ERRATA

Page	Line	Should Read
3	4th from bottom	results
35	2nd from bottom	sending
38	bottom	arcs........and
57	14	$D(z,Q_\eta') \leq \underline{\lim}_{p\to\infty} D(z_p,Q_\eta') \leq \underline{\lim}_{p\to\infty} D(z_p,Q) \leq K^2 < +\infty$,
81	2	$\{\alpha'_{i_s}\}$
81	3	$\{\alpha''_{i_r}\}$
81	bottom	$z(\alpha''_{i_r})$
94	2nd from bottom	$F^2(\zeta,J_\zeta) \to F^2(z_0,J_{z_0})$
103	4	$\sum_{i=1}^{3}\{\lVert\zeta^i_{n_u} - z^i_{n_u}\rVert L_2 \lVert\zeta^i_{n_v}\rVert L_2 + \lVert\zeta^i_{n_v} - z^i_{n_v}\rVert L_2 \lVert z^i_{n_u}\rVert L_2\}$
106	3,5	semicontinuous
109	10	condition
126	entry 9.	Surface Area, Princeton University Press

THE UNIVERSITY OF MICHIGAN
COLLEGE OF LITERATURE, SCIENCE, AND THE ARTS
Department of Mathematics

Technical Report

FREE BOUNDARY PROBLEMS IN THE CALCULUS OF VARIATIONS

Leonard J. Lipkin

ORA Project 07100

under contract with:

NATIONAL SCIENCE FOUNDATION
GRANT NO. GP-3920
WASHINGTON, D.C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION ANN ARBOR

April 1965

This report was also a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in The University of Michigan, 1965.

ACKNOWLEDGMENTS

I wish to thank Professor Lamberto Cesari for his teaching and guidance throughout the preparation of this dissertation. I also wish to thank Professor Maxwell O. Reade for his reading of the manuscript. There are others, far too numerous to mention, who have made profound contributions to my education, both at Oberlin College and at The University of Michigan. To these people I express my deepest appreciation.

This research was supported in part by National Science Foundation Grant GP-57 at The University of Michigan.

TABLE OF CONTENTS

ABSTRACT

The purpose of this thesis is to study the problem of existence of an absolute minimum for two-dimensional integrals of the calculus of variations when the admissible surfaces satisfy certain variable boundary conditions. R. Courant has studied the existence theory for the special case of the Dirichlet integral, but for general integrals only some classical necessary conditions are known.

For the non-parametric problem, we study integrals of the form $I(z) = \int_B f(x,y,z,z_x,z_y)dxdy$, where the integrand is continuous, convex in (p,q), and satisfies the usual growth conditions. For admissible functions we take those which are continuous on a closed disk B, absolutely continuous in the sense of Tonelli on the interior of B, which map a fixed subarc γ of the boundary B^* of B onto a fixed arc Γ in E^3, and such that the values on the complement of γ in B^* are bounded by fixed constants. Thus, these are surfaces whose boundaries have a portion spanning Γ, and the complementary portion is free on a fixed finite cylinder. By a convenient generalization of a process of L. Tonelli for smoothing surfaces, we prove that there is a function in this class which yields an absolute minimum for the integral. We are then able to extend this result to the case in which the boundaries of admissible surfaces are completely free in a fixed cylinder.

Next we consider the same problem when the cylinder is replaced by a "capstan" surface, that is, a surface of revolution generated by a portion of a curve of the shape of, say, a branch of a hyperbola. The admissible surfaces are now defined on different Jordan domains contained in the disk B and containing a fixed disk D. We apply the Carathéodory theory of conformal mappings on variable domains to convert a minimizing sequence into a new minimizing sequence of functions all defined on the unit disk. We show that one of these sequences admits a uniformly convergent subsequence, and then by a conformal mapping we show that the image of the limiting function is again admissible, it is defined on the kernel domain, and it yields an absolute minimum.

In the parametric problem we study integrals of the form $I_O(z,G) = \int_G F(z,J)dG$, where $z(u,v)$ is a vector (surface) in E^3, J is the corresponding Jacobian vector, and the function $F(z,J)$ is a usual parametric integrand. We let τ be a fixed torus, or similar manifold, in E^3, and for admissible vectors we take continuous ones in the Sobolev space $W_2^1(G)$ which span τ and "cover the hole" of the torus. Since Courant has shown that a minimizing vector need not have a continuous trace on the fixed manifold, we say that a surface spans τ if for every sequence of points in the parameter domain approaching a point of the boundary, the corresponding sequence of points on the surface approaches τ. In order to say precisely what it means to "cover the hole," we prescribe a topological linking condition.

For the existence proof, we introduce a class of sequences $\{z_n\}$ of surfaces which are admissible, except that their boundaries approach τ in the limit rather than lie on τ. The number $\delta = \inf (\underline{\lim}_{n\to\infty} I_0(z_n,G))$, where the infimum is taken over all these sequences, is (possibly) smaller than the infimum of $I_0(z,G)$ over all admissible surfaces. We introduce an integral $I(z,G)$, following C. B. Morrey, which dominates $I_0(z,G)$ and agrees in case the surface is quasi-conformal. Minimizing $I(z,G)$ in suitable classes $\mathcal{F}(K)$, we obtain a sequence $\{z_K\}$ of vectors such that $I(z_K,G) \to \delta$ and $I_0(z_K,G) \to \delta$. We are then able to show that a subsequence $\{z_p\}$ converges to an admissible vector z_0. Appealing to a lower semicontinuity theorem of L. Cesari and L. Turner, z_0 yields an absolute minimum for $I_0(z,G)$.

INTRODUCTION

The problem which will be discussed in this thesis is that of proving the existence of minimizing functions for integrals of the calculus of variations when the boundary values for admissible functions are partly fixed and partly free, or totally free in a fixed manifold. More specifically, in the non-parametric case, consider a finite (closed) cylinder over a circle B^* and a set of functions $z = z(x, y)$ defined on the disk B bounded by B^*, with the requirement that the graph of z traces out a continuous curve lying in the cylinder. Then we seek a class of such functions with the property that a given integral of the form

$$I(z) = \int_B F(x, y, z, z_x, z_y)\, dx\, dy$$

assumes an absolute minimum in this class.

Alternatively, on a fixed subarc γ of the boundary of B we may prescribe continuous boundary values which each function $z = z(x, y)$ is to assume, and on the complementary arc we allow the values of $z(x, y)$ to be continuous, but otherwise free in the cylinder.

Instead of a cylinder, one may ask the same questions for, say, a capstan-shaped surface (Chapter I, Definition 3.1), or for portions of such surfaces.

In the parametric case we admit more general manifolds: for example, a torus, or a deformation of a cylinder or torus. Here, of course, we seek minimizing vectors in appropriate classes, and the integrals studied have the form

$$I(z, G) = \int_G F(z, J)\, du\, dv \quad ,$$

where $z(u, v) = (z^1(u, v), z^2(u, v), z^3(u, v))$, (u, v) in G, and $J = (J^1, J^2, J^3)$ is the vector of Jacobians relative to the mapping $z = z(u, v)$.

These problems have been studied from the classical point of view for a long time, that is, studied under the assumption that a minimum exists. The first work goes back to Gauss, and the expression for the first variation was originally obtained by Poisson (1833) for problems of this type. In O. Bolza's book, Vorlesungen über Variationsrechnung, there appears a brief study of necessary conditions of two-dimensional problems with variable boundaries, and there is also a guide to the early literature (page 668).

H. A. Simmons [28] obtained expressions for both the first and second variations in the case of non-parametric surfaces whose boundaries are free in a capstan-shaped manifold, and he extended his results to higher dimensional problems [29].

E. A. Nordhaus [24] generalized the problem studied by Simmons to a Bolza problem. He obtained expressions for the first and second variations, and also the transversality conditions.

For the existence theory in the case of parametric surfaces, Riemann and Schwarz asked for simply connected surfaces of relative minimum area, whose boundaries consist of one or more segments and one or more arcs on given planes. The existence theorem for minimal surfaces with partially free and partially fixed boundaries was given by R. Courant [10], and for totally free boundaries by R. Courant and N. Davids [12], and N. Davids [14]. Most of this material is summarized in the book by R. Courant [11].

One notices that although some necessary conditions are known for free boundary problems, the existence theory for general non-parametric integrals has not been studied, and little is known about the parametric problem except for the case of minimal surfaces where the integral in question is the Dirichlet integral

$$D(z, G) = \int_G (z_x^2 + z_y^2)\, dx\, dy \quad .$$

We now describe the results obtained on existence theory for general integrals in this thesis. If G is a domain in the plane, we shall denote by $\bar{G}$ its closure, by G^* its boundary, and by G^o its interior.

In Chapter I we treat the non-parametric problem in two forms. In the first we consider the class of functions $z = z(x, y)$ which are continuous and ACT (Chapter I, Definition 1.1) on a closed Jordan domain $\overline{G}$, which assume given continuous boundary values on an arc γ of the boundary G^*, and whose values on the complementary arc are bounded, but otherwise free; and therefore the corresponding portion of the boundary of the surface $S : z = z(x, y)$ lies on the cylinder over G^*. We show that there is a function in this class which minimizes the integral

$$I(z, G) = \int_G F(x, y, z, z_x, z_y)\, dx\, dy \quad , \tag{0.1}$$

where $F(x, y, z, p, q)$ is continuous in (x, y, z, p, q), convex in (p, q), and satisfies the conditions

$$F(x, y, z, p, q) \geq \nu + \mu (p^2 + q^2) \quad , \qquad \mu > 0 \quad ,$$

$$F(x, y, z, 0, 0) = f(x, y) \quad .$$

In order to prove the existence theorem, we need to show that there is a minimizing sequence which converges uniformly on the closed domain $\overline{G}$ to an admissible function. To this end, we select a minimizing sequence $\{z_n(x, y)\}$, $n = 1, 2, 3, \ldots,$ and then define a convenient generalization of a leveling process due to L. Tonelli. By this process we obtain a new minimizing sequence in which the functions do not oscillate too badly. We are then able to show that the sequence is equicontinuous on the boundary, the

Dirichlet integrals are uniformly bounded, consequently the functions are equicontinuous on $\bar{G}$, and there is a convergent subsequence. Using a lower semicontinuity theorem of L. Tonelli [31] and L. Turner [32], we prove that the limiting function yields an absolute minimum for the integral.

In the second form of the non-parametric problem, we consider those functions $z(x,y)$ defined on a closed Jordan domain $\bar{G}_z$ (depending upon z) whose boundary contains a fixed arc γ, such that $z(x,y)$ is continuous and ACT on $\bar{G}_z$, assumes given boundary values on γ, and such that the points $(x, y, z(x,y))$, (x,y) in $G^* - \gamma$, lie on a fixed capstan surface. In such a class of functions we wish to minimize the integral (0.1). This means that the problem is twofold: we must find an admissible domain G and an admissible function $z(x,y)$ defined on G which yields an absolute minimum for $I(z,G)$. The procedure is to begin with a minimizing sequence $\{z_n(x,y), (x,y) \in \bar{G}_n\}$, $n = 1, 2, \ldots,$ apply a leveling process as before, and to map each domain G_n conformally onto the unit disk B by a mapping $(u_n(x,y), v_n(x,y))$, arranging things so that three distinct fixed points of γ are mapped on three distinct fixed points of B^* (the same points for each n). We then prove that the vector functions

$R_n(u,v) = (x_n(u,v), y_n(u,v), z_n(x_n(u,v), y_n(u,v)))$ are equicontinuous on B^*, where $x_n(u,v)$ and $y_n(u,v)$ are the inverses of $u_n(x,y)$

and $v_n(x, y)$, respectively. Next we are able to apply the Carathéodory theory of conformal mappings onto variable domains and we produce a limiting conformal map $(x(u, v), y(u, v))$ of B which we prove to be a homeomorphism of $\overline{B}$ onto an admissible domain $\overline{G}$. After showing that the functions $z_n(x_n(u, v), y_n(u, v))$ actually converge uniformly on $\overline{B}$ to a function $z(u, v)$, we prove that the function $g(x, y) = z(u(x, y), v(x, y))$ defined on $\overline{G}$ yields an absolute minimum for the integral.

For both forms of the non-parametric problem we extend these results to prove the existence of minimizing functions when the boundaries are completely free on the fixed cylindrical and capstan surfaces.

In Chapter II we treat the parametric problem for the integral

$$I_0(z, Q) = \int_Q F(z, J)\, du\, dv \quad ,$$

where $z(u, v)$ is a vector in Euclidean 3-dimensional space, E^3, $J = (J^1, J^2, J^3)$ is the vector of Jacobians, and $F(z, J)$ is continuous in (z, J), convex in J, positively homogeneous of degree 1 in J, a Lipschitz function in z and in J, and satisfies the relation

$$m|J| \leq F(z, J) \leq M|J| \quad , \qquad 0 < m \leq M \quad .$$

Let $\mathcal{T}$ be a fixed torus in E^3. (This replaces the capstan surface considered before. $\mathcal{T}$ could actually be a rather general manifold of which the torus and capstan surface are special cases.) For admissible vectors we turn to the space W_2^1 of S. L. Sobolev [30] (or the space P_2 of J. W. Calkin [2] and C. B. Morrey [19]). Specifically, a vector $z(u, v)$ defined on the unit square Q is admissible if it is of class $W_2^1(Q)$, continuous of the interior of Q, and if the surface $S : z = z(u, v)$ "spans the fixed torus $\mathcal{T}$ and covers the hole of $\mathcal{T}$."

The last part of the admissibility condition needs a precise formulation. In order to define what is meant by "covering the hole," we prescribe a topological linking condition. Let H be a simple closed curve linking the solid torus. Then we may say that the surface $S : z = z(u, v)$ "covers the hole" if there is a boundary strip $Q_h \subset Q$ such that every simple closed curve lying in Q_h, which is homotopic to the boundary of Q, is mapped by $z = z(u, v)$ onto a continuous curve in E^3 linking H.

Deciding what it shall mean for a surface $S : z = z(u, v)$ to "span $\mathcal{T}$" is a more delicate problem. Courant [11, p. 220] has given an example which shows that minimizing surfaces for some free boundary problems (specifically, for minimal surfaces) do not always have a continuous trace on the fixed manifold. Therefore, in order to formulate a reasonable problem, we shall say that the boundary of $\mathbf{S} : z = z(u,v)$ lies on $\mathcal{T}$, or $\mathbf{S}$ spans $\mathcal{T}$, if the shortest distance of $z(u,v)$ from points of $\mathcal{T}$ approaches 0 whenever (u,v) approaches (u_0, v_0), $(u,v) \in Q^o$, $(u_0 v_0) \in Q^*$.

The lower semicontinuity theorem which we want to use (L. Cesari [6], L. Turner [32]) requires uniform convergence of continuous vectors of class $W_2^1(Q)$. However, the example of Courant shows that we cannot in general obtain uniform convergence on $\overline{Q}$. Therefore, our reasoning will have to rely on interior properties only, and we shall have to settle for uniform convergence on closed subdomains of Q (the compact-open topology). Even when such convergence is obtained, the difficult problems remain of proving that the limiting surface "covers the hole" and spans $\mathcal{T}$. The result which guarantees that the limiting surface spans $\mathcal{T}$ is proved in Section 2 of Chapter II. It essentially says that if a sequence of continuous vectors of class W_2^1, whose boundary values lie on a manifold M, converges uniformly on every closed subdomain of Q, and if the norms in the space W_2^1 are uniformly bounded, then the boundary values of the limit vector also lie on M. The linking is proved by a variant of an argument of Courant [11].

For the initial step in the existence proof we introduce, as does C. B. Morrey [21], the integral

$$I(z,G) = \int_G \sqrt{F^2(z,J) + \frac{(M+m)^2}{4}\left[\left(\frac{E-G}{2}\right)^2 + F^2\right]}\, du\, dv \quad ,$$

where E, F, G in the integrand represent the usual fundamental quantities of the surface. This integral dominates $I_0(z,G)$ and agrees with $I_0(z,G)$ in case the vector $z(u,v)$ is quasi-conformal (Chapter II, Definition 3.4). The integral $I(z,G)$ is shown to satisfy the inequality

$$\frac{m}{2} D(z,G) \leq I(z,G) \leq \frac{M}{2} D(z,G) \quad ,$$

where

$$D(z,G) = \int_G (|z_u|^2 + |z_v|^2) du\, dv \quad .$$

Therefore, a sequence $\{z_n\}$ for which $I(z_n, Q)$ is uniformly bounded will have a subsequence which converges weakly in the W_2^1 norm (Section 4). In Sections 3 and 4 we enlarge the class of minimizing sequences to a class of generalized minimizing sequences. Thus

$$\delta = \inf (\liminf_{n \to \infty} I_0(z_n, Q)) \quad ,$$

where the infimum is taken over this larger class, is less than or equal to the original infimum. We show that there is a generalized minimizing sequence $\{\zeta_n\}$ of vectors of class C^∞ such that

$$\lim_{n \to \infty} I_0(\zeta_n, Q) = \delta \qquad \text{and} \quad \lim_{n \to \infty} I(\zeta_n, Q) = \delta \quad .$$

In Section 5 we introduce the class $\mathcal{F}(K)$ of admissible vectors of class $W_2^1(Q)$ with

$$J(z, Q) = \int_Q (|z_{uu}|^2 + 2|z_{uv}|^2 + |z_{vv}|^2)\, du\, dv \leq K \quad ,$$

and which map a fixed segment σ into a closed set bounded away from the torus. We show that $I(z, Q)$ attains a minimum in $\mathcal{F}(K)$, and applying the result of Section 4, we show that $\lim_{k \to \infty} I(z_k, Q) = \delta$, where z_k is the minimizing function in $\mathcal{F}(K)$. Using a "Dirichlet growth theorem" of C. B. Morrey, we are able to prove that a subsequence of $\{z_k\}$, say, $\{z_p\}$, converges uniformly on closed subdomains of $Q^o - \sigma$ and weakly in $W_2^1(Q)$. We then prove that the limiting vector z is actually continuous (and of class $W_2^1(Q)$) on all of Q^o, and apply the lemma stated earlier that the boundary of $S : z = z(u, v)$ lies on the torus $\mathcal{T}$. Finally we prove that the boundary of S links the fixed curve H, and z is the required minimizing vector, since z is admissible and

$$I_0(z, Q) \leq \delta \leq \inf I_0(\zeta, Q) \quad ,$$

where the infimum is taken over the class of all admissible vectors.

CHAPTER I

THE NON-PARAMETRIC PROBLEM

1. Basic Definitions and Theorems

In this chapter we shall be dealing with non-parametric surfaces, that is, with functions $z = z(x,y)$ defined on an open set G of the Euclidean 2-dimensional plane.

Definition 1.1. A function $z = z(x,y)$ defined on G is said to be absolutely continuous in the sense of Tonelli in G, or ACT, provided $z(x,y)$ is continuous in G and

i) for almost all $\bar{x}$ the function $z(\bar{x},y)$ of y alone is absolutely continuous in each closed interval contained in the set $G(\bar{x}) = \{(x,y) \in G : x = \bar{x}\}$;

ii) for almost all $\bar{y}$ the function $z(x,\bar{y})$ of x alone is absolutely continuous in each closed interval contained in the set $G(\bar{y}) = \{(x,y) \in G : y = \bar{y}\}$; and

iii) the partial derivatives $p = \frac{\partial}{\partial x} z$, $q = \frac{\partial}{\partial y} z$, which exist almost everywhere in G are summable in G.

As is usual in variational problems employing the direct methods, we shall need a theorem of closure for certain classes of functions. The theorem which will be used in this chapter is the following one.

Theorem 1.1. (Tonelli [31, § 2, No. 11]) Let $\{z_n(w)\}$, $w \in G$, $n = 1, 2, 3, \ldots$, be any sequence of continuous ACT functions in the bounded open set G, let $\{z_n\}$ converge to $z(w)$ uniformly in each closed bounded set $H \subset G$, where $z(w)$ is a given function defined in G, and let the partial derivatives $p_n = \frac{\partial}{\partial x} z_n$, $q_n = \frac{\partial}{\partial y} z_n$, be L_α-integrable in G, $\alpha > 1$, and $\int_G |p_n|^\alpha + |q_n|^\alpha)\, dx\, dy \leq M < +\infty$, M a given constant.

Then z is continuous and ACT in G, the partial derivatives $p = \frac{\partial}{\partial x} z$, $q = \frac{\partial}{\partial y} z$ are L_α-integrable in G, and

$$\int_G |p|^\alpha \, dx\, dy \leq \underline{\lim}_{n\to\infty} \int_G |p_n|^\alpha \, dx\, dy \quad ,$$

$$\int_G |q|^\alpha \, dx\, dy \leq \underline{\lim}_{n\to\infty} \int_G |q_n|^\alpha \, dx\, dy \quad .$$

In the problems to be considered in this chapter, the admissible functions will be continuous, and therefore we shall eventually be faced with the task of proving that certain minimizing sequences are equicontinuous. We shall now describe a procedure, due to Tonelli [31, § 1, No. 6], which will be applicable in the existence theorems which we shall prove. This procedure replaces certain sequences of functions by other sequences, and in our cases the new sequence turns out to be equicontinuous.

Let $f(x,y)$ be a function continuous on $\bar{G}$, the closure of G, and ACT in G. Let n be a fixed integer, and let z_j, $z_0 < z_1 < \ldots < z_N$, be all the numbers of the form $\frac{k}{n}$, $k = 0, \pm 1, \pm 2, \ldots,$ such that the planes $P_j : z = z_j$ intersect the surface $z = f(x,y)$, $(x,y) \in \bar{G}$.

The set S_0 of the points (x,y) in G where $f(x,y) = z_0$ is open, and we consider only those components $g_{01}, g_{02}, \ldots,$ on whose boundary g^*_{0s} the function f has the constant value z_0. Then we denote by f_0 the function which is equal to z_0 in each set g_{0s}, $s = 1, 2, \ldots,$ and is equal to f otherwise. Thus $f_0 = f$ on G^*, and f_0 is continuous on $\bar{G}$ and ACT on G.

We now repeat the same process on f_0 using the plane P_1. We obtain a new function f_1 with $f_1 = f_0 = f$ on G^*, which is continuous on $\bar{G}$ and ACT on G. Repeating this process N times we obtain a function $\bar{f}$ with $\bar{f} = f$ on G^*, which is continuous on $\bar{G}$ and ACT on G, and we shall say that $\bar{f}$ <u>has been obtained from</u> f <u>by a</u> $1/n$ <u>leveling</u>. If a function f has been obtained by this process, then we say that f <u>is</u> $1/n$ <u>leveled</u>.

For the equicontinuity theorems which we shall use, we need a restriction on the type of domain G.

Definition 1.2. A domain G satisfies condition (α) provided there is a number $d_0 > 0$ such that for every point $P_0 \in G^*$ and every square Q_ℓ of center P_0, sides parallel to the x- and y-axes, and side length 2ℓ with $\ell \leq d_0$, we have $Q_\ell^* \cap G^* \neq \emptyset$.

