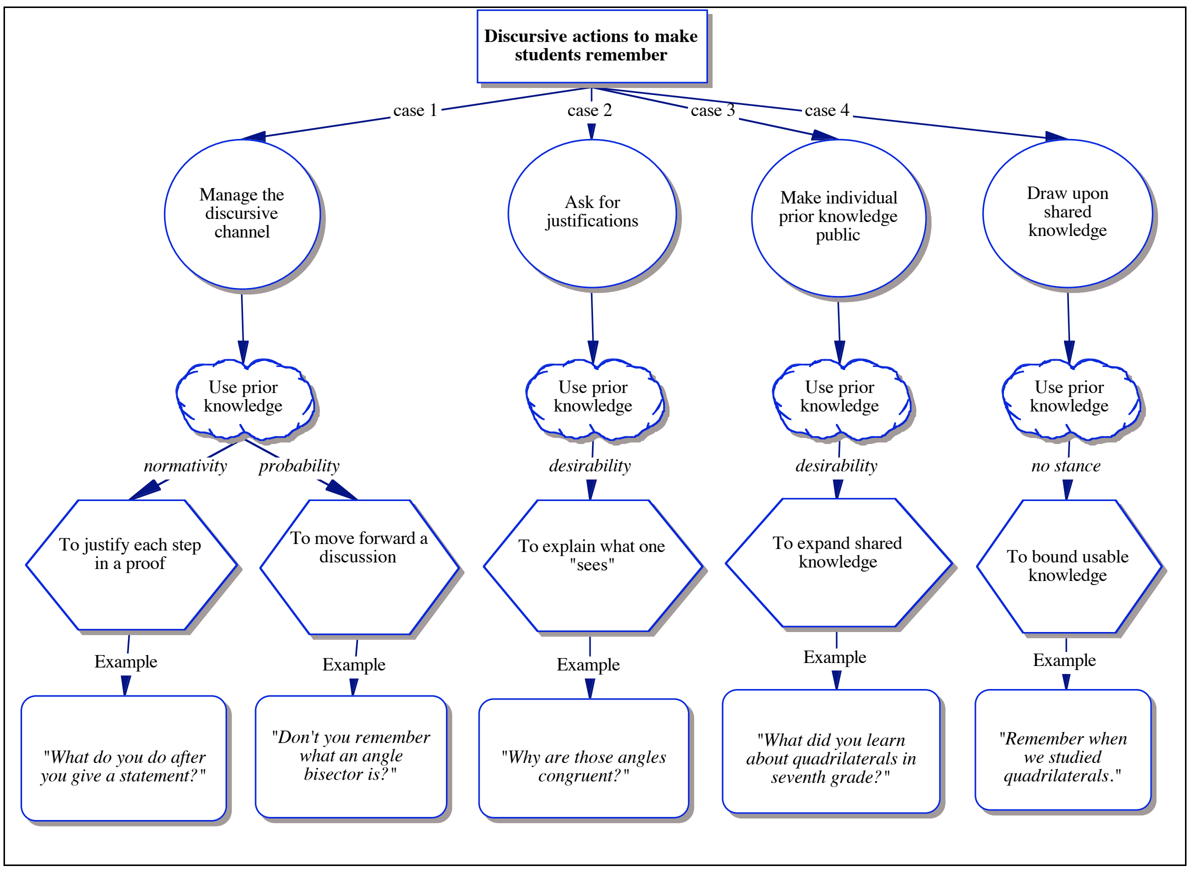

Figure 26. Possible discursive actions to manage students' prior knowledge.

Figure 26 includes a representation of the discursive actions I have presented in this section. These are actions that a teacher could do while students are working on a mathematical task to manage students' prior knowledge: managing the discursive channel, asking students for justifications, making individual prior knowledge public, and drawing upon shared knowledge. These actions have different purposes. Some are related to making a mathematical argument by taking into account what students already know. Other actions have other rationales. For example, one purpose of managing the discursive channel is to move forward a discussion. This purpose is related to the need to sustain an instructional activity. So, the underlying rationale for these actions could be instructional or could be mathematical. Now, in all these actions, the teacher is responsible for triggering prior knowledge and for making students use that prior knowledge as they work on a problem by means of a discursive action. That is, students use prior knowledge in response to a discursive action by the teacher. In doing so, teachers perform a tactical move. In the next section I give more evidence for tactical moves of a teacher when deciding what ought to be memorable as students work on a mathematical task.

Teachers' Actions to Organize What Students Should Remember in the Future While Students are Working on a Task

In the previous section I discussed teachers' actions while students are working on the problem to draw upon students' prior knowledge. In this section I discuss teachers' actions also when they are working on a problem but with the purpose of shaping students' future memories. That is, what actions teachers hold themselves accountable to do while students are working on a task in order to organize what students should remember in the future. This is relevant to this study because the question of how teachers handle students' prior knowledge is two-fold. On the one hand, students do possess prior knowledge that they bring to bear when working on a problem. Students, on the other hand, produce new knowledge as they work on a problem. But students may not be aware that they are producing new knowledge without the teacher's help. Some of that knowledge ought to be memorable, according to the teacher's expectation. But if the teacher does not do anything, this knowledge might be lost. I demonstrate in this section that a teacher's actions to shape what students should remember are important to make students responsible to call upon that knowledge in the future.

To study the question of how teachers control what students should remember in the future while students are working on a task, I looked for data where participants talked about actions done to a diagram. Since the diagram is an important element in teaching geometry and it can keep a record of what is done with a problem, it seemed promising to examine participants' comments about changes to a diagram. By changing a diagram, students may have available an alternative solution, different than the one that the teacher had anticipated. In particular, I looked for intervals where participants reacted to the video of the proof of the claim that the angle bisectors of a rectangle make a square. That video is important because in that video, Ms. Keating filters students' ideas about possible changes to the diagram that could lead to a different kind of solution than the one that the class worked on. From 38 intervals where participants discussed that video, there were 23 intervals with comments about actions to the diagram. First, I present participants' perspectives on how to deal with students' ideas when working on a problem, especially, for the case when students come up with ideas that the teacher had not anticipated earlier. Then, I discuss how participants report that they make use of diagrams in order to control what students should remember.

With this discussion of teachers' actions regarding future memories, I show a teacher's work in shaping the collective memory of the class. I will argue that a teacher can filter the operations students do with a diagram, and thus, control the resources students can call upon to work on a problem, making them forget some solutions and making them remember others. So, not everything is memorable from students' work on a mathematical task. The collective memory includes those things that the teacher wants students to remember, even though students may want to remember other things. By being deliberate about what things students should remember and what things students should forget, teachers control the resources students bring to bear and the operations do when working on a mathematical task.

Incorporating Students' Ideas for Solving a Problem

A teacher's work of managing students' memories as they work on a problem involves more than drawing from students' prior knowledge. A teacher also manages memories that students should hold in the future from their work on a problem. In the focus group sessions, this issue surfaced when participants debated whether or not to consider students' ideas for solving a problem. In particular, participants debated about considering students' ideas when they had not anticipated these ideas prior to watching the video episode from Ms. Keating's class. Participants reported a teacher's tension between following students' unexpected solutions and having students work on a solution already known by the teacher.

It is conceivable that while students were working on a problem, they might come up with solutions that are novel for the teacher. So, the teacher might have to make instructional decisions based upon students' ideas, even when the teacher had not anticipated them. There could be risks associated with a teacher's decision to invest class time considering an idea that he or she does not know where it leads to. Alternative solutions that are novel for the teacher could lead to "dead ends." A student might propose an idea for solving a problem that does not appear to be connected to its solution yet, in spite of its potential usefulness. Moreover, a student might propose an idea that when followed it actually leads to an incorrect conclusion, but the teacher may have been unable to pinpoint earlier the error involved. As a consequence, the teacher could be leading students to follow an incorrect path for solving the problem.

I show that, from the participants' perspective, teachers tend to control the kind of work that students do on a problem, because of these potential difficulties when dealing with students' unexpected solutions. According to participants, teachers tend to prevent students from taking a path that does not lead to a known solution. I argue that with actions intended to filter students' unanticipated ideas out, the teacher shapes the collective memory of the class. In particular, teaching actions geared towards disregarding resources and operations that should not be memorable are crucial for shaping the collective memory. This is especially the case for changes to diagrams, where some operations such as adding an auxiliary line could lead to alternative solutions that a teacher may not want students to remember because they would lack of resources to solve a problem with the new version of the diagram.

In the focus group sessions, participants reacted to a video of Ms. Keating's class where the class was proving that the angle bisectors of a rectangle make a square. As I said earlier, the video shows a student at the board, Jackie, doing the proof with the input of other students in the class. Throughout the video episode, from the back of the room, Ms. Keating makes comments about the ideas that students propose for the proof. Ms. Keating controls the operations that students could do to a diagram. In particular, Ms. Keating questions a suggestion proposed by different students in the class (including Jackie) of drawing an auxiliary line. Ms. Keating asks students to state their reason for adding an auxiliary line. However, students do not make explicit how to use the auxiliary line in their solution. Eventually, Jackie does not add the auxiliary line and continues to pursue a solution to the problem that does not require the auxiliary line. This interaction between Ms. Keating and the students around the decision of adding an auxiliary line illustrates how a teacher might manage students' ideas as they work on a problem.

While participants in the focus group sessions showed curiosity for the solution proposed by the students, they also pondered whether students had a compelling argument for adding an auxiliary line. In TWP040505-51, Alice made comments about Jackie's decision to withdraw her request to draw a diagonal. When thinking through the problem while watching the video, it was difficult for Alice to anticipate a solution that would make use of the auxiliary line. However, she started to formulate a plan for a proof that would make use of the auxiliary line.

Alice made her comments approximately two hours and twenty minutes into the three-hour session. The interval starts with Alice's comments in response to a question by the moderator. Throughout the interval, Alice and Nathan answered questions by the moderator and the researcher, using as reference a handout with a copy of the diagram in the video from Ms. Keating's class. Alice and Nathan are the main speakers in this interval, which lasted three minutes and 15 seconds. The transcript of the exchange between the moderator and Alice at the beginning of interval 51 reads:

Turn #	**Speaker**	**Turn**
404.	Moderator	But um, but in this episode, in one, at one point Jackie wanted to draw a diagonal, and, and then, the teacher asked, "why do you want to draw this for?" and things like that. So in a way it's like when you're grading those tests and they come up with an idea that you were not expecting at the moment. What do you do about that? You know?
405.	Alice	Well, except, bringing up that point, I was kind of curious about that 'cause I thought Jackie backed away. And I was disappointed in Jackie's reaction. I wanted to see what she was gonna do with it. Um… 'cause I think there's something to that, you know, to draw the diagonal. If you can get them— and I was trying to—you know, it was fast paced so I couldn't think through the problem, but can you get those diagonals to intersect at right angles, and you got yourself a square? So, um, you know, I just thought, well gee, the question was posed in such a way that it intimidated the student. Whether it was intended to be that way, I don't know. But it did seem to intimidate the student.

(TWP040505-51)

Alice disclosed some of the tensions involved in a teacher's decision to make use of students' unexpected ideas when working on a problem. A student could come up with an idea that is potentially useful to solve a problem. However, at the moment of presenting that idea, it could be difficult for a teacher to determine whether the idea is useful or not. In this case, Alice suggested that by drawing the diagonals of the figure that results from the intersection of the angle bisectors, and then by proving that those diagonals are perpendicular, one could prove that the figure is a square (see Figure 27). Yet, the correctness of this solution was still uncertain for her, even after having watched the video. Alice showed disappointment for Jackie's decision to retreat from her idea of drawing an auxiliary line. She identified Jackie's decision as an effect of Ms. Keating's questions. From Alice's perspective, Ms. Keating's questions intimidated Jackie. Consequently, Jackie did not pursue her idea of drawing the auxiliary line, even when this could have been a fruitful path towards solving the problem.

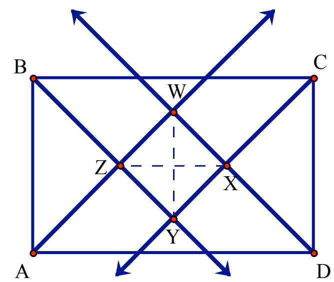

Figure 27. A rectangle, its angle bisectors, and the diagonals of the resulting figure.

During the same focus group session, TWP040505, another participant, Nathan, elaborated on tensions that emerge when students propose an unanticipated solution. Nathan alluded to the tension between attempting possible solutions to a problem and making sure that the class engages in the discussion of a known solution. According to Nathan, there is a risk in pursuing an unexpected idea to solve a problem in that the class could invest too much time working on the problem without arriving at a solution. Nathan's comments illustrate how a teacher's decisions at a moment in time can affect how long it would take to solve a problem in class.

Nathan’s comments also belong to interval 51, approximately two hours and twenty minutes into the session. Participants had been commenting on Ms. Keating’s decision not to draw the auxiliary line that students had suggested. Nathan’s comments are at the end of the interval. The transcript of a part of the interval including Nathan’s comments follows.

Turn #	Speaker	Turn
414.	Researcher	Somebody even says, "wouldn't that be the altitude?" [Pause 7 sec]
415.	Nathan	I think that we always have the end in mind. And we'd like to get multiple solutions and all this, as teachers, but, still there is, there's always that tug to say, "well, no, no, that's not gonna work." And you don't say it that way but, you say, "uh, does anybody, does anybody think that's not gonna work?" [Laughter from the group.] You know. And then somebody says, "yeah, I don't think it's gonna work." And you go from there. But I would say if you have time, and I think it's generally worth the time, let her do that. And somebody else will correct her without you having to say—without you saying anything at all, or they won't correct her and she'll get to a dead end or somebody will help her along the way. I mean, in general, as long as you have a few more minutes, let her do it.
416.	Alice	[inaudible] the evil [Nathan: Oh totally.], the problem is the time.

(TWP040505-51)

From a participant's perspective, a teacher could condition the decision to incorporate into class discussions a proposed idea for solving a problem by considering two main factors: a teacher's anticipation for the usefulness of that idea and how much class time is available at the moment when the new idea is proposed. When Nathan said, "we always have the end in mind," Nathan stated a teacher's usual concern about the duration of the class. This concern could lead a teacher to resist exploring alternative solutions that make use of an unanticipated idea. I take that "the end" here is a reference to the end of the class, since he said later, "if you have time" and "as long as you have a few minutes." These other comments add to my interpretation that he is considering how much time is left before the end of the class.

Nathan's choice of the word "tug" emphasized how a teacher would make a strong effort to pull in students away from an unexpected path. Nathan showed *desirability* of toward teaching students multiple solutions to problems. Nathan said, "And **we'd like** to get to multiple solutions." However, he used the conjunction "but" to preface an exception to showing multiple solutions when a new idea does not appear to be leading towards solving a problem. Nathan said, "But I would say if you have time, and I think it's generally worth the time, let her do that." He said that letting students continue with their proposed solution is a valuable investment of time. However, Nathan's choice of the word "generally" signaled that he may usually consider a student's idea but in some cases he would not. Therefore, Nathan would question the usefulness of following a students' idea when there is not much class time left. Alice concurred with Nathan in that time constraints affect teachers' decisions to discuss alternative solutions.

Nathan described possible teaching moves to prevent students from using an idea that the teacher finds futile in solving a problem. According to Nathan, a teacher could say directly to students that their proposed idea is not going to work. Or, a teacher could indirectly cue students by making use of the opinion of other students in class who judge the new idea as useless for solving the problem. From Nathan's linguistic choices one could conclude that if a teacher asks the class whether an idea is going to work or not, the teacher depends upon the likelihood that a student would realize that the idea leads to a "dead end." The evaluative stance that could be associated with Nathan's comments is that of *probability,* signaled by his use of "anybody" and "somebody" to denote the likelihood that a student in class would either question or expose the usefulness of an idea. One could translate the probability assessment as, *When cued by one teacher,* (a) *it is possible that a student would say that the proposed solution is not going to work,* (b) *it is possible that a student would correct another student who is presenting a wrong solution on the board*, and (c) *it is possible that a student would assist another student who hits a "dead end" when presenting a wrong solution on the board.* So, even though other alternatives are conceivable, such as the possibility that the solution would work or the possibility that students would think that the solution works when it does not, Nathan seemed to have noticed only these three alternatives. In the three assertions, Nathan used the class as the Chorus in ancient Greek tragedies, the voice of reason, judging the potential value of unanticipated solutions, especially when a solution appears to be incorrect. Also, a teacher could manipulate the class to echo the teacher's judgment.

According to Nathan, a teacher could make use of and manipulate other students' reactions to a proposed solution in a moment of opportunity. By making use of students' reactions a teacher could decide at that point in time whether to pursue an unanticipated solution or not. For a teacher, this decision could be timely—a turning point—because the subsequent development of the lesson could change by taking on an unanticipated idea. Hence, from a participant's perspective, teachers' timely moves are crucial for sieving out unanticipated solutions from the collective memory of the class. At the moment when a new idea is proposed, a teacher has the opportunity to decide whether the class should consider that new idea or not. If the teacher were to follow up on the new idea, the class might remember that idea as they work on the problem. But if the teacher were to disregard the new idea, then it might not become part of what students would be accountable to remember.

In contrast with Nathan's expectation that a student prompted by the teacher would question a given answer, a participant in another focus group session said that she would not count on the probability that one student would question a proposed solution. In TWP031506-62, the researcher asked participants to compare two possible ways to prove that the angle bisectors of a rectangle make a square. One solution makes use of the segment addition postulate and does not involve drawing an auxiliary line. The other solution requires drawing auxiliary lines to prove that the diagonals of the avowed square are perpendicular. Jillian argued that no student would question the collinearity of the points that results from drawing an auxiliary line (see Figure 28). Jillian compared the case of a student who may question a proposed solution with her own experience coming into the focus group session. At the beginning of the session, other participants questioned a solution Jillian had thought of for the angle bisectors problem. Upon reflection about her experience, Jillian said that it is difficult for a teacher to consider every possible case when solving a problem.

Interval 62 starts approximately12 minutes after the end of the viewing of the video for the proof of the claim that the angle bisectors of a rectangle make a square. At this point in the session, participants had been discussing the possibility of allowing students to draw the auxiliary line, as requested in the video. They had sketched a proof with the auxiliary line, so they had two alternative proofs: one using the segment addition postulate (and without an auxiliary line) and another one using the altitude of one of the triangles (with the auxiliary line). The interval lasted one minute and 45 seconds. The part of the transcript where Jillian anticipated what students would do reads:

Turn #	**Speaker**	**Turn**
629.	Researcher	You don't think there is any particularly…anything better about segment addition than about the (drawing)
630.	Jillian	(This) one seems easier.
631.	Unidentifiable speaker	[Inaudible]
632.	Jillian	And, you know, there's not one kid who is gonna question that that line doesn't go through those points and hit, I mean, that's fine. Just…you know. And I, that's the part that, it was just

		like in problem when I tried it before I came in here. 'Cause I was so sure that this would work, but I wasn't sure they were gonna hit. And I didn't know what to do with that. And that's always that [gesture moving the right hand side by side as unsure], you know, have you thought about every case?

(TWP031506-62)

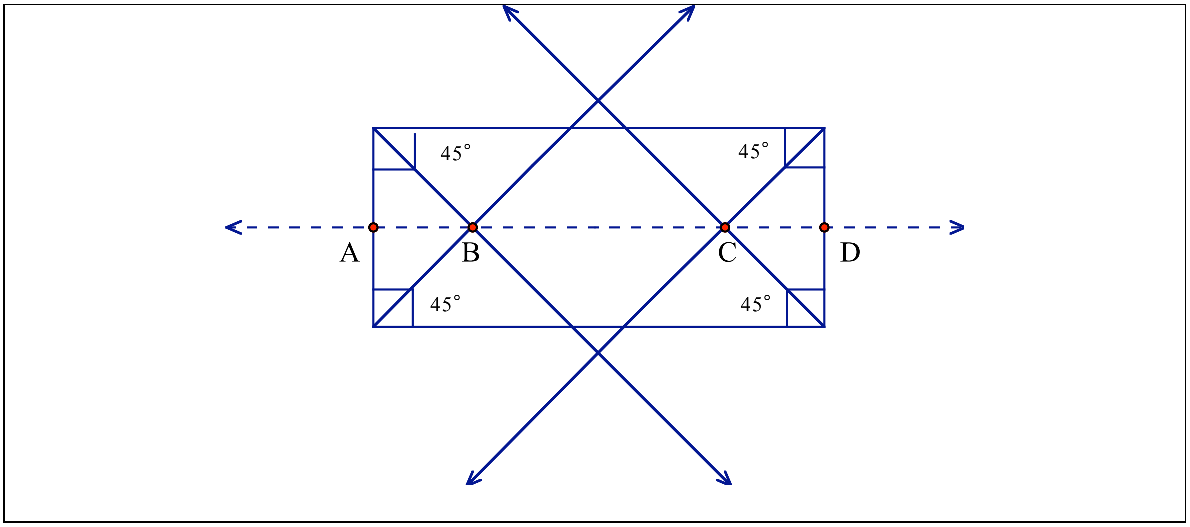

Figure 28. A rectangle, its angle bisectors, and an auxiliary line.

Jillian's comments illustrate how a teacher's reaction depends upon the likelihood that a student would judge an idea appropriate. Jillian did not expect that a student would question the collinearity of the midpoints of two opposite sides of a rectangle and two opposite vertices of the figure made by the intersection of the angle bisectors of the rectangle. Jillian said, "And, you know**, there's not one kid** who is gonna question that that line doesn't go through those points and hit, I mean, that's fine." In this case, when no student questions a proposed solution, a teacher cannot make use of the voice of a student to ask whether drawing an auxiliary line is a valid step or not. That is, a teacher cannot count on the existence of a student who would evaluate the possibility of having an auxiliary line with the needed characteristics. A student who questions a proposed solution would give a teacher an opportunity to make his or her concerns public and thus, evaluate the appropriateness of a strategy in finding the solution to a problem. Possible concerns regarding a solution raised by students could be helpful, especially when the correctness of the solution is uncertain.

A teacher's decision to reject an unanticipated idea—whether this idea is questioned by other students in the class or not—could be justified by other reasons besides lack of time, as Nathan had stated. By considering an unanticipated idea, a teacher could lose control of the new path that the alternative solution could take on. It is plausible that a teacher who takes on an unanticipated solution would not be able to make connections with the known solution, thus, reaching class time without giving closure to the problem. Following Nathan's concerns in the TWP040505 session about the lack of class time to consider unanticipated ideas, Lucille elaborated on a teacher's decision—to try out a novel idea for solving a problem or to persist with a solution already known. According to Lucille, the cost for a teacher who decides to take on an unanticipated solution could be to lose his or her credibility among students. Therefore, in dealing with students' unanticipated ideas, a timely move for a teacher could be to postpone the discussion of an unanticipated solution. The next day, the teacher could bring about that idea to the class, after the teacher has worked on the problem outside of class time.

Lucille's turn belongs to the end of interval 52, shortly after Nathan's comments about a teacher's decision to let a student draw an auxiliary line depending upon what students say. Interval 52 lasted approximately 1 minute and 30 seconds and it is characterized by Lucille's long monologue. The transcript of Lucille's turn follows.

Turn #	**Speaker**	**Turn**
419.	Lucille	I think, if the student brings up something that… you hadn't thought about, and you, and you're not… like, prepared to think about it, or

		you need the time to think about it, sometimes I think it's just for saving face on the teacher's part to just, you know, "Well, let's move along with the way I know works." It's a lot safer. If it's okay to not know, and look at it again the next day if the relationship is such that it's okay for the teacher to say, "I'm not really sure. Let's see if that did work." Then that's, that's okay. But I think you've gotta have a really good learning classroom in terms of, "It's okay for me still to be learning and it's okay for you to be learning. And, yet, I can still be the, the teacher."

(TWP040505-52)

Lucille brought about a new theme regarding how a teacher may manage the uncertainty of unanticipated solutions during class discussions. She said that for teachers to work with an unanticipated idea at the moment when it surfaces in class discussions, they would have had to establish a safe environment much earlier in their classroom where students perceive teachers to be learners of mathematics, just as they themselves are. Within this environment, a teacher has the responsibility to teach despite the acknowledgement that he or she is still a learner of mathematics. Lucille animated the voice of a teacher and said, "It's okay for me still to be learning and it's okay for you to be learning. And, yet, I can still be the, the teacher." She used "can" to show a teacher's "capability" (Eggins & Slade, 1996, p. 107) even when the teacher is not sure about a solution. Moreover, she used "still" repeatedly to amplify the force of attitudes (Martin & Rose, 2003, p. 38), by stating that a teacher who is learning continues to be "the teacher." Therefore, from a participant's perspective, a teacher who takes on a student's unanticipated idea to solve a problem could be risking his or her status in the class. By postponing the discussion of an unanticipated idea, a teacher could focus students' attention to the known solution. As a result, one could expect that students might not need to remember the unanticipated idea until the teacher has tried out this idea on his or her own.

Lucille's alternative for dealing with an unanticipated idea illustrates teachers' apparent hesitancy to follow students' lead at the moment of working on a problem in class. She explained that there should be a tolerant environment for teachers to do mathematical work with their students. According to Lucille, the learning process could benefit both, the teacher and the student, through their interactions. A teacher could control students' work on the problem, making them work on those solutions that are known to him or her. If so, the solution would not be uncertain and the relationship between the teacher and the students might remain unaffected. Instead of taking on an unanticipated solution, the teacher might make students invest time in a solution that is correct. When Lucille said, "Well, let's move along with the way I know works," she demonstrated a teacher's possible decision to disregard a student's unexpected idea in order to pursue a solution known by the teacher. That is, a teacher would take the unexpected idea as "forgettable" until he or she finds a solution later.

In sum, according to participants, teachers face a tension between letting students pursue multiple solutions and presenting a solution that the teacher already knows. The tension lies on valuing students' input and investing class time working on a solution that is correct. Teachers do not want to lead the class to a wrong path by spending time on a solution that does not work. So, teachers filter ahead of time unanticipated solutions either by preventing students from going along unexpected paths, or by having students voice opinions against a proposed solution. One way to filter students' ideas is by keeping control of the diagram. The diagram on the board displays and keeps a record of those things that students should remember about a problem while working on the problem. At the same time, the diagram on the board could enable students to foresee the solution to a problem. If teachers cannot anticipate where that solution is leading to, they are likely hesitate to let students alter diagrams. Therefore, at the moment when a student proposes a new idea, it is conceivable that a teacher might have to estimate whether the suggestion will work or not, in order to make a decision: follow the student or follow the teacher's plan. At that moment, a teacher's anticipation of a solution is crucial to decide if the idea should become memorable.

