Step Options and Forward Contacts

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Abstract

Motivated by risk management problems with barrier options, we propose a flexible modification of the standard knock-out and knock-in provisions and introduce a new family of path-dependent derivatives: *step options, forward contracts and swaps.* They are parametrized by a *finite knock-out (knock-in) rate*, r_B. For a down-and-out step option, its payoff at expiration is defined as the payoff of an otherwise identical vanilla option discounted by the *knock-out factor* $\exp(-r_B\tau_B^-)$ (continuous compounding) or $Max(1 - R_B\,\tau_B^-, 0)$ (no compounding), where τ_B^- is the total time during the life of the contract the price of the underlying asset is lower than the pre-specified barrier level (*occupation time*). The stochastic model of step options is that of Brownian motion with killing at finite rate r_B below the barrier level. We derive *closed-form* pricing formulas for step options and forwards with any knock-out rate $0 \leq r_B < \infty$. In the limit of zero knock-out rate, step options coincide with vanilla options. In the limit of an infinitely high knock-out rate, they coincide with standard barrier options, thus continuously interpolating between the standard vanilla and barrier contracts. A remarkable property of step options is that for any finite knock-out rate both the step option's value and delta are continuous functions of the underlying price at the barrier, and the option's gamma undergoes a finite jump (step) at the barrier. As a result, they can be continuously hedged by trading the underlying asset and borrowing. These risk management properties make step options attractive "no-regrets" alternatives to standard barrier options both for buyers and sellers. As a by-product, we derive a dynamic almost-replicating trading strategy for standard barrier options by considering a replicating strategy for a step option with arbitrarily high but finite knock-out rate. Step forwards and swaps possess interesting principal amortizing properties. Closed-form solutions are also obtained for *perpetual step options* independent of time. Discrete-time approximations (Monte Carlo simulation and the Hull and White extension of the CRR binomial model) are also developed for step options. Finally, a general class of derivative securities contingent on occupation times is considered, closed-form pricing formulas are derived, and applications to the modeling of credit risk and mortgage-backed securities are suggested.

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1 Introduction

1.1 Risk Management Problems with Barrier Options

Barrier options have become increasingly popular over the last several years in over-the-counter options markets. A large variety of barrier options are currently traded in foreign exchange, equity and fixed-income markets. The popular knock-out options are extinguished (knocked out) when the price of the underlying asset hits a pre-specified price level (barrier) from above (below) for down-and-out (up-and-out) options. Closed-form pricing formulas for barrier options were obtained by Merton (1973) for a down-and-out call and Rubinstein and Reiner (1991) for all eight types of barrier options (see also Derman and Kani (1993) and Rich (1994) for details). Malz (1995) reports that transactions in FX knock-out options have increased to between two and twelve percent of all currency options trading by earlier 1995 from a negligible share just two years ago. Knock-outs are also popular in equity and fixed income markets (e.g., barrier swaptions).

Knock-out options are often attractive to investors because they are cheaper than vanilla contracts. If an investor believes that it is unlikely that the underlying asset price will fall lower than a certain price level (support), he could consider adding a knock-out provision to his option with the barrier set at the support level, thus reducing the premium payment. The reduction in premium by allowing for a knock-out provision in the contract can be very substantial, especially when volatility is high. By buying a barrier option, the investor can eliminate paying for those scenarios he feels are unlikely. Barrier options also allow one to execute various complex hedging and trading strategies, matching investors' hedging needs more closely than similar standard options.

However, these benefits come at a significant cost. The *discontinuity* at the barrier inherent in knock-out contracts creates a number of serious problems both for buyers and sellers of barrier options. An erroneous short-term price movement through the barrier can extinguish the option, leaving the buyer without his position. Even if the investor is generally right on the market direction, *an accidental price spike can lead to the loss of the entire investment.* Because of the discontinuous payoff, investors must be very precise in choosing the barrier; a slight mistake can be very costly. Furthermore, when large positions of knock-out options with the same barrier level are accumulated in the market, traders can drive the price of the underlying to the barrier, thus creating massive losses by triggering the knock-outs.

This is vividly illustrated by the recent events in the foreign exchange markets. According to the Wall Street Journal[2], "... knock-out options can roil even the mammoth foreign-exchange markets for brief periods." WSJ continues, "Most foreign-exchange traders now take it for granted that once in a while you will get a little extra kick in the price movement from a large number of options in the market." "For example, David D. Hale,

[2] *Do Knock-out Options Need to Be Knocked Out?*, Wall Street Journal, May 5, 1995.

chief economist at Kemper Financial Cos. in Chicago, notes that in the past year, many Japanese exporters moved to hedge against a falling dollar with currency options. Confident at the time that the dollar would fall no further than 95 yen, the exporters chose options that would knock out at that level. Once the dollar plunged through 95 yen early last month, they lost everything, he says. The dollar then tumbled as the Japanese companies, which had lost their hedges, scrambled to cover their large exposures by dumping dollars." "Making matters more volatile, dealers say that pitched battles often erupt around knock-out barriers, with traders hollering across the trading floor of looming billion-dollar transactions. ... In three or four minutes it is all over. But in that time every trade gets sucked into the vortex." As a result, according to Derivatives Week [3], "Some U.S. players are keen to include a statement (*in the standardized trade confirmation for barrier options - V.L.*) alerting counterparties to that fact they may be involved in other trades that could move the market and extinguish the barrier..."

This situation prompted some market participants to appeal to regulators on the necessity to regulate knock-out options. George Soros went as far as to suggest a ban on knock-out options. He says in his recent book, Soros (1995): "Recent experiences indicate that the so-called knock-out options are particularly pernicious in this regard. They relate to ordinary options the way crack relates to cocaine. *Do you think they should be banned?* Yes. I would not have said that a few months ago, when I testified before Congress, but we have had a veritable crash in currency markets since then. As I have said before, knock-out options played the same role in the 1995 yen explosion as portfolio insurance did in the stock market crash of 1987, and for the very same reason. Portfolio insurance was subsequently rendered inoperable by the introduction of the so-called circuit breakers. *Something similar needs to be done now with knock-out options*[4]."

The barrier option's delta is discontinuous at the barrier, thus creating serious hedging problems for options sellers. To hedge barrier options, dealers establish positions in a series of standard vanilla options which provide a good hedge for a wide range of underlying prices. However, when the underlying nears the barrier level, these static hedges need to be rebalanced, which results in a flurry of trading activity in vanilla options. This, in turn, results in further trading activity in the underlying asset as dealers who sold vanilla options to hedgers of knock-out options need to dynamically hedge their exposure (see Malz (1995)). This increases market volatility around popular barrier levels and increases the cost of hedging barrier options.

Another striking example of a problem created by the discrete nature of barrier options is provided by the recent events in the Venezuelan bond market.[5] [6] A group of hedge funds managed by Steinhardt and Leitner purchased from Merrill Lynch and other dealers $500

[3] *Barrier Options Standard Waits on Risk Disclosure*, Derivatives Week, November 6, 1995.

[4] Emphasis ours.

[5] *Funds, Merrill Battle over Venezuelan Bonds*, Wall Street Journal, February 15, 1995.

[6] I am grateful to Jim Bodurtha for bringing this example to my attention.

milion of barrier (up-and-in) puts on Venezuelan bonds with the strike of 45 cents on the dollar. In order for the puts to be activated, underlying bonds should have traded above the barrier set at 51 cents at some time prior to puts' maturity. An obvious conflict of interests between the counterparties, the funds who needed to drive the prices higher to trigger the barrier and the dealers who needed the prices to stay below the barrier to avoid the liability, resulted in a fierce battle to control Venezuelan bond prices. Wall Street Journal reports: "At one point during the clash, the screen flashed a 'bid' price of 51 1/8, indicating a willing buyer at that price. Shortly afterwards, however, Euro Brokers flashed 'error' on its screens, negating that particular transaction. At that point, whether or not the knockin options had been triggered was academic: the puts' 45 exercise price was far below the market, meaning that they could not be exercised at a profit. A few days later, however, it mattered a great deal: Mexico devalued its peso, torpedoing emerging-market bonds. By Jan. 10, Venezuelan par bonds had plunged 24cents on the dollar. Suddenly the knock-ins were potentially very valuable, if they had, in fact, been activated. But had they? By January, that was a matter of mighty dispute. Merrill Lynch vehemently disagreed. The 51 1/8 bid that had flickered on the Euro Brokers screen 'was one of two things: Either someone made a mistake or it was an attempt at manipulation'..." This prompted the SEC to start an investigation into the alleged price manipulation in Venezuelan bond market by US institutions.

This discussion suggests that it is desirable *to modify the barrier provision in such a way as to retain as much of the premium savings afforded by the standard barrier provision as possible, but at the same time to achieve continuity of both the option's payoff and delta at the barrier.* This would alleviate many of the risk management problems with standard barriers.

One way to reduce the knock-out risk of barrier options is to include a partial knock-out provision for only a part of the option's life (knock-out window) (see Heynen and Kat (1994)). This reduces the risk of being knocked-out at the cost of raising the premium. However, it does not change the inherently discontinuous character of barrier options, and the risk of being knocked out by a short-term price spike during the knock-out window is still present.

Another solution, the so-called continuous range or soft barrier options, was recently proposed by Hart and Ross (1994). They proposed to include a continuous barrier range from B_{min} to B_{max}, rather than a single discrete barrier level. Such an option would knock-out gradually as the underlying price gets into the barrier range.

1.2 Regularizing Barrier Contracts By Introducing Finite Knock-Out Rates

In this paper we propose an alternative solution to the barrier options problem. The stochastic model of barrier options is that of Brownian motion *instantaneously killed* as

soon as a Brownian particle reaches the barrier level B (see Ito and McKean (1974), Karlin and Taylor (1981) and Karatzas and Shreve (1992)). Then the natural solution is to consider Brownian motion *with killing at finite rate* r_B below the barrier B. That is, the probability that a Brownian particle survives until time T is

$$\exp\left(-\int_0^T r_B\theta(B-S(t'))\right)dt',$$

where $\theta(x)$ is the *Heaviside step function* (defined by Eq.(2.4)). The integrand $r_B\theta(B-S)$ is called *step potential*, and the time integral is the *occupation time* below the barrier, τ_B^-, times the killing rate r_B.[7] This exponential can be interpreted as a *knock-out discount factor with the finite knock-out rate* r_B below the barrier. The introduction of the finite knock-out rate will *regularize* the problem of barrier options and effectively play a role of the *circuit breakers* called for by Soros by making the option's payoff and delta continuous at the barrier and thus alleviating many of the hedging problems with standard barriers.

We also extend this construction to other derivatives and suggest a new design for forward contracts and swaps. These new contracts have very interesting *principal amortizing* properties that make them potentially attractive hedging instruments in FX and fixed income markets.

The rest of this paper is devoted to the development of pricing methodology for this family of derivatives parametrized by knock-out rates. We call them *step derivatives: step options, forwards and swaps.* They could also be called *gradual knock-out (knock-in) contracts.* The paper is organized as follows. In Section 2, we define step options and forward contracts by their payoffs at expiration and discuss their qualitative properties. For example, a down-and-out step call is defined by its payoff at expiration:

$$\exp\left(-r_B\tau_B^-\right)\ Max(S_T-K,0).$$

In Section 3, we develop continuous-time theory of step options and forwards and obtain the closed-form pricing formulas by means of the Green's functions technique. In Section 4, we obtain closed-form formulas for step options sensitivities and discuss their hedging properties. We also show how a dynamic almost-replicating strategy for barrier options can be formulated by considering a dynamic replicating strategy for step options with arbitrarily high but finite knock-out rate. In Section 5, a general class of derivative securities based on occupation times is considered and analytical pricing formulas are derived. In Section 6, we consider *linear step options* with the knock-out factor *linear* in occupation time (no compounding):

$$Max(1-R_B\tau_B^-,0)Max(S_T-K,0).$$

[7]Another interpretation in quantum mechanics is that of a quantum particle in the infinitely high potential barrier. Here we suggest to consider a *finite* rather than the *infinite* potential barrier, or a *step potential of finite height* (see, e.g., Messiah (1961) and Landau and Lifshits (1965)).

In Section 7, we develop discrete-time models for pricing occupation time derivatives. Both Monte Carlo simulation and the Hull and White path-dependent extension of the standard CRR binomial tree are discussed. In Section 8, numerical examples are given and properties of step options are analyzed. Section 9 summarizes our results and discusses further research directions. In Appendix A we give a brief overview of the Wiener-Feynman path integrals and Green's functions approach to options pricing. Appendices B and C contain some mathematical formulas used in the derivation of our closed-form pricing formulas. In Appendix D, we obtain static perpetual solutions to the step options PDE, perpetual step options. Appendix E gives a brief summary of barrier forwards.

2 A New Family of Risk Management Products

2.1 Step Options: A No-Regrets Alternative to Barrier Options

To re-state the problem, we wish to modify the barrier provision to retain as much of the premium savings as possible, and at the same time eliminate the discontinuities at the barrier. Consider a standard call with strike price K and time to expiration $\tau = T - t$ (t and T denote the contract inception and expiration times, respectively). A down-and-out provision renders the option worthless as soon as the underlying price hits a pre-specified price level (barrier) B. Accordingly, the payoff of a down-and-out call at expiration can be written as (*do* stands for down-and-out)

$$C_B^{do}(S_T, T) = \mathbf{1}_{\{L_T > B\}}\, Max(S_T - K, 0), \tag{2.1}$$

where S_T is the underlying price at expiration, L_T is the lowest price of the underlying asset between the inception of the contract t and expiration T, $L_T = \min_{t' \in [t,T]} S(t')$, and $\mathbf{1}_{\{L_T > B\}}$ is the indicator function equal to one if $L_T > B$ (the barrier is never hit) and zero otherwise.

We modify the payoff at expiration by introducing a finite *knock-out rate* r_B and defining the payoff by the formula

$$C_{B,r_B}^{do}(S_T, T) = \exp(-r_B \tau_B^-)\, Max(S_T - K, 0), \tag{2.2}$$

where τ_B^- is *the total time during the life of the option the underlying price is lower than the barrier level* B. It is called *occupation time* of the underlying price process (see, e.g., Ito and McKean (1974), Karlin and Taylor (1981) and Karatzas and Shreve (1992)). We decompose the lifetime of the option $\tau = T - t$ as follows: $\tau = \tau_B^- + \tau_B^+$, where τ_B^- is the total time spent below the barrier and τ_B^+ — above the barrier. Figure 1 illustrates the calculation of τ_B^-. We call the discount factor $\exp(-r_B \tau_B^-)$ *knock-out factor*. It is defined on price paths $\{S(t'), t \le t' \le T\}$ and can be represented in the form:

$$\tau_B^- = \int_t^T \theta(B - S(t'))dt', \;\; \exp(-r_B \tau_B^-) = \exp\left(-\int_t^T r_B\, \theta(B - S(t'))dt'\right), \tag{2.3}$$

where $\theta(x)$ is *the Heaviside step function* defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{2.4}$$

The integrand $r_B\theta(B - S)$ is called *step potential.* It has the shape of a step of the staircase of height r_B (see Figure 2). Accordingly, we call a one-parameter family of path-dependent options defined by the payoff (2.2) *step options.* It is easy to see that the step option payoff (2.2) tends to the payoff of an otherwise identical standard barrier option (2.1) in the limit of an infinitely high knock-out rate

$$\lim_{r_B \to \infty} e^{-r_B \tau_B^-} = \begin{cases} 1 & \text{if } \tau_B^- = 0 \\ 0 & \text{if } \tau_B^- > 0 \end{cases} \tag{2.5}$$

and coincides with the standard European call payoff $Max(S_T - K, 0)$ in the limit of zero knock-out rate

$$\lim_{r_B \to 0} e^{-r_B \tau_B^-} = 1, \tag{2.6}$$

thus continuously interpolating between otherwise identical vanilla and down-and-out contracts.

The other three types of step options, up-and-out step call, down-and-out step put and up-and-out step put, are defined similarly by their payoffs (C (P) stands for call (put) and $uo(do)$ for up-and-out (down-and-out) option, respectively):

$$C^{uo}_{B,r_B}(S_T, T) = \exp(-r_B \tau_B^+)\, Max(S_T - K, 0), \tag{2.7a}$$

$$P^{do}_{B,r_B}(S_T, T) = \exp(-r_B \tau_B^-)\, Max(K - S_T, 0), \tag{2.7b}$$

$$P^{uo}_{B,r_B}(S_T, T) = \exp(-r_B \tau_B^+)\, Max(K - S_T, 0). \tag{2.7c}$$

A *knock-in step option* is defined complimentary to the corresponding knock-out option so that their sum is equivalent to an otherwise identical vanilla contract. The payoffs at expiration for down-and-in step calls, up-and-in step calls, down-and-in step puts and up-and-in step puts are, respectively (i stands for in- options):

$$C^{di}_{B,r_B}(S_T, T) = (1 - \exp(-r_B \tau_B^-))\, Max(S_T - K, 0), \tag{2.8a}$$

$$C^{ui}_{B,r_B}(S_T, T) = (1 - \exp(-r_B \tau_B^+))\, Max(S_T - K, 0), \tag{2.8b}$$

$$P^{di}_{B,r_B}(S_T, T) = (1 - \exp(-r_B \tau_B^-))\, Max(K - S_T, 0), \tag{2.8c}$$

$$P^{ui}_{B,r_B}(S_T, T) = (1 - \exp(-r_B \tau_B^+))\, Max(K - S_T, 0). \tag{2.8d}$$

Consequently, it is enough to consider out- options only — in- options can be obtained synthetically.

A holder of a down-and-out (up-and-out) step option is penalized at the *continuously compounded penalty rate* r_B for the time the underlying price spent below (above) the barrier during the option's lifetime. For a standard knock-out option, the knock-out penalty rate is infinitely high and the entire option payoff is *instantaneously* lost by the option holder as a *knock-out penalty* should the underlying price hit the barrier even momentarily. For any finite knock-out rate, however, it takes some time below (above) the barrier to reduce the option payoff to close to zero: the option knocks out *gradually.* We define an *effective knock-out time*, $\mathcal{T}_B^-$ ($\mathcal{T}_B^+$), as the time below (above) the barrier needed to reduce the terminal payoff of a down-and-out (up-and-out) step option by ninety percent

$$\exp(-r_B \mathcal{T}_B) = 0.1, \; \mathcal{T}_B = \frac{\ln 10}{r_B}. \tag{2.9}$$

That is, the holder of a down-and-out step call will receive only ten percent of the payoff of an otherwise identical vanilla call at expiration if the asset spends time $\mathcal{T}_B^-$ below the barrier during the life of the contract, $\tau_B^- = \mathcal{T}_B^-$. Another useful measure of knock-out speed is a *single-day knock-out factor* β_B,

$$\beta_B = \exp(-r_B/250) \tag{2.10}$$

(we assume 250 trading days per year, and rate r_B is *continuously compounded*).[8] This is a factor by which the payoff is discounted for every trading day the underlying asset spends below the barrier. Obviously, the single-day knock-out factor is zero for barrier options and unity for vanilla options. It continuously interpolates between zero and unity depending on the knock-out rate r_B for step options. The payoff (2.2) can be re-written as

$$C^{do}_{B,r_B}(S_T, T) = (\beta_B)^{n_B^-} \, Max(S_T - K, 0), \tag{2.11}$$

where n_B^- is the total number of trading days the underlying spent below the barrier during the option's lifetime, $\tau_B^- = n_B^-/250$ (τ_B^- and n_B^- are occupation times measured in years and in trading days, respectively).

One of the major advantages of step options is the ability to structure contracts with any knock-out rate desired by investors. An investor just needs to specify a desired knock-out factor β_B or effective knock-out time $\mathcal{T}_B$. The corresponding knock-out rate r_B can then be readily determined from Eqs.(2.9-10). This affords significant flexibility. By choosing a single-day knock-out factor greater than zero, the option buyer assures himself that his option will never lose its entire value due to a short-term price movement. In fact, an investor can customize the option in a way that will most properly match his belief about the future price behavior of the underlying.

To illustrate, suppose an investor wants to purchase a call. Suppose further that he also holds a view that the underlying is unlikely to fall below the barrier B. To reduce

[8] It is established practice in options markets to exclude weekends for the purpose of barrier monitoring.

the premium, he may elect to include a knock-out barrier provision in the contract. With the availability of step options, he is no longer caught in the dilemma of choosing between the premium savings of a barrier option accompanied by the risk of loss of his entire investment due to a whipsaw, or paying the full premium for the standard vanilla call. He is allowed to set his own risk/reward profile by selecting an appropriate knock-out rate for his option corresponding to his risk aversion and the degree of confidence in the barrier not being hit during the option's lifetime.

At the same time, it is more advantageous for the dealer to sell a step option rather than the standard barrier option since, as we will see, the step option delta is continuous at the barrier, thus allowing to hedge by executing a continuous dynamic replicating strategy. Thus, *step options with finite knock-out rates (gradual knock-outs) have significant risk management advantages both for the buyer and for the seller.*

Let us also mention another potential positive effect of introducing step options into the marketplace. Since different market participants will select different knock-out rates, even though they may all set the barrier at the same obvious support or resistance level, no short-term manipulation by traders will result in massive simultaneous knock-outs. This would reduce the volatility around obvious barrier levels.

2.2 Step Forwards and Swaps: Principal-Amortizing Hedging Products

Similar to step options, *step forwards and swaps* can be designed so that their principal is reduced by *an amortization factor based on the occupation time.* A down-and-out *step forward* with delivery price K and the barrier B is defined so that its payoff at expiration is:

$$f^{do}_{B,r_B}(S_T,T) = \exp(-r_B\tau^-_B)(S_T - K), \tag{2.12}$$

where $\exp(-r_B\tau^-_B)$ is the same *amortizing knock-out factor* as for the down-and-out step option. An up-and-out step forward is defined similarly:

$$f^{uo}_{B,r_B}(S_T,T) = \exp(-r_B\tau^+_B)(S_T - K). \tag{2.13}$$

Step forwards possess an interesting property of *amortizing the principal for each unit of time the underlying is below (above) the barrier.* A crucial feature of this contract design is that a step forward coincides with the standard forward contract in the zero knock-out limit $r_B \to 0$, and with the barrier forward in the limit $r_B \to \infty$ (a *barrier forward* is a forward contract that ceases to exist as soon as the underlying hits the barrier, see Appendix E).

Knock-in step forwards are designed similar to knock-in step options by requiring that the sum of complimentary in- and out- contracts is equivalent to an otherwise identical vanilla forward:

$$f^{di}_{B,r_B}(S_T,T) = (1 - \exp(-r_B\tau^-_B))(S_T - K), \tag{2.14a}$$

$$f^{ui}_{B,r_B}(S_T,T) = (1-\exp(-r_B\tau^+_B))(S_T-K). \qquad (2.14b)$$

Step swaps can be designed as multi-period step forwards (portfolios of step forwards). They are very similar to index-amortizing swaps (see, e.g., Jarrow (1996)). A *step swap* is an amortizing swap (say, receive fixed and pay floating) in which the principal is reduced by an amortizing schedule based on the occupation time the spot rate spends below a pre-specified threshold barrier level. Similar to standard index-amortizing swaps, *step swaps may be used to partially hedge against the prepayment risk of mortgage-backed securities.* We will explore this interesting application in a separate paper.