Thus, every Jordan domain satisfies condition (α), but the set $G : 0 < x^2 + y^2 < 1$, for example, does not.

We are now able to state our main equicontinuity theorem.

Theorem 1.2. (Tonelli [31]) If G is a bounded open set satisfying condition (α), if $\{z_n(x,y)\}$, $(x,y) \in \bar{G}$, is a sequence of functions which are continuous on $\bar{G}$, equicontinuous on G^*, ACT in G, with

$$\int_G (|p_n|^2 + |q_n|^2)\, dx\, dy \leq A < +\infty , \quad n = 1, 2, \ldots ,$$

for some constant A, and if each function z_n is $1/n$ leveled, then the functions z_n are equicontinuous on $\bar{G}$.

The final bit of preparation needed before stating and attacking our problem is a lower semicontinuity theorem for our integrals. There have recently been some very general theorems of this type given [22, 26], but the convergence of the functions which is used is not enough to guarantee that the limiting function is continuous.

Therefore, in this chapter we may just as well use a theorem of lower semicontinuity with respect to uniform convergence. The theorem is essentially that of Tonelli [31], but we use the more general form due to Turner [32].

Theorem 1.3. Let G be a bounded open set and let $f(x,y,z,p,q)$ be a continuous function of (x,y,z,p,q) for $(x,y) \in G$ and all z, p, q. Assume that

i) $f(x,y,z,p,q) \geq N$ for some real constant N and all $(x,y,z,p,q) \in G \times E^3$;

ii) for every $M > 0$ there are positive numbers α, μ, and L such that

$$f(x,y,z,p,q) \geq \mu(|p|^{1+\alpha} + |q|^{1+\alpha})$$

for all $(x,y,z,p,q) \in G \times E^3$, with $|z| \leq M$, $|p| + |q| \geq L$; and

iii) f is convex in (p,q).

Let $\mathcal{C}$ be the class of all functions $z(x,y)$ which are continuous and ACT on G and for which

$$-\infty < I(z) = \int_G f(x,y,z,z_x,z_y)\,dx\,dy < +\infty .$$

Then $I(z)$ is lower semicontinuous on $\mathcal{C}$ with respect to uniform convergence.

We complete this section with a lemma which will be useful in proving the equicontinuity of certain sequences of functions.

Lemma 1.1. If $u(t)$, $a \leq t \leq b$, is an absolutely continuous function in $[a,b]$ whose derivative $u'(t)$ is L_β-integrable in $[a,b]$ for some $\beta > 1$, and if $u(t)$ has an oscillation $> \sigma$ in $[a,b]$, then

$$\int_a^b |u'(t)|^\beta \, dt > \sigma^\beta / (b-a)^{\beta-1} \quad .$$

Proof. By the Schwarz inequality we have

$$\sigma \leq \int_a^b |u'(t)| \, dt \leq \left(\int_a^b |u'(t)|^\beta \, dt \right)^{1/\beta} (b-a)^{(\beta-1)/\beta} \quad ,$$

and the lemma now follows immediately.

2. The First Free Boundary Problem

In this section we shall state and prove an existence theorem in the case of a partially free and partially fixed boundary. We begin with the admissible functions defined on the closed unit disk $\overline{B}$, then generalize to any Jordan domain, and finally reach a result concerning totally free boundaries.

We shall consider the problem of minimizing the integral

$$I(z) = \int_B F(x, y, z_x, z_y)\, dx\, dy \quad ,$$

where we take as the elements of the class $\mathcal{C}$ of admissible functions $z = z(x, y)$ those functions which are continuous on $\overline{B}$, ACT on B, and which take on prescribed continuous boundary values $\zeta = \zeta(\theta)$, $\theta_1 \leq \theta \leq \theta_2$, $\theta_2 - \theta_1 \leq 2\pi$, on the arc $\gamma : [\theta_1 \leq \theta \leq \theta_2,\ r = 1]$. Furthermore, there are constants a, b, $a \leq b$, such that for all (x, y) on the boundary arc K complementary to γ, $z = z(x, y)$ satisfies the relation $a \leq z(x, y) \leq b$. Thus, the portion of the boundary of the surface $S : z = z(x, y)$, $(x, y) \in \overline{B}$, which corresponds to the arc K, traces out a continuous curve on a finite (closed) cylinder over B^*. We assume that there is at least one admissible function for which the integral in question is finite.

We now make an extension of Tonelli's $1/n$ leveling process. Let $z(x, y)$ be an admissible function, and assume that it is already $1/n$ leveled. We let $P_j : z = z_j$, $j = 0, 1, \ldots, N$, be the same planes used in the first leveling operation. Just as before, the set S_0 of the points of B where $z(x,y) \neq z_0$ is open, but now we consider those components $g_{01}, g_{02}, \ldots,$ on whose boundary g^*_{0s} the function $z(x,y)$ has constant value z_0 <u>or</u> whose boundary g^*_{0s} intersects only the arc K (and <u>not</u> γ) and such that $z(x,y)$ has constant value z_0 on $g^*_{0s} \cap B^o$. Then we denote by $z_0(x,y)$ the function which is equal to z_0 on each set g_{0s}, and equal to $z(x,y)$ otherwise. Thus $z_0(x,y)$ is still continuous and ACT, $z_0 = z$ on the arc γ, and on K we still have $a \leq z_0(x,y) \leq b$.

Now we repeat the process using the planes $P_1, \ldots, P_N$ in succession finally obtaining an admissible function $\ddot{z}(x,y)$.

<u>Theorem 2.1</u>. Let $F(x,y,p,q)$ be continuous for all $(x,y) \in \bar{B}$ and all (p,q), let $F(x,y,p,q)$ be convex in (p,q), and assume that $F(x,y,0,0) = 0$ and $F(x,y,p,q) \geq (p^2 + q^2)$. If there is an admissible function z for which $I(z) = \int_B F(x,y,z_x,z_y)\,dx\,dy$ is finite, then the integral $I(z)$ assumes an absolute minimum in the class $\mathcal{C}$.

<u>Proof</u>. Let $L = \inf \int_B F(x,y,z_x,z_y)\,dx\,dy$, where the infimum is taken over the class of admissible functions. By hypothesis, $0 \leq L < +\infty$. Let $\{z_n\}$ be a minimizing sequence, and we may well assume that $L \leq I(z_n) \leq L + 1/n \leq L + 1$. Now we apply to each z_n our generalized $1/n$ leveling process and obtain a new sequence, which we again call $\{z_n\}$. This new sequence is again

a minimizing sequence since $F(x, y, z_x, z_y)$ is non-negative, $F(x, y, 0, 0) = 0$, and the leveling process reduces the absolute value of z_x and z_y to 0 over the sets on which the leveling takes place. We shall first prove that the functions z_n are equicontinuous on the boundary B^*. Since B^* is a compact set, it suffices to prove that given any $\varepsilon > 0$ and any point $w_0 \in B^*$ there is a number $r_0 > 0$ such that for any circle C of center w_0 and radius $r \leq r_0$, every function z_n has an oscillation $\leq \varepsilon$ on $B^* \cap C$. Actually, given $\varepsilon > 0$ and w_0, it is enough to prove this for some $r_0 > 0$ and all $n \geq N$, N a fixed number. If w_0 is an interior point of γ, there is nothing to prove since all z_n take on the same values along γ. Thus we consider the two cases where w_0 is an interior point of K, and where w_0 is an endpoint of K. The disk B satisfies condition (α), and we let h_1 be the number relative to B and this condition (Definition 1.2).

<u>Case I.</u> Assume that w_0 is an interior point of K. Let h_0 be the smallest of the numbers h_1 and the two distances from w_0 to the endpoints of K. We shall prove that the equicontinuity property mentioned above holds for $r_0 = \min[\frac{1}{2}h_0, \frac{1}{2}h_0 \exp(8^3 \varepsilon^{-2}(L+2))]$, $N = 8/\varepsilon$. Suppose this is not true. Then there is a function z_n, $n \geq N$, whose oscillation in the circle C_0 of center w_0 and radius r_0 is $> \varepsilon$. Then there are two points $w_1, w_2 \in C_0 \cap B^*$ such

that $|z_n(w_1) - z_n(w_2)| > \varepsilon$. Then, say, $z_n(w_1) > z_n(w_0) + \varepsilon/2$, and we may assume that $z_n(w_2) < z_n(w_0) < z_n(w_1)$, for there will surely be some point w in C_0 with this property, and without loss of generality, it may be w_0. Let Q_ℓ denote the closed square of center w_0, sides parallel to the x- and y-axes, and side length 2ℓ, $\ell_0 = r_0 \leq \ell \leq \ell_1 = h_0/2$. Let α, β be the two points of intersection of Q_ℓ^* with K. Now since $|z_n(w_1) - z_n(w_2)| > \varepsilon$, we must have either $|z_n(\alpha) - z_n(w_2)| > \varepsilon/2$ or $|z_n(\alpha) - z_n(w_1)| > \varepsilon/2$.

(a). Suppose that $|z_n(\alpha) - z_n(w_2)| > \varepsilon/2$.

1. Assume $z_n(w_2) > z_n(\alpha) + \varepsilon/2$. Then we claim that there is a point τ on Q_ℓ^* such that $z_n(\tau) > z_n(w_0) + \varepsilon/4$. For if not, then for all $w \in Q_\ell^* \cap B$ we have $z_n(w) \leq z_n(w_0) + \varepsilon/4$, while $z_n(w_1) > z_n(w_0) + \varepsilon/2$. Hence

$$z_n(w_1) > z_n(w_0) + \varepsilon/2 = [z_n(w_0) + \varepsilon/4] + \varepsilon/4 \geq z_n(w) + \varepsilon/4$$

for all $w \in Q_\ell^* \cap B$. Thus if $\{P_k\}$ are the z-coordinates of the $1/n$ leveling planes, we have for some j

$$z_n(w) < P_j < P_{j+1} < z_n(w_1) \quad ,$$

since $1/n \leq 1/N = \varepsilon/8$. But this contradicts the fact that z_n is $1/n$ leveled in our generalized sense. Thus there is a point τ on Q_ℓ^* such that $z_n(\tau) > z_n(w_0) + \varepsilon/4$. Now we already have

$z_n(\alpha) + \varepsilon/2 < z_n(w_2) < z_n(w_0)$, and hence $z_n(\alpha) < z_n(w_0) + \varepsilon/4$. Therefore, z_n has an oscillation $> \varepsilon/8$ on $Q^*_\ell \cap B$, $\ell_0 \leq \ell \leq \ell_1$.

2. Assume that $z_n(\alpha) > z_n(w_2) + \varepsilon/2$. Then we claim that there is a point $\tau \in Q^*_\ell \cap B$ with $z_n(\tau) < z_n(w_2) + \varepsilon/4$. If not, for all $w \in Q^*_\ell \cap B$ we have $z_n(w) \geq z_n(w_2) + \varepsilon/4$. Then there is a j such that just as before $z_n(w_2) < P_j < P_{j+1} < z_n(w)$. But this again contradicts the fact that z_n is $1/n$ leveled. Therefore, there is a point $\tau \in Q^*_\ell \cap B$ such that $z_n(\tau) < z_n(w_2) + \varepsilon/4$. Hence, just as before, $z_n(\alpha) > [z_n(w_2) + \varepsilon/4] + \varepsilon/4 > z_n(\tau) + \varepsilon/4$, and so z_n has an oscillation $> \varepsilon/8$ on $Q^*_\ell \cap B$.

(b). Suppose that $|z_n(\alpha) - z_n(w_1)| > \varepsilon/2$. Then precisely as in case (a), parts 1 and 2 above, we see that z_n has an oscillation $> \varepsilon/8$ on $Q^*_\ell \cap B$.

Therefore by Lemma 1.1, for almost all ℓ we have

$$\int_{Q^*_\ell} F(x,y,z_{nx},z_{ny})ds \geq \int_{Q^*_\ell} (z_{nx}^2 + z_{ny}^2)ds \geq (\varepsilon/8)^2(8\ell)^{-1} \quad .$$

Integrating with respect to ℓ in $[\ell_0, \ell_1]$ we have

$$L + 1 \geq \int_B F(x, y, z_{nx}, z_{ny})\,dx\,dy \geq \int_B (z_{nx}^2 + z_{ny}^2)\,dx\,dy$$

$$\geq (\varepsilon/8)^2 \int_{\ell_0}^{\ell_1} (8\ell)^{-1}\,d\ell = 8^{-3}\varepsilon^2 \log \frac{\ell_1}{\ell_0} .$$

Therefore,

$$L + 1 \geq 8^{-3}\varepsilon^2 \log\left(\frac{2^{-1}h_0}{r_0}\right) \geq 8^{-3}\varepsilon^2 \log(\exp(8^3\varepsilon^{-2}(L + 2))) = L + 2 ,$$

a contradiction. This proves the equicontinuity at every point w_0 interior to K.

Case II. Assume that w_0 is an endpoint of the arc K (and hence also an endpoint of the arc γ). Since all the functions z_n agree on γ (and at w_0), there is a number $h > 0$ such that every z_n has an oscillation $< \varepsilon/8$ in $C \cap \gamma$, where C is any circle with center w_0 and radius $r \leq h$. If h_0 is the number relative to condition (α), we may assume that $h_0 \leq h$. We shall again prove that the equicontinuity statement at the beginning of this proof is true for

$$r_0 = \min\left[\tfrac{1}{2}h_0, \tfrac{1}{2}h_0 \exp(8^3\varepsilon^{-2}(L + 2))\right] , \qquad N = 8/\varepsilon .$$

Indeed, suppose it is not true. Then there is a function z_n, $n \geq N$, which has an oscillation $> \varepsilon$ on $C \cap (\gamma \cup K)$ for some circle C with center w_0 and radius $r \leq r_0$, and we may assume that $r = r_0$. Then there are two points $w_1, w_2 \in C \cap (\gamma \cup K)$ with $|z_n(w_1) - z_n(w_2)| > \varepsilon$, and hence for at least one of these points, say w_1, we must have $|z_n(w_1) - z_n(w_0)| > \varepsilon/2$. Let us assume that $z_n(w_1) > z_n(w_0) + \varepsilon/2$. We again consider all squares Q_ℓ with center w_0, sides parallel to the x- and y-axes, and side length 2ℓ, $r_0 = \ell_0 \leq \ell \leq \ell_1 = h_0/2$. All these squares are contained in a circle C_1 with center w_0 and radius h_0. Also, the boundary Q^*_ℓ of Q_ℓ must meet γ in exactly one point and K in exactly one point. If α is the point of intersection of Q^* with γ, we know that $z_n(\alpha) < z_n(w_0) + \varepsilon/8$ (see page 22).

We now claim that there is a point $\tau \in Q^*_\ell \cap B$ such that $z_n(\tau) > z_n(w_0) + \varepsilon/4$. If this were not the case, we would have $z_n(w) \leq z_n(w_0) + \varepsilon/4$. for all $w \in Q^*_\ell \cap B$, while $z_n(w_1) > z_n(w_0) + \varepsilon/2$. We draw an arc from w_0 to Q^*_ℓ so close to the arc γ that for all w on this new arc, $z_n(w) < z_n(w_0) + \varepsilon/8$. This, of course, is possible by the continuity of z_n. Then we have

$$z_n(w_1) > [z_n(w_0) + \varepsilon/4] + \varepsilon/4 > z_n(w) + \varepsilon/4 \quad ,$$

and, therefore, there is a number j as before with

$$z_n(w) < P_j < P_{j+1} < z_n(w_1) \quad ,$$

contradicting the fact that z_n is $1/n$ leveled in our generalized sense. Thus there is a point $\tau \in Q^*_\ell \cap B$ with $z_n(\tau) > z_n(w_0) + \varepsilon/4$, while $z_n(\alpha) < z_n(w_0) + \varepsilon/8$. Hence z_n has an oscillation $> \varepsilon/8$ on $Q^*_\ell \cap B$ for every ℓ satisfying $\ell_0 \leq \ell \leq \ell_1$. Now exactly as before we are led to the contradiction $L + 1 \geq L + 2$.

Therefore, we conclude that the functions z_n are equicontinuous on the boundary of the disk B. Thus by Theorem 1.2, the functions z_n are equicontinuous on $\bar{B}$. By Ascoli's Theorem, there is a subsequence $\{z_{n_k}\}$ which converges uniformly on $\bar{B}$ to a function z which is continuous on $\bar{B}$ and takes on the prescribed values $\zeta = \zeta(\theta)$ on γ. Now by Theorem 1.1 we can conclude that z is admissible, since for every n,

$$\int_B (z_{nx}^2 + z_{ny}^2)\, dx\, dy \leq L + 1 \quad .$$

By Theorem 1.3, the functional $I(z)$ is lower semicontinuous, so

$$I(z) \leq \underline{\lim}_{k\to\infty} I(z_{n_k}) = L \quad .$$

But since z is in our admissible class $\mathcal{C}$, $I(z) \geq L$. Finally, $I(z) = L$, and z is the desired minimizing function. This proves the theorem.

We now state a slightly more general theorem than the preceding one. We consider the class $\mathcal{C}$ of functions $z(x,y)$ defined on a closed Jordan domain $\overline{G}$, which are continuous on $\overline{G}$ and ACT on G. Furthermore, each $z(x,y)$ takes on prescribed continuous boundary values $\zeta = \zeta(x,y)$ on an arc γ of the boundary G, and on the complementary arc K we have $a \leq z(x,y) \leq b$ for constants a, b, with $a < b$.

<u>Theorem 2.2</u>. Let $F(x,y,z,p,q)$ be continuous in (x,y,z,p,q) on $\overline{G} \times E^3$, convex in (p,q), and assume that there are constants $\mu > 0$, ν real, such that $F(x,y,z,p,q) \geq \nu + \mu(p^2 + q^2)$. Suppose further that there is a continuous function $f(x,y)$ such that $F(x,y,z,0,0) = f(x,y)$, and that there is at least one function $z(x,y)$ in the class $\mathcal{C}$ described above such that

$$I(z) = \int_G F(x,y,z,z_x,z_y)\,dx\,dy$$

is finite. Then the functional $I(z)$ has an absolute minimum in $\mathcal{C}$.

<u>Proof</u>. First note that

$$I(z) \geq \nu \text{ measure } (G) + \mu \int_G (z_x^2 + z_y^2)\,dx\,dy \geq \nu \text{ measure } (G)$$

for all z in the class $\mathcal{C}$ for which $I(z)$ is finite. Hence,

$$L = \inf_{z \in \mathcal{C}} I(z) > -\infty \quad ,$$

and since by hypothesis there is a function z in $\mathcal{C}$ with $I(z)$ finite, we have $-\infty < L < +\infty$.

Let $\{\bar{z}_n\}$ be a minimizing sequence. To this sequence we apply our generalized $1/n$ leveling process. The new sequence, $\{z_n\}$, again consists of members of the class $\mathcal{C}$. Furthermore, it is a minimizing sequence. Indeed, if we assume that

$$L \leq I(\bar{z}_n) \leq L + 1/n \leq L + 1 \quad ,$$

then since $F(x,y,z_n,z_{n_x},z_{n_y}) \leq F(x,y,\bar{z}_n,\bar{z}_{n_x},\bar{z}_{n_y})$ (using the fact that $F(x,y,z,0,0) = f(x,y)$), we have

$$L \leq I(z_n) \leq I(\bar{z}_n) \leq L + 1/n \leq L + 1 \quad .$$

Since G is a Jordan domain, it satisfies condition (α). Therefore, exactly as in the proof of Theorem 2.1, except that we replace

$$r_0 = \min[\tfrac{1}{2}h_0, \tfrac{1}{2}h_0 \exp(8^3 \varepsilon^{-2}(L + 2))]$$

by

$$r_0 = \min[\tfrac{1}{2}h_0, \tfrac{1}{2}h_0 \exp(8^3 \varepsilon^{-2} \mu^{-1}(L + 2 - \nu\,\text{meas}(G)))] ,$$

we conclude that the functions z_n are equicontinuous on G^*.

Since

$$I(z_n) \geq \nu \text{ meas } (G) + \mu \int_G (z_{n_x}^2 + z_{n_y}^2)\, dx\, dy \quad ,$$

we have

$$\int_G (z_{n_x}^2 + z_{n_y}^2)\, dx\, dy \leq \mu^{-1}(I(z_n) - \nu \text{ meas } (G))$$

$$\leq \mu^{-1}(L + 1 - \nu \text{ meas } (G)) = A \quad ,$$

where A is a constant. Therefore, by Theorem 1.2 the functions z_n are equicontinuous on $\bar{G}$, and by Ascoli's Theorem there is a subsequence $\{z_{n_k}\}$ which converges uniformly to a continuous function z on $\bar{G}$. Furthermore, by Theorem 1.1 z is admissible (that is, in the class $\mathcal{C}$), and by Theorem 1.3 the functional $I(z)$ is lower semicontinuous. Hence, $I(z) \leq \underline{\lim}_{k\to\infty} I(z_{n_k}) = L$, while at the same time $I(z) \geq L$. Therefore, $I(z) = L$, and z yields an absolute minimum in the class $\mathcal{C}$. This proves the theorem.

The results of Theorems 2.1 and 2.2 and their proofs enable us to prove the existence of a minimizing function for the problem in which the boundary values are left completely free in the cylinder over the boundary G^* of the Jordan domain G, where we, of course, take only a finite (closed) cylinder. Let $\mathcal{C}$ be the class of all functions $z(x,y)$ defined on $\bar{G}$, continuous on $\bar{G}$, and ACT on G, whose boundary values lie in the finite cylinder S given by $\{(x,y,z) : (x,y) \in G^*, \ a \leq z \leq b\}$.

Theorem 2.3. Under the same hypotheses as in Theorem 2.2, if the class $\mathcal{C}$ contains a function z such that $I(z)$ is finite, then the functional $I(z)$ assumes an absolute minimum in the class $\mathcal{C}$.

Proof. Let $\{z_n\}$ be a minimizing sequence. We may assume that

$$L = \inf_{z \in \mathcal{C}} I(z) \leq I(z_n) \leq L + 1/n \leq L + 1 \quad .$$

We divide the boundary G^* into two Jordan arcs α, β such that α and β have only their endpoints in common and $\alpha \cup \beta = G^*$. We may do this so that neither arc is degenerate. For the moment, let n be fixed. We consider the class $\mathcal{C}'$ of all functions z which are continuous on $\overline{G}$, ACT on G, and take on the values $z_n(x,y)$ for all $(x,y) \in \alpha$. Then by Theorem 2.2 there is a function $\bar{z}_n(x,y)$ in this class which minimizes $I(z)$ among all functions in $\mathcal{C}'$. Therefore, $I(\bar{z}_n) \leq I(z_n)$, and $\bar{z}_n$ is still in the class $\mathcal{C}$. Therefore,

$$L \leq I(\bar{z}_n) \leq I(z_n) \leq L + 1/n \leq L + 1 \quad ,$$

so the sequence $\{\bar{z}_n\}$ is again a minimizing sequence. However, each $\bar{z}_n$ is $1/n$ leveled in the generalized sense with respect to the fixed values on the arc α. Indeed, if one of them were not, we could apply the leveling process and obtain a function z_{0n} such that $I(z_{0n}) \leq I(\bar{z}_n)$. Now exactly the same proof as in Theorem 2.1 shows that the functions $\bar{z}_n$ are equicontinuous on β.