Using Diagrams to Keep a Memory of Students' Work

One way for a teacher to organize students' memories about their work on a problem is by means of diagrams. A teacher can make use of diagrams to keep track of students' ongoing work on a problem as they find a solution. That is, students and teachers can keep an open record of their work on a problem by annotating a diagram with the addition of new features such as markings. Moreover, a teacher might make use of the diagram to filter claims that are useful towards solving a problem from those that are not useful. I use data from the focus group sessions to provide examples of four diagram-related actions that teachers consider plausible for them to do to keep a memory of students' work: altering the diagram, inspecting the diagram, drawing a separate diagram, and organizing a set of diagrams on the board.

Altering a Diagram

In the video for the proof of claim that the angle bisectors of a rectangle make a square, Ms. Keating keeps control of the diagram by preventing students from adding an auxiliary line. Students propose to add an auxiliary line to the diagram on several occasions. However, Ms. Keating deters students from doing so by asking them to give a rationale for modifying the diagram. In contrast, there are other kinds of additions to the diagram that are not remarked upon (or resisted) by Ms. Keating. As Jackie works on the problem, she adds markings to the diagram to keep track of those things they had proven earlier. Ms. Keating does not question Jackie's decision to add markings to the diagram. On the contrary, adding markings to the diagram to record what had been proven already seems to be a usual practice in Ms. Keating's class.[66] The two distinct actions—drawing an auxiliary line and marking congruent parts—had two different reactions from Ms. Keating: disapproval and approval. Similarly, participants in the focus groups did not comment on Jackie's actions of marking the diagram as she had worked on the problem. However, participants discussed whether any teacher would let students draw an auxiliary line.[67]

In the TWP040505 session, Alice commented that the decision of adding an auxiliary line is contingent on what the teacher assumes that students know at the moment about the solution to the problem. Alice summarized what the class had proven that far and predicted the steps that would follow, if one were to add an auxiliary line. Alice's comments belong to the beginning of interval 53. This interval lasted approximately 4 minutes. Throughout the interval, the researcher led the discussion. Alice, Lucille, Nathan, and Mara answered questions by the researcher. The interval

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[66] Herbst and Brach (2006) have argued that adding marks to a diagram as one works on a proof is a usual practice allocated to the student within the situation of doing proofs. So, Jackie's actions conform to the norms of doing proofs.

[67] The contrast between teachers' disapproval to let students add an auxiliary line to a diagram against teachers' approval to let students add markings to a diagram is usual within the situation of "doing proofs" (Herbst, 2002; Herbst & Brach, 2006).

starts with the researcher's references to a previous comment by Alice and with Alice's response. The transcript of this exchange between the researcher and Alice reads:

Turn #	Speaker	Turn
420.	Researcher	I was taken bu-, um, by Alice's comment that, well, if you got these two diagonals to be perpendicular, you already would have a square. That seemed like a, [Alice: Well.] like an interesting reasoning on the fly that the teacher could use, you know, to, to decide, "hey, should I go on with this?" You know.
421.	Alice	Well, and that's because we already, they had already discovered that the- all the opposite angles were 90-degrees. We knew it was a rectangle but now if we can get those diagonals to be perpendicular, we've got a square. So, um, I just—I was kinda disappointed as I said, that it didn't go, that they didn't take that step.

(TWP040505-53)

In this interval, Alice placed the decision of drawing an auxiliary line within the timeline of the lesson. Alice used temporal markers when she said "already" and when she chose to use the past perfect tense to summarize what the class had found out by then. Alice said that at the moment when there were suggestions about adding an auxiliary line, the class had already concluded that all the angles of the quadrilateral made by the angle bisectors of a rectangle were 90-degrees. From this result, the class could deduce that the figure was a rectangle. At this point of the lesson, the class needed to answer whether the figure resulting from the intersection of the angle bisectors of a rectangle was a square. One way to prove that the resulting figure was a square required proving that all sides were congruent. Instead, Alice conceived of an alternative way to prove that the quadrilateral was a square by proving that its diagonals were perpendicular. So, Alice saw a potential value in drawing an auxiliary line.

Alice's willingness to take on the students' suggestions to add an auxiliary line was somewhat uncharacteristic among the positions taken by participants in the focus group sessions. Most participants said that they would caution students against drawing an auxiliary line if there was not a clear justification for the usefulness of altering a diagram in reaching a solution. For example, in the TWP031506 session, Denis animated what he reportedly says to his students when they want to draw in an auxiliary line. His comment came about after a discussion about a possible solution for the problem of proving that the angle bisectors of a rectangle make a square using an auxiliary line. This interval, interval 59, starts approximately two hours and fifteen minutes into the three-hour session. After Denis' response to the moderator, participants went back to consider the solution with the auxiliary line upon Jillian's request. The transcript of the researcher's question and Denis' response reads:

Turn #	**Speaker**	**Turn**
605.	Researcher	I wonder, well we're still thinking about this, how helpful it is to draw in this diagonal? Do you see yourself- if a student is at the board, and you're standing there, and the student wants to add this diagonal, and you're not sure whether this is going to be helpful or not. Are you gonna allow him to just draw it in or kind of, eh, lead them in a different way?
606.		[3 seconds pause]
607.	Denis	I always caution the kids to, "before you draw in something new, ask yourself, is it gonna help you?" And most kids will say "yes." And then, I'm gonna ask, "show me." And the minute they can't, I'm gonna say, "take it off." I mean, and we know, and, and we know it would help, it could help them, if they have the right reasoning. But if they ca—if they find a place where they get lost, they get confused, you, you have to tell them, you have to be willing to say, "Oops, that's a mistake, let's try something else." And, and also you make sure that they don't get frustrated.
608.	Jillian	That, that's what I thought. Can you walk that through again? The, the order to prove like--

(TWP031506-59)

In this interval, the researcher posed a question regarding a teacher's decision to let a student at the board draw an auxiliary line. Participants did not answer the researcher's question immediately. Then, Denis gave a response. Denis' comments illustrate that changes to a diagram—especially, in the case of adding auxiliary lines—must have a teacher's approval. Otherwise, a teacher runs the risk of confusing and frustrating students by leading them to an incorrect solution.

Denis used projective clauses to show that he controls what students add to a diagram. According to Denis, students should justify how changes to the diagram are helpful for solving a problem prior to making changes to the diagram. If he discovers that students are making a mistake, then he lets students forget that solution and encourages them to try something else. Even though there are moments when making mistakes may seem unavoidable, from Denis' comments I infer that it is *undesirable* for a teacher to allow students to get confused and frustrated. Denis showed the undesirability of a teacher provoking students' confusion when he said, "And, and also you [the teacher] make sure that they [students] don't get frustrated." So, Denis expects students to know what the purpose of proposed changes to a diagram is. If students do not know, then Denis thinks appropriate to prevent them from changing the diagram. In his comments, Denis did not mention whether the teacher's knowledge of a solution with the altered diagram would influence his actions. Instead, Denis held students responsible for providing the justifications for altering the diagram.

In his example, Denis illustrated how he reportedly uses the diagram to control what solutions to a problem become memorable in his class. According to Denis, he usually asks students about the usefulness of adding an auxiliary line. His question is timely, because he asks for consideration about whether the change will be useful before making any changes to the diagram. Denis holds students responsible for knowing why they would want to alter the diagram. In Denis' class, any changes to a diagram would therefore have a temporary status, until there is a justification for making that change. Denis would ask students to take away auxiliary lines for which a justification cannot be produced. Denis animated how he would say, "take it off," to a student who could justify the addition of an auxiliary line. In his decision to take away an auxiliary line, Denis would use the diagram to make students forget about a proposed solution that had not been justified.

Denis' comments illustrate how a teacher could ask for different kinds of justifications to change a diagram. In a geometry class, justifications are usually used for backing up statements in a proof. However, the kind of justification needed to alter a diagram is not one about justifying steps in a proof. According to Denis, an important question when changing a diagram is whether those changes are convenient and helpful to solve the mathematical problem at hand. If changes to a diagram are not warranted by a strategy to solve the problem, then according to Denis it would be inconvenient to change the diagram.

I find Chazan and Lueke's (in press) distinction between *strategic justifications* and *legal justifications* for steps taken to solve a problem to be helpful constructs for understanding the kinds of justifications a teacher could ask for when students propose to change a diagram. In Denis' example, strategic justifications involve statements about the usefulness of making changes to a diagram with the aim of solving the problem. In contrast, a teacher could ask students to give legal justifications for a step taken in solving a problem. A legal justification would involve justifying that the alteration planned to a diagram can be done. I present two examples to illustrate the difference between these two kinds of justifications. One can draw the diagonal of a quadrilateral because of the postulate stating that two points determine a unique line—this is a legal justification. However, the decision of drawing that diagonal may or may not lead towards solving a problem—one would need a strategic justification for that. Alternatively, one would want to draw a line passing through four points that appear to be collinear. There is a strategic justification to draw this line to solve a problem (see Figure 29)—if one could prove that such line is the angle bisector of one of the angles of the triangle made by a side of the rectangle and two of the angle bisectors then the line would also bisect an angle in the rectangle inside. It would necessarily bisect the opposite angles in the rectangle also, and thus that rectangle would be a square. However, the question of whether such a line exists is a mathematical question that requires a legal justification. One would need to show that the four points are collinear prior to drawing the auxiliary line, or to show that a line passing through two of them includes the other two. So, in the second example one could give a strategic justification because drawing that auxiliary line could be useful to solve a problem. However, the possibility of drawing such a line requires a legal justification.

Both legal and strategic justifications may involve memories from prior knowledge. In the case of the examples I presented, a legal justification would require students to remember the postulate that warrants the possibility of drawing a line given two points. In the case of a strategic justification, one could remember the strategy of drawing the auxiliary line for solving another problem before, and apply this strategy to a new problem. Whether the purpose for such an alteration to the diagram is similar or different to what one remembers from a previous problem is something that students would need to decide at the moment of changing the diagram. In geometry, there are some alterations to diagrams that become routine. For example, drawing the radius of a circle, adding the diagonals of a quadrilateral, and finding the altitude of a trapezoid are all cases where students get used to altering the diagram in particular ways to solve particular problems. These alterations, while requiring strategic and legal justifications, could be so engrained into the collective memory of the class that they may not need to be justified anymore. However, one could expect that the first time a teacher introduces to students these alterations, the justifications—legal and strategic—would be called upon.

Inspecting the Diagram

Another action to a diagram that teachers could do to control what kind of things students would remember in the future has to do with asking students to inspect the diagram. In the TWP031506 session, Holly showed concerns about allowing students to draw an auxiliary line. According to Holly, if a teacher were to allow students to draw an auxiliary line to solve the problem of the angle bisectors of a rectangle, students would assume that they could draw in auxiliary lines for other problems thereafter. So, according to Holly, a teacher should be prudent about letting students add new elements to a diagram.

Holly's comments came about in interval 62. The interval is located approximately 12 minutes after the viewing of the video of the proof for the claim that the angle bisectors of a rectangle make a square, and approximately two hours and twenty minutes into the 3-hour session. I referred to this interval earlier with the example of a teacher's lack of expectation that a student would question whether the auxiliary line passes through collinear points stated by Jillian. Holly's comments followed Jillian's. After Holly's comments, Jillian took other long turns to talk about her perception of students' wanting to solve problems quickly. Here, I only present the transcript of the part of the interval that includes Holly's comments.

Turn #	**Speaker**	**Turn**
633.	Researcher	Yeah, but, but, as, as a teacher, I mean, precisely because they, they aren't going to question that that line is parallel wouldn't you worry about that?
634.	Jillian	Yes. [Holly nods.]
635.	Researcher	[Soft laughter.] Because they are not going to see the reason to prove it.
636.	Jillian	Yes.
637.	Holly	That would be my biggest concern [Jillian: Absolutely.] in letting them do the, the altitude or connecting those two points is that–I think that they'll just keep jumping and they may not see that, that, "yeah, it works and that's great and…" in this, in this case it's gonna work just fine for us, but I don't want them to assume that for every other proof we do. Unless that's the goal…I mean, like, I have told my kids that if the diagram—if all the things that are marked look correct, then you ought to look for things that also look correct, because probably your diagram is fairly accurate. Where, you know, if the measurements don't look correct, then your diagram is probably not an accurate representation of what's happening. So, I guess it depends on if we want a direct proof, then we can't jump like that. But if I say, "well, sure, you know these things all look correct and that's where we're trying to get," then I wouldn't be so worried about it.

(TWP031506-62)

Here, Holly showed a negative stance toward the decision of letting students draw an auxiliary line: *Students would not understand that the operation of drawing an auxiliary line works for this problem and not for all proofs.* Holly said, "I think that they'll just keep jumping and they may not see that, that, 'yeah, it works and that's great and…' in this, **in this case** it's gonna work just fine for us, but **I don't want them to assume that for every other proof we do**." In the future, students could potentially make the mistake of drawing an auxiliary line when it is not useful to solve a problem. Students then would take for granted attributes of the diagram and consequently, would make a mistake assuming that they can draw an auxiliary line when they cannot.

Instead, Holly would want students to remember another kind of operation with a diagram: *to look for things that look correct.* Holly said, "I have told my kids that if the diagram—if all the things that are marked look correct, then **you ought to look for things that also look correct**, because probably your diagram is fairly accurate." Holly expects students to rely on visual perception at least to come up with plausible statements. Students' visual perception would guide them to use an accurate diagram to work through possible solutions to a problem. Thus, it is important for Holly that students would have an accurate diagram. Otherwise, their visual perception could lead them astray. Holly said, "if the measurements **don't look correct**, then your diagram is probably **not an accurate representation** of what's happening." Therefore, Holly asks students to combine their intuitions about the diagram with the empirical evidence students get from the diagram through measurements. In the case that the measurements did not confirm what they believe is true about a diagram, then students would need to question the accuracy of their diagram.

Holly's comments about the kind of work that she would expect students to do suggest that students would be following the *descriptive* mode of interaction with diagrams (Herbst, 2004, p. 131). In this mode of interaction, students are expected to read the signs in the diagram and use their interpretation of those signs to make statements about geometric objects. Those statements would then become part of a proof. According to Herbst, the descriptive mode of interaction "takes away from students some important aspects of the activity of proving, especially those connected with the reasoned building of a conjecture, or the use of argument to find out what could be true" (Herbst, 2004, p. 131). In contrast, the *generative* mode of interaction with diagrams is associated with students' work of making reasoned conjectures in geometry. The generative mode of interaction involves students' use of a diagram to make statements about what the relationship between geometric objects should be. Holly's comments suggest that she would not encourage students to do an operation that is characteristic of the generative mode of interaction: to add an auxiliary line. In Holly's example, the teacher carries the responsibility for teaching students the operation of decoding signs in the diagram, instead of letting the student alter the diagram on his or her own.

According to Holly, a teacher's decision to allow students to draw an auxiliary line could have detrimental effects in the future. When working on a problem, students would remember the operation of adding auxiliary lines to a diagram. However, students may not be aware of whether the auxiliary line they are adding is something that they can do only for special cases rather than for any case. It is conceivable that a teacher would remind students that they are working on a special case so as to prevent students from drawing an auxiliary line when solving other problems. However, Holly's comments at this point of the discussion did not consider this alternative. A teacher would not need to remind students that they are working with a special case or not if they did not draw an auxiliary line. Similar to Denis' example, Holly gave an example where a teacher would not ask students to give legal justifications for changing a diagram.

Further discussions in the same focus group session led Holly to articulate another reason for why it would be undesirable to add auxiliary lines without knowing how to use them to solve a problem: *The diagram would become too complicated for students to figure out the usefulness of an auxiliary line.* Holly described a case where the class is uncertain about how to solve a problem and working with the diagram could be a means for finding a solution. However, premature changes to the diagram—before one knows how these changes would help to find a solution—might hinder the possibility of using visual perception to get cues about a possible solution.

The interval lasted approximately 3:30 minutes and took place in the last half hour of the three-hour session. The interval is characterized by Holly's and Stan's long turns, without interruptions by other participants, to answer a question by the moderator. The turns where the moderator and Holly spoke read:

Turn #	**Speaker**	**Turn**
644.	Moderator	But Minnie was saying, and the way you were saying that, these are the things that you do like at the end of the class, right? Like once the kids, once the kids go and you stare at the board and they are like, "yeah, that may work." Or once you do your plan, then you say, "let me try what you did, Johnny." You know. But this was like on the spot. That was like ch—So, what, what do you think about that?
645.	Holly	I'm usually very honest with my kids. I usually say, "I'm not sure. I'm not sure where this is gonna take us." I have, I usually tell them, I have an idea of where we can go so that we will get an answer that we are looking for or we will be able to prove it. I don't know where we are going with this, or get us there or not. My concern is that we may add a lot to this, and usually they just punk out and go, "forget it then, if you don't know either I'm not gonna do it and it's too hard." [Stan: Right.] But I try, and maybe I don't, maybe it's a bad choice, but it's been my strategy in the past, just to be honest and say, "I'm just not sure." And I'd be happy to look at it in my own time and revisit it next week if you'd like that solution. But I typically won't—I don't necessarily encourage them to go in that direction, I just let them know that if they do go in that direction we

		are going al-, like all going together, as a, as a collective we are not sure what's gonna happen next.

(TWP031506-64)

Holly said that she *usually* warns her students if she is going to take on an unanticipated idea. She would say, "I **don't know** where we are going with this, or get us there or not," or she would say, "I'm just not sure." These warnings allow Holly to label as uncertain those ideas that she has not studied in depth yet. She would rather study the plausibility of a solution that uses the unanticipated idea on her own and bring up that solution to class later. Holly's comments are similar to those made by Lucille (and discussed in the previous section) in that they expect a teacher would make the decision to postpone an unanticipated idea, disregarding those ideas from the memory of the class. Additionally, a teacher would make students remember those ideas that are useful in relation to the solution the teacher already knows.

If Holly were to let students add an auxiliary line, she would make a disclaimer to the class by saying that the solution is uncertain. Yet, Holly said that adding an auxiliary line is something that she would not *usually* do. Holly said, "But I **typically** won't—I don't **necessarily** encourage them to go in that direction." Therefore, Holly claims that a teacher's *usual* decision would be to control changes to a diagram proposed by students, especially when there are not justifications for those changes.

At the moment when an unanticipated idea is presented, Holly said that she would hold all students responsible for trying out the proposed solution "as a collective," even when she is not sure about its usefulness. Since she had cautioned students about the uncertainty of that solution, the status of their work on the problem would be temporary and contingent on whether they get a solution or not. With that disclaimer, Holly said she would grant permission to the class to alter the diagram in order to pursue a possible solution.

Another reason for a teacher to filter changes to a diagram, besides the lack of justifications, is that alterations to the diagram could prevent students from finding a solution. Holly was concerned that if students add too many auxiliary lines to the diagram, they would not be able to visualize a solution. Holly said, "My concern is that **we may add a lot to this**, and usually they just **punk out and go**, 'forget it then, if you don't know either I'm not gonna do it and it's too hard'." Students could get frustrated if they were not able to solve the problem. Moreover, students could understand a teacher's uncertainty as a sign of the difficulty of the problem, and consequently, quit working on a problem if the teacher could not figure the problem out either.

According to Holly, there could be negative consequences if a teacher did not prevent students from adding things to a diagram for which they do not have a strategic justification. Students might not be able to see a solution when too many new objects had been added to a diagram. Holly said that by altering a diagram, without any purpose, it could be hard for students to find a solution that would make use of the different elements added to the diagram. So, students would judge the problem as too complex and become frustrated about it. Moreover, if students perceived that the teacher is having difficulties to solve the problem, they could conclude that it would be less likely for them to find a solution. As a result, students might disengage and quit from working on the problem. Thus, using a diagram to keep a record of all the proposed ideas for solving a problem could confuse students, because remembering all the alternatives proposed might prevent them from foreseeing a solution. This emergent story where students get confused and eventually quit from working on a problem describes what a teacher might perceive from students' actions. It is conceivable that students actual be interested by the problem, but that the teacher would interpret students' actions as confusion and disengagement from the problem. So a teacher might take students' actions as feedback to make further teaching decisions, assuming that students are confused and disengaged.

The action of inspecting the diagram is an example of an action that allows a teacher to filter changes to a diagram. By inspecting the diagram, a teacher might pace decisions to change the diagram. In other words, a teacher would not record on the diagram all of the proposed changes, but instead pause, requesting students to consider whether there are strategic justifications for the changes. Accepted changes to a diagram would become part of the collective memory of the class. Students would keep those changes in memory in order to continue to pursue the solution to a problem. In contrast, the class would forget proposed changes that were not incorporated to a diagram.

Drawing a Separate Diagram

In the previous sections, I have presented alternatives proposed by the participants to prevent students from drawing an auxiliary line to a diagram that is on the board. According to participants, another way to avoid changes to the original diagram would be to make a separate diagram where students could incorporate changes. This idea was suggested in the TWP050306 session, interval 45. The interval occurred approximately three and a half minutes after the video had been shown. In this interval, Robin dominated the discussion by modeling with gestures what she would do with the diagram on the board. Robin suggested that some of the students who could be initially convinced of the plausibility of using an auxiliary line to solve the angle bisectors problem would change their opinion when they had a chance to study a diagram that included the auxiliary line. Robin stated that the feedback that students get from the diagram could become memorable later, because they would remember that drawing an auxiliary line did not lead to a plausible solution. The transcript of the interval follows.

Turn #	Speaker	Turn
784.	Researcher 1	So then, what about these kids that wanted to draw the, the diagonal of the figure in the middle?
785.	Robin	I probably would have either gone up to the board or had the kid make another diagram and say, “okay, put them in there” because visually then, they would then see that “hey, that’s not gonna get me anywhere.” Some kids can’t visualize geometry. I think that’s one of the, the big things that you have to overcome at the beginning of the year. There are some kids that just do not visualize it, and you have to work at—I had to work at visualizing geometry. I had a rotten geometry teacher in high school and I got As because he gave us the, the, problems in the book on the test and I memorized how to do them and then spit ‘em back. So you have kids who conceptually cannot see—some kids can look at this diagram and mentally in there heads draw the diagonals and see it doesn’t work, but I’d actually, for the kids who can’t see that, have somebody, on another diagram because you don’t wanna mess this one up [gestures to paper on table], have them draw them in and then they can see visually, “hey, that doesn’t get me anywhere.”
786.	Researcher 2	So even if, so even in a case where you know something’s not gonna lead them anywhere, it’s uh, I mean, it’s okay for (them to put it on the board and…) and to follow the argument to see where you get stuck (and)…
787.	Robin	(Sure, it’s okay because, right.) (Right), because otherwise they are always going to—well some of my students are always going to think, “well that would have worked if she had let me.”
788.	Alex	(Mm-hmm.)
789.	Jenna	(Yeah…)

(TWP050306-45)

Here, Robin suggested to draw a separate diagram, different from the one that was already on the board. Robin stated that the new diagram would be helpful for students who cannot visualize that a solution with the auxiliary line would not be possible.[68] She said, "but I'd actually, for the kids who can't see that, have somebody, on **another diagram"** (turn 785). The second diagram would allow students to examine the alternative of drawing an auxiliary line, without changing the original diagram. Robin had negative reactions towards changing the original diagram when she said, "because you don't wanna **mess** this one [the original diagram] **up**" (turn 785). The choice of the word "mess up" underscores the negative consequences of changing the diagram. One could infer that the second diagram would have a temporary status, because it was drawn with the purpose of trying out a different solution.

Another reason for drawing a second diagram, without altering the original one, is to make it memorable for students that the proposed solution with an auxiliary line does not work. First, Robin would use the new diagram to convince students that the proposed solution should be discarded. When she said, "visually then, they would then see that 'hey, that's not gonna get me anywhere'" (turn 785), she was counting on students to realize that the auxiliary line does not lead them towards solving the problem. Secondly, Robin would expect students to remember later that the proposed idea of drawing an auxiliary line did not lead to a solution. Robin said, "because otherwise they [the students] are always going to—well some of my students are always going to think, 'well that would have worked if she [the teacher] had let me'" (turn 787). By impersonating the voice of a student in disbelief that the proposed solution works, Robin suggested that

[68] Here, Robin referred to the diagonal of the original rectangle as the auxiliary line that was not helpful in solving the problem (see Figure 30). Later in the session, there is evidence of Robin referring to the diagonals of the rectangle as the auxiliary line. This auxiliary line is different from the one proposed by Jackie and other students in the video, which was the diagonal of the figure made by the angle bisectors of a rectangle.

some students need a visual confirmation before discarding a proposed solution. A teacher would make it memorable that the proposed solution does not work by using a visual image.

There are two evaluative stances pertaining to students' work with the diagram to conceive of a solution to the angle bisectors problem: *probability* and *comprehensibility*. One could translate the probability statement as: *It is possible that students will understand the solution*. The statement regarding the comprehensibility of a solution could be written as: *It is understandable that the solution is incorrect once one sees the diagram.* The two evaluative stances complement each other because the diagram becomes the means by which students—especially those students who cannot visualize a diagram unless they see it—make sense of a particular solution to the problem. Once students get to see the diagram, then they can evaluate the solution as incorrect.

Further discussion in the focus group session led Robin to clarify that she anticipated students to be confused with which diagonal to draw (TWP050306-46). Even though in the video episode Jackie had been referring to the diagonal of the figure made by the angle bisectors, Robin noticed that other students could have been considering the diagonal of the rectangle. According to Robin, a teacher should clarify what geometrical objects they are referring to when there is some ambiguity. Also, by drawing the diagonal of the rectangle on the board, students would have noticed that this auxiliary line would not lead them to solve the problem.

The transcript from TWP050306-46, where Robin talked about students' possible understanding of the solution, follows. In this interval, participants looked at handouts of possible students solutions and gave their reactions. Robin and Jenna dominated this interval, commenting about each other's ideas.

Turn #	**Speaker**	**Turn**
790.	Researcher 1	(Well), in, in solutions, uh C and D, those are solutions were, that were based on actually drawing that line, uh -- of course there may be something wrong with this, we just transcribed work that was done by students, but um…the…you know the…there is -- they seem to have gotten some mileage out of that idea right?
791.	Robin	But they were talking about drawing diagonals. Which diagonal are you talking about? See (when I first heard)--
792.	Researcher 1	(*X*, *XC*)--
793.	Researcher 2	(The diagonal of the…])
794.	Robin	Well I understand what you're saying [to Researcher 1] but to the, to the kids in the class, when somebody says, "draw diagonals" they're gonna think from *B* to *D* and from *A* to *C*. (Very first thing.)
795.	Researcher 1	(Well that, what) Jackie, Jackie outlined, the one in the middle, you know, uh, with, she (chose the [draw])--
796.	Robin	(Right, I understand), but there are gonna be kids when originally somebody says, "draw the diagonals", in their head, (they're gonna use) *B* and *D* and *A* and *C*. (Their gonna draw) the diagonals of the original figure. I want somebody draw that on the board and let them see that that diagonal, those diagonals are gonna do you zero good.
797.	Researcher 1	(Oh I see.) Okay. (I see.) [Inaudible] angle?)

