Minimum Diameter Stalactites

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ABSTRACT

Assuming that stalactites grow into the aqueous space available to them at their tips, it is shown how the pendant drop controls the smallest possible equilibrium diameter of a soda-straw stalactite. By dimensional analysis, it is shown that there exists a characteristic Bond Number, Bo = $\rho g d^2/\sigma$, which determines their diameter. From experiments on drops formed on glass capillary tubes of different sizes, it is found that the Bond number for minimum diameter stalactites is Bo = 3.50. This gives a soda-straw diameter of 5.1 mm under ordinary conditions, agreeing with existing observations. Finally, it is shown that the diameter of a *non-equilibrium* stalactite should converge, with growth, in an exponential manner to the minimum equilibrium diameter.

Stalactites in caves are of great interest to the public, to cavers and to speleologists. They are seen as objects of beauty, as subjects for fanciful imagination, as mineralogical curiosities, and as indicators of factors of the cave environment. Except for facets of mineralogy and crystal structure, there does not appear to be much complexity to the story of stalactite morphology. This may be why their literature is relatively scanty. Moore (1962) treated the subject historically and mineralogically, and illustrated the basic features of stalactite crystal structure and growth. The "soda straw" stalactite, illustrated in Figure 1, is the simplest form: a tube of nearly uniform diameter, deposited from a pendant drop. Both Moore (1962) and Goodman (1966) say that the diameter of this tube is equal to the diameter of a drop of water, which seems rather obvious—until it is pointed out that the size of a drop of water depends upon the diameter of the tube from which it hangs. It is my purpose here to explore this "paradox" and to suggest some controlling factors in determining the smallest possible equilibrium diameter of a stalactite. Limiting consideration to the minimum diameter stalactite must necessarily reduce the problem of stalactite morphology to its simplest level. I hope that this may serve as a point of departure for future quantitative studies.

The first feature of soda-straw stalactite growth that seems evident from Figure 1, from the illustrations of Moore (1962), Goodman (1965, 1966), and from the many photographs that have appeared in various publications, is that the growing crystals at the tip are constrained to form within the boundary of the drop surface. That is, they do not appear to distort the shape of the drop by pushing against its surface from within, nor do they penetrate the interface. This observation is not contradicted by the observations of Went (1969), who found that fungus mycelium may guide some stalactite growth. I will therefore assume that the drop surface is a boundary for crystal growth and, conversely, that crystal growth does not directly affect the drop shape except as it determines the size of the tube from which the drop hangs.

The factors affecting the shape of a drop hanging from a rod or tube are the volume of liquid in the drop, its density, the acceleration of gravity, the diameter of the tube, and the surface tension of the liquid. The shapes that may occur are varied and it is useful to observe these on tubes of different diameters. For this purpose, drops were formed slowly on the ends of capillary tubes

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that had been cut off and polished. Figure 2 shows the apparatus used to carry out these experiments.