3 Continuous-Time Theory and Closed-Form Pricing Formulas

3.1 Step Calls

3.1.1 Risk-Neutral Pricing

To value step options we assume the underlying asset price follows a geometric Brownian motion with constant drift rate m and volatility σ, $dS = mSdt + \sigma Sdz$, there is no payout on the underlying, and we live in the Black-Scholes world with constant continuously compounded risk-free interest rate r. According to the risk-neutral valuation approach (see, e.g., Hull (1996), Duffie (1996) and Jarrow and Turnbull (1996)), at the inception of the contract t the present value of a down-and-out step call with payoff (2.2) is given by its discounted average over the risk-neutral measure conditional on the initial asset price S at time t:

$$C^{do}_{B,r_B}(S,t) = e^{-r\tau}E_{(t,S)}\left[e^{-r_B\tau^-_B}\,Max(S_T-K,0)\right]$$

$$= e^{-r\tau}E_{(t,S)}\left[\exp\left(-\int_t^T r_B\theta(B-S_{t'})dt'\right)\,Max(S_T-K,0)\right]. \qquad (3.1)$$

The average (3.1) can be conveniently represented as a *Wiener-Feynman path integral* over all possible price paths from the initial price S at inception t to the final price S_T at expiration T (see Dash (1989), (1993), Eydeland (1994), Esmailzadeh (1995) and Linetsky (1996) for applications of path integrals to options pricing). Introducing a new variable $x(t')$ for the logarithm of the asset price, $x(t') = \ln S(t')$ ($x = \ln S$ and $x_T = \ln S_T$), we can re-write (3.1) as follows (see Appendix A for details on risk-neutral pricing, Wiener-Feynman path integrals and Green's functions)

$$C^{do}_{B,r_B}(S,t) = e^{-r\tau}\int_{\ln K}^{\infty}(e^{x_T}-K)$$

$$\times\left(\int_{x(t)=x}^{x(T)=x_T}\exp\left(-\int_t^T(\mathcal{L}_{BS}+r_B\theta(b-x_{t'}))dt'\right)\mathcal{D}x_{t'}\right)dx_T\,, \qquad (3.2)$$

where $\mathcal{L}_{BS}$ is the Black-Scholes Lagrangian function for the risk-neutral process $dx = \mu dt + \sigma dz$ with the risk-neutral drift rate μ and volatility σ (t' parametrizes paths from t to T):

$$\mathcal{L}_{BS} = \frac{1}{2\sigma^2}(\dot{x} - \mu)^2, \ \dot{x}(t') := \frac{dx}{dt'}, \tag{3.3}$$

$$\mu = r - \frac{1}{2}\sigma^2, \tag{3.4}$$

and b is the logarithm of the barrier B, $b = \ln B$, and, obviously, $\theta(B - S) = \theta(b - x)$.

3.1.2 The Step PDE and its Feynman-Kac Solution

Eq.(3.2) is the *Feynman-Kac representation* of a unique solution to the Cauchy problem for the Black-Scholes partial differential equation with *step potential*

$$\frac{\sigma^2}{2}S^2\frac{\partial^2 C^{do}_{B,r_B}}{\partial S^2} + rS\frac{\partial C^{do}_{B,r_B}}{\partial S} - (r_B\theta(B - S) + r)\, C^{do}_{B,r_B} = -\frac{\partial C^{do}_{B,r_B}}{\partial t} \tag{3.5}$$

and terminal condition (2.2). The presence of the step potential effectively means that, while above the barrier the risk-free rate is r and there are no dividends on the underlying, below the barrier an effective risk-free rate is $(r + r_B)$ and continuous dividend yield is r_B.

By noting that

$$\int_t^T \mathcal{L}_{BS}\, dt' = \int_t^T \mathcal{L}_0\, dt' - \frac{\mu}{\sigma^2}(x_T - x) + \frac{\mu^2\tau}{2\sigma^2}, \tag{3.6}$$

where $\mathcal{L}_0$ is the Lagrangian function for a process with zero drift $dx = \sigma dz$ (martingale),

$$\mathcal{L}_0 = \frac{1}{2\sigma^2}\dot{x}^2, \tag{3.7}$$

Eq.(3.2) can be reduced to

$$C^{do}_{B,r_B}(S,t) = e^{-r\tau}\int_{\ln K}^{\infty}(e^{x_T} - K)e^{\frac{\mu}{\sigma^2}(x_T - x) - \frac{\mu^2\tau}{2\sigma^2}}\mathcal{K}^-_{B,r_B}(x_T, x; \tau)\, dx_T\,. \tag{3.8}$$

3.1.3 The Green's Function

Here $\mathcal{K}^-_{B,r_B}$ is the *Green's function* (transition probability density) for zero-drift Brownian motion with killing at finite rate r_B below the barrier:

$$\mathcal{K}^-_{B,r_B}(x_T, x; \tau) = \int_{x(t)=x}^{x(T)=x_T} \exp\left(-\int_t^T (\mathcal{L}_0 + r_B\theta(b - x_{t'}))dt'\right)\mathcal{D}x_{t'}. \tag{3.9}$$

It is the Feynman-Kac representation of the fundamental solution of the step PDE with zero drift

$$\frac{\sigma^2}{2}\frac{\partial^2\mathcal{K}^-_{B,r_B}}{\partial x^2} - r_B\theta(b - x)\mathcal{K}^-_{B,r_B} = \frac{\partial\mathcal{K}^-_{B,r_B}}{\partial\tau} \tag{3.10a}$$

and initial condition

$$\mathcal{K}^-_{B,r_B}(x_T, x; 0) = \delta(x_T - x), \tag{3.10b}$$

where $\delta(x_T - x)$ is the Dirac delta function.

The Green's function must also satisfy two additional *continuity boundary conditions* at the barrier $x = b$,

$$\lim_{\epsilon \to 0_+} \left(\mathcal{K}^-_{B,r_B}(x_T, b - \epsilon; \tau) - \mathcal{K}^-_{B,r_B}(x_T, b + \epsilon; \tau) \right) = 0, \tag{3.11a}$$

$$\lim_{\epsilon \to 0_+} \left(\frac{\partial \mathcal{K}^-_{B,r_B}}{\partial x}(x_T, b - \epsilon; \tau) - \frac{\partial \mathcal{K}^-_{B,r_B}}{\partial x}(x_T, b + \epsilon; \tau) \right) = 0, \tag{3.11b}$$

to insure that the second derivative $\frac{\partial^2 \mathcal{K}^-_{B,r_B}}{\partial x^2}$ is piece-wise continuous (it has a finite jump of height proportional to r_B at the barrier $x = b$ due to the presence of the step function in Eq.(3.10a)). The boundary conditions (3.11) simply state that both the Green's function and its first derivative are continuous functions at the barrier for any finite knock-out rate. Consequently, the step option's value and delta are also continuous at the barrier for any finite knock-out rate. *The continuity boundary conditions are a key to hedging properties of step options, allowing for the existence of a continuous dynamic replication strategy.*

3.1.4 Solving the Boundary Value Problem for the Resolvent

To solve the PDE problem (3.9)-(3.11), we introduce a resolvent kernel by taking the Laplace transform

$$G^-_{B,r_B}(x_T, x; s) = \int_0^\infty e^{-s\tau} \mathcal{K}^-_{B,r_B}(x_T, x; \tau) d\tau. \tag{3.12}$$

Note that the integration in Eq.(3.8) is performed over the interval $\ln K \leq x_T < \infty$, so we need only consider the region where $x_T > b$, since down-and-out calls are usually structured so that $B < K$ and $b < \ln K \leq x_T$. The resolvent kernel satisfies the following ordinary differential equations with boundary conditions:

Region I: $x > b$, $x_T > b$

$$\frac{\sigma^2}{2} \frac{\partial^2 G^{-I}_{B,r_B}}{\partial x^2} - s G^{-I}_{B,r_B} = -\delta(x_T - x), \tag{3.13a}$$

supplemented by the asymptotic boundary condition

$$\lim_{x \to \infty} G^{-I}_{B,r_B}(x_T, x; s) = 0. \tag{3.13b}$$

Region II: $x < b$, $x_T > b$

$$\frac{\sigma^2}{2} \frac{\partial^2 G^{-II}_{B,r_B}}{\partial x^2} - (s + r_B) G^{-II}_{B,r_B} = 0, \tag{3.14a}$$

supplemented by the asymptotic boundary condition

$$\lim_{x\to-\infty} G^{-II}_{B,r_B}(x_T,x;s)=0. \tag{3.14b}$$

The continuity boundary conditions (3.11a,b) glue solutions in the regions I and II together at the barrier separating the regions:

$$G^{-I}_{B,r_B}(x_T,b;s)=G^{-II}_{B,r_B}(x_T,b;s), \tag{3.15a}$$

$$\frac{\partial G^{-I}_{B,r_B}}{\partial x}(x_T,b;s)=\frac{\partial G^{-II}_{B,r_B}}{\partial x}(x_T,b;s). \tag{3.15b}$$

Now it is a standard task to solve problem (3.13)-(3.15). Solutions of the corresponding homogeneous equations are $\exp\left(\pm x\sqrt{2s}/\sigma\right)$ and $\exp\left(\pm x\sqrt{2(s+r_B)}/\sigma\right)$. Imposing the boundary conditions and carefully solving for integration constants, we arrive at the following solution for the resolvent kernel:

$$G^{-I}_{B,r_B}(x_T,x;s)=\frac{1}{\sigma\sqrt{2s}}\left(\exp(-|y_T-y|\sqrt{2s})-\mathcal{R}_{r_B}(s)\exp(-|y_T+y|\sqrt{2s})\right), \tag{3.16a}$$

$$G^{-II}_{B,r_B}(x_T,x;s)=\frac{1}{\sigma\sqrt{2s}}\mathcal{T}_{r_B}(s)\exp\left(y\sqrt{2(s+r_B)}-y_T\sqrt{2s}\right), \tag{3.16b}$$

where we have introduced new variables

$$y=\frac{x-b}{\sigma}=\frac{1}{\sigma}\ln\left(\frac{S}{B}\right),\ y_T=\frac{x_T-b}{\sigma}=\frac{1}{\sigma}\ln\left(\frac{S_T}{B}\right), \tag{3.17}$$

and $\mathcal{R}$ and $\mathcal{T}$ are *reflection* and *transmission* coefficients, respectively,

$$\mathcal{R}_{r_B}(s)=\frac{\sqrt{s+r_B}-\sqrt{s}}{\sqrt{s+r_B}+\sqrt{s}}, \tag{3.18a}$$

$$\mathcal{T}_{r_B}(s)=1-\mathcal{R}_{r_B}(s)=\frac{2\sqrt{s}}{\sqrt{s+r_B}+\sqrt{s}}. \tag{3.18b}$$

One can readily check that this solution is indeed continuous at the barrier $x=b$ $(y=0)$ for any finite r_B. The asymptotic limits are:

Standard vanilla call

$$r_B\to 0:\ \mathcal{T}_0(s)=1\,,\ \mathcal{R}_0(s)=0, \tag{3.19a}$$

Standard down-and-out call

$$r_B\to\infty:\ \mathcal{T}_\infty(s)=0\,,\ \mathcal{R}_\infty(s)=1. \tag{3.19b}$$

This solution is well known in quantum mechanics where it describes tunneling of a quantum particle through the potential barrier of finite height. The coefficient $\mathcal{R}$ gives an amplitude of the wave reflected from the barrier, while $\mathcal{T}$ - of the wave transmitted through the barrier (see, e.g., Messiah (1961) and Landau and Lifshits (1965)). We retain this terminology and call $\mathcal{R}$ and $\mathcal{T}$ reflection and transmission coefficients.

3.1.5 Inverting the Laplace Transform

Now, closed-form expressions for the transition probability density are readily obtained by inverting the Laplace transform, and are given in terms of convolution integrals (see Appendix B for derivation)

Region I: $x > b$, $x_T > b$

$$\mathcal{K}^{-I}_{B,r_B}(x_T, x; \tau) = \mathcal{K}^{-}_{B}(x_T, x; \tau) \tag{3.20a}$$
$$+ \int_0^\tau \frac{(1 - e^{-r_B(\tau-\tau')})|y_T + y|}{2\pi\sigma r_B(\tau-\tau')^{\frac{3}{2}}(\tau')^{\frac{3}{2}}} \exp\left(-\frac{(y_T+y)^2}{2\tau'}\right) d\tau'$$

Region II:$x < b$, $x_T > b$

$$\mathcal{K}^{-II}_{B,r_B}(x_T, x; \tau) = \int_0^\tau \frac{(1 - e^{-r_B(\tau-\tau')})\left[y_T\tau'(\tau-\tau'-y^2) + y(\tau-\tau')(\tau'-y_T^2)\right]}{2\pi\sigma r_B(\tau-\tau')^{\frac{5}{2}}(\tau')^{\frac{5}{2}}}$$
$$\times \exp\left(-\frac{y_T^2}{2\tau'} - \frac{y^2}{2(\tau-\tau')}\right) d\tau', \tag{3.20b}$$

where y and y_T are defined by Eq.(3.17), and $\mathcal{K}^-_B$ is the standard transition probability density for Brownian motion with absorbing barrier at the level b used in pricing down-and-out options which, by the reflection principle, is equal to the difference of two normal densities

$$\mathcal{K}^-_B(x_T, x; \tau) = \mathcal{K}(x_T, x; \tau) - \mathcal{K}(x_T, 2b - x; \tau), \tag{3.21a}$$

$$\mathcal{K}(x_T, x; \tau) = \frac{1}{\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{(x_T - x)^2}{2\sigma^2\tau}\right). \tag{3.21b}$$

One sees that the density (3.20a) consists of two summands: a standard barrier density and a *step premium* density expressed as a convolution integral.

3.1.6 Closed-Form Pricing Formulas

Now it is a matter of tedious but straightforward algebra to simplify the expression (3.8) with the density (3.20) (see Appendix C), and we arrive at the final result, a closed-form pricing formula for the down-and-out step call:

Region I: Above the barrier, $S \geq B$

$$C^{do,I}_{B,r_B}(S,t) = C^{do}_B(S,t) \tag{3.22}$$
$$+ \left(\frac{B}{S}\right)^\gamma \int_0^\tau \frac{\left(1 - e^{-r_B(\tau-\tau')}\right) e^{-\kappa(\tau-\tau')}}{\sqrt{2\pi}\, r_B\,(\tau-\tau')^{3/2}} \left(\nu_2 \left(\frac{B^2}{S}\right) N(d_4') - \nu_1 e^{-r\tau'} K N(d_3')\right) d\tau',$$

where $C_B^{do}(S,t)$ is the standard down-and-out call

$$C_B^{do}(S,t) = C(S,t) - \left(\frac{B}{S}\right)^{\gamma} C\left(\frac{B^2}{S}, t\right), \tag{3.23}$$

$C(S,t)$ is the standard vanilla call

$$C(S,t) = SN(d_2) - e^{-r\tau}KN(d_1), \tag{3.24}$$

and we have introduced the following notations:

$$\mu = r - \frac{\sigma^2}{2},\ \gamma = \frac{2\mu}{\sigma^2} = \frac{2r}{\sigma^2} - 1,\ \kappa = r + \frac{\mu^2}{2\sigma^2} = \frac{1}{2}\left(r + \frac{r^2}{\sigma^2} + \frac{\sigma^2}{4}\right), \tag{3.25a}$$

$$\nu_1 = \frac{\mu}{\sigma} = \frac{r}{\sigma} - \frac{\sigma}{2},\ \nu_2 = \nu_1 + \sigma = \frac{r}{\sigma} + \frac{\sigma}{2}, \tag{3.25b}$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \mu\tau}{\sigma\sqrt{\tau}},\ d_2 = d_1 + \sigma\sqrt{\tau},\ d_3' = \frac{\ln\left(\frac{B^2}{SK}\right) + \mu\tau'}{\sigma\sqrt{\tau'}},\ d_4' = d_3' + \sigma\sqrt{\tau'}. \tag{3.25c}$$

The $N(x)$ and $n(x)$ are the cumulative standard normal distribution functions and its density, respectively,

$$N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy,\ n(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}. \tag{3.26}$$

Region II: Below the barrier, $S < B$

$$C_{B,r_B}^{do,II}(S,t) = \left(\frac{B}{S}\right)^{\frac{\gamma}{2}} \int_0^{\tau} \frac{\left(1 - e^{-r_B(\tau-\tau')}\right) e^{-\kappa(\tau-\tau')}}{\sqrt{2\pi}\, r_B\, (\tau-\tau')^{3/2}} \tag{3.27}$$

$$\times \left\{\nu_1\rho_1(y)e^{-r\tau'}KN(d_5') - \nu_2\rho_2(y)BN(d_6') - \sigma y B(\tau')^{-\frac{1}{2}}n(d_6')\right\} \exp\left(-\frac{y^2}{2(\tau-\tau')}\right) d\tau',$$

where we have introduced the following notations (recall that $y = \frac{1}{\sigma}\ln\left(\frac{S}{B}\right)$):

$$\rho_1(y) = \frac{y^2}{\tau-\tau'} + \nu_1 y - 1,\ \rho_2(y) = \rho_1(y) + \sigma y = \frac{y^2}{\tau-\tau'} + \nu_2 y - 1, \tag{3.28}$$

$$d_5' = \frac{\ln\left(\frac{B}{K}\right) + \mu\tau'}{\sigma\sqrt{\tau'}},\ d_6' = d_5' + \sigma\sqrt{\tau'}. \tag{3.29}$$

We see from Eq.(3.22) that step option price consists of two parts: the standard down-and-out call and a *step premium* investors have to pay for the privilege of having the option *knock out gradually with the pre-specified finite rate* r_B, rather than instantly. The higher the knock-out rate, the lower the premium.

One can readily check that at the barrier $S = B$ the two solutions C^I and C^{II} are indeed equal,

$$C_{B,r_B}^{do,I}(B,t) = C_{B,r_B}^{do,II}(B,t). \tag{3.30}$$

Closed-form pricing formulas for other types of out- options are derived similarly.

3.1.7 Asymptotic Limits $r_B \to 0$ and $r_B \to \infty$

In the limit $r_B \to \infty$ the option knocks out instantly as soon as the barrier is hit, and the step premium is equal to zero,

$$\lim_{r_B \to \infty} C^{do}_{B,r_B}(S,t) = C^{do}_B(S,t). \tag{3.31}$$

The lower the knock-out rate, the higher the premium, and in the limit of zero knock-out rate, the premium is the highest and the step option coincides with an otherwise identical vanilla option,

$$\lim_{r_B \to 0} C^{do}_{B,r_B}(S,t) = C(S,t). \tag{3.32}$$

The knock-out rate *controls the trade-off between premium savings and knock-out speed.*

3.1.8 Seasoned Step Options

Eqs.(3.22) and (3.27) price a newly written step option at the inception of the contract t. Suppose now we want to value a *seasoned* step option at some time t' during the life of the contract, $t < t' < T$. We divide the occupation time from inception t to expiration T into two parts:

$$\tau^-_B(t,T) = \tau^-_B(t,t') + \tau^-_B(t',T),$$

where $\tau^-_B(t,t')$ is the occupation time accumulated to date t', and $\tau^-_B(t',T)$ is yet unknown. Since the underlying asset prices between t and t' are already fixed, one can readily determine $\tau^-_B(t,t')$ by looking at the price history available to date. Then to value the seasoned step option at time t' one must discount the value of an otherwise identical newly written option, given by (3.22) or (3.27) depending on where the underlying is with respect to the barrier at time t', by the already fixed knock-out factor $\exp(-r_B\tau^-_B(t,t'))$:

$$e^{-r_B\tau^-_B(t,t')} C^{do}_{B,r_B}(S_{t'},t'). \tag{3.33}$$

3.2 Step Forwards

3.2.1 Step Forward Price and the Present Value of a Step Forward Contract

Step forwards are analyzed similarly. The delivery price of a down-and-out step forward contract maturing at time T is set so that the contract has zero present value at inception t_0:

$$f^{do}_{B,r_B}(S_0,t_0) = e^{-r(T-t_0)} E_{(t_0,S_0)}\left[\exp(-r_B\tau^-_B)(S_T - K)\right] \tag{3.34}$$

$$= e^{-r(T-t_0)}\left\{E_{(t_0,S_0)}\left[\exp(-r_B\tau^-_B)S_T\right] - KE_{(t_0,S_0)}\left[\exp(-r_B\tau^-_B)\right]\right\} = 0.$$

We call this delivery price *step forward price*:

$$K = F^{do}_{B,r_B}(S_0,t_0) = \frac{E_{(t_0,S_0)}\left[\exp(-r_B\tau^-_B)S_T\right]}{E_{(t_0,S_0)}\left[\exp(-r_B\tau^-_B)\right]} \tag{3.35}$$

The present value of a seasoned step forward contract at some date t during the life of the contract, $t_0 < t < T$, is given by ($S = S(t)$)

$$f^{do}_{B,r_B}(S,t) = e^{-r(T-t)-r_B\tau^-_B(t_0,t)}\left\{E_{(t,S)}\left[\exp(-r_B\tau^-_B(t,T))S_T\right]\right. \tag{3.36}$$

$$\left. -KE_{(t,S)}\left[\exp(-r_B\tau^-_B(t,T))\right]\right\},$$

where K is the delivery price (3.35) that was set at inception t_0. Here $\tau^-_B(t_0,t)$ and $\tau^-_B(t,T)$ are the occupation times below the barrier during the already passed time period $t-t_0$ (fixed prices) and the future time period $T-t$ (yet unknown prices), respectively.

3.2.2 The Averages $E_{(t,S)}\left[\exp(-r_B\tau^-_B)\,S_T\right]$ and $E_{(t,S)}\left[\exp(-r_B\tau^-_B)\right]$

To calculate the step forward price (3.35) and to find the present value of a step forward contract after inception (3.36), we need to calculate two averages: $E_{(t,S)}\left[\exp(-r_B\tau^-_B)\,S_T\right]$ and $E_{(t,S)}\left[\exp(-r_B\tau^-_B)\right]$. The calculation methodology is the same as for down-and-out step options and relies on the Feynman-Kac formula:

$$E_{(t,S)}\left[\exp(-r_B\tau^-_B)\right] = \int_{-\infty}^{\infty} e^{\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\,\mathcal{K}^-_{B,r_B}(x_T,x;\tau)\,dx_T\,, \tag{3.37}$$

and

$$E_{(t,S)}\left[\exp(-r_B\tau^-_B)\,S_T\right] = \int_{-\infty}^{\infty} e^{x_T+\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\,\mathcal{K}^-_{B,r_B}(x_T,x;\tau)\,dx_T\,, \tag{3.38}$$

where $\mathcal{K}^-_{B,r_B}$ is the same Green's function (transition density) as for the down-and-out step call.

