Steps to the Barrier

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Technical Report 96-19

December 1996

IOE Technical Report 96-19

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Abstract

Motivated by risk management problems with barrier options, we propose a flexible modification of the standard knock-out provision and introduce a new family of path-dependent options: *step options*. They are parametrized by a *finite knock-out rate* and can be thought of as *gradual knock-out options*. The terminal payoff is discounted with knock-out rate based on the occupation time below a pre-specified barrier level. Standard barrier options are recovered in the limit of an infinitely high knock-out rate.

December 1996

[1] *Acknowledgements.* The author thanks John Birge, Jim Bodurtha, Larry Eisenberg, Vladimir Finkelstein and Eric Reiner for useful discussions and encouragement.

They (knock-out options) relate to ordinary options the way crack relates to cocaine. Do you think they should be banned? *Yes. I would not have said that a few months ago, when I testified before Congress, but we have had a veritable crash in currency markets since then. As I have said before, knock-out options played the same role in the 1995 yen explosion as portfolio insurance did in the stock market crash of 1987, and for the very same reason. Portfolio insurance was subsequently rendered inoperable by the introduction of the so-called circuit breakers. Something similar needs to be done now with knock-out options.*
—George Soros (1995, p. 107)

1 Introduction

Barrier options have become increasingly popular over the last several years in over-the-counter options markets. A large variety of barrier options are currently traded in foreign exchange, equity and fixed-income markets. The popular knock-out options are extinguished (knocked out) when the price of the underlying asset hits a pre-specified price level (barrier) from above (below) for down-and-out (up-and-out) options. Closed-form pricing formulas for all eight types of barrier options are given by Rubinstein and Reiner (1991). Knock-out options are often attractive to investors because they are cheaper than vanilla contracts. By including a barrier provision in the option contract, an investor can eliminate paying for those scenarios he feels are unlikely, and the reduction in premium can be very substantial, especially when volatility is high.

However, these benefits come at a significant cost. The *discontinuity* at the barrier inherent in knock-out contracts creates serious risk management problems both for option buyers and sellers. An erroneous price movement through the barrier can extinguish the option, leaving the buyer without his position. Even if the investor is generally right on market direction, *an accidental price spike can lead to the loss of his entire investment.* Furthermore, when large positions of barrier options with the same barrier level are accumulated in the market, traders can drive the underlying to the barrier, thus creating massive losses by triggering the barriers.

This is vividly illustrated by the recent events in the foreign exchange market. According to the Wall Street Journal[2], "... knock-out options can roil even the mammoth foreign-exchange markets for brief periods." WSJ continues, "Most foreign-exchange traders now take it for granted that once in a while you will get a little extra kick in the price movement from a large number of options in the market." "For example, David D. Hale, chief economist at Kemper Financial Cos. in Chicago, notes that in the past year, many Japanese exporters moved to hedge against a falling dollar with currency options. Confident at the time that the dollar would fall no further than 95 yen, the exporters chose options that would knock out at that level. Once the dollar plunged through 95 yen

[2] *Do Knock-out Options Need to Be Knocked Out?*, Wall Street Journal, May 5, 1995.

early last month, they lost everything, he says. The dollar then tumbled as the Japanese companies, which had lost their hedges, scrambled to cover their large exposures by dumping dollars." "Making matters more volatile, dealers say that pitched battles often erupt around knock-out barriers, with traders hollering across the trading floor of looming billion-dollar transactions. ... In three or four minutes it is all over. But in that time every trade gets sucked into the vortex." This situation prompted some market participants to appeal to regulators on the necessity to regulate knock-out options. George Soros went as far as to suggest a ban on knock-out options. An excerpt from his recent book Soros (1995) serves as an epigraph to this paper.

The barrier option's delta is discontinuous at the barrier, thus creating serious hedging problems for options sellers as well. To hedge barrier options, dealers establish positions in a series of standard vanilla options which provide a good hedge for a wide range of underlying prices. However, when the underlying nears the barrier level, these static hedges need to be rebalanced, which results in a flurry of trading activity in vanilla options. This, in turn, results in further trading activity in the underlying asset as dealers who sold vanilla options to hedgers of knock-out options need to dynamically hedge their exposure (see Malz (1995)). This increases market volatility around popular barrier levels and increases the cost of hedging barrier options.

Thus it is desirable *to modify the barrier provision in such a way as to retain as much of the premium savings afforded by the standard barrier provision as possible, but at the same time to achieve continuity of both the option's payoff and delta at the barrier.* This would alleviate many of the risk management problems with standard barriers.