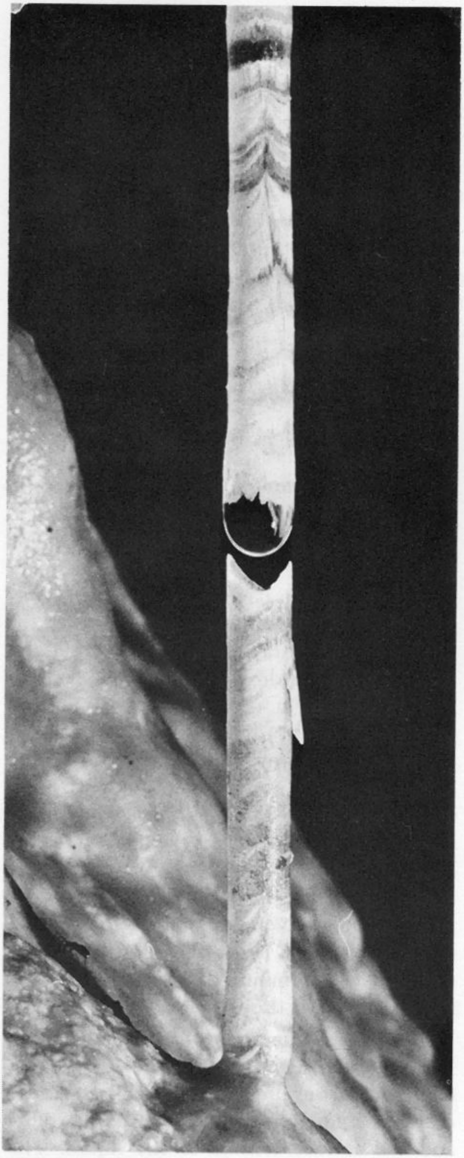

Fig. 1. *Soda-straw* stalactite. The original stalactite apparently broke or dissolved at one point, an extremely unusual event. Photo by Carl Kunath.

Drops were formed on tips made from various diameters of glass capillary tubing. The tubing was connected to a burette, allowing the adjustment and measurement of the water flow. The tip was enclosed within a bottle while in use in order to maintain a water-saturated atmosphere. Microscope slides cemented to the inside and outside of one side of the (square) bottle gave an optically undistorted view of the tip and of its pendant drop. The system was illuminated through an interposed heat-absorbing solution (copper sulfate). The image of the tip and drop was focused upon a flat surface (and enlarged about seven-fold) by a lens and mirror. By placing photographic paper in the plane of the projected image, photographs (negatives) of the drops could be obtained at various stages in their growth. These are shown in Figure 3.

From left to right are shown water drops forming on tips having diameters of 0.311 cm, 0.497 cm and 0.728 cm. Time, and hence drop volume, increases from top to bottom in each column of pictures. We see that on a "small" tip, the drop becomes larger in its maximum diameter than the tip itself, prior to forming a "neck", and at the middle stage bulges outward from its line of attachment to the tip. At this stage, if crystals were growing on the tip, they could grow into the drop and partly outward, *increasing the tip diameter*. Under the conditions of the experiment, therefore, this tube would be smaller than the final minimum size of a soda straw. In speaking here of "minimum diameter", I will generally mean the smallest *equilibrium* diameter, although it is possible for a stalactite to commence growth at a smaller diameter if the initial drop size is controlled by a small enough ceiling projection.

On the larger tip (0.728 cm), the drop always hangs in such a way that the drop surface slopes inward. The middle picture shows the drop at the condition of minimum inward slope. If the drop surface constrains crystal growth, as has been assumed, this tube is too large and growth would lead to a decreasing diameter. The intermediate diameter tube (0.497 cm) appar-