Note, however, that the integration over x_T is performed from $-\infty$ to ∞ and the logarithm of the barrier $b = \ln B$ is now inside the integration interval. Thus, we need all four densities $\mathcal{K}^{-I}$, $\mathcal{K}^{-II}$, $\mathcal{K}^{-III}$ and $\mathcal{K}^{-IV}$. The first two are given by Eqs.(3.20). To find the former two we again employ Laplace transform. Resolvent kernels satisfy the following ordinary differential equations:

Region III: $x > b$, $x_T < b$

$$\frac{\sigma^2}{2}\frac{\partial^2 G^{-III}_{B,r_B}}{\partial x^2} - sG^{-III}_{B,r_B} = 0, \tag{3.39a}$$

$$\lim_{x\to\infty} G^{-III}_{B,r_B}(x_T,x;s) = 0. \tag{3.39b}$$

Region IV: $x < b$, $x_T < b$

$$\frac{\sigma^2}{2}\frac{\partial^2 G^{-IV}_{B,r_B}}{\partial x^2} - (s+r_B)G^{-IV}_{B,r_B} = -\delta(x_T-x), \tag{3.40a}$$

$$\lim_{x\to-\infty} G^{-IV}_{B,r_B}(x_T,x;s) = 0. \tag{3.40b}$$

Continuity boundary conditions to glue the solutions G^{-III} and G^{-IV} together at the barrier are the same as in Eq.(3.15), and we obtain:

$$G^{-III}_{B,r_B}(x_T, x; s) = \frac{\sqrt{2}}{\sigma(\sqrt{s+r_B}+\sqrt{s})} \exp\left(y_T\sqrt{2(s+r_B)} - y\sqrt{2s}\right), \tag{3.41a}$$

$$G^{-IV}_{B,r_B}(x_T, x; s) \tag{3.41b}$$

$$= \frac{1}{\sigma\sqrt{2(s+r_B)}}\left\{\exp\left(-|y_T - y|\sqrt{2(s+r_B)}\right) + \mathcal{R}_{r_B}(s)\exp\left(-|y_T + y|\sqrt{2(s+r_B)}\right)\right\}.$$

Inverting the Laplace transform yields

$$\mathcal{K}^{-IV}_{B,r_B}(x_T, x; \tau) = e^{-r_B\tau}\,\mathcal{K}^{-}_{B}(x_T, x; \tau) \tag{3.42a}$$

$$+\int_0^\tau \frac{e^{-r_B\tau'}(1-e^{-r_B(\tau-\tau')})|y_T+y|}{2\pi\sigma r_B(\tau-\tau')^{\frac{3}{2}}(\tau')^{\frac{3}{2}}}\exp\left(-\frac{(y_T+y)^2}{2\tau'}\right)d\tau',$$

and

$$\mathcal{K}^{-III}_{B,r_B}(x_T, x; \tau) = \mathcal{K}^{-II}_{B,r_B}(x, x_T; \tau), \tag{3.42b}$$

where $\mathcal{K}^{-}_{B}$ is the standard down-and-out density (3.21).

Then the averages (3.37), (3.38) are given by the integrals:

Region I: $S \geq B$

$$E_{(t,S)}\left[\exp(-r_B\tau^-_B)\right] = \int_{-\infty}^{b} e^{\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\,\mathcal{K}^{-III}_{B,r_B}(x_T, x; \tau)\,dx_T$$

$$+\int_b^{\infty} e^{\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\,\mathcal{K}^{-I}_{B,r_B}(x_T, x; \tau)\,dx_T, \tag{3.43a}$$

and

$$E_{(t,S)}\left[\exp(-r_B\tau^-_B)\,S_T\right] = \int_{-\infty}^{b} e^{x_T+\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\,\mathcal{K}^{-III}_{B,r_B}(x_T, x; \tau)\,dx_T$$

$$+\int_b^{\infty} e^{x_T+\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\,\mathcal{K}^{-I}_{B,r_B}(x_T, x; \tau)\,dx_T; \tag{3.43b}$$

and

Region II: $S < B$

$$E_{(t,S)}\left[\exp(-r_B\tau^-_B)\right] = \int_{-\infty}^{b} e^{\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\mathcal{K}^{-IV}_{B,r_B}(x_T, x; \tau)\,dx_T$$

$$+\int_b^{\infty} e^{\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\mathcal{K}^{-II}_{B,r_B}(x_T, x; \tau)\,dx_T, \tag{3.44a}$$

and

$$E_{(t,S)}\left[\exp(-r_B\tau_B^-)\,S_T\right] = \int_{-\infty}^{b} e^{x_T+\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\mathcal{K}^{-IV}_{B,r_B}(x_T,x;\tau)\,dx_T$$

$$+\int_{b}^{\infty} e^{x_T+\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\mathcal{K}^{-II}_{B,r_B}(x_T,x;\tau)\,dx_T. \qquad (3.44b)$$

Substituting these integrals back into Eq.(3.35), we obtain a closed-form solution for the down-and-out step forward price:

$$F^{do}_{B,r_B}(S_0,t_0) = e^{r(T-t_0)}S_0\,\mathcal{D}^{do}_{B,r_B}(S_0,t_0) = \mathcal{D}^{do}_{B,r_B}(S_0,t_0)\,F_0, \qquad (3.45)$$

where $F_0 = e^{r(T-t_0)}\,S_0$ is the standard forward price and the factor $\mathcal{D}^{do}_{B,r_B}(S_0,t_0)$ is given by:

$$\mathcal{D}^{do}_{B,r_B}(S_0,t_0) = \frac{E_{(t_0,S_0)}\left[\exp(-r_B\tau_B^-)S_T\right]}{E_{(t_0,S_0)}\left[S_T\right]E_{(t_0,S_0)}\left[\exp(-r_B\tau_B^-)\right]}, \qquad (3.46)$$

and $E_{(t_0,S_0)}\left[S_T\right] = e^{r(T-t_0)}S_0$ in the risk-neutral world.

The present value of a seasoned step forward contract at some time t, $t_0 < t \le T$, is then given by Eq.(3.36), where K is the delivery price set at time t_0.

3.2.3 Asymptotic Limits

Asymptotic properties of step forwards are similar to those of step options. In the zero knock-out rate limit the forward price F and the present value of a forward contract f coincide with those of an otherwise identical vanilla forward:

$$\lim_{r_B\to 0} D^{do}_{B,r_B}(S,t) = 1,\ \lim_{r_B\to 0} F^{do}_{B,r_B}(S,t) = e^{r\tau}S, \qquad (3.47a)$$

$$\lim_{r_B\to 0} f^{do}_{B,r_B}(S,t) = S - e^{-r\tau}K. \qquad (3.47b)$$

In the infinite knock-out rate limit, they tend to those of the standard barrier forward (see Appendix E for a summary of barrier forwards).

Another crucial property of step forwards is that the forward price and the present value of a step forward contract are continuous at the barrier:

$$F^{do,I}_{B,r_B}(B,t) = F^{do,II}_{B,r_B}(B,t),\ f^{do,I}_{B,r_B}(B,t) = f^{do,II}_{B,r_B}(B,t). \qquad (3.48)$$

Up-and-out, down-and-in and up-and-in forwards are analyzed similarly.

3.3 Step Puts and the Put-Call-Forward Parity

To value step puts with the payoff (2.7b) we just need to note an obvious identity — a *Put-Call-Forward Parity* for step options and forwards:

$$C^{do}_{B,r_B}(S,t) - P^{do}_{B,r_B}(S,t) = f^{do}_{B,r_B}(S,t). \tag{3.49}$$

Here f is a long step forward position with delivery price K equal to the strike of the call and the put, and with the same maturity T:

$$f^{do}_{B,r_B}(S,t) = e^{-r\tau}\left\{E_{(t,S)}\left[\exp(-r_B\tau_B^-)S_T\right] - KE_{(t,S)}\left[\exp(-r_B\tau_B^-)\right]\right\}. \tag{3.50}$$

The parity is satisfied for the payoffs (2.2), (2.7b) and (2.16) at expiration T, so it must hold for the present values at time t as well. This is a generalization of the standard put-call-forward parity for vanilla contracts

$$C - P = f = S - e^{-r\tau}K \tag{3.51}$$

and for barrier contracts (see Appendix E). Thus for step puts we have

$$P^{do}_{B,r_B}(S,t) = C^{do}_{B,r_B}(S,t) - f^{do}_{B,r_B}(S,t). \tag{3.52}$$

Up-and-out step puts are priced similarly:

$$P^{uo}_{B,r_B}(S,t) = C^{uo}_{B,r_B}(S,t) - f^{uo}_{B,r_B}(S,t). \tag{3.53}$$

4 Dynamic Replication of Step and Barrier Options

Closed-form expressions for the delta can be readily obtained by differentiating (3.22), (3.27) with respect to the underlying price S. For down-and-out calls we have:

Region I: Above the barrier, $S > B$

$$\Delta^{do,I}_{B,r_B}(S,t) = \frac{\partial C^{do,I}_{B,r_B}}{\partial S} = \Delta^{do}_B(S,t) - \frac{1}{S}\left(\frac{B}{S}\right)^{\gamma}\int_0^{\tau}\frac{\left(1-e^{-r_B(\tau-\tau')}\right)e^{-\kappa(\tau-\tau')}}{\sqrt{2\pi}\,r_B\,(\tau-\tau')^{3/2}}$$

$$\times\left(\frac{B^2}{S}\left[(\gamma+1)\nu_2 N(d_4') + (\tau')^{-\frac{1}{2}}n(d_4')\right] - e^{-r\tau'}\gamma\nu_1 K N(d_3')\right)d\tau', \tag{4.1}$$

where $\Delta^{do}_B(S,t)$ is a standard down-and-out call delta (d_3 and d_4 are defined by Eq.(3.42a))

$$\Delta^{do}_B(S,t) = N(d_2) + \frac{1}{S}\left(\frac{B}{S}\right)^{\gamma}\left[(\gamma+1)\left(\frac{B^2}{S}\right)N(d_4) - \gamma K e^{-r\tau}N(d_3)\right], \tag{4.2}$$

$$d_3 = \frac{\ln\left(\frac{B^2}{SK}\right) + \mu\tau}{\sigma\sqrt{\tau}},\quad d_4 = d_3 + \sigma\sqrt{\tau}.$$

Region II: Below the barrier, $S < B$

$$\Delta^{do,II}_{B,r_B}(S,t) = \frac{\partial C^{do,II}_{B,r_B}}{\partial S} = \frac{1}{S}\left(\frac{B}{S}\right)^{\frac{\gamma}{2}} \int_0^\tau \frac{\left(1-e^{-r_B(\tau-\tau')}\right)e^{-\kappa(\tau-\tau')}}{\sqrt{2\pi}\, r_B\,(\tau-\tau')^{3/2}}$$

$$\times \left[\nu_2\lambda_2(y)BN(d_6') + \rho_1(y)B(\tau')^{-\frac{1}{2}}n(d_6') - \nu_1\lambda_1(y)e^{-r\tau'}KN(d_5')\right]$$

$$\times \exp\left(-\frac{y^2}{2(\tau-\tau')}\right)d\tau', \tag{4.3}$$

where

$$\lambda_1(y) = \phi\rho_1(y) - \frac{1}{\sigma}\left(\frac{2y}{\tau-\tau'} + \nu_1\right),\ \lambda_2(y) = \phi\rho_2(y) - \frac{1}{\sigma}\left(\frac{2y}{\tau-\tau'} + \nu_2\right),$$

$$\phi = \frac{\gamma}{2} + \frac{y}{\sigma(\tau-\tau')}. \tag{4.4}$$

The delta of a seasoned option at some time t' during the life of the contract, $t < t' < T$, is obtained by discounting similar to Eq.(3.33)

$$e^{-r_B\tau_B^-(t,t')}\Delta^{do}_{B,r_B}(S_{t'},t'), \tag{4.5}$$

where $\Delta^{do}_{B,r_B}(S_{t'},t')$ is the delta of a newly written option at time t'.

The step option's delta is continuous at the barrier for any finite r_B due to the continuity boundary condition for the first derivative of the density (3.11b):

$$\lim_{\epsilon\to 0_+}\left(\Delta^{do,I}_{B,r_B}(B+\epsilon,t) - \Delta^{do,II}_{B,r_B}(B-\epsilon,t)\right) = 0, \tag{4.6}$$

allowing to dynamically replicate step options by continuously trading the underlying asset and borrowing. Although continuous everywhere, the delta does have a kink at the barrier $S = B$. Hence, the step option's gamma undergoes a finite jump (step) at the barrier proportional to the knock-out rate (see Eq.(3.5)):

$$\lim_{\epsilon\to 0_+}\left(\Gamma^{do,I}_{B,r_B}(B+\epsilon,t) - \Gamma^{do,II}_{B,r_B}(B-\epsilon,t)\right) = \frac{2r_B}{\sigma^2}\left(\frac{C^{do}_{B,r_B}(B,t)}{B^2}\right), \tag{4.7}$$

but its value is finite for $S = B$ in contrast to the standard barrier case. In the barrier limit $r_B \to \infty$ this jump is infinite, while in the limit of zero knock-out rate the right-hand side of Eq.(4.7) vanishes and the gamma is continuous.

Now consider a standard down-and-out call $C^{do}_B(S,t)$. According to the asymptotic property (3.31) it can be approximated by a down-and-out step call with the large but finite knock-out rate r_B. Consequently, a barrier option can be *approximately hedged* by executing a dynamic replicating strategy for an otherwise identical step option. The higher

the knock-out rate r_B, the closer the strategy approximates the barrier option. Theoretically, any degree of accuracy can be attained by choosing high enough r_B. Thus, barrier options can be *almost* replicated by this dynamic trading strategy[9]. Another method of hedging barrier options is static replication developed recently by Bowie and Carr (1994) and Derman, Ergener and Kani (1995).

5 General Occupation Time Derivatives

5.1 Risk-Neutral Pricing

Our discussion in this paper can be generalized as follows. Consider a claim contingent on the terminal asset price S_T as well as occupation time τ_B^- with the payoff at expiration

$$C_F(S_T, T) = F(S_T, \tau_B^-) \tag{5.1}$$

for a given function F.

The price at the inception of the contract t is given by the discounted Feynman-Kac formula (see Appendix A and Linetsky (1996)):

$$C_F(S,t) = e^{-r\tau} E_{(t,S)}[F(S_T, \tau_B^-)]$$

$$= e^{-r\tau} \int_{-\infty}^{\infty} e^{\frac{\mu}{\sigma^2}(x_T - x) - \frac{\mu^2 \tau}{2\sigma^2}} \left(\int_{x(t)=x}^{x(T)=x_T} F(e^{x_T}, \tau_B^-) e^{-\int_t^T \mathcal{L}_0 dt'} \mathcal{D}x_{t'} \right) dx_T, \tag{5.2}$$

where $\mathcal{L}_0$ is the Lagrangian for the martingale $dx = \sigma dz$, $\mathcal{L}_0 = \frac{1}{2\sigma^2}\dot{x}^2$ (all dependence on the drift μ is factorized into the exponential $\exp\left(\frac{\mu}{\sigma^2}(x_T - x) - \frac{\mu^2 \tau}{2\sigma^2}\right)$). To calculate the path integral

$$\int_{x(t)=x}^{x(T)=x_T} F(e^{x_T}, \tau_B^-) e^{-\int_t^T \mathcal{L}_0 dt'} \mathcal{D}x_{t'}, \tag{5.3}$$

we can transform the payoff function as follows. First, introduce an auxiliary variable τ', $0 \leq \tau' \leq \tau$:

$$F(e^{x_T}, \tau_B^-) = \int_0^\tau \delta(\tau' - \tau_B^-) F(e^{x_T}, \tau') d\tau'. \tag{5.4}$$

Next, the delta function can be represented as an inverse Laplace transform

$$\delta(\tau' - \tau_B^-) = \mathcal{L}_{\tau'}^{-1}\left[e^{-s\tau_B^-}\right], \tag{5.5}$$

so that

$$F(e^{x_T}, \tau_B^-) = \int_0^\tau \mathcal{L}_{\tau'}^{-1}\left[e^{-s\tau_B^-}\right] F(e^{x_T}, \tau') d\tau'. \tag{5.6}$$

[9]The notion of *almost replicability* was introduced by Chriss and Ong (1995) in the context of digital options.

Substituting this into the path integral (5.3) we arrive at

$$\int_{x(t)=x}^{x(T)=x_T} \left(\int_0^\tau \mathcal{L}_{\tau'}^{-1} \left[e^{-s\tau_B^-} \right] F(e^{x_T}, \tau') d\tau' \right) e^{-\int_t^T \mathcal{L}_0 dt'} \mathcal{D}x_{t'}$$

$$= \int_0^\tau F(e^{x_T}, \tau') \mathcal{P}_B^-(x_T, \tau', \tau | x) d\tau', \tag{5.7}$$

where $\mathcal{P}_B^-(x_T, \tau', \tau|x)$ is the joint probability density for the terminal state x_T and occupation time τ_B^- (variable τ') at expiration T conditional on the initial state x at inception t:

$$\mathcal{P}_B^-(x_T, \tau', \tau|x) = \mathcal{L}_{\tau'}^{-1} \left[\int_{x(t)=x}^{x(T)=x_T} \exp\left(-\int_t^T \left[\mathcal{L}_0 + s\theta(b - x_{t'}) \right] dt' \right) \mathcal{D}x_{t'} \right]$$

$$= \mathcal{L}_{\tau'}^{-1} \left[\mathcal{K}_{B,s}^-(x_T, x; \tau) \right]. \tag{5.8}$$

It is just an inverse Laplace transform of our Green's function for down-and-out step options with respect to the knock-out rate (note that the Laplace variable s in Eq.(5.8) plays a role of the knock-out rate r_B in Eq.(3.9)).

Finally, substituting this back into the pricing formula (5.2) we arrive at

$$C_F(S,t) = e^{-r\tau} \int_{-\infty}^{\infty} \int_0^\tau F(e^{x_T}, \tau') e^{\frac{\mu}{\sigma^2}(x_T - x) - \frac{\mu^2 \tau}{2\sigma^2}} \mathcal{P}_B^-(x_T, \tau', \tau|x) \, d\tau' dx_T. \tag{5.9}$$

5.2 The Joint Probability Density of x_T and τ_B^-

The Laplace transform in (5.8) can be easily inverted by using (B.6), and the results for the density in all four regions I, II, III and IV are:

Region I: $x > b$, $x_T > b$

$$\mathcal{P}_B^{-I}(x_T, \tau', \tau|x) = \mathcal{L}_{\tau'}^{-1} \left[\mathcal{K}_{B,s}^{-I}(x_T, x; \tau) \right]$$

$$= \delta(\tau') \mathcal{K}_B^-(x_T, x; \tau) + \int_0^{\tau - \tau'} \frac{|y_T + y|}{2\pi\sigma(\tau - \tau'')^{\frac{3}{2}}(\tau'')^{\frac{3}{2}}} \exp\left(-\frac{(y_T + y)^2}{2\tau''} \right) d\tau''; \tag{5.10a}$$

Region II: $x < b$, $x_T > b$

$$\mathcal{P}_B^{-II}(x_T, \tau', \tau|x) = \mathcal{L}_{\tau'}^{-1} \left[\mathcal{K}_{B,s}^{-II}(x_T, x; \tau) \right] \tag{5.10b}$$

$$= \int_0^{\tau - \tau'} \frac{[y_T \tau''(\tau - \tau'' - y^2) + y(\tau - \tau'')(\tau'' - y_T^2)]}{2\pi\sigma(\tau - \tau'')^{\frac{5}{2}}(\tau'')^{\frac{5}{2}}} \exp\left(-\frac{y_T^2}{2\tau''} - \frac{y^2}{2(\tau - \tau'')} \right) d\tau'';$$

Region III: $x > b$, $x_T < b$

$$\mathcal{P}_B^{-III}(x_T, \tau', \tau|x) = \mathcal{L}_{\tau'}^{-1} \left[\mathcal{K}_{B,s}^{-III}(x_T, x; \tau) \right] = \mathcal{P}_B^{-II}(x, \tau', \tau|x_T); \tag{5.10c}$$

Region IV: $x < b$, $x_T < b$

$$\mathcal{P}_B^{-IV}(x_T, \tau', \tau | x) = \mathcal{L}_{\tau'}^{-1}\left[\mathcal{K}_{B,s}^{-IV}(x_T, x; \tau)\right]$$

$$= \delta(\tau' - \tau)\,\mathcal{K}_B^-(x_T, x; \tau) + \int_0^{\tau'} \frac{|y_T + y|}{2\pi\sigma(\tau - \tau'')^{\frac{3}{2}}(\tau'')^{\frac{3}{2}}} \exp\left(-\frac{(y_T + y)^2}{2\tau''}\right) d\tau''. \qquad (5.10d)$$

Formulas (5.9) and (5.10) allow us to price any occupation time derivative with payoffs of the form $F(S_T, \tau_B^-)$.

5.3 General Step Options

Consider a call with the payoff

$$C_f(S_T, T) = f(\tau_B^-)\, Max(S_T - K, 0), \qquad (5.11)$$

where f is a given function of the occupation time τ_B^-. Our down-and-out step call (2.2) is a particularly simple example with $f = exp(-r_B \tau_B^-)$. It is also the most fundamental one in a sense that others can be expressed through it as follows:

$$C_f(S, t) = e^{-r\tau} \int_{\ln K}^{\infty} \int_0^{\tau} f(\tau')(e^{x_T} - K) e^{\frac{\mu}{\sigma^2}(x_T - x) - \frac{\mu^2 \tau}{2\sigma^2}} \mathcal{L}_{\tau'}^{-1}\left[\mathcal{K}_{B,s}^-(x_T, x; \tau)\right] d\tau' dx_T$$

$$= \int_0^{\tau} f(\tau') \mathcal{L}_{\tau'}^{-1}\left[C_{B,s}^{do}(S, t)\right] d\tau'. \qquad (5.12)$$

Thus, any other option of the form (5.11) is expressed in terms of our basic *exponential step option*, and we will call other payoffs of the form (5.11) *step options* as well, specifying the function f.