2 Step Options: A No-Regrets Alternative to Barrier Options

Consider a standard call with strike K and time to expiration $\tau = T - t$ (t and T denote the contract inception and expiration times, respectively). A down-and-out provision renders the option worthless as soon as the underlying price hits a pre-specified barrier B. Accordingly, the payoff of a down-and-out call can be written as[3]

$$DAO(S_T, T) = \mathbf{1}_{\{L_T > B\}} \, Max(S_T - K, 0), \tag{1}$$

where S_T is the underlying price at expiration, L_T is the lowest price of the underlying between inception t and expiration T, and $\mathbf{1}_{\{L_T > B\}}$ is an indicator function equal to one if $L_T > B$ and zero otherwise.

The stochastic model of barrier options is that of Brownian motion *instantaneously killed*

[3]For brevity in this paper we discuss down-and-out calls only. Further details are given in Linetsky (1996a).

as soon as the barrier is hit. To eliminate the discontinuity, we suggest to consider Brownian motion *with killing at finite rate* below the barrier (see, e.g., Karlin and Taylor (1981)) [4]. We modify the payoff at expiration by introducing a finite *knock-out rate* r_B and defining the payoff by the formula (Linetsky (1996a))

$$\exp(-r_B\tau_B^-)\,Max(S_T-K,0), \tag{2}$$

where τ_B^- is *the total time during the life of the option the underlying price is lower than the barrier level B*. It is called *occupation time* of the underlying process (see, e.g., Karlin and Taylor (1981)). Figure 1 illustrates the calculation of τ_B^-. At expiration, first the payoff of an otherwise identical vanilla option is determined, and then discounted at continuously compounded rate r_B for the time the underlying spent below the barrier during the option's lifetime. We call the discount factor $\exp(-r_B\tau_B^-)$ *knock-out factor.* It is defined on price paths and can be represented in the form:

$$\tau_B^- = \int_t^T \theta(B-S(t'))dt',\ \exp(-r_B\tau_B^-) = \exp\left(-\int_t^T r_B\,\theta(B-S(t'))dt'\right), \tag{3}$$

where $\theta(x)$ is *the Heaviside step function* defined by

$$\theta(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \tag{4}$$

The integrand $r_B\theta(B-S)$ is called *step potential.* It has the shape of a step of the staircase of height r_B (Figure 2). Accordingly, we call a one-parameter family of path-dependent options with the payoff (2) *step options.* It is easy to see that the payoff (2) coincides with the standard European call $Max(S_T-K,0)$ in the limit of zero knock-out rate, and tends to the payoff of an otherwise identical barrier option (1) in the limit of an infinitely high knock-out rate. Thus we have a sequence of step options approaching the barrier option - hence the title for this paper. [5]

A holder of a down-and-out step option is penalized at rate r_B for the time the underlying price is below the barrier. For a standard knock-out, the knock-out rate is infinitely high and the entire option payoff is *instantaneously* lost by the option holder should the underlying price hit the barrier even momentarily. For any finite knock-out rate, however, it takes some time below the barrier to reduce the option payoff to close to zero: the option knocks out *gradually.* We define an *effective knock-out time*, $\mathcal{T}_B^-$, as time below the barrier needed to reduce the terminal payoff of a down-and-out step option by 90%, $\exp(-r_B\mathcal{T}_B^-) = 0.1$. That is, the holder of a down-and-out step call receives only 10% of

[4]Another interpretation in quantum mechanics is that of a quantum particle in the infinitely high potential barrier. Here we suggest to consider a *finite* rather than the *infinite* potential barrier, or a *step potential of finite height* (see, e.g., Messiah, A. (1961): *Quantum Mechanics*, Vol.1, North-Holland Publishing, Amsterdam, or Landau, L.D. and E.M. Lifshits (1965): *Quantum Mechanics*, 2nd Ed., Pergamon, Oxford).

[5]I am grateful to Jim Bodurtha for suggesting this title.

the payoff of an otherwise identical vanilla call at expiration if the asset spent time $\mathcal{T}_B^-$ below the barrier during the life of the contract. Another useful measure of knock-out speed is a *single-day knock-out factor* β_B, $\beta_B = \exp(-r_B/250)$ (we assume 250 trading days per year). This is a factor by which the terminal payoff is discounted for every trading day the underlying spends below the barrier. Obviously, β_B is zero for barrier options and unity for vanilla options.