(TWP050306-46)

Here, one of the researchers pointed to some samples of students' work. These included solutions for the problem of the angle bisectors of a rectangle using an auxiliary line (see Figure 29). The researcher was referring to the diagonal of the quadrilateral made by the angle bisectors (line *XZ* in Figure 29).

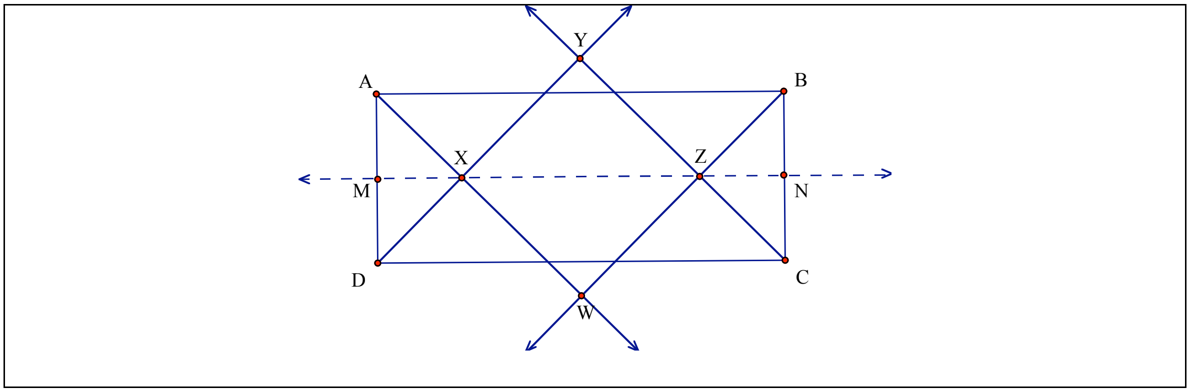

Figure 29. A rectangle, its angle bisectors, and an auxiliary line.

However, Robin worried that other students would not consider that diagonal. Instead, students would assume that the diagonals of the rectangle (*AC* and *BD* in Figure 30) would be helpful to prove that the angle bisectors of a rectangle make a square. Therefore, if a teacher wanted students to make a proposed solution memorable, the teacher should confirm that they all agree upon the visual representation that corresponds to that proposed solution.[69]

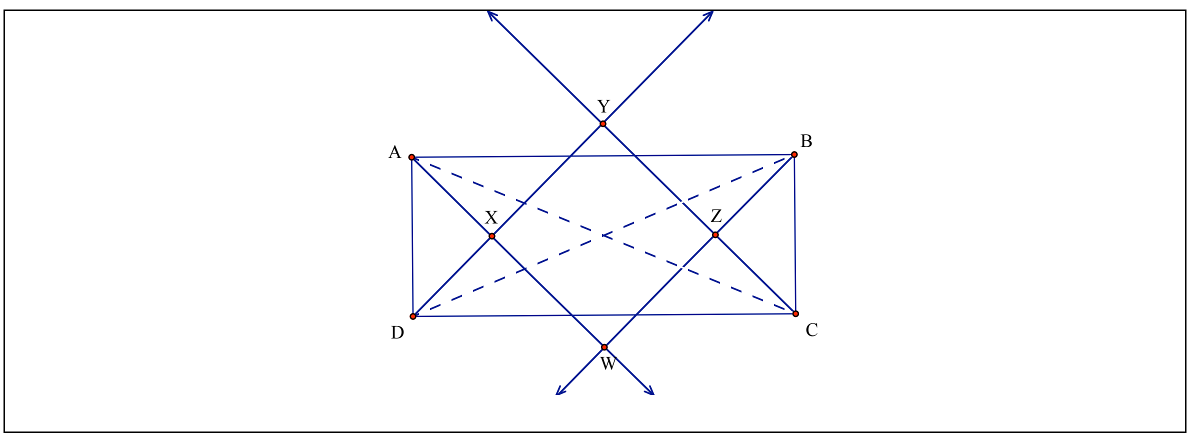

Figure 30. The auxiliary lines, *AC* and *BD,* are the diagonals of rectangle *ABCD.*

[69] It is possible to do a proof for the claim that the angle bisectors of a rectangle make a square using the diagonals of the rectangle by making an argument relying on symmetry. However, this option was not discussed in the focus group session.

From the participants' perspective, by drawing an alternative diagram where students could incorporate changes while leaving the original diagram intact, a teacher would accomplish two goals. One goal is to have students explore alternative solutions without changing the path already taken in solving the problem. If the alternative solution did not work, students could discard the alternative solution proposed. Then, students would continue to work on the original problem, which would have been kept intact within the memory of the class because there would not have been any changes to the original diagram. That is, by leaving the original diagram intact a teacher might keep a record in the collective memory of the class. So, according to participants, teachers could use the original diagram to keep a record of the class' work with the problem before exploring an alternative solution.

Another goal for drawing a new diagram would be to make memorable the instances in which one tried an alternative that did not work. By trying out an alternative solution using a separate diagram, teachers could attend to those students who are skeptical about rejecting a solution without trying it out. According to participants, those students would have a visual image to remind them later that the proposed solution did not work. Once there was evidence that a proposed solution had not worked, the class could forget this detour and come back to the original problem.

Organizing a Set of Diagrams on the Board

So far I have presented three actions to use a diagram to keep a memory of students' work in the future: altering a diagram, inspecting the diagram, and drawing a separate diagram. With these actions, teachers make students remember something, but they also make them forget particular solutions that the teacher does not want to be part of the collective memory of the class. Another teaching move for making students forget an alternative solution involves the spatial arrangement of diagrams on the board. Jenna took a long turn of speech to talk about this teaching move at the end of TWP050306-46 (the same interval where Robin drawing a separate diagram, mentioned earlier). Jenna suggested that drawing a diagram for a proposed solution on the side, next to the original diagram, would cue students that their proposed solution is incorrect. Jenna described how she uses the spatial organization of diagrams on the board to make students realize which solutions are plausible and which are not. The transcript of Jenna's turn follows.

Turn #	**Speaker**	**Turn**
798.	Jenna	(So y'know) like that teacher said, she said, "well what diagonals are you talking about, what (are you talking about there?)" And, so, you could say, "oh, maybe, why don't you draw that off to the side and we'll see if we're gonna go somewhere" (and then) transfer it on there. Course, then, you are sort of—that's always a question for me, to, whether to do that, and pull it off to the side or not because I think it's a big clue for students to think, "it's not going anywhere, that's why she's having us draw it off to the side." 'Cause then we're never gonna use it." So maybe let 'em draw it in there and then, erase it and redraw. Y'know but I, um…or use a basic overhead, y'know, put it on an overhead, project it, y'know, have the y'know, the basic, like this [holds up paper] on there and put another one over top and let 'em draw whatever they want on it and then you can say, "okay, does that look like it's going somewhere? No? Well okay, let's take that off and put …" Y'know. But, um…but overall yeah, I think it's very important because they, they hold onto something, unless you have thoroughly proven to them, it doesn't work, sometimes they'll hold on to it, that idea even if you say, "no, let's go in this direction, let's try this." If that's—they have that in their head, that, like you said, that that was gonna work, and you didn't let 'em go there somehow to prove that it didn't, it's always in the back of their mind.

(TWP050306-46)

By controlling the organization of diagrams on the board, a teacher could cue students about what they should remember when working on a problem. Jenna's decision to draw an alternative diagram to the side of the one that already existed is intended to have students question the value of the alternative diagram. She enacted what students could possibly think when she said, "I think it's a big clue for students to think, ' it's not going anywhere, that's why she's having us draw it off to the side.'" Students would realize that if she were to endorse the new idea, she would have incorporated this idea into the existing diagram. Otherwise, those students might remember that another solution was possible. Jenna said, "it's [the proposed solution] always in the back of their mind." Therefore, by drawing an alternative diagram, those students who might be skeptical about discarding the proposed solution could be convinced that that solution does not work and would not bring up that solution for the class to remember it.

In contrast with Robin's and Jenna's suggestions that the diagram is a means for students to realize that a solution does not work, a teacher could also use a diagram as a means showing that or finding out whether that a solution actually works. Participants in the TWP040505 session commented that Ms. Keating had requested students to give a rationale in order to add the auxiliary line. The moderator asked whether students' work on a diagram could lead them to figure out something that they did not know at the moment. Mara and Alice agreed to let students add the auxiliary line in. However, participants did not elaborate on this alternative. By the end of this interval, Nathan commented that he would let a student draw an auxiliary line in until students reached the point where they could not provide a rationale for altering the diagram.

In a previous section I already presented parts of TWP040505-53. In what follows, I present the full transcript of the interval, which lasted approximately 3:45 minutes.

Turn #	Speaker	Turn
420.	Researcher	I was taken by um, by Alice's comment that, well, if you got these two diagonals to be perpendicular, you already would have a square. [Alice: Well.] Seems like a, like an interesting reasoning on the fly that the teacher could use to, to decide, "should I go on with this?" You know.
421.	Alice	Well, and that’s because we already, they had already discovered that the- all the opposite angles were 90-degrees. We knew it was a rectangle but now if we can get those diagonals to be perpendicular, we’ve got a square. So, um, I just—I was kinda disappointed as I said, that it didn’t go, that they didn’t take that step.
422.	Researcher	And I guess what you were saying now is that you might not actually have that thought (at the moment).
423.	Lucille	(If you didn’t see) it at the time and you’re, and you’re trying to direct it to finish up at a certain time, and someone brings it up instead of engaging that or saying “here's case one in terms of description, here’s case two, in terms of the description of what’s going on.” You might just say, “Oh, okay, but let’s go back to this.” You might redirect it just because that’s what you’re ready to or, you know, prepared to discuss.
424.	Researcher	So you wouldn’t know that the comment is relevant at all?
425.	Lucille	You might not.
426.	Researcher	If you didn’t have that, like that’s, that's how it’s relevant. Because if I can get that the two of them are perpendicular then, I get a square?
427.	Moderator	So--
428.	Mara	But, she did ask him [Moderator: Yeah.] that. She said, "okay if you put that diagonal- you put a segment in there if you have a reason for- what are you going to do with it?" And she did not really have a reason [Unidentifiable speaker: That’s true.] [Alice: Right, right.] she was just sticking it in there.
429.	Moderator	(But would) you scribble stuff?
430.	Researcher	(So if…)
431.	Moderator	Would you encourage students to scribble stuff and maybe you would come up with the reason eventually?
432.	Alice	Right.
433.	Mara	Yeah, I would, yeah.
434.	Researcher	So, you would rather have, had them, “yeah, draw it,” and let’s see if somebody else has a reason?
435.	Mara	Right.
436.	Researcher	Now, if, if you would have known this is useful to do, but the student didn’t come up with a reason why to do it, would you still say, “well, if you don’t have a reason don’t do it.”

437.	Nathan	(4 sec pause) I think I would always go, go, go until, you know, once again, it's within reason, it's time, it's whatever, but I would always go until somebody else said, "yes or no." Ss- said, "Okay, I see where this is going." Or said, "ok…" And you know, after some wait-time. Nobody said anything about what that segment might do, then, "so, okay, so let's not do that."
438.	Researcher	On the other hand, somebody who really was interested in going orderly towards the argument, and had seen the, the segment addition argument is quite compelling, actually. So you would say, "Well, this is-this isn't really relevant for segment addition." So, if I go there, then they may get lost, you know, they get, you know, outside of the focus that what I want them to do is to compare this segment with this segment.
439.	Lucille	But again it goes in what did they have previous knowledge of. If they didn't have the previous knowledge, that to be a rhombi that the diagonals would, um, intersect at a perpendicular angle, and they, what- and then what they knew was similar tr-, er, congruent triangles, and segment addition, and all of that, then I could see-I mean that would be a good thing to pull out...later. So, again, I think it's "where you are starting from?"
440.	Moderator	So could you use this as an excuse and then say later, "Oh, what if we had this."
441.	Lucille	Could you say that again?
442.	Moderator	Do you use this problem as an excuse to get [Lucille: Something else] that property of rhombi.
443.	Researcher	Like in the spirit of saying, if I was going to use this unit to teach quadrilaterals--
444.	Lucille	Yes, so, it opens the door for that discussion. Sure.

(TWP040505-53)

I claim that teachers have to manage a tension between using a diagram on the board as an object for the class to think with and using a diagram on the board to present a solution that is previously known by the teacher. On the one hand, Alice considered Jackie's initiative to draw an auxiliary line desirable and was disappointed that Ms. Keating did not follow through this idea (turn 421). Alice and Mara also said that they would encourage students to add things to a diagram, expecting students to come up with a reason for making those changes to a diagram later (turns 432 and 433). On the other hand, Lucille stated her preference to choose a solution that a teacher knows beforehand.

Lucille said, "You might just say, 'Oh, okay, but let's go back to this [a known solution].' You might redirect it just because that's [a known solution] what you're ready to or, you know, prepared to discuss" (turn 423). So, as Lucille animated the voice of a teacher who has to face an alternative for which he or she is not ready for, Lucille exposed the apparent tendency for a teacher to ignore an unexpected solution and to reconsider the original diagram.

One way for a teacher to deal with a diagram on the board is to add new elements to the diagram only when these are useful towards solving a problem and when students can justify the usefulness of making a change. Mara recalled a moment in the video when Ms. Keating had asked Jackie if there was a reason for adding an auxiliary line, but Jackie had not been able to provide a justification for changing the diagram (turn 428). Mara posed a critique to Jackie when she said, "And she did not really have a reason, she was **just sticking** it [the auxiliary line] in there." Mara contrasted Jackie's decision to add an auxiliary line by "just sticking it in there," with the process of having a justification or a "reason" for altering the diagram. From Mara's comment one could infer that it would be *undesirable* for a teacher to change a diagram that is on the board without justifying the usefulness of the change for finding a solution.

A class usually takes the diagram on the board as contributing to the official solution to the problem—as what should be remembered. Similar to Lucille's, Nathan's reaction to the video of Ms. Keating's class shows a teacher's responsibility to filter plausible solutions (turn 437). In particular, a teacher would ask questions to students that could justify a change to the diagram on the board at the moment when that change is proposed. Nathan stated three conditions for letting students alter a diagram: (a) if students' suggestions are "within reason," (b) if there is enough class time, and (c) "until somebody else said, 'yes or no'" (turn 437). These conditions proposed by Nathan suggest that a teacher would put to test a student's proposal to change a diagram, reluctant to incorporate immediately any alterations to the diagram. Finally, if nobody in class could justify the proposed changes to the diagram, a teacher would not let students change the diagram. Nathan's description illustrates a case where the teacher controls the diagram and makes the decision to incorporate proposed changes at a particular moment, if those changes are justified.

According to participants, teachers might also make use of the diagram on the board to keep a record of students' ideas as they work through a problem. In an interval where participants discussed strategies for dealing with overlapping diagrams, Nathan suggested that Ms. Keating could have made a separate diagram, taking the overlapping pieces apart, instead of providing an oral summary of the solution. Nathan noticed that Ms. Keating clarified what had been said about overlapping triangles in the figure by pointing to specific parts of the diagram. However, he would have used a new diagram, with the same labels as the old one. Then, he would have asked students to mark known things in the new diagram, keeping a record of what they had discovered thus far.

The interval is approximately two and a half minutes long. The moderator asked a question in the last 15 minutes of the session and continued to ask questions throughout. The transcript of the interval where Nathan made his comments follows.

Turn #	**Speaker**	**Turn**
497.	Moderator	I have a question. It's something different. Because, um, some of the old, not old, but, textbooks there is a section of overlapping triangles. And they talk about, well if you have a figure with lots of triangles you should separate them, or draw them, at some point it's,

		you know, you have this triangle here, this other one overlapping. And it was hard to see, you know, which triangle are we talking about. I'm wondering how do you deal with this, when that happens in the classroom, what do you do, in that situation when you have all these triangles overlapping.
498.	Lucille	Using different colored chalk or markers… Just to distinguish.
499.	Alice	I pull them apart as well.
500.	Moderator	Do you draw them separately?
501.	Alice	I've done both. I've done the markers. If it's not too challenging. But if it's a challenging drawing I pull them, the triangles apart.
502.	Moderator	Do you think this was a problem here of having those, that diagram of those overlapping…?
503.	Nathan	It was for a moment but, but because she literally kind of said, you know, "this," she explained [inaudible] to the class however many were listening [inaudible], but she used her hand and said [draws with hands and fingers in the air] "this is this and this is an isosceles triangle, and this is that same triangle, and there's these-" she used the term "little pieces," or some kid in the class did. But, I think that they got that.
504.	Moderator	So that worked out.
505.	Nathan	Yeah. Overall, I say, separate them too and I guess when I'm working one on one with kids in my class, it depends on the kid. Like if a kid is just hitting the wall, I say, "listen, you need to take this triangle, *ABC* or whatever, draw that, and draw this one, and label everything you know." Dive in that way.

(TWP040505-59)

Here, Nathan said that there was a moment in the video when students were confused because of the overlapping parts of the diagram. According to Nathan, Ms. Keating had given an oral explanation in which she matched different parts of the diagram with claims that had been made about it (turn 503). Ms. Keating made a summary of the argument made to solve the problem, by pointing to relevant parts of the diagram. Ms. Keating's oral explanation regarding what they had discovered that far about those overlapping parts could have helped students who were confused. However, Nathan said that there are times when oral explanations are not enough for students. From Nathan's comments one could conclude that at those moments it would be *desirable* for a teacher to separate a diagram with overlapping parts into different diagrams, with labels that would point to parts that are common to both diagrams (turn 505). This teaching move of separating a diagram with overlapping parts into different diagrams would especially helpful when students were having difficulties with a problem. Nathan said, "Like if a kid is just hitting the wall, I say, 'listen, you need to take this triangle, *ABC* or whatever, draw that, and draw this one, and label everything you know'" (turn 505). Nathan's request for a frustrated student to label everything known, suggests the intention of using a diagram to record prior work that is useful in finding a solution. The labels and the markings would keep this prior work in memory, even when doing a separate diagram.[70]

Summary

In this section I have presented teachers' actions with diagrams as examples of those actions that teachers do to organize what students should remember in the future when students are working on a task. The diagram at the board is a contested place where teachers and students transact the work that they do on a problem. The analysis above shows that teachers perceive it as a contested place as well. Teachers hold themselves responsible for taking control of the diagram and for filtering any proposed changes. A teacher uses a diagram to pursue a particular solution which is usually known to the teacher. The diagram is a way to represent those solutions that are worthy of becoming part of the collective memory of the class. However, timely actions to a diagram could have the dual role of recording past claims about the diagram and also of

[70] Nathan did not comment on the difficulty to record the sequence of markings in a diagram with the labels and the markings. We have gathered and examined evidence from geometry teachers reacting to this difficulty in a different set of study group sessions (González & Herbst, 2008).

making students remember those claims in the future. In that sense, the diagram becomes a place where to record the ongoing work on a problem, as the class moves towards finding a solution.

According to participants, teachers also exercise control on the diagram when they disregard those solutions that students ought to forget. In particular, a diagram could be the means by which a teacher makes memorable the moment at which they arrive at an incorrect solution that they should exclude from the memory of the class. A teacher's decision to control the spatial organization of diagrams on the board could be timely; by controlling when to draw a separate diagram a teacher could suggest to students whether a solution is correct or not.

The strategy of adding an auxiliary line could be a useful operation for solving other problems in the future. However, students may not be aware of the conditions that are necessary to draw an auxiliary line. According to the participants, a teacher who allowed students to draw auxiliary lines for solving a problem could run the risk of having students reproduce this operation in the future, for other problems for which it would not be strategic to draw auxiliary lines or for which they would not be legally entitled to do so. Thus, participants perceive that they must be aware of the consequences of letting students perform some operations with diagrams because students may remember these operations in the future, even when these operations are not applicable.

If students suggest changes to a diagram, they are accountable for giving an overview of what the solution of the problem would be. The teacher (or the students) would assess that bid to solve the problem before making any changes to the diagram. A teacher would assign value to students' ideas for making changes to the diagram according to the prior knowledge that students would draw upon in their proposed solution. Moreover, the new solution would be deemed valuable according to the work that has been done for solving the problem so far. A teacher would need to consider how the proposed change to the diagram adds to what they already know about solving this problem at that particular moment. If the teacher cannot anticipate the value of a proposed change to the diagram, then the teacher would lead the class to work on a solution that is already known by the teacher. However, if the teacher decides to add features to the diagram prematurely, without having a justification for how this change would help solve the problem, the teacher could alter significantly the diagram of the original problem. By using an altered diagram, it would be difficult for the teacher to trace back their work prior to the moment when they considered an alternative solution, because there are no records that would keep prior work within the memory of the class.

According to participants, a teacher has the responsibility of organizing students' ideas as they work through the problem, so that they can build upon each other's insights. The diagram at the board could provide the means for teachers to summarize the work they have done so far. At times, when the diagram is too complicated, teachers may need to use other resources to clarify students' ideas. For example, a teacher could provide oral explanations that take parts of the diagram one by one; or the teacher could draw new diagrams that separate overlapping pieces into different parts. These moves scaffold students' work on a problem by helping them keep in memory what they have already solved, and put in perspective those steps ahead for solving the problem. These moves can be timely, because the teacher might decide to make changes to a diagram in response to students' confusion when the diagram is too complex. Consequently, a teacher's work with diagrams would be helpful to keep a record of what has been discussed so far within the collective memory of the class.

From my analysis, I have found that in the didactical contract of the geometry class, a teacher is responsible for organizing the knowledge that students should remember in the future. Students are not responsible for deciding what they should remember on their own. On the contrary, teachers assume the responsibility of shaping what students should know, and in doing so, teachers hold themselves accountable to shape the collective memory of the class. As students work on a problem, teachers expect to prevent students from remembering ideas that are not supposed to be memorable, with actions that aim at cueing students to think that those ideas are not valuable. For example, if there was an alternative solution with a diagram that the teacher does not endorse, he or she might draw a separate diagram. With this action, teachers think that students would get the hint that the alternative solution is not as valuable as the one under consideration. So, instead of incorporating a solution to the diagram that is already on the board, a teacher who draws a separate diagram expects to send the message to his or her students that the new solution is something they could forget. It is possible for a teacher to do some other actions rather than to draw a separate diagram. One could conceive of a case where the teacher incorporated all ideas proposed to the diagram that is already on the board and let students decide what to remember and what to forget. In this case, students would be responsible for their memories. However, it seems as if the control of what should be memorable and what should be disregarded is something that a teacher assumes responsibility to do.

A teacher's decisions regarding whether to incorporate or not changes to a diagram can be timely. Teachers could consider what students suggest, and could use those suggestions to decide whether or not to change the diagram at that moment. One could conceive that decisions about changes to the diagram are strategic in that a teacher had already studied many possibilities for altering the diagram that would give a substantially different solution to the problem. However, it seems as if teachers had already selected a preferred solution and in that selection there is a strategic decision. Then, the decision whether to consider or not other alternatives proposed by students, is a tactical one, taking each of those suggestions as a moment of opportunity to change what the class should remember about the problem in the future.

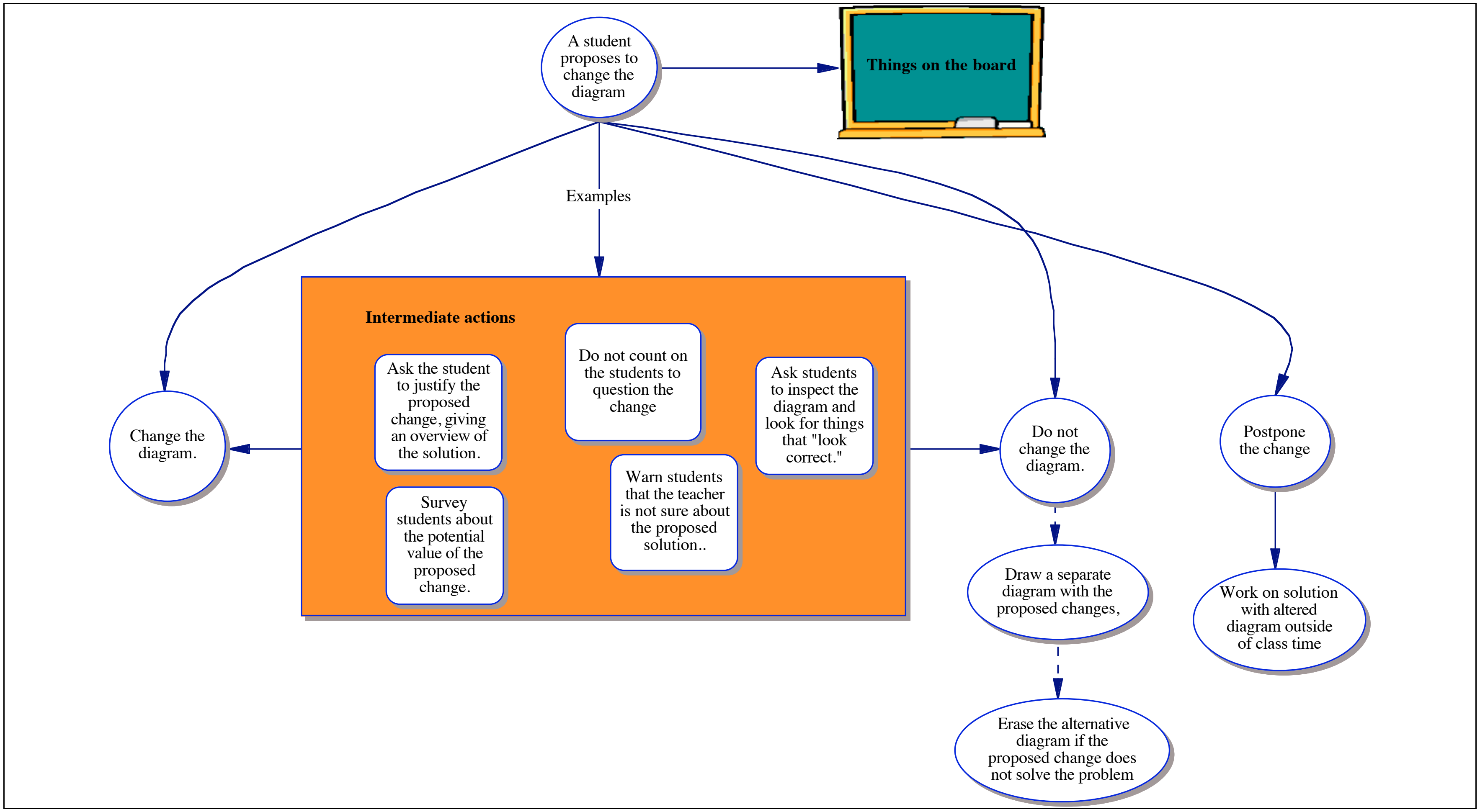

Figure 31. Teaching actions make students remember or forget proposed changes to a diagram.