Using our results (3.22) and (3.27) for exponential step options we derive general pricing formulas by inverting the Laplace transform in Eq.(5.12):

Region I: Above the barrier, $S > B$

$$C_f^I(S, t) = f(0)\, C_B^{do}(S, t) \qquad (5.13)$$

$$+ \left(\frac{B}{S}\right)^{\gamma} \int_0^{\tau} \frac{F(\tau - \tau') e^{-\kappa(\tau - \tau')}}{\sqrt{2\pi}\,(\tau - \tau')^{3/2}} \left(\nu_2 \left(\frac{B^2}{S}\right) N(d_4') - \nu_1 e^{-r\tau'} K N(d_3')\right) d\tau',$$

Region II: Below the barrier, $S < B$

$$C_f^{II}(S, t) = \left(\frac{B}{S}\right)^{\frac{\gamma}{2}} \int_0^{\tau} \frac{F(\tau - \tau')\, e^{-\kappa(\tau - \tau')}}{\sqrt{2\pi}\,(\tau - \tau')^{3/2}} \qquad (5.14)$$

$$\times \left\{\nu_1 \rho_1(y) e^{-r\tau'} K N(d_5') - \nu_2 \rho_2(y) B N(d_6') - \sigma y B(\tau')^{-\frac{1}{2}} n(d_6')\right\} \exp\left(-\frac{y^2}{2(\tau - \tau')}\right) d\tau',$$

where the function $F(\tau - \tau')$ is obtained from f in (5.11) as follows:

$$F(\tau - \tau') = \int_0^{\tau} f(\tau'') \mathcal{L}_{\tau''}^{-1} \left[\frac{1 - e^{-s(\tau - \tau')}}{s} \right] d\tau'' = \int_0^{\tau - \tau'} f(\tau'') d\tau''. \tag{5.15}$$

For the exponential step call with payoff (2.2) we have:

$$F(\tau - \tau') = \int_0^{\tau - \tau'} e^{-r_B \tau''} d\tau'' = \frac{1 - e^{-r_B(\tau - \tau')}}{r_B}. \tag{5.16}$$

5.4 Delayed Barrier Options

As a simple application of the formulas (5.13-15), consider the following payoff:

$$\theta(\lambda\tau - \tau_B^-) \, Max(S_T - K, 0), \tag{5.17}$$

where λ is a given constant, $0 \leq \lambda \leq 1$. Here $f(\tau') = \theta(\lambda\tau - \tau')$ and

$$F(\tau - \tau') = \begin{cases} \lambda\tau & \text{if } \tau' \leq (1 - \lambda)\tau \\ \tau - \tau' & \text{if } \tau' > (1 - \lambda)\tau \end{cases} \tag{5.18}$$

This option is a down-and-out call which knocks out when the occupation time below the barrier exceeds $\lambda\tau$. Reportedly, this type of *delayed barrier options* have been trading over the counter for some time. It should be mentioned that introducing the delay before the option knocks out does not solve the main problem with barrier options: discontinuity of the payoff and delta. The payoff (5.17) is discontinuous as a function of occupation time: when the occupation time to date t gets close to $\lambda\tau$, the delayed barrier option behaves like a standard barrier option and its delta is discontinuous at $\tau_B^- = \lambda\tau$. In contrast, our step options knock out *gradually and continuously* and the delta is continuous everywhere as a function of the current price, as well as the occupation time to date t (for seasoned options).

5.5 Derivatives Contingent on the Occupation Time Between Two Barriers and on Several Occupation Times

Our discussion can be straightforwardly extended to derivatives contingent on the occupation time $\tau_{[B_1, B_2]}$ inside the range between two barriers B_1 and B_2 by noting that

$$\tau_{[B_1, B_2]} = \tau_{B_2}^- - \tau_{B_1}^-. \tag{5.19}$$

A popular family of path-dependent *range products* (such as range notes and corridor options; see, e.g., Pechtl (1995) and Turnbull (1995)) provide examples of this type of derivatives. In the simplest case, the holder of a range note is paid a fixed dollar amount, D, for every time unit (say, a trading day) the underlying stays within the pre-specified range $B_1 < S < B_2$. The payoff of this instrument is proportional to the occupation time

$D\,\tau_{[B_1,B_2]}$.

A further generalization is provided by derivatives with payoffs contingent on several occupation times: $F(S_T, \tau^-_{B_1}, \tau^-_{B_2}, \ldots, \tau^-_{B_n})$. A chief example is a multi-step option with n successive barriers with the payoff:

$$\exp\left(-\sum_{i=1}^{n} r_{B_i}\tau^-_{B_i}\right) Max(S_T - K, 0),\ B_1 < B_2 < \ldots < B_n. \tag{5.20}$$

The multi-step model can be applied to modeling *credit risk* (credit downgrades) and *mortgage-backed securities* (threshold prepayment levels).

6 Linear Step Options

6.1 Contract Design

The specific choice of the exponential knock-out factor $f = \exp(-r_B\tau^-_B)$ in Eq.(5.11) corresponds to *continuous compounding* at knock-out rate $(-r_B)$ below the barrier. Another practically meaningful choice is *simple interest without compounding*:

$$f(\tau^-_B) = Max(1 - R_B\,\tau^-_B, 0). \tag{6.1}$$

This is a *simple discount factor* $(1 - R_B\,\tau^-_B)$ with knock-out rate R_B and linear in occupation time.[10] A down-and-out *linear* step call is then defined similar to the *exponential* step call (2.2) (l in *ldo* stands for linear):

$$C^{ldo}_{B,R_B}(S_T, T) = Max(1 - R_B\,\tau^-_B, 0)\,Max(S_T - K, 0). \tag{6.2}$$

The optionality in occupation time (the Max function in Eq.(6.1)) is needed to limit the option buyer's liability to not more than the premium paid for the option. Thus, linear step options are options on the underlying price, as well as occupation time.

A *knock-out time* $\mathcal{T}^-_B$ for linear step options is defined as the minimum occupation time below the barrier needed to reduce the option payoff to zero:

$$1 - R_B\,\mathcal{T}^-_B = 0,\ \mathcal{T}^-_B = \frac{1}{R_B}. \tag{6.3}$$

Another useful quantity is a *knock-out ratio* of the knock-out time to the total lifetime of the option:

$$\lambda_B = \frac{\mathcal{T}^-_B}{\tau},\ R_B = \frac{1}{\lambda_B\tau}. \tag{6.4}$$

[10] I am grateful to Vladimir Finkelstein and Eric Reiner for suggesting to consider this payoff structure.

The λ_B can be selected anywhere in the interval between zero (standard down-and-out call – the option knocks out instantly as soon as the barrier is hit) and infinity (an infinite amount of time below the barrier is needed to extinguish the option):

$$\lim_{\lambda_B \to 0} Max\left(1 - \frac{\tau_B^-}{\lambda_B \tau}, 0\right) = \begin{cases} 1 & \text{if } \tau_B^- = 0 \\ 0 & \text{if } \tau_B^- > 0 \end{cases} \tag{6.5a}$$

$$\lim_{\lambda_B \to \infty} Max\left(1 - \frac{\tau_B^-}{\lambda_B \tau}, 0\right) = 1. \tag{6.5b}$$

One should note that, given the same fraction of option payoff lost in the first trading day below the barrier, the linear step option will knock out faster thereafter than an otherwise identical exponential one. Indeed, the linear and exponential knock-out factors are: $(1 - n_B^- \rho_B)$ and $(\beta_B)^{n_B^-}$, where β_B is the single-day knock-out factor for the exponential option, $\rho_B = R_B/250$ is the knock-out rate per trading day for the linear option, and n_B^- is the total number of trading days below the barrier during the life of the option. To compare the two structures, let us choose β_B and ρ_B so that $\beta_B = 1 - \rho_B$, i.e. the fractions of the terminal payoff lost in the first trading day below the barrier are equal in both cases. Then, after n_B^- trading days below the barrier we have: $(1 - n_B^- \rho_B) < (\beta_B)^{n_B^-}$. Figure 6 illustrates this situation. If we select $\rho_B = 0.2$, the linear option will lose its entire premium in five days, while the exponential one will lose its premium slower and slower due to the effect of compounding. This may create undesirable accounting problems. Suppose a six month exponential option is structured with the effective knock-out time of five days. This means that after the first five trading days below the barrier the option loses ninety percent of its premium. However, there may still be several months to expiration, and both the buyer and the seller will have to carry on their books this low-priced tail left over from the exponential knock-out. Often, the parties will elect to close out the position before expiration to avoid the accounting hustle. In contrast, the linear knock-out option will lose all of its premium over the five days below the barrier, leaving no low priced tails to worry about.

The other seven types of linear step options are defined similarly by their payoffs:

$$C^{luo}_{B,R_B}(S_T, T) = Max(1 - R_B\, \tau_B^+, 0)\, Max(S_T - K, 0), \tag{6.6a}$$

$$P^{ldo}_{B,R_B}(S_T, T) = Max(1 - R_B\, \tau_B^-, 0)\, Max(K - S_T, 0), \tag{6.6b}$$

$$P^{luo}_{B,R_B}(S_T, T) = Max(1 - R_B\, \tau_B^+, 0)\, Max(K - S_T, 0), \tag{6.6c}$$

$$\begin{aligned} C^{ldi}_{B,R_B}(S_T, T) &= \left(1 - Max(1 - R_B\, \tau_B^-, 0)\right)\, Max(S_T - K, 0) \\ &= Min(R_B\, \tau_B^-, 1)\, Max(S_T - K, 0), \end{aligned} \tag{6.6d}$$

$$\begin{aligned} C^{lui}_{B,R_B}(S_T, T) &= \left(1 - Max(1 - R_B\, \tau_B^+, 0)\right)\, Max(S_T - K, 0) \\ &= Min(R_B\, \tau_B^+, 1)\, Max(S_T - K, 0), \end{aligned} \tag{6.6e}$$

$$P^{ldi}_{B,R_B}(S_T,T) = Min(R_B\,\tau^-_B,1)\,Max(K-S_T,0), \tag{6.6f}$$

$$P^{lui}_{B,R_B}(S_T,T) = Min(R_B\,\tau^+_B,1)\,Max(K-S_T,0). \tag{6.6g}$$

And the four types of forwards are:

$$f^{luo}_{B,R_B}(S_T,T) = Max(1-R_B\,\tau^+_B,0)\,(S_T-K), \tag{6.7a}$$

$$f^{ldo}_{B,R_B}(S_T,T) = Max(1-R_B\,\tau^-_B,0)\,(S_T-K), \tag{6.7b}$$

$$f^{lui}_{B,R_B}(S_T,T) = Min(R_B\,\tau^+_B,1)\,(S_T-K), \tag{6.7c}$$

$$f^{ldi}_{B,R_B}(S_T,T) = Min(R_B\,\tau^-_B,1)\,(S_T-K). \tag{6.7d}$$

6.2 Pricing and Hedging

Closed-form pricing formulas for the linear step call are given by Eqs.(5.13-14), where $f(\tau') = Max(1-R_B\,\tau',0)$, $f(0)=1$, and the function $F(\tau-\tau')$ is found by integration:

$$F(\tau-\tau') = \int_0^{\tau-\tau'} Max(1-R_B\,\tau'',0)\,d\tau'' \tag{6.8}$$

$$= \begin{cases} \frac{1}{2R_B}, & 0 \le \tau' \le \tau - \frac{1}{R_B} \\ (\tau-\tau')\left[1-\frac{R_B}{2}(\tau-\tau')\right], & \tau-\frac{1}{R_B} < \tau' \le \tau \end{cases}$$

Pricing formulas (5.13-14), (6.8) are as easy to implement numerically as Eqs.(3.22), (3.27) for the exponential case.

The delta of a linear step call is given by the same analytical formulas (4.1-3) as for the exponential case, where one only needs to substitute:

$$\frac{1-e^{-r_B(\tau-\tau')}}{r_B} \to F(\tau-\tau'), \tag{6.9}$$

where the function $F(\tau-\tau')$ is given by Eq.(6.8). Just as in the exponential case, the delta is continuous everywhere.

Seasoned linear step options at some time t' during the life of the contract are priced by dividing the occupation time from inception t to maturity T into two parts: $\tau^-_B(t,T) = \tau^-_B(t,t') + \tau^-_B(t',T)$, where $\tau^-_B(t,t')$ is the occupation time already known to date t', and $\tau^-_B(t',T)$ is yet unknown. Then the payoff of a seasoned option can be re-written as follows:

$$Max(1-R_B\,\tau^-_B(t,t')-R_B\,\tau^-_B(t',T),0) = \phi_B\,Max(1-R'_B\,\tau^-_B,0), \tag{6.10}$$

where

$$\phi_B = 1-R_B\,\tau^-_B(t,t'),\quad R'_B = \frac{R_B}{1-R_B\,\tau^-_B(t,t')}. \tag{6.11}$$

Thus a *seasoned* linear step option can be priced in the same way as the quantity ϕ_B of *newly written* options with the adjusted knock-out rate R'_B.

7 Discrete-Time Approximations for Occupation Time Derivatives

7.1 Monte Carlo Simulation

One can also price occupation time derivatives in discrete time either by Monte Carlo simulation or the lattice or finite-difference methods. In the Monte Carlo simulation approach (Boyle 1976), one generates a large number M of price paths with N discrete time steps h, $h = (T-t)/N$, from the contract inception t to expiration T. Price paths $\{S_i^k, i = 0,1,\ldots,N\}$, $k = 1,2,\ldots,M$ (upper index counts paths, lower index – time steps), follow a discretized risk-neutral geometric Brownian motion (disregarding the terms of higher orders in h)

$$S_{i+1} = (1 + r\,h + \epsilon_{i+1}\sigma\sqrt{h})S_i\,,\; i = 0,1,\ldots,N. \tag{7.1}$$

For the k-th price path $\{S^k\}$, let n_B^{-k} be equal to the total number of time points between t and T that the asset price S_i^k is less than or equal to the barrier B (there are a total of $N+1$ time points t_i, $t_i = t + ih$, and $0 \le n_B^{-k} \le N+1$), i.e.

$$n_B^{-k} = \sum_{i=0}^{N} \theta(B - S_i^k), \tag{7.2}$$

where θ is the Heaviside step function (2.4). It is the occupation time measured in time units of h, $\tau_B^- = n_B^- h$. Then the *knock-out factor* for the kth path is $(\beta(h))^{n_B^{-k}}$, where $\beta(h)$, $\beta(h) = e^{-r_B h}$, is the *knock-out factor for a single time period h*. Now the down-and-out exponential step call price is given by the discounted average of payoffs over all price paths:

$$C^{do}_{B,r_B}(S,t) = \frac{e^{-r\tau}}{M}\sum_{k=1}^{M}(\beta(h))^{n_B^{-k}} Max(S_T^k - K, 0), \tag{7.3}$$

where S_T^k is the terminal asset price of the kth path. This is a discretized version of the continuous-time average (3.1). The knock-out discount factors enter as *path weights* in the average. For the standard vanilla call $\beta(h) = 1$ and all weights are equal to one. For the standard down-and-out call $\beta(h) = 0$ and all price paths that hit the barrier are killed and enter with zero weights. For a step call with $0 < r_B < \infty$, each path is weighted according to discrete occupation time, the total number of discrete time points t_i the underlying price is less than or equal to the barrier price. Up-and-out calls are priced similarly. One only needs to reverse the sign of the argument of the Heaviside functions in (7.2). More general occupation time derivatives are valued by substituting the payoff $F(S_T^k, n_B^{-k}\,h)$ in Eq.(7.3).

7.2 Binomial Model

In the binomial framework, occupation time derivatives can be valued by employing the Hull and White (1993) extension of the standard Cox-Ross-Rubinstein (1979) binomial

model (see also Cox and Rubinstein (1985)).

To fix our notation, first consider the standard CRR binomial tree. The entire time interval to expiration is divided into N equal time periods bounded by $N+1$ equally spaced discrete time points $t_i = t + ih$, $i = 0, 1, \ldots, N$, $h = \tau/N$. At any particular time point t_i, the underlying price S_i can go either up to $S_{i+1} = uS_i$ with the probability p or down to $S_{i+1} = dS_i$ with the probability $(1-p)$, such that

$$u = e^{\sigma\sqrt{h}},\ d = e^{-\sigma\sqrt{h}},\ p = \frac{e^{rh} - d}{u - d}. \tag{7.4}$$

Nodes of the binomial tree are labeled by the pairs (i,j), $i = 0, 1, \ldots, N, j = 0, 1, \ldots, i$, and the underlying price at the node (i,j) is $S_{i,j} = u^j d^{i-j} S = u^{2j-i} S$, where S is the initial asset price at the node $(0,0)$. Then the backward induction can be employed to calculate the price of the call as a discounted expected payoff at expiration. If one knows the option prices $C_{i+1,j+1}$ and $C_{i+1,j}$ at the nodes $(i+1, j+1)$ and $(i+1, j)$, respectively, one can derive the price at the preceeding node $C_{i,j}$ by discounting the average:

$$C_{i,j} = e^{-rh}\left(pC_{i+1,j+1} + (1-p)C_{i+1,j}\right). \tag{7.5}$$

Starting from the known payoffs $C_{N,j} = Max(S_{N,j} - K, 0)$ corresponding to the terminal underlying prices $S_{N,j} = u^{2j-i}S$ and going backwards through the tree, one arrives at the "closed-form" binomial pricing formula

$$C = e^{-r\tau} \sum_{j=0}^{N} \left(\begin{array}{c} N \\ j \end{array} \right) p^j (1-p)^{N-j} C_{N,j}. \tag{7.6}$$

Hull and White (1993) extended the CRR binomial model to value path-dependent options. The discretized occupation time n_B^- plays a role of the second *path-dependent* state variable in the Hull and White method. More precisely, the second state variable in addition to the underlying price S_i is the current occupation time $n(i)$[11] equal to the total number of observed price points below the barrier on the path segment from inception t to date t_i:

$$n(i) = \sum_{j=0}^{i} \theta(B - S_j),\ \tau_B^-(t, t_i) = n(i)h. \tag{7.7}$$

It satisfies both requirements for the Hull and White method to be computationally feasible. It is possible to compute $n(i+1)$ by knowing $n(i)$ and S_{i+1},

$$n(i+1) = n(i) + \theta\left(B - S_{i+1}\right), \tag{7.8}$$

and the number of possible alternative values at time step t_i in the binomial tree with N time steps is never greater than $i+1$ ($n(i) = i+1$ corresponds to the extreme case of all discrete prices being less than or equal to the barrier price, $S_j \leq B$, $j = 0, 1, 2, \ldots, i$).

[11] To lighten the notation we drop the indexes B and - in the notation $n_B^-(i)$.

Consider a node (i,j). The occupation time $n(i,j)$ accumulated up to this node can take a number of alternative values depending on the price history from $(0,0)$ to (i,j):

$$n(i,j) = n(i,j)_{min}, n(i,j)_{min}+1, \ldots, n(i,j)_{max}. \tag{7.9}$$

Here $n(i,j)_{min}$ and $n(i,j)_{max}$ are minimum and maximum possible values of occupation time at the node (i,j). They are determined by the location of the node with respect to the barrier as follows (in this Section we assume that the initial price S at the node (0,0) is above the barrier, $S > B$):

Node (i,j) above the barrier, $i - 2j < m_B$

$$n(i,j)_{min} = 0,\ n(i,j)_{max} = \begin{cases} 0 & \text{if } i-j \le m_B \\ 2(i-j-m_B)+1 & \text{if } i-j > m_B \end{cases} \tag{7.10a}$$

Node (i,j) below or on the barrier, $i - 2j \ge m_B$

$$n(i,j)_{min} = i - 2j - m_B + 1,\ n(i,j)_{max} = i - m_B + 1. \tag{7.10b}$$

Here m_B denotes a minimum number of down jumps in a row needed to get the underlying price below the barrier starting from (0,0), so that $d^{m_B}S \le B < d^{m_B-1}S$, where $d = \exp(-\sigma\sqrt{h})$. Thus m_B is the smallest integer larger than $\frac{\ln(S/B)}{\sigma\sqrt{h}}$. To obtain (7.10) one needs to consider two paths from (0,0) to the node (i,j) with the minimum and maximum possible number of discrete prices below or on the barrier.

Now we can develop the path-dependent backward induction algorithm to find the value of any occupation time derivative with the payoff (6.1). First, all possible alternative payoffs at the node (N,j) depending on the price history of the underlying are:

$$C_{N,j;n} = F(u^{2j-N}S, n(N,j)), \tag{7.11}$$

where $n(N,j) = n(N,j)_{min}, \ldots, n(N,j)_{max}$, and $n(N,j)_{min}$ and $n(N,j)_{max}$ are given by (7.10). Next, suppose we know all the values $\{C_{i,j;n},\ j = 0,1,\ldots,i,\ n = n(i,j)_{min}, \ldots, n(i,j)_{max}\}$ at time t_i. Then we can find all values $\{C_{i-1,j;n},\ j = 0,1,\ldots,i-1, n = n(i-1,j)_{min}, \ldots, n(i-1,j)_{max}\}$ at preceeding time t_{i-1}:

$$C_{i-1,j;n} = \begin{cases} e^{-rh}\left(pC_{i,j+1;n} + (1-p)C_{i,j;n}\right) & \text{if } i-2j < m_B \\ e^{-rh}\left(pC_{i,j+1;n} + (1-p)C_{i,j;n+1}\right) & \text{if } i-2j = m_B, m_B+1 \\ e^{-rh}\left(pC_{i,j+1;n+1} + (1-p)C_{i,j;n+1}\right) & \text{if } i-2j > m_B+1 \end{cases}. \tag{7.12}$$

Implementing this backward induction through the tree yields the value of the occupation time derivative $C_F(S,t) \equiv C_{0,0;0}$ at the contract inception.

For our exponential step options with the payoff $\exp(-r_B\tau_B^-)$ the binomial algorithm has a closed-form solution similar to the CRR formula (7.6). Indeed, in this special case we actually *do not need the second state variable.* The step option price satisfies the step PDE (3.5). The only complication is the step potential term $r_B\theta(B - S_{t'})$ killing the Brownian motion at rate r_B below the barrier. Thus, we can price step options by using the standard CRR tree and discounting the nodes below the barrier at rate r_B:

$$C_{i-1,j} = \begin{cases} e^{-rh}\left(pC_{i,j+1} + (1-p)C_{i,j}\right) & \text{if } i - 2j < m_B \\ e^{-(r+r_B)h}\left(pC_{i,j+1} + (1-p)C_{i,j}\right) & \text{if } i - 2j \geq m_B \end{cases}. \tag{7.13}$$

This algorithm can be solved in closed form similar to Eq.(7.6)

$$C^{do}_{B,r_B} = e^{-r\tau}\sum_{j=0}^{N} P_j^N(\beta(h))\, p^j(1-p)^{N-j}C_{N,j}, \tag{7.14}$$

where $P_j^N(\beta(h))$ are polynomials in $\beta(h)$

$$P_j^N(\beta(h)) = \sum_{n=n_{min}}^{n=n_{max}} P_{j,n}^N\, \beta(h)^n, \tag{7.15}$$

and $P_{j,n}^N$ are combinatorial coefficients equal to the total number of distinct paths from (0,0) to (N, j) through the tree with the total number of n discrete prices below the barrier B. They satisfy the following forward recursion

$$P_{j,n}^{N+1} = \begin{cases} P_{j,n}^N + P_{j-1,n}^N & \text{if } N - 2j < m_B - 1 \\ P_{j,n-1}^N + P_{j-1,n-1}^N & \text{if } N - 2j \geq m_B - 1 \end{cases}, \tag{7.16}$$

and their sum over n reduces to the binomial coefficient

$$\sum_{n=n_{min}}^{n=n_{max}} P_{j,n}^N = \begin{pmatrix} N \\ j \end{pmatrix}. \tag{7.17}$$

To illustrate the general algorithm and further develop our intuition with step options, let us consider the simplest non-trivial example: a three-period binomial tree for at-the-money options, $S = K$, shown in Figure 3. At each node (i, j) all possible values of $n(i, j)$ are given.[12] First, all the possible payoffs at expiration are:

$$\begin{gathered} C_{3,3;0} = S(u^3 - 1); \\ C_{3,2;0} = S(u-1),\ C_{3,2;1} = \beta(h)S(u-1); \\ C_{3,1;n} = 0,\ n = 1,2,3; \\ C_{3,0;3} = 0. \end{gathered} \tag{7.18}$$

[12] Although we can price step options by using the standard tree with one state variable, here we choose to illustrate the general Hull-White algorithm which can be used to price any occupation time derivative.