One of the major advantages of step options is the ability to structure contracts with any desired knock-out rate. By choosing a finite rate, an option buyer assures himself that the option will never lose its entire value due to a short-term price movement. An investor can customize the option by selecting an appropriate knock-out rate according to his risk aversion and the degree of confidence in the barrier not being hit during the option's lifetime. At the same time, it is more advantageous for the dealer to sell a step option rather than the standard barrier option, since the step option delta is continuous at the barrier allowing to hedge by continuously trading the underlying. Thus, *step options with finite knock-out rates (gradual knock-outs) have significant risk management advantages both for the buyer and for the seller.* Let us also mention another potential positive effect of introducing step options into the marketplace. Since different market participants will select different knock-out rates, even though they may all set the barrier at the same obvious support or resistance level, no short-term manipulation by traders will result in massive simultaneous knock-outs. This would reduce the volatility around barrier levels.

One may also wish to consider other payoff structures of the form:

$$f(\tau_B^-)\, Max(S_T - K, 0), \tag{5}$$

where f is a given discount function of the occupation time (we assume $f(0) = 1$). We call options with the payoff (5) *general step options.* Our *exponential step call* (2) is a particularly simple example with $f = exp(-r_B\tau_B^-)$. It corresponds to continuous compounding at knock-out rate $(-r_B)$ below the barrier. Another practically interesting choice is simple interest without compounding[6]:

$$Max(1 - R_B\tau_B^-, 0)\, Max(S_T - K, 0). \tag{6}$$

We call options with the payoff (6) *linear step options.* The optionality in occupation time is needed to limit the option buyer's liability to not more than the premium paid for the option. Thus, linear step options are options both on the terminal asset price, as well as occupation time. A *knock-out time* $\mathcal{T}_B^-$ for the linear step option is defined as the minimum occupation time below the barrier needed to reduce the option payoff to zero: $1 - R_B\mathcal{T}_B^- = 0$, $\mathcal{T}_B^- = 1/R_B$. The option buyer just needs to specify the desired knock-out time as a fraction of time to maturity.

[6]I am grateful to Vladimir Finkelstein (J.P. Morgan) and Eric Reiner (UBS) for suggesting to consider this payoff structure

3 Closed-Form Pricing Formulas

To value step options we assume the underlying asset price follows a geometric Brownian motion with constant drift rate and volatility σ, there is no payout on the underlying, and we live in the Black-Scholes world with constant continuously compounded risk-free interest rate r. According to the risk-neutral valuation approach (see, e.g., Hull (1996) and Duffie (1996)), at inception t the present value of an option with payoff (5) is given by the discounted average over the risk-neutral measure conditional on the initial asset price S at time t:

$$e^{-r\tau} E_{(t,S)} \left[f(\tau_B^-) \, Max(S_T - K, 0) \right]. \tag{7}$$

In Linetsky (1996a) we calculated this average in closed form for any function f by employing the Feynman-Kac approach for pricing path-dependent derivatives developed in Linetsky (1996b). First, we derive a closed-form expression for the joint probability density $\mathcal{P}(S_T, \tau', \tau | S)$ of the terminal asset price and occupation time $\tau' = \tau_B^-$ conditional on the initial price S at time t. Then the average (7) reduces to:

$$e^{-r\tau} \int_0^\infty \int_0^\tau f(\tau') \, Max(S_T - K, 0) \mathcal{P}(S_T, \tau', \tau | S) \, d\tau' dS_T. \tag{8}$$

The resulting closed-form pricing formula for the down-and-out step call is (Linetsky (1996a)): <u>(I) initial price is above the barrier, $S \geq B$</u>

$$Step(S,t) = DAO(S,t) \tag{9}$$

$$+ \left(\frac{B}{S}\right)^\gamma \int_0^\tau \frac{F(\tau - \tau')e^{-\alpha(\tau-\tau')}}{\sqrt{2\pi}(\tau - \tau')^{3/2}} \left(\nu_2 \left(\frac{B^2}{S}\right) N(d_4') - \nu_1 e^{-r\tau'} K N(d_3') \right) d\tau',$$

where $DAO(S,t)$ is the standard down-and-out call (Rubinstein and Reiner (1991))

$$DAO(S,t) = C(S,t) - \left(\frac{B}{S}\right)^\gamma C\left(\frac{B^2}{S}, t\right), \; C(S,t) = SN(d_2) - e^{-r\tau} K N(d_1), \tag{10}$$

$C(S,t)$ - vanilla call given by the Black-Scholes formula, $N(x)$ and $n(x)$ are the cumulative standard normal distribution function and its density, respectively, and we introduced the following notations:

$$\mu = r - \frac{\sigma^2}{2}, \; \gamma = \frac{2\mu}{\sigma^2}, \; \alpha = r + \frac{\mu^2}{2\sigma^2}, \; \nu_1 = \frac{\mu}{\sigma}, \; \nu_2 = \nu_1 + \sigma, \tag{11}$$