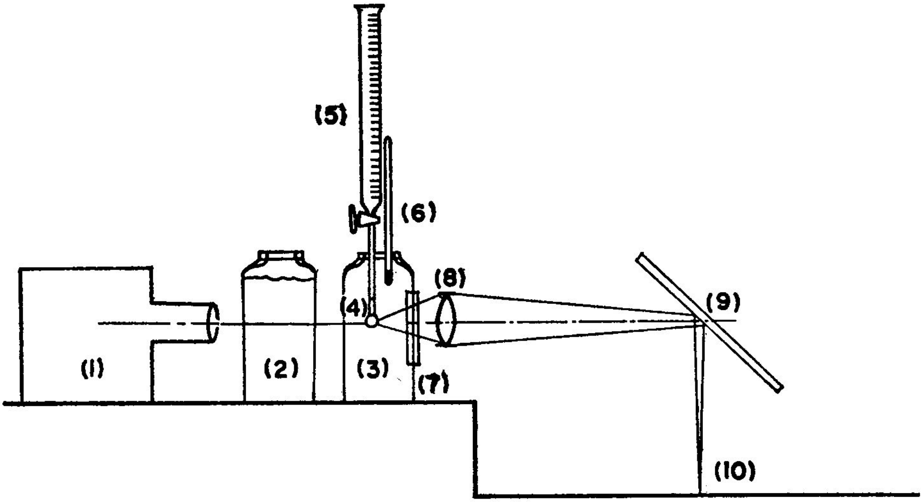

Fig. 2. Experimental apparatus. (1) light source; (2) heat absorber; (3) bottle in which drops are formed; (4) capillary tip and pendant drop; (5) burette; (6) thermometer; (7) microscope slides cemented to bottle side; (8) lens; (9) mirror; (10) surface on which image of tip and drop is brought to focus.

ently has nearly that diameter at which the maximum expansion of the pendant drop leads only to a vertical drop surface at the point of attachment. Under this condition, crystals can only grow vertically downward and the tube diameter will be maintained. This should be the condition for the maintenance of the *minimum equilibrium diameter stalactite.*

The included angle the drop surface makes at the point of attachment of the drop to the tip will be called θ, as shown in Figure 4. As the volume of the liquid in the drop increases, θ may be seen first to increase (Figure 3), then to attain a maximum value, θ_m, and finally to decrease until the drop falls from the tip. θ_m obviously depends upon tip diameter d. The factors determining the drop shape which have already been mentioned must be the same factors determining the angle θ. There must, therefore, exist a functional relation-

$$\theta = f\,(d, \rho, g, \sigma, v) \qquad (1)$$

ship where ρ is the fluid density (g/cm^3), g the acceleration of gravity (cm/sec^2), σ the surface tension of the fluid (g/sec^2), and v the drop volume (cm^3). Because the physics of pendant drops is well known, it is theoretically possible to calculate this relationship from first principles (Adamson, 1967) but this involves complex numerical calculations and has not been done for the present situation. It is simpler, as will be seen, to proceed experimentally.

Dimensional analysis (see Catchpole and Fulford, 1966, for references) then constrains the form of Equation (1) to one involving only dimensionless groups, such as those in

$$\theta = f\left(\frac{d^2\rho g}{\sigma}, \frac{d^3}{v}\right) \qquad (2)$$

The first group in the function has been called both a Bond number and an Eötvös number (Catchpole and Fulford, 1966). The use of the former name is older and will be adopted here. Let

$$Bo = \frac{d^2\rho g}{\sigma} \qquad (3)$$

which represents a dimensionless ratioing of gravitational to surface-tension forces. The condition for θ_m is, then,

$$\left.\frac{\partial \theta}{\partial v}\right|_{\theta = \theta_m} = -\frac{d^3}{v^2}\frac{\partial f}{\partial (d^3/v)} = 0 \qquad (4)$$