Next, going backwards through the tree option prices at each node are calculated according to (7.12):

$$C_{2,2;0} = e^{-rh}S\left(p(u^3-1)+(1-p)(u-1)\right),$$
$$C_{2,1;0} = e^{-rh}Sp(u-1), C_{2,1;1} = e^{-rh}\beta(h)Sp(u-1), \tag{7.19}$$
$$C_{2,0;2} = 0,$$
$$C_{1,1;0} = e^{-2rh}S\left(p^2(u^3-1)+2p(1-p)(u-1)\right),$$
$$C_{1,0;1} = e^{-2rh}\beta(h)Sp^2(S-1).$$

Finally, the price of the step call is (Eq.(7.14) for $N=3$):

$$C_{0,0;0} = e^{-r\tau}S\left(p^3(u^3-1)+p^2(1-p)(u-1)(2+\beta(h))\right), \beta(h) = e^{-r_B h}. \tag{7.20}$$

The answer depends on the single-period knock-out factor $\beta(h)$. This simple formula is a discrete version of our closed-form continuous-time solution (3.22) and it captures some important qualitative properties of step options. For $\beta(h)=1$, Eq.(7.20) reduces to the standard vanilla call in the three-period binomial model. For $\beta(h)=0$, it is the standard down-and-out call.

Note that, similar to the case of standard barrier options, this binomial procedure is computationally unstable with respect to the choice of the number of time periods N in the tree (see Boyle and Lau (1994) and Cheuk and Vorst (1996)). An optimal choice of N minimizing the computation error must be selected carefully. The problem is that in the binomial tree the barrier B will in general lie somewhere between the two horizontal layers of nodes. The calculation error depends on the position of the barrier with respect to the layers. To avoid these problems it is more appropriate to use the numerical PDE approach. In the case of step options, one just needs to solve numerically the step PDE (3.5). The finite-difference grid should be arranged so that one of the layers of nodes lies *exactly* on the barrier. The finite-difference PDE approach is also the method of choice for multi-step models.

7.3 Discrete Monitoring of the Barrier

In practice, many barrier options are structured with discrete monitoring of the barrier, e.g., by comparing daily closing prices to the barrier. Step option contracts can also be structured with discrete monitoring of occupation time. Indeed, let Δt be the monitoring interval and L — the total number of times the underlying price is compared with the barrier during the life of the option τ, $\tau = L\Delta t$. Then the terminal payoff of the exponential step call is given by

$$C^{do}_{B,r_B}(S_T,T) = (\beta(\Delta t))^{n_B} Max(S_T-K,0), \beta(\Delta t) = e^{-r_B\Delta t}. \tag{7.21}$$

Here $\beta(\Delta t)$ is the knock-out factor for a single monitoring period Δt and n_B^- is the number of times the underlying is below the barrier at the fixing time, $0 \leq n_B^- \leq L$ (discrete occupation time measured in the units of Δt). In the Monte Carlo and binomial schemes, one needs to keep track of how many times the underlying asset is below the barrier during the monitoring time points.

For barrier options, Broadie, Glaserman and Kou (1995) showed that, with good accuracy, one can still use the closed-form pricing formulas for continuous barrier options to price discretely monitored ones, if one shifts the barrier B away from the price S by multiplying the barrier by a certain factor. It is interesting if a similar approximation can also be used for occupation time derivatives.

8 Numerical Examples

Our closed-form pricing formulas are easy to implement numerically with essentially any degree of accuracy. All computations in the following examples were performed in Maple V on the Pentium PC and took only seconds per option. Generally, computation times depend on the numerical algorithm used to compute the convolution integrals in (3.22), (3.27) and the desired accuracy. Table 1 and Figures 4 and 5 show dependence of the down-and-out exponential step call value and delta on the single-day knock-out factor β_B and knock-out rate r_B. One sees that step options interpolate between vanilla call (for $\beta_B = 1$, $r_B = 0$) and barrier call (for $\beta_B = 0$, $r_B = \infty$) as β_B and r_B vary. The knock-out rate acts as a control parameter controlling the tradeoff between the premium savings and knock-out speed. It allows one to structure step options to best match the desired risk/reward profile and market outlook. Similarly, Table 2 and Figures 7 and 8 show dependence of the down-and-out linear step call value and delta on the knock-out rate ρ_B per trading day ($\rho_B = R_B/250$). The dotted horizontal lines on Figures 7 and 8 are asymptotic values in the standard barrier limit $R_B \to \infty$ ($DAO = 4.9958$ and $\Delta = 0.9932$).

Table 3 and Figures 9 and 10 show vanilla, exponential step, linear step and barrier call values and deltas as functions of the underlying asset price S. Figure 9 shows that the option value holds well when the underlying asset falls slightly below the barrier, but deteriorates quickly as the underlying continues to fall further, as the probability of getting back up above the barrier decreases and the expected value of occupation time below the barrier increases. Figure 10 illustrates continuity of the step option's delta at the barrier.

We have also implemented in C numerical approximations discussed in Section 7. The great advantage of closed-form solutions is computation speed. However, in practice one will need to resort to numerical approximations if one wishes to include the implied volatility surface, as well as other real world complications.

β_B	r_B	$\mathcal{T}_B^-$	C^{do}_{B,r_B}	Δ^{do}_{B,r_B}	β_B	r_B	$\mathcal{T}_B^-$	C^{do}_{B,r_B}	Δ^{do}_{B,r_B}
0.000	∞	0.00	4.9958	0.9932	0.525	161.09	3.57	7.6621	0.9361
0.025	992.22	0.62	6.1650	0.9690	0.550	149.46	3.85	7.7550	0.9340
0.050	748.93	0.77	6.2884	0.9663	0.575	138.35	4.16	7.8539	0.9317
0.075	647.57	0.89	6.3819	0.9643	0.600	127.7	4.51	7.9595	0.9293
0.100	575.65	1.00	6.4623	0.9626	0.625	117.50	4.90	8.0731	0.9266
0.125	519.86	1.12	6.5356	0.9610	0.650	107.70	5.35	8.1960	0.9236
0.150	474.28	1.21	6.6044	0.9596	0.675	98.26	5.86	8.3298	0.9206
0.175	435.74	1.32	6.6706	0.9581	0.700	89.17	6.46	8.4767	0.9171
0.200	402.36	1.43	6.7351	0.9567	0.725	80.34	7.16	8.6394	0.9133
0.225	372.91	1.54	6.7988	0.9553	0.750	71.92	8.00	8.8214	0.9089
0.250	346.57	1.66	6.8623	0.9539	0.775	63.72	9.03	9.0274	0.9039
0.275	322.75	1.78	6.9260	0.9525	0.800	55.76	10.32	9.2639	0.8980
0.300	300.99	1.91	6.9904	0.9511	0.825	48.09	11.97	9.5403	0.8911
0.325	280.98	2.05	7.0559	0.9497	0.850	40.63	14.17	9.8705	0.8827
0.350	262.46	2.19	7.1227	0.9482	0.875	33.38	17.24	10.2762	0.8722
0.375	245.21	2.35	7.1913	0.9467	0.900	26.34	21.85	10.7942	0.8583
0.400	229.07	2.51	7.2620	0.9451	0.925	19.49	29.53	11.4918	0.8389
0.425	213.92	2.69	7.3352	0.9435	0.950	12.82	44.89	12.5130	0.8085
0.450	199.63	2.88	7.4114	0.9417	0.975	6.33	90.95	14.2252	0.7519
0.475	186.11	3.09	7.4904	0.9400	1.000	0	∞	17.8551	0.6068
0.500	173.29	3.32	7.5743	0.9381					

Table 1: Down-and-out exponential step call value and delta as functions of the single-day knock-out factor β_B ($\beta_B = \exp(-r_B/250)$). Option parameters: $S = 100$, $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months).

ρ_B	R_B	$\mathcal{T}_B^-$	C_{B,r_B}^{ldo}	Δ_{B,r_B}^{ldo}	ρ_B	R_B	$\mathcal{T}_B^-$	C_{B,r_B}^{ldo}	Δ_{B,r_B}^{ldo}
0.000	0.00	∞	17.8551	0.6068	0.525	131.25	1.90	7.2671	0.9453
0.025	6.25	40.00	13.2122	0.7917	0.550	137.50	1.82	7.2180	0.9463
0.050	12.50	20.00	11.3723	0.8452	0.575	143.75	1.74	7.1720	0.9474
0.075	18.75	13.33	10.4117	0.8706	0.600	150.00	1.67	7.1288	0.9483
0.100	25.00	10.00	9.7953	0.8862	0.625	156.25	1.60	7.0881	0.9492
0.125	31.25	8.00	9.3557	0.8970	0.650	162.50	1.54	7.0497	0.9500
0.150	37.50	6.67	9.0212	0.9050	0.675	168.75	1.48	7.0133	0.9508
0.175	43.75	5.71	8.7554	0.9113	0.700	175.00	1.43	6.9789	0.9516
0.200	50.00	5.00	8.5373	0.9164	0.725	181.25	1.38	6.9461	0.9523
0.225	56.25	4.44	8.3542	0.9207	0.750	187.50	1.33	6.9150	0.9530
0.250	62.50	4.00	8.1974	0.9243	0.775	193.75	1.29	6.8854	0.9536
0.275	68.75	3.64	8.0612	0.9274	0.800	200.00	1.25	6.8571	0.9542
0.300	75.00	3.33	7.9414	0.9302	0.825	206.25	1.21	6.8300	0.9548
0.325	81.25	3.08	7.8350	0.9326	0.850	212.50	1.18	6.8042	0.9554
0.350	87.50	2.86	7.7395	0.9347	0.875	218.75	1.14	6.7794	0.9559
0.375	93.75	2.67	7.6532	0.9367	0.900	225.00	1.11	6.7556	0.9564
0.400	100.00	2.50	7.5748	0.9384	0.925	231.25	1.08	6.7327	0.9569
0.425	106.25	2.35	7.5030	0.9400	0.950	237.50	1.05	6.7107	0.9574
0.450	112.50	2.22	7.4371	0.9415	0.975	243.75	1.03	6.6896	0.9579
0.475	118.75	2.11	7.3762	0.9429	1.000	250.00	1.00	6.6692	0.9583
0.500	125.00	2.00	7.3197	0.9441	∞	∞	0.00	4.9958	0.9932

Table 2: Down-and-out linear step call value and delta as functions of the knock-out rate ρ_B per trading day ($\rho_B = R_B/250$). Option parameters: $S = 100$, $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months).

S	C	Δ	C^{do}_{B,r_B}	Δ^{do}_{B,r_B}	C^{ldo}_{B,r_B}	Δ^{ldo}_{B,r_B}	C^{do}_{B}	Δ^{do}_{B}
85	9.8517	0.4554	1.6062	0.2376	0.7200	0.1730	0	0
86	10.3126	0.4664	1.8607	0.2720	0.9115	0.2108	0	0
87	10.7844	0.4772	2.1518	0.3109	1.1438	0.2548	0	0
88	11.2670	0.4879	2.4842	0.3549	1.4234	0.3055	0	0
89	11.7602	0.4986	2.8634	0.4044	1.7572	0.3634	0	0
90	12.2641	0.5091	3.2951	0.4602	2.1528	0.4291	0	0
91	12.7783	0.5194	3.7860	0.5229	2.6182	0.5031	0	0
92	13.3029	0.5297	4.3434	0.5933	3.1619	0.5858	0	0
93	13.8377	0.5398	4.9755	0.6723	3.7929	0.6778	0	0
94	14.3825	0.5498	5.6912	0.7608	4.5206	0.7793	0	0
95	14.9373	0.5597	6.5008	0.8598	5.3548	0.8908	0	1.0058
96	15.5019	0.5694	7.3603	0.8591	6.2450	0.8895	1.0044	1.0029
97	16.0760	0.5790	8.2192	0.8587	7.1339	0.8884	2.0060	1.0003
98	16.6597	0.5884	9.0777	0.8584	8.0218	0.8875	3.0049	0.9977
99	17.2528	0.5977	9.9360	0.8583	8.9089	0.8867	4.0015	0.9954
100	17.8551	0.6068	10.7942	0.8583	9.7953	0.8862	4.9958	0.9932
101	18.4664	0.6158	11.6526	0.8585	10.6812	0.8857	5.9879	0.9911
102	19.0867	0.6247	12.5113	0.8589	11.5668	0.8855	6.9780	0.9892
103	19.7157	0.6333	13.3704	0.8594	12.4522	0.8853	7.9662	0.9873
104	20.3533	0.6419	14.2301	0.8600	13.3375	0.8853	8.9527	0.9857
105	20.9994	0.6503	15.0904	0.8607	14.2229	0.8855	9.9376	0.9841

Table 3: Vanilla, down-and-out exponential step, linear step and barrier call values and deltas as functions of the underlying asset price S. Option parameters: $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months). Exponential step call parameters: $\beta_B = 0.9$ ($r_B = 26.34$, $\mathcal{T}_B^- = 21.85$ trading days). Linear step call parameters: $\rho_B = 0.1$ ($R_B = 25$, $\mathcal{T}_B^- = 10$ trading days, $\beta_B = 0.9$).

9 Summary and Conclusion

In this paper we studied a family of path-dependent derivatives: *step options, forward contracts and swaps.* They are parametrized by the *knock-out rate*, r_B, and their payoff at expiration is defined as the payoff of otherwise identical vanilla derivatives discounted by the *knock-out factor* $\exp(-r_B\tau_B)$ (exponential knock-out) or $Max(1 - R_B\,\tau_B, 0)$ (linear knock-out), where τ_B is the occupation time bellow (above) the barrier level B for down (up) step contracts. We derived closed-form pricing formulas for step options and forwards with any knock-out rate $0 \leq r_B < \infty$. In the limit of zero knock-out rate, step options coincide with vanilla options. In the limit of an infinitely high knock-out rate, they coincide with standard barrier options, thus continuously interpolating between vanilla and barrier contracts.

A crucial property of step options is that, unlike barrier options, for any finite knock-out rate $r_B < \infty$ the step option's delta is a continuous function of the underlying price at the barrier. As a result, they can be continuously hedged by executing a dynamic trading strategy. As a by-product, we derive a dynamic almost-replicating trading strategy for standard barrier options by considering the replicating strategy for a step option with arbitrarily high but finite knock-out rate.

An ever increasing number of new financial derivative products appear every year in over-the-counter derivative markets. *Is the introduction of yet another family of derivatives justified?* We believe that their hedging properties make step options, forward contracts and swaps a potentially valuable addition to the derivatives risk management toolkit. Standard barrier options have grown increasingly popular in recent years. Investors and corporate hedgers are attracted by significant premium savings. However, because of the inherent discontinuity at the barrier, barrier options pose serious risk management problems both for buyers and sellers. Step options, on the other hand, offer the desired premium savings and at the same time retain continuity of the delta. Thus they may be *an attractive alternative to standard barrier options* in a number of situations.

On the other hand, step forwards and swaps possess interesting principal amortizing features that could make them attractive hedging products in FX and fixed-income markets, where the need to amortize the principal based on occupation times may arise. Especially promising applications may include hedging foreign currency cash flows where the principal payment in the foreign currency is itself subject to additional business risk related to the exchange rate, as well as hedging mortgage-backed securities.

Our second motivation for this research was provided by the interesting properties of step options and forwards themselves. Their payoff structure with the discount factor determined by occupation time of the underlying price process, as well as the existence of an elegant closed-form solution to this valuation problem, single them out as in some sense fundamental structures from the point of view of financial economics. Similar to

their other path-dependent relatives: Asian, lookback and barrier contracts, their payoff is based on the value of a second path-dependent state variable. While in the case of Asian, lookback and barrier contracts these are average prices and the minimum or maximum price achieved to date, in the case in question the second state variable is occupation time.

We also believe that the stochastic model of Brownian motion with killing at finite rate below the barrier level will also find further interesting applications in finance. It may serve as an attractive first approximation in many problems due to the availability of the closed-form solution. In particular, we are currently working on applications to modeling *credit risk and credit risky securities* and *mortgage prepayments and mortgage-backed securities.* We will discuss these applications in subsequent papers.

Appendix

A Risk-Neutral Valuation, Wiener-Feynman Path Integrals and Green's Functions

Path integrals[13] constitute one of the basic tool of modern quantum physics. They were introduced in physics by Richard Feynman in 1942 in his Ph.D. thesis on path integral formulation of quantum mechanics (Feynman (1942), (1948), Feynman and Hibbs (1965), Kac (1949), (1951), Simon (1979), Glimm and Jaffe (1981)). In classical deterministic physics, time evolution of dynamical systems is governed by the Least Action Principle. Classical equations of motion, such as Newton's equations, can be viewed as the Euler-Lagrange equations for a minimum of a certain *action functional*, a time integral of the Lagrangian function defining the dynamical system. Their deterministic solutions, trajectories of the classical dynamical system, minimize the action functional (*the least action principle*). In quantum, i.e. probabilistic, physics, one talks about probabilities of different paths a quantum (stochastic) dynamical system can take. One defines a measure on the set of all possible paths from the initial state x_i to the final state x_f of the quantum (stochastic) dynamical system, and expectation values (averages) of various quantities dependent on paths are given by *path integrals* over all possible paths from x_i to x_f (path integrals are also called sums over histories, as well as functional integrals, as the integration is performed over a set of functions (paths)). The classical action functional is evaluated to a real number on each path, and the exponential of the negative of this number gives a weight of the path in the path integral. *A path integral is defined as a limit of the sequence of finite-dimensional multiple integrals*, in a much the same way as the Riemannian integral is defined as a limit of the sequence of finite sums. The path integral representation of averages can also be obtained directly as the Feynman-Kac solution to the partial differential equation describing the time evolution of the quantum (stochastic) dynamical system (the Schrodinger equation in quantum mechanics or diffu-

[13]For a more detailed discussion see Linetsky (1996).

sion (Kolmogorov) equation in the theory of stochastic processes).

In finance, the fundamental principle is the absence of arbitrage (Ross (1976), Cox and Ross (1976), Harrison and Kreps (1979), Harrison and Pliska (1981), Merton (1990), Duffie (1996)). In finance it plays a role similar to the least action principle and the energy conservation law in natural sciences. Accordingly, similar to physical dynamical systems, one can introduce Lagrangian functions and action functionals for financial models. Since financial models are stochastic, expectation values of various quantities contingent upon price paths (financial derivatives) are given by path integrals, where the action functional for the underlying (risk-neutral) price process defines a measure on the set of all paths. Averages satisfy the Black-Scholes partial differential equation, which is a finance counterpart of the Schrodinger equation of quantum mechanics, and the risk-neutral valuation formula is interpreted as the Feynman-Kac representation of the PDE solution. Thus, *the path-integral formalism provides a natural bridge between the risk-neutral martingale pricing and the arbitrage-free PDE-based pricing.*

To the best of our knowledge, applications of path integrals and related techniques from quantum physics to finance were first systematically developed in the eighties by Jan Dash (1989), (1993). See also Esmailzadeh (1995) for applications to path-dependent options and Eydeland (1994) for applications to fixed-income derivatives and interesting numerical algorithms. This approach is also very close to the semigroup pricing developed by Garman (1985), as path integrals provide a natural representation for pricing semigroup kernels. See the monograph Duffie (1996) for the Feynman-Kac approach in finance and further references.

Let us first review the Black-Scholes model. A path-independent option is defined by its payoff at expiration $\mathcal{O}_F(S_T, T) = F(S_T)$, where S_T is the terminal asset price and $F(S_T)$ is a given function of S_T. We assume we live in the Black-Scholes world. Then the present value of the option $\mathcal{O}_F(S,t)$ at the inception of the contract t satisfies the Black-Scholes PDE

$$\frac{\sigma^2}{2}S^2\frac{\partial^2\mathcal{O}_F}{\partial S^2} + rS\frac{\partial\mathcal{O}_F}{\partial S} - r\mathcal{O}_F = -\frac{\partial\mathcal{O}_F}{\partial t}. \qquad (A.1)$$

Introducing a new variable $x = \ln S$, Eq.(A.1) reduces to

$$\frac{\sigma^2}{2}\frac{\partial^2\mathcal{O}_F}{\partial x^2} + \mu\frac{\partial\mathcal{O}_F}{\partial x} - r\mathcal{O}_F = -\frac{\partial\mathcal{O}_F}{\partial t},\ \mu = r - \sigma^2/2,\ \mathcal{O}_F(e^{x_T}, T) = F(e^{x_T}). \qquad (A.2)$$

A unique solution to the Cauchy problem (A.2) is given by the Feynman-Kac formula

$$\mathcal{O}_F(S,t) = e^{-r\tau}E_{(t,S)}\left[F(S_T)\right]$$

$$= e^{-r\tau}\int_{-\infty}^{\infty}\left(\int_{x(t)=x}^{x(T)=x_T} F\left(e^{x_T}\right)e^{-A_{BS}[x(t')]}\mathcal{D}x(t')\right)dx_T, \qquad (A.3)$$

where $A_{BS}[x(t')]$ is the Black-Scholes action functional defined on price paths $\{x(t'), t \leq t' \leq T\}$ as a time integral of the Black-Scholes Lagrangian function

$$A_{BS}[x(t')] = \int_t^T \mathcal{L}_{BS}\, dt',\ \mathcal{L}_{BS} = \frac{1}{2\sigma^2}(\dot{x}(t') - \mu)^2. \tag{A.4}$$

According to Feynman, the path integral in (A.3) is defined as follows. First, paths are discretized. Time to expiration τ is discretized into N equal time steps Δt bounded by $N+1$ equally spaced time points $t_i = t + i\Delta t$, $i = 0, 1, \ldots, N$, $\Delta t = (T-t)/N$. Discrete prices at these time points are denoted by $S_i = S(t_i)$ ($x_i = x(t_i)$ for the logarithms). The discretized action functional becomes a function of $N+1$ variables x_i ($x_0 \equiv x$, $x_N \equiv x_T$)

$$A_{BS}(x_i) = \frac{\mu^2\tau}{2\sigma^2} - \frac{\mu}{\sigma^2}(x_T - x) + \frac{1}{2\sigma^2\Delta t}\sum_{i=0}^{N-1}(x_{i+1} - x_i)^2. \tag{A.5}$$

Now, the path integral over all paths from the initial state $x(t)$ to the final state x_T is defined as a limit of the sequence of finite-dimensional multiple integrals, just as the standard Riemannian integral is defined as a limit of the sequence of finite sums:

$$\int_{x(t)=x}^{x(T)=x_T} F\left(e^{x_T}\right) e^{-A_{BS}[x(t')]}\mathcal{D}x(t')$$

$$:= \lim_{N\to\infty} \underbrace{\int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty}}_{N-1} F(e^{x_T}) e^{-A_{BS}(x_i)} \frac{dx_1}{\sqrt{2\pi\sigma^2\Delta t}} \ldots \frac{dx_{N-1}}{\sqrt{2\pi\sigma^2\Delta t}}. \tag{A.6}$$

Path integrals are especially useful for pricing *path-dependent derivatives*. Consider a *path-dependent* option defined by the payoff

$$\mathcal{O}_F(T) = F[S(t')], \tag{A.7}$$

where $F[S(t')]$ is a given *functional* on price paths $\{S(t'), t \leq t' \leq T\}$, rather than a function dependent just on the terminal asset price.