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \mu\tau}{\sigma\sqrt{\tau}}, \; d_2 = d_1 + \sigma\sqrt{\tau}, \; d_3' = \frac{\ln\left(\frac{B^2}{SK}\right) + \mu\tau'}{\sigma\sqrt{\tau'}}, \; d_4' = d_3' + \sigma\sqrt{\tau'};$$

<u>(II) initial price is below the barrier, $S < B$</u>

$$Step(S,t) = \left(\frac{B}{S}\right)^{\frac{\gamma}{2}} \int_0^\tau \frac{F(\tau - \tau')e^{-\alpha(\tau-\tau')}}{\sqrt{2\pi}(\tau - \tau')^{3/2}} \tag{12}$$

$$\times \left\{ \nu_1 \rho_1(y) e^{-r\tau'} K N(d_5') - \nu_2 \rho_2(y) B N(d_6') - \sigma y B (\tau')^{-\frac{1}{2}} n(d_6') \right\} \exp\left(-\frac{y^2}{2(\tau - \tau')} \right) d\tau',$$

where we introduced the following notations:

$$y = \frac{1}{\sigma} \ln\left(\frac{S}{B}\right), \; \rho_1(y) = \frac{y^2}{\tau - \tau'} + \nu_1 y - 1, \; \rho_2(y) = \rho_1(y) + \sigma y, \tag{13}$$

$$d_5' = \frac{\ln\left(\frac{B}{K}\right) + \mu \tau'}{\sigma\sqrt{\tau'}}, \; d_6' = d_5' + \sigma\sqrt{\tau'}.$$

The function $F(\tau - \tau')$ in (9), (12) is obtained from the discount function f in (5) by integration:

$$F(\tau - \tau') = \int_0^{\tau - \tau'} f(\tau'')\, d\tau'' \tag{14}$$

and for the exponential (2) and linear (6) step options it is given by:

$$F_{exp}(\tau - \tau') = \frac{1 - e^{-r_B(\tau - \tau')}}{r_B}, \tag{15}$$

$$F_{lin}(\tau - \tau') = \begin{cases} \frac{1}{2R_B}, & 0 \le \tau' \le \tau - \frac{1}{R_B} \\ (\tau - \tau')\left[1 - \frac{R_B}{2}(\tau - \tau')\right], & \tau - \frac{1}{R_B} < \tau' \le \tau \end{cases} \tag{16}$$

We see from Eq.(9) that the step option price consists of two parts: the standard down-and-out call and a *step premium* investors have to pay for the privilege of having the option *knock out gradually with pre-specified finite rate.* The higher the knock-out rate, the lower the premium. In the limit $r_B \to \infty$ ($R_B \to \infty$) the option knocks out instantly as soon as the barrier is hit, and the step premium is equal to zero. The lower the knock-out rate, the higher the premium, and in the limit of zero knock-out rate the premium is the highest and the step option coincides with an otherwise identical vanilla option. The knock-out rate *controls the trade-off between premium savings and knock-out speed.*

A crucial property of step options is that, unlike standard barrier options, *the delta is continuous at the barrier for any finite knock-out rate* r_B (Linetsky (1996a)), allowing to dynamically replicate step options by continuously trading the underlying asset and borrowing.

Our closed-form pricing formulas are easy to implement numerically with essentially any degree of accuracy. All computations in the following example were performed in Maple V on the Pentium PC and took only seconds per option. The option parameters in our example are: $K = 100$, $B = 95$, $\tau = 0.5$ (six months), $\sigma = 0.6$ and $r = 0.05$. The continuously compounded knock-out rate r_B for the exponential step call is chosen so that the single-day knock-out factor is $\beta_B = 0.9$ ($r_B = -250 \ln \beta_B = 26.34$), i.e. 10% of the payoff is lost in the first trading day below the barrier. The corresponding effective knock-out time is about 22 days, i.e., taking into account the effect of compounding, 90%

S	C	Δ	$EStep$	Δ	$LStep$	Δ	DAO	Δ
85	9.8517	0.4554	1.6062	0.2376	0.7200	0.1730	0	0
90	12.2641	0.5091	3.2951	0.4602	2.1528	0.4291	0	0
95	14.9373	0.5597	6.5008	0.8598	5.3548	0.8908	0	1.0058
100	17.8551	0.6068	10.7942	0.8583	9.7953	0.8862	4.9958	0.9932
105	20.9994	0.6503	15.0904	0.8607	14.2229	0.8855	9.9376	0.9841

Table 1: Call values and deltas as functions of the underlying price S.