Figure 31 summarizes actions that a teacher could do to handle a suggestion of changing a diagram. According to the participants, there are actions that aim at making students remember the altered diagram in the future, whereas there are other actions that aim at making students forget the suggestion or the altered diagram. For example, a teacher could draw a separate diagram with the proposed change, and keep the memory of the original diagram intact. Alternatively, a teacher could postpone the change until he or she works on a solution using the proposed change. Figure 30 also shows other intermediate actions that the participants mentioned for a teacher to decide whether to make the change or not. These intermediate actions include surveying the class, warning students that she is not sure about the solution, asking students to inspect the diagram, not counting on students to question the change, and asking students to justify the change by giving an overview of the solution. With intermediate actions, a teacher uses the feedback from students to make other decisions about what should be memorable when students are working on a mathematical task. Since, for the most part, teachers keep control of the things on the board then teachers assume the responsibility for filtering what is memorable and what is not. Things on the board are a longer lasting record than spoken words. Things on the board may stay for a while, keeping a memory of what has been said or done. A diagram is one of the things on the board that a teacher can manipulate in order to make students remember or forget something.

Teachers' Actions to Control What Should Be Memorable After Students' Work on a Task Has Ended

In this section I present actions that teachers hold themselves accountable to do in order to make students remember something from their work on a problem in the future. At the same time, there are other actions that teachers do in order to exclude students' memories, making them forget about things and events that are not perceived as valuable by teachers. From my analysis of the conversations in focus group sessions, I identify two examples of such actions: a teacher's work naming those theorems that should be remembered and a teacher's decision to prevent students from remembering a diagram that represents something incorrect. While the first activity illustrates teachers' actions geared towards making something memorable, the second activity shows teaching moves towards preventing students from remembering something that happened in their class. The contrast between these two activities is of interest here in examining how teachers control the collective memory. When students achieve the goal of a mathematical task, a teacher has to sieve out things and event that should be memorable from that work. I show the work of the teacher that, according to the participants, is involved in shaping future memories.

Naming Theorems that Should Be Memorable

The issue of naming theorems surfaced in 2 out of a total of 313 intervals in all focus group sessions. The act of naming theorems is important because the names chosen could suggest how valuable the work being done is, and in particular, the name might identify what about the work ought to be remembered. For example, within the situation of *installing theorems*, Herbst and Nachlieli (2007) found that it is important to give a name to theorems because these names will be cited later in the "reasons" column as a justification for a statement when doing a proof.

I show data from the focus group sessions that illustrates different possibilities that participants considered on how to name theorems, including whether to name theorems after students' names or by using standard names as in geometry textbooks.

Teachers would argue that by coining idiosyncratic names, theorems and the events surrounding the act of naming that theorem could be more memorable for particular students or for their own classes. On the other hand, standard names could have added value outside a specific geometry class, because students would be able to remember things that they studied in the geometry course in future mathematics classes. Both perspectives underscore the importance of naming theorems when introducing new knowledge into the collective memory of the class.

TWP052405-37 illustrates the kinds of discussions held around the act of naming theorems. In this interval, Lynne and Karen debated about the value of names for theorems. Lynne worried that by assigning idiosyncratic names to theorems, following the names of students in a class, students would not recognize those theorems later as the ones they have already studied. By naming theorems after students in a class, the names of the theorems would be too particular to students' experiences in their own geometry class and disconnected from the way that knowledge is kept in the discipline. Yet, Karen wanted to use the naming of theorems as an opportunity to teach students about a greater mathematical practice. According to Karen, part of the work of doing mathematics involves assigning names to propositions and some names are arbitrary.

The interval started approximately an hour and a half into the session and it lasted approximately three and a half minutes. The activity that characterizes this interval is Lynne’s and Karen’s reaction to a slide presented by the moderator. The moderator had altered the order of his slideshow presentation to share two quotations, one was called the "G-Bay quote." The "G-Bay quote" was based upon a comment from a teacher who participated in a prior *Teaching with Problems* session. This participant had shared an anecdote about the value of producing new knowledge in his class (see Figure 32). The quote makes references to "e-Bay," an American website which allows users to put goods up for auction. Instead of using an "e," the quotation uses "G" to denote "Geometry."

What's good about this problem?
"We could use a problem like this for students to develop and prove their own original theorems, instead of just proving things that are already known. I always tell them — this is a little joke we have — that if we come up with some new properties, we can sell them on G-Bay." — a participant in the previous focus group session

Figure 32. The "G-Bay" quote.

The transcript of the interval where Lynne and Karen held this discussion follows.

Turn #	Speaker	Turn
386.	Moderator	Okay, so I wanted to show you two quotes now, from previous participants in focus groups like this one who saw kind of sampler and worked on the same - um we saw that one already, hold on…that's what happens when you change the order in the middle…there you go! So this is a participant from a previous session we did a couple of months ago. And I think we have heard similar comments already from the people in this group that this is an opportunity for students to come up with something on their own, rather than just prove something that other people already know about.
387.	Nicola	[Laughs.]
388.	Moderator	The G-bay joke is the one that I stole for my classroom. Would anybody like to add to this or take away anything from this? I mean, how closely is this a reflection of the, of your feelings. Are there opinions that you would disagree with in this or would like to add on to it?

389.		[Silence.]
390.	Moderator	People are comfortable with this quote?
391.	Lynne	Part of the quotation that scares me is something that I saw in other mathematics classrooms where the kids came up with their own distinct mathematical language and there was Jack's theorem and Sarah's theorem and that little part about prove their own original theorems, not that any of those were bad but then they had a totally different language when they approach a textbook, or a student from a different classroom, or a different school, or move on to a different higher level math class they are not going to have their right vocabulary. So, that's one of the things that jumped out at me about that quotation.
392.	Moderator	So you want to make sure that the language that their speaking is--
393.	Lynne	Somehow ties back into the mathematical language that's accepted.
394.	Moderator	Are there other worries along the same lines, issues of sort of meshing up with you know, what the book says and what other schools do and make sure that you are not completely off the map?
395.	Karen	Well, I think that, that the languages morph over time anyway. I mean, that things, you know, the way I was taught geometry, thirty-some years ago, this is, is not the way, doesn't have the same languages in it that we have now. And I think it is an important part of the language, of the language of math, for kids to understand, this is something that someone made up. And it's a constructed language. So, to a certain extent I'd like to have them make up their names for properties. And then as we go through the year we get to the place where we know which ones we made up and which ones are the official names. But because it's that, being part of this adventure of creating language from our, from reason rather than just by chance, people talking and umph whatever comes up as a word. But [inaudible] them to be able to say, "we want like to call it that way because it makes sense that way." I like that and I like um then, you know, then we use both words for it.

(TWP052405-37)

Lynne voiced her opinion after several invitations from the moderator for teachers to speak up. Lynne prefaced her comment by stating that there was something "scary" about the "G-Bay quote." Lynne had a negative stance towards the practice of letting students coin names for theorems after students' names. She distanced herself from this practice by saying that this is something she has seen in other classrooms. In order to assuage her critique to the practice of naming theorems after students' names, Lynne said, “not that any of those were bad." Then, she used "but" to state what she considered to be troublesome from this practice: It is possible that students would not get to recognize these theorems later in other contexts besides their own geometry class (turn 391).

Lynne's comment about tying students' vocabulary "to the mathematical language that’s accepted" suggests that a teacher should utilize the same names for theorems that are used in the mathematics curriculum, so that these theorems become recognizable and usable in the future. For Lynne, having the "right vocabulary" would allow students to communicate with others beyond their geometry class. The theorems from the geometry class would then become more marketable. Theorems from the geometry class would appear to be useful in other experiences outside of the geometry class. Therefore, names could cue students to remember what they have learned in their geometry class, increasing the value of the theorems learnt in class.

In response to Lynne, Karen argued in favor of naming theorems after students in a class (turn 395). Karen stated that many students are not aware of the arbitrariness of names in mathematics. She suggested that the experience of letting students make up names for theorems could open up their opportunities to learn how mathematical language is constructed in practice. However, Karen did not disregard Lynne's idea of giving a standard name to theorems. Karen said, "And then as we go through the year we get to the point where we get to the official name." So, according to Karen, a teacher should eventually use canonical names for theorems. One could expect that students would ultimately be responsible for remembering the canonical names for theorems, instead of the idiosyncratic names. Therefore, the underlying norm that one could conclude from Karen's and Lynne's responses to the prompts given in the session is that a teacher should give an official name to a theorem.

Even though Lynne and Karen held different perspectives as to when to give official names to theorems—immediately after presenting a new theorem or later on in the year—they agreed that a teacher should use standard names for theorems. In the future, they expect that students remember what they have learned in their geometry class. According to participants, by naming theorems with the same names used for theorems in the mathematics curriculum (and not after particular students' names), teachers would enable students to make use of the theorems learnt in the geometry class in other contexts besides the geometry class. Thus, the value of students' memories would go beyond the time boundaries of the geometry course.

Names could be a way for a teacher to organize the new knowledge produced by the class. Names for theorems encode some sort of classification because the name of a theorem could cue students about the work that has already been done. Textbooks include names for some theorems such as "Isosceles Triangle Theorem" (Clemens, O'Daffer, Cooney, & Dossey, 1992, p. 177), "Exterior Angle Theorem" (Boyd et al., 1998, p. 190), or "Triangle Proportionality Theorem" (Jurgensen, Brown, & Jurgensen, 1994, p. 269). In the case of theorems for which there is not a standard name in textbooks, students would need to give out the whole statement of the theorem or some sort of representation of the theorem. The name of a theorem, especially when it is an unconventional name, emphasizes some ideas and hides others. For example, it is not obvious for the name "base angles theorem" that the theorem applies only to isosceles triangles; from the name, it is not obvious to remember how to prove it. Teachers could have students remember a theorem by having them write an iconic representation of the statement of the theorem.[71] Another unconventional name is students' names. However, by using students' names to label theorems, there is not a way to relate students' work with theorems in textbooks. Thus, the value of students' work does not cash into a recognizable statement of a theorem.

In a geometry class, theorems' names are mostly used when doing proofs. The name of a theorem would appear in the "reasons" column, backing up assertions about a figure in the "statements" column. So, for example, one may claim that a pair of angles is congruent in the "statements" column." Then, to the right of that statement, there will be a justification in the reasons column, such as "base angles theorem." Without a name (or an icon), the column with reasons of a theorem would be empty. A theorem named after a student's name would obscure the mathematical justification underlying a particular claim in the proof. Students would have to remember on their own that a particular theorem named after a student is related to a particular statement in the proof. In contrast, a theorem with a name that points to the concepts involved in the proof would give pointers to help students remember.

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[71] Within the data corpus of videos of geometry teaching collected by GRIP, we have examples of teachers who allow students to summarize the statement of a theorem using iconic representations in a two-column proof. For example, to denote that the base angles of an isosceles triangle are congruent, teachers let students write a diagram of a triangle with markings for two congruent sides, an arrow pointing to the right ($\mapsto$), and a diagram of the same isosceles triangle with markings to denote that the two base angles are congruent. Another example is the use of letters of the alphabet, "Z," "C," and "F" to make students remember theorems about pairs of angles congruent made by a line that intersects two parallel lines (Herbst, Hsu, Chen, González, & Jeppsen, 2007).

Preventing Students from Remembering Diagrams that Represent Something Incorrect

In the focus group sessions participants discussed the potential usefulness of "wrong" diagrams in teaching with a problem. Participants' comments about the possibility that students might be contemplating a diagram that represents something incorrect are examples of how a teacher might help students encode memories for the future. In general, participants revealed concerns about losing control for what students could remember. They opposed the idea of presenting a diagram that represents something incorrect to their class. Participants said that it is likely that students would get to remember things that are incorrect if these were emphasized in class, even if they were corrected later.

I looked for intervals where participants commented on the possibility of presenting a diagram that represents something incorrect with the prompt of the angle bisectors of a kite. There were 6 such intervals. I selected TWP052405-60. In the last half hour of the three-hour session, the moderator had distributed a handout with possible drawings to represent the angle bisectors of a kite (Figure 33). The handout complicates the possibility of choosing the correct alternative that the angle bisectors of a kite are concurrent (option #1) by presenting other alternatives where the angle bisectors are not concurrent. The angle bisectors of a kite present an interesting case where two of the angle bisectors coincide with a diagonal. Three of the diagrams in the handout have a diagonal as the angle bisector, whereas one diagram shows incorrectly that the four angle bisectors are not the diagonals (option #3). One of the diagrams does not show the intersection points of all angle bisectors (option #4), so students who do not take the initiative of extending the rays could say that a pair of angle bisectors never intersect.

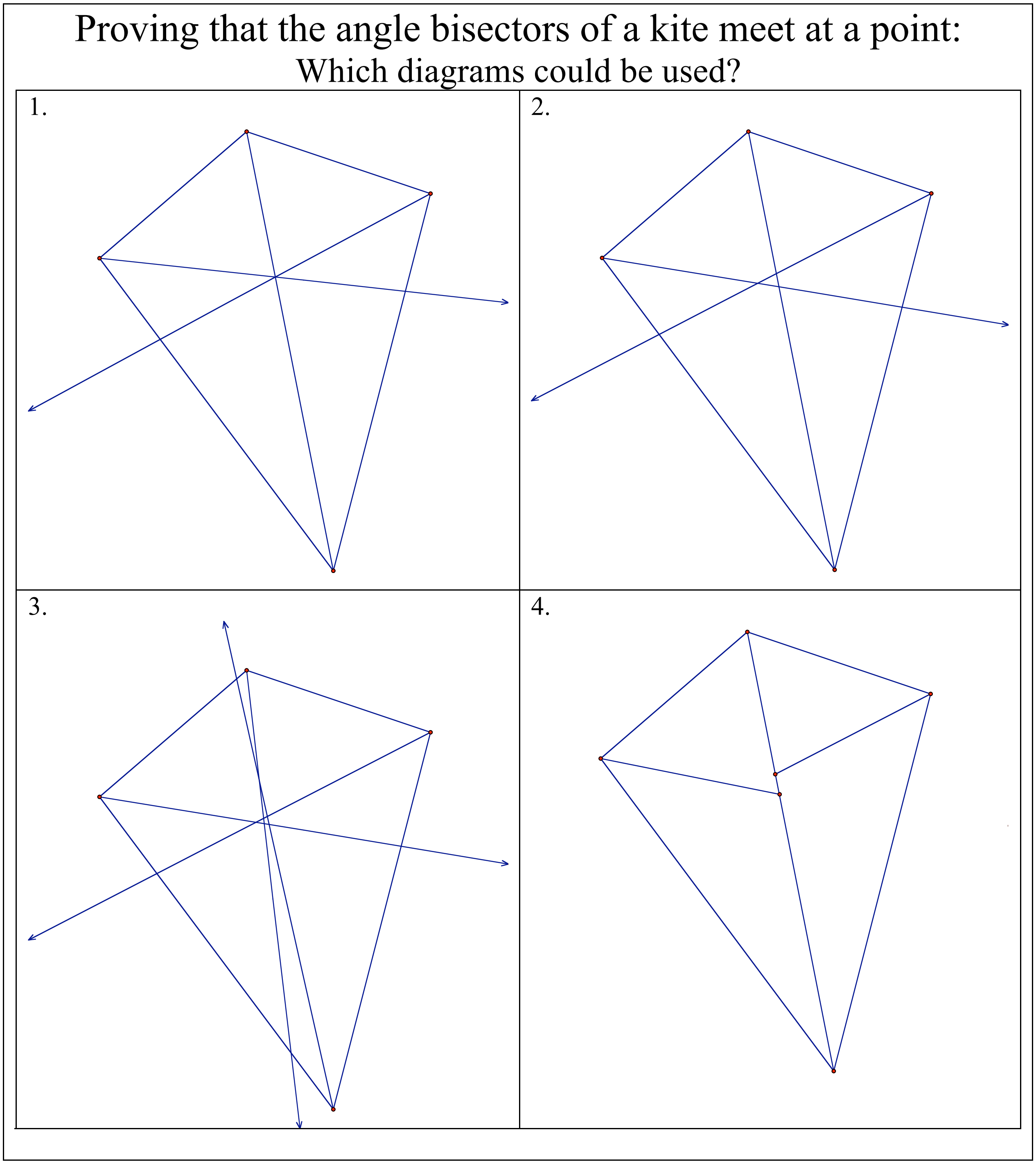

Figure 33. Angle bisectors of a kite.

A teacher could make use of these diagrams that show something incorrect as an opportunity to teach properties of the angle bisectors of a kite that otherwise could be taken for granted. The fact that a diagonal of a kite bisects two of its angles could be an opportunity to teach students that the same geometric object could be defined differently, according to what one is looking for. The diagram of a kite and its angle bisectors, where the angle bisectors make a small triangle inside the kite, could be a useful resource to prove by contradiction that the angle bisectors of a kite are concurrent. These ideas illustrate how a teacher could make use of diagrams that showcase an impossible case to teach something, such as the properties of the angle bisectors of kites.

In the interval I selected, approximately six minutes after the moderator had distributed the handout with diagrams, a researcher asked participants how they would feel about giving students a diagram that represented something incorrect. Lynne repaired this seemingly odd move of giving students such a diagram by saying she would cue students to pick the correct one. Then, Karen shared her concerns about using an incorrect diagram: She worried that the following day students would remember the incorrect diagram as opposed to the correct one. The transcription of this discussion follows.

Turn #	**Speaker**	**Turn**
643.	Researcher	But in terms of you giving the picture to the students, would it be, would you feel at ease doing that, like giving either of those pictures that don't uh, that don't depict the situation that really exists?
644.	Lynne	I think you could ask which on these pictures shows the true situation. And then justify your reason. [Researcher: Mm-hmm].
645.	Karen	If you give them, you know, I would feel more comfortable if I gave them, you know, a choice with a right one in there. So, you know, it's like "pick out the right one and why." In terms of just giving it to them, because of the way that attention goes, and sometimes you only get them for two minutes, out of the fifty-five minutes, they might remember the wrong one [Nicola: Mm-hmm.] and say "you told me this" [Participants laugh]. So, because, I mean, so, it's sort of in the context of, "pick the right one," that,

		that's amusing and interesting. In terms of, "let me draw this picture up here." I, if I had one of these pictures I would say, "what's wrong with my picture to begin with?" Or, "do you see anything wrong with the picture?" So that, they get it in their heads pretty quickly that there might be something wrong with the picture.
646.	Researcher	So you would tell them that there is something wrong. They just need to figure out what it is.
647.	Karen	I would feel more comfortable if I were drawing a sketch and drew a wrong one and then ask them then, "tell me, what's wrong with it."
648.	Researcher	And if they were going to be the ones that draw the picture and say you roam around the room and you see that many of them are coming up with number two, how would you handle that?
649.	Karen	Well, then, then I think I would start to say, "okay, so, there's your sketch um, what can you, what can you prove about it? Because your sketch is saying it doesn't happen. So, can you show that it doesn't happen?"
650.	Researcher	Okay.

(TWP052405-60)

In this interval, Lynne and Karen hesitated to say that they would give a "wrong" diagram to their students. They suggested using different diagrams, including a correct one where the angle bisectors of a kite are concurrent. Then, they would ask students to choose which one shows a "true situation." Another suggestion by Karen was to draw the incorrect diagram on the board and ask students directly to identify what is "wrong" with the diagram. Both scenarios proposed, using multiple diagrams or just one diagram, are similar in that a teacher would cue students to look for what is incorrect about a diagram, whenever a diagram representing something incorrect is within the alternatives.

According to participants, there is a potential risk in using incorrect diagrams: Students would remember what is incorrect instead of what is correct. Karen anticipated the problem that by showing only a diagram that is purposely incorrect, students might get to remember that diagram instead of the correct one (turn 645). Nicola expressed agreement with Karen. The laughter of other teachers in the session was a humorous reaction to Karen's impersonation of a student (turn 645).

Karen elaborated on what a teacher should do to manage a diagram that represents an incorrect idea. Whenever there is such a diagram—either drawn by the teacher or drawn by a student—the teacher should pose questions to make students realize that there is something incorrect. From a participant's perspective, the teacher is responsible for helping students find mistakes in a diagram. Otherwise, students would take a diagram that is incorrect as correct. By stating that there is something incorrect, teachers prevent students from remembering something incorrect as correct.

Karen argued that students' attention span is limited. It may be the case that students pay attention to class discussion at the moment when the teacher is presenting an incorrect diagram and consequently students would take the incorrect diagram as true. Therefore, a teacher is responsible for clarifying potential errors on a timely manner in order to make students remember only those results that are correct.

Analysis of Participation of Karen's comments in turn 645 gives more evidence to support the claim that a teacher is responsible to shape students' memories (see Table 34). From this analysis, I identify distinct Processes where the teacher is the agent and where the students are the agents. Table 35 shows clauses that involve the teacher and the students. From a total of 24 clauses, 10 of those clauses involve the teacher demanding some action from the students either directly or indirectly (Table 35). In these clauses, the teacher asks students to do something, or says something to the students (namely, clauses 1, 3, 5, 6, 7, 9, 12, 15, 20, and 21). For example, the teacher gives students the problem (clauses 1, 3, and 7), gives instructions for students to select the correct diagram (clauses 5, 15, 20, and 21), asks students to justify their answer (clause 6), gets students' attention (clause 9), and gives explanations to the students (clause 12). These set of actions show that, according to the participants, the teacher is responsible for guiding students' actions, especially actions where students use their prior knowledge.

Table 34

Analysis of participation (TWP052405-60, turn 645)[72]

	Clauses stated by Karen	**Participants**			**Process**	**Circumstance**
			The teacher	**The students**		
1.	If you give them		• *you (the teacher)*	• **them (the students)**	• give	
2.	I would feel more comfortable		• *I (the teacher)*	•	• would feel more comfortable	
3.	if I gave them, you know, a choice with a right one in there.	• a choice with a right one in there	• *I (the teacher)*	• **them (the students)**	• gave	
4.	So, you know, it is like					
5.	"pick out the right one	• the right one"		• **("you, students)**	• "pick out"	
6.	and why?"	• "why"		• **("you,**	• ("say")	

[72] Terms in parenthesis are my interpretation of Karen's references. Terms in brackets are those that Karen elided. I use italics for Participants that involve the teacher and bold for Participants that involve the students.

	Clauses stated by Karen	Participants			Process	Circumstance
			The teacher	**The students**		
				students)		
7.	In terms of just giving it to them	• it (the wrong diagram)	• *(the teacher)*	• **them (the students)**	• giving / to	
8.	because of the way that attention goes			• the way **(students')** attention	• goes	
9.	and sometimes you only get them for two minutes, out of the fifty-five minutes[73]	•	• *you (a teacher)*	• **them (the students)**	• get (attention)	• sometimes • two minutes, out of the fifty-five minutes
10.	they might remember the wrong one	• the wrong one (diagram)		• **they (the students)**	• might remember	

[73] I take that the Process of getting attention is one where the teacher is the active agent, controlling what students pay attention to.

	Clauses stated by Karen	Participants			Process	Circumstance
			The teacher	**The students**		
11.	and say,			• **(students)**	• say	
12.	"you told me this."[74]	• this" (an incorrect claim)	• *"you (the teacher)*	• **me (the student)**	• "told"	
13.	So, because, I mean					
14.	so, it's sort of in the context of,					
15.	"pick the right one,"	• the right one (diagram)		• **(you, the students)**	• "pick"	
16.	that, that's amusing	• that (picking the right diagram)			• is amusing	
17.	and interesting.	• (that, picking the right diagram)			• (is) interesting	
18.	In terms of, "let me draw this picture up here."	• "this picture"	• *"me" (the teacher)*		• "let/draw"	• "up here"
19.	I, if I had one	• one of these	• *I (the*		• had	

[74] Even though in this case Karen was enacting the voice of a student who claims that the teacher said something to the class in the past, I take the teacher as an active agent of clause 12. In clause11, the student is actively engaged in the present, saying something to the teacher. In clause 12, the teacher was active in the past, as reported by the student.

	Clauses stated by Karen	Participants			Process	Circumstance
			The teacher	**The students**		
	of these pictures	pictures	*teacher)*			
20.	I would say,		• *I (the teacher)*		• would say	
21.	"what's wrong with my picture to begin with?"	• *"what"* • *"wrong with my picture"*			• "is"	• "to begin with"
22.	Or, "do you see anything wrong with the picture?"	• anything wrong with the picture"		• **you (the student)**	• "do/see"	
23.	So that, they get it in their heads pretty quickly	• it (X)[75]		• **they (the students)**	• get in **their (students')** heads	• pretty quickly
24.	that (X =) there might be something wrong with the picture.	• something wrong with the picture			• there might be	

[75] This is a complex figure, which I broke down into two. "X" is a place-holder for something that is expanded in the following figure.

From a total of 24 clauses, 3 of those clauses involve the students as agents (clauses 10, 11, and 22). The Processes associated with students, when they are active agents, are to *remember* (clause 10), to *say* something to the teacher (clause 11), and to *get* something in their heads (clause 22). Besides the action of saying something to the teacher, the other two actions are mental actions, which result as a consequence of something that the teacher said. For example, students "might remember" what the teacher had avowedly said in the past (clauses 10 and 11), and students would "get in their heads" a teacher's question about the diagram. This illustrates how students' memories of prior knowledge could be prompted by what the teacher says in class. In particular, according to Karen, students will remember claims that the teacher says about the diagram.