Next, for each n, we may replace $\bar{z}_n$ by the function $\bar{\bar{z}}_n$ which minimizes $I(z)$ among all those admissible functions which agree with $\bar{z}_n$ along the arc β. As before,

$$L \leq I(\bar{\bar{z}}_n) \leq I(\bar{z}_n) \leq L + 1/n \leq L + 1 \quad ,$$

and we may assume the functions $\bar{\bar{z}}_n$ to be $1/n$ leveled. The proof of Theorem 2.1 shows that the functions $\bar{\bar{z}}_n$ are equicontinuous on α. Therefore, these functions are equicontinuous on G^*, and using the fact that the cylinder is finite, we may reason exactly as before. Thus we see that there is an admissible function z such that $I(z) = L$, that is, $I(z)$ takes on an absolute minimum in the class $\mathcal{C}$. This proves the theorem.

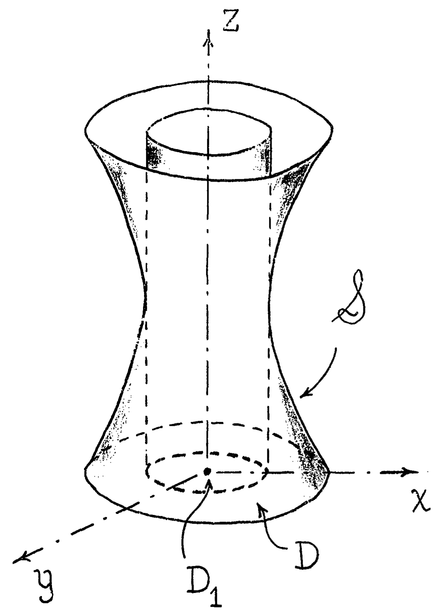
3. The Free Boundary Problem for a Capstan Surface

Definition 3.1. By a capstan surface $\mathcal{S}$ we shall mean a surface of revolution generated by revolving a curve C in the (x, z)-plane about the z-axis, where C can be represented by a twice continuously differentiable function $x = f(z)$, $a \leq z \leq b$, $f(z) > 0$, having the properties that $f'(z)$ has precisely one zero $z = \frac{(a+b)}{2}$, $f(z)$ is symmetric in the line $z = \frac{(a+b)}{2}$ and $f''(z) > 0$ for all z in the interval $a \leq z \leq b$. Such a surface may be represented parametrically by the equations

$$x = X(u, v) \ , \qquad y = Y(u, v) \ , \qquad z = Z(u, v) \ ,$$

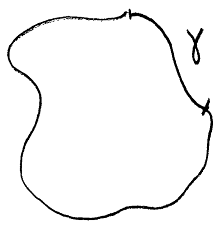
where $0 \leq u \leq \eta$, $-\tau \leq v \leq +\tau$. The functions $X(u, v)$, $Y(u, v)$, $Z(u, v)$, are assumed to be at least of class C^2, and $X(u, v)$, $Y(u, v)$, are assumed to have positive real period η in the variable u for values of v in the interval $-\tau \leq v \leq +\tau$.

We take the (x, y)-plane as the base plane for $\mathcal{S}$, and we denote by D^* the intersection of $\mathcal{S}$ with the (x, y)-plane. Thus, D^* is a circle, and we denote by D the disk it bounds. Let D_1 be the disk contained in D with the property that the cylinder over D_1^* is tangent to $\mathcal{S}$.

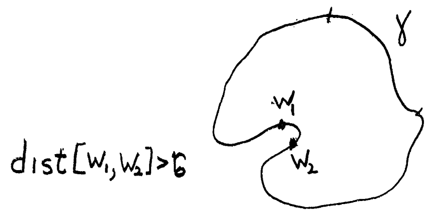


We shall be interested in Jordan domains G with the property that G^* contains a fixed Jordan arc γ lying in the annulus bounded by the circles D_1^* and D^* and satisfying the relation $D_1 \subset G \subset D$. It will be necessary to impose a certain restriction on the boundary G^* of G. This restriction is embodied in the following condition.

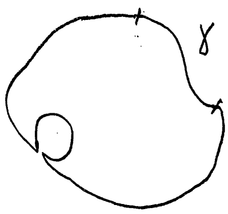
<u>Condition</u> $S(r_0, a_0, b_0)$. Let r_0, a_0, b_0 be given real $r_0 > 0$, $\pi/2 \leq a_0$, $\pi/2 \leq b_0$. The domain G is said to satisfy <u>condition</u> $S(r_0, a_0, b_0)$ if for every point $w \in G^* - \gamma$ there are circular sectors S_{a_0}, S_{b_0} each with vertex at w and radius r_0, having vertex angles a_0 and b_0, respectively, such that one of the sectors lies completely interior to G and one lies completely exterior to G (except for the point w). The domains in figures 1 and 2 below satisfy condition $S(r_0, a_0, b_0)$ with $a_0 = b_0 = \pi/2$, while the domains in figures 3 and 4 do not satisfy the condition for this set of numbers.



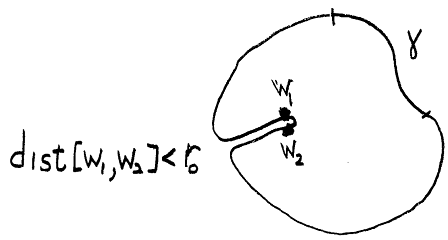
1



2



3



4

In the existence proof which follows, we shall use as one of our main tools the theory of convergence of conformal mappings onto variable domains (the Carathéodory theory). We first recall the definition of the kernel of a sequence of domains and then state the Carathéodory convergence theorem.

Definition 3.1. Let $\{B_n\}$ be a sequence of domains in the complex plane, each B_n containing a fixed point, say, 0. Then the kernel $B = K\{B_n\}$ is defined by the properties:

(a) If $\bigcap_n B_n$ contains no neighborhood of 0, then $B = \{0\}$;

(b) If $\bigcap_n B_n$ contains a neighborhood of 0, then B is a domain such that

(i) $0 \in B$;

(ii) If E is any compact subset of B, then for n sufficiently large, $E \subset B_n$;

(iii) If D is a domain satisfying (i) and (ii), then $D \subset B$.

Such a domain always exists. Furthermore, if $\{B_{n_k}\}$ is a subsequence of $\{B_n\}$, if $B = K\{B_n\}$, $B' = K\{B_{n_k}\}$, then $B' \supset B$. If actually $B' = B$ for all subsequences $\{B_{n_k}\}$, then we say that the sequence of domains $\{B_n\}$ converges to the domain B.

We shall now state a convergence theorem which will be of use to us.

Lemma 3.1. Let $\{f_n(w)\}$ be a sequence of conformal mappings defined on the disk B, let $f_n(B) = G_n$, $D_1 \subset G_n \subset D$ for all n and two fixed concentric disks D_1 and D. If the sequence $\{f_n\}$ converges uniformly on $\bar{B}$ to a function f, then f is a conformal mapping of B onto the domain $G = K\{G_n\}$, and the sequence of domains $\{G_n\}$ converges to the kernel G. Furthermore, on every compact subset of G we have

$$\lim_{n\to\infty} f_n^{-1}(z) = f^{-1}(z)$$

uniformly.

Remark. This is a slightly different statement of the Carathéodory theorem than is usually found, for example, in Goluzin [16, p. 46] or Carathéodory [3]. The essential reason is that the usual statement is a necessary and sufficient condition, and it includes a normalization of the form $f_n(0) = 0$, $f_n'(0) > 0$. But since we are assuming uniform convergence of the conformal mappings (and hence the limit function is conformal) along with the uniform boundedness of the domains G_n, we do not need to make assumptions of this type. The proof of Lemma 3.1 may be obtained from the two references noted above or more conveniently from the very general theorems of F. W. Gehring [15].

We wish to consider functions $z = z(x,y)$ defined on a Jordan domain G_z (depending upon z) with $D_1 \subset G_z \not\subset D$, and having the property that the set of points $C : (x, y, z(x,y)), (x,y) \in G_z^*$, lies on the capstan surface $\mathcal{S}$. That is, the surface $S : z = z(x,y)$, $(x,y) \in \bar{G}_z$, has its boundary on the capstan surface.

We let γ be a fixed Jordan arc lying in the annular region bounded by the concentric circles D_1^* and D^*, and let $\zeta = \zeta(x,y)$ be a given continuous function defined on γ, whose graph lies on $\mathcal{S}$. Denote by $\mathcal{C}$ the class of all functions $z = z(x,y)$, defined on some Jordan domain G_z whose boundary G_z^* lies in the domain $\bar{D} - D_1$ and contains the arc γ, such that $z(x,y)$ is continuous on $\bar{G}_z$, ACT on G_z, agrees with $\zeta(x,y)$ on γ, and the points $(x,y,z(x,y)), (x,y) \in G_z^*$, lie on $\mathcal{S}$. We further assume that each domain G_z satisfies condition $S(r_0, a_0, b_0)$ for some fixed numbers r_0, a_0, b_0.

Theorem 3.1. Let $F(x,y,p,q)$ satisfy the same conditions as in Theorem 2.1. Assume that there is at least one function z in $\mathcal{C}$ such that $I(z) = \int_{G_z} F(x,y,z_x,z_y)\,dx\,dy$ is finite. Then the functional $I(z)$ has an absolute minimum in $\mathcal{C}$.

Proof. Let $\{\bar{z}_n\}$ be a minimizing sequence, where the domain of $\bar{z}_n$ is denoted by G_n. We may assume that

$$L = \inf_{z \in \mathcal{C}} I(z) \leq I(\bar{z}_n) \leq L + 1/n \leq L + 1 \quad .$$

We apply Tonelli's $1/n$ leveling process described in Section 1 (not our generalized process) to obtain a new minimizing sequence $\{z_n\}$, each z_n having the same domain G_n as did $\bar{z}_n$. By the Riemann mapping theorem, for each n there exists an analytic univalent function f_n mapping the interior of G_n onto the interior of the unit disk B in the $w = (u, v)$-plane, with f_n^{-1} also analytic and univalent, mapping B onto G_n. These functions f_n, as we know from the Carathéodory extension theory, may be extended to continuous functions on the closure of the domains G_n.

Let Q_i, $i = 1, 2, 3$, be three distinct fixed points on the fixed arc γ, Q_1 and Q_3 being the endpoints, and let $P_{ni} = z_n(Q_i)$, $i = 1, 2, 3$. The triples $(f_n(Q_1), f_n(Q_2), f_n(Q_3))$ define one or the other of the two orientations of B^*. One of the orientations occurs infinitely many times. Make such a choice of orientation, and extract a subsequence, again called $\{f_n\}$, such that the orientation is the same for all n. Let (w_1, w_2, w_3) be a triple of 3 distinct fixed points on B^* which defines the same orientation as above. Then there is, for each n, a unique linear fractional transformation $s_n(w)$ of $\bar{B}$ onto itself sending $f_n(Q_i)$ onto w_i, $i = 1, 2, 3$. Let $h_n(w) = f_n^{-1}(s_n^{-1}(w))$. Then $h_n(w)$ is, for each n, a one-to-one conformal mapping of $\bar{B}$ onto $\bar{G}_n$, sending w_i onto Q_i, $i = 1, 2, 3$. Furthermore, for each n, $z_n(h_n(w_i)) = P_{ni}$, $i = 1, 2, 3$. If we write $h_n(w) = (x_n(w), y_n(w))$

and $z_n(w) = z_n(x_n(w), y_n(w))$, we have a sequence of vectors $R_n(w) = (x_n(w), y_n(w), z_n(w))$ mapping $\overline{B}$ into E^3, such that the points $R_n(w)$, $w \in B^*$, lie on $\mathscr{A}$, and $R_n(w_i) = P_{ni}$, $i = 1, 2, 3$. By the definition of the points P_{ni} we have

$$\inf_n \text{[mutual distances of the } P_{ni}] = d_0 > 0 \quad .$$

We now assert that the vectors $R_n(w)$ are equicontinuous on B^*. For suppose that this is not the case. Then there is a number $d > 0$ and a sequence of pairs of points w'_{nm}, w''_{nm}, $m = 1, 2, \ldots$, of B^* with $|w'_{nm} - w''_{nm}| \to 0$ as $m \to \infty$, and

$$|R_m(w'_{nm}) - R_m(w''_{nm})| \geq d \qquad \text{for all } m = 1, 2, \ldots .$$

By a convenient extraction of a subsequence and a renaming, we may assume that the sequence $[nm]$ is the sequence $[m]$, that the sequence $\{w'_n\}$ converges to a point $w_0 \in B^*$, and that $|w'_n - w_0| < 1/n$, $|w''_n - w_0| < 1/n$, $n = 1, 2, \ldots$. Now the point w_0 is interior to one of the arcs defined by w_1, w_2, w_3, say, to that arc $\widehat{w_1 w_3}$ which contains w_2. We may assume that all of the points w'_n, w''_n are also interior to the same arc, and that the points in question are oriented (w_1, w'_n, w''_n, w_3). The vector functions $R_n(w)$ are one-to-one on B^*, and the images under R_n of the two disjoint arcs $\widehat{w_3 w_1}$ and $\widehat{w'_n w''_n}$ are two disjoint arcs $\widehat{P_{n3} P_{n1}}$ and $\widehat{P'_n P''_n}$ of diameters $\geq d_0$ and $\geq d$, respectively.

Now let $\varepsilon > 0$ be given. Choose N so large that $8/N < \varepsilon$, and choose $\delta_1 < 1/N$. For the "fixed arc," that is, the arc which is common to all the curves $C_n : R = R_n(w)$, $w \in B^*$, there is a number $\delta_2 > 0$ such that any two points on the fixed arc which have a distance less than δ_2 lie on a subarc of diameter less than ε (property of a continuous arc). Choose $\bar{\delta} = \min [\delta_1, \delta_2,$ diameter of fixed arc, $\delta_{\mathcal{S}}]$, where $\delta_{\mathcal{S}}$ arises as follows. Given any $\varepsilon > 0$ there is a number $\delta_{\mathcal{S}} > 0$ such that if P, Q are points of the capstan surface $\mathcal{S}$ of distance less than $\delta_{\mathcal{S}}$, if α is any arc lying in $\mathcal{S}$ joining P and Q such that the projection of α on the (x, y)-plane is contained wholly in one of the two sectors (possibly degenerate) of D determined by the center of D and the projections of P and Q, then the diameter of the projection of α on the (x, y)-plane is less than $\varepsilon/2$.

Now we claim that there is a number δ, $0 < \delta < \bar{\delta}$ such that for $n \geq N$, any two points of C_n at a distance less than δ belong to a subarc of diameter $< \varepsilon$. Suppose, on the contrary, that this is not the case. Then there is a sequence of points P_{n1}, P_{n2}, $n \geq N$, such that dist $(P_{n1}, P_{n2}) \to 0$ as $n \to \infty$, while each of two arcs λ_{n1}, λ_{n2} with endpoints P_{n1}, P_{n2}, $\lambda_{n1} \cup \lambda_{n2} = C_n$, has diameter $> \varepsilon$. Since dist $(P_{n1}, P_{n2}) \to 0$, we may assume that dist $(P_{n1}, P_{n2}) < 1/n$, and thus if ζ_{n1}, ζ_{n2} are, respectively, the projections of P_{n1}, P_{n2} on the (x, y)-plane, then

$|\zeta_{n1} - \zeta_{n2}| = \text{dist}[\zeta_{n1}, \zeta_{n2}] < 1/n$. Now by the choice of $\bar{\delta}$, ζ_{n1} and ζ_{n2} cannot both lie on the fixed arc γ.

Case I. Suppose that ζ_{n1}, ζ_{n2} are both on the arc complementary to γ. Let us make the convention that λ_{n1} does not contain the fixed part of C_n. Since diameter $\lambda_{n1} > \varepsilon$, there is a point P_{n0} on λ_{n1} such that if ζ_{n0} is its projection on the (x, y)-plane $|\zeta_{n0} - \zeta_{n2}| \geq \eta > 0$, $|\zeta_{n0} - \zeta_{n1}| > 0$ for some constant η depending only on the curvature of the capstan surface. However, if we let $n \to \infty$ then $|\zeta_{n1} - \zeta_{n2}| \to 0$, and this contradicts the fact that the domain G_n satisfies condition $S(r_0, a_0, b_0)$.

Case II. Suppose that ζ_{n1} lies on the arc complementary to γ, and ζ_{n2} lies on γ. Let ζ_0 be the endpoint of γ on the smaller of the two arcs bounded by ζ_{n1} and ζ_{n2}. Since $|\zeta_{n1} - \zeta_{n2}| \to 0$ as $n \to \infty$ we must have $|\zeta_{n1} - \zeta_0| \to 0$ as $n \to \infty$ and $|\zeta_{n2} - \zeta_0| \to 0$ as $n \to \infty$. We may choose n so large that the fixed arc on the capstan surface between P_0 and P_{n2} has diameter $< \varepsilon/2$ (by the property of the continuous curve). Then the diameter of the arc $\overset{\frown}{P_{n1}P_0}$ is $> \varepsilon/2$ for all n. Now we may reason exactly as in case I. This proves the assertion.

Now we take $\varepsilon = \min[d_0, d]$ and $\delta > 0$ as above. Then we claim that the arcs $\overset{\frown}{P_{n1}P'_n}$ and $\overset{\frown}{P''_nP_{n3}}$ of C_n have a mutual distance $\geq \delta$. If not, there would be points P', P'' of the first two arcs at a distance $< \delta$ and therefore a subarc of C_n of diameter $< \varepsilon$ containing P' and P''. Since C_n is simple, this arc must contain either $\widehat{P'_n P''_n}$ or $\widehat{P_3 P_1}$, both of diameter $\geq \varepsilon$, a contradiction.

Let s_0 be the minimum distance of w_0 from w_1 and w_3. Let us consider the family of arcs β of circumferences with center w_0 and radius r, $s_n \leq r \leq s_0$, $s_n = 1/n$, contained in B with endpoints on B^*. The images of the endpoints α, τ of these arcs on C_n under the mapping R_n lie on $\widehat{P_1 P'_n}$ and $\widehat{P''_n P_3}$, and hence at a mutual distance $\geq \delta$. Introduce polar coordinates (r, θ) with w_0 as pole. The region E covered by the arcs β is given by $[s_n \leq r \leq s_0;\ \theta_1(r) \leq \theta \leq \theta_2(r)]$. Recalling that $R_n(w) = (x_n(w), y_n(w), z_n(w))$, where x_n and y_n are the real and imaginary parts of a conformal mapping, we see that

$$\int_B (x_{n_u}^2(u,v) + x_{n_v}^2(u,v))\,du\,dv = \int_B (y_{n_u}^2(u,v) + y_{n_v}^2(u,v))\,du\,dv$$

$$= \int_{G_n} dx\,dy \leq \text{area of } D = A \ ,$$

since each G_n is contained in the disk D, and since the above integrands are expressions for the Jacobian of the mapping $(x_n(w), y_n(w))$.

Therefore, we have

$$L + 1 + 2A \geq I(z_n) + 2A \geq \int_{G_n} (z_{n_x}^2 + z_{n_y}^2)\,dx\,dy + 2A$$

$$\geq \int_B (x_{n_u}^2 + y_{n_u}^2 + z_{n_u}^2 + x_{n_v}^2 + y_{n_v}^2 + z_{n_v}^2)\,du\,dv$$

$$\geq \int_{s_n}^{s_0} \int_{\theta_1}^{\theta_2} (x_{n_r}^2 + y_{n_r}^2 + z_{n_r}^2 + r^{-2}[x_{n_\theta}^2 + y_{n_\theta}^2 + z_{n_\theta}^2])\,r\,dr\,d\theta$$

$$\geq \int_{s_n}^{s_0} r^{-1}dr \int_{\theta_1}^{\theta_2} (x_{n_\theta}^2 + y_{n_\theta}^2 + z_{n_\theta}^2)\,d\theta$$

But we also have

$$\delta \leq \int_{\theta_1}^{\theta_2} (|x_{n_\theta}| + |y_{n_\theta}| + |z_{n_\theta}|)\,d\theta < \sqrt{\pi}\left\{\int_{\theta_1}^{\theta_2} (x_{n_\theta}^2 + y_{n_\theta}^2 + z_{n_\theta}^2)\,d\theta\right\}^{\frac{1}{2}}$$

or

$$\int_{\theta_1}^{\theta_2} (x_{n_\theta}^2 + y_{n_\theta}^2 + z_{n_\theta}^2)\,d\theta \geq \frac{\delta^2}{\pi} \quad .$$

Combining this with the inequality at the top of this page, we obtain

$$L + 1 + 2A \geq \frac{\delta^2}{\pi} \log(ns_0) \quad ,$$

a contradiction, since the right hand side approaches $+\infty$ as n approaches $+\infty$. Therefore, the functions $R_n(w)$ are equicontinuous on B^*.

The equicontinuity of the vector functions $R_n(w)$ on B^* implies that the mappings $h_n(w) = (x_n(w), y_n(w))$ are equicontinuous on B^*. However, each $h_n(w)$ is an analytic function on B and continuous on $\bar{B}$. Hence by the maximum-modulus principle, the functions $h_n(w)$ are equicontinuous on $\bar{B}$. Therefore, a subsequence $\{h_{n_k}\}$ converges uniformly on $\bar{B}$ to a function $h(w)$ which is analytic on B and continuous on $\bar{B}$. Since each h_{n_k} maps the three distinct points w_i, $i = 1, 2, 3$, of B^* onto the three distinct points Q_i, $i = 1, 2, 3$, respectively, of γ, and since $h(w)$ is continuous and $h(w_i) = Q_i$, $i = 1, 2, 3$, $h(w)$ cannot be constant on B. Therefore, $h(w)$ is a one-to-one analytic function on B [16, p. 9], continuous on $\bar{B}$, and $h(w)$ maps one of the two arcs $\alpha_1 = \widehat{w_1 w_2 w_3}$, $\alpha_2 = \widehat{w_3 w_1 w_2}$ onto the arc γ in a one-to-one manner. We may choose the subsequence $\{h_{n_k}\}$ so that each member maps, say, the arc $\alpha = \alpha_1$ onto the arc γ. Then $h(w)$ also maps α onto γ.

