

# Minimum Diameter Stalagmites

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## ABSTRACT

The theory of Franke setting forth the factors controlling equilibrium stalagmite diameter at high drip rates is extended to the low-flow situation in which a minimum diameter is obtained. It is shown that the minimum cross-sectional area for a stalagmite must be determined by the ratio of incident drop volume to the thickness of the water film at the apex. Reasonable values for these quantities predict a minimum diameter of about 3 cm, close to that observed. An approximate model, the primary feature of which is the repetitive transient relaxation of the tip-growth of a stalagmite between drop impacts, is used to bridge between the high and the low flow regimes. The importance of presently little-known factors involving drop impact, mixing, crystallization from solution, and film flow in determining equilibrium stalagmite morphology are brought out.

## INTRODUCTION

In a previous paper (Curl, 1972) the question of the smallest possible stalactite diameter was discussed and its answer was shown to involve the forces of surface tension and gravity. It is apparent that the identical factors cannot be operable in the complementary speleothem, the stalagmite, if only because of the absence of the phenomenon of the pendant drop. In addition, the normal range of diameters of stalagmites never includes any below about an inch—considerably larger than the 5 mm or so diameter of soda-straw stalactites. What, then, determines the lower limit on the size of a stalagmite?

Interest in stalagmite size goes back many years and has revolved primarily around the possibility of using such information for dating cave deposits and artifacts. The early observations are mentioned by Allison (1923), who attempted an ambitious classification of stalagmites into 32 "types" according to the formative factors of drip rate, air circulation, relative humidity, temperature, and solution concentration. This was all based on the observation of five stalagmites deposited from water that had leached lime from concrete. Some of Allison's conclusions seem reasonable, but his complete

theory has little physical or chemical basis and has not been fruitful. The smallest of his study stalagmites had a diameter of 3.5 cm. His smallest theoretical type would have a diameter of 5 millimeters, but this has not been observed in nature. Allison's most useful conclusion (actually, a hypothesis) was that *symmetry (meaning uniform diameter) in a stalagmite indicates constant growth conditions*, and that such stalagmites are growing vertically without an attendant increase in diameter (as shown in Figure 3).

Hendrix (1950), writing about caves for a general audience, attempted to present Allison's theory. He introduced the better term "equilibrium diameter" to describe the growth of stalagmites with a constant diameter. Through an obscure argument he concluded that

$$d = K \sqrt{\frac{q}{E}} \quad (1)$$

where  $d$  is the equilibrium diameter,  $K$  a constant,  $q$  the flow rate ("drip rate") of depositing solution, and  $E$  the "evaporation rate". As will be seen, this is similar to the "modern" theory for one regime of stalagmite growth.

Figure 1 shows examples of near-equilibrium stalagmite growth. Because the diameters of several adjacent *nearly* constant diameter stalagmites do differ, the conclu-

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Figure 1. Nearly uniform stalagmites in Medvedia Cave, Czechoslovakia. Photo by Vojtech Benický.

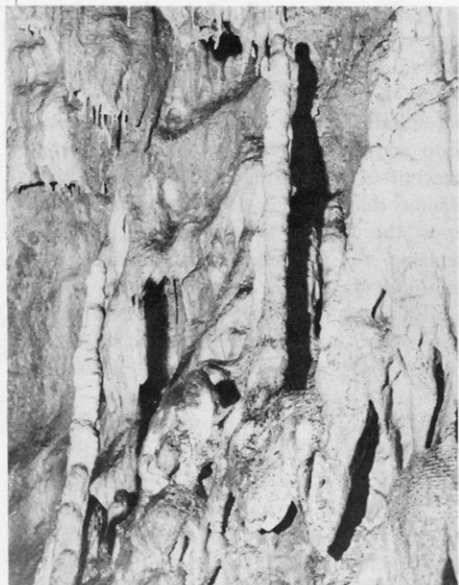


Figure 2. Center-right; Stalagmite in Le Grotte Rouchambou, Belgium, the variation in diameter of which is reported in text.

sion seems inescapable that constant diameter implies constant (if different) growth conditions. This theme, and its consequences, has been most thoroughly developed by Franke (1961 *ff*), who derived his interest primarily from studies of paleo-chronology and climatology.