of the payoff is lost if the underlying spends 22 trading days below the barrier during the contract's life. For comparison, the simple knock-out rate R_B for the linear step call is also chosen so that 10% of the payoff is lost in the first day below the barrier, $R_B = 250(1 - \beta_B) = 25$. Since in this case there is no compounding, the linear option knocks out faster and is extinguished after 10 days below the barrier. Table 1 and Figures 3 and 4 show vanilla (C), exponential step (EStep), linear step (LStep) and standard barrier (DAO) call values and deltas as functions of the underlying price S. Figure 3 shows that the step option value holds well when the underlying falls slightly below the barrier, but deteriorates quickly as the underlying continues to fall further, as the probability of getting back up above the barrier decreases and expected value of the occupation time below barrier increases. Figure 4 illustrates continuity of the step option delta at the barrier.

In summary, we believe that their hedging properties make step options a potentially valuable addition to the derivatives toolkit. They may offer *an attractive alternative to standard barrier options* in a number of situations. The finite knock-out rate eliminates the discontinuity and effectively plays a role of the *circuit breakers* called for by Soros by making the option's payoff and delta continuous at the barrier and thus alleviating many of the risk management problems with standard barriers.

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4 List of Figures

Figure 1. Calculation of the total time the asset price spends below the pre-specified barrier level B.

Figure 2. The step potential.

Figure 9. Vanilla, down-and-out exponential step, linear step and barrier call values as functions of the current asset price S. Option parameters: $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months). The exponential step call parameters: $\beta_B = 0.9$ ($r_B = 26.34$, $\mathcal{T}_B^- = 21.85$ trading days). The linear step call parameters: $\rho_B = 0.1$ ($R_B = 25$, $\mathcal{T}_B^- = 10$ trading days, $\beta_B = 0.9$).

Figure 10. Vanilla, down-and-out exponential step, linear step and barrier call deltas as functions of the current asset price S. Option parameters: $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months). Exponential step parameters: $\beta_B = 0.9$ ($r_B = 26.34$, $\mathcal{T}_B^- = 21.85$ trading days). Linear step parameters: $\rho_B = 0.1$ ($R_B = 25$, $\mathcal{T}_B^- = 10$ trading days, $\beta_B = 0.9$).