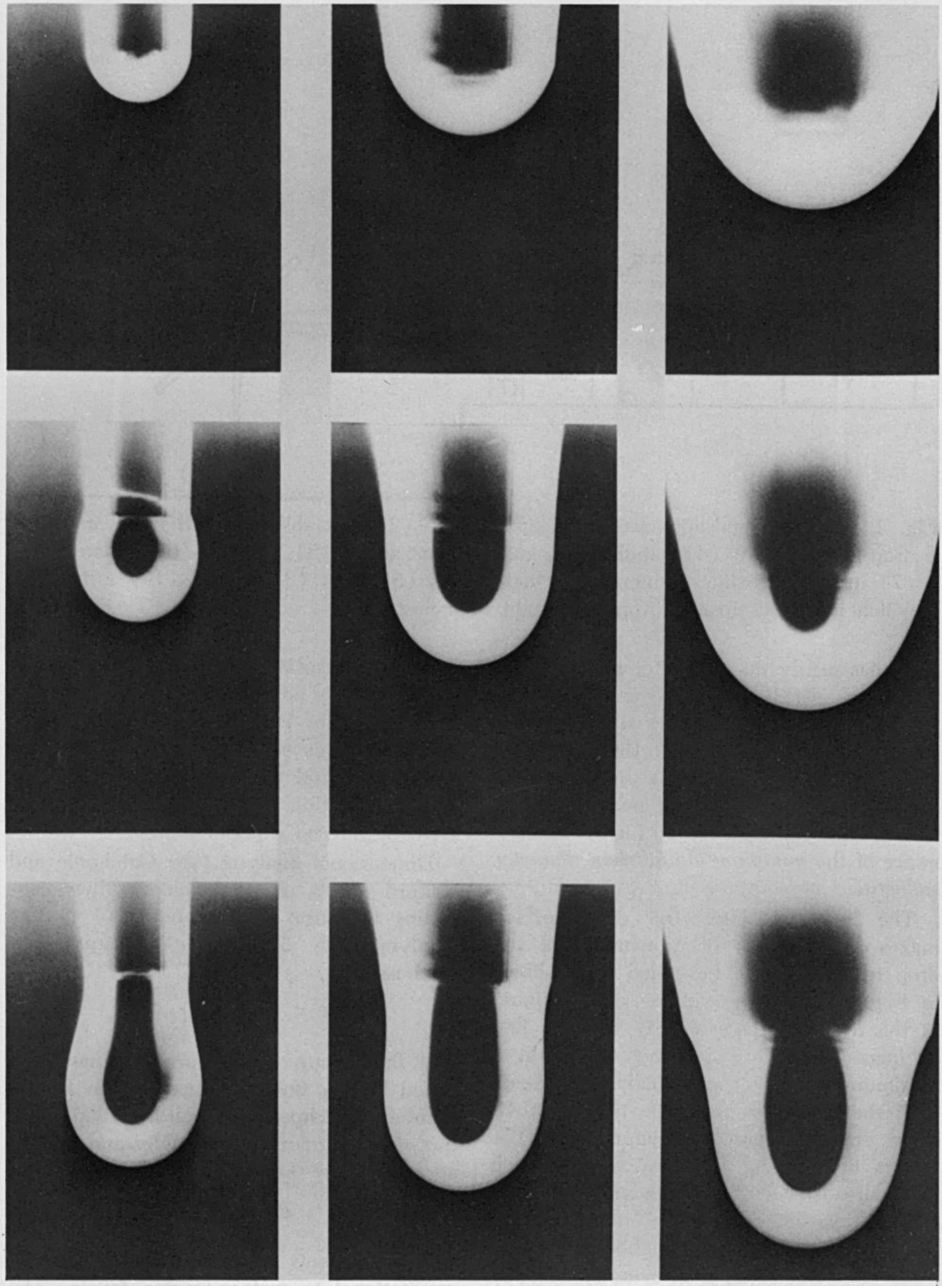

Fig. 3. Images of drops. Columns (left to right): tips of 0.311, 0.497 and 0.728 cm. Rows (top to bottom): liquid remaining after drop detachment, pendant drop at $\theta = \theta_m$, pendant drop just prior to detachment.