It should be noted that, although the stalagmites in Figure 1 are individually of moderately constant diameter, there is some variation. Figure 2 shows a 70 cm high stalagmite in Le Grotte Rouchambou (Belgium) the diameter of which varies as follows: base, 8.9 cm; 20 cm up, 7.0 cm; 30 cm, 4.9 cm; 40 cm, 6.4 cm; 50 cm, 5.8 cm; and at the top, 4.8 cm. This represents more than a two-fold variation in cross-sectional area, despite the relatively uniform appearance. Nevertheless, the concept of a uniform, equilibrium stalagmite is a useful abstraction.

#### FRANKE'S THEORY

Figure 3 (after Franke, 1965) shows successive stages in the growth of an equi-

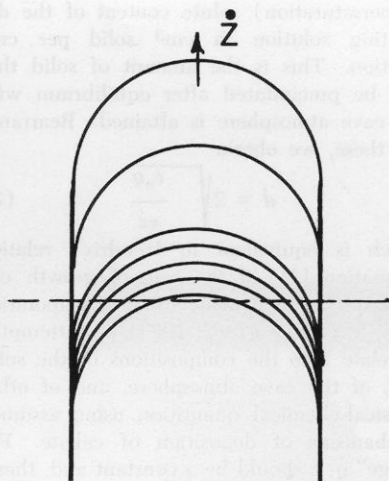


Figure 3. Successive stages in the development of an equilibrium stalagmite, shown in vertical section. The dotted line indicates a cross section that would show "growth rings" narrower near the outside. The equilibrium stalagmite grows upward at velocity  $\dot{z}$ .

librium-diameter stalagmite. As Franke has pointed out, cross sections show "growth" rings that are more closely spaced at greater distances from the axis simply because of the orientation of the developing "caps". Vertical sectioning of natural stalagmites has disclosed that there develop a succession of "caps" (often demarked by impurity layers) that taper to nothing as they descend down the side. This means, first, that all of the solute in drops falling on the stalagmite, in excess of saturation, can be deposited before the residual water runs off (or evaporates) and second, that the rate of *vertical* growth is the same at every radius from the axis of the stalagmite. (For this to be true, of course, the rate of crystal deposition must decrease as the slope of the surface becomes greater.) It then follows (Franke, 1962, 1963) that

$$\dot{z}A = \dot{z} \frac{\pi d^2}{4} = c_0 q \quad (2)$$

where  $\dot{z}$  is the rate of growth in height,  $A$  the cross-sectional area, and  $c_0$  the available

(supersaturation) solute content of the depositing solution in  $\text{cm}^3$  solid per  $\text{cm}^3$  solution. This is the amount of solid that will be precipitated after equilibrium with the cave atmosphere is attained. Rearranging these, we obtain

$$d = 2\sqrt{\frac{c_0 q}{\pi \dot{z}}} \quad (3)$$

which is equivalent to Hendrix's relation (Equation [1]) if the rate of growth can be taken as equivalent to the "evaporation rate" *E*. Franke (1962, 1963) has attempted to relate  $\dot{z}$  to the compositions of the solution, of the cave atmosphere, and of other physical-chemical quantities, using assumed mechanisms of deposition of calcite. For "large"  $q$ ,  $\dot{z}$  should be a constant and, therefore, the cross-sectional area of the equilibrium stalagmite should increase linearly with increasing flow.

What happens at low flows? Equation (3) says that  $d$  becomes zero when  $q$  goes to zero, all other things being constant. Since this is not observed in nature (there seems to exist a minimum diameter for equilibrium stalagmites) "other things" must not remain constant. In particular,  $\dot{z}$  must become proportional to  $c_0 q$  as  $q$  becomes small in order for a minimum diameter to exist. The reasons for this may now be developed to give a theory for *minimum diameter stalagmites*. As was the case with stalactites, an exploration of the limiting case should simplify the general "stalagmite" problem and thus help to clarify some of the factors at work.

#### MINIMUM DIAMETER

At the time that the data on the stalagmite shown in Figure 2 were obtained, the diameters of a number of others of small diameter (and usually of relatively small height, also) in the same cave were also measured. The values from a "random" sampling were 3.1, 3.7, 4.4, 4.9, 5.1, 5.6 and 5.9 cm. Three centimeters, thus, is about the minimum observed diameter, at least under the conditions in that cave.