We denote by G the image of B under the mapping h. Thus $G = h(B)$ is a simply connected domain, $D \supset G \supset D_1$, and G^* contains the arc γ. By Lemma 3.1 we conclude that $G = K\{G_{n_k}\}$ and the sequence $\{G_{n_k}\}$ converges to G.

Moreover, $\lim_{k\to\infty} h^{-1}_{n_k}(\zeta) = h^{-1}(\zeta)$ uniformly on compact subsets of G. We now replace the sequence $[n_k]$ by the sequence $[n]$ for the sake of simplicity.

Next, we see that h maps B^* onto G^*. Indeed, if $\zeta^* \in G^*$ and $\lim_{k\to\infty} \zeta_k = \zeta^*$, $\zeta_k \in G$, then the points $\{h^{-1}(\zeta_k)\}$ have an accumulation point $w^* \in \overline{B}$. Then a subsequence, again called $\{h^{-1}(\zeta_k)\}$, converges to w^*. Since h(w) is a continuous mapping on $\overline{B}$

$$\zeta^* = \lim_{k\to\infty} \zeta_k = \lim_{k\to\infty} h(h^{-1}(\zeta_k)) = h(\lim_{k\to\infty} h^{-1}(\zeta_k)) = h(w^*) \quad .$$

But since h(w) is a homeomorphism on B, w^* must be a point of B^*. Therefore, h maps B^* onto G^*.

Now we shall show that h is actually a one-to-one mapping of B^* onto G^* (and therefore of $\overline{B}$ onto $\overline{G}$). Then we can conclude that G is a Jordan domain, since h will be a homeomorphism on $\overline{B}$.

Suppose, on the contrary, that there are distinct points w′, w″ on B^* such that h(w′) = h(w″), and we may assume that h is not constant on the arc $\overset{\frown}{w'w''} = \beta$ which does not contain the arc $\alpha = h^{-1}(\gamma)$. (We know by the "three point lemma" [11, p. 103] that h is a one-to-one mapping of α onto γ.) Then there is a point w_0 interior to the arc β such that $h(w_0) \neq h(w')$. By the uniform convergence of the functions h_n toward h, we may choose $N > 0$ so large that for every $n \geq N$ we have

$$|h_n(w) - h(w)| < r_0/4 \quad , \quad w = w', \ w'', \ w_0 \ ,$$

where r_0 is the number relative to condition $S(r_0, a_0, b_0)$. Consequently,

$$|h_n(w') - h_n(w'')| < r_0/2 \quad ,$$

while $|h_n(w') - h_n(w_0)|$ and $|h_n(w'') - h_n(w_0)|$ are bounded away from 0. But this contradicts the condition $S(r_0, a_0, b_0)$.

Therefore, h is a one-to-one mapping of $\overline{B}$ onto $\overline{G}$, and G is a Jordan domain satisfying condition $S(r_0, a_0, b_0)$.

Also from the equicontinuity of the vectors $R_n(w)$ on B^* follows the equicontinuity on B^* of the functions $z_n(w) = z_n(h_n(w))$. Since $h_n(w)$ is conformal, $z_n(w)$ is ACT on B. If necessary, we may apply Tonelli's $1/n$ leveling process obtaining a new sequence, again called $\{z_n\}$, which is equicontinuous on B^*, ACT on B, and furthermore

$$L + 1 \geq \int_{G_n} F(x, y, z_{n_x}, z_{n_y}) dx\, dy$$

$$\geq \int_B (z_{n_u}^2 + z_{n_v}^2) du\, dv \quad .$$

Thus by Theorem 1.2 the functions $z = z_n(w)$ are equicontinuous on $\overline{B}$, and so we may choose a subsequence, again called $\{z_n\}$, which converges uniformly on $\overline{B}$ to a continuous and ACT function $z(w)$.

Now we write

$$g_n(x,y) = z_n(h_n^{-1}(x,y)) \quad , \qquad g(x,y) = z(h^{-1}(x,y)) \quad ,$$

$(x,y) \in G = K\{G_n\}$. Thus we have shown that the functions $g_n(x,y)$ converge uniformly on $\overline{G}$ to a continuous and ACT function $g(x,y)$, and furthermore, the sequence $\{g_n\}$ is a minimizing sequence. The values of $g(x,y)$ on the arc γ are such that the points $(x,y,g(x,y))$, $(x,y) \in \gamma$, lie on the capstan surface. Thus, g is in the class $\mathcal{C}$ of admissible functions. By the lower semicontinuity theorem

$$I(g) \leq \underline{\lim}_{n \to \infty} I(g_n) = L \quad ,$$

but since g is in $\mathcal{C}$, $I(g) \geq L$. Therefore, $I(g) = L$ and the theorem is proved.

We may now make the extension to slightly more general integrands just as in Theorem 2.2.

<u>Theorem 3.2</u>. Let $F(x,y,z,p,q)$ satisfy the same conditions as in Theorem 2.2. Then $I(z) = \int_{G_z} F(x,y,z,z_x,z_y)\,dx\,dy$ assumes an absolute minimum in the class $\mathcal{C}$ (as long as there is a function $z \in \mathcal{C}$ such that $I(z)$ is finite).

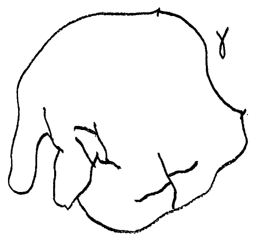
Proof. The proof of this extension is reduced to the case of Theorem 3.1 just as in the earlier proof of Theorem 2.2.

We may now use the final extension in Section 2 to prove the analogous existence theorem for boundaries totally free on a (closed) finite capstan surface. For this case we let $\mathcal{C}$ denote the class of functions $z(x,y)$ defined on a Jordan domain G_z, $D_1 \subset G_z \subset D$, such that $z(x,y)$ is continuous on $\overline{G}_z$, ACT on G_z, and such that the points $(x, y, z(x,y))$, $(x,y) \in G_z^*$, lie on $\mathcal{S}$. Let each G_z satisfy condition $S(r_0, a_0, b_0)$.

Theorem 3.3. Under the same hypotheses as Theorem 2.3, the functional $I(z)$ assumes an absolute minimum in the class $\mathcal{C}$.

Proof. We let $\{z_n\}$ be a minimizing sequence, where z_n is defined on the domain G_n, and let γ_n be a connected subarc of G_n^* with endpoints P_{n1} and P_{n3} lying, respectively, on the positive y-axis and the negative y-axis, with γ_n lying in the right half-plane. Then there is also a point P_{n2} which intersects the positive x-axis. We now proceed as in the proof of Theorem 3.1. by mapping the three distinct points P_{n1}, P_{n2}, P_{n3} onto three distinct points w_1, w_2, w_3 of the unit circle B^*. The proof of Theorem 3.1 may now be repeated, except that we no longer need to make a special argument about equicontinuity at endpoints of a "fixed arc." This proves the theorem.

Remark. It is clear that the theorems in this section are not really restricted to the case of a capstan surface which is a surface of revolution. The only features which we actually used were the fact that the curvature was bounded away from 0 and the fact that the projection of each admissible surface $S : z = z(x, y)$ onto the (x, y)-plane is a Jordan domain G_z containing a fixed disk D_1 and contained in another fixed disk D, and satisfying condition $S(r_0, a_0, b_0)$. Without a condition like $S(r_0, a_0, b_0)$ it is conceivable that a limiting (minimizing) domain has the form of Figure 5. If we were to enlarge the class of admissible domains to, say, simply connected domains, we would not be able to extend the Riemann mapping functions, used in the proof, to continuous functions on the closed domains.



5

CHAPTER II

THE PARAMETRIC PROBLEM

1. Basic Definitions and Theorems

In this chapter we shall consider the problem of minimizing the integral

$$(1.1) \qquad I_0(z, G) = \int_G F(z, J)\,du\,dv \quad ,$$

$$z(u,v) = (z^1(u,v),\ z^2(u,v),\ z^3(u,v)) \quad , \qquad (u,v) \in G \quad ,$$

$$J = (J^1, J^2, J^3) \quad , \qquad J^1 = \begin{vmatrix} z^2_u & z^3_u \\ z^2_v & z^3_v \end{vmatrix} \quad ,$$

$$J^2 = \begin{vmatrix} z^3_u & z^1_u \\ z^3_v & z^1_v \end{vmatrix} \quad , \qquad J^3 = \begin{vmatrix} z^1_u & z^2_u \\ z^1_v & z^2_v \end{vmatrix} \quad ,$$

where the integrand $F(z, J)$ satisfies the conditions:

(i) F is continuous in (z, J) for all (z, J) ;

(ii) F is positively homogeneous of degree 1 in J ;

(iii) F is convex in J for each fixed z ;

(iv) there are numbers m, M, $0 < m \leq M$, such that

$$m|J| \leq F(z, J) \leq M|J|$$

for all (z, J) ; and

(v) there are numbers $L_1, L_2 > 0$, such that for all (z, J)

$$|F(z_1, J) - F(z_2, J)| \leq L_1 |z_1 - z_2| ,$$

$$|F(z, J_1) - F(z, J_2)| \leq L_2 |J_1 - J_2| .$$

It is known that properties (i) and (ii) imply that the integral $I_0(z, G)$ is independent of the representation (z, G) of the Fréchet surface (L. Cesari [5]). Thus we shall consider all surfaces (vectors) to be defined on the unit square Q. It is also known that this integral is lower semicontinuous with respect to uniform convergence of continuous vectors of class $W_2^1(Q)$, Definition 1.1 (L. Cesari [6], L. Turner [32]),

The existence proof is mainly a generalization of the technique of C. B. Morrey [21] used for the proof of existence of a minimizing function of the parametric problem for surfaces spanning a fixed Jordan curve in space; that result was first obtained for general integrands independently by L. Cesari [8], J. M. Danskin [3], and A. G. Sigalov [27]. We also incorporate the notions of R. Courant with regard to the formulation of the free boundary problem.

We shall now give a brief discussion of the basic definitions and properties of the function spaces $W_p^m(Q)$, the so-called Sobolev spaces. The study and systematic use of these spaces in potential theory, partial differential equations, and calculus of variations has been developed by many authors. Their use goes back to the work of B. Levi, and important contributions have been made by L. Tonelli, G. C. Evans, J. W. Calkin, C. B. Morrey, S. L. Sobolev, and others. A sizeable list of references may be found in Morrey [22]. The basic properties of these spaces may be found in [2, 19, 20, 22, 23, 30].

Definition 1.1. A function z is of class $W_p^1(Q)$, $p \geq 1$, if z is of class $L_p(Q)$ and there exist functions h_1, h_2 also of class $L_p(Q)$, such that

$$\int_Q g(u,v)h_i(u,v)\,du\,dv = -\int_Q g_i(u,v)z(u,v)\,du\,dv \quad , \quad i = 1, 2 ,$$

for all functions g of class C^1 with compact support in Q, where $g_1 = g_u$, $g_2 = g_v$.

The functions h_i are uniquely determined up to null functions, and furthermore, if z is of class $W_p^1(Q)$ and $z^* = z$ almost everywhere, then z^* is also of class $W_p^1(Q)$ and the same functions h_i will serve for z^* in the above definition. The functions h_i are called the generalized derivatives of the function z, and we shall write $z_u = h_1$, $z_v = h_2$, and call z_u and z_v the derivatives of z. In the language of the theory of distributions, the distribution derivative of the function z (as a distribution) is a function of class $L_p(Q)$. Since we shall be concerned with vector functions, we shall say that a vector is of class $W_p^1(Q)$ if each component is in that class.

<u>Definition 1.2</u>. The <u>space</u> $W_p^1(Q)$ consists of equivalence classes of functions of class $W_p^1(Q)$ under the equivalence relation of equality almost everywhere.

<u>Definition 1.3</u>. <u>A function</u> z <u>is of class</u> $W_p^m(Q)$ if z is of class $W_p^1(Q)$ and if each of its generalized derivatives up to order m - 1 is of class $W_p^1(Q)$. Analogous to Definition 1.2 above, we define the <u>space</u> $W_p^m(Q)$.

By introducing the norms

$$\|z\|^2_{W_2^1} = \int_Q |z|^2 du\, dv + \int_Q (|z_u|^2 + |z_v|^2) du\, dv$$

$$= L_2(z, Q) + D(z, Q)$$

and

$$\|z\|^2_{W_2^2} = L_2(z, Q) + D(z, Q) + \int_Q (|z_{uu}|^2 + 2|z_{uv}|^2 + |z_{vv}|^2) du\, dv$$

$$= L_2(z, Q) + D(z, Q) + J(z, Q) \quad ,$$

the spaces $W_2^1(Q)$ and $W_2^2(Q)$ become, respectively, Banach spaces (in fact, Hilbert spaces).

Theorem 1.1. If $z \in W_p^1(Q)$, $p \geq 1$, then z has a representative $\bar{z}$ which is absolutely continuous along almost all lines parallel to the coordinate axes, and the partial derivatives of $\bar{z}$ are representatives of the generalized derivatives of z. If $\bar{z}$ is a function of class $L_p(Q)$, $p \geq 1$, which is absolutely continuous on almost all lines parallel to the coordinate axes, and if each first partial derivative of $\bar{z}$ is of class $L_p(Q)$, then $\bar{z}$ is of class $W_p^1(Q)$.

Theorem 1.2. Let $z \in W_p^1(Q)$. Then

(a) there exists a sequence $\{z_n\}$ of Lipschitz functions such that $z_n \to z$ in $W_p^1(Q)$;

(b) there is a function $\phi \in L_p(Q^*)$ such that $z_n \to \phi$ in $L_p(Q^*)$ for every sequence as in (a) ;

(c) if $z \in W_p^m(Q)$, then there exists a sequence $\{z_n\}$ of functions of class $C^\infty(\bar{Q})$ such that $z_n \to z$ in $W_p^m(Q)$; and

(d) if $z \in W_p^m(Q)$, then $\phi \in W_p^{m-1}(Q^*)$; if also $z_n \to z$ in $W_p^m(Q)$, then $\phi_n \to \phi$ in $W_p^{m-1}(Q^*)$.

Theorem 1.3. (a) If $p > 1$, then bounded families in $W_p^m(Q)$ are conditionally compact with respect to weak convergence in $W_p^m(Q)$;

(b) if $z_n \to z$ weakly in $W_p^m(Q)$, then $z_n \to z$ strongly in $W_p^{m-1}(Q)$ and $\phi_n \to \phi$ in $W_p^{m-1}(Q^*)$; and

(c) if $mp > 2$, then every function $z \in W_p^m(Q)$ is continuous (i.e., is equivalent to a continuous function); moreover, any set $\{z\}$ of functions in $W_p^m(Q)$, $mp > 2$, with uniformly bounded norms, is a compact set in the space of continuous functions.

Definition 1.4. A function ϕ is a Friedrichs mollifier (or a mollifier) if ϕ is of class $C^\infty(E^2)$, $\phi(u,v) \geq 0$, ϕ has compact support in the unit disk B, and $\int_B \phi(u,v)\,du\,dv = 1$.

Definition 1.5. If z is locally summable on an open set G, we define its ϕ-mollified function z_r by

$$z_r(w) = \int_{B(w,r)} z(x)\phi_r^*(x-w)dx \quad , \qquad w \in G_r = \{w \in G : B(w,r) \subset G\} \quad ,$$

where $B(w,r)$ is the disk with center w and radius r, $w = (u,v)$, $x = (x_1, x_2)$, and $\phi_r^*(y) = r^{-2}\phi(r^{-1}y)$, $y = (y_1, y_2)$.

Theorem 1.4. Suppose $z \in L_p$, ϕ is a mollifier, and z_r denotes its ϕ-mollified function. Then

(a) $z_r \in C^{\infty}(E^2)$ and its derivatives are formed by differentiating under the integral sign ;

(b) $z_r \to z$ almost everywhere as $r \to 0$; if z is continuous, the convergence is uniform ;

(c) $\|z_r\|_{L_p} \leq \|z\|_{L_p}$;

(d) $z_r \to z$ in $L_p(E^2)$;

(e) if $z \in W_p^m$ and h is a generalized derivative of $D^k z$ of order $\leq m$, then $D^k z_r = h_r$, so that all such $D^k z_r \to D^k z$ in $L_p(E^2)$ as $r \to 0$;

(f) if $z \in W_p^m(G)$, then (e) holds for $w \in G_r$ and the convergence in (e) holds on each compact subset of G.

Most of these results will be stated in the text which follows as they are needed. We have stated all the definitions and theorems in this section in terms of functions of 2 independent variables, that is, functions defined on a subset of E^2. However, all that has been said holds for functions of n independent variables, except that Theorem 1.3 (c) holds for $mp > n$, and r^{-2} must be replaced by r^{-n} in the definition of $\phi_r^*(y)$. Also we remark that any function in class $L_p(G)$ becomes a function in class $L_p(E^2)$ by simply defining it to be identically 0 outside of G.

For notational convenience, we shall write P_2 for W_2^1, P_2' for the class of functions in P_2 which are absolutely continuous along almost all lines parallel to the coordinate axes, P_2'' for those functions of class P_2 which are continuous, and H^2 for W_2^2.

We propose to consider surfaces (vectors) whose boundaries are free on fixed manifolds. R. Courant has shown [11, p. 220] that there are smooth vectors which minimize Dirichlet's integral but whose boundaries are not continuous curves. Although there are some conditions known under which minimizing surfaces are continuous along, and up to, the boundary (see, for example, [10, 17]), we shall use the torus as our manifold for the sake of convenience, but we shall produce a general existence proof which holds even in case the manifold is the one of Courant's example. Thus, we shall make a precise definition of the concept of a surface having its boundary on a manifold.

Definition 1.6. [11, p. 202] Let $\mathcal{M}$ be a (topologically) closed manifold (i.e., closed connected point set) in the 3-dimensional space E^3, and let $\rho_{\mathcal{M}}[z(u,v)]$ denote the shortest distance from the point $z(u,v)$ to the manifold $\mathcal{M}$. If z is a vector defined on $\overline{Q}$, we say that the image of Q^* under z, or $z(Q^*)$, lies on $\mathcal{M}$ if $(u,v) \to (u_0, v_0)$, $(u,v) \in Q^o$, $(u_0, v_0) \in Q^*$, implies that $\rho_{\mathcal{M}}[z(u,v)] \to 0$, for all such (u,v), (u_0, v_0). (We shall sometimes say that "z lies on $\mathcal{M}$," or "the boundary of z lies on $\mathcal{M}$.")

Let $\mathcal{T}$ be a solid torus in $\mathbf{E}^3$, and let H be a fixed circle linking $\mathcal{T}$ and situated so that for every point p on H

$$\text{dist}(p, \mathcal{T}) = d = \text{constant} > 0 \quad ,$$

and we take d small relative to the diameter of $\mathcal{T}$. Our manifold will really be the surface of $\mathcal{T}$, and the only reason for mentioning the solid torus above was to prescribe the correct homology class for H.

<u>Definition 1.7</u>. We say that z <u>links</u> H, <u>or</u> $z(Q^*)$ <u>links</u> H, if there is a number $r > 0$ such that for every closed curve C, homotopic to Q^*, lying in the strip $\mathbf{S}_r \subset Q$ adjacent to Q^* having width r, z(C) is a closed curve linking H.

For all the notions of topological linking and intersection, we refer to Alexandroff-Hopf [1, pp. 413-425].

2. The Main Lemma Concerning Boundary Values

We first state a lemma of Reshetnyak [13, p. 747].

Lemma 2.1. Let z be a vector of class $P_2(Q)$, defined on $\overline{Q}$ and continuous on Q^*. Then there exists a sequence $\{\zeta_n(u,v)\}$ of continuous piecewise-linear vectors which are non-degenerate in any triangle $\Delta \subset Q$, and having the properties that as $n \to \infty$, $\{\zeta_n\}$ converges strongly in $\mathbf{P}_2$ to the vector z, while on Q^* $\{\zeta_n\}$ converges uniformly to z. Moreover, if $z(u,v)$ is continuous on Q, then $\{\zeta_n\}$ converges to z uniformly on every closed subset $F \subset Q$ whose boundary does not contain a point of Q^*. Furthermore, there is actually a sequence of continuously differentiable vectors satisfying all the conditions above.

The goal now is to prove a type of closure theorem for admissible vectors, which will be instrumental in our treatment of free boundaries. We remark that $\rho_{\mathcal{M}}[z(u,v)] \to 0$ uniformly as $(u,v) \to Q^*$ if the boundary of z is on $\mathcal{M}$ [11, p. 202]. In the following lemma we shall take Q to be the entire upper half-plane for the sake of simplicity. By a conformal mapping, the conclusion will hold for the case of the square.

Let $\overline{Q}$ be the upper half-plane $v \geq 0$, and denote by Q'_η the set $\{(u,v) : v \geq \eta\}$, $\eta > 0$. We shall call $P_2''(Q')$ the class of all vectors which are in $P_2''(Q'_\eta)$ for every $\eta > 0$ (i.e., in $P_2(Q)$ and continuous on Q^o).

Lemma 2.2. Let $\{z_p\}$ be a sequence of vectors of class $H^2(Q')$ which converge uniformly on each Q'_η to a vector z of class $P_2''(Q')$, and assume that the image of $Q^* : [-\infty < u < +\infty; v = 0]$ under each z_p lies on the manifold $\mathfrak{M}$. Suppose that there is a constant K, $0 < K < +\infty$, such that $\|z_p\|_{P_2(Q'_\eta)} \leq K^2$ for all p and all $\eta > 0$. Then the image of Q^* under z also lies on $\mathfrak{M}$.

Proof. Since for each p and each $Q'_\eta \subset Q$ we have $D(z_p, Q'_\eta) \leq K^2$, letting $\eta \to 0$ we obtain $D(z_p, Q) \leq K^2$. By the lower semicontinuity of $D(z, G)$ with respect to weak convergence in $P_2(G)$ [20, Chapter III, § 4], we have

$$D(z, Q'_\eta) \leq \varliminf_{p \to \infty} D(z_p, Q'_\eta) \leq \varliminf_{p \to \infty} D(z_p, Q) \leq K^2 < \infty ,$$

so that letting $\eta \to 0$ we find $D(z, Q) \leq K^2$.

Let $\varepsilon > 0$ be given. Then we may choose a number h, $0 < h < \varepsilon$, such that in the strip $Q_h : [-\infty < u < +\infty ; 0 \leq v \leq 2h]$ we have $D(z, Q_h) < \varepsilon^4$.

We recall that since z_p and z are in class $P_2(Q')$, each is equivalent, respectively, to a function $\bar{z}_p$, $\bar{z}$ which is absolutely continuous on almost all horizontal and almost all vertical lines. We may assume (since all z_p, z are continuous) [2, p. 181] that $\bar{z}_p = z$, $\bar{z} = z$, and that the line $v = h$ is a line of absolute continuity for all the functions concerned.