Suppose the payoff functional F can be represented in the form

$$F = f(S_T)\exp\left(-I[x(t')]\right), \tag{A.8a}$$

where $f(S_T)$ depends only on the terminal asset price S_T, and I is a time integral

$$I[x(t')] = \int_t^T V(x(t'), t')dt' \tag{A.8b}$$

of some potential $V(x, t')$. The payoff of our down-and-out step call is of this general form. Then the Feynman-Kac formula takes the form

$$\mathcal{O}_F(S, t) = e^{-r\tau}\int_{-\infty}^{\infty} f(e^{x_T}) e^{\frac{\mu}{\sigma^2}(x_T - x) - \frac{\mu^2\tau}{2\sigma^2}} \mathcal{K}_V(x_T, T|x, t)dx_T, \tag{A.9}$$

where $\mathcal{K}_V$ is the Green's function (transition density) for zero-drift Brownian motion with killing at rate $V(x,t')$

$$\mathcal{K}_V(x_T,T|x,t) = \int_{x(t)=x}^{x(T)=x_T} \exp\left(-\int_t^T (\mathcal{L}_0 + V)dt'\right) \mathcal{D}x(t'). \qquad (A.10)$$

This is the Feynman-Kac representation of the fundamental solution of diffusion PDE with potential $V(x,t)$

$$\frac{\sigma^2}{2}\frac{\partial^2 \mathcal{K}_V}{\partial x^2} - V(x,t)\mathcal{K}_V = \frac{\partial \mathcal{K}_V}{\partial \tau} \qquad (A.11a)$$

and initial condition

$$\mathcal{K}_V(x_T,T|x,T) = \delta(x_T - x). \qquad (A.11b)$$

Then the option price (A.9) satisfies the PDE with potential

$$\frac{\sigma^2}{2}\frac{\partial^2 \mathcal{O}_F}{\partial x^2} + \mu\frac{\partial \mathcal{O}_F}{\partial x} - (r + V(x,t))\mathcal{O}_F = -\frac{\partial \mathcal{O}_F}{\partial t}. \qquad (A.12)$$

This equation can be interpreted as the Black-Scholes equation with an effective state-dependent risk-free rate $r + V(x,t)$ and continuous dividend yield $V(x,t)$. In Linetsky (1996) we extend this Feynman-Kac methodology to more general payoffs $F(S_T, I^i)$, where F is a given function of the terminal asset price and $\{I^i\}$ are a set of time integrals on price paths. In Section 6 of this paper this methodology is applied to occupation time derivatives.

It is easy to compute the Green's function (A.10) when $V = 0$ directly from the definition (A.6). The finite-dimensional integral is Gaussian and is evaluated exactly by successive integration on x_i yielding the normal density:

$$\lim_{N\to\infty} \underbrace{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{N-1} \exp\left(-\frac{1}{2\sigma^2\Delta t}\sum_{i=0}^{N-1}(x_{i+1}-x_i)^2\right) \frac{dx_1}{\sqrt{2\pi\sigma^2\Delta t}} \cdots \frac{dx_{N-1}}{\sqrt{2\pi\sigma^2\Delta t}}$$

$$= \frac{1}{\sqrt{2\pi\sigma^2\tau}} \exp\left(-\frac{(x_T - x)^2}{2\sigma^2\tau}\right). \qquad (A.13)$$

When V is non-zero, it is still possible in some cases to find a closed-form solution for the Green's function, as in the case of step options in this paper. In Linetsky (1996) more examples are given. Finally, there exist a variety of numerical procedures for evaluating path integrals when no analytical solutions are available. Monte Carlo simulation has been especially popular for evaluating difficult path integrals.

Now consider a general D-dimensional diffusion process x^μ, $\mu = 1,2,\ldots,D$,

$$dx^\mu = a^\mu dt + \sum_{a=1}^{D} \sigma_a^\mu dz^a,\ a^\mu = a^\mu(\mathbf{x},t),\ \sigma_a^\mu = \sigma_a^\mu(\mathbf{x},t), \qquad (A.14)$$

where dz^a, $a = 1, 2, \ldots, D$, are standard uncorrelated Wiener processes, $E\left[dz^a dz^b\right] = \delta^{ab} dt$ (δ^{ab} is the Kroeneker symbol, $\delta^{ab} = 1$ if $a = b$ and zero otherwise). Suppose the risk-free rate is $r(\mathbf{x}, t)$, Eq.(A.14) describes a general D-dimensional risk-neutral price process with the risk-neutral drift (boldface letters $\mathbf{x}$ denote D-dimensional vectors) $a^\mu(\mathbf{x}, t) = r(\mathbf{x}, t)x^\mu - D^\mu(\mathbf{x}, t)$ (D^μ are dividends), and the option's payoff at expiration is path-dependent and $\mathcal{O}_F(T) = F[\mathbf{x}(t')]$ for a given functional F. Then the present value $\mathcal{O}_F(t)$ is given by the Feynman-Kac formula

$$\mathcal{O}_F(t) = \int_{\mathbf{R}^D} \left(\int_{\mathbf{x}(t)=\mathbf{x}}^{\mathbf{x}(T)=\mathbf{x}_T} F[\mathbf{x}(t')] e^{-A[\mathbf{x}(t')]} \mathcal{D}\mathbf{x}(t') \right) d^D \mathbf{x}_T. \qquad (A.15)$$

Here A is the action functional $A = \int_t^T \mathcal{L}\, dt'$, with the Lagrangian function for the process (A.14) given by (see, e.g., Langouche, Roekaerts and Tirapegui (1980) and (1982), and Freidlin (1985))

$$\mathcal{L} = \frac{1}{2} \sum_{\mu,\nu=1}^{D} g_{\mu\nu}(\mathbf{x}, t')(\dot{x}^\mu(t') - a^\mu(\mathbf{x}, t'))(\dot{x}^\nu(t') - a^\nu(\mathbf{x}, t')) + r(\mathbf{x}, t'), \qquad (A.16)$$

where $g_{\mu\nu} = g_{\nu\mu}$ is an inverse of the variance-covariance matrix $g^{\mu\nu}$, $\sum_{\rho=1}^{D} g_{\mu\rho} g^{\rho\nu} = \delta_\mu^\nu$, and $g^{\mu\nu} = \sum_{a=1}^{D} \sigma_a^\mu \sigma_a^\nu$. The general multi-asset path-integral (A.15) is defined as a limit of the sequence of finite-dimensional multiple integrals similar to the one-factor example. A discretized action functional is given by

$$A(\mathbf{x}_i) = \frac{1}{2} \sum_{i=0}^{N-1} \sum_{\mu,\nu=1}^{D} g_{\mu\nu}(\mathbf{x}_i, t_i) \left(\frac{x_{i+1}^\mu - x_i^\mu}{\Delta t} - a^\mu(\mathbf{x}_i, t_i) \right) \left(\frac{x_{i+1}^\mu - x_i^\mu}{\Delta t} - a^\nu(\mathbf{x}_i, t_i) \right) \Delta t$$

$$+ \sum_{i=0}^{N-1} r(\mathbf{x}_i, t_i) \Delta t, \qquad (A.17)$$

and

$$\int_{\mathbf{x}(t)=\mathbf{x}}^{\mathbf{x}(T)=\mathbf{x}_T} F[\mathbf{x}(t')] e^{-A[\mathbf{x}(t')]} \mathcal{D}\mathbf{x}(t') \qquad (A.18)$$

$$:= \lim_{N\to\infty} \underbrace{\int_{\mathbf{R}^D} \ldots \int_{\mathbf{R}^D}}_{N-1} F(\mathbf{x}_i) \exp(-A(\mathbf{x}_i)) \prod_{i=1}^{N-1} \frac{d^D \mathbf{x}_i}{\sqrt{(2\pi)^D det(g^{\mu\nu}(\mathbf{x}_i, t_i))\Delta t}}.$$

The determinant $det(g^{\mu\nu}(\mathbf{x}_i, t_i))$ of the variance-covariance matrix $g^{\mu\nu}(\mathbf{x}_i, t_i)$ appearing in the square roots defines the integration measure over intermediate points $\mathbf{x}_i$. This discretization scheme is called *pre-point discretization* and is consistent with the Ito's calculus.

For path-independent options, when the payoff depends only on the terminal prices $\mathbf{x}_T$, $\mathcal{O}_F(T) = F(\mathbf{x}_T)$, the option value satisfies the backward Kolmogorov PDE

$$\mathcal{H}\, \mathcal{O}_F = -\frac{\partial \mathcal{O}_F}{\partial t}, \qquad (A.20)$$

where $\mathcal{H}$ is a second order differential operator (generator of the diffusion process (A.14) also called Hamiltonian (energy) operator in quantum mechanics)

$$\mathcal{H} = \frac{1}{2}\sum_{\mu,\nu=1}^{D} g^{\mu\nu}(\mathbf{x},t)\frac{\partial^2}{\partial x^\mu \partial x^\nu} + \sum_{\mu=1}^{D} a^\mu(\mathbf{x},t)\frac{\partial}{\partial x^\mu} - r(\mathbf{x},t). \tag{A.21}$$

The general Feynman-Kac formula (A.15) is a powerful and versatile tool for obtaining both closed-form and approximate solutions to financial derivatives valuation problems. A number of powerful techniques are available to calculate path integrals. These include changes of state variables and time (Cameron-Martin-Girsanov theorem), semiclassical expansion and WKB approximation, perturbation expansion in the powers of volatility parameter, Monte Carlo simulation and finite-difference schemes. See Linetsky (1996) for more details.

B Laplace Transform

The following inverse transforms are used to calculate Green's functions (Abramowitz and Stegun (1965)):

$$\mathcal{L}_\tau^{-1}[1] = \delta(\tau), \tag{B.1}$$

$$\mathcal{L}_\tau^{-1}\left[\frac{1}{\sqrt{s}}e^{-a\sqrt{s}}\right] = \frac{1}{\sqrt{\pi\tau}}\exp\left(-\frac{a^2}{4\tau}\right), a \geq 0, \tag{B.2}$$

$$\mathcal{L}_\tau^{-1}\left[e^{-a\sqrt{s}}\right] = \frac{a}{2\sqrt{\pi}\tau^{3/2}}\exp\left(-\frac{a^2}{4\tau}\right), a > 0, \tag{B.3}$$

$$\mathcal{L}_\tau^{-1}\left[\sqrt{s}e^{-a\sqrt{s}}\right] = \frac{(a^2-2\tau)}{4\sqrt{\pi}\tau^{5/2}}\exp\left(-\frac{a^2}{4\tau}\right), a \geq 0, \tag{B.4}$$

$$\mathcal{L}_\tau^{-1}\left[\frac{1}{\sqrt{s+a}+\sqrt{s}}\right] = \mathcal{L}_\tau^{-1}\left[\frac{\sqrt{s+a}-\sqrt{s}}{a}\right] = \frac{1-e^{-a\tau}}{2a\sqrt{\pi}\tau^{3/2}}, a \geq 0, \tag{B.5}$$

$$\mathcal{L}_\tau^{-1}\left[\frac{e^{-as}}{s}\right] = \theta(\tau - a), \tag{B.6}$$

as well as linearity, translation and convolution properties

$$\mathcal{L}_\tau^{-1}[aF + bG] = a\mathcal{L}_\tau^{-1}[F] + b\mathcal{L}_\tau^{-1}[G], \tag{B.7}$$

$$\mathcal{L}_\tau^{-1}[F(s+a)] = e^{-a\tau}\mathcal{L}_\tau^{-1}[F(s)], \tag{B.8}$$

$$\mathcal{L}_\tau^{-1}[F(s)G(s)] = \int_0^\tau f(\tau')g(\tau-\tau')d\tau'. \tag{B.9}$$

To obtain (3.20a), we first write

$$G^{-I}_{B,r_B}(x_T,x;s) = G^{-}_{B}(x_T,x;s) + \frac{\sqrt{2}}{\sigma\left(\sqrt{s+a}+\sqrt{s}\right)}\exp\left(-|y+y_T|\sqrt{2s}\right), \qquad (B.10)$$

where G^{-}_{B} is the standard down-and-out resolvent for Brownian motion with absorbing barrier

$$G^{-}_{B}(x_T,x;s) = \frac{1}{\sigma\sqrt{2s}}\left(\exp(-|y-y_T|\sqrt{2s}) - \exp(-|y+y_T|\sqrt{2s})\right). \qquad (B.11)$$

Then Eq.(3.20a) is obtained by using Eqs.(B.3) and (B.5).

To obtain Eq.(3.20b), we use the convolution property

$$\mathcal{L}^{-1}_{\tau}\left[\frac{\sqrt{2}}{\sigma r_B}\left(\sqrt{s+r_B}-\sqrt{s}\right)\exp\left(y\sqrt{2(s+r_B)}-y_T\sqrt{2s}\right)\right] \qquad (B.12)$$

$$= \int_0^{\tau}\left(\mathcal{L}^{-1}_{\tau}\left[\sqrt{s+r_B}e^{y\sqrt{2(s+r_B)}}\right]\mathcal{L}^{-1}_{\tau}\left[e^{-y_T\sqrt{2s}}\right] - \mathcal{L}^{-1}_{\tau}\left[e^{y\sqrt{2(s+r_B)}}\right]\mathcal{L}^{-1}_{\tau}\left[\sqrt{s}e^{-y_T\sqrt{2s}}\right]\right)d\tau'$$

$$= \int_0^{\tau}\frac{e^{-r_B(\tau-\tau')}\left[y_T\tau'(y^2-\tau+\tau')+y(\tau-\tau')(y_T^2-\tau')\right]}{2\pi\sigma r_B(\tau-\tau')^{\frac{5}{2}}\tau'^{\frac{5}{2}}}\exp\left(-\frac{y_T^2}{2\tau'}-\frac{y^2}{2(\tau-\tau')}\right)d\tau'.$$

Notice that as y tends to zero ($x \to b$), the integrand becomes singular at $\tau' \to \tau$. A leading singularity $(\tau-\tau')^{-\frac{3}{2}}$ as $y \to 0$ and $\tau' \to \tau$ is the zero-order term in the exponential, $\exp(-r_B(\tau-\tau')) = 1 - r_B(\tau-\tau') + \ldots$ However, it is easy to see that the corresponding integral vanishes identically for all $y < 0$

$$\int_0^{\tau}\frac{\left[y_T\tau'(y^2-\tau+\tau')+y(\tau-\tau')(y_T^2-\tau')\right]}{2\pi\sigma r_B(\tau-\tau')^{\frac{5}{2}}\tau'^{\frac{5}{2}}}\exp\left(-\frac{y_T^2}{2\tau'}-\frac{y^2}{2(\tau-\tau')}\right)d\tau' \equiv 0. \qquad (B.13)$$

This identity is established by taking the Laplace transform as in Eq.(B.11) of both sides of the identity $(\sqrt{s}\exp(y\sqrt{2s}))(\exp(-y_T\sqrt{2s})) - (\exp(y\sqrt{2s}))(\sqrt{s}\exp(-y_T\sqrt{2s})) \equiv 0$. To prevent numerical instabilities due to round-off errors during the computation of the convolution integral, we explicitly subtract the integrand in Eq.(B.13) from the integrand in Eq.(B.12) and finally arrive at Eq.(3.20b). This integral converges for all $y \leq 0$.

C Calculation of Integrals

Eq.(3.22)

To obtain the expression (3.22) from Eqs.(3.8) and (3.20a), one needs to simplify the integral

$$I = e^{-r\tau}\int_{\ln K}^{\infty}(e^{x_T}-K)e^{\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}} \qquad (C.1)$$

$$\times \left\{ \int_0^\tau \frac{g(\tau-\tau')(y_T+y)}{\sqrt{2\pi}\sigma\tau'^{\frac{3}{2}}} \exp\left(-\frac{(y+y_T)^2}{2\tau'}\right) d\tau' \right\} dx_T,$$

where

$$g(\tau-\tau') = \frac{1-e^{-r_B(\tau-\tau')}}{\sqrt{2\pi}r_B(\tau-\tau')^{\frac{3}{2}}}. \tag{C.2}$$

First, introduce a new integration variable

$$z = \frac{\mu\tau' - \sigma(y+y_T)}{\sigma\sqrt{\tau'}} = \frac{\mu\tau' + 2b - x - x_T}{\sigma\sqrt{\tau'}}. \tag{C.3}$$

After some algebra, the integral reduces to

$$I = \left(\frac{B}{S}\right)^\gamma \int_0^\tau g(\tau-\tau')e^{-\kappa(\tau-\tau')}Jd\tau', \tag{C.4}$$

where

$$J = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_3'} \left(\left(\frac{B^2}{S}\right) e^{-\sigma\sqrt{\tau'}z - \sigma^2\tau'/2} - e^{-r\tau'}K \right) (\nu_1 - (\tau')^{-\frac{1}{2}}z) e^{-\frac{z^2}{2}} dz, \tag{C.5}$$

and ν_1, κ and d_3' are defined in Eq.(3.25). The integral J is calculated as follows

$$J = \left(\frac{B^2}{S}\right)\left(\nu_1 J_1 - (\tau')^{-\frac{1}{2}}J_2\right) - \nu_1 e^{-r\tau'}KJ_3 + (\tau')^{-\frac{1}{2}}e^{-r\tau'}KJ_4, \tag{C.6}$$

where

$$J_1 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_3'} \exp\left(-\sigma\sqrt{\tau'}z - \frac{\sigma^2\tau'}{2} - \frac{z^2}{2}\right) dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_3'} \exp\left(-\frac{(z+\sigma\sqrt{\tau'})^2}{2}\right) dz = N(d_4'), \tag{C.7a}$$

$$J_2 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_3'} z \exp\left(-\sigma\sqrt{\tau'}z - \frac{\sigma^2\tau'}{2} - \frac{z^2}{2}\right) dz \tag{C.7b}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_4'} (z' - \sigma\sqrt{\tau'}) e^{-\frac{z'^2}{2}} dz' = -\sigma\sqrt{\tau'}N(d_4') - n(d_4'),$$

$$J_3 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_3'} e^{-\frac{z^2}{2}} dz = N(d_3'), \tag{C.7c}$$

$$J_4 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_3'} z\, e^{-\frac{z^2}{2}} dz = -n(d_3'). \tag{C.7d}$$

Substituting J_i back into Eq.(C.6) and simplifying the result by using an obvious identity

$$\left(\frac{B^2}{S}\right) n(d_4') - e^{-r\tau'}Kn(d_3') = 0, \tag{C.8}$$

we arrive at

$$J = \nu_2 \left(\frac{B^2}{S}\right) N(d_4') - \nu_1 e^{-r\tau'} K N(d_3'). \tag{C.9}$$

Finally, substituting this back into Eq.(C.4) proves Eq.(3.22).

Eq.(3.27)

Eq.(3.27) is proved similarly. We need to simplify the integral

$$I = e^{-r\tau} \int_{\ln K}^{\infty} (e^{x_T} - K) e^{\frac{\mu}{\sigma^2}(x_T - x) - \frac{\mu^2 \tau}{2\sigma^2}} \tag{C.10}$$

$$\times \left\{ \int_0^{\tau} \frac{h(\tau - \tau')}{\sqrt{2\pi}\sigma \tau'^{\frac{3}{2}}} f(y_T, y) \exp\left(-\frac{y_T^2}{2\tau'}\right) d\tau' \right\} dx_T,$$

where

$$h(\tau - \tau') = \frac{(e^{-r_B(\tau - \tau')} - 1)}{\sqrt{2\pi} r_B (\tau - \tau')^{\frac{3}{2}}} \exp\left(-\frac{y^2}{2(\tau - \tau')}\right) \tag{C.11}$$

and

$$f(y_T, y) = y_T \left(\frac{y^2}{\tau - \tau'} - 1\right) + y \left(\frac{y_T^2}{\tau'} - 1\right). \tag{C.12}$$

Again, introducing a new integration variable

$$z = \frac{\mu\tau' - \sigma y_T}{\sigma\sqrt{\tau'}} = \frac{\mu\tau' + b - x_T}{\sigma\sqrt{\tau'}}, \tag{C.13}$$

the integral reduces to:

$$I = \left(\frac{B}{S}\right)^{\frac{\gamma}{2}} \int_0^{\tau} h(\tau - \tau') e^{-\kappa(\tau - \tau')} J d\tau', \tag{C.14}$$

$$J = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_5'} \left(B\epsilon^{-\sigma\sqrt{\tau'}z - \sigma^2\tau'/2} - e^{-r\tau'} K\right) (a_2 z^2 - a_1 z + a_0) e^{-\frac{z^2}{2}} dz \tag{C.15}$$

$$= a_2 J_2 - a_1 J_1 + a_0 J_0,$$

where coefficients a_i are given by

$$a_0 = \nu_1 \left(\frac{y^2}{\tau - \tau'} - 1\right) - y \left(\frac{1}{\tau'} - \nu_1^2\right), \tag{C.16a}$$

$$a_1 = \frac{1}{\sqrt{\tau'}} \left(\frac{y^2}{\tau - \tau'} + 2\nu_1 y - 1\right), \; a_2 = \frac{y}{\tau'}, \tag{C.16b}$$

and the integrals J_i are calculated as follows

$$J_0 = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_5'} \left(B\epsilon^{-\sigma z\sqrt{\tau'} - \sigma^2\tau'/2} - e^{-r\tau'} K\right) e^{-\frac{z^2}{2}} dz \tag{C.17a}$$

$$= BN(d_6') - e^{-r\tau'}KN(d_5'),$$

$$J_1 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_5'}\left(Be^{-\sigma z\sqrt{\tau'}-\sigma^2\tau'/2} - e^{-r\tau'}K\right)ze^{-\frac{z^2}{2}}dz \qquad (C.17b)$$

$$= -\sigma\sqrt{\tau'}BN(d_6'),$$

$$J_2 = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d_5'}\left(Be^{-\sigma z\sqrt{\tau'}-\sigma^2\tau'/2} - e^{-r\tau'}K\right)z^2e^{-\frac{z^2}{2}}dz$$

$$= B\left[(1+\sigma^2\tau')N(d_6') + \sigma\sqrt{\tau'}n(d_6')\right] - e^{-r\tau'}KN(d_5'). \qquad (C.17c)$$

To calculate J_2, the following integral is used

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d}x^2e^{-\frac{x^2}{2}}dx = N(d) - dn(d). \qquad (C.18)$$

Substituting (C.16) and (C.17) into Eq.(C.5) one finally arrives at

$$J = \nu_2\rho_2(y)BN(d_6') - \nu_1\rho_1(y)e^{-r\tau'}KN(d_5') + \sigma yB(\tau')^{-\frac{1}{2}}n(d_6'), \qquad (C.19)$$

where $\rho_1(y)$ and $\rho_2(y)$ are given by Eq.(3.28). Substituting it back into (C.14) proves Eq.(3.27).