The flow upon a stalagmite is, of course, not continuous but in the form of drops and,

as  $q$  becomes small, these drops fall at longer and longer intervals. When the time interval becomes great enough, the initial solute in excess of saturation will be totally precipitated on the stalagmite before another drop arrives. In order to develop a simple concept of what would then happen, let us assume that when a drop of volume  $v$  falls upon the top of the stalagmite, it locally replaces the depleted solution there with solution of source supersaturation  $c_0$ . In addition, assume that the thickness of the thin water layer on the stalagmite top,  $\delta$ , is moderately constant after the "splash" is over. The actual structure of the "water layer" is quite complicated as stalagmite surfaces are not perfectly smooth (Franke, 1968), but let us overlook this complexity.

With these assumptions, the growth in height *per drop* will be  $\delta c_0$ . This can be seen by realizing that, per unit area at the stalagmite apex, this is the *volume* of solid that will be precipitated—on this same unit area. Since equilibrium growth is being assumed, this will also be the vertical height grown at every radius on the stalagmite top. Because the volume of dissolved minerals that arrives in one drop is  $v c_0$ , and the volume finally deposited must be  $\delta c_0 A_m$ , where  $A_m$  is the minimum equilibrium cross-sectional area, there results  $v = \delta A_m$ , or

$$A_m = \frac{\pi d^2}{4} = \frac{v}{\delta} \quad (4)$$

Subject to the assumptions that have been made, the minimum cross-sectional area of a stalagmite is just the incident drop volume divided by the thickness of the water film on the stalagmite top, and does not depend upon the composition of the solution or other factors! In terms of diameter,

$$d = 2\sqrt{\frac{v}{\pi \delta}} \quad (5)$$

The simplicity of this is astonishing. Looking more closely, we see that  $\delta$  is itself not simple, as it depends on the aforementioned surface roughness and, hence, upon microscopic details of crystallization, and also upon the mechanics of the impact and drainage of a drop falling on such a surface.

For a 3 cm stalagmite (about the observed minimum diameter) and drops of volume  $v = 0.075$  cc (the volume of a drop falling from a minimum diameter stalactite developed at  $10^\circ\text{C}$ , using the drop-weight method of Harkins and Brown [1919]; see also Curl [1972]), Equation (5) gives a value of  $\delta$  of 0.011 cm (.004 inches). This is not unreasonable, but no experimental or other theoretical values for it are available. With this degree of "success", it will be worthwhile to investigate extensions of the model and, in particular, to relax some of the assumptions and to extend the model to intermediate flows where neither Equation (3) nor (5) should apply.

It should be noted, that in both of the above cases it was necessary to consider only what happened on the apex of the stalagmite where drops fall; the assumption of equilibrium size (and shape) then, in effect, carried the rest of the stalagmite along. Morphological details of stalagmite "caps" would require a consideration of radial effects of flow and deposition which is beyond the scope of this paper. It will be worthwhile later, however, to consider how "splash cups" affect our results as these obviously increase  $\delta$  considerably and might appear to negate Equation (5). If  $\delta = 1$  cm and  $v = .05$  cm<sup>3</sup>,  $d = 0.8$  cm, which is somewhat incompatible.

#### TRANSITION—THEORY

When a drop falls upon a stalagmite top, it splashes and some portion of the drop volume, plus some portion of the liquid already present, may splash off the stalagmite and not fall back upon it. This represents a partial loss in the solid available for deposition on the stalagmite. Let the fraction of the incident drop that remains on the stalagmite be  $\phi_1$ , which will be called the *first splash function*. It must depend on, at least,  $d$ ,  $\delta$ ,  $h$  (the height from which the drop fell), gravity, and fluid properties (including surface tension). The first splash function has not been previously measured, but it probably increases with increasing  $d$  or  $\delta$  and decreases with increasing  $h$ . The

actual solid that is available to the stalagmite from each drop is, then,  $\phi_1 v c_0$ .

When a drop falls into a liquid layer, a mixing process occurs and the resulting solution, at the point of impact, will attain some average supersaturation composition between that of the incident drop ( $c_0$ ) and that of the solution present there at the time of impact,  $c$ . Defining  $\phi_2$ , which will be called the *second splash function*, as the fractional contribution of the incident drop to the final concentration after mixing,  $c'$ , we may write

$$c' = (1-\phi_2)c + \phi_2 c_0 \quad (6)$$

The parameter  $\phi_2$  must be a function of, at least,  $v$ ,  $\delta$ ,  $h$ , and fluid properties. It has never been measured, but it would presumably increase with increasing  $v$  or decreasing  $\delta$ .