Let $S_h : [h/3 \leq v \leq 2h \; ; -1 < u < +1] = \overline{M}$. In this strip we still have $D(z, S_h) < \varepsilon^4$. By Lemma 2.1 there is a continuously differentiable function $\zeta(u,v)$ defined on S_h such that

$$\text{(i)} \quad \| z - \zeta \|^2_{P_2(M)} < \varepsilon^4 \qquad ,$$

$$\text{(ii)} \quad | z(u,v) - \zeta(u,v) | < \varepsilon^2 \quad , \qquad \text{for all} \quad (u,v) \in M^* \quad ,$$

$$\text{(iii)} \quad | z(u,v) - \zeta(u,v) | < \varepsilon \quad , \qquad \text{for all} \quad (u,v) \in \overline{F} \quad ,$$

where $\overline{F} = [h/3 + \tau \leq v \leq 2h - \tau; -1 + h/3 < u < 1 - h/3]$, τ a fixed number, $0 < \tau < h/8$.

Next, let $\phi_\eta(u,v)$ be a Friedrichs mollifier (Definition 1.4), i.e., ϕ_η is of class C^∞, $\phi_\eta \geq 0$, $\phi_\eta(u,v) = 0$ for $|(u,v)| \geq \eta$, $\iint \phi_\eta(u,v)\,du\,dv = 1$. By writing $\phi_\eta(u,v)$ we really mean the vector
E^2
$(\phi_\eta, \phi_\eta, \phi_\eta)$, and when we write $z\phi_\eta$ we really mean the vector $(z^1(u,v)\phi_\eta(u,v), z^2(u,v)\phi_\eta(u,v), z^3(u,v)\phi_\eta(u,v))$. We shall always take $\eta < h/6$.

Define

$$r_\eta(u,v) = \int_{h/2}^{3h/2} \int_{-1+h/3}^{1-h/3} \zeta_u(x,y)\phi_\eta(u - x, \; v - y)\,dx\,dy \quad .$$

Then we know (Section 1) that $r_\eta(u,v)$ is a C^∞ vector and $r_\eta(u,v) \to \zeta_u(u,v)$ uniformly as $\eta \to 0$. Thus we may choose η so small that

$$\text{(iv)} \quad |r_\eta(u,v) - \zeta_u(u,v)| < \varepsilon^2 \quad .$$

$$\text{Let} \quad J(r_\eta, G) = \iint_G (|r_{\eta_{uu}}|^2 + 2|r_{\eta_{uv}}|^2 + |r_{\eta_{vv}}|^2) du\, dv \quad ,$$

$G = [-1 + h/3 \leq u \leq 1 - h/3 \,;\, h/3 + \tau \leq v \leq 5h/3\,]$. Let $\bar{h} = h/3 + \tau$. Then $J(r_\eta, G) < +\infty$. We may choose a number h', $\bar{h} < h' \leq 5h/3$, such that

$$\text{(v)} \quad D(r_\eta, G_1) < \varepsilon^4 \quad , \quad G_1 : [-1 + h/3 \leq u \leq 1 - h/3; \bar{h} \leq v \leq h'] \; .$$

Then we still have

$$\text{(vi)} \quad D(z, G_1) < \varepsilon^4 \quad .$$

Let $h'' = \bar{h} + \frac{1}{2}(h' - \bar{h})$. By the uniform convergence of z_p toward z, there is a number $N = N(\varepsilon, h'')$ such that for all $p \geq N$

(2.1) $$|z_p(u,h'') - z(u,h'')| < \varepsilon \quad , \quad -1 \leq u \leq 1 \quad .$$

We now obtain the following chain of inequalities:

$$\left|\int_0^u \zeta_u(u,h'')du\right| \leq \left|\frac{1}{h'-\bar{h}}\int_{\bar{h}}^{h'}\int_0^u \zeta_u(u,h'')du\,dv\right|$$

$$\leq \frac{1}{h'-\bar{h}}\int_{\bar{h}}^{h'}\int_0^{|u|} |\zeta_u(u,h'')|\,du\,dv$$

$$\leq \frac{1}{h'-\bar{h}}\int_{\bar{h}}^{h'}\int_0^{|u|} |\zeta_u(u,h'') - \zeta_u(u,v)|\,du\,dv \tag{2.2}$$

$$+\frac{1}{h'-\bar{h}}\int_{\bar{h}}^{h'}\int_0^{|u|} |\zeta_u(u,v)|\,du\,dv \quad .$$

First consider the second member of the right-hand side of (2.2):

$$\frac{1}{h'-\bar{h}}\int_{\bar{h}}^{h'}\int_0^{|u|} |\zeta_u(u,v)|\,du\,dv \leq \frac{1}{h'-\bar{h}}\left\{\int_{\bar{h}}^{h'}\int_0^{|u|} du\,dv\right\}^{\frac{1}{2}}\left\{\int_{\bar{h}}^{h'}\int_0^{|u|} |\zeta_u(u,v)|^2 du\,dv\right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{h'-\bar{h}}\left\{(h'-\bar{h})|u|\right\}^{\frac{1}{2}}\left\{D(\zeta, G_1)\right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{\ell}\left\{\ell\,|u|\right\}^{\frac{1}{2}}\left\{\sqrt{D(\zeta - z, G_1)} + \sqrt{D(z, G_1)}\right\}$$

$$\leq \frac{|u|}{\ell}^{\frac{1}{2}}\; 2\,\varepsilon^2 \quad , \tag{2.3}$$

as follows by (i) and (vi), where $\ell = h' - \bar{h}$.

Next we obtain

$$\frac{1}{\ell}\int_{\bar h}^{h'}\int_0^{|u|}|r_\eta(u,h'') - r_\eta(u,v)|\,du\,dv = \frac{1}{\ell}\int_{\bar h}^{h'}\int_0^{|u|}\left|\int_v^{h''} r_{\eta_v}(u,t)dt\right|du\,dv$$

$$\leq \frac{1}{\ell}\left\{\int_{\bar h}^{h''}\int_0^{|u|}\left|\int_v^{h''} r_{\eta_v}(u,t)dt\right|du\,dv + \int_{h''}^{h'}\int_0^{|u|}\left|\int_{h''}^{v} r_{\eta_v}(u,t)dt\right|du\,dv\right\}$$

$$\leq \frac{1}{\ell}\left\{\int_{\bar h}^{h''}\int_0^{|u|}\int_v^{h''}|r_{\eta_v}(u,t)|\,dt\,du\,dv + \int_{h''}^{h'}\int_0^{|u|}\int_{h''}^{v}|r_{\eta_v}(u,t)|\,dt\,du\,dv\right\}$$

$$\leq \frac{1}{\ell}\left\{\int_{\bar h}^{h''}\int_0^{|u|}\int_{\bar h}^{h''}|r_{\eta_v}(u,t)|\,dt\,du\,dv + \int_{h''}^{h'}\int_0^{|u|}\int_{h''}^{h'}|r_{\eta_v}(u,t)|\,dt\,du\,dv\right\}$$

$$\leq \frac{1}{\ell}\left\{(h''-\bar h)\int_0^{|u|}\int_{\bar h}^{h''}|r_{\eta_v}(u,t)|\,dt\,du + (h'-h'')\int_0^{|u|}\int_{h''}^{h'}|r_{\eta_v}(u,t)|\,dt\,du\right\}$$

$$\leq \frac{(h''-\bar h)}{\ell}\int_0^{|u|}\int_{\bar h}^{h'}|r_{\eta_v}(u,t)|\,dt\,du \qquad (\text{since } h''-\bar h = h'-h'')$$

$$\leq \frac{1}{2}\int_0^{|u|}\int_{\bar h}^{h'}|r_{\eta_v}(u,t)|\,dt\,du \qquad (\text{since } \ell = h'-\bar h)$$

$$\leq \frac{1}{2}\left\{\int_0^{|u|}\int_{\bar h}^{h'}dt\,du\right\}^{\frac{1}{2}}\left\{\int_0^{|u|}\int_{\bar h}^{h'}|r_{\eta_v}(u,t)|\,dt\,du\right\}^{\frac{1}{2}}$$

$$(2.4) \qquad \leq \frac{1}{2}\left\{|u|(h'-\bar h)\right\}^{\frac{1}{2}}\left\{D(r_\eta, G_1)\right\}^{\frac{1}{2}} \leq \frac{(\ell|u|)^{\frac{1}{2}}}{2}\,\varepsilon^2 \quad .$$

Now using (iv) we obtain

$$\frac{1}{\ell}\int_{\bar{h}}^{h'}\int_0^{|u|} |\zeta_u(u,h'') - \zeta_u(u,v)|\,du\,dv \le \frac{1}{\ell}\int_{\bar{h}}^{h'}\int_0^{|u|} |\zeta_u(u,h'') - r_\eta(u,h'')|\,du\,dv$$

$$+\frac{1}{\ell}\int_{\bar{h}}^{h'}\int_0^{|u|} |r_\eta(u,h'') - r_\eta(u,v)|\,du\,dv$$

$$+\frac{1}{\ell}\int_{\bar{h}}^{h'}\int_0^{|u|} |r_\eta(u,v) - \zeta_u(u,v)|\,du\,dv$$

$$\le \frac{1}{\ell}\,\ell|u|\,\varepsilon^2 + \frac{(\ell|u|)^{\frac{1}{2}}}{2}\varepsilon^2 + \frac{1}{\ell}\,\ell|u|\,\varepsilon^2$$

(2.5) $$\le (2|u| + \frac{(\ell|u|)^{\frac{1}{2}}}{2})\varepsilon^2$$

Now from (2.2), (2.4), (2.5), we see that

$$|\int_0^u \zeta_u(u,h'')du| \le (2|u| + \frac{(\ell|u|)^{\frac{1}{2}}}{2})\varepsilon^2 + 2(\frac{|u|}{\ell})^{\frac{1}{2}}\varepsilon^2$$

$$\le [2|u| + |u|^{\frac{1}{2}}(\frac{\ell}{4})^{\frac{1}{2}} + |u|^{\frac{1}{2}}(\frac{4}{\ell})^{\frac{1}{2}}]\,\varepsilon^2$$

(2.6)

$$\le [2|u|^{\frac{1}{2}} + |u|^{\frac{1}{2}}(\frac{\ell}{4})^{\frac{1}{2}} + |u|^{\frac{1}{2}}(\frac{4}{\ell})^{\frac{1}{2}}]\,\varepsilon^3, \quad \text{since } |u| \le 1,$$

(2.6) $$\le |u|^{\frac{1}{2}}[2 + (\frac{\ell}{4})^{\frac{1}{2}} + (\frac{4}{\ell})^{\frac{1}{2}}]\varepsilon^2 .$$

Therefore, for all u, $-1 \le u \le 1$, we have (since $\zeta(u,v)$ is of class C^1), (2.7) $$|\zeta(u,h'') - \zeta(0,h'')| \leq |u|^{\frac{1}{2}} [2 + (\tfrac{\ell}{4})^{\frac{1}{2}} + (\tfrac{4}{\ell})^{\frac{1}{2}}] \varepsilon^2 \quad .$$

Hence for $|u|^{\frac{1}{2}} < \dfrac{1}{[2 + (\frac{\ell}{4})^{\frac{1}{2}} + (\frac{4}{\ell})^{\frac{1}{2}}]\,\varepsilon} = \delta^{\frac{1}{2}}$, or

(2.8) $$|u| < \frac{1}{[2 + (\frac{\ell}{4})^{\frac{1}{2}} + (\frac{4}{\ell})^{\frac{1}{2}}]^2 \varepsilon^2} = \delta$$

we obtain

(2.9) $$|\zeta(u,h'') - \zeta(0,h'')| < \varepsilon \quad .$$

Now combining (2.9) with (iii), we get

(2.10) $$\begin{aligned} |z(u,h'') - z(0,h'')| &\leq |z(u,h'') - \zeta(u,h'')| + |\zeta(u,h'') - \zeta|0,h'')| \\ &\quad + |\zeta(0,h'') - z(0,h'')| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon \quad . \end{aligned}$$

Finally, from (2.1) and (2.10) we conclude that for $|u| < \delta$ and $p \geq N$

(2.11) $$\begin{aligned} |z(0,h'') - z_p(u,h'')| &\leq |z(0,h'') - z(u,h'')| + |z(u,h'') - z_p(u,h'')| \\ &< 3\varepsilon + \varepsilon = 4\varepsilon \quad . \end{aligned}$$

Next, let $p \geq N$ be fixed. Since $z_p \in H^2(Q')$ we have $z_{p_v} \in P_2(Q')$, and furthermore $z_{p_v} \in P_2(Q_h)$ on the strip Q_h of height 2h. Thus there is a sequence $\{\zeta_m(u,v)\}$ of functions of class $C^1(Q_h)$ such that

(2.12) $$\| \zeta_m - z_{p_v} \|_{P_2(Q_h)} \to 0 \quad \text{as} \quad m \to \infty \quad .$$

We may assume that this norm is $< \varepsilon^2$ for all m.

Let $G_m(u) = \int_0^{h''} |\zeta_m(u,v)|^2 dv$. Then $G_m(u)$ is of class C^1 on the interval $[-1 \le u \le 1]$. We may apply the mean value theorem to $\int_0^u G_m(t)dt$ on the interval $[0 \le u \le \delta]$. Thus there is a number u_m, $0 \le u_m \le \delta$, such that

$$\text{(2.13)} \qquad \int_0^{\delta} G_m(t)dt - \int_0^{0} G_m(t)dt = \delta G_m(u_m) = \int_0^{\delta} G_m(t)dt \quad ,$$

where u_m depends upon m . We have

$$\left\{\int_0^{\delta}\int_0^{h''} |\zeta_m(t,v)|^2 dv\, dt\right\}^{\frac{1}{2}} \le \left\{\int_0^{\delta}\int_0^{h''} |\zeta_m - z_{p_v}|^2\right\}^{\frac{1}{2}} + \left\{\int_0^{\delta}\int_0^{h''} |z_{p_v}|^2\right\}^{\frac{1}{2}}$$

$$< \varepsilon + K \quad ,$$

so $\int_0^{\delta}\int_0^{h''} |\zeta_m(t,v)|^2 dv\, dt < \varepsilon^2 + 2\varepsilon K + K^2$. Thus

$$\varepsilon^2 + 2\varepsilon K + K^2 > \int_0^{\delta}\int_0^{h''} |\zeta_m(t,v)|^2 dv\, dt = \int_0^{\delta} G_m(t)dt$$

$$= \delta G_m(u_m)$$

$$= \delta \int_0^{h''} |\zeta_m(u_m, v)|^2 dv \quad ,$$

and hence

$$(2.14) \qquad \int_0^{h''} |\zeta_m(u_m, v)|^2 dv \le \frac{\varepsilon^2 + 2\varepsilon K + K^2}{\delta} = \frac{(\varepsilon + K)^2}{\delta} \quad .$$

Now the sequence of numbers $\{u_m\}$ is bounded, so there is a subsequence $\{u_{m_i}\}$ which converges to a number u_0, $0 \le u_0 \le \delta$. Rename this subsequence $\{u_m\}$ and assume that $u_m \le u_0$ for every m, $u_m \uparrow u_0$, and $|u_m - u_0| < 1/m$. (There is no loss of generality here because of the nature of the proof which follows.) We have

$$(2.15) \qquad \left\{ \int_0^{h''} |z_{p_v}(u_0, v)|^2 dv \right\}^{\frac{1}{2}} \le \left\{ \int_0^{h''} |z_{p_v}(u_0, v) - \zeta_m(u_0, v)|^2 dv \right\}^{\frac{1}{2}}$$

$$+ \left\{ \int_0^{h''} |\zeta_m(u_0, v) - \zeta_m(u_m, v)|^2 dv \right\}^{\frac{1}{2}}$$

$$+ \left\{ \int_0^{h''} |\zeta_m(u_m, v)|^2 dv \right\}^{\frac{1}{2}} \quad .$$

Let us examine the second integral on the right-hand side.

$$\left\{\int_0^{h''} |\zeta_m(u_0,v) - \zeta_m(u_m,v)|^2 dv\right\}^{\frac{1}{2}} \leq \left\{\int_0^{h''} \left|\int_{u_m}^{u_0} \zeta_{m_u}(u,v)du\right|^2 dv\right\}^{\frac{1}{2}}$$

$$\leq \left\{\int_0^{h''}\int_{u_m}^{u_0} |\zeta_{m_u}|^2 du\, dv\right\}^{\frac{1}{2}}$$

$$\leq \left\{\int_0^{h''}\int_{u_m}^{u_0} |\zeta_{m_u} - z_{p_{vu}}|^2 du\, dv\right\}^{\frac{1}{2}}$$

$$+ \left\{\int_0^{h''}\int_{u_m}^{u_0} |z_{p_{vu}}|^2 du\, dv\right\}^{\frac{1}{2}}$$

$$\leq \left\{\iint_{Q_h} |\zeta_{m_u} - z_{p_{vu}}|^2 du\, dv\right\}^{\frac{1}{2}} + \left\{\int_0^{h''}\int_{u_m}^{u_0} |z_{p_{vu}}|^2 du\, dv\right\}^{\frac{1}{2}}.$$

Since $\zeta_m \to z_{p_v}$ in the P_2 norm, there exists an M_1 such that $m \geq M_1$ implies $\left\{\iint_{Q_h} |\zeta_{m_u} - z_{p_{vu}}|^2 du\, dv\right\}^{\frac{1}{2}} < \varepsilon^2/6$, and

by the absolute continuity of the integral (and the fact that $u_m \uparrow u_0$) there is an M_2 such that $m \geq M_2$ implies

$\left\{\int_0^{h''}\int_{u_m}^{u_0} |z_{p_{vu}}|^2 du\, dv\right\}^{\frac{1}{2}} < \varepsilon^2/6$. Thus for $m \geq \max[M_1, M_2]$ we have

$$(2.17) \qquad \left\{ \int_0^{h''} |\zeta_m(u_0, v) - \zeta_m(u_m, v)|^2 dv \right\}^{\frac{1}{2}} < \varepsilon^2/3 \quad .$$

Now examine the first integral on the right-hand side of (2.15). First, we know that $\zeta_m(u, v) \to z_{p_v}(u, v)$ almost everywhere on Q_h. Let $E \subset Q_h$ be the exceptional set. Then measure of $(E) \equiv \mu(E) = 0$, and $Q_h = E \cup B$, $B \cap E = \emptyset$. Then for almost all $\bar{u} \in [-1, 1]$, the set $I_{\bar{u}} = \{(\bar{u}, v) : 0 \leq v \leq 2h\}$ has intersection with B the set $B_{\bar{u}} = I_{\bar{u}} \cap B$, and $|B_{\bar{u}}| > 0$ where $|\ldots|$ denotes one-dimensional Lebesgue measure. In fact, for almost all $\bar{u}$, $|B_{\bar{u}}| = 2h$. Assume first that $\bar{u} = u_0$ is one of these points. Then $|B_{u_0}| = 2h > 0$ and $\zeta_m(u_0, v) \to z_{p_v}(u_0, v)$ for almost all $v \in [0, 2h]$. Therefore,

$$F_m(v) = |z_{p_v}(u_0, v) - \zeta_m(u_0, v)|^2 \to 0$$

almost everywhere on $[0, 2h]$ as $m \to \infty$. Then we know that $F_m(v) \to 0$ almost uniformly on $[0, 2h]$. That is, for every $\eta > 0$ there exists a measurable set $H \subset I_{2h} = [0, 2h]$ such that $|I_{2h} - H| < \eta$ and $\lim_{m\to\infty} F_m(v) = 0$ uniformly on H. Let us write $- H \equiv I_{2h} - H$.

Then $\lim_{m\to\infty} \int_H F_m(v)dv = 0$. Thus there is a number M_3 such that $m \geq M_3$ implies

$$(2.18) \qquad \left\{ \int_H F_m(v)dv \right\}^{\frac{1}{2}} = \left\{ \int_H |z_{p_v}(u_0, v) - \zeta_m(u_0, v)|^2 dv \right\}^{\frac{1}{2}} < \varepsilon^2/6 \ .$$

Next,

$$\left\{\int_{-H}^{} |z_{p_v}(u_0,v) - \zeta_m(u_0,v)|^2 dv\right\}^{\frac{1}{2}} = \left\{\int_0^1 \int_{-H} |z_{p_v}(u_0,v) - \zeta_m(u_0,v)|^2 dv\, du\right\}^{\frac{1}{2}}$$

$$\leq \left\{\int_0^1 \int_{-H} |z_{p_v}(u_0,v) - z_{p_v}(u,v)|^2 dv\, du\right\}^{\frac{1}{2}}$$

$$(2.19) \qquad + \left\{\int_0^1 \int_{-H} |z_{p_v}(u,v) - \zeta_m(u,v)|^2 dv\, du\right\}^{\frac{1}{2}}$$

$$+ \left\{\int_0^1 \int_{-H} |\zeta_m(u,v) - \zeta_m(u_0,v)|^2 dv\, du\right\}^{\frac{1}{2}} .$$

Examining the first integral on the right we see that

$$\left\{\int_0^1 \int_{-H} |z_{p_v}(u_0,v) - z_{p_v}(u,v)|^2 dv\, du\right\}^{\frac{1}{2}} = \left\{\int_0^1 \int_{-H} \left|\int_u^{u_0} z_{p_{vu}}(u,v) du\right|^2 dv\, du\right\}^{\frac{1}{2}}$$

$$= \left\{\int_0^{u_0} \int_{-H} \left|\int_u^{u_0} z_{p_{vu}} du\right|^2 dv du + \int_{u_0}^1 \int_{-H} \left|\int_u^{u_0} z_{p_{vu}} du\right|^2 dv\, du\right\}^{\frac{1}{2}}$$

$$\leq \left\{\int_0^{u_0} \int_{-H} \left[\int_u^{u_0} |z_{p_{vu}}|^2 du\right] dv\, du + \int_{u_0}^1 \int_{-H} \left[\int_{u_0}^u |z_{p_{vu}}|^2 du\right] dv\, du\right\}^{\frac{1}{2}}$$

$$\leq \left\{\int_0^{u_0} \int_{-H} \left[\int_0^{u_0} |z_{p_{vu}}|^2 du\right] dv\, du + \int_{u_0}^1 \int_{-H} \left[\int_{u_0}^1 |z_{p_{vu}}|^2 du\right] dv\, du\right\}^{\frac{1}{2}}$$

$$(2.20) \qquad \leq \left\{u_0 \int_{-H} \int_0^{u_0} |z_{p_{vu}}|^2 du\, dv + (1 - u_0) \int_{-H} \int_{u_0}^1 |z_{p_{vu}}|^2 du\, dv\right\}^{\frac{1}{2}} .$$

Now since $z_p \in H^2(Q')$ there is a number $\eta_1 > 0$ such that if $|-H| < \eta_1$, then the whole right-hand side of (2.20) is $< \frac{\varepsilon^2}{18}$.