D Perpetual Step Options

The down-and-out step PDE (3.5) also admits an interesting time-independent solution which we call *perpetual step option*. Consider a time-independent problem with continuous dividend yield q:

Region I: Above the barrier, $S \geq B$

$$\frac{\sigma^2}{2}S^2\frac{\partial^2 C^{do,I}_{B,r_B}}{\partial S^2} + (r-q)S\frac{\partial C^{do,I}_{B,r_B}}{\partial S} - rC^{do,I}_{B,r_B} = 0, \qquad (D.1)$$

with the boundary condition[14]

$$C^{do,I}_{B,r_B} \to S^{\lambda_+ +1} \ as \ S \to \infty. \qquad (D.2)$$

Region II: Below the barrier, $S \leq B$

$$\frac{\sigma^2}{2}S^2\frac{\partial^2 C^{do,II}_{B,r_B}}{\partial S^2} + (r-q)S\frac{\partial C^{do,II}_{B,r_B}}{\partial S} - (r+r_B)C^{do,II}_{B,r_B} = 0, \qquad (D.3)$$

with the boundary condition

$$\lim_{S\to 0} C^{do,II}_{B,r_B} = 0. \qquad (D.4)$$

[14] $S^{\lambda_+ +1}$ is a stationary solution of the standard Black-Scholes PDE with continuous dividend yield q.

Continuity boundary conditions to patch both solutions together at the barrier are

$$C^{I}_{B,r_B}(B) = C^{do,II}_{B,r_B}(B), \tag{D.5a}$$

$$\frac{\partial C^{do,I}_{B,r_B}}{\partial S}(B) = \frac{\partial C^{do,II}_{B,r_B}}{\partial S}(B). \tag{D.5b}$$

A time-independent solution for the perpetual down-and-out step option as a function of the underlying asset price is:

$$C^{do,I}_{B,r_B}(S) = S^{\lambda_+ + 1}\left(1 - \mathcal{R}\left(\frac{B}{S}\right)^{\lambda_+ - \lambda_-}\right), \; S > B, \tag{D.6a}$$

$$C^{do,II}_{B,r_B}(S) = \mathcal{T} S^{\lambda_+ + 1}\left(\frac{B}{S}\right)^{\lambda_+ - \lambda_3}, \; S \leq B, \tag{D.6b}$$

where $\mathcal{R}$ and $\mathcal{T}$ are *reflection and transmission* coefficients

$$\mathcal{R} = \frac{\lambda_3 - \lambda_+}{\lambda_3 - \lambda_-}, \; \mathcal{T} = 1 - \mathcal{R} = \frac{\lambda_+ - \lambda_-}{\lambda_3 - \lambda_-}, \tag{D.7}$$

and

$$\lambda_\pm = -\rho \pm \sqrt{\rho^2 + 2q/\sigma^2}, \; \lambda_3 = -\rho + \sqrt{\rho^2 + 2(q + r_B)/\sigma^2}, \; \rho = \frac{(r-q)}{\sigma^2} + \frac{1}{2}. \tag{D.8}$$

This solution interpolates between the stationary solution of the Black-Scholes PDE with dividend yield q when knock-out rate is zero

$$\lim_{r_B \to 0} C^{do}_{B,r_B}(S) = S^{\lambda_+ + 1}, \tag{D.9}$$

and the standard perpetual barrier option when knock-out rate is infinite (see Ingersoll (1987) pages 371-3 for pricing barrier perpetuities)

$$\lim_{r_B \to \infty} C^{do,I}_{B,r_B} = S^{\lambda_+ + 1}\left(1 - \left(\frac{B}{S}\right)^{\lambda_+ - \lambda_-}\right), \; S > B, \tag{D.10a}$$

$$\lim_{r_B \to \infty} C^{do,II}_{B,r_B} = 0, \; S \leq B. \tag{D.10b}$$

An up-and-out perpetual step option is treated similarly.

E Barrier Forwards

The barrier forward is a forward contract with delivery price K maturing at time T with the attached barrier provision. For example, a down-and-out forward is the forward contract that ceases to exist (extinguishes) as soon as the underlying spot price hits the pre-specified barrier B from above. Thus the payoff of a down-and-out forward is:

$$f^{do}_B(S_T, T) = \mathbf{1}_{\{L_T > B\}}(S_T - K), \tag{E.1}$$

where $\mathbf{1}_{\{L_T>B\}}$ is the barrier indicator defined in Section 2.1. The present value of a barrier forward at some time t' between the inception of the contract at time t and expiration T is then

$$f_B^{do}(S',t') = e^{-r\tau'}\left\{E_{(t',S')}\left[\mathbf{1}_{\{L_T>B\}}\,S_T\right] - KE_{(t',S')}\left[\mathbf{1}_{\{L_T>B\}}\right]\right\}. \qquad (E.2)$$

The barrier forward price is a delivery price such that the barrier forward contract has zero present value at inception $t' = t$:

$$K = F_B^{do}(S,t) = \frac{E_{(t,S)}\left[\mathbf{1}_{\{L_T>B\}}\,S_T\right]}{E_{(t,S)}\left[\mathbf{1}_{\{L_B>B\}}\right]} \qquad (E.3)$$

Thus we need the two averages $E_{(t,S)}\left[\mathbf{1}_{\{L_B>B\}}\,S_T\right]$ and $E_{(t,S)}\left[\mathbf{1}_{\{L_T>B\}}\right]$ (the first average is just the probability to hit the barrier B during time interval $[t,T]$, given the price S at time t). They can be calculated as integrals ($\mathcal{K}_B^-$ is the down-and-out density (3.21))

$$E_{(t,S)}\left[\mathbf{1}_{\{L_T>B\}}\,S_T\right] = \int_b^\infty e^{x_T+\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\,\mathcal{K}_B^-(x_T,x;\tau)\,dx_T$$

$$= e^{r\tau}S\left(N(z_2) - \left(\frac{B}{S}\right)^{\gamma+2}N(z_4)\right), \qquad (E.4)$$

and

$$E_{(t,S)}\left[\mathbf{1}_{\{L_T>B\}}\right] = \int_b^\infty e^{\frac{\mu}{\sigma^2}(x_T-x)-\frac{\mu^2\tau}{2\sigma^2}}\,\mathcal{K}_B^-(x_T,x;\tau)\,dx_T = N(z_1) - \left(\frac{B}{S}\right)^{\gamma}N(z_3), \qquad (E.5)$$

where

$$z_1 = \frac{\ln\left(\frac{S}{B}\right)+\mu\tau}{\sigma\sqrt{\tau}},\ z_2 = z_1+\sigma\sqrt{\tau},\ z_3 = \frac{\ln\left(\frac{B}{S}\right)+\mu\tau}{\sigma\sqrt{\tau}},\ z_4 = z_3+\sigma\sqrt{\tau}. \qquad (E.6)$$

Substituting these results into Eqs.(E.3) and (E.2) we find

$$K = F_B^{do}(S,t) = e^{r\tau}S\left(\frac{N(z_2)-\left(\frac{B}{S}\right)^{\gamma+2}N(z_4)}{N(z_1)-\left(\frac{B}{S}\right)^{\gamma}N(z_3)}\right), \qquad (E.7)$$

and

$$f_B^{do}(S',t') = S'\left(N(z_2') - \left(\frac{B}{S'}\right)^{\gamma+2}N(z_4')\right) - e^{-r\tau'}K\left(N(z_1') - \left(\frac{B}{S'}\right)^{\gamma}N(z_3')\right), \qquad (E.8)$$

where S' and t' are substituted in (E.6) in place of S and t.

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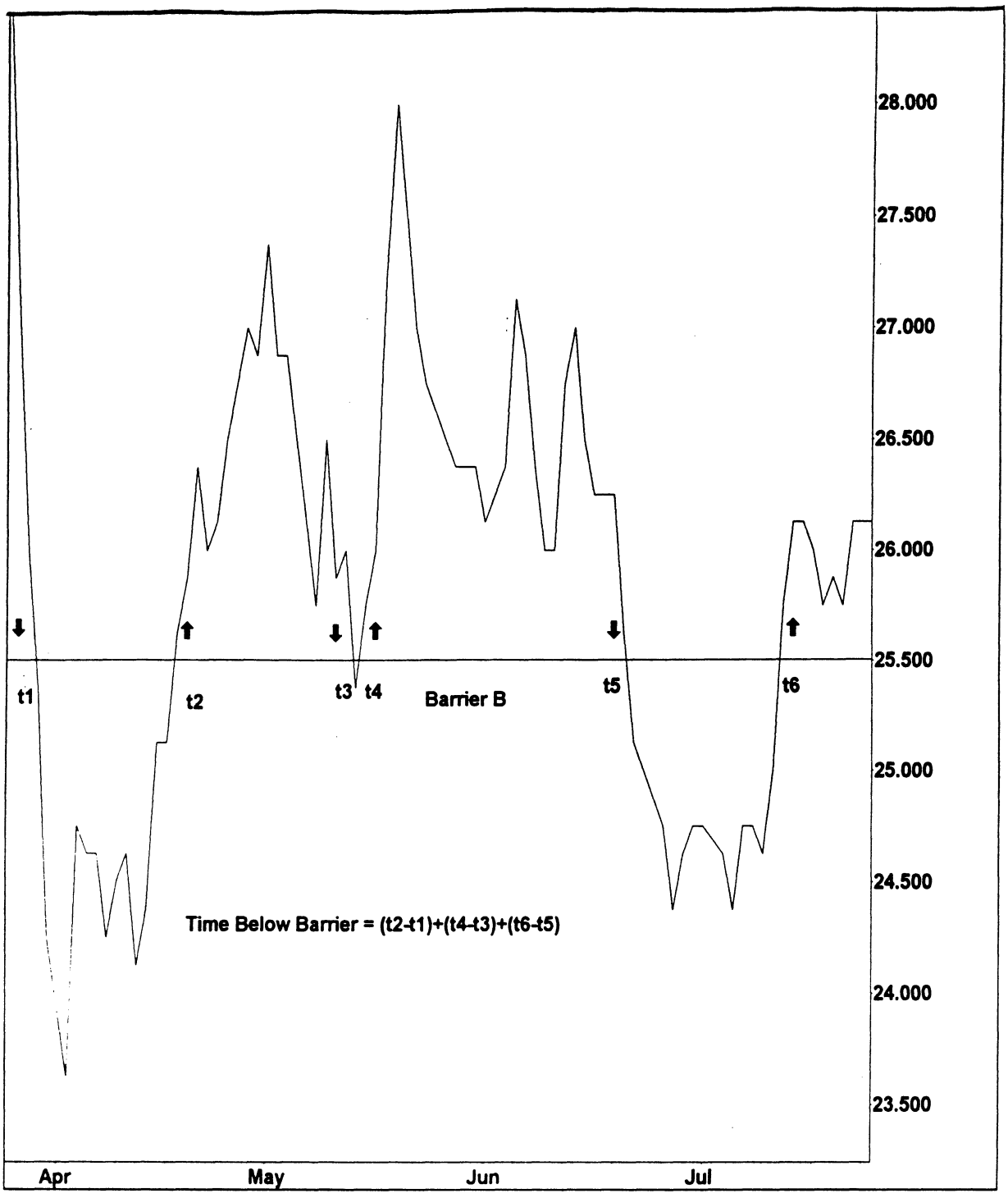

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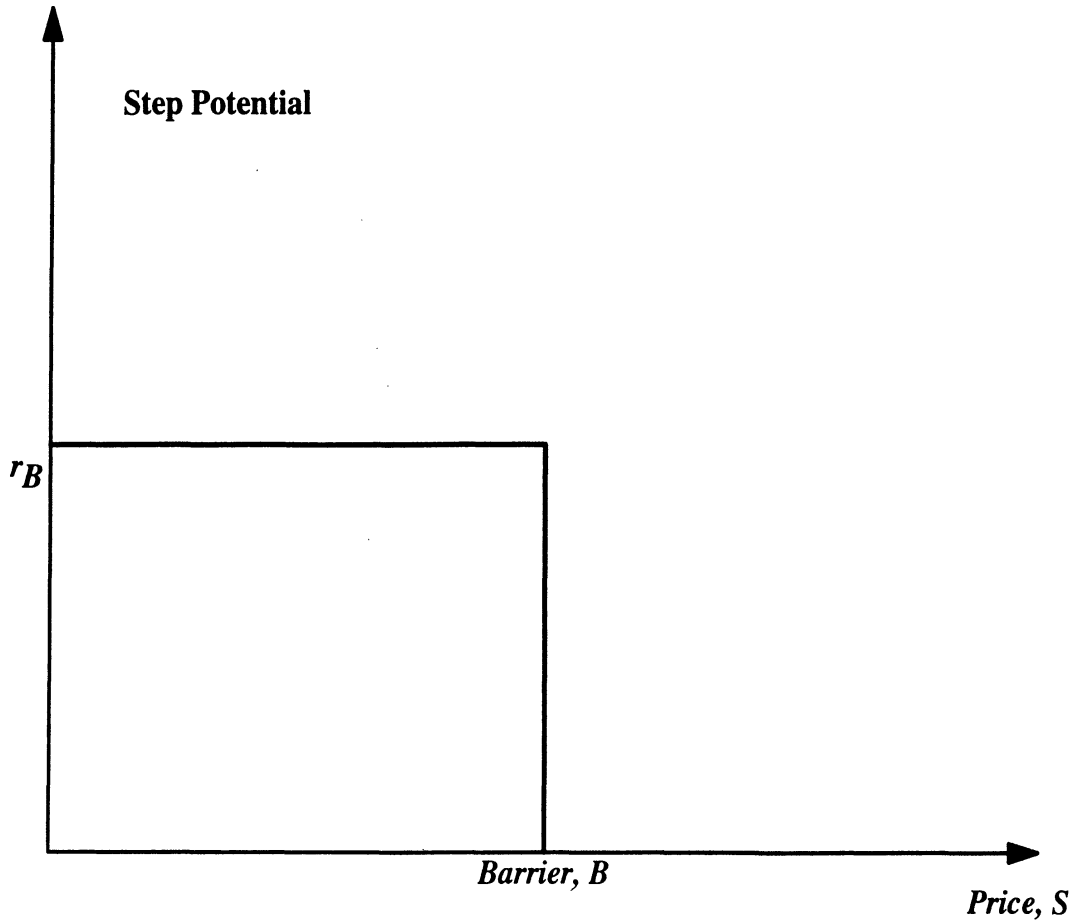

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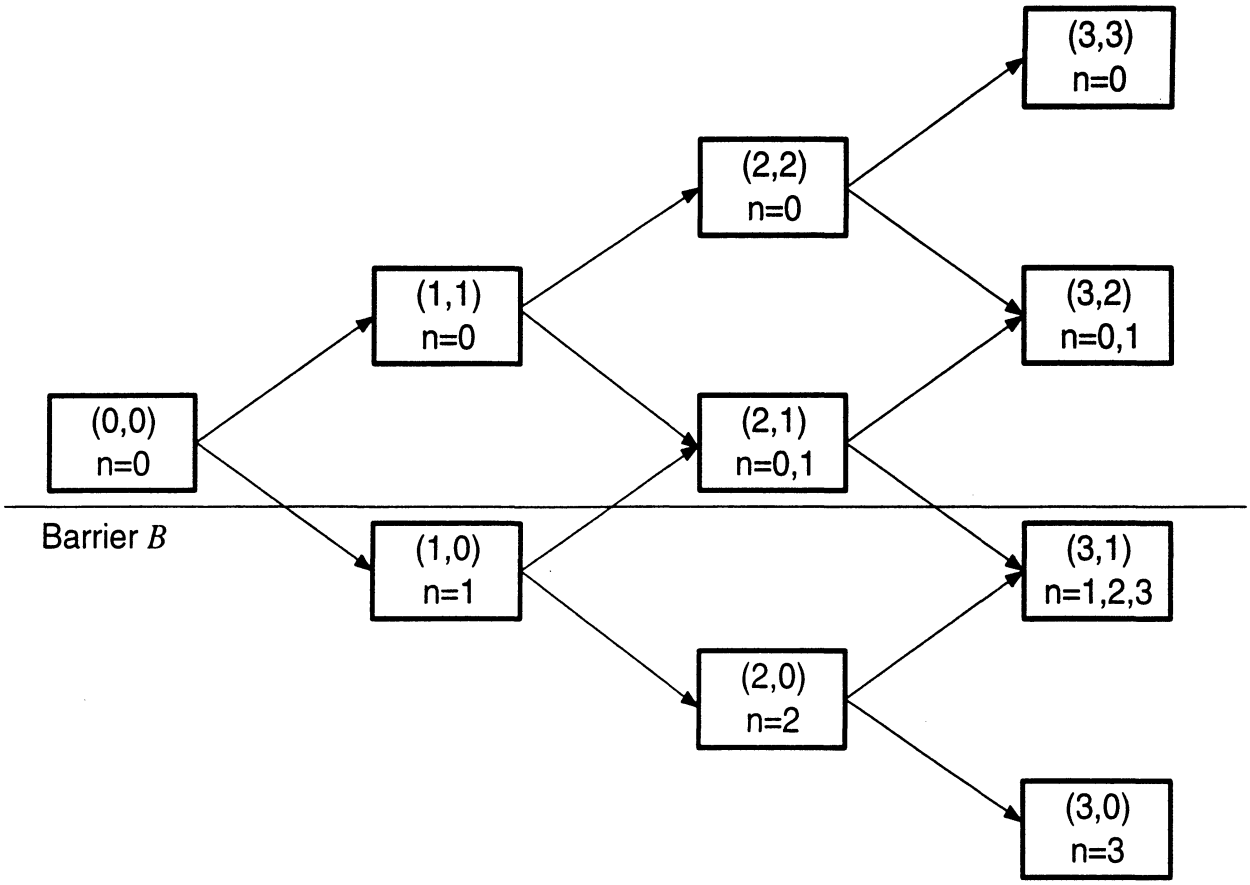

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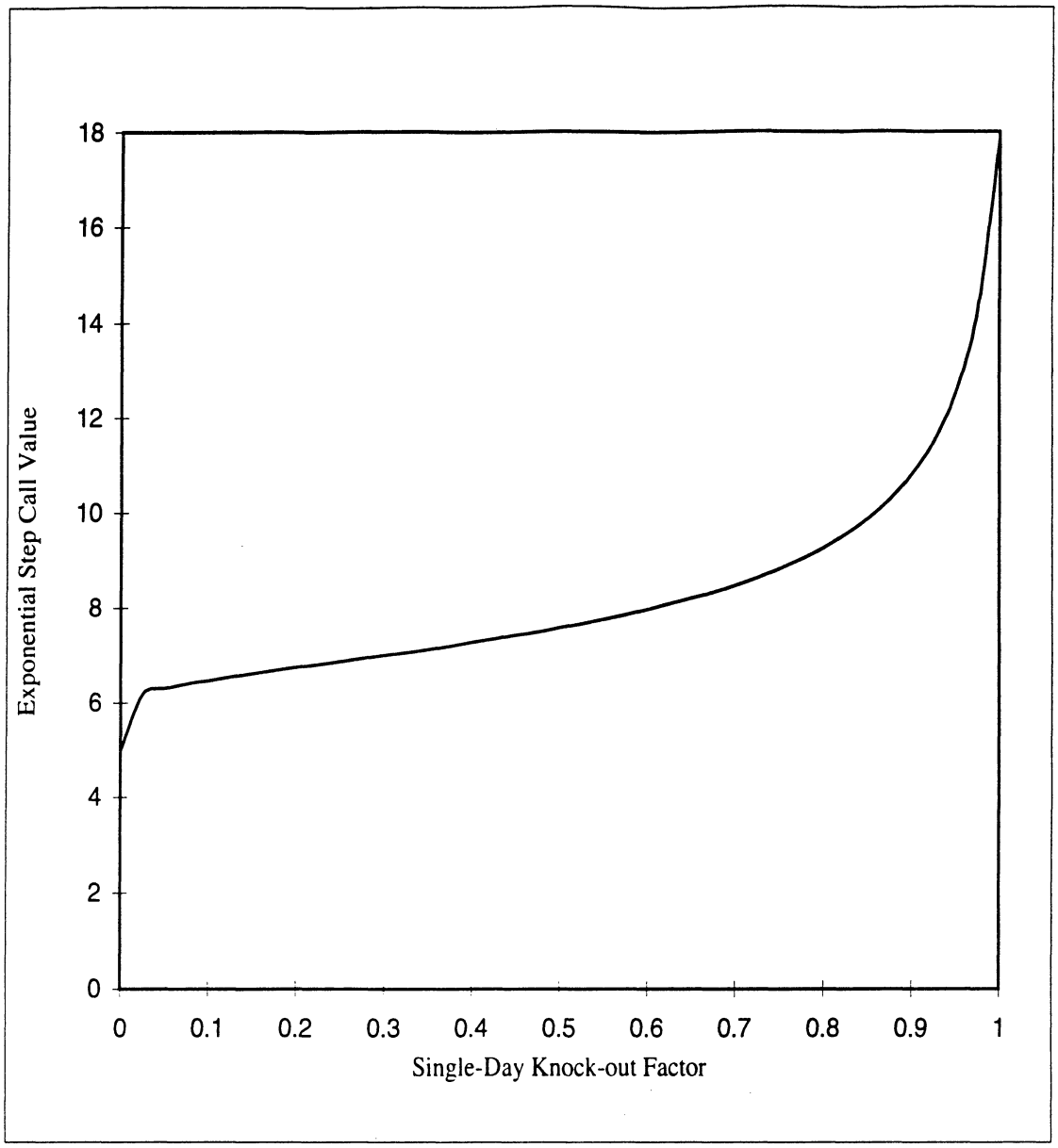