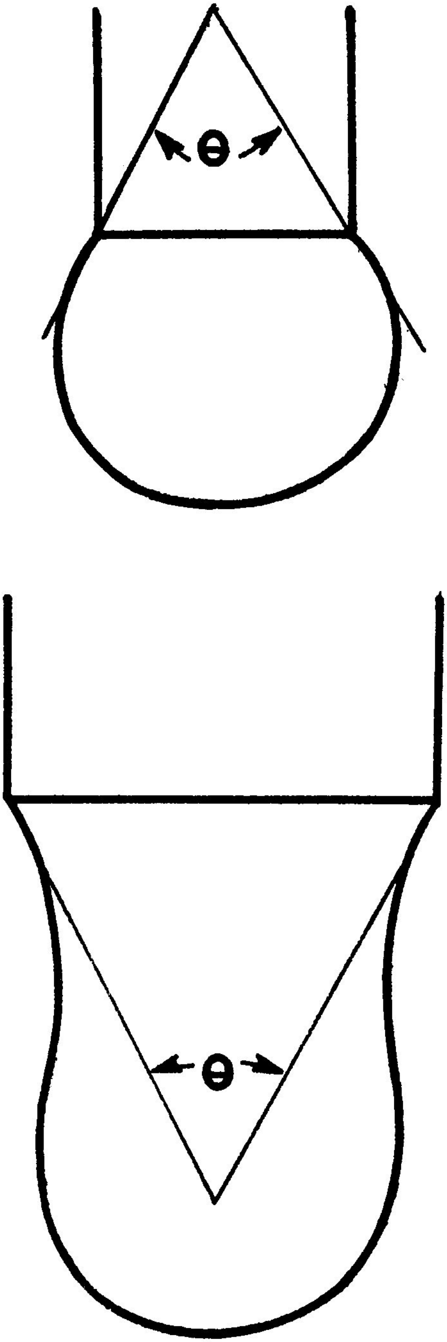

Fig. 4. Definition of angle θ. Above: θ positive. Below: θ negative.

which determines another relationship between Bo and d^3/v. That is, imposing the maximization introduces an additional relationship between the variables θ, Bo and d^3/v, allowing one to be eliminated. We conclude, therefore, that θ_m must be a function of the Bond number alone.

$$\theta_m = f\ (Bo) \qquad (5)$$

The problem, then, comes down to determining the value of Bo for which $\theta_m = 0$ (i.e., when the drop surface hangs vertically at its point of attachment, at maximum θ). When this value of Bo is known, a specification of fluid density and surface tension, and the local acceleration of gravity, determine the associated minimum equilibrium diameter.

Using the apparatus shown in Figure 2, θ_m was determined on a tip of a given size by allowing drops to form slowly (about two per minute), one after the other, and eventually to fall under their own weight. θ was followed on the projected image by holding the edges of pieces of paper tangent to the drop surface at its contact with the tip until the maximum angle was reached. θ_m was measured for two or more consecutive drops at the beginning of a series of about 60 drops, and again at the end. In this way, a standard error of measurement of θ_m could be estimated and confidence intervals (95% C. I.) evaluated. In addition, the average volume of the drops that fell from the tip was determined by measuring the total volume used from the burette and knowing the number of drops formed. The surface tension of the water was then determined using the drop-weight method of Harkins and Brown (1919). The density of water was taken as 0.998 g/cm^3 at 22°C and the acceleration of gravity as 981 cm^2/sec. The results of these measurements are shown in Table 1 and in Figure 5.

Surface tension is sensitive to temperature changes and impurities in the solution. It therefore was determined in the course of the experiments rather than assumed to be the value reported in handbooks. Tap water was used. The temperature in all experiments was about 21°-22°C. The measured values of σ are, happily enough, close to the reported value for pure water of 72.4 g/sec^2 at 22°C. An error analysis, including the effects of errors in the measurements of d and v, indicates a maximum error of about 3% in the determination of the Bond numbers.

In Figure 5, θ_m and Bo are plotted as $tan(\theta_m/2)$ versus $\sqrt{Bo}$. It was found, by trial and error, that this produced an essentially straight line plot. The point where the line intersects the abscissa represents the value of the "minimum stalactite diameter" Bond number, $\sqrt{Bo}$ = 1.87, or Bo = 3.50. The equation of the line, which will be used later, is given by

$$tan(\theta_m/2) = -0.455\ (\sqrt{Bo} - 1.870 \qquad (6)$$

Knowing this minimum equilibrium diameter Bond number, we are able to calculate the minimum d for different circumstances and even for different materials. Examples for "soda straws", icicles and lava stalactites are shown in Table 2. The values are reasonable, although few actual measurements have been reported. Goodman (1966) shows two specimens, both almost exactly 0.50 cm in diameter, but does not report the temperature. The surface tension might also depend upon surfactants in solution. No data is available on the composition of lava stalactites, so the properties given in the Table are the density of plagioclase and the surface tension of a nominal blast furnace slag as given by Elliott, *et al* (1963). Lava stalactites are illustrated by Hicks (1950).