By the same steps as in (2.20),

$$\text{(2.21)} \qquad \left\{ \int_0^1 \int_{-H} |\zeta_m(u,v) - \zeta_m(u_0,v)|^2 dv\, du \right\}^{\frac{1}{2}}$$

$$\leq \left\{ u_0 \int_{-H} \int_0^{u_0} |\zeta_{m_u}|^2 du\, dv + (1 - u_0) \int_{-H} \int_{u_0}^1 |\zeta_{m_u}|^2 du\, dv \right\}^{\frac{1}{2}} .$$

But since $\{\zeta_m\}$ converges in the $P_2(Q_h)$ norm to z_{p_v} and since the $P_2(Q_h)$ norm of z_{p_v} is finite, it follows that the $P_2(Q_h)$ norm of ζ_m is finite for each m and, in fact, uniformly bounded for all m. Therefore, there is a number $\eta_2 > 0$ such that if $|-H| < \eta_2$ then the whole right-hand side of (2.21) is $< \frac{\varepsilon^2}{18}$.

Finally, by the convergence of the sequence $\{\zeta_m\}$ in norm to z_{p_v}, there is a number M_4 such that $m \geq M_4$ implies

$$\text{(2.22)} \qquad \left\{ \int_0^1 \int_{-H} |z_{p_v}(u,v) - \zeta_m(u,v)|^2 dv\, du \right\}^{\frac{1}{2}} \leq \| z_{p_v} - \zeta_m \|_{P_2(Q_h)} < \frac{\varepsilon^2}{18} .$$

Now let $\eta = \min[\eta_1, \eta_2]$, $M = \max[M_1, M_2, M_3, M_4]$. Then there is a subset $H \subset I_{2h}$, $|-H| < \eta$, such that $m \geq M$ implies

$$\text{(2.23)} \qquad \left\{ \int_{-H} |z_{p_v}(u_0,v) - \{\zeta_m(u_0,v)|^2 dv \right\}^{\frac{1}{2}} < \frac{\varepsilon^2}{18} + \frac{\varepsilon^2}{18} + \frac{\varepsilon^2}{18} = \frac{\varepsilon^2}{6} .$$

Combining this with (2.22) we obtain

$$(2.24)\qquad \Big\{\int_0^{h''} |z_{p_v}(u_0, v) - \zeta_m(u_0, v)|^2 dv\Big\}^{\frac{1}{2}}$$

$$\leq \Big\{ \int_{-H} |z_{p_v}(u_0, v) - \zeta_m(u_0, v)|^2 dv$$

$$+ \int_{-H} |z_{p_v}(u_0, v) - \zeta_m(u_0, v)|^2 dv\Big\}^{\frac{1}{2}}$$

$$< \Big\{ \frac{\varepsilon^4}{36} + \frac{\varepsilon^4}{36} \Big\}^{\frac{1}{2}}$$

$$< \frac{\varepsilon^2}{3} \qquad .$$

We now combine (2.14), (2.15), (2.17), (2.24) to get that for $m \geq M$

$$(2.25)\qquad \Big\{\int_0^{h''} |z_{p_v}(u_0, v)|^2 dv\Big\}^{\frac{1}{2}} < \frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3} + \frac{\varepsilon + K}{\delta^{\frac{1}{2}}} \quad .$$

Now suppose that $\bar{u} = u_0$ is not one of the points for which $|B_{u_0}| = 2h$. Then we may choose a point u_1 so close to u_0, $u_1 < u_0$, that

$$(2.26)\qquad \Big\{ \int_0^{h''} \int_{u_1}^{u_0} |z_{p_{vu}}|^2 du\, dv\Big\}^{\frac{1}{2}} < \frac{\varepsilon^2}{6} \quad .$$

We must show that (2.17) holds now. Since $u_m \uparrow u_0$, for m sufficiently large, say, $m \geq M'$, we have $u_1 < u_m < u_0$, and so

$$\left\{\int_0^{h''}\int_{u_1}^{u_m} |z_{p_{vu}}|^2 du\,dv\right\}^{\frac{1}{2}} < \frac{\varepsilon^2}{6}.$$ The rest of the inequality (2.16)

holds now with u_0 replaced by u_1. Therefore, (2.17) holds with u_0 replaced by u_1. We note that if u_0 is replaced by u_1 in (2.24), the inequality still holds except that we may have to choose H differently. Further, (2.14) does not depend upon u_0 at all. Therefore, replacing u_0 by u_1 in (2.25) we obtain

$$(2.27) \qquad \left\{\int_0^{h''} |z_{p_v}(u_j, v)|^2 dv\right\} < \frac{\varepsilon^2}{3} + \frac{\varepsilon^3}{3} + \frac{\varepsilon + K}{\delta^{\frac{1}{2}}}$$

for either $j = 0$ or $j = 1$. We now replace u_j by u_0, where u_0 is whichever point u_0 or u_1 yields (2.27). We point out that u_0 depends upon p. From (2.27) we have

$$(2.28) \qquad |z_p(u_0, h'') - z_p(u_0, 0)| = \left|\int_0^{h''} z_{p_v}(u_0, v)dv\right|$$

$$\leq \int_0^{h''} |z_{p_v}(u_0, v)|dv$$

$$\leq \sqrt{h''}\left\{\int_0^{h''} |z_{p_v}(u_0, v)|^2 dv\right\}^{\frac{1}{2}}$$

$$\leq \sqrt{h''}\left\{\frac{\varepsilon^2}{3} + \frac{\varepsilon^2}{3} + \frac{\varepsilon + K}{\delta^{\frac{1}{2}}}\right\}^{\frac{1}{2}} .$$

Before proceeding further, we make the following observations. Let $\zeta_p(u,v) = z_p(u, v+\delta_p)$ where δ_p is a positive constant. Then the vector $\zeta_p(u,v)$ is continuous for $v = 0$, and the values $\zeta_p(u,0)$ lie on a curve M_p whose distance from the manifold can be made arbitrarily small if δ_p is chosen sufficiently small. Now as $\delta_p \to 0$, $\zeta_p(u,v) \to z_p(u,v)$, so we may prove the lemma by writing z_p instead of ζ_p and $\mathcal{M}$ instead of M_p. Note that with this convention, $z_p(u,v)$ has continuous boundary values on the line $v = 0$. Therefore, $z_p(u_0, 0)$ lies on $\mathcal{M}$.

Hence by the triangle inequality and (2.28) we have

$$\text{(2.29)} \qquad \rho_{\mathcal{M}}[z_p(u_0,h'')] \leq |z_p(u_0,h'') - z_p(u_0,0)| + \rho_{\mathcal{M}}[z_p(u_0,0)]$$

$$\leq \sqrt{h''}\left\{\frac{2\varepsilon^2}{3} + \frac{\varepsilon + K}{\delta^{\frac{1}{2}}}\right\}^{\frac{1}{2}} + 0 \quad .$$

From (2.11) we have for $p \geq N$, since $|u_0| < \delta$,

$$|z_p(u_0,h'') - z(0,h'')| < 4\varepsilon \quad .$$

By the triangle inequality,

$$\text{(2.30)} \qquad \rho_{\mathcal{M}}[z(0,h'')] \leq |z(0,h'') - z_p(u_0,h'')| + \rho_{\mathcal{M}}[z_p(u_0,h'')]$$

$$\leq 4\varepsilon + \sqrt{h''}\left\{\frac{2\varepsilon^2}{3} + \frac{\varepsilon + K}{\delta^{\frac{1}{2}}}\right\}^{\frac{1}{2}} \quad ,$$

and the right-hand side does not depend upon p.

Now we recall that

$$\delta^{\frac{1}{2}} = \frac{1}{[2 + (\frac{\ell}{4})^{\frac{1}{2}} + (\frac{4}{\ell})^{\frac{1}{2}}]\,\varepsilon} \quad , \quad \ell = h' - \bar{h},\ h'' = \bar{h} + \tfrac{1}{2}(h' - \bar{h}),\ h < \varepsilon .$$

The second term on the right-hand side of (2.30) yields

$$\text{(2.31)} \qquad \sqrt{h''}\left\{\frac{2\varepsilon^2}{3} + \frac{\varepsilon + K}{\delta^{\frac{1}{2}}}\right\}^{\frac{1}{2}} = \left\{\frac{2\varepsilon^2}{3}\sqrt{h''} + (\varepsilon + K)\varepsilon[2 + (\tfrac{\ell}{4})^{\frac{1}{2}} + (\tfrac{4}{\ell})^{\frac{1}{2}}]\sqrt{h''}\right\}$$

$$\leq \left\{\frac{2}{3}\varepsilon^{5/2} + (\varepsilon + K)\varepsilon[2\sqrt{h''} + (\frac{h''\ell}{4})^{\frac{1}{2}} + (\frac{4h''}{\ell})^{\frac{1}{2}}]\right\}$$

$$\leq \left\{\frac{2}{3}\varepsilon^{5/2} + (\varepsilon + K)\varepsilon[2\varepsilon^{\frac{1}{2}} + \varepsilon/2 + \sqrt{8}]\right\} .$$

Therefore, $\rho_{\mathfrak{M}}[z(0,h'')]$ can be made uniformly small. Since $(0,h'')$ was merely a convenient point on the line $v = h''$, we conclude that the same estimate holds for any point (u,h''), $-\infty < u < +\infty$. This proves the lemma.

<u>Corollary 2.2.1</u>. Lemma 2.2 is true if we assume the weaker condition that $z_p \in P_2''(Q')$.

<u>Proof</u>. The only place in which $z_p \in H^2(Q')$ is used is to obtain relation (2.8); namely, there exists a number u_0 such that

$$|z_p(u_0,h'') - z_p(u_0,0)| \leq \sqrt{h''}\left\{\frac{2\varepsilon^2}{3} + \frac{\varepsilon + K}{\delta^{\frac{1}{2}}}\right\}^{\frac{1}{2}} = R \quad .$$

This relation is used in (2.29) to obtain

$$\rho_{\mathfrak{M}}[z_p(u_0,h'')] \leq R \quad .$$

Now since z_p lies on the manifold $\mathcal{M}$, there is a number d, $0 < d < \frac{h''}{2}$, such that $\rho_{\mathcal{M}}[z_p(u_0, d)] < \varepsilon$, where d depends upon p. Let $\overline{D} = Q'_{d/2}$. Then letting $\zeta_p(u,v) = (z_p * \phi_\eta)(u,v)$, where ϕ_η is a mollifier and η is sufficiently small, $\zeta_p \in C^\infty(\overline{D})$, and hence $\zeta_p \in H^2(\overline{D})$, and we have

$$|z_p(u,v) - \zeta_p(u,v)| < \varepsilon \quad \text{for all} \quad (u,v) \in \overline{D} \quad .$$

Now we may repeat the reasoning from (2.12) to (2.28), except that all integrals involved with respect to v have the form $\int_{d/2}^{h''} (\ldots)dv$ instead of $\int_0^{h''} (\ldots)dv$. Then we conclude that there is a number u_0 such that

$$|\zeta_p(u_0, h'') - \zeta_p(u_0, d)| \leq R \quad .$$

Therefore,

$$\text{(2.28')} \qquad |z_p(u_0,h'') - z_p(u_0,d)| \leq |z_p(u_0,h'') - \zeta_p(u_0,h'')|$$
$$+ |\zeta_p(u_0,h'') - \zeta_p(u_0,d)|$$
$$+ |\zeta_p(u_0,d) - z_p(u_0,d)|$$
$$\leq 2\varepsilon + R \quad .$$

Finally,

$$\text{(2.29')} \qquad \rho_{\mathcal{M}}[z_p(u_0,h'')] \leq |z_p(u_0,h'') - z_p(u_0,d)| + \rho_{\mathcal{M}}[z_p(u_0,d)]$$
$$\leq 2\varepsilon + R + \varepsilon = 3\varepsilon + R \quad .$$

From here we may repeat the reasoning from (2.28) to (2.31) which gives the conclusion that $z(Q^*)$ lies on $\mathfrak{M}$. This proves the corollary.

Corollary 2.2.2. Lemma 2.2 remains true under the weakened conditions:

(1) The boundaries of the z_p are on manifolds M_p, where M_p tends to a continuous manifold $\mathfrak{M}$ in the sense that the greatest distance from points on M_p to points on $\mathfrak{M}$ tends to 0 as $p \to \infty$;

(2) $z_p \in P_2''(Q')$.

Proof. We must show that, under the remaining assumptions of the lemma, the relation

$$\rho_{M_p}[z_p(u,v)] \to 0 \qquad \text{as} \quad v \to 0$$

implies

$$\rho_{\mathfrak{M}}[z(u,v)] \to 0 \qquad \text{as} \quad v \to 0 \quad .$$

Let $g(M_p, \mathfrak{M})$ be the greatest distance from any point of M_p to $\mathfrak{M}$. By the triangle inequality we have

$$\rho_{\mathfrak{M}}[z(u,v)] \leq \rho_{M_p}[z(u,v)] + g(M_p,\mathfrak{M}) \quad ,$$

and hence

$$\rho_{\mathfrak{M}}[z(u,v)] \leq \underline{\lim}_{p\to\infty} \rho_{M_p}[z(u,v)] \quad .$$

Therefore it is sufficient to investigate $\rho_{M_p}[z(u,v)]$. M_p plays the role of $\mathfrak{M}$ in the lemma. Relation (2.31) shows that the bound on $\rho_{M_p}[z(u,v)]$ is independent of p. Therefore, the same bound holds for $\rho_{\mathcal{M}}[z(u,v)]$. But this bound can be made arbitrarily small, uniformly in h''. This completes the proof.

3. Some Lemmas Concerning Quasi-conformal Representations of Surfaces

In this section we shall define A-admissible vectors for our problem and then show that each one of these may be replaced by a vector with a quasi-conformal representation which is again A-admissible, and such that the new vector does not increase the integral I_0.

Definition 3.1. A vector z defined on $\overline{Q}$ is called A-admissible if

(i) $z \in P_2''(\overline{Q})$;

(ii) $z(Q^*)$ lies on a manifold whose greatest distance from the torus $\mathcal{T}$ is $< d/4$, $d = \text{dist}\,[H, \mathcal{T}]$;

(iii) $z(Q^*)$ links H ;

(iv) $I_0(z, Q) < +\infty$.

We now state some definitions and a representation theorem from surface area theory. These may be found in Cesari [9, pp. 472-486; 8, pp. 266-271].

Let $S : z = z(u,v)$, $(u,v) \in \overline{Q}$, be a continuous surface (vector, mapping). For each point $z_0 \in E^3$ which is in the graph of S, we denote by

$$S^{-1}(z_0) = \{ w = (u,v) \in \overline{Q} : z(w) = z_0 \} .$$

The set $S^{-1}(z)$, $z \in [S]$ = graph of S, is a closed subset of $\overline{Q}$, and hence its components γ are subcontinua of Q (possibly single points of $\overline{Q}$). Let us denote by G the collection of all continua $\gamma \in \overline{Q}$ which are components of at least one set $S^{-1}(z)$. The collection G has the following properties:

(i) each point $w \in \overline{Q}$ belongs to one and only one continuum γ of G;

(ii) G is the collection of maximal continua of $\overline{Q}$ on which the vector $z = z(w)$ is constant;

(iii) the collection G is an upper semi-continuous decomposition of $\overline{Q}$.

Definition 3.2. A surface $S : z = z(w)$, $w \in \overline{Q}$, is called a base surface if for any continuum $\gamma \in G$ the open set $E^2 - \gamma$ is connected.

Definition 3.3. A surface $S : z = z(w)$, $w \in \overline{Q}$, is called non-degenerate if

(i) for any continuum $\gamma \in G$ the open set $Q^o - \gamma \cap Q^o$ is connected;

(ii) $\gamma \supset Q^*$ for some $\gamma \in G$ implies $\gamma = \overline{Q}$.

The property in the definition of base surface and the two properties in the definition of non-degenerate surface are invariant under Fréchet equivalence.

Definition 3.4. A representation $S : z = z(w)$, $w \in \overline{Q}$, is called quasi-conformal if $z \in P_2''(\overline{Q})$ and $z \in P_2'(Q)$, and if $E = G$, $F = 0$ almost every where in Q, where

$$E = \sum_{i=1}^{3} z_u^i(w), \quad G = \sum_{i=1}^{3} z_v^i(w), \quad F = \sum_{i=1}^{3} z_u^i(w) z_v^i(w),$$

$z(w) = (z^1(w), z^2(w), z^3(w))$.

Lemma 3.1. (C. B. Morrey [18]; L. Cesari [9, p. 484]) Every non-degenerate surface S with finite Lebesgue area has a quasi-conformal representation $S : z = z(w)$, $w \in \overline{Q}$. For any representation of this kind we have

$$\text{Area } (S) = \int_Q |J| \, dw = \int_Q \sqrt{EG - F^2} \, dw = \tfrac{1}{2} \int_Q (E + G) = \tfrac{1}{2} D(z, Q).$$

Let $z \in P_2''(\overline{Q})$, let $S : z = z(w)$ be a base surface, and let $G = \{g\}$ be the upper semicontinuous collection of maximal continua on which $z(w)$ is constant. Let $\{g\}^*$ be the collection of all those $g \in G$ such that $g \cap Q^* \neq \emptyset$. Finally, let $\overline{F} = \{w \in \overline{Q} : w \in g, \ g \in \{g\}^*\}$.

Lemma 3.2. [4, p. 907]. $\overline{F}$ is closed.

Let $H = \overline{Q} - \overline{F}$. Since $Q^* \subset \overline{F}$ and $\overline{Q} = Q + Q^*$, then $H = Q - Q\overline{F}$, and therefore H is open in the plane. Let $\{\alpha_i\}$ be its (open) components, and denote by α_i^* the boundary of α_i. Then there are finitely many or countably many α_i, and each $p \in \alpha_i^*$ belongs to a continuum $g \in \{g\}^*$.

Lemma 3.3. [4, p. 907]. Each α_i is simply connected.

Lemma 3.4. [4, p. 919] Let n be any positive integer. Then there exist at most finitely many α_i such that $\text{diam}\{z(\alpha_i)\} \geq \frac{1}{n}$.

Now let z be an **A**-admissible base surface. Then by Lemma 3.4 there are finitely many components α_i, $i = 1, 2, \ldots, N$, such that $\text{diam}\{z(\alpha_i)\} > \frac{3d}{4}$. (Note that there must be at least one such α_i, or else z would not be **A**-admissible.)

Lemma 3.5. If $\text{diam}\{z(\alpha_j)\} < \frac{3d}{4}$, then the intersection number of $z(\alpha_j)$ with H is 0.

Proof. A necessary condition that $z(\alpha_j)$ have a non-zero intersection number with H is that some point of $z(\alpha_j)$ touch H. But

$$\text{diam}\{z(\alpha_j)\} = \text{diam}\{z(\bar{\alpha}_j)\} < \frac{3d}{4} \quad ,$$

and since $\alpha_j^* \cap Q^* = \emptyset$,

$$\text{dist}[z(\alpha_j^* \cap Q^*), \quad] < \frac{d}{4} \quad .$$

Thus no point of $z(\alpha_j)$ can touch H, and the intersection number is 0.

Now we separate $\{\alpha_i\}$, $i = 1, 2, \ldots, N$, into two mutually exclusive subcollections $\mathcal{A} = \{\alpha'_{i_s}\}$, $s = 1, \ldots, K$, and $\mathcal{B} = \{\alpha''_{i_r}\}$, $r = 1, \ldots, L$, $L + K = N$. The collection $\mathcal{A}$ is defined in the following way. Let P_0 be the center of gravity of $\mathcal{T}$, let $\Pi(H)$ be the plane containing the circle H, and let Π' be the (unique) plane passing through P_0, through the center h_0 of H, and perpendicular to $\Pi(H)$. Then $\alpha_i \in \mathcal{A}$ if and only if for any plane Π passing through P_0 and perpendicular to Π' we have

$$\Pi \cap z(\alpha_i^*) \neq \emptyset \quad .$$

It is possible that $\mathcal{B} = \emptyset$, but $\mathcal{A} \neq \emptyset$ since z is A-admissible.

<u>Lemma 3.6</u>. The intersection number of $z(\alpha''_{i_r})$ with H is 0 for all $\alpha''_{i_r} \in \mathcal{B}$.

<u>Proof</u>. Since $\alpha''_{i_r} \in \mathcal{B}$ there is a plane Π satisfying the above conditions with $\Pi \cap z(\alpha''^*_{i_r}) = \emptyset$. We have

$$\text{maximum dist}[z(\alpha''^*_{i_r}), \mathcal{T}] \leq \max \text{dist}[z(Q^*), \mathcal{T}] < \frac{d}{4} \quad ,$$

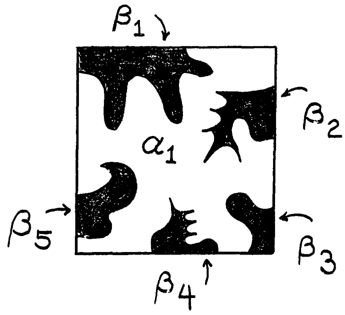
and d was assumed small relative to the diameter of $\mathcal{T}$. Therefore, $z(\alpha''^*_{i_r})$ is not linked with H and so the intersection number of $z(\alpha''^*_{i_r})$ with H is 0.

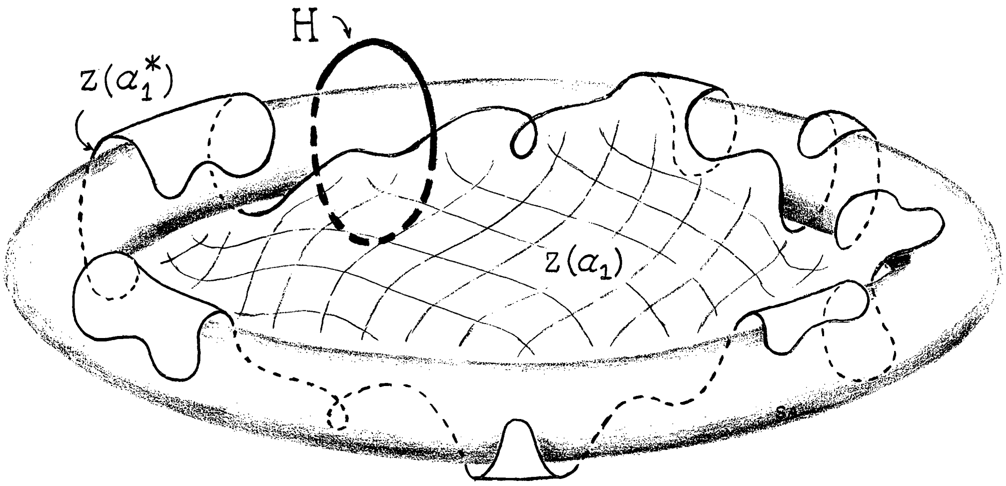
By Lemmas 3.5 and 3.6, and by the hypothesis on z,

$$\text{linking number}\{z(Q^*), H\} = \sum_{s=1}^{k} \text{intersection number}(z(\alpha'_{i_s}), H) \neq 0 .$$

Thus for one of the α'_{i_s}, call it α_1, the intersection number of $z(\alpha_1)$ with H is $\neq 0$. Furthermore, $z(\alpha_1^*)$ "winds around the hole of the torus" since for any plane Π satisfying the conditions on page 81, $\Pi \cap z(\alpha_1^*) \neq \emptyset$.