Between drops, the deposition of solid depletes the solution of solute and  $c$  falls. It was the rate of this deposition process that Franke (1962, 1963, 1965) attempted to model. He considered processes controlled by disequilibrium either of dissolved carbon dioxide or of calcium ion, but the process has not been studied experimentally. For the present purposes, it will be sufficient to make several approximations in order to help bridge between the high flow and low flow limits.

First, assume that the rate of growth,  $\dot{z}$ , is proportional to the remaining supersaturation,  $c$ .

$$\dot{z} = kc \quad (7)$$

This "linearizes" the problem. The constant  $k$  may be defined as the maximum rate of growth (at  $c_0$ ) divided by  $c_0$ ; i.e.,  $k = \dot{z}_{\max}/c_0$ . This assumption is most likely to be allowable if carbon dioxide equilibrium with the ambient atmosphere is attained by a drop prior to its falling upon the stalagmite. That is, the drop is initially at the maximum supersaturation with respect to calcite deposition. It is possible, however, for the solution to be initially aggressive because of the presence of excessive carbon dioxide in solution. Equation (7) clearly would not apply in such a situation, which would permit some initial dissolution prior

to final deposition. In effect, a one solute system is being assumed; the complexities of the rates of simultaneous gaseous exchange with the atmosphere and of the dissolution and crystallization of calcite, involving a two-solute system, are not understood and are beyond an adequate treatment at this time.

Second, assume that the liquid layer on the top of the stalagmite is *well mixed* in depth, that is to say, that it is of uniform composition throughout its thickness  $\delta$ , but is not necessarily so radially. A material balance on the solute in solution in the liquid layer at the stalagmite apex, in excess of saturation, then gives

$$\delta \frac{dc}{dt} = -\dot{z} = -kc \quad (8)$$

Writing an equation of this type or, equivalently, making the assumptions leading to it, is the same as assuming that the supersaturation falls in an exponential manner (the form of the solution for Equation [8]) between the arrival of drops, with a "time constant"  $\tau = \delta/k$ . Either  $\tau$  or  $k$  may be taken as the characteristic rate parameter for this approximation to the supersaturation-relaxation process.

If drops of volume  $v$  and supersaturation  $c_0$  arrive at intervals of  $t'$  (in which case the average flow rate would be  $\bar{q} = v/t'$ ), we may apply Equation (8) in the interval. The general solution to this is

$$c = a \exp\left(-\frac{kt}{\delta}\right) = a \exp(-t/\tau) \quad (9)$$

where  $a$  is a constant of integration. The concentration pattern for the liquid at the stalagmite apex resulting from the arrival of consecutive drops is shown in Figure 4. If we assume that a drop arrives at  $t = 0$ , it then mixes with the solution already present according to Equation (6). That mixture, however, is the result of the concentration relaxation over the interval  $t'$  following the arrival of the previous drop. Therefore, for the terms in Equation (6) we may write, at  $t = t'$ ,

$$c = a \exp(-t'/\tau) \quad (10)$$

and at  $t = 0$ , but after arrival of a new drop,

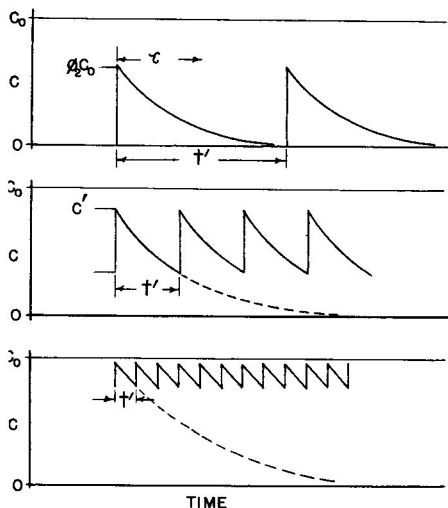


Figure 4. Time histories of the supersaturation at the apex of a stalagmite for different drop rates.  $\phi_2$  about 0.6. The time and concentration scales are arbitrary. Top:  $t'$  much greater than  $\tau$ ;  $c$  relaxes to zero and the minimum diameter results. Middle:  $t'$  comparable to  $\tau$ . Transition between high and low flow regimes. Bottom:  $t'$  small compared to  $\tau$ .  $c$  approaches  $c_0$  and maximum growth rate is approached in the "high flow" regime. The dashed curves indicate the composition relaxation in the absence of subsequent drops.