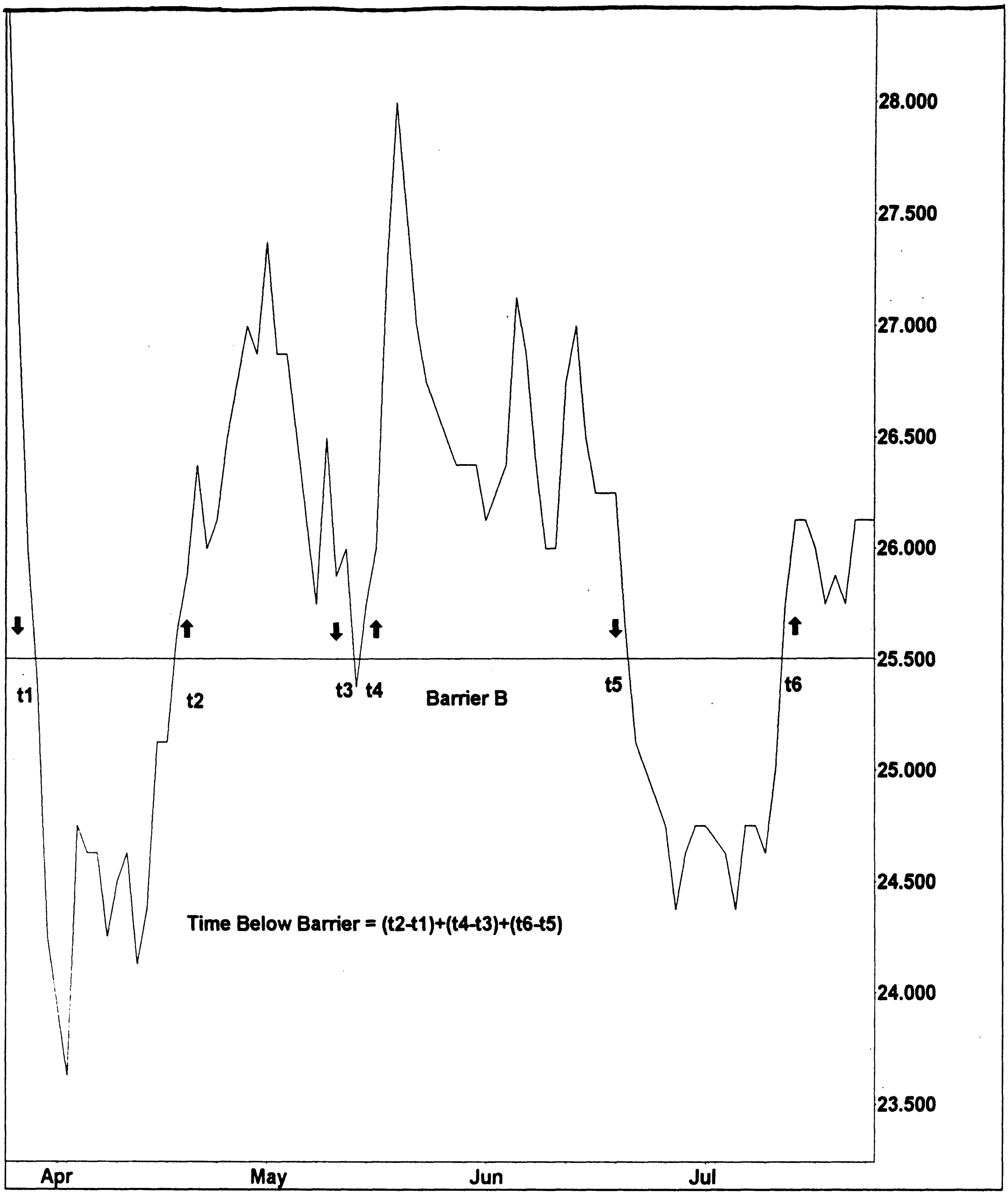

Figure 1. Calculation of occupation time below a specified barrier level *B*.

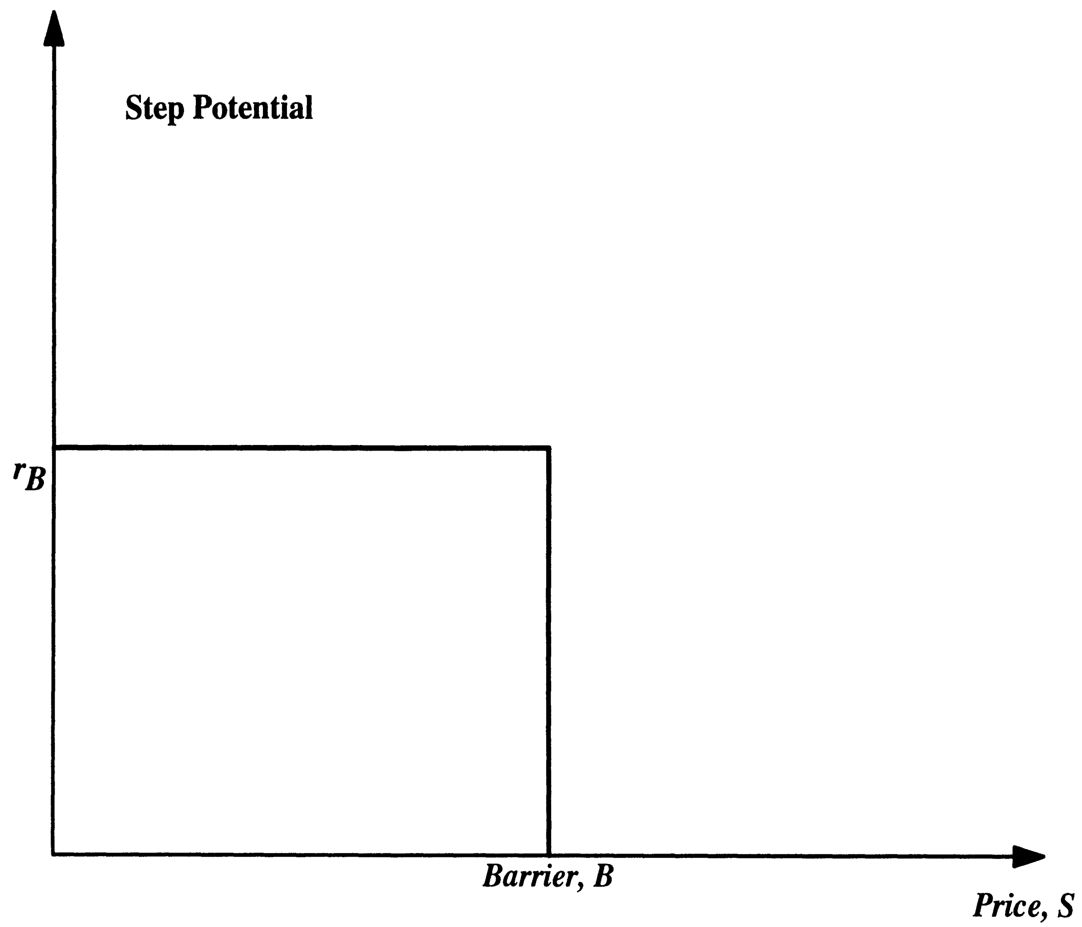

Figure 2. The step potential.