We mention also the well-known fact that since α_1 is a bounded, open, connected, simply connected set in the plane, α_1^* is closed and connected.





We shall now describe a process for cutting off the "pinches" from $S : z = z(w)$, $w \in \overline{Q}$, leaving an **A**-admissible surface $\overline{S} : z = \bar{z}(w)$, $w \in \overline{Q}$, which is non-degenerate. Such a surface $\overline{S}$ always has a quasi-conformal representation (Lemma 3.1).

The complement of $\bar{\alpha}_1$ in Q, $\mathcal{C}(\bar{\alpha}_1)$, is open in $\overline{Q}$. Thus $\mathcal{C}(\bar{\alpha}_1) = \bigcup_{i \in I} \beta_i$, β_i open in $\overline{Q}$, I a finite or countable index set. We notice that $\beta_i^* \subset \alpha_i^*$, and moreover $\beta_i^* \subset g_i \in \{g_i\}^*$. Therefore, z is constant on $\beta_i^* \subset g_i$.

Now we define a vector $\bar{z}(w)$, $w \in \overline{Q}$, as follows:

$$\bar{z}(w) = \begin{cases} z(w), & w \in \bar{\alpha}_1 \\ z(\beta_i^*), & w \in \bar{\beta}_i \end{cases} \qquad (3.1)$$

Let $I_{\bar{u}} = \{(u,v) : u = \bar{u},\ 0 \leq v \leq 1\}$, $\bar{u} \in [0,1]$. That is, $I_{\bar{u}}$ is the intersection of the line $L_{\bar{u}} : u = \bar{u}$ with $\overline{Q}$. Let

$\mathcal{O}_{\bar{u}} = I_{\bar{u}} \cap \mathcal{C}(\bar{\alpha}_1)$, an open linear set, and let $F_{\bar{u}} = I_{\bar{u}} \cap \bar{\alpha}_1$, a closed linear set. Then $\mathcal{O}_{\bar{u}} \cap F_{\bar{u}} = \emptyset$, and $I_{\bar{u}} = \mathcal{O}_{\bar{u}} \cup F_{\bar{u}}$. The set $\mathcal{O}_{\bar{u}}$ may be written as $\mathcal{O}_{\bar{u}} = \bigcup_{i \in I} J_i$, J_i disjoint open (intervals) components of $\mathcal{O}_{\bar{u}}$, I a finite or countable index set. $F_{\bar{u}}$ may be written as $F_{\bar{u}} = \bigcup_{\lambda \in \Lambda} D_\lambda$, D_λ disjoint (closed) components of $F_{\bar{u}}$. We note that $z(w) =$ constant $= c_i$ for all $w \in J_i$; for if J_i contains points in β_i and β_j, $i \neq j$, it must contain a point of β_i^* (since β_i, β_j are components), and since $\beta_i^* \subset \alpha_1^*$ it also contains a point of $\bar{\alpha}_1$, a contradiction.

We recall that $z \in P_2''(\overline{Q})$, and therefore z is equivalent to a function $z_0 \in P_2'(Q)$. Since z is continuous, we may take $z = z_0 \in P_2''(\overline{Q})$ and $z \in P_2'(\overline{Q})$ [2, p. 181].

Lemma 3.7. If z is absolutely continuous on the segment $I_{\bar{u}} : [u = \bar{u},\ 0 \leq v \leq 1]$, then $\bar{z}$ is absolutely continuous on $I_{\bar{u}}$. Similarly, $\bar{z}$ is absolutely continuous on $I_{\bar{v}}$.

Proof. Let $\varepsilon > 0$ be given. Then there exists a number $\delta > 0$ such that for every finite collection $\{(v_{i-1}, v_i)\}$, $i = 1, 2, \ldots, N$, $v_0 < v_1 \leq v_2 < v_3 \leq \ldots < v_N$, of non-overlapping intervals with $\sum_{i=1}^{N} |v_i - v_{i-1}| < \delta$ it follows that $\sum_{i=1}^{N} |z(\bar{u}, v) - z(\bar{u}, v_{i-1})| < \varepsilon$. For brevity we shall suppress $\bar{u}$ in the argument of z, $\bar{u}$ being fixed throughout.

Case 1. $v_i, v_{i-1} \in F_{\bar{u}}$. Then $z(v_i) = \bar{z}(v_i)$, $z(v_{i-1}) = \bar{z}(v_{i-1})$, and hence $|\bar{z}(v_{i-1}) - \bar{z}(v_i)| = |z(v_{i-1}) - z(v_i)|$.

Case 2. $v_i, v_{i-1} \in \mathcal{O}_{\bar{u}}$. If both are in β_k for some k, then $|\bar{z}(v_i) - \bar{z}(v_{i-1})| = |c_k - c_k| = 0 \leq |z(v_i) - z(v_{i-1})|$. If $v_i \in \beta_k$, $v_{i-1} \in \beta_\ell$, then $v_i \in J_r$, $v_{i-1} \in J_s$, $r \neq s$. Let $J_r = (a_r, b_r)$, $J_s = (a_s, b_s)$. Then

$$v_i < b_s \leq a_r < v_i \quad ,$$

and $b_s \in \beta_\ell^*$, $a_r \in \beta_k^*$. Therefore,

$$|\bar{z}(v_i) - \bar{z}(v_{i-1})| = |z(a_r) - z(b_s)| \quad ,$$

$$|b_s - a_r| \leq |v_i - v_{i-1}| \quad .$$

Case 3. $v_i \in \mathcal{O}_{\bar{u}}$, $v_{i-1} \in F_{\bar{u}}$. Then $v_i \in J_r \subset \beta_k$ for some r, k, and $\bar{z}(v_{i-1}) = z(v_{i-1})$. We let $J_r = (a_r, b_r)$, so $v_{i-1} < a_r < v_i < b_r$, and $a_r \in \beta_k^*$. Thus

$$|\bar{z}(v_i) - \bar{z}(v_{i-1})| = |z(a_r) - z(v_{i-1})| \quad , \text{ and}$$

$$|a_r - v_{i-1}| \leq |v_i - v_{i-1}| \quad .$$

For $v_i \in F_{\bar{u}}$, $v_{i-1} \in \mathcal{O}_{\bar{u}}$, the analogous result holds.

Now combining cases 1, 2, 3, we obtain

$$\sum_{i=1}^{N} |\bar{z}(v_i) - \bar{z}(v_{i-1})| \leq \sum_{i_1=1}^{N_1} |z(v_{i_1-1}) - z(v_{i_1})| + \sum_{i_2=1}^{N_2} |z(a_{i_2}) - z(b_{i_2})|$$

$$+ \sum_{i_3=1}^{N_3} |z(a_{i_3}) - z(v_{i_3})| \quad .$$

By rearranging and renaming on the right-hand side, we obtain a sum of the form

$$\sum_{i=1}^{N} |z(v_i') - z(v_{i-1}')| \quad , \qquad \text{where } \{(v_{i-1}', v_i')\} \ , \ i = 1, \ldots, N,$$

is a collection of non-overlapping intervals with $\sum_{i=1}^{N} |v_i' - v_{i-1}'| < \delta$.

Thus the right-hand side is $< \varepsilon$, and this proves the absolute continuity of $\bar{z}$ on $I_{\bar{u}}$. The proof is the same for $I_{\bar{v}}$.

Lemma 3.8. $\bar{z} \in P_2(Q)$.

Proof. $\bar{Q} = \bar{\alpha}_1 \cup \complement(\bar{\alpha}_1)$. Let $w \in \bar{\alpha}_1$; then $\bar{z}(w) = z(w)$, so $|\bar{z}(w)|^2 = |z(w)|^2$. Next let $w \in \complement(\bar{\alpha}_1)$. By our cutting process, $\bar{z}(w)$ is a point of $z(Q^*)$, and $|z(Q^*)|$ is uniformly bounded. Therefore, $|z(w)|^2$ is uniformly bounded on $\complement(\bar{\alpha}_1)$, and hence $\bar{z} \in L_2(Q)$.

By Lemma 3.7, $\bar{z}_u, \bar{z}_v$ exist a.e. in Q. We must show that each is in $L_2(Q)$. Let $w_0 = (u_0, v_0)$ be a point at which $\bar{z}_u$ and z_u are both defined.

Case 1. Assume $w_0 \in \alpha_1$. Then $z(w) \equiv \bar{z}(w)$ in a neighborhood of w_0. Therefore, $\bar{z}_u(w_0) = z_u(w_0)$.

Case 2. Assume $w_0 \in \beta_i$ for some i. Then $\bar{z}(w) \equiv c_i$ in a neighborhood of w_0. Therefore, $\bar{z}_u(w_0) = 0$.

Case 3. Assume $w_0 \in \beta_i^*$. Then $\bar{z}(w_0) = z(w_0) = c_i$.

(i) If $(u_0 + h, v_0) \in \bar{\beta}_i$, then $|\bar{z}(u_0 + h, v_0) - \bar{z}(u_0, v_0)| = |c_i - c_i| = 0$.

(ii) If $(u_0 + h, v_0) \in \bar{\alpha}_i$, then $\bar{z}(u_0 + h, v_0) = z(u_0 + h, v_0)$ and hence

$$|\bar{z}(u_0 + h, v_0) - \bar{z}(u_0, v_0)| = |z(u_0 + h, v_0) - z(u_0, v_0)| .$$

(iii) Let $(u_0 + h, v_0) \in \beta_j$, $j \neq i$, and assume $h > 0$.

Now $L_{v_0} \cap \beta_j = I_{v_0}$, $L_{v_0} : v = v_0$, is an open linear set, and $(u_0 + h, v_0)$ belongs to one of its components, say J_1, an open interval. Let $(u_0 + h_1, v_0)$ be the left-hand endpoint of J_1; so $0 < h_1 < h$. Then $(u_0 + h_1, v_0) \in \beta_j^*$, so

$$|\bar{z}(u_0 + h, v_0) - \bar{z}(u_0, v_0)| = |\bar{z}(u_0 + h, v_0) - z(u_0, v_0)|$$

$$\leq |z(u_0, v_0) - z(u_0 + h_1, v_0)| + |z(u_0 + h_1, v_0) - \bar{z}(u_0 + h, v_0)|$$

$$\leq |z(u_0, v_0) - z(u_0 + h_1, v_0)| + |z(u_0 + h_1, v_0) - z(u_0 + h_1, v_0)|$$

$$\leq |z(u_0, v_0) - z(u_0 + h_1, v_0)| \quad ,$$

and $h_1 \to 0$ as $h \to 0$. Moreover, since $0 < h_1 < h$,

$$\frac{|\bar{z}(u_0 + h, v_0) - \bar{z}(u_0, v_0)|}{h} \leq \frac{|z(u_0 + h_1, v_0) - z(u_0, v_0)|}{h_1} \quad .$$

The analogous argument holds for $h < 0$.

Now combining cases 1, 2, 3, we see that the absolute values of the difference quotients for $\bar{z}$ are bounded, point by point, by the absolute values of the difference quotients for z. Since the derivatives are known to exist almost everywhere in Q, we have that a.e. in Q

$$|\bar{z}_u(w)| \leq |z_u(w)| \quad , \quad |\bar{z}_v(w)| \leq |z_v(w)| \quad .$$

Since $z_u, z_v \in L_2(Q)$, so are $\bar{z}_u, \bar{z}_v$. Therefore, we conclude that $\bar{z} \in P_2(Q)$, and moreover, $\bar{z} \in P_2'(Q)$.

Corollary 3.8.1. $\bar{z} \in P_2''(\bar{Q})$.

Proof. $z(w)$ was continuous on $\bar{Q}$, and so we may write

$$|\bar{z}(u_0 + h, v_0 + k) - \bar{z}(u_0, v_0)| \leq |\bar{z}(u_0 + h, v_0 + k) - \bar{z}(u_0 + h, v_0)|$$
$$+ |\bar{z}(u_0 + h, v_0) - \bar{z}(u_0, v_0)|$$

and repeat cases 1, 2, 3 in the proof of the Lemma 3.8. to conclude that the right-hand side approaches 0 as $(h, k) \to 0$. Therefore, $\bar{z}$ is continuous, and therefore, $\bar{z} \in P_2''(\bar{Q})$.

Therefore, we have shown that for any A-admissible vector z which represents a base surface **S**, we may choose a vector $\bar{z}$ which is also A-admissible and such that the surface $\bar{S} : z = \bar{z}(w)$ is non-degenerate. Moreover, since $|\bar{J}| = 0$ for all $w \notin \bar{\alpha}_1$, $\bar{J}$ being the Jacobian vector for $\bar{z}$, and since

$$I_0(\zeta, Q) \leq M \int_Q |J| \quad ,$$ we see that

$I_0(\bar{z}, Q) \leq I_0(z, Q)$, and $[\bar{S}] \subset [S]$, where $[S]$ denotes the point set of the surface S. Since $\bar{S}$ is non-degenerate, and since it has finite Lebesgue area, it has a quasi-conformal representation (Lemma 3.1) ζ which is again an A-admissible vector. By the invariance of I_0 under change of representation, we have $I_0(\zeta, Q) = I_0(\bar{z}, Q) \leq I_0(z, Q)$.

Lemma 3.9. [8, p. 271] Let $S : z = z(w)$, $w \in \bar{Q}$, be a surface of class $P_2''(\bar{Q})$. Then there is a base surface $S_0 : z = z_0(w)$, $w \in \bar{Q}$, such that $\partial S = \partial S_0$, $[S_0] \subset [S]$, $z_0(w) \in P_2''(\bar{Q})$, and $I_0(z_0, Q) \leq I_0(z, Q)$.

We collect the main results of this section.

Theorem 3.1. Let $z = z(w)$, $w \in \overline{Q}$, be an A-admissible vector. Then there exists an A-admissible vector $z = \overline{z}(w)$, $w \in \overline{Q}$, such that $I_0(\overline{z}, Q) \leq I_0(z, Q)$, $[\overline{z}] \subset [z]$, and $\overline{z}(w)$ is quasi-conformal, i.e., $\overline{E} = \overline{G}$, $\overline{F} = 0$, a.e. in Q.

4. Admissible Vectors and Minimizing Sequences

Definition 4.1. A vector $z = z(w)$, $w \in Q$, is admissible if

(i) $z \in P_2''(Q^o)$;

(ii) $z(Q^*)$ lies on the torus $\mathcal{T}$;

(iii) $z(Q^*)$ links H ;

(iv) $I_0(z, Q) < +\infty$.

Let

(4.1) $$L = \inf I_0(z, Q) \quad ,$$

where the infimum is taken over the class of admissible vectors. Our variational problem is to show that there is an admissible vector z such that $I_0(z, Q) = L$.

Definition 4.2. A sequence of vectors $\{z_p\}$ is called an admissible sequence if each z_p satisfies the conditions of Definition 4.1, except that $z_p(Q^*)$ is not necessarily on $\mathcal{T}$, but on a manifold $\mathcal{M}_p$ which approaches $\mathcal{T}$ as $p \to \infty$ in the sense that the greatest distance of points of $\mathcal{M}_p$ from $\mathcal{T}$ tends to 0 as $p \to \infty$.

Let

(4.2) $$\delta = \inf (\liminf_{p \to \infty} I_0(z, Q)) \quad ,$$

where the infimum is over all admissible sequences.

<u>Definition 4.3</u>. An admissible sequence $\{z_p\}$ for which $I_0(z_p, Q) \to \delta$ is called a <u>generalized minimizing sequence</u>.

<u>Remarks</u>.

(1) Obviously $\delta \leq L$.

(2) We assume, as usual, that there is at least one admissible vector z with $I_0(z, Q) < +\infty$.

<u>Lemma 4.1</u>. Every generalized minimizing sequence (g.m.s.) $\{z_p\}$ may be replaced by a g.m.s. $\{\overline{z}_p\}$ such that each $\overline{z}_p$ is continuous on $\overline{Q}$.

<u>Proof</u>. For each p we choose a concentric square $Q_p : [0 < \tau_p \leq u \leq 1 - \tau_p ; \tau_p \leq v \leq 1 - \tau_p]$, where τ_p is chosen so small that the (continuous) curve $z_p(Q_p^*)$ links H and lies at a distance from $\mathcal{M}_p$ (and hence from $\mathcal{T}$) going to 0 as $p \to \infty$. This is possible since $\{z_p\}$ is a g.m.s. By a conformal mapping $u = u_p(x, y)$, $v = v_p(x, y)$, we may map Q_p onto Q and obtain a vector

$$\overline{z}_p(x, y) \equiv z_p(u_p(x, y), v_p(x, y))$$

defined on $\overline{Q}$ and of class $P_2''(\overline{Q})$. The integral I_0 is known to be independent of representation, and so $I_0(\overline{z}_p, Q) = I_0(z_p, Q_p) \leq I_0(z_p, Q)$. But $I_0(z_p, Q) \to \delta$, and hence $\overline{\lim\limits_{p \to \infty}} I_0(\overline{z}_p, Q) \leq \delta$. On the other hand, $\{\overline{z}_p\}$ is an admissible sequence, so $\underline{\lim\limits_{p \to \infty}} I_0(z_p, Q) \geq \delta$.

Thus $\lim\limits_{p \to \infty} I_0(\overline{z}_p, Q) = \delta$. Therefore, we may choose the sequence

$\{\bar{z}_p\}$ such that $I_0(\bar{z}_p, Q) \to \delta$.

Let us now introduce the integral [21, p. 571]

(4.3) $$I(z,G) = \iint_G f(z,p,q)\,du\,dv \quad ,$$

where $p = (z_u^1, z_u^2, z_u^3)$, $q = (z_v^1, z_v^2, z_v^3)$, and

(4.4) $$f^2(z,p,q) \equiv F^2(z,J) + \left(\frac{M+m}{2}\right)^2 \left[\left(\frac{E-G}{2}\right)^2 + F^2\right] , \quad f \geq 0 ,$$

$E = |p|^2$, $G = |q|^2$, $F = p \cdot q$, and we note that $J = p \times q$, where "$\cdot$" denotes scalar product and "$\times$" denotes vector product. Then $f(z,p,q)$ is as smooth as $F(z,J) = F(z, p \times q)$, and moreover

(4.5) $$\frac{m}{2}(|p|^2 + |q|^2) \leq f(z,p,q) \leq \frac{M}{2}(|p|^2 + |q|^2) \quad .$$

To see that (4.5) holds, we recall that $|p \times q|^2 = |p|^2|q|^2 - (p \cdot q)^2$, and by assumption,

$$F(z,J) = F(z, p \times q) \leq M|J| = M|p \times q| \quad .$$

Then we have

$$\begin{aligned} f^2(z, p \times q) &= F^2(z, p \times q) + \left(\frac{M+m}{2}\right)^2 \left[\left(\frac{E-G}{2}\right)^2 + F^2\right] \\ &\leq M^2|p \times q|^2 + M^2\left[\left(\frac{|p|^2 - |q|^2}{2}\right)^2 + (p \cdot q)^2\right] \\ &\leq \frac{M^2}{4}(4|p|^2|q|^2 + |p|^4 - 2|p|^2|q|^2 + |q|^4) \\ &\leq \frac{M^2}{4}(|p|^2 + |q|^2)^2 \quad . \end{aligned}$$

This demonstrates half of (4·5), and the verification for the other half is similar. As a consequence we see that

$$\frac{m}{2} D(z,G) \le I(z,G) \le \frac{M}{2} D(z,G) \quad , \tag{4.6}$$

and if z is quasi-conformal,

$$I(z,G) = I_0(z,G) \quad .$$

The basic idea behind the introduction of the integral I is that we must always have

$$I_0(z,G) \le I(z,G) \tag{4.7}$$

in addition to (4.6). Moreover, we see that

$$EG - F^2 + \left\{ \left(\frac{E-G}{2}\right)^2 + F^2 \right\} = |J|^2 + \left\{ \left(\frac{E-G}{2}\right)^2 + F^2 \right\}$$

$$= \left(\frac{E+G}{2}\right)^2 \quad ,$$

and so

$$\left\{ \left(\frac{E-G}{2}\right)^2 + F^2 \right\}^{\frac{1}{2}} = \left\{ \left(\frac{E+G}{2}\right)^2 - |J|^2 \right\}^{\frac{1}{2}} .$$

Hence we may loosely say that the integral $I(z,G)$ gives $I_0(z,G)$ plus "the average over G of the amount z misses being quasi-conformal." Furthermore, it has the feature that boundedness of $I(z,G)$ is equivalent to boundedness of $D(z,G)$.

Lemma 4.2. $I(z,G)$ is invariant with respect to conformal change of variables.

<u>Proof</u>. Let $\zeta(w) = \zeta(u,v) = x(u,v) + iy(u,v)$, $w = u + iv$, be a conformal mapping of the (Jordan) domain $\mathbf{A}$ in the $w = u + iv$ plane onto a (Jordan) domain B in the $x + iy$ plane. Write $z_0(u,v) = z(x(u,v), y(u,v))$, $z \in P_2''(B)$. Let $\overline{E}, \overline{F}, \overline{G}$ correspond to z, and let E, F, G correspond to z_0. Then

$$I(z,B) = \iint_B \sqrt{F^2(z,J_z) + \left(\frac{M+m}{2}\right)^2\left[\left(\frac{\overline{E}-\overline{G}}{2}\right)^2 + \overline{F}^2\right]}dx\,dy$$

$$= \iint_B \sqrt{F^2(z,J_z) + \left(\frac{M+m}{2}\right)^2\left[\left(\frac{\overline{E}+\overline{G}}{2}\right)^2 - |J_z|^2\right]}\ dx\,dy \quad .$$

But the integrals

$$\iint_B F(z,J)\ , \quad \iint_B (\overline{E}+\overline{G})\ , \quad \iint_B |J|\ ,$$

are known to be invariant. Therefore, the right-hand side of the expression above may be written

$$\iint_A \sqrt{F^2(\zeta,J_\zeta)\ |\zeta'(w)|^{-4} + \left(\frac{M+m}{2}\right)^2\left[\left(\frac{E-G}{2}\right)^2 + F^2|\zeta'(w)|^{-4}\right]}|\zeta'(w)|^2 du\,dv$$

$$= I(z_0, A) \quad .$$

Combining Lemma 4.1 and Theorem 3.1 we see that there is a generalized minimizing sequence $\{z_n\}$ such that $z_n \in P_2''(\overline{Q})$, $z_n \in P_2'(Q)$, and z_n is quasi-conformal.