Table 35

Figures that involve the teacher and the students (TWP052405-60, turn 645)

# of clause	**Clause**	**Participant (teacher or students)**	**Process**	**Participant (teacher or students)**	**Does the teacher demand students' actions?**
1	If you give them	*you (the teacher)*	give	**them (the students)**	yes
2	I would feel more comfortable	*I (the teacher)*	would feel more comfortable		no
3	If I gave them, you know, a choice with a right one in there.	*I (the teacher)*	gave	**them (the students)**	yes
5	"pick out the right one	**("you," students)**	"pick out"		yes
6	and why?"	**("you,**	("say")		yes

# of clause	Clause	Participant (teacher or students)	Process	Participant (teacher or students)	Does the teacher demand students' actions?
		students)			
7	In terms of just giving it to them	*(the teacher)*	giving / to	**them (the students)**	yes
8	because of the way that attention goes				no
9	And sometimes you only get them for two minutes, out of the fifty-five minutes	*you (a teacher)*	get (attention)	**them (the students)**	yes
10	they might remember the wrong one	**they (the students)**	might remember		no
11	and say, "you told me this."	**(students)** *"you (the teacher)* **me (the student)** this" (an incorrect claim)	say "told"		no
12	"you told me this."	*"you (the teacher)*	told	**me (the student)**	yes
13	So, because, I mean				no
14	so, it's sort of in the context of,				no
15	"pick the right one,"	**(you, the students)**	"pick"		yes
16	that, that's amusing	that (picking the wrong diagram)	is amusing		no
17	and interesting.	(that, picking the wrong diagram	is interesting		no
18	In terms of, "let me draw this	*"me" (the teacher)*	"let/draw"		no

# of clause	Clause	Participant (teacher or students)	Process	Participant (teacher or students)	Does the teacher demand students' actions?
	picture up here."				
19	I, if I had one of these pictures	*I (the teacher)*	had		no
20	I would say, "what's wrong with my picture to begin with?"	*I (the teacher)*	would say		yes
21	Or, "do you see anything wrong with the picture?"	**you (the student)**	"do/see"		yes
22	So that, they get it in their heads pretty quickly	**they (the students)**	get in **their (students')** heads		no
23	that there might be something wrong with the picture.	There something wrong with the picture	might be		no

The teacher and the students have distinct responsibilities when using a diagram that represents something incorrect. These responsibilities are evident in three possibly storylines which could be interpreted from Karen's comments. In Table 36 I present how the events of each story unfold, with implicit steps noted in brackets.

Table 36

Emergent stories (TWP052405-60, turn 645)

Moment	Story 1	Story 2	Story 3
1	The teacher gives out the wrong diagram to students without telling them that the diagram is wrong.	The teacher gives out a handout with several diagrams, including the correct one and others that are wrong.	The teacher gives out the wrong diagram.
2	Students pay scattered attention in class and some get to see the wrong diagram,	The teacher asks students to select the correct diagram and to	The teacher asks students to identify what is

	but ignore the observation that the diagram is wrong.	justify their answers.	wrong with the diagram.
3	The next day, students remember the wrong diagram as correct.		Students find what is wrong.
4	[The teacher tells students that the diagram is wrong.]		
5	The students claim that the teacher told them that this diagram was correct.		

In the first story, the teacher loses control of students' memories. Students say that the teacher made incorrect claims about the diagram. Students confront the teacher and remind the teacher about previous assertions regarding the diagram made in class. Thus, even when the teacher could consider that students should not remember incorrect claims about the diagram, it haunts the teacher the possibility that students would consider these claims true, and also remember them. For the teacher, it would be too late to alter what students would remember, because students would already remember the “wrong” diagram as correct.

In the second and the third stories, the teacher controls what students do with the diagram by asking them to find something incorrect in the diagram. Moreover, the teacher prevents students to remember something that is “wrong” by having students realize quickly that the diagram is incorrect. The teacher is an active agent in most of the figures that contribute to these two stories. In contrast, students are passively following the teacher's instructions. The teacher makes a timely decision to ask students to find something incorrect about the diagram. This question prevents students from remembering things that are incorrect. Since the teacher asked students to find something “wrong” in the diagram, students will presumably not consider claims made about that diagram as something they should remember.

The data suggests that there is a norm in a geometry class, which implicitly states that a teacher should not present to students diagrams or claims that are purposely incorrect. In the interval, there is confirming evidence for this norm. The evaluative stance towards the story where a teacher presents a “wrong” diagram without stating that there is something incorrect about a diagram (the first story) was of *undesirability*. In contrast, the evaluative stance towards the stories (the second and the third stories) where the teacher prompts students to find out that the diagram is incorrect was of *probability*. In presenting the second and the third stories, Karen prefaced her preference for cueing students to find something incorrect in a diagram by saying, "I **would** say." Karen used projective clauses (e.g., "pick out the right one and why") to enact the voice of a teacher who instructs students how to make use of the diagram. Those projective clauses are a repair of a breach of a norm in a geometry class: A teacher should not give students a diagram that represents something incorrect.

Karen described students' usual actions when she justified her concerns about not giving students enough cues to notice that there is something incorrect. According to Karen, students pay attention to class every so often. Therefore students would miss out important information about the incorrect diagram. This information could lead students to find out what is incorrect about the diagram. So, by using a “wrong” diagram, a teacher could mislead students into believing that the diagram is correct.

The probe used in the focus group session asked participants to consider different diagrams, including diagrams that represent something incorrect, for students to produce a proof for the problem that the angle bisectors of a kite meet at a point. The product of that task would be to do a proof. However, in two of the alternative stories proposed by Karen (the second and the third stories), she changed the task so that students would not have to produce a proof. The products of these new tasks consist of selecting the correct diagram and justifying their choice, or stating what is incorrect with a given incorrect diagram.

In the three stories, Karen modified the original task, which required students to do a proof. Alternatively, the products of the new tasks, modified by Karen, are claims about the incorrect diagram. In the first story, the teacher would not signal to students that the diagram is incorrect. As a consequence, students could take those claims about the diagram as true when they are not. Karen described a possible clash because students had assumed that the diagram was correct. In the other two stories, the teacher would suggest to students that the diagram is "wrong." As a result, students would not be responsible for remembering claims about the "wrong" diagram; students would be responsible for remembering that the diagram is incorrect. Also, students would be responsible for remembering possible explanations about what is incorrect about the diagram and why. The teacher's intervention to ask students to find something out about the diagram, at the moment of presenting the diagram, increases the potential for students to achieve valuable products that ought to be memorable.

In the three stories, the teacher acts as if the product of the task is the learning stake. Alternatively, the teacher could let students work with a "wrong" diagram and then, perform some sort of action to let students know what they should have learnt from working on a mathematical task. Or, the teacher could pose a sequence of tasks, leading students to identify what they should have learnt. However, it seems that the memories associated with discussions around diagrams that represent something incorrect would be too strong to be corrected by other actions. The change in the task proposed in the focus group, which required students to use the "wrong" diagram to do a proof, demonstrates that using those "wrong" diagrams is difficult for teachers.

The hypothesis of the existence of a collective memory could help explain the underlying rationale in these statements about students' memories. It seems as if the positive value of learning from a diagram that represents something incorrect (Borasi, 1994) is counterbalanced by the danger of having students remember a mistaken idea. Participants did not propose an alternative story where students would remember what is incorrect about the diagram. On the contrary, the only story that includes the student action of remembering has the negative consequence that students would remember something incorrect instead of something that is correct (story 1). The other two stories (stories 2 and 3) change the mathematical task by signaling to students that the diagram is "wrong." Consequently, these two stories could be regarded as marginal stories that comply with the request proposed in the focus group session without really using the "wrong" diagram as an opportunity to teach properties of kites. Then, I conclude that the collective memory of the class should only include ideas that are correct and approved by the teacher. The events surrounding the discovery of an idea that is incorrect could be misleading because it is unusual for a teacher to spend too much time discussing something that is incorrect. So, according to participants, what might become memorable for students is the incorrect idea instead of the events leading towards discovering that the idea is incorrect. From participants' perspective it seems that teachers do not expect that students would keep a memory of these events as part of the collective memory of the class. On the contrary, these events would confuse students. So it is preferable if students were to forget their work with “wrong” diagrams.

In sum, I contend that a teacher's work of shaping the collective memory of the class is threatened by using diagrams that represent ideas that are incorrect. According to participants, the potential value of using those diagrams when *teaching with a problem* becomes problematic. Students' claims about a “wrong” diagram do not exchange for memorable products, unless the teacher makes timely decisions to proclaim that the diagram is incorrect. Therefore, if a teacher decided to use “wrong” diagrams, the teacher should make it explicit to students that the diagram is incorrect. By stressing that the diagram represents an incorrect idea, a teacher would prevent students from remembering something incorrect as correct.

Summary

In this section I have presented examples of teachers’ perspectives on actions geared towards shaping the collective memory of the class. On the one hand, teachers’ deliberate moves to name theorems according to names already established in the geometry curriculum could allow them to assign some value to the work of students on problems, and also to make this work memorable and usable in the future. So if while working on a problem students produced an assertion that happened to be a theorem that students should learn, teachers would give the official name of the theorem as stated in the curriculum. Within the geometry course, names of theorems are useful to spell out the reasons for statements in a proof (Herbst & Nachlieli, 2007). Beyond the geometry course, students would be able to remember things learnt as part of the geometry curriculum, without associating these concepts with particular instances in their particular geometry class.

On the other hand, a teacher's decision to exclude memories of students' work with "wrong" diagrams calls the attention to teachers' perceived dangers of having students remember something that is incorrect, instead of something that is correct. Teachers assume that students will remember "wrong" diagrams as correct. Accordingly, participants denounced the possibility that a teacher would present a diagram that represented an incorrect idea in class without giving students enough cues to stress that the diagram is incorrect. Instead of using a "wrong" diagram as a means for students to find out something new, teachers would cue students to remember that the diagram is incorrect (and at times to say what is incorrect about the diagram) before disregarding the diagram. The presumption that a "wrong" diagram could be used precipitated teachers' deliberation, which reflects the intention to sieve out what students should remember from their work on a problem. While one could conceive of the use of diagrams that represent impossible or incorrect ideas as a mathematical means to get new knowledge about properties of a geometric figure, teachers perceive the use of those diagrams as dangerous. One reason stated by participants is that students usually pay attention to class selectively, and thus, they may not listen to a teacher's explanation for why the diagram is incorrect. Another reason implied by participants is that the case of presenting "wrong" diagrams to students (without saying that these are incorrect) is so unusual that students would take any diagram presented in class as correct.

In sum, at the end (or during) of students' work on a problem, teachers consider appropriate to perform actions to emphasize what should be remembered or to prevent students from remembering something. These are actions that students would not do on their own. Even when students may attempt to decide what should be remembered on their own (such as the case of having students give an ad hoc name to a theorem), teachers might need to correct those actions to make sure that students remember things that they have officially approved as part of the collective memory of the class. This control of students' memories would allow teachers to sieve out memories of things and events that they do not perceive to be valuable for the future.

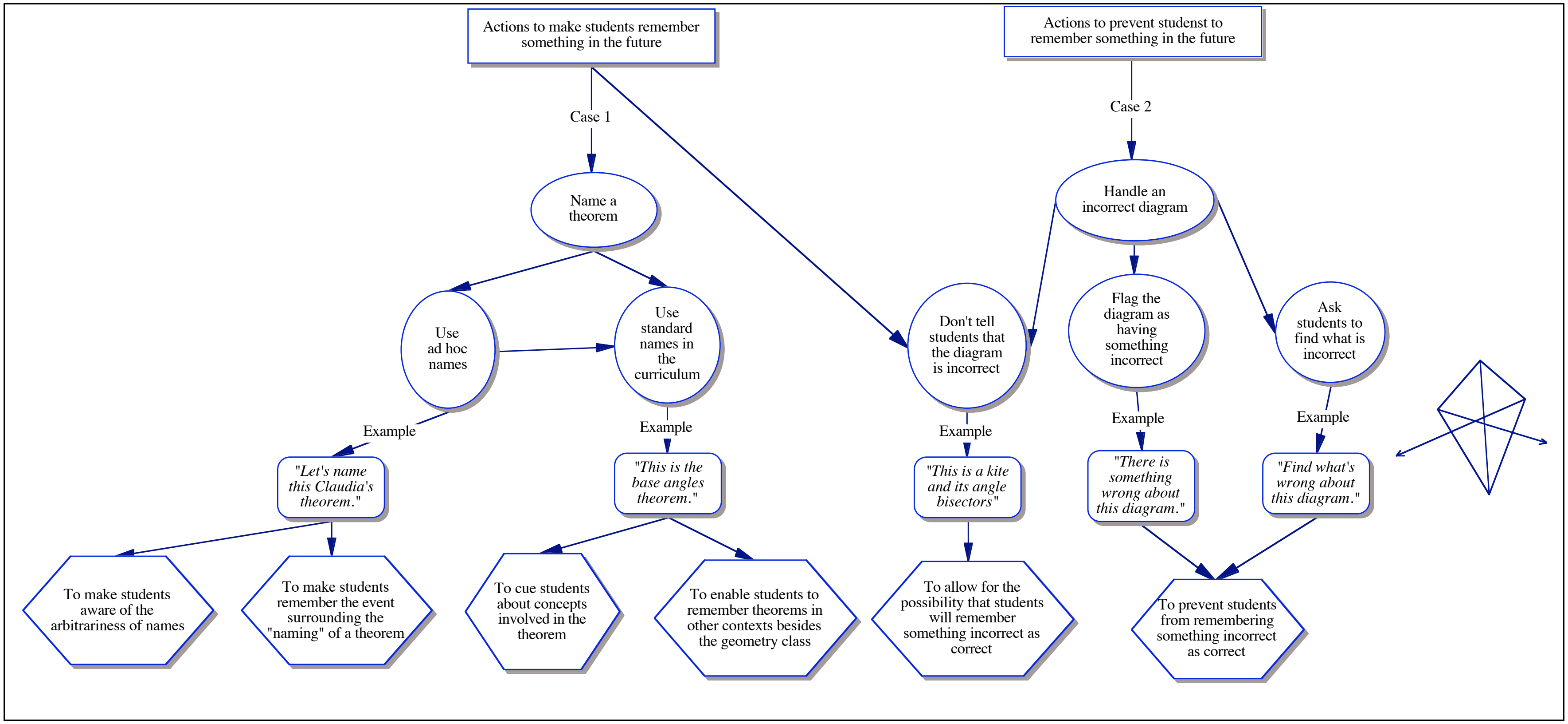

Figure 34. Possible teaching actions to make students remember or forget something in the future.

Figure 34 summarizes actions after students' work on a task has ended to make students remember something and actions to make students forget something in two cases: naming theorems and giving incorrect diagrams. There are two possibilities for naming a theorem: a teacher could allow an idiosyncratic name for a theorem (e.g., after a student's names) or else give an official name to the theorem. When a teacher gives an idiosyncratic name to a theorem, students could learn about the arbitrariness of names for theorems in mathematics. But, even when naming a theorem after a student's name, participants reported that they would eventually give the theorem its official name. While naming theorems after a student's name could make the event of naming the theorem memorable, the use of standard names would help students to remember the theorem in the future. Moreover, with its official name, students could connect the new theorem with concepts associated with that theorem. In the case of handling an incorrect diagram, teachers add other sorts of actions to make students forget that diagram, such as flagging the diagram or asking students to find what is incorrect. When teachers do not signal to students that the diagram is incorrect, participants reported that there is a risk that students will remember in the future the incorrect diagram as correct. So, even though an incorrect diagram could provoke fruitful mathematical discussions, participants did not give a positive purpose to the action of using an incorrect diagram, concerned about losing control of students' memories.

Conclusions

In this chapter I examined the question of how teachers perceive and appreciate alternative ways of managing students' prior knowledge when using a problem to teach new geometric content. I studied this question by studying what teachers had to say about teaching actions to manage students' prior knowledge at specific moments in the enactment of a mathematical task—before, during, and after students work on a task.

From my analysis of conversations in focus group sessions where participants considered the possibility of teaching properties of quadrilaterals with the angle bisectors problem, I highlight three aspects of a teachers' work when teaching with a problem. First, I consider teachers' and students' responsibilities regarding prior knowledge (as seen by teachers). Then, I discuss differences between tactical and strategic actions by the teacher (as recognized by teachers). Finally, I elaborate on the problem of using prior memories and of creating memories for the future.

In the division of labor on how to make use of students' prior knowledge when working on a problem, the data suggests that teachers have more responsibilities than students in identifying the resources needed to work on a problem. Teachers are expected to use the statement of the problem to cue students about the things that students should remember before they work on a problem. As students are working on a problem, teachers are expected to scaffold students' work by means of controlling instructional discourse and controlling the diagram on the board. For example, teachers may ask for justifications for students to support their answers, teachers may make individual prior knowledge public as to make sure that students share the same resource for working on a problem, and teachers may filter changes to diagrams according to solutions teachers prefer. After students' work on a problem, teachers are expected to take responsibilities for emphasizing what students should remember in the future and for excluding what they should not remember. With actions such as naming theorems and preventing students from remembering things that are incorrect, teachers intentionally shape what students should remember, sometimes at the expense of excluding actual memories. Teachers may control what students should remember in the future, holding students accountable for drawing from memories that the teacher has already approved.

There are some actions that students could do on their own. For example, students may decide to draw a diagram to work on a problem, or select the particular tools that they may use to perform some operations with diagrams. Students may also have individual memories of resources that they could apply to solve a problem. For example, students may bring about memories of geometric concepts studied in the past. However, for the most part, participants reported that they would filter the results of students' actions. In doing so, a teacher would establish a difference between what individual students may remember and what all students, as a collective, should remember. That is, a student may remember something on his or her own. However, if those memories are in conflict with what the teacher needs students to remember, then those memories may not become publicly accepted in class.

An important finding from my analysis of records from focus group sessions is that the actions that teachers expect to have to perform while students work on a problem differ in that some are strategic and others are tactical. Strategic actions involve controls for planning these actions beforehand. Also, strategic actions are not contingent on the observation of students' work on a problem. For example, a strategic action that teachers do is to set up a mathematical task by presenting to students the statement of the problem. This action may require pondering and anticipating what students might do with a problem before students actually work with the problem. In contrast, a tactical action is contingent on students' actions. An example is that of providing scaffolds as students' work on a problem. A teacher would have to react to students' work on a problem to provide more or less resources than the ones hinted with the statement of the problem, according to what students do. This difference between strategic and tactical actions could be helpful for teachers to decide what elements to consider as they plan a lesson that involves teaching with a problem, and also what actions could help them to gather more information about students' work to make immediate decisions.

Finally, the problem of managing students' memories lies at the intersection of two different temporalities. On the one hand, a teacher has to deal with students' memories of prior knowledge—either from their geometry class or from other sources outside of the geometry class. There are past things or events that a teacher may want to filter when allowing students to use resources and operations that they know. On the other hand, a teacher may use the opportunity to work on a problem to forge memories for the future. So, there are things and events related to students' work on a problem that ought to be memorable, whereas other things and events could be disregarded or forgettable. This is important because the problem of how teachers handle students' memories is not a problem of the past, but a problem of how the past becomes relevant in the present and in the future. By filtering what students should remember from the past, and also what students should remember in the future, teachers manage the complexities of dealing with individual and collective memories in a class.

However, there is a potential risk if a teacher were to allow individual students to use any memories of their prior knowledge: Teachers could lose control of what resources students would make use of. Moreover, it is possible that not all students would possess the same memories of prior knowledge, because their individual memories may be different than what the teacher expects the collective to have. So, it seems safer for a teacher to assume the responsibility of making students remember relevant prior knowledge so as to control what kind of resources they should deploy when solving a problem.

CHAPTER 6

CONCLUSION

At the beginning of this dissertation I asked the question: *How can the hypothesis that the teacher creates a collective memory of the class be demonstrated in the case of a teacher who attempts to teach geometry with problems?* This question focused this study of teaching on the activity of teaching with a problem and on high school geometry teaching. From previous research there is evidence that the activity of using a problem to teach new mathematical ideas is difficult for a teacher (Herbst, 2006; Lampert, 2001). The geometry class, conceived in the curriculum as an opportunity to teach students about deductive reasoning, provides a setting for investigating how teachers manage students' prior knowledge, especially because students have learned geometry earlier but much of that content is taught again. So I took the geometry class and the activity of teaching with a problem as a case to study a broader phenomenon on how teachers shape a class's collective memory.

I have proposed the construct of the *collective memory of the mathematics class* with the purpose of studying how teachers manage students' prior knowledge. According to Kilpatrick, "A researcher makes a contribution to our field by providing us with alternative constructs to work with that illuminate our world in a new way, and not simply by piling up a mass of data and results" (1981, p. 27). In the two studies I conducted, the construct of the collective memory enabled me to explain teachers' underlying rationality for making decisions about how to manage students' prior knowledge. The hypothesis of the collective memory states that teachers manufacture a privileged representation of the past—a discourse of the past—as the orthodox **collective memory of a class**. With the aim of creating this collective memory, a teacher makes moves to make students remember (or forget) things and events from the class. In the following sections I discuss the research questions, the complementary nature of the conclusions of the studies, and the implications.

Revisiting the Research Questions

In the first chapter I asked four research questions:

1. How does a teacher hold students accountable for remembering what they ought to know in order to do a problem?
2. How does student participation feature in a teacher's management of the collective memory?
3. What do teachers hold themselves accountable for doing to make students remember or forget past things and events when using a problem to teach something new?
4. How do teachers perceive and appreciate alternative ways of managing students' prior knowledge when using a problem to teach new geometric content?

The first study was geared towards answering the first two research questions by showing a case study of teaching with problems with high demands on prior knowledge to do the work. From this study, I propose a model for the boundaries of usable prior knowledge when teaching with a problem. This model suggests that a teacher activates and intends students to use their memories of prior knowledge from other mathematics classes and from shared experiences besides the geometry class. The model also suggests that a teacher provokes students to anticipate new knowledge when teaching with a problem. The second study intended to answer the last two research questions by studying practitioners' reactions to the presumption of teaching with a problem. As a result, I propose a model for the study of teaching actions to manage students' prior knowledge. This model gives a catalogue of teaching actions for activating memories and for deauthorizing memories at different moments in the enactment of a task.

A Model for the Boundaries of Usable Prior Knowledge

In the first study I presented evidence for a teacher's actions to shape the collective memory of the class. My analysis of the case of a teacher teaching a problem-based unit on quadrilaterals demonstrated that the activities of the unit pushed the teacher to make apparent the knowledge that they could use to work on a mathematical task. The teacher had to cope with difficulties that resulted from changes in the boundaries for usable knowledge when teaching with a problem. These boundaries include distinctions between knowledge from a remote past (before the geometry class), prior knowledge acquired in the geometry class, and future knowledge.

When playing the Guess my Quadrilateral game, students could rely on prior knowledge that they individually held, but that they did not necessarily share. In addition, the play of game did not involve the class in producing proofs for properties of special quadrilaterals. Figure 35 shows that the game activated students' prior knowledge about properties of quadrilaterals from prior mathematics classes or from shared experiences. The teacher allowed students to use prior knowledge of properties of quadrilaterals with the purpose of finding the hidden quadrilateral. However, the next day, when students assumed properties of rectangles, the teacher did not allow students to draw upon those memories. Thus, the Guess my Quadrilateral game, as an activity of teaching with a problem, required students to use prior knowledge that the teacher would not allow students to use later, when doing a proof.

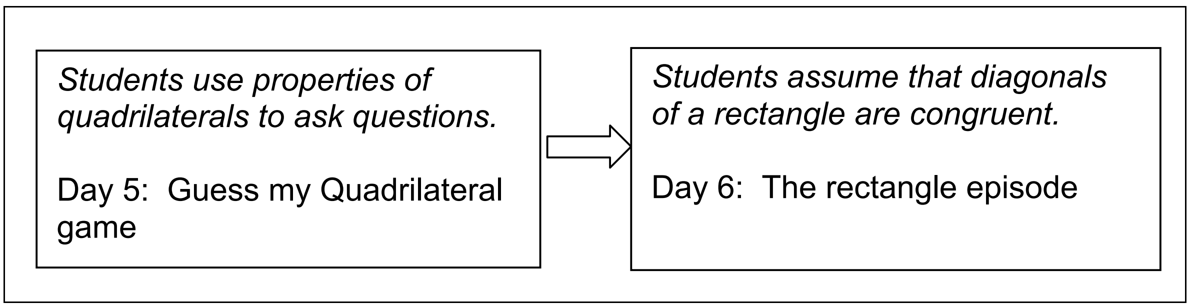

Figure 35. A teacher's activation of individual memories of prior knowledge.

While students worked with midpoint quadrilaterals during the unit, they gathered evidence for a conjecture: Midpoint quadrilaterals are always parallelograms. Students had the opportunity to make important arguments if they relied on a conjecture that they had made, but had not proven. The proof of this conjecture uses the medial-line theorem. This theorem had not been proven in class yet. The activities in the unit provoked the need of proving the medial-line theorem, thus, anticipating knowledge of memories yet to come. Figure 36 shows that, in the kite episode, students wanted to use a conjecture when proving that the midpoint quadrilateral of a kite is a rectangle. However, the teacher did not allow them to use that conjecture, because it had not been proven. To prove the conjecture, the teacher would have to allow students to anticipate knowledge they did not have yet. The memory of the medial-line theorem is something that students would possess in the future, but at that moment students did not know.

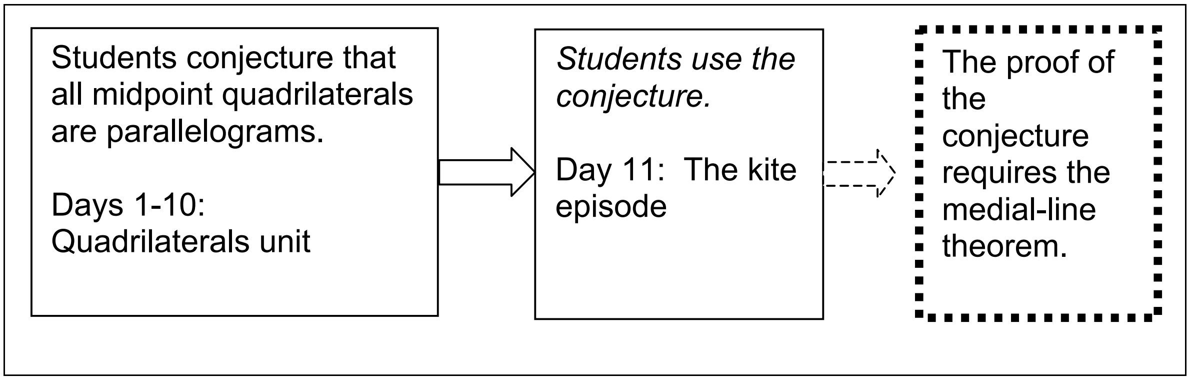

Figure 36. A teacher's work provoking anticipations of a theorem.

The difficulties that the teacher faced when asking students to do a proof in the rectangle episode and in the kite episode made apparent the need for having an official record of the knowledge of the class, a collective memory. From my analysis of these two episodes I concluded that the teacher's usual reliance on practices that set boundaries for what can be remembered—limiting the shared memory of the class to the immediate past of the geometry class—sustains the work of doing proofs. Usually the teacher does not allow students to use knowledge from prior mathematics classes or to anticipate knowledge; students only have resources from the geometry class to do proofs. However, the activity of teaching with a problem involves activating students' prior memories and also provoking the need for anticipating knowledge. Therefore, *doing proofs* and *teaching geometry with problems* involve different kinds of resources with regards to the prior knowledge students could make use of when working on a problem.