$$c' = a = (1 - \phi_2) a \exp(-t'/\tau) + \phi_2 c_0 \quad (11)$$

Solving these two relations for  $a$ , and inserting the result into Equation (9), and that into Equation (8), there results

$$\dot{z} = \frac{k\phi_2 c_0 \exp(-t'/\tau)}{(1 - (1 - \phi_2) \exp(-t'/\tau))}, \quad 0 < t < t' \quad (12)$$

which gives the transient rate of growth between drop impacts. Averaging this over the interval  $t'$ , we obtain the average rate of growth

$$\dot{z} = \frac{\delta\phi_2 c_0 (1 - \exp(-t'/\tau))}{t' (1 - (1 - \phi_2) \exp(-t'/\tau))} \quad (13)$$

Referring again to Figure 4, in which Equation (10) is shown graphically, the effect

of drip rate upon the composition pattern is shown. At low flows,  $c$  relaxes completely to zero. At high flows, it approaches a constant value of  $c_0$ . These are the limits of the simplified theories first discussed. Since the average rate of growth must also be proportional to the arrival rate of solute,  $\dot{z} = \phi_1 c_0 \bar{q}/A$ , it follows that

$$A = \frac{\pi d^2}{4} = \frac{v \phi_1 \bar{q} (1 - (1 - \phi_2) \exp(-\beta))}{\delta \phi_2 (1 - \exp(-\beta))} \quad (14)$$

where  $\beta = t'/\tau$  ( $= kt'/\delta = v/\bar{q} \tau$ ) is the dimensionless parameter controlling the transition from the "high" to the "low" flow behavior. It is, of course, just the ratio of the interval between drops to the characteristic relaxation time of composition. Equation (14) has for high and low limits,

$$\bar{q} \rightarrow \infty: A = \frac{\phi_1 \bar{q}}{k} = \frac{\phi_1 c_0 \bar{q}}{\dot{z}_{\max}} \quad (15)$$

and

$$\bar{q} \rightarrow 0: A_m = \frac{\phi_1 v}{\phi_2 \delta} \quad (16)$$

which are just the original, simple relations with the splash functions introduced.

In Figure 5 is plotted  $A/A_m$  versus  $\phi_2/\beta$  ( $= \phi_2 \tau \bar{q}/v$ ) for various  $\phi_2$ . The choice of variables is such that the two asymptotes for high and for low  $\bar{q}$  are a single pair of straight lines. The intersection of the asymptotes at  $\beta = \phi_2$  marks the transition between the two regimes. It is, in fact, not a bad approximation to use the asymptotes themselves, at least for  $\phi_2$  near 1.0, when the transition between the regimes is moderately abrupt. For example, at  $\beta = \phi_2$ ,  $A/A_m = 1.58$ , which represents an increase in the minimum diameter of only about 25%. One implication is that when stalagmites get to diameters over 6 cm (assuming the minimum is about 3 cm), they probably are well into the high flow regime.

It is also apparent that the theoretical transition between the two regimes is rather independent of the details of the model used. Recognizing that the two asymptotes are represented by  $A/A_m$  equal to 1.0 at low  $\phi_2/\beta$  or by  $\phi_2/\beta$  itself at high values, an arbitrary

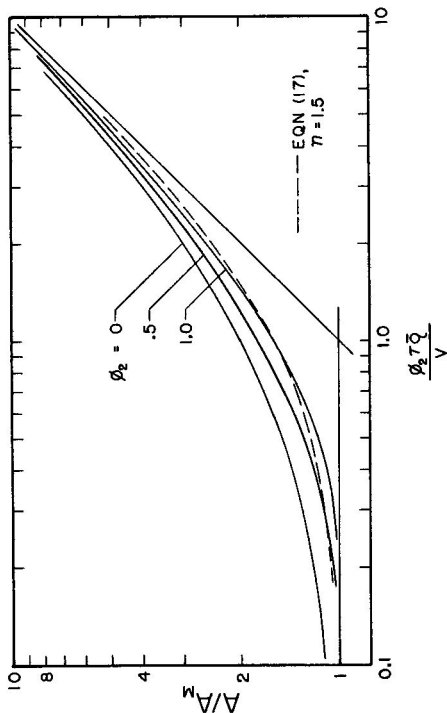


Figure 5. Stalagmite cross-section area, relative to the minimum, versus flow rate (nondimensionalized). The low and high flow asymptotes intersect at (1,1). An empirical transition relation, Equation (17), with  $n = 1.5$ , is shown by the dashed curve.