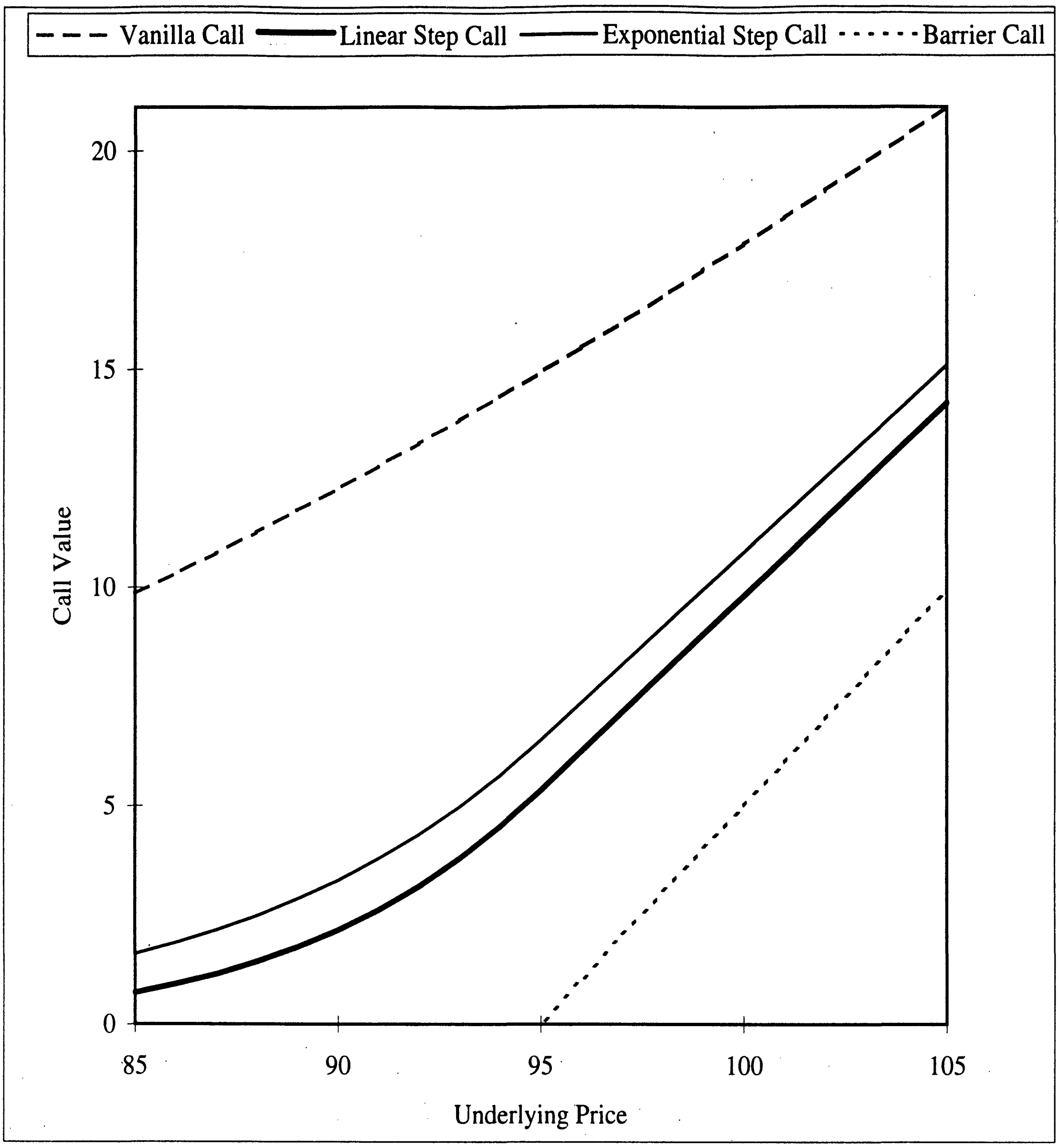

Figure 3. Vanilla, down-and-out exponential step, linear step and barrier call values as functions of the current asset price S. Option parameters: $S = 100$, $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months). Exponential step call parameters: $\beta_B = 0.9$ ($r_B = 26.34$, $T_B = 21.85$ trading days). Linear step call parameters: $\rho_B = 0.1$ ($R_B = 25$, $T_B = 10$ trading days, $\beta_B = 0.9$).