Figure 37 shows a representation of the boundaries for usable knowledge when *doing proofs* based upon my reading of prior research (Herbst & Brach, 2006). In this diagram, there are clear distinctions between the knowledge that teachers activate and the knowledge that students may possess but that the teacher does not expect them to use. Knowledge from the geometry class is available for students to use and it is appropriate for students to use, with more emphasis on the immediate knowledge from the geometry class. In contrast, knowledge from previous mathematics classes, prior knowledge from shared experiences, and future theorems that students may anticipate are not part of the usable knowledge when *doing proofs.* Thus, in the *situation* of doing proofs, teachers activate knowledge from the geometry class, and in particular, immediate knowledge from the geometry class. As a result, students use knowledge from the geometry class, disregarding knowledge from other sources such as previous mathematics classes or shared experiences.

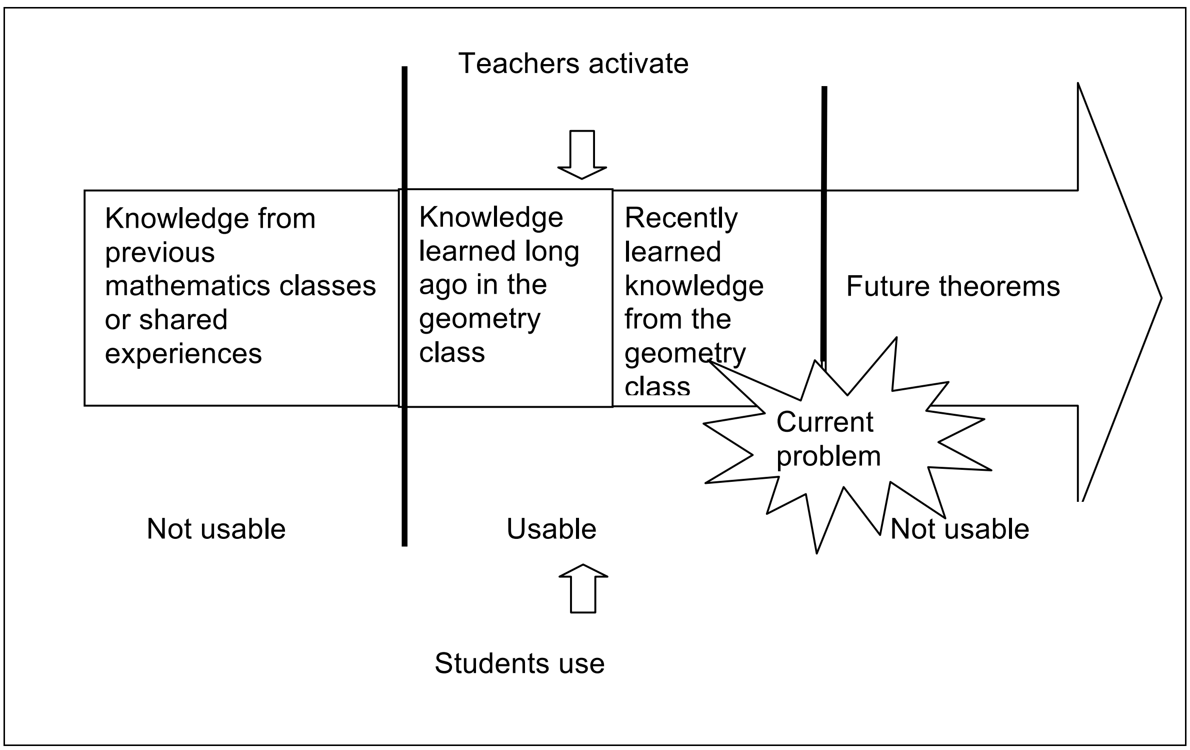

Figure 37. Boundaries for usable knowledge when *doing proofs* in the geometry class.

In contrast with the situation of doing proofs, the activity of teaching with a problem has more permeable boundaries for the usable knowledge. Teachers have to activate students' memories of prior knowledge from previous mathematics classes, from shared experiences in everyday life, and from the geometry class. In addition, teachers provoke the need for future theorems by assigning problems that would require students to anticipate knowledge. The model to describe the boundaries for usable knowledge when *teaching with a problem* shows that students have more knowledge available to use than when *doing proofs* (Figure 38). This difference is relevant because it suggests that a teacher's work managing students' prior knowledge would be different in the two activities: When teaching with a problem a teacher has to do more work to manage the trespassing of boundaries for usable knowledge than when *doing proofs*.

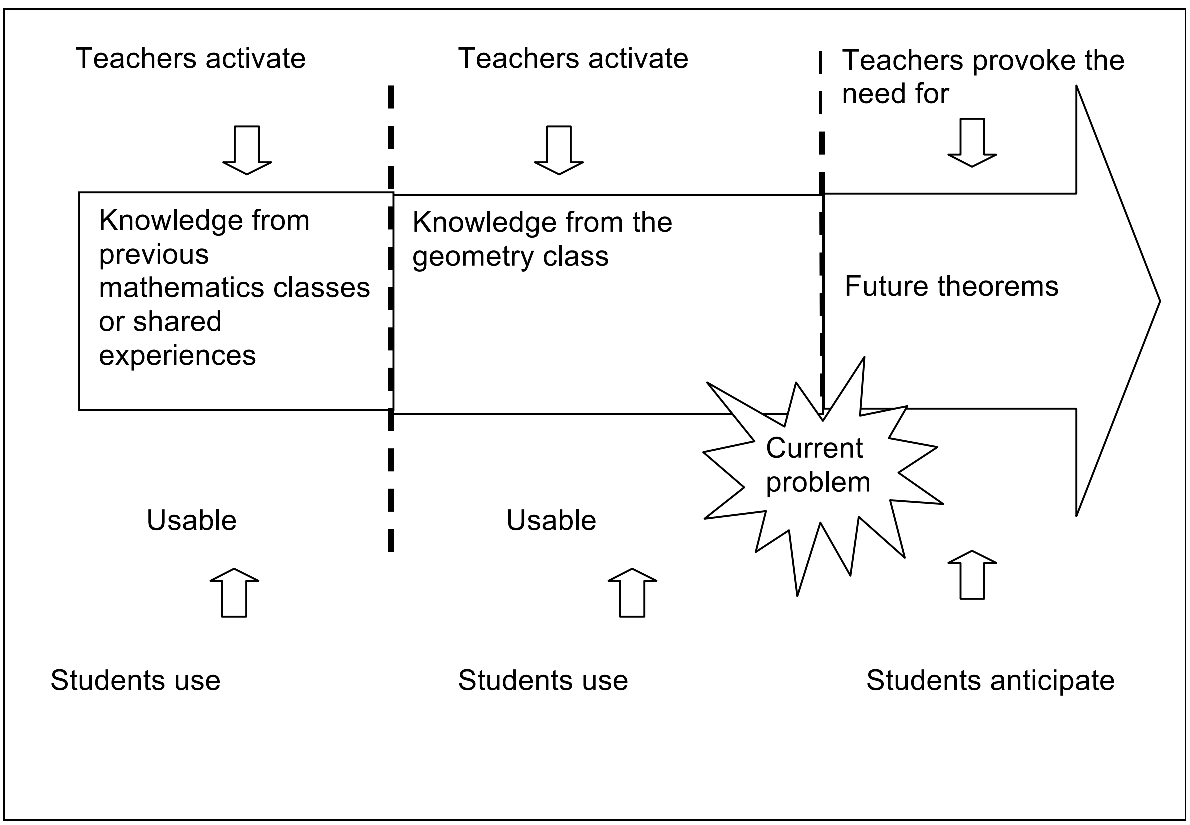

Figure 38. Boundaries for usable knowledge when teaching with a problem.

In answer to the second research question, how does student participation feature in a teacher's management of the collective memory, I contend that when teaching with a problem teachers activate students' prior knowledge that might have not been officially introduced in the geometry class. Students' individual and unofficial memories are useful, but they can create difficulties. On the one hand, a teacher draws upon students' memories, but, on the other hand, some of the memories may be incomplete, incorrect, or not shared. As a result, compared to the situation of *doing proofs,* when *teaching with a problem* a teacher has to do more work transforming individual prior knowledge into public knowledge that all students should share.

Difficulties for a Teacher to Manage Students' Prior Knowledge

I propose that there are three main reasons why managing students' prior knowledge is more complicated in the activity of teaching with a problem than when students are doing a proof: students' individual memories of prior knowledge are not necessarily shared, a teacher loses control of teaching by following the usual sequence of topics, and there are alterations in the means by which new knowledge is taken as such. These three difficulties were apparent in the case study of the teacher teaching the quadrilaterals unit and help illustrate the bigger point on the challenges that teachers face when teaching with a problem. In the following sections I discuss these three difficulties.

The Availability of Knowledge to All Students in the Class

During the play of the Guess my Quadrilateral game, one group mentioned the diagonals of a quadrilateral in the questions they asked. Based upon this evidence, four students in the class had made a connection between the diagonals of a quadrilateral and the properties of a quadrilateral. However, an analysis of students' responses to a homework exercise showed that most students had applied that diagonals of a rectangle are congruent in solving that exercise. One could infer that most students had knowledge about that property of diagonals of a rectangle. But this knowledge was not public in class yet, since the teacher had not installed this property after the play of the game.

In the usual didactical contract of the geometry class, a teacher is responsible for teaching students properties of special quadrilaterals that are related to the diagonals of that quadrilateral. It could be the case that other students knew about properties of diagonals during the play of the game. But from the record of all questions asked by students in that class there was not enough evidence for the teacher to assume that all students possessed knowledge about properties of diagonals of a quadrilateral. So the teacher had the challenge of bringing about a discussion about the diagonals of a special quadrilateral to the whole class, drawing upon the prior knowledge that some students apparently possessed. The teacher also had the challenge of considering the prior knowledge other students may have possessed about diagonals of a quadrilateral, but that they did not show in the play of the game. Thus, the game activated students' prior knowledge about quadrilaterals—including prior knowledge about properties of the diagonals—that not all students necessarily shared. As a result, the teacher needed to make that individual prior knowledge public and shared knowledge for the class. The homework assignment was insufficient to give students the right to know. The discussion of the homework problem gave the teacher an opportunity to make knowledge about properties of diagonals public and shared.

The Guess my Quadrilateral game brought about discussions about properties of quadrilaterals. However, these discussions did not necessarily involve all students in the class. Some students may have recalled some properties and other students may have recalled other properties. The homework problem asked students to use properties of quadrilaterals, including that diagonals of a rectangle are congruent. But regardless of whether or not some students knew that diagonals of a rectangle are congruent, the teacher had not had the opportunity to teach that property to all students in the class.

The difficulty of coping with individual prior knowledge that students do not necessarily share illustrates how a teacher manages the tension between teaching individual students and the whole class. Lampert (2001, p. 426-427) has referred to this problem as the "social complexities of practice." The teacher tries to manage paying attention to individual students and to all students in a public setting. I argue that by making deliberate moves to shape the collective memory of a class a teacher copes with this tension and provides an alternative to the problem of having too many individual memories that are not necessarily shared.

The Order in which New Knowledge Is Introduced

A second reason for teachers' difficulty to manage students' prior knowledge is that teachers lose control of establishing the usual sequence of topics when teaching with a problem. As I have said before, in the geometry curriculum, the introduction of definitions and propositions follows deductively from prior definitions and propositions. Lampert (1993, p. 154) has reported that geometry teachers do not alter that sequence of topics already established by the curriculum. However, in teaching with a problem, the ideas needed to work on a problem do not necessarily appear in the same order established in the curriculum. Therefore, a teacher would have to manage a tension between following a predetermined order of topics or following the flow of the class discussion about the problem.

This tension was apparent in the kite episode. Students had gathered evidence to support the conjecture that midpoint quadrilaterals are always parallelograms. In the design of the unit, we had conceived that activities surrounding the question about midpoint quadrilaterals could be an opportunity to bring about the medial-line theorem. In the kite episode there was a need to prove the theorem in order to prove the conjecture. However, instead of taking the kite episode as an opportunity to introduce the theorem, the teacher settled for a minor result that did not require them to use the theorem. The teacher's actions confirm that in the didactical contract of the geometry class it is inappropriate to anticipate knowledge and to alter the usual sequence of topics predetermined in the curriculum.

An alternative to teaching geometry following a predetermined order of topics is to follow the path of a problem. Kenneth Henderson (1947) had suggested this approach to geometry in an article entitled "Hypotheses as a point of departure in geometry teaching." He argued that as an alternative to follow a textbook a teacher could use class discussions as a vehicle to come up with definitions, theorems, and proofs of the theorems. He said that this method would enable students to experience the nature of mathematics and also to develop reasoning skills. Henderson's suggestion is consistent with an idea proposed by Lampert (2001) regarding teaching with problems as the teaching of "conceptual fields." Lampert draws on Vergnaud's (1988) theory of conceptual fields to propose that the activity of teaching with problems involves clustering topics by means of problems. This might involve a teacher in changing the usual order of topics established in the curriculum or in teaching topics simultaneously. This kind of teaching could be difficult because it would require for a teacher to have a deep knowledge of the subject matter and of how different topics are connected. In addition, a teacher would need to anticipate students' ideas for solving a problem and incorporate those ideas in the teaching of a new concept. By assuming the responsibility for when new theorems are installed as part of the collective memory of the class, a teacher controls the order in which new topics are presented in the geometry class.

The Means by which New Knowledge Is Introduced

A third reason why managing students' prior knowledge when teaching with a problem could be difficult for a teacher is because of possible alterations to the means by which new knowledge is installed in a class. There is evidence that teachers of geometry do not necessarily prove all the theorems that they teach (Miyakawa & Herbst, 2007). However, since a proposition is a theorem only if it can be proved, there is an expectation that new theorems will be given a proof.

In the rectangle episode there was evidence that the teacher controlled the means by which new knowledge was installed in the class. For the teacher, it was not enough whether students had been able to solve the homework exercise that used a theorem about properties of rectangles. The class discussion in the rectangle episode allowed her to install a theorem by means of providing a proof. The Guess my Quadrilateral game had exposed students to discussions about properties of quadrilaterals. However, during the play of the game the class did not produce proofs of those properties. The teacher controlled the way by which new knowledge was introduced in class when she decided to do a proof of a property that students had assumed true when solving the homework exercise. Similarly, the kite episode shows an example where the teacher did not allow students to draw upon a conjecture that had not been proven in class. The work of teaching with a problem involved anticipating knowledge of a theorem. But the teacher had lost control of the means by which new knowledge was introduced in class when students wanted to assume that the conjecture was true in the proof.

When a teacher shapes the collective memory, the teacher controls more than the sequence of topics but also the way in which a new piece of knowledge acquires status to become memorable. The actions surrounding the installation of a theorem, including the possibility of proving the theorem, are a way for the teacher to show how new knowledge is taken as such. The collective memory of the class includes concepts and propositions that a teacher has made official in some way, including by means of doing a proof.

Three Tensions Related to the Management of Prior Knowledge in Teaching Geometry

I propose that geometry teachers face three tensions when managing students' prior knowledge. Figure 39 shows a diagram of three tensions that influence the content of the collective memory and that make it challenging for a teacher to manage students' prior knowledge when teaching geometry with a problem. The first tension is related to the question of who possesses knowledge in the geometry class. Teachers face the tension of considering students' individual prior knowledge and taking into account the collective knowledge of the class. As a consequence, the availability of knowledge to all students in the class affects the content of the collective memory. A teacher might decide to set boundaries to the knowledge of the class if new knowledge is not publicly shared.

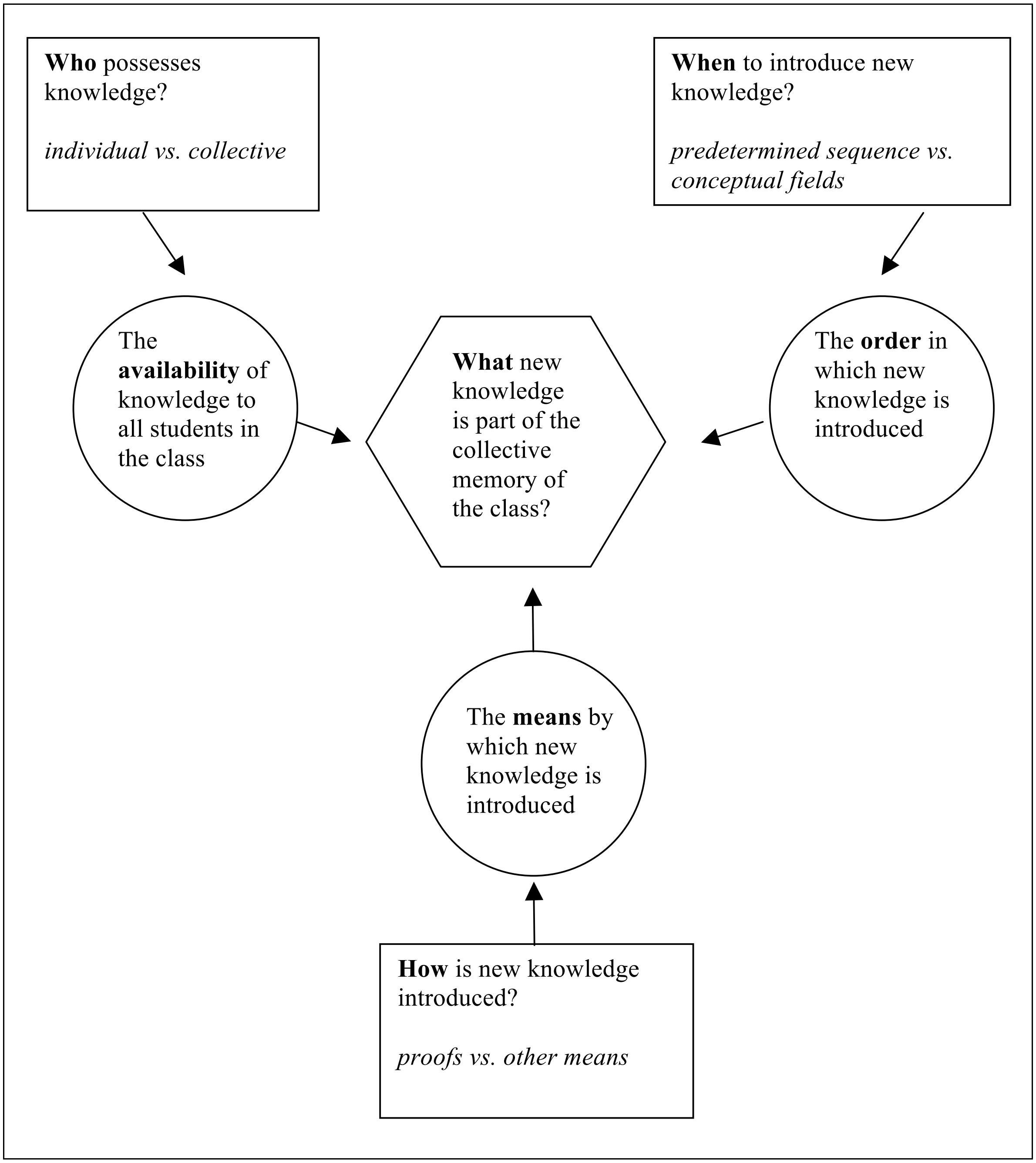

Figure 39. Tensions that influence the content of the collective memory of a class.

A second tension that influences the content of the collective memory has to do with the question of when to introduce new knowledge. Teachers could follow a curriculum with a predetermined sequence of topics. Teachers, on the other hand, could teach topics according to conceptual fields. The order in which new topics are introduced affects the content of the collective memory. By setting boundaries for the usable knowledge in a class the teacher also keeps control of the sequence of topics taught.

Finally, the question of how new knowledge is introduced is important because it brings about the tension of introducing new knowledge by means of a proof or by other means. Since the geometry course has been conceived as an opportunity to teach students about deductive reasoning (González & Herbst, 2006), teachers carry the responsibility of introducing new knowledge by means of a proof whenever possible. A teacher can control what new knowledge is to become part of the collective memory if that new knowledge does not come about in ways that the teacher would consider appropriate. All these difficulties add to the question of what knowledge is introduced as part of the collective memory of the class by means of teaching geometry with a problem.

In sum, the case study of a teacher using a problem to teach geometry allows for some conjectures. First, the kind of use of prior knowledge when *teaching with a problem* is very different than usual ways of managing prior knowledge in other stable activities such as *doing proofs.* The boundaries for usable prior knowledge in the activity of teaching geometry with a problem are more permeable than in other more stable activities in the geometry class such as doing proofs. Secondly, teachers assume responsibility for activating students' prior knowledge, but also for preventing students from using prior knowledge. Teachers assume control of the order of topics to be introduced in the collective memory, of the means by which these topics are introduced, and of making individual prior knowledge public. With these actions teachers shape the content of the memories that students could have available when working on a problem. Therefore a teacher's work of managing students' prior knowledge involves creating and shaping the collective memory of the class. The second study provides more insight about how teachers get to do so.

A Catalogue of Possible Teaching Actions to Manage Students' Prior Knowledge

The second study provides the beginnings of a catalogue of teaching actions towards making students remember or forget something as they work on a mathematical task. These actions are not comprehensive but examples of the kinds of actions that a teacher might do in order to manage students' prior knowledge at different moments in the enactment of a task. I documented two different kinds of actions: actions to manage students' prior knowledge and actions to shape ideas to be remembered in the future. This distinction is important because actions pertaining the collective memory are related to past memories and also to future memories that a teacher may want students to remember.

With the aim of studying relations between teachers' actions and students' expected actions, I created a model that allowed me to classify teaching actions at different moments of the enactment of a task (Figure 40). The unit of time, within instructional time, is the time to work on a mathematical task. During that time, there are actions that could happen before, during, or after students work on a problem. The model shows the actions of two actors: the teacher and the students. Even though the teacher and the students share responsibilities for some actions, most of the time the teacher is the one who makes final decisions, especially when those actions are public.

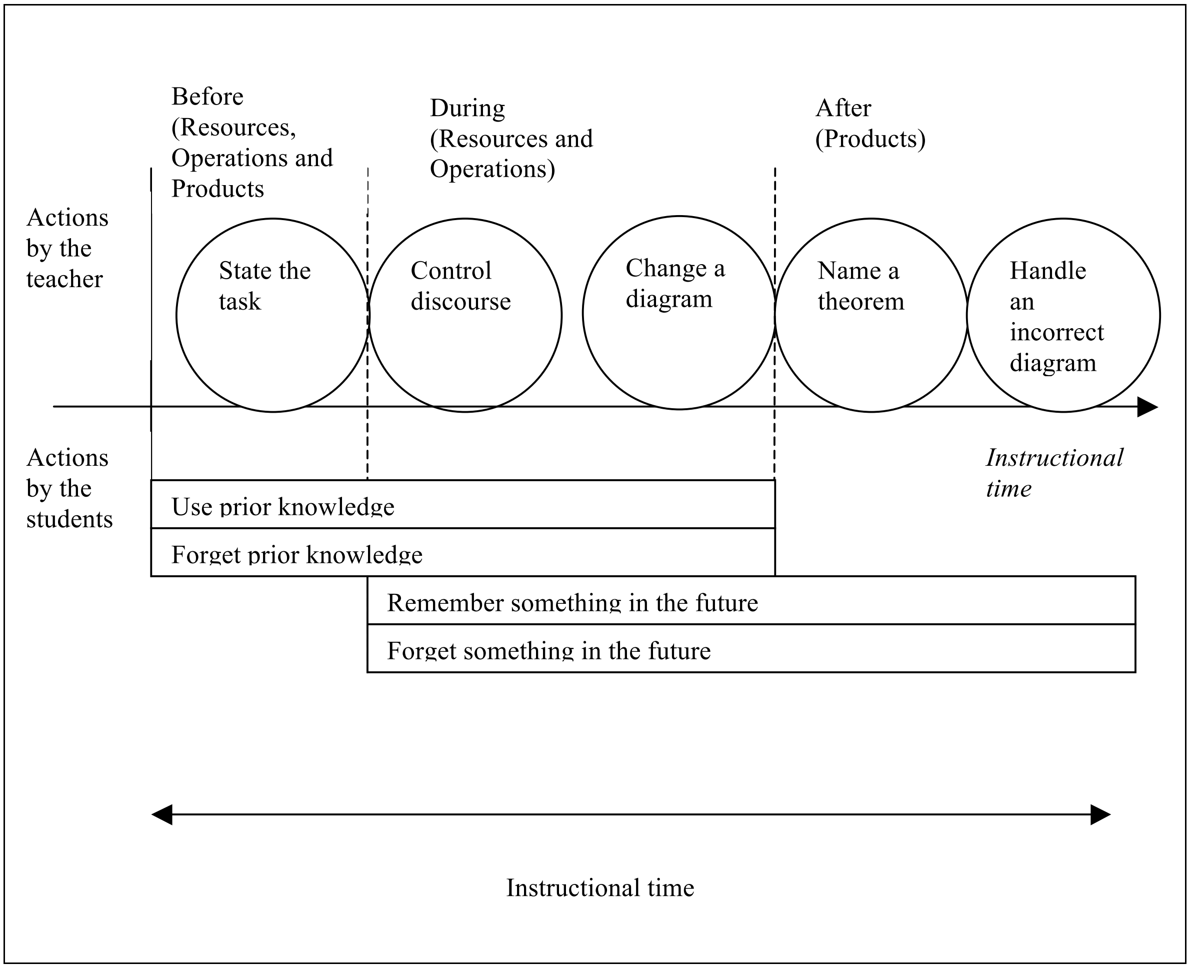

Figure 40. Model for teaching actions to manage students' prior knowledge.

I draw upon the model of a mathematical task as a set of *resources* and *operations* that students use to achieve a *product* (Doyle, 1988; Herbst, 2003, 2006). I use this model to conceive of teaching actions that use any of these elements—resources, operations, and products. For example, when stating a task, a teacher may remind students about the resources and the operations they need to achieve a task, as well as make explicit the products of the task. Then during students' work on a problem, a teacher may control whether or not to provide the resources and the operations for a task.

The teacher may do this by controlling discussions or by changing the diagram on the board. After the task has concluded, the teacher assumes the responsibility to emphasize what ought to be memorable and for preventing students from keeping memories of things and events that they should forget. This action is important to connect students' current work with topics that ought to be covered in the curriculum. For example, a teacher may use the name of a theorem to emphasize the memory of the concepts involved in the theorem.