TABLE 1. Experimental results

d (cm)	θ_m (°)	95% C.I. (°)	σ (g/sec^2)	Bo
0.311	37.1	±1.91	71.6	1.32
0.378	24.2	±0.94	72.7	1.92
0.497	1.8	±0.59	72.6	3.33
0.592	−16.6	±1.78	72.5	4.73
0.728	−39.1	±1.02	71.9	7.22

TABLE 2. Predicted minimum diameters

Type	Temp. (°C)	σ (g/sec^2)	ρ (g/cm^3)	d (earth) (cm)	d (moon) (cm)
Soda-straw	10	74.2	1.0	0.51	1.24
Icicle	0	75.6	1.0	0.52	1.26
Lava	1400	400	2.6	0.74	1.79

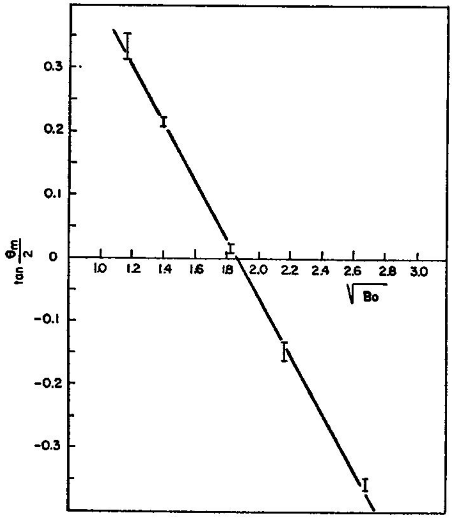

Fig. 5. Measured pendant-angle *versus* Bond number. Choice of ordinate and abscissa scales gives a straight-line plot.

The growth both of icicles and of lava stalactites are by external flow. We therefore would expect that most examples would be larger than the minimum diameter, but the estimate for d is, nevertheless, a lower limit on their size. Because the minimum size in all cases depends upon the acceleration of gravity, moon stalactites, growing in a gravitational field of g = 167 cm/sec^2, should be over twice as large, (if they exist!).

Several factors in the process of drop formation, detachment, and calcite stalactite growth may introduce some variation in "minimum" diameters. Of course, direct deposition on the sides of growing stalactites will make them bigger, but this requires access of solutions supersaturated with calcite to outer surfaces. Film flow from above is probably the most common process, but it is also possible for a pendant drop to enclose the entire tip and, in effect, to hang from a point on the side of the stalactite somewhat above the end. This is encouraged by the tube being too small—that is, if θ is positive. This is actually seen in Figure 3 for the middle stage of drop formation on the 0.311 cm tip; the light line above the end of the tip is the actual line along which the drop is hanging. (This phenomenon was avoided in the subsequent experiments by slightly "contaminating" the sides of the smaller glass tips with a very thin film of grease.) The net effect would be to enhance the rate of attainment of the "stable" size.

The hypothesis presented here, that a stalactite will grow into the largest aqueous space available to it, and therefore follows the drop surface at θ_m, does not take into account the fact that θ is at or near θ_m only part of the time. This does not seem to be any particular difficulty as any growth at an angle of θ_m establishes the new rim from which the drop hangs, even if at smaller values of θ. There is, however, a possibility that θ may momentarily exceed what we have defined as θ_m; when a drop is detached and falls, the remaining fluid rebounds and it is possible, although this was not ascertained in the experiments, that for a few milliseconds θ is greater than the steady θ_m. The question, then, is whether the observed θ_m is the determining factor, or whether some type of time-averaged θ must be taken into account. This will depend upon the details of the attachment of a drop to a stalactite end, which may be expected to be somewhat different those of its attachment to a smooth glass tip. An additional factor is that the rims of stalactites often are serrated with growing crystals. The extent to which either of these disturbs the relatively simple hypothesis presented here is not known, although the agreement between the prediction in Table 2 and Goodman's observation (1966) is reassuring.