empirical transition equation, based on no model whatsoever, might be chosen, such as

$$\frac{A}{A_m} = (1 + (\phi_2/\beta)^n)^{\frac{1}{n}} \quad (17)$$

(where  $n$  is a "fitting" parameter) that is correct both at the low and at the high flow limits. It can be shown that when  $\phi_2 = 0$ , Equation (14) gives this exactly with  $n = 1.0$ , while with  $n = 1.5$ , Equation (17) is a good approximation to  $\phi_2 = 1.0$  (Figure 5). The point here is that the use of more-correct non-linear models in place of Equation (7) would change only the details, not the general transition behavior, of Equation (14). It remains, however, that

there exist no confirmed predictive methods for  $\dot{z}$  as a function of  $c$  or of  $\delta$ .

If it were possible to estimate a value for  $\tau$ , it would be possible to determine the drop rate at which a stalagmite goes into its minimum diameter mode. Whatever the composition relaxation time is, it is true that if the time between drops much exceeds this, the minimum diameter should result.

For drops falling short distances,  $\phi_1$  may approach 1.0 and  $\phi_2$  will probably be somewhat less than 1.0. The original elementary estimate for  $A_m$ , therefore, is itself a lower limit on possible minima. For drops falling from a great height,  $\phi_1$  might be considerably less than 1.0, while  $\phi_2$  should not change much. The result is to decrease the effective size of the drop that would otherwise fall from an *equilibrium stalactite*, which should result in yet smaller stalagmites. That such do not occur below water sources at a great height probably results from the larger random spread of the "impact points" for such drops, which would broaden the resulting stalagmite. This has not been studied.

#### SPLASH CUPS

Some stalagmites have cup-shaped tops, often referred to as *splash-cups*. Allison (1923) said that the "depth of the splash cup is increased by high evaporation, low concentration and rapid drip." There seem to be no particular reasons why this should be so. The falling drops could have excess carbon dioxide in solution and be thereby temporarily aggressive upon impact, thus cutting a depression in the top of a stalagmite, but this alone does not explain splash-cups; in the state of equilibrium growth, the bottom of the splash cup must be growing upward at the *same rate* as the rim of the cup and therefore, both points must have the same average supersaturation, or at least the same deposition rate. To have an equilibrium splash-cup, it is necessary only that there be a local minimum in the deposition rate at a radius lying between the center and rim of the cup. This would be

a consequence of complex processes of mixing and splashing, interacting with (and ultimately forming) the shape of the top of the stalagmite. This is all beyond the scope of the present paper.

One point is relevant, however: as has been mentioned, a splash cup increases  $\delta$  and, therefore, would appear to decrease the equilibrium minimum diameter. That such does not occur naturally to a dramatic degree must be related to the evident fact that  $\phi_2$ , the second (mixing) splash function, must also decrease as  $\delta$  increases. The previous arguments can, in fact, be inverted and the conclusion reached from the existence of a minimum stalagmite diameter of about 3 cm that  $\phi_2$  eventually must become approximately inversely proportional to  $\delta$ . This would prevent the denominator of Equation (16) from changing in the presence of a significant splash-cup.

#### CONCLUSIONS

In comparing the present study with the conclusions previously deduced for minimum diameter stalactites, we see how different are the mechanisms in the two cases. The latter were controlled by the static balance between gravity and surface tension, while stalagmites apparently respond to dynamic processes of supply and drainage of incident drops and crystallization kinetics. It is stimulating to the imagination how much physics, chemistry and physical-chemistry is encompassed by something as "simple" as a stalactite-stalagmite pair.

The present analysis raises a number of questions—which may be its primary virtue. The two splash functions,  $\phi_1$  and  $\phi_2$ , are ripe subjects for experimental study, as are the kinetics of calcite precipitation from thin films and the film-drainage process of speleothem surfaces. Clearly, all of these are involved in the general problem of speleothem morphology; that they appear immediately when even the most simplified situation is considered—the minimum diameter stalagmite—suggests the important role they will play in future studies.



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## Errata

Curl, R. L., Minimum Diameter Stalagmites  
Bull. Nat. Speleological Soc., 1973, 35(1): pp 1-9.

p. 6. eqn. 9. close brackets, viz.  $= a \exp(-t / \tau)$

p. 6. eqn. 11. argument of exponential term is.  $\exp(-t' / \tau)$