The second study allows for three conjectures. First, teachers assume responsibility for triggering the prior knowledge students should use to work on a problem. Secondly, teachers assume responsibility for shaping what from the students' work on a problem should be memorable. These two conjectures underscore that teachers hold themselves accountable for making students remember. Finally, teachers assume responsibility for performing tactical moves in reaction to students' work on a problem in order to activate students' prior knowledge and to make things memorable in the future. Most tactical actions happen while students work on a problem. These actions involve how a teacher makes use of students' prior knowledge to inform further decisions in teaching. Some of these actions are timely because they happen at particular moments in response to students' actions. Other actions are strategic in that they do not require teachers to respond to moment-by-moment interactions, but could be controlled in anticipation of a task.

The distinction between strategic and tactical actions is important in describing the work of managing students' prior knowledge when teaching with a problem. Tactical actions are important when teaching with problems because a teacher uses students' work in order to make further decisions. Strategic actions are things that the teacher can plan for in advance. From the results of the second study I have identified levers that a teacher can control with the aim of shaping the collective memory. These levers include the statement of the task, discourse, the diagram on the board, and the name of a theorem. While the list is not meant to be comprehensive, the elements of the list allow for identifying differences between tactical and strategic actions that a teacher could accomplish when manipulating these levers. The levers identify possibilities for a teacher to perform strategic and tactical actions. For example, stating a task is a strategic action in that the teacher can control beforehand what resources, operations, and products should be explicit in the statement of the task and in what form. However, if the teacher were to modify the statement of the task in response of students' reaction—students might say that they do not remember a term or they might ask for more specifics about how to perform an operation—then, the teacher would be performing a tactical move.

Another example about the distinction between strategic and tactical actions is related to the conclusions of the first study. The first study illustrates that the work of teaching with problems involves teaching conceptual fields as Lampert (2001). So, if a teacher were to teach properties of quadrilaterals with the angle bisectors problem, the teacher would have to be strategic about the kinds of concepts involved in teaching with that problem. However, the connections between those concepts could require tactical moves. According to students' work a teacher would decide whether it is possible to establish connections between different concepts with the problem. As an illustration, Figure 41 shows different concepts related to the angle bisectors problem. If a teacher were to identify these concepts before students were to work on the problem, this would be a strategic action. The arrows identify possible connections that the teacher would have to do, using tactical actions. When performing these tactical actions, the teacher would have to take into account students' work on the problem to make decisions about what elements from the work on the problem should be memorable.

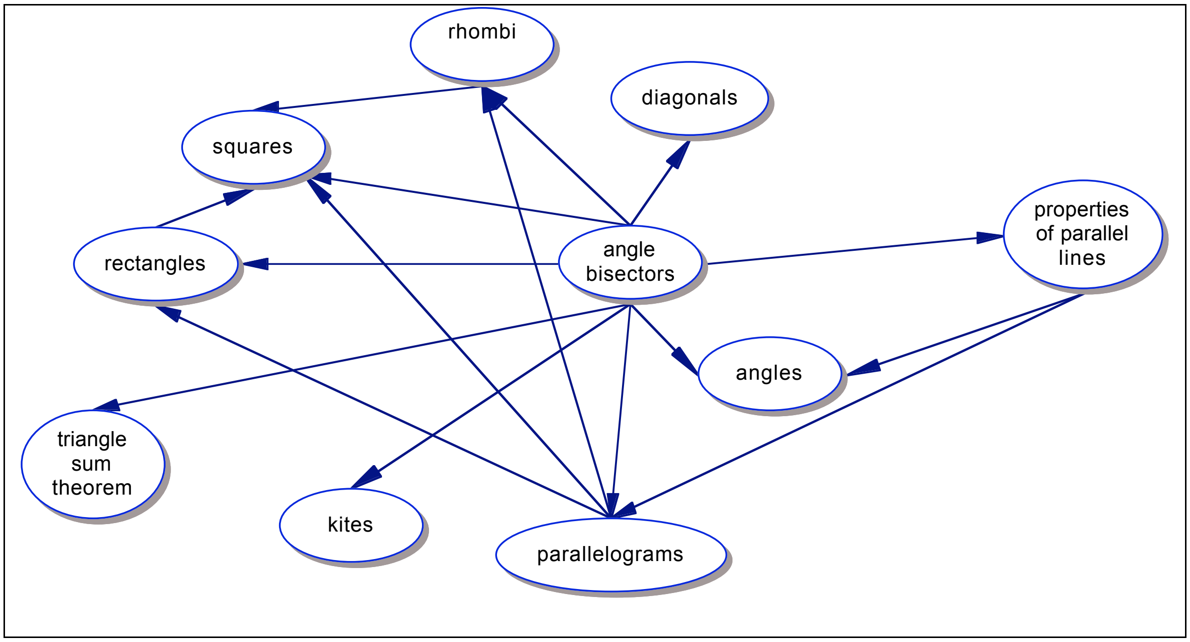

Figure 41. Concepts in teaching with the angle bisectors problem.

In general, teachers combine tactical and strategic moves to shape the collective memory. Strategic moves involve actions that a teacher could anticipate and plan for, especially before students start working on a mathematical task. Tactical moves involve actions that respond to students' work on a problem. In response to students' work, a teacher may activate prior knowledge or may prevent students from relying upon some memories.

The levers are useful when considering what elements a teacher could control in the enactment of the task to make students remember something from the past or to make something memorable in the future. For example, by controlling discourse a teacher could ask for justifications that would require students to use prior knowledge. Also, by controlling the diagrams on the board, a teacher could enable students to make connections between properties of different geometric figures. The intermediate actions to handle a diagram that I identified in chapter five are examples of tactical work. Strategic work happens at other times, for example at the moment of the design of a task, when a teacher aims at the knowledge at stake.

In conclusion, the second study provides more information about how teachers perceive they could shape the collective memory of the class by performing strategic and tactical moves. A difficulty of teaching with problems is that it requires the teacher to rely heavily on tactical moves. Teaching with problems requires tactical decision-making to handle students' memories of prior knowledge and to decide about what from the work of the problem should be memorable in the future. The levers allow for a better understanding of what a teacher perceives they could do at particular moments in the enactment of the task to shape the collective memory of the class, thus enabling students to work on a problem as an opportunity to learn something new.

The Hypothesis of the Collective Memory of the Mathematics Class

I formulated the hypothesis of the collective memory of the mathematics class with the purpose of studying how teachers manage students' prior knowledge. The hypothesis was useful to explain the rationality involved in a teacher's work when teaching with a problem. In shaping the collective memory, teachers do two kinds of actions. On the one hand, they incorporate new knowledge into the collective memory of the class. Teachers, on the other hand, set boundaries to what students are entitled to remember. The collective memory of the class is dynamic: it changes according to what the teacher perceives students should remember and according to the new knowledge introduced in the class. The teacher assumes the responsibility of deciding the content of the collective memory of the class.

The levers provide the means for the teacher to shape the collective memory of the class at particular moments in the enactment of a task. The levers are important because they give teachers opportunities to respond to student work, in particular by performing some tactical moves that incorporate the feedback from students. In my work I provide a catalogue of possible teaching actions to manage students' prior knowledge. It is not a comprehensive catalogue, but it includes actions that are plausible and reasonable from the perspective of teachers.

One could ask for reasons that a teacher may have for managing students' prior knowledge by means of creating a collective memory of the class. A possible answer could be related to conflicts between the organization of knowledge and how new experiences with a problem unfold. Guy Brousseau (1997) has noted that in reporting findings from the work on the problem, the traces of the work on the problem have to disappear. Brousseau says,

> [The mathematician] must conceal the reasons which led her in these directions and the personal influences which guided success. One must skillfully contextualize even ordinary remarks, while avoiding trivialities. One must look, too, for the most general theory within which the results remain valid. Thus, the

> producer of knowledge depersonalizes, decontextualizes and detemporalizes her results as much as possible." (Brousseau, 1997, p. 22)

In these remarks, Brousseau speaks about a contrast between the work of solving a problem and the work of producing new knowledge when reporting the results encountered through that problem solving. The mathematician needs to decontextualize, depersonalize, and detemporalize in order to make knowledge from the work on the problem. One could expect that a teacher who teaches with a problem would have to do something more in addition to guiding students' work with a problem. The teacher would need to decontextualize, depersonalize, and detemporalize students' experiences with the problem for students to learn mathematics.

A teacher's work to create a collective memory could have the aim of resolving some of the tensions provoked by producing knowledge that is personal but ought to be depersonalized; that emerged in a particular context, but that ought to be decontextualized; and that is connected to a particular temporal sequence of events, but that should remain detemporalized. While students may not be aware of the distinction between the new knowledge that they have produced and the conditions by which that new knowledge was produced, a teacher could make those distinctions clearer by manipulating students' memories.

Two Studies on A Teacher's Management of Students' Prior Knowledge

The two studies on a teacher's management of students' prior knowledge show two ways to study the same problem from different perspectives: an *etic* perspective with the analysis by an observer of a teacher's work and an *emic* perspective with the analysis of comments about teaching actions by practitioners. Both studies are descriptive in that

I describe teaching actions to manage students' prior knowledge. The first study provides evidence of a teacher who, in managing students' prior knowledge when teaching with a problem, shaped the memory of the class. The second study provides a catalogue of possible teaching actions that teachers consider available for them to manage students' prior knowledge during the enactment of a task.

The two studies are complementary. The first study is a proof of existence of the possibility that a teacher shapes the collective memory of the class to repair a breach provoked by the changes in the organization of knowledge spurred by the unit. The second study incorporates the voices of several geometry teachers who were immersed in the possibility of teaching with a problem. Their comments in reactions to the prompts show what it would take to manage students' prior knowledge if they were to teach with a problem.

Methodological Choices for the Analysis of Data

The choice of Systemic Functional Linguistics (SFL) for the analysis of data allowed me to study the question of how teachers manage students' prior knowledge in the two studies. The first study required examining prior knowledge in terms of the themes of the conversation. I used SFL to demonstrate that the teacher and the students were working on a proof. Since this proof was not written on the board in the usual two-column proof format, it is not obvious to an observer that they were doing a proof. I study the content of the proof to see what prior knowledge the teacher allowed students to use. The focus on ideational meanings allowed me to show how the teacher and the students used prior knowledge in the content of their talk. Then, the focus on attitudinal meanings enabled me to identify disputes between the teacher and the students about the prior knowledge that they could use in the proof.

In the second study SFL was useful to understand how, in their conversations about instruction, teachers who participated in the focus group sessions talked about prior knowledge. The focus on ideational meanings allowed me to examine prior knowledge as a theme in the content of their talk and also the division of labor between the teacher and the students with respect to managing prior knowledge. Then, by examining participants' attitudinal meanings I distilled their evaluations of possible teaching actions to manage students' prior knowledge.

Implications

Implications for the Study of Teaching

The literature on teacher thinking (Clark & Peterson, 1986) has been useful in understanding teachers' choices and the factors affecting those choices. One difference between this literature and the studies in this dissertation is the experimental set up. For example, some studies on teachers' decision-making were based upon stimulated recall or on the analysis of teachers' writings on journals. In contrast, data[76] collected for this dissertation proceeded from breaching experiments in instruction.

In ethnomethodology, breaching experiments confront individuals with unexpected changes to usual practices. In their reactions to those changes, individuals make their expectations explicit. In relation to breaching experiments, Mehan and Wood (1975) have said, "People interact without listing the rules of conduct…When the reality

[76] P. Herbst, Project Director of GRIP at the University of Michigan, designed the experiments in this dissertation.

is disrupted, the interactional activity structuring the reality becomes visible" (p. 24). For example, Garfinkel (1963/2005) reported a breaching experiment where individuals were asked questions about the meaning of something that they had said and that could be assumed to be shared knowledge in a conversation. Those individuals reacted with surprise to the change in their expectations about the conversation, at times showing what they had expected instead. The experimental nature of the studies in this dissertation allowed me to see how, in a breaching experiment, participants would repair changes in the usual way in which they manage students' prior knowledge by deliberately shaping the collective memory of the class.

The replacement unit on quadrilaterals was designed as an "instructional experiment" (Herbst, 2006) because it introduced changes to the usual practices in Megan's class by means of using a problem to teach a unit. Moreover, the unit required students to anticipate a theorem, the medial-line theorem, that had not been stated or proved in class yet. In that instructional experiment, the teacher had to cope with changes to her usual teaching of quadrilaterals. In doing so, she made evident the need to shape the collective memory of the class with the purpose of doing a proof.

In the focus group sessions, video episodes of classroom instruction were used as probes to provoke conversations about teaching (Herbst & Chazan, 2003). Participants of those focus group sessions were usual teachers of geometry, immersed into the possibility of teaching with a problem, which they did not routinely do. The videos were analogous to a breaching experiment in that they were intended to provoke reactions that would make overt teaching actions to manage students' prior knowledge. In their reactions, participants represent the response of the collective of practitioners—how practitioners would expect to take on the activity of teaching with a problem. In contrast with one-on-one interviews, in these focus groups participants had the chance to agree or disagree with each other. Moreover, participants were compelled to talk about what might have seemed to be appropriate for other practitioners to hear in a professional setting. Thus, in my analysis of the data I look for what is normative from the perspective of practitioners.

Another difference between the literature in teachers' decision-making and the studies in this dissertation is in the nature of the explanations for teachers' actions. Clark and Peterson (1985) concluded that models proposed for teachers' interactive decision-making "should reflect the definition of interactive decision making as a deliberate choice to implement a specific action rather than a choice of actions from several activities" (p. 277). Also, they said that a model for teachers' decision-making should include considerations other than teachers' ideas about students. In this study, the explanation of teachers' actions is given in terms of the *practical rationality* underlying those actions (Herbst & Chazan, 2003). So, one implication is that the problems teachers face when teaching with a problem are characteristic of the practice of teaching.

In the second study, the actions that teachers propose to cope with the problems of practice go beyond individual choices or individual beliefs, and respond to ways in which practitioners perceive and appreciate teaching actions. This is a distinction between the studies in this dissertation and the literature on beliefs (Cooney & Shealy, 1997; Cooney, Shealy, & Arvold, 1998; Isenberg, 1990; Pajares, 1992). I do not assume that an individual teacher's perspectives about teaching are constant in their practice, but that these perspectives could change according to particular circumstances in teaching, specifically according to the tasks, situations, and contracts in which they work. In addition, the literature on beliefs explains teachers' actions as a consequence of teachers' individual beliefs about big ideas such as mathematics, teaching, and learning. I propose an alternative to explanations of teaching actions in terms of teachers' beliefs by circumscribing the analysis of teaching actions to the circumstances surrounding those actions. The studies in this dissertation illustrate that teachers' decisions are related to the mathematical content that they teach and the activity of teaching that they do in their class.

The studies also focus on an object of study alternative to teacher knowledge. The issues that surfaced when managing students' prior knowledge are not about the mathematical knowledge that teachers possess. The problems that teachers need to manage are part of the problems of practice. Despite the fact that teachers would need to have a good understanding of the subject matter in order to teach with a problem, there is evidence in the first study that the teacher, with a good understanding of the mathematical ideas, had to handle difficulties because of changes in the organization of knowledge in the class. In the second study teachers conceived of alternatives ways of doing a proof for the claim that the angle bisectors of a rectangle make a square. However, the decision not to allow students to make changes in a diagram was guided by more than the knowledge of the proof; teachers considered students' work on the problem and how their current actions with a diagram would have an effect in the organization of knowledge.

A possible follow up to this study could explore the question of whether teaching actions to manage students' prior knowledge are contract-specific. For example, in an integrated mathematics course, one could expect that the boundaries of usable knowledge may be different because teachers may not need to alienate all the geometry that students have learned before. Likewise, in teaching Algebra 2 a teacher may have to rely upon students' prior knowledge in Algebra 1. A possible research question is whether the curriculum of other mathematics courses takes into account students' prior knowledge from previous mathematics courses. The contrast with the geometry course could be useful for understanding teaching actions for managing students' prior knowledge involved in different mathematics classes.

Implications for Research

This dissertation is about how teachers manage students' prior knowledge by focusing on the mathematical content of their work. The methodology uses ways for identifying references to students' prior knowledge when having discussions in mathematics classrooms and when having conversations about mathematics teaching. In my analysis I show that teachers hold different stances towards students' prior knowledge according to the content of the mathematical ideas involved. Therefore, further studies could take into account the content of mathematical ideas when investigating what prior knowledge teachers allow students to make use of.

The analysis of the rectangle episode in the first study is a methodological contribution of the dissertation, showing how to analyze oral arguments using conjunctions. This is important because, as a result of the analysis, I was able to demonstrate that the teacher and the students were doing a proof, even though the proof was not written on the board. That is, an oral performance could have the same discursive properties as a proof on the board. In spite of the fact that a proof was not written, the oral argument worked as a proof because the conjunctions were useful to organize the structure of the argument. It would be interesting to apply the analysis of conjunctions to other classroom interactions where arguments are made, with the purpose of having a better understanding of how teachers and students make oral arguments in mathematics classrooms.

Similarly, the analysis of the rectangle and the kite episodes involved showing how the same geometric object was visualized differently at different points in the argument. This is a case where speakers used resources from language to preserve cohesion. Speakers were talking about the same geometric figure in different ways. Through my analysis, I present cohesion chains that enabled speakers to preserve continuity and coherence in their conversation about the geometric figure. Methodologically, this contribution opens up the opportunities for understanding how one could show how teachers and students transform geometric objects in a conversation with the mediation of a diagram. The proofs involved working with the same diagram and visualizing that diagram in different ways. The analysis showed how cohesion was preserved in order to do those proofs.

The second study provides insights about the actions that teachers perform to manage students' prior knowledge at particular moments during the enactment of a mathematical task. I identify strategic and tactical actions that teachers could perform to activate prior knowledge and also to prevent students from relying upon knowledge that had not been officially introduced in class. The "levers" describe elements that a teacher could manipulate at different moments and for different purposes during the enactment of a task. These levers are helpful to describe how a teacher prompts students' prior knowledge.

A possible question for future research is: *How are strategic and tactical actions that aim at activating prior knowledge connected?* For example, it is possible that, before a lesson, a teacher would plan a lesson including strategic moves for activating students' prior knowledge. In that plan, the teacher could be anticipating students' work in the lesson, expecting that the manipulation of one lever (e.g., providing a diagram as a resource in the statement of the task) would make students remember prior knowledge needed to work on a problem. However, in the enactment of the lesson, there could be other things that the teacher would have to do—tactical moves—because the strategic action was insufficient to make students remember. One could investigate how one action (strategic or tactical) calls for another action.

In this dissertation, by studying teachers' perspectives on action, I have discovered that what I called levers are useful for teachers to observe teaching actions regarding the management of students' prior knowledge. The levers describe things that a teacher can act on. That is, means of control to mange students' prior knowledge. The new question would involve a description of what teachers' awareness of the levers could do for them when engaged in teaching. This is different from the studies in this dissertation. For the analysis of the first study the notion of the levers was not available to collect data because there was not an argument to anticipate that these levers would be evident to practitioners. The results of the second study and the notion of the levers are useful tools for collecting data of a new empirical study. In this new study one could observe teachers plan and conduct a lesson. From these observations one could examine, with the idea of the levers, how teachers incorporate tactical and strategic actions to take into account students' prior knowledge.

Implications for Teacher Education

A difficulty of managing students' prior knowledge when teaching with a problem is that teachers should act tactically, based upon their observations of students' work. These tactical moves would require knowledge about students and knowledge about the content for teachers to act at particular moments in enactment of a task. Professional development activities could be built around the idea of the levers to look at actions to activate prior knowledge in the enactment of a task.

For example, one could examine a lesson, based upon the model of Japanese lesson study, where participants would look at videos and see what the teacher can do to activate memories at different moments of the enactment of a task, using the levers. Lesson study is a Japanese professional development activity (Fernandez, 2002; Lewis, Perry, & Murata, 2006; Stigler & Hiebert, 1999). Teachers set some specific goals for students to achieve in a lesson in terms of curriculum guidelines and standards. Teachers use those goals to collaborate on designing a lesson plan. This lesson plan includes anticipations of students' work during the lesson and specifics about the observations they would do during the lesson. During the enactment of the lesson, teachers observe the lesson and collect data. These observations are useful for a post-lesson discussion where teachers discuss the data collected and re-design the lesson. Their work on that lesson becomes the basis for another cycle of lesson study.

Researchers have noted that, in adaptations of Japanese lesson study in the United States, American teachers have had difficulties making research questions and focusing their observations on students' work (Fernandez, Cannon, & Chokshi, 2003). In addition, American teachers have had difficulties using their observations to study a lesson. The levers could be used as categories for teachers to examine videos of a lesson. Because the levers are visible categories for practitioners, they could help them in studying how to use prior knowledge in their teaching. That is, the levers could be entry points for having conversations with teachers about their practice. For example, since the diagram is one of the levers, teachers could be asked to formulate a goal for the lesson related to how students use the diagram to remember something or to make something memorable in the future. Then, in their observations about the lesson, teachers could pay attention to how students make use of the diagram and how the teacher supports students' work with the diagram. So, an agenda for teachers' professional development could benefit from having teachers analyze their work by means of the levers. The levers could enable teachers to attend to students' opportunities to learn that result from using the levers to manage students' prior knowledge.

The levers could also provide means for teachers to talk about teaching. In reviewing literature on teaching with a focus on practitioner knowledge, Hiebert, Gallimore, and Stigler (2002) have identified as one of its features that practitioner knowledge is *linked with practice* and is also *detailed*, *concrete*, and *specific*. The levers could give a language for teachers to identify and to discuss specific teaching actions in relation to the **content** that they teach, and in relation to specific **moments** in the enactment of a task. Teachers can use the levers to publicly share their knowledge about teaching, thus making a transition from practitioner knowledge to professional knowledge.

Another implication for teacher education is related to the methodological aspects of this study. The methodology I apply to the analysis of an oral mathematical argument—conjunction analysis and the analysis of "being" statements—could be developed further as a methodological contribution for the teaching of mathematics. The rectangle episode illustrates the case of an experienced teacher's use of linguistic resources in guiding students to make a mathematical argument. Novice teachers could benefit from an awareness of how to use conjunctions with the purpose of enabling students to improve their communication skills in mathematics. Since language is a resource available for all teachers, this finding could be relevant for investing resources in teachers' use of language to teach mathematics.

Implications for Policymakers

Reform documents in mathematics education have asked teachers to teach with problems because it is a way to get students to do authentic mathematics and for students to become a different kind of student (NCTM, 2000). However, teaching with problems is difficult. This study gives a more complicated picture of what it takes for a teacher to teach with problems. The study also provides some ideas about what kinds of things teachers could do in order to manage students' prior knowledge when teaching with a problem, with the idea of the levers.

The kind of resources that teachers need to teach with a problem involve more than the individual skills that a teacher may possess. Teachers need other resources to attend to students' mathematical learning in ways where they could make use of students' prior knowledge. Lampert, Boerst, and Graziani (in press) report the work of teachers at Italiaidea Center for Italian Language and Culture studies as an example of a place that provides the infrastructure for teachers to adapt their lessons according to their observations of student work. Teachers in that school have a shared collection of activities for their lessons. These activities allow teachers to pay attention to student learning because they are connected to the kinds of skills that they want to teach their students. The activities are useful because teachers can compare their anticipation of students' work with the work that students actually do. So the activities enable teachers to have a common language about students' work in terms of their performance in those activities. In addition, the school's day schedule allows for moments when teachers could share their insights about their students with other colleagues. These opportunities are especially useful when students move from one class to another, because teachers share information about the prior knowledge that individual students' possess, with the expectation that the new class would support students in strengthening the skills that they lack of.

The traditional geometry curriculum does not support well teaching with problems. Teachers would need special curricular support to engage in this work—this could be provided by means of special professional development opportunities. Resources to support instructional needs could also help. For example a teacher could have available concept maps with information about how different concepts are connected to students' work on a problem. A teacher could also use anticipations of students' work on a problem in terms of the ideas that could come about, the errors that students could make, and alternatives for the order in which ideas could be developed. Teachers would need a curriculum with problems that are sufficiently rich to cover the content that they are supposed to teach. So, from the perspective of curriculum development, the elaboration of problems to teach the content of the curriculum, anticipations of students' work with those problems, and samples of possible ways in which teachers could handle discussions of the problems would be useful artifacts to prepare teachers to perform tactical moves when teaching with a problem.

Concluding Remarks

In 1903, John Dewey published an article entitled *The psychological and the logical in teaching geometry.* In this article he talked about the prior knowledge that students have of geometric concepts because of their experiences in the world and how the logic of geometry requires students to experience these concepts differently. He proposed to foster connections between students' experiences and the logical thinking that geometric knowledge affords:

> More than any other one thing it would seem as if the high-school pupil, in particular, were are the point where his greatest need is neither merely intuitive nor strictly demonstrative geometry, but rather skill in moving back and forth from the concrete situations of experience to their abstracts in geometric statements. (Dewey, 1903, p. 398)

More than 100 years later, these words have resonance for the teaching of geometry. Problem-based instruction could provide the means for teachers to merge students' experiences in the world and the abstract knowledge of geometry.

There is an expectation for teachers to engage in problem-based instruction. The idea of teaching with a problem is not just to do a problem, but to use the problem as an opportunity to teach new content that will satisfy the *didactical contract* of the geometry class. There is also an expectation for teachers to draw upon individual students' prior knowledge. Studies on learning recommend teachers to activate students' prior knowledge (Brown, Bransford, Ferrara, & Campione, 1983). But it is complicated for a teacher to activate the prior knowledge that individual students have. Teachers would have to cope with the individual memories of many students. As a way to manage students' prior knowledge, teachers use a representation of the knowledge of the class: the collective memory of the class.

A teacher's work to shape the collective memory of the class involves tactical and strategic actions when students work on a mathematical task. While strategic actions could be planned beforehand, tactical actions require giving feedback to students' work. Tactical actions are part of the complexities of teaching because it could be hard for a teacher to anticipate how to use student work at a particular moment in the enactment of a task. The kind of tactical work that is required in teaching has been characterized as improvisation (King, 2001; Sawyer, 2004). By improvisation it is not meant to elide teachers from some standards on performance. On the contrary, improvisation requires knowledge of a catalogue of possible moves, of the context where some moves could be performed, and of the timing for performing such moves. This view of teaching as a performance involves a deep understanding of the students, of the subject matter, of the curriculum, and of the possibilities of teaching actions. If the reform pushes teachers to engage in problem-based instruction, there should be more support for teachers to learn how to perform tactical moves at the moment of engaging students in a mathematical task.