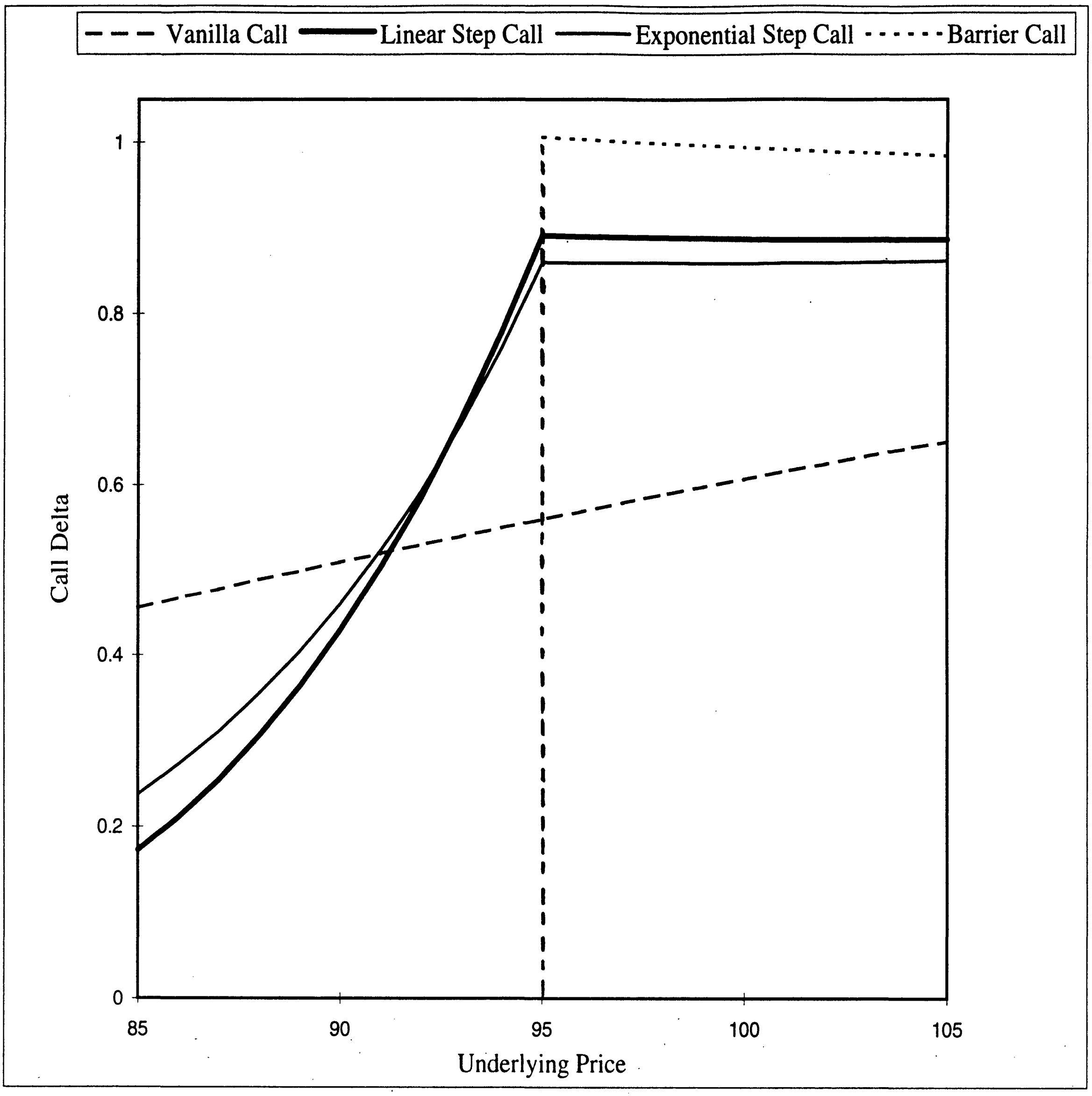

Figure 4. Vanilla, down-and-out exponential step, linear step and barrier call deltas as functions of the current asset price S. Option parameters: $S = 100$, $K = 100$, $B = 95$, $\sigma = 0.6$, $r = 0.05$, $\tau = 0.5$ (six months). Exponential step call parameters: $\beta_B = 0.9$ ($r_B = 26.34$, $T_B = 21.85$ trading days). Linear step call parameters: $\rho_B = 0.1$ ($R_B = 25$, $T_B = 10$ trading days, $\beta_B = 0.9$).