The variation in diameter of a *non-equilibrium* stalactite may be described in a cylindrical coordinate system by the dependence of the radius r upon the axial distance z, as shown in Figure 6. Assuming that a stalactite does grow as hypothesized, at an included angle of θ_m, we may derive a relation for the way in which a stalactite larger or smaller than the equilibrium diameter will approach the latter value. The identity between $tan(\theta_m/2)$ and the slope of the surface with respect to

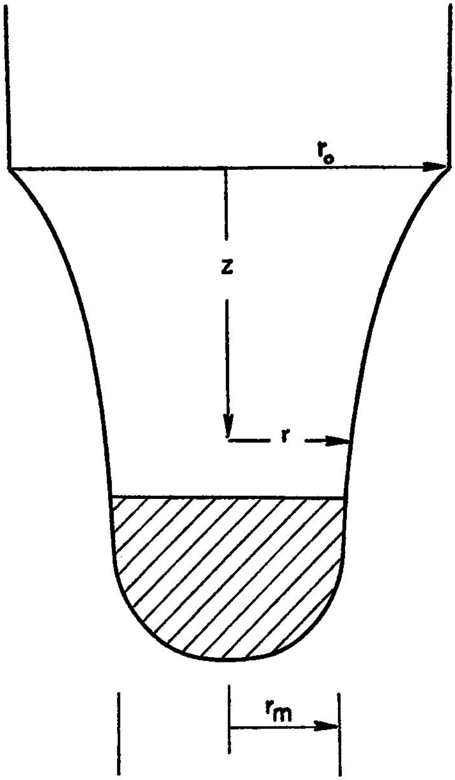

Fig. 6. Convergence of stalactite to equilibrium diameter when initially $r_o = 2r_m$ (pendant drop shaded).

the stalactite axis allows us to write, using Equation (6) and letting $d = 2r$.

$$tan(\theta_m/2) = \frac{dr}{dz} = -0.91\left(\sqrt{\frac{\rho g}{\sigma}}\, r - 0.935\right) \quad (7)$$

which is, fortuitously, a linear differential equation for r. Lettering r_m be the stable radius (obtainable from $Bo = 3.50$) and r_o the initial radius, the solution to Equation (7) is

$$\frac{r}{r_m} = 1 - \left(1 - \frac{r_o}{r_m}\right) \exp\left(-0.85\,\frac{z}{r_m}\right) \quad (8)$$

This states that the radius approaches the final radius exponentially. The characteristic "relaxation" distance is $r_m/0.85 = 1.176\ r_m$, which is the distance in which the *departure* from r_m decreases by the factor $e^{-1} = 0.368$. The result is shown in Figure

(6) for $r_0 = 2r_m$. It is not unusual to see new soda straws developing from the larger end of a previously broken stalactite tip and the above exponential contraction would seem to be a reasonable description, but there are no measurements available from which to ascertain whether the observed relaxation distance is actually 1.176 r_m. This would be a useful check of the present theory.

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Errata

Curl, R. L., Minimum Diameter Stalactites,
Bull. Nat. Speleological Soc., 1972, 34(4) pp 129-136

p. 131, eqn (4). Derivative should be in brackets, viz.

$$\left(\frac{\delta\theta}{\delta v}\right)_{\theta=\theta_m}$$

p. 134, eqn (6). Close brackets after – 1.870.

p. 135, eqn (7). Should read

$$\tan\left(\theta_m / 2\right) = \frac{dr}{dz} = -0.91\left(\sqrt{\frac{\rho g}{\sigma}}\, r - 0.935\right)$$

p. 135. Two lines below eqn (7):

Letting r_m be the……

p. 135. col. 1, line 30:

somewhat different than those of…..