APPENDICES

APPENDIX A

TITLES OF CHAPTERS IN THE GEOMETRY TEXBOOK USED IN MEGAN'S CLASS

Chapter 1: Discovering points, lines, plans, and angles

Chapter 2: Connecting reasoning and proof

Chapter 3: Using perpendicular and parallel lines

Chapter 4: Identifying congruent triangles

Chapter 5: Applying congruent triangles

Chapter 6: Exploring quadrilaterals

Chapter 7: Connecting proportion and similarity

Chapter 8: Applying right triangles and trigonometry

Chapter 9: Analyzing circles

Chapter 10: Exploring polygons and area

Chapter 11: Investigating surface area and volume

Chapter 12: Continuing coordinate geometry

Chapter 13: Investigating loci and coordinate transformations

(Boyd, Burrill, Cummins, Kanold, & Malloy, 1998)

APPENDIX B

TRANSCRIPTION OF THE VIDEO EPISODE

The Bisectors of A Rectangle Make A Square

1. Megan: Jack, Jack what shape do you have?
2. Jack: Square.
3. Megan: The inside you got a square?
4. Jack: Yes.
5. Megan: What's the outside?
6. Jack: A rectangle
7. Megan: Okay, how many other groups got that if you have a square you get, or a rectangle, you get a square on the inside? Okay, what I would like to do is talk about why some of these might be true, let's look at somebody's that has that on their screen. Who has that on their screen already and can bring it up here? Dewey, bring it here. We'll look at his calculator. Okay and we might talk about why his looks different than some people's too.
8. Callie: Mine's in the middle
9. Megan: Okay, okay, whose is in the middle? Okay you know what I'm going to mess with Dewey's picture then and we'll see if we can make it more like Callie and Jackie's. Cause Jackie, you had it inside too, didn't you?
10. Jackie: No, I was just looking at Callie's.
11. Megan: Okay, you got a rectangle, okay you can see it does look like it's still a square. Who has a picture that looks more like this? Who got one that looks more like this? Just their group? Okay, Anthony, you did? Say that you wanted to prove that that thing was a square. Okay, how could I maybe go about showing that if I start with a rectangle that that inside thing will always be a square? Dewey, you got an idea?

12. Dewey: Use the measure tool.
13. Megan: No. But that is a good idea to start with, to see is it really a square, maybe you could do that just to get a feel for, does it seem like my hypothesis is true…Anthony, what would I have to show to show that it's a square? What properties do squares have that I could use here?
14. Anthony: All the sides are equal. All the angles are 90.
15. Megan: Okay, does anybody see a way that I could get any of those things, I could get one of Anthony's conditions, either that all the sides are equal or that all the angles are 90. Jackie?
16. Jackie: We know that this is 45.
17. Megan: Why?
18. Jackie: Because this is an angle bisector. Geez, you scared me! And this is 45 then we know that um, this is 45
19. Megan: Why?
20. Jackie: Because it's an angle bisector! And this is 45. So that means that this is 90 degrees right here, so like totaling, and then this triangle is 180 degrees minus 90 and this would be 90 degrees and this and this are vertical so they'd be 90. And this and this is supplementary so this is 90 and this is supplementary to that so this is 90 and this is 90 and this is 90.
21. Megan: Now what is it Jackie?
22. Jackie: So now it's a rectangle. Or it can be a square.
23. Megan: How can it be a square?
24. Jackie: Cause it has four 90 degrees angles.
25. Megan: So it's a rectangle, for sure, it might be a square.
26. Jackie: And um, don't we know that like the angle bisectors are like congruent? Or?
27. Megan: The diagonals of a rectangle are congruent we know that.
28. Jackie: So, are these, can I draw diagonals in would that help or no?
29. Megan: Okay, let's think about that, if you're going to put an extra line in, usually you like have a, some kind of goal for why you're putting the line in, so

if you're going to put the diagonal in Jackie, what would your goal be, why would you be doing that? Dewey, you got an idea?

30. Dewey: Yeah, um, you could use the, you could just draw one in say from *X* to *Z* and then you know it would bisect the angles and then you could do um, then the two angles in the top triangle would be equal
31. Megan: Okay wait. Does that help me? Jack, you thought you had an idea, what were you going to do?
32. Jack: Well, if you had a line from *Y*, from *W* to *Y*, and then you proved that those two triangles were congruent you could say that *WX* is congruent because…
33. Megan: And that was Dewey's idea earlier, Dewey said, he said draw in this one.
34. Samuel: Isn't that the altitude?
35. Megan: Altitude of what?
36. Samuel: Of *AB* or *CD*…never mind
37. Megan: Who else has something? Who else sees some way that I could prove…?
38. Jackie: Okay, ah, I got it, if this is congruent to this because it's an isosceles triangle then uh, and this is 45 and we know this is 90 then this has to be 45
39. Megan: That little triangle?
40. Jackie: Yeah, Does that help? This would be 45 and this is congruent to that, so if this is congruent to that and then this is congruent to that, then that means that this is congruent to this you know
41. Megan: Okay, stop for a minute. Do you see those little marks you have on there on, my eyes are bad, I have to get closer, on *WZ*, do you see that *WZ* isn't equal to *AZ*?
42. Jackie: Where's *AZ*?

    Megan: That whole thing, you put little marks.
43. Jackie: No, they are congruent.
44. Megan: No, there's still a little piece there from *W* out to the edge. But what do I know about those little pieces?
45. Jackie: These are congruent

46. Megan: Okay, now Michael thinks they are. Michael said those little pieces are congruent. Okay, why are those little pieces congruent, Michael you got a reason?
47. Michael: Well no I was actually talking about…
48. Megan: But Michael's way is perfectly good, he's looking at this little triangle saying that's 90 so what do you know about this little triangle?
49. Jackie: They're isosceles.
50. Megan: It's isosceles. Okay, um, does that help me? Who else sees some way that I could prove that this little piece is equal to some of these other ones? Cause I can do that.
51. Samuel: You can do the exact same thing with the bottom one…
52. Megan: These little triangles, why would that help me?
53. Samuel: I have no idea.
54. Jackie: Because you subtract it from there and then you find out if the other triangle is the same, yeah, segment addition postulate.
55. Megan: I'm going to use segment addition. Okay, let's start doing this. Let's prove this things a square…Okay, we're gonna, we're not gonna write out all this stuff about the angles being 90 because people already said a pretty convincing argument so we already know that uh, I'm gonna put numbers in here, angle 1, 2, and 3, we're gonna have all these steps in between. Angle 1 equals angle 2 equals angle 3 equals angle 4 equals 90…Okay, now we're going for the sides so Jackie start talking…Jackie went with triangle *ABZ* is equal to triangle *DXC* and she used angle-side-angle. Okay, so now Jackie proved these two triangles are congruent…*AZ* is congruent to *XC*.
56. Jackie: Wait, let's not look at those triangles, we're gonna look at *BAZ* and *BZ*—at the end of the thing…
57. Megan: I do want to say, I want you to pay attention because Jackie does know what she's doing, Jackie has a really good idea here…
58. Jackie: So we know this whole thing is congruent to that which is congruent to this…no, wait, I'm trying to say that if you have this and you subtract that, these ones, then you get this.

59. Megan: Okay, sit down I’m gonna say that in English. Okay, we’re not gonna try and write steps we’re just gonna talk about what Jackie is saying…She can prove that these two segments are equal, she can prove these two segments are equal and she can use that same argument on these two segments. Jackie was saying, she can show that all these big segments here are equal, okay where’s the other one? Here’s one, here’s one, and now what Dewey’s saying is, can’t you just subtract off all these little equal triangles? Yes, this is what Jackie is saying. And Callie, Callie said we’ve gotta use segment addition I can show that these two little segments are equal and these two little segments are equal because I have little isosceles triangles so I can subtract off these little equal parts and get that the sides are equal. Okay, we’re killing this problem, we’re stopping here…

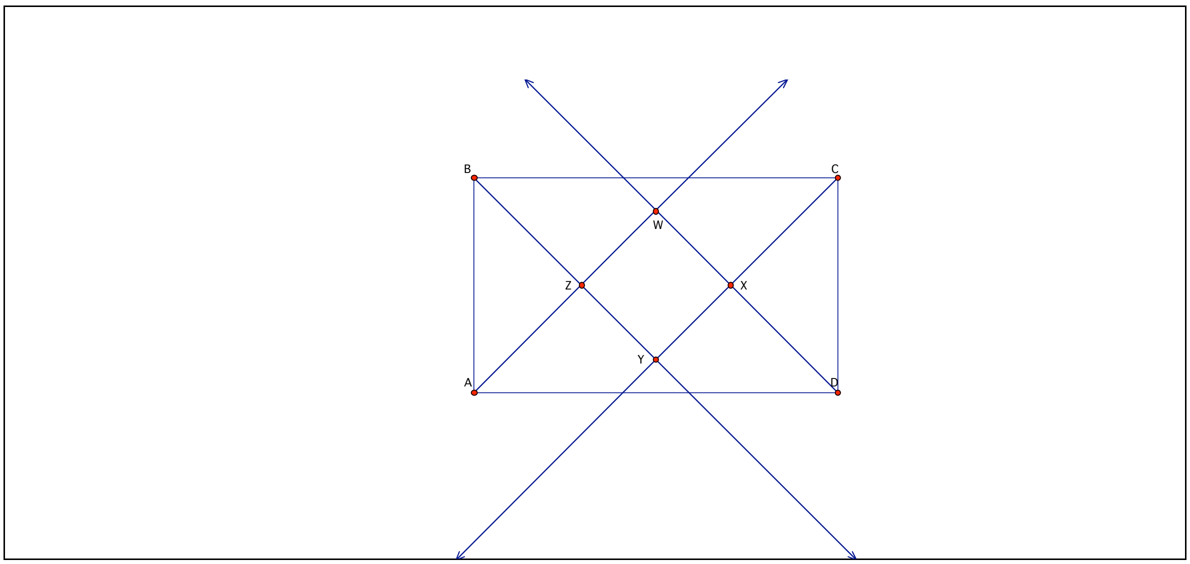

Figure 42. Diagram in the video episode the bisectors of a rectangle make a square.

REFERENCES

REFERENCES

Achugar, M. (2004). The events and actors of 11 September 2001 as seen from Uruguay: Analysis of daily newspaper editorials. *Discourse and Society, 15*(2-3), 291-320.

Ball, D.L., & Bass, H. (2000). Making believe: The collective construction of public mathematical knowledge in the elementary classroom. In D. Phillips, (Ed.), *Yearbook of the National Society for the Study of Education, Constructivism in Education*, (pp. 193-224). Chicago: University of Chicago Press.

Borasi, R. (1994). Capitalizing on errors as "springboards for inquiry": A teaching experiment. *Journal for Research in Mathematics Education, 25*(2), 166-208.

Borko, H., Atwood, N., & Shavelson, R. (1979). Teachers' decision making. In P. Peterson and H. Walberg, H. (Eds.), *Research on Teaching: Concepts, findings and implication* (pp. 136-160). *Berkeley, CA: McCutchan.*

Boyd, C., Burrill, G., Cummins, J., Kanold, T., & Malloy, C. (1998). *Geometry.* Westerville, OH: Glencoe.

Brach, C. (2004). *Learning to remember: Mathematics teaching practice and students' memory.* Unpublished manuscript. University of Michigan.

Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des Mathematiques 1970-1990* (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Trans.). Dordrecht, The Netherlands: Kluwer.

Brousseau, G., & Centeno, J. (1991). Rôle de la mémoire didactique de l'enseignant. *Recherches en Didactique de Mathématiques, 11*(2-3), 167-210.

Brown, A. L., Bransford, J. D., Ferrara, R. A., & Campione, J. C. (1983). Learning, remembering, and understanding. In J. H. Flavell & E. M. Markman (Eds.), *Handbook of child psychology: Vol. 3. Cognitive development* (4th ed., p. 78-166). New York: Wiley.

Chazan, D. (1995). *Where do student conjectures come from? Empirical exploration in mathematics classes.* National Center for Research on Teacher Learning, Craft Paper 95-8. East Lansing, MI: Michigan State University.

Chazan, D., & Ball, D. (1999). Beyond being told not to tell. *For the Learning of Mathematics, 19*(2), 2-10.

Chazan, D., & Lueke, H. M. (in press). Exploring the tensions between disciplinary knowledge and school mathematics: Implications for reasoning and proof in school mathematics. In D. Stylianou, E. Knuth, & M. Blanton (Eds.), *Teaching and learning mathematical proof across the grades.* New York: Routledge.

Chevallard, Y. (1985). *La Transposition Didactique: Du savoir savant au savoir enseigné*. Grenoble, France: La Pensée Sauvage.

Christiansen, I. M. (1997). When negotiation of meaning is also negotiation of task. *Educational Studies in Mathematics, 34,* 1-25.

Christie, F. (2002). *Classroom discourse analysis: A functional perspective.* London: Continuum.

Clark, C. M. & Peterson, P. L. (1986). Teachers' thought processes. In M. C. Wittrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 255-296). New York: Macmillan.

Clemens, S. R., O'Daffer, P. G., Cooney, T. J., & Dossey, J. A. (1992). *Geometry.* Menlo Park, CA: Addison-Wesley.

Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119-142.

Cooney, T. J., & Shealy, B. E. (1997). On understanding the structure of teachers' beliefs and their relationship to change. In E. Fennema & B.S. Nelson (Eds.), *The mathematics teachers in transition* (pp. 87-109). Mahwah: Lawrence Erlbaum.

Cooney, T. J., Shealy, B. E., & Arvold, B. (1998). Conceptualizing belief structures of preservice secondary mathematics teachers. *Journal for Research in Mathematics Education, 29*(3), 306-333.

Cox, D. A. (1994). Introduction to Fermat's last theorem. *American Mathematically Monthly, 101* (1), 3-14.

Dewey, J. (1903). The psychological and the logical in teaching geometry. *Educational Review, 25,* 387-399.

Dochy, F., Segers, M., & Buehl, M. (1999). The relation between assessment practices and outcomes of studies: The case of research on prior knowledge. *Review of Educational Research, 69*(2), 145-186.

Douglas, M. (1986). *How institutions think.* Syracuse, NY: Syracuse University Press.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist, 23*(2), 167-180.

Eggins, S., & Slade, D. (1996). *Analysing casual conversation.* London: Cassell.

Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics Teaching*: *The State of the Art* (pp. 249-254). London: Falmer Press.

Erickson, F. (2004). *Talk and social theory*. Malden, MA: Polity.

Fenstermacher, G. D. (1985). Philosophy of research on teaching: Three aspects. In M.C. Wittrock (Ed.), *Handbook of research on teaching* (pp. 37-49). New York: Simon & Schuster Macmillan.

Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. *Journal of Teacher Education, 53*(5), 393-405.

Fernandez, C., Cannon, J., & Chokshi, S. (2003). A US-Japan lesson study collaboration reveals critical lenses for examining practice. *Teaching and Teacher Education, 19*, 171-185.

Flückiger, A. (2005). Macro-situation and numerical knowledge building: The role of pupil's didactic memory in classroom interactions. *Educational Studies in Mathematics, 59,* 59-84.

Garfinkel, H. (2005). A conception of and experiments with "trust" as a condition of concerted stable actions. In J. O'Brien (Ed.), *The production of reality: Essays and readings on social interaction*. Thousand Oaks, CA: Pine Forge Press. (Original work published in 1963)

Garfinkel, H. (1967). *Studies in ethnomethodology*. Englewoods Cliffs, NJ: Prentice-Hall.

Goffman, E. (1986). *Frame analysis*. Boston, MA: Northeastern University Press. (Original work published in 1974)

González, G., & Herbst,, P. (2006). Competing arguments for the geometry course: Why were American high school students supposed to study geometry in the twentieth century? *International Journal for the History of Mathematics Education, 1*(1), 7-33.

González, G. & Herbst, P. (2008, March). *How teachers of geometry use diagrams as repository of the collective memory of a class.* Paper presented at the American Educational Research Association annual meeting, New York, NY.

Goodwin, C. (1987). Forgetfulness as an interactive resource. *Social Psychology Quarterly, 50*(2), 115-130.

Grabiner, J. V. (1974). Is mathematical truth time-dependent? *American Mathematically Monthly, 81,* 354-365.

Greenhoot, A. F. (2000). Remembering and understanding: The effects of changes in underlying knowledge on children's recollections. *Child Development 71*(5), 1309-1328.

Halbwachs, M. (1992). *On collective memory*. Chicago: The University of Chicago Press.

Halliday, M. A. K., & Hasan, R. (1976). *Cohesion in English*. London: Longman.

Halliday, M. A. K., & Matthiessen, C. (2004). *An Introduction to Functional Grammar* (3rd ed.). London: Hodder Arnold.

Hasan, R., & Cloran, C. (1990). A sociolinguistic interpretation of everyday talk between mothers and children. In M. A. K. Halliday, J. Gibbons, & H. Nicholas (Eds.), *Learning, keeping, and using language* (pp. 16-21). Selected papers from the 8th World Congress of Applied Linguistics, Sydney, August, 1987. Amsterdam/Philadelphia: John Benjamins Publishing Company.

Henderson, K. B. (1947). Hypotheses as a point of departure in geometry teaching. *Mathematics Teacher, 40*(4), 176-177.

Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education, 28*(5), 524-549.

Herbst, P. (2002). Engaging students in proving: A double bind on the teacher. *Journal for Research in Mathematics Education, 33* (3), 176-203.

Herbst, P. (2003). Using novel tasks in teaching mathematics: Three tensions affecting the work of the teacher. *American Educational Research Journal, 40*(1), 197-238.

Herbst, P. (2004). Interactions with diagrams and the making of reasoned conjectures in geometry. *Zentralblatt für Didaktik der Mathematik 36*(5), 129-139.

Herbst, P. (2005). Knowing about "equal area" while proving a claim about equal areas. *Recherches en Didactique des Mathématiques, 25*, 11-56.

Herbst, P. (2006). Teaching geometry with problems: Negotiating instructional situations and mathematical tasks. *Journal for Research in Mathematics Education, 37*(4), 313-347.

Herbst, P., & Brach, C. (2006). Proving and "doing proofs" in high school geometry classes: What is "it" that is going on for students and how do they make sense of it? *Cognition and Instruction, 24*(1), 73-122.

Herbst, P., & Chazan, D. (2003). Exploring the practical rationality of mathematics teaching through conversations about videotaped episodes: the case of engaging students in proving. *For the Learning of Mathematics, 23*(1), 2-14.

Herbst, P., Chazan, D., & Nachlieli, T. (2007, April). *Toward an Experimental Paradigm for the Study of Mathematics Teaching: The Case of "Installing a Theorem."* Paper presented at the American Educational Research Association annual meeting, Chicago, IL.

Herbst, P., Chen, C., Weiss, M., & González, G., with Nachlieli, T., Hamlin, M., & Brach, C. (in press). "Doing proofs" in geometry classrooms. In D. Stylianou, E. Knuth, & M. Blanton (Eds.), *Teaching and learning mathematical proof across the grades.* New York: Routledge.

Herbst, P., Hsu, H. Y., Chen, C. L., González, G., & Jeppsen, A. (2007, April). *Conceptions of "figure" in high school geometry instruction.* Poster presented at the American Educational Research Association annual meeting, Chicago, IL.

Herbst, P., & Nachlieli, T. (2007, April). *Studying the practical rationality of mathematics teaching: What goes into installing a theorem in geometry?* Paper presented at the American Educational Research Association annual meeting, Chicago, IL.

Herbst, P., Hsu, H. Y, Weiss, M., Chen, C. L., & González, G. (2008). *Instructional situations and students' opportunities to reason in the high school geometry class.* Manuscript in preparation, University of Michigan, Ann Arbor.

Hiebert, J., Gallimore, R., & Stigler, J. W. (2002). A knowledge base for the teaching profession: What would it look like and how can we get one? *Educational Researcher, 31*(5), 3-15.

Isenberg, J. P. (1990). Teachers' thinking and beliefs and classroom practice. *Childhood Education, 66*(5), 322-327.

Jacobs, J. K., Hiebert, J., Givvin, K. B., Hollingsworth, H., Garnier, H., & Wearne, D. (2006). Does eighth-grade mathematics teaching in the United States align with the NCTM "Standards?" Results from the TIMSS 1995 and 1999 video studies. *Journal for Research in Mathematics Education, 37*(1), 5-32.

Jacobs, J. K., & Morita, E. (2002). Japanese and American teachers' evaluations of videotaped mathematics lessons. *Journal for Research in Mathematics Education, 33*(3), 154-175.

Johnson, R. J., Greenhoot, A. F., Glisky, E., & McCloskey, L. A. (2005). The relations among abuse, depression, and adolescence autobiographical memory. *Journal of Clinical Child and Adolescent Psychology, 34*(2), 235-247.

Jurgensen, R. G., Brown, R. G., & Jurgensen, J. W. (1994). *Geometry.* Boston, MA: Houghton Mifflin.

Kilpatrick, J. (1981). The reasonable ineffectiveness of research in mathematics education. *For the Learning of Mathematics, 2*(2), 22-29.

King, K. D. (2001). Conceptually-oriented mathematics teacher development: Improvisation as a metaphor. *For the Learning of Mathematics, 21*(3), 9-15.

King, P. M. (2000). Learning to make reflective judgments. *New Directions for Teaching and Learning, 82,* 15-26.

Kitcher, P. (1983). *The nature of mathematical knowledge.* New York: Oxford University Press.

Lampert, M. (1993). Teacher's thinking about students' thinking about geometry: The effects of new teaching tools. In J. L. Schwartz, M. Yerushalmy, & B. Wilson (Eds.), *The Geometric Supposer: What is it a case of?* (pp. 143-177). Hillsdale, NJ: Erlbaum.

Lampert, M. (2001). *Teaching problems and the problems of teaching.* New Haven, CT: Yale.

Lampert, M., Boerst, T., & Graziani, F. (in press). Organizational resources in the service of school-wide ambitious teaching practice. *Teachers College Record.*

Lemke, J. (1998). Resources for attitudinal meaning: Evaluative orientations in text semantics. *Functions of Language, 5*(1), 33-56.

Lemke, J. (2000). Across the scales of time: Artifacts, activities, and meaning in ecosocial systems. *Mind, Culture, and Activity, 7*(4), 273-290.

Lewis, C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*(3), 3-14.

Martin, J. R., & Rose, D. (2007). *Working with discourse: Meaning beyond the clause* (2nd ed). London: Continuum.

Martin, J. R., & Rose, D. (2003). *Working with discourse: Meaning beyond the clause.* London: Continuum.

Martin, J. R., & White, P. R. R. (2005). *The language of evaluation: Appraisal in English.* New York: Palgrave Macmillan.

Mehan, H. & Wood, H. (1975). *The reality of ethnomethodology.* Malabar, FL: Krieger.

Middleton, D. (1997). The social organization of conversational remembering: Experience as individual and collective concerns. *Mind, Culture, and Activity, 4*(2), 71-85.

Middleton, D., & Brown, S. D. (2005). *The social psychology of experience: Studies in remembering and forgetting.* London: SAGE.

Miyakawa T., & Herbst P. (2007). The nature and role of proof when installing theorems: the perspective of geometry teachers. In J. H. Woo, H. C. Lew, K. S. Park & D. Y. Seo (Eds.), *Proceedings of the 31st Conference of the International Group for the Psychology of Mathematics Education* (Vol.3, pp. 281-288), Seoul, Korea: PME.

Nachlieli, T., & Herbst, P. (2006) *Facilitating encounters among teachers with representations of teaching: Two registers*. Unpublished manuscript, University of Michigan.

Nachlieli, T., & Herbst, P. with González, G. (in press). Seeing a colleague encourage a student to make an assumption while proving: What teachers put to play in casting an episode of geometry instruction. *Journal for Research in Mathematics Education*.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics.* Reston, VA: National Council of Teachers of Mathematics.

National Research Council. (2000). *How people learn: Brain, mind, experience, and school,* J. Bransford, J., Pellegrino, D. Berliner, M. Cooney, A. Eisenkraft, H. Ginsburg, et al. (Eds.). Washington, DC: National Academy Press.

Nuthall, G. (2000a). The anatomy of memory in the classroom: Understanding how students acquire memory processes from classroom activities in science and social studies units. *American Educational Research Journal, 37*(1), 247-304.

Nuthall, G. (2000b). The role of memory in the acquisition and retention of knowledge in science and social studies units. *Cognition and Instruction, 18*(1), 83-139.

O'Halloran, K. I. (2005). *Mathematical discourse: Language, symbolism, and visual images.* London: Continuum.

Oliver, P. N. (2001). Pierre Varignon and the parallelogram theorem. *Mathematics Teacher, 94*(4), 316-319.

Pajares, M. F. (1992). Teachers' beliefs and educational research: Cleaning up a messy construct. *Review of Educational Research, 62*(3), 307-332.

Perrin-Glorian, M. J. (1993). Questions mathématiques soulevées à partir de l'enseignement des mathématiques dans les classes faibles. *Recherches en didactique des mathématiques, 13*(1.2), 95-118.

Santagata, R., & Stigler, J. W. (2000). Teaching mathematics: Italian lessons from a cross-cultural perspective. *Mathematics Thinking and Learning, 2*(3), 191-208.

Sawyer, R. K. (2004). Creative teaching: Improvisation in the constructivist classroom. *Educational Researcher, 33*(2), 12-20.

Saylor, M. M., Baird, J. A., & Gallerani, C. (2006). Telling others what's new: Preschoolers' adherence to the given-new contract. *Journal of Cognition and Development, 7*(3), 341-379.

Shavelson, R. J. (1981). Research on teachers' pedagogical thoughts, judgments, decisions, and behavior. *Review of Educational Research, 51*(4), 455-498.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455-488.

Stigler, J. W., & Hiebert, J. (1999). *The teaching gap.* New York: The Free Press.

Vergnaud, G. (1988). Multiplicative structures. In J. Hiebert & M. Behr (Eds.), *Number concepts and operations in the middle school* (pp. 41-52). Reston, VA: NCTM.

Wagner, D., & Herbel-Eisenmann, B. (2008). "Just don't": The suppression and invitation of dialogue in the mathematics classroom. *Educational Studies in Mathematics, 67,* 143-157.

Weiss, M. & Herbst, P. (2007, April). *Every single little proof they do, you could call it a theorem: Translation between abstract concepts and concrete objects in the Geometry classroom.* Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.

Wertsch, J. V. (1998). *Mind as action.* New York: Oxford University Press.

Wertsch, J. V. (2002). *Voices of collective remembering.* Cambridge: Cambridge University Press.

Williams, G. (2001). Literacy pedagogy prior to schooling: Relations between social positioning and semantic variation. In A. Morais, I. Neves, B. Davies, & H. Daniels (Eds.), *Towards a sociology of pedagogy* (pp. 17-45). New York: Peter Lang.

Williams, G. (2005). *Memory, literacy, and the ontogenesis of vertical discourse.* Paper presented at the International Systemic Functional Linguistics Congress, Sidney.

Wilson, S. (2003). *California dreaming: Reforming mathematics education.* New Haven, CT: Yale University Press.