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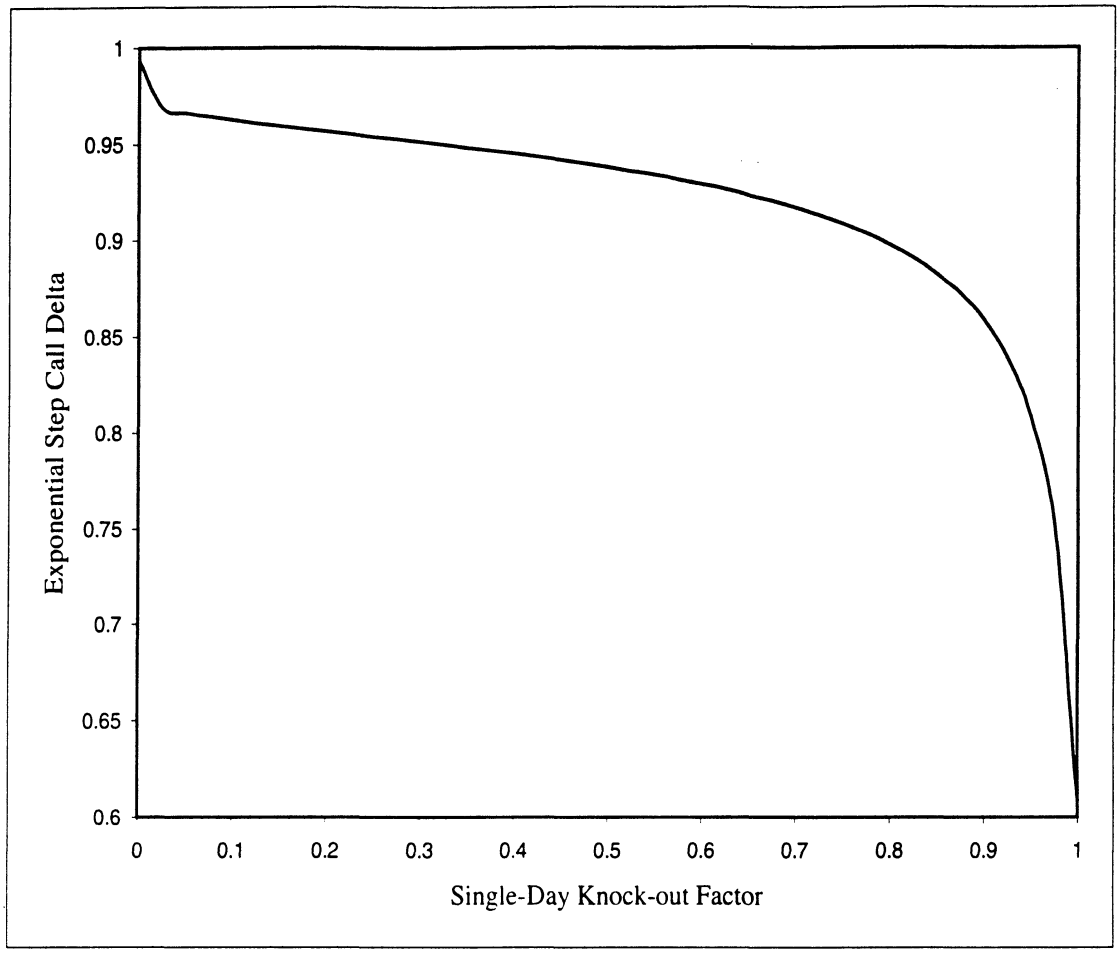

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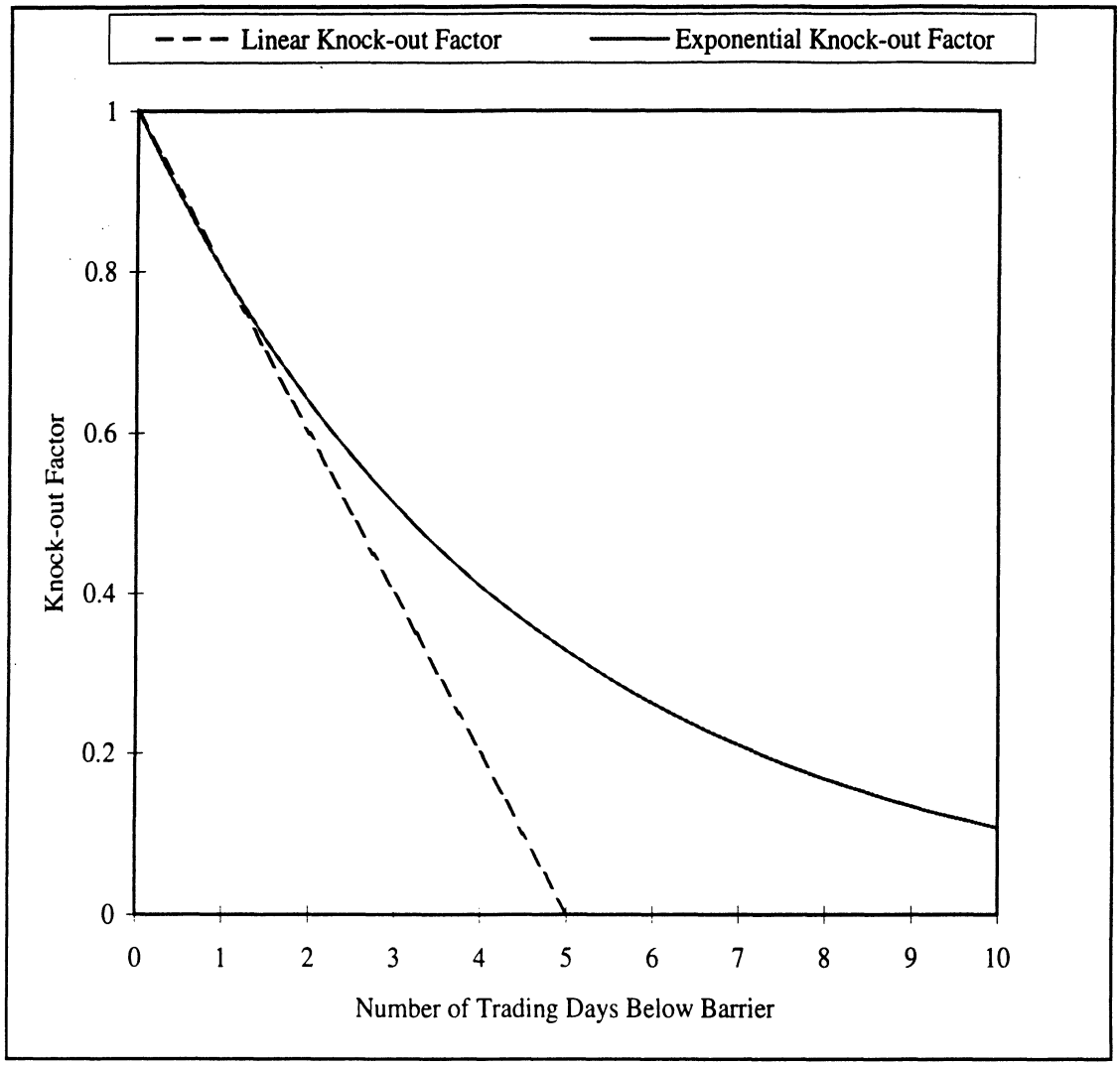

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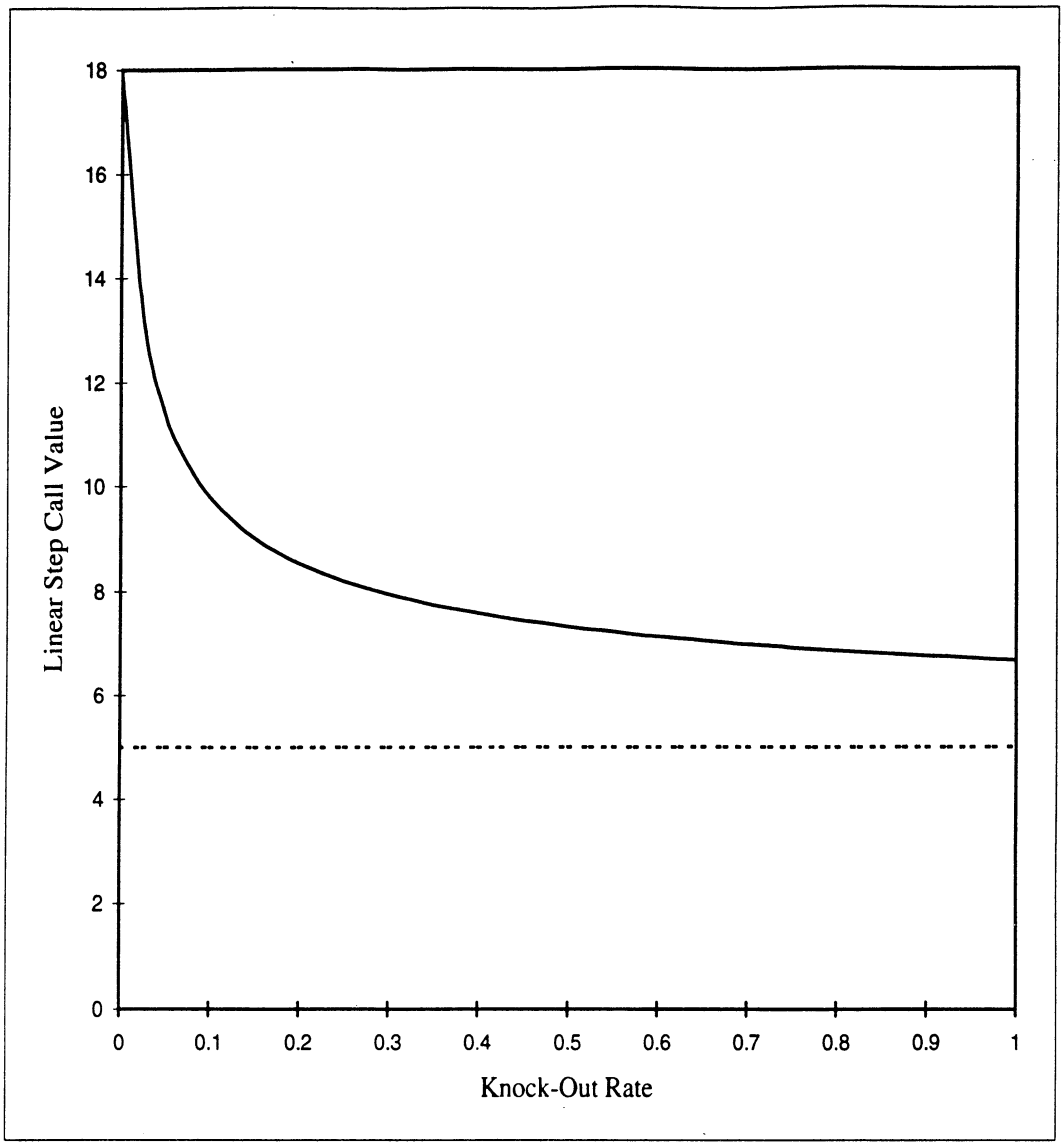

Figure 7. Down-and-out linear step call value as a function of the knock-out rate ρ_B per trading day ($\rho_B = R_B/250$). Option parameters: $S = 100$, $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months).

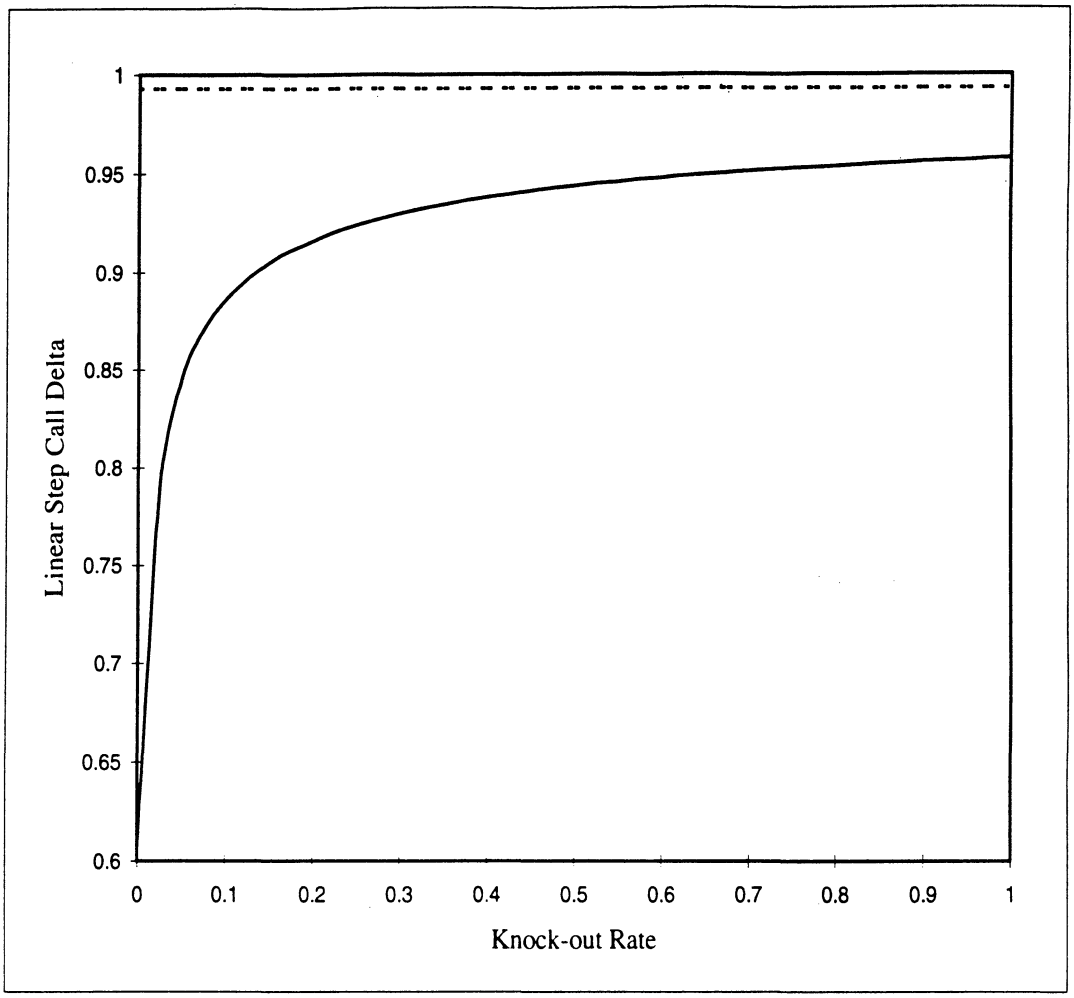

Figure 8. Down-and-out linear step call delta as a function of the knock-out rate ρ_B per trading day. Option parameters: $S = 100$, $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months).

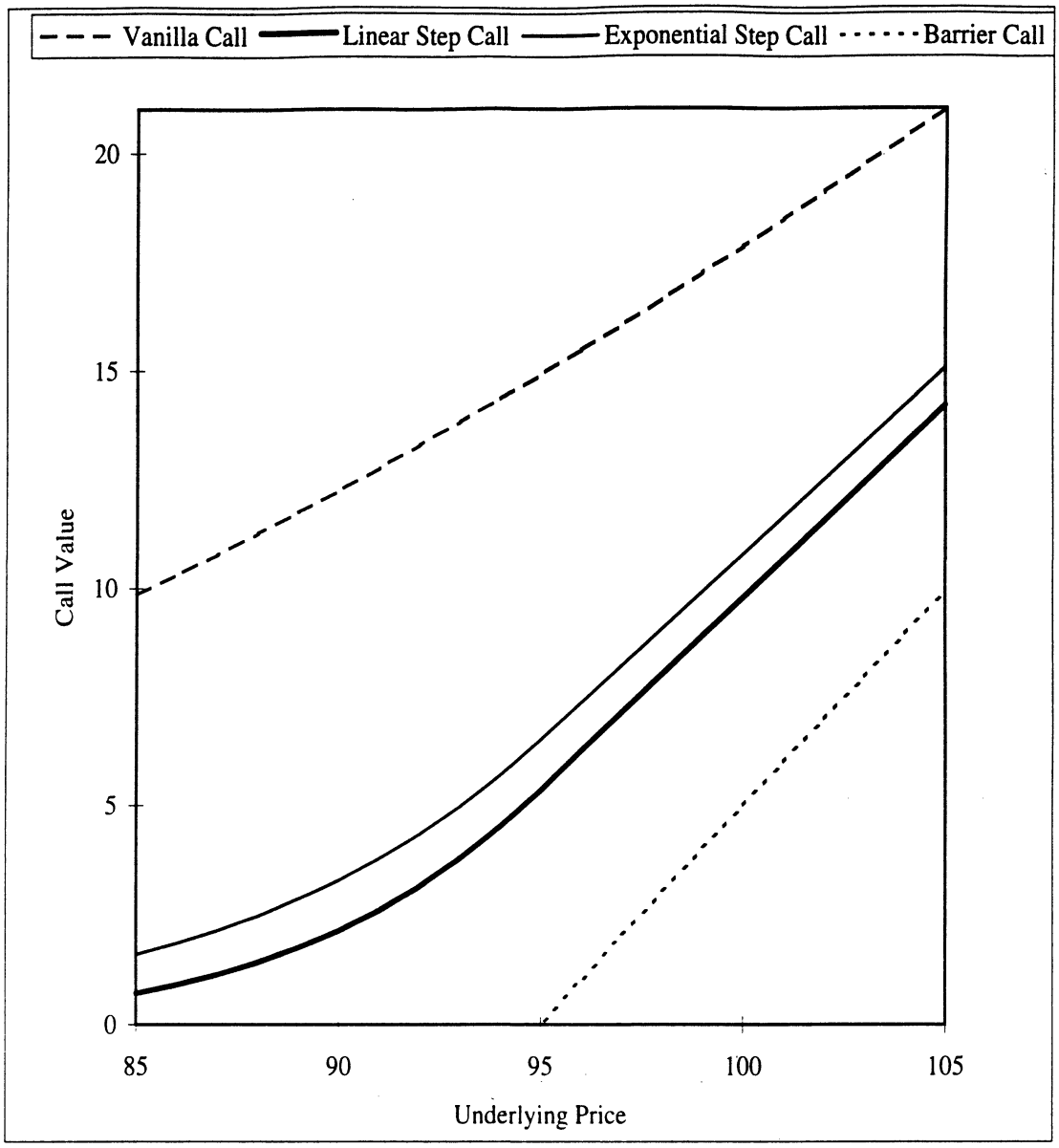

Figure 9. Vanilla, down-and-out exponential step, linear step and barrier call values as functions of the current asset price S. Option parameters: $S = 100$, $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months). Exponential step call parameters: $\beta_B = 0.9$ ($r_B = 26.34$, $T_B = 21.85$ trading days). Linear step call parameters: $\rho_B = 0.1$ ($R_B = 25$, $T_B = 10$ trading days, $\beta_B = 0.9$).

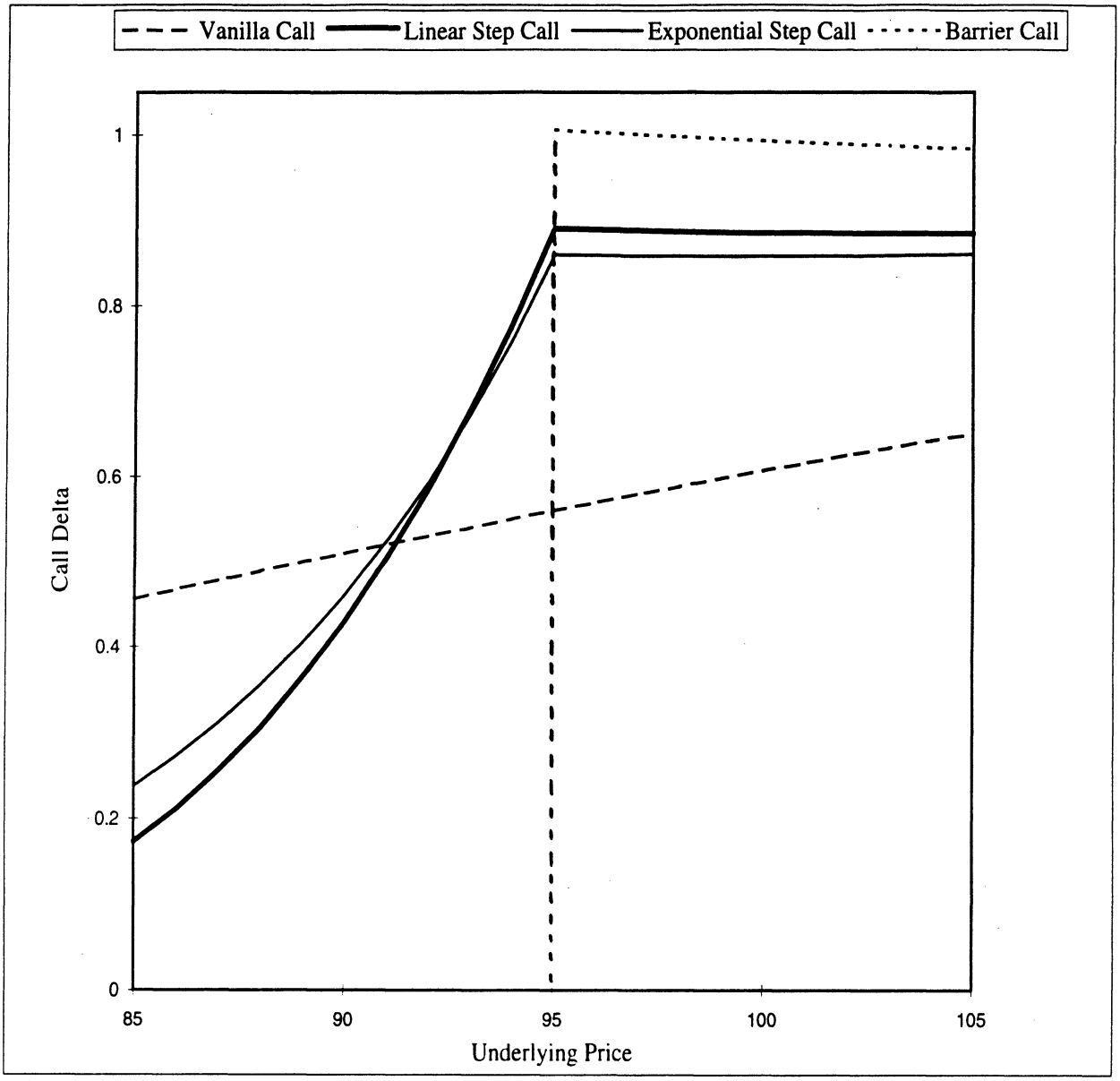

Figure 10. Vanilla, down-and-out exponential step, linear step and barrier call deltas as functions of the current asset price S. Option parameters: $S = 100$, $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months). Exponential step call parameters: $\beta_B = 0.9$ ($r_B = 26.34$, $T_B = 21.85$ trading days). Linear step call parameters: $\rho_B = 0.1$ ($R_B = 25$, $T_B = 10$ trading days, $\beta_B = 0.9$).

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