Let F be a closed solid cylinder in E^3 with the properties that the distance from F to $\mathcal{T}$ is a positive number τ, and if S is any surface whose boundary has greatest distance from $\mathcal{T}$ less than $\tau/2$ and whose boundary links H, then any line parallel to a generator of F has a non-zero intersection number with S.

Since the boundary of z_n links H, there is a point $w_n \in Q^o$ which is mapped by z_n into the interior of F, and a neighborhood U_n of w_n whose closure is mapped into F. Among all such neighborhoods U_n we choose one of maximal size, in the sense that for every line parallel to a generator of F there is a point $w_n \in U_n$ which maps onto this line. The continuity of z_n assures the existence of such an open set U_n. In such a U_n, there is a point $\overline{w}_n$ such that $z_n(\overline{w}_n)$ lies on the axis of F. Now by a conformal mapping of Q onto itself we may send $\overline{w}_n$ onto the center P_0 of Q. Then the neighborhood U_n is mapped onto a neighborhood U_n of P_0. Let us assume this has already been done and that z_n is the resulting function. Then z_n is still quasi-conformal, and the values of $I_0(z_n, Q)$ and $I(z_n, Q)$ have remained unchanged.

Let σ be the segment in Q which is parallel to the v-axis, has P_0 as midpoint, and has length, say, $1/8$. Then a portion of

σ is contained in our distinguished neighborhood of P_0. If σ lies completely in this neighborhood, then the image $z_n(\sigma)$ lies completely in F. If σ does not lie completely in this neighborhood, then we make the following change. A portion of σ, say σ_n, lies in U_n. We shall map Q onto itself by a diffeomorphism d_n so that outside a thin rectangular strip R_n about σ the mapping is constant, the segment σ_n is stretched onto the segment σ, and such that if $\zeta_n(w) \equiv z_n(d_n^{-1}(w))$, then $I(\zeta_n, R_n) < 1/n$, $I(z_n, R_n) < 1/n$. Hence ζ_n maps σ into F, and since d_n is a diffeomorphism, $I_0(\zeta_n, Q) = I_0(z_n, Q)$. Therefore, we have

$$\begin{aligned} 0 < I(\zeta_n, Q) - I(z_n, Q) &= I(z_n, Q-R_n) + I(\zeta_n, R_n) - I(z_n, Q-R_n) - I(z_n, R_n) \\ &= I(\zeta_n, R_n) - I(z_n, R_n) \\ &\leq 1/n \quad . \end{aligned}$$

Since $I(z_n, Q) \to \delta$ and $I_0(z_n Q) \to \delta$, we still have $I(\zeta_n, Q) \to \delta$ and $I_0(\zeta_n, Q) = I_0(z_n, Q) \to \delta$.

Therefore, replacing ζ_n by z_n, we have a minimizing sequence $\{z_n\}$ of vectors which map σ into F, and which are quasi-conformal except on a rectangle R_n which may be chosen so that its width approaches 0 as $n \to \infty$.

The sequence of numbers $\{I(z_n, Q)\}$ is bounded. By relation (4.6) this implies that $\{D(z_n, Q)\}$ is bounded, and by [23, § 2] so is $\{L_2(z_n, Q) = \int_Q |z_n|^2\}$. Let R be a bound for these numbers.

Recall that $\overline{Q}$ is the square with vertices $(0,0)$, $(0,1)$, $(1,1)$, $(1,0)$. We now extend the domain of definition for all the z_n to the square $\overline{Q}_1$ with vertices $(-1,-1)$, $(-1,2)$, $(2,2)$, $(2,-1)$ by reflection in the sides of the square $\overline{Q}$, and then by reflection in the sides of the four resulting squares. The new vector, which we again call z_n, is clearly still in class $P_2''(\overline{Q}_1)$ and quasi-conformal.

Let ϕ_ρ be a Friedrichs mollifier (Section 1). Then

(i) $\phi_\rho \in C^\infty$ and has compact support $|w| \leq \rho$;

(ii) $\phi_\rho \geq 0$;

(iii) $\displaystyle\int_{B_0(\rho)} \phi_\rho(u,v)\,du\,dv = 1$, $B_0(\rho) = \{ w : |w| \leq \rho \}$.

We shall always take $\rho < 1/8$. For each n we form the ϕ_ρ-mollified function

$$z_n^\rho(u,v) \equiv (z_n * \phi_\rho)(u,v) = \iint_{B_{(u,v)}(\rho)} z_n(\xi,\eta)\,\phi_\rho(u-\xi, v-\eta)\,d\xi\,d\eta ,$$

where "$z_n \phi_\rho$" means the vector $(z_n^1\phi_\rho, z_n^2\phi_\rho, z_n^3\phi_\rho)$, and "$z_n^\rho$" means the vector $(z_n^1 * \phi_\rho, z_n^2 * \phi_\rho, z_n^3 * \phi_\rho)$.

The following facts about z_n^ρ are known [10, p. 14]:

(a) $z_n^\rho \in C^\infty$;

(b) $z_n^\rho \to z_n$ uniformly on $\overline{Q}$ as $\rho \to 0$;

(c) $(z_n^\rho)_u = (z_{n_u})^\rho$, $(z_n^\rho)_v = (z_{n_v})^\rho$;

(d) $\| z_{n_u}^\rho - z_{n_u} \|_{L_2} \to 0$ and $\| z_{n_v}^\rho - z_{n_v} \|_{L_2} \to 0$ as $\rho \to 0$.

For each n we choose $\rho = \rho(n)$ so small that

(α) $\| z_{n_u}^\rho - z_{n_u} \|_{L_2} = \varepsilon_n^1 < 1/n$, $\| z_{n_v}^\rho - z_{n_v} \|_{L_2} = \varepsilon_n^2 < 1/n$;

(β) $| z_n^\rho(u,v) - z_n(u,v) | \leq (1/n)\,(d_n/4)$,

where $d_n/4$ is the greatest distance from points of $z_n(Q^*)$ to the torus $\mathcal{T}$, $d_n \to 0$ as $n \to \infty$, and $d_n < d = \text{dist}[H, \mathcal{T}\,]$, for all n.

Now we set $\zeta_n(u,v) = z_n^\rho(u,v)$, and we consider the domain of ζ_n to be only $\overline{Q}$. Then $\zeta_n(Q^*)$ is a continuous (in fact, C^∞) curve linking H (since it may be continuously deformed into $z_n(Q^*)$ without touching H), and the greatest distance from $\zeta_n(Q^*)$ to approaches 0 as $n \to \infty$. Thus the new sequence $\{\zeta_n\}$ is an admissible sequence. Furthermore, if $\rho = \rho(n)$ is chosen sufficiently small, the segment σ is mapped by ζ_n into the closed set F (page 96). We shall show that this new sequence is actually a generalized minimizing sequence.

Lemma 4.3. $I_0(\zeta_n, Q) \to \delta$ as $n \to \infty$.

<u>Proof.</u>

$$|I_0(\zeta_n, Q) - I_0(z_n, Q)| = \left|\int_Q [F(\zeta_n, J_{\zeta_n}) - F(z_n, J_{z_n})]du\,dv\right|$$

$$\leq \int_Q |F(\zeta_n, J_{\zeta_n}) - F(\zeta_n, J_{z_n})| + \int_Q |F(\zeta_n, J_{z_n}) - F(z_n, J_{z_n})|$$

(4.8) $$\leq L_2 \int_Q |J_{\zeta_n} - J_{z_n}| + L_1 \int_Q |\zeta_n - z_n| \qquad \text{(relation 1.1 (v))}$$

$$\leq L_2 \int_Q |J_{\zeta_n} - J_{z_n}| + L_1 \ (1/n)(d_n/4) \text{ meas}(Q).$$

We obtain

$$|J^1_{\zeta_n} - J^1_{z_n}| = \left| \begin{vmatrix} \zeta^2_{n_u} & \zeta^3_{n_u} \\ \zeta^2_{n_v} & \zeta^3_{n_v} \end{vmatrix} - \begin{vmatrix} z^2_{n_u} & z^3_{n_u} \\ z^2_{n_v} & z^3_{n_v} \end{vmatrix} \right|$$

$$= |\zeta^2_{n_u}\zeta^3_{n_v} - \zeta^2_{n_v}\zeta^3_{n_u} - z^2_{n_u}z^3_{n_v} + z^2_{n_v}z^3_{n_u}|$$

$$\leq |\zeta^2_{n_u}\zeta^3_{n_v} - z^2_{n_u}z^3_{n_v}| + |\zeta^2_{n_v}\zeta^3_{n_u} - z^2_{n_v}z^3_{n_u}|$$

(4.9) $$\leq |(\zeta^3_{n_u} - z^2_{n_u})\zeta^3_{n_v} + (\zeta^3_{n_v} - z^3_{n_v})z^2_{n_u}|$$

$$+ |(\zeta^3_{n_u} - z^3_{n_u})\zeta^2_{n_v} + (\zeta^2_{n_v} - z^2_{n_v})z^3_{n_u}|$$

$$\leq |\zeta^2_{n_u} - z^2_{n_u}|\,|\zeta^3_{n_v}| + |\zeta^3_{n_v} - z^3_{n_v}|\,|z^2_{n_u}|$$

$$+ |\zeta^3_{n_u} - z^3_{n_u}|\,|\zeta^2_{n_v}| + |\zeta^2_{n_v} - z^2_{n_v}|\,|z^3_{n_u}| \quad .$$

Now

$$\int_Q |\zeta^2_{n_u} - z^2_{n_u}|\,|\zeta^3_{n_v}| \leq \|\zeta^2_{n_u} - z^2_{n_u}\|_{L_2} \|\zeta^3_{n_v}\|_{L_2}$$

(4.10)
$$\leq \varepsilon_n (\|\zeta^3_{n_u} - z^3_{n_u}\|_{L_2} + \|z^3_{n_v}\|_{L_2})$$

$$\leq \varepsilon_n(\varepsilon_n + R) \quad ,$$

where $\varepsilon_n = \max(\varepsilon^1_n, \varepsilon^2_n)$, page 98, and R is the bound given on page 96. The same inequality holds for the other three members of (4.9). Thus

$$\int_Q |J^1_{\zeta_n} - J^1_{z_n}| \leq 4\varepsilon_n(\varepsilon_n + R) \quad .$$

Since $|J| = |(J^1, J^2, J^3)| \leq |J^1| + |J^2| + |J^3|$, it follows that

(4.11)
$$\int_Q |J_{\zeta_n} - J_{z_n}| \leq 12\,\varepsilon_n(\varepsilon_n + R) \quad .$$

From (4.8) and (4.11) we obtain

(4.12)
$$|I_0(\zeta_n, Q) - I_0(z_n, Q)| \leq L_2\,(12\,\varepsilon_n)(\varepsilon_n + R) + L_1\,(1/n)(d_n/4) \;,$$

and the right-hand side approaches 0 as $n \to \infty$. Since $I_0(z_n, Q) \to \delta$, we have $I_0(\zeta_n, Q) \to \delta$. This completes the proof.

We let $\overline{E}$, $\overline{G}$, $\overline{F}$ correspond to z_n, and E, G, F to ζ_n. Since z_n was quasi-conformal on $Q - R_n$, $\overline{E} = \overline{G}$, $\overline{F} = 0$ almost everywhere on $Q - R_n$ (page 96). While ζ_n need not be quasi-conformal, as $n \to \infty$ ζ_n becomes "nearly quasi-conformal" as the next lemma shows.

<u>Lemma 4.4</u>. $I(\zeta_n, Q) \to \delta$ as $n \to \infty$.

<u>Proof</u>.

$$I(\zeta_n, Q) = \int_Q \sqrt{F^2(\zeta_n, J) + \left(\frac{M+m}{2}\right)^2 \left[\left(\frac{E-G}{2}\right)^2 + F^2\right]}$$

(4.13)

$$\leq \int_Q F(\zeta_n, J) + \left(\frac{M+m}{2}\right)\left\{\int_Q \left|\frac{E-G}{2}\right| + \int_Q |F|\right\} \quad .$$

Using the fact that z_n is quasi-conformal on $Q_n = Q - R_n$

$$\int_{Q_n} |E - G| = \int_{Q_n} |E - \overline{E} + \overline{G} - G|$$

(4.14)

$$\leq \int_{Q_n} |E - \overline{E}| + \int_{Q_n} |\overline{G} - G| \quad .$$

We obtain successively,

$$\int_{Q_n} |E - \overline{E}| = \int_{Q_n} \left| \sum_{i=1}^{3} [(\zeta^i_{n_u})^2 - (z^i_{n_u})^2] \right|$$

$$\leq \sum_{i=1}^{3} \int_{Q_n} |(\zeta^i_{n_u})^2 - (z^i_{n_u})^2|$$

$$\leq \sum_{i=1}^{3} \int_{Q_n} |\zeta^i_{n_u} - z^i_{n_u}|\,|\zeta^i_{n_u} + z^i_{n_u}|$$

(4.14a)

$$\leq \sum_{i=1}^{3} \|\zeta^i_{n_u} - z^i_{n_u}\|_{L_2} \|\zeta^i_{n_u} + z^i_{n_u}\|_{L_2}$$

$$\leq \sum_{i=1}^{3} \varepsilon_n (\|\zeta^i_{n_u} - z^i_{n_u}\|_{L_2} + 2\|z^i_{n_u}\|_{L_2})$$

$$\leq \sum_{i=1}^{3} \varepsilon_n(\varepsilon_n + 2R) = 3\varepsilon_n(\varepsilon_n + 2R) \quad ,$$

ε_n, R as before. Similarly,

(4.14b) $$\int_{Q_n} |\overline{G} - G| \leq 3\varepsilon_n(\varepsilon_n + 2R) \quad .$$

Next,

$$\int_{Q_n} |F| = \int_{Q_n} |F - \overline{F}| \leq \sum_{i=1}^{3} \int_{Q_n} | \zeta^i_{n_u} \zeta^i_{n_v} - z^i_{n_u} z^i_{n_v} |$$

$$\leq \sum_{i=1}^{3} \int_{Q_n} | (\zeta^i_{n_u} - z^i_{n_u}) \zeta^i_{n_v} + (\zeta^i_{n_v} - z^i_{n_v}) z^i_{n_u} |$$

(4.15) $$\leq \sum_{i=1}^{3} \{ \int_{Q_n} | \zeta^i_{n_u} - z^i_{n_u} | \, | \zeta^i_{n_v} | + \int_{Q_n} | \zeta^i_{n_v} - z^i_{n_v} | \, | z^i_{n_u} | \}$$

$$\leq \sum_{i=1}^{3} \{ \| \zeta^i_{n_u} - z^i_{n_u} \|_{L_2} \| \zeta^i_{n_v} \|_{L_2} + \| \zeta^i_{n_v} - z^i_{n_v} \|_{L_2} \| z^i_{n_u} \|_{L_2} \}$$

$$\leq \sum_{i=1}^{3} \{ \varepsilon_n (\varepsilon_n + R) + \varepsilon_n R \} = 3 \varepsilon_n (\varepsilon_n + 2R) \quad .$$

Combining (4.14a), (4.14b), (4.15) and substituting in (4.4),

$$\left(\frac{M+m}{2}\right) \{ \int_{Q_n} \left| \frac{E-G}{2} \right| + \int_{Q_n} |F| \} \leq \left(\frac{M+m}{2}\right) \{ \tfrac{1}{2} \, 6 \varepsilon_n (\varepsilon_n + 2R) + 3 \varepsilon_n (\varepsilon_n + 2R) \}$$

$$\leq \left(\frac{M+m}{2}\right) \{ 6 \varepsilon_n (\varepsilon_n + 2R) \} \quad .$$

From (4.13) we obtain

$$I(\zeta_n, Q) \leq \int_{Q_n} F(\zeta_n, J) + \varepsilon_n [6 \left(\frac{M+m}{2}\right)(\varepsilon_n + 2R)]$$

(4.16)

$$\leq I_0(\zeta_n, Q_n) + \varepsilon_n [6 \left(\frac{M+m}{2}\right)(\varepsilon_n + 2R)].$$

By Lemma 4.3 and the fact that $\varepsilon_n \to 0$ as $n \to \infty$, the right-hand side of (4.16) approaches δ as $n \to \infty$. Therefore, $\underline{\lim}_{n\to\infty} I(\zeta_n, Q_n) \leq \delta$. But $I_0(\zeta_n, Q_n) \leq I(\zeta_n, Q_n)$, so $\underline{\lim}_{n\to\infty} I(\zeta_n, Q_n) \geq \underline{\lim}_{n\to\infty} I_0(\zeta_n, Q_n) = \lim_{n\to\infty} I_0(\zeta_n, Q_n) = \delta$. Thus for any $\varepsilon > 0$ we may choose n so large that $\delta - \varepsilon/2 < I_0(\zeta_n, Q_n) \leq I(\zeta_n, Q_n) \leq I_0(\zeta_n, Q_n) + \varepsilon/2 < \delta + \varepsilon$, that is, $I(\zeta_n, Q_n) \to \delta$ as $n \to \infty$, and since $\text{meas}(Q_n) \to \text{meas}\, Q$, $I(\zeta_n, Q) \to \delta$ as $n \to \infty$.

We collect the main results of this section.

Theorem 4.1. There exists a generalized minimizing sequence $\{\zeta_n\}$ of C^∞ vectors for $I_0(\zeta, Q)$. Furthermore, $I(\zeta_n, Q) \to \delta$ as $n \to \infty$, and the segment σ is mapped into the closed set F for every n.

5. The Existence Proof

Let $\{\zeta_n\}$ be a sequence given by Theorem 4.1. Denote by ρ_n the greatest distance from $\zeta_n(Q^*)$ (i.e., the greatest distance of the manifold $\mathcal{M}_n$ on which $\zeta_n(Q^*)$ lies) to $\mathcal{T}$. Then $\rho_n \to 0$ as $n \to \infty$. For the moment we fix n. Let $\mathcal{F}(K)$ be the class of all vectors $z \in H^2(Q)$ having the properties

(i) $z(Q^*)$ links H ;

(ii) the greatest distance from $z(Q^*)$ to $\mathcal{T}$ is $\leq \rho_n$;

(iii) The segment σ (as before) is mapped by z into the closed set F;

(iv) $$J(z, Q) = \int_Q (|z_{uu}|^2 + 2|z_{uv}|^2 + |z_{vv}|^2)\,du\,dv \leq K.$$

Then for K sufficiently large, $\zeta_n \in$ (K).

Theorem 5.1. There exists a vector $z_K \in \mathcal{F}(K)$ (K sufficiently large) such that $I(z, Q)$ is minimized by z_K among all vectors in $\mathcal{F}(K)$.

Proof. $\{z_p\}$ be a minimizing sequence for I in $\mathcal{F}(K)$. Then by the remark on page 96 and the definition of the norm in H^2, page 50, the H^2 norms of the vectors z_p are uniformly bounded. Thus, by the Sobolev imbedding theorem, there is a subsequence $\{z_{p_m}\}$ which converges uniformly on $\overline{Q}$ and weakly in $H^2(Q)$ to a vector $z_K \in H^2(Q)$. For simplicity, assume $[p_m] = [p]$. Weak convergence in $H^2(Q)$ implies weak convergence in $L_2(Q)$ of the second derivative and strong convergence in $L_2(Q)$ of the first derivatives [23, pp. 99, 100]. Since $J(z, Q)$ is lower semi-continuous with respect to this convergence, we have

$$J(z_K, Q) \leq \varliminf_{p\to\infty} J(z_p, Q) \leq K \quad ,$$

and since $I(z, Q)$ is lower semi-continuous,

$$I(z_K, Q) \leq \varliminf_{p\to\infty} I(z_p, Q) \quad .$$

Therefore, if z_K behaves properly, it will be the desired minimizing function.

First, since each z_p links H and since $z_p(Q^*)$ is bounded away from H, the uniform convergence of z_p toward z_K on $\overline{Q}$ implies that $z_K(Q^*)$ may be continuously deformed into $z_p(Q^*)$ without touching H. Thus z_K is linked.

Secondly, by the uniform convergence of z_p toward z_K on $\overline{Q}$, given $\varepsilon > 0$ there is a number P such that $p \geq P$ implies $|z_p(w) - z_K(w)| < \varepsilon$ for all $w \in \overline{Q}$. Thus

$$\begin{aligned} \operatorname{dist}[z_K(w), \mathcal{T}] &\leq \operatorname{dist}[z_p(w), \mathcal{T}] + |z_p(w) - z_K(w)| \\ &\leq \rho_n + \varepsilon \quad , \qquad \text{all } w \in Q^* . \end{aligned}$$

But $\varepsilon > 0$ was arbitrary, so $\operatorname{dist}[z_K, \mathcal{T}] \leq \rho_n$.

Finally, the uniform convergence again gives the fact that σ is mapped into the closed set F. Therefore, z_K is a minimizing function in $\mathcal{F}(K)$.

Theorem 5.2. For each n we may choose a vector z_{K_n} as above, $K_n < K_{n+1} < \ldots$, such that $\lim_{n\to\infty} I(z_{K_n}, Q) = \delta$.

Proof. For each n, choose K_n so that $\zeta_n \in \mathcal{F}(K_n)$. (See page 105). Since z_{K_n} is a minimizing function for $I(z, Q)$ in $\mathcal{F}(K_n)$, we have

$$I(z_{K_n}, Q) \leq I(\zeta_n, Q) \quad .$$

The sequence $\{z_{K_n}\}$ is an admissible sequence for I_0, so

$$\delta \leq \underline{\lim}_{n\to\infty} I_0(z_{K_n}, Q) \leq \underline{\lim}_{n\to\infty} I(z_{K_n}, Q) \leq \underline{\lim}_{n\to\infty} I(\zeta_n, Q) \quad ,$$

but

$$\underline{\lim}_{n\to\infty} I(\zeta_n, Q) = \lim_{n\to\infty} I(\zeta_n, Q) = \delta .$$

Thus

$$\lim_{n\to\infty} I(z_{K_n}, Q) = \delta \quad .$$

We shall now begin the process of showing that a subsequence of $\{z_{K_n}\}$ converges to a minimizing vector for our problem.

Lemma 5.1. [23, p. 100] Let $z \in P_2'(G)$ and let

$$\int_{B(w_0, r)} (|z_u|^2 + |z_v|^2)\,du\,dv \leq A(r/a)^{2\mu} \quad , \quad 0 < r \leq a,\ 0 < \mu < 1 \ ,$$

for every $w_0 \in G$, where a is the distance from w_0 to G^*. Then

$$|z(w_1) - z(w_2)| \leq N(\frac{|w_1 - w_2|}{a})^{\mu} \quad , \quad 0 \leq |w_1 - w_2| \leq a \;,$$

for every pair of points $w_1, w_2 \in G$ such that every point of the segment joining w_1 and w_2 is at a distance $\geq a$ from G^*, and $N = \mu^{-1} A^{\frac{1}{2}} 2^{1-\mu} 3^{-\frac{1}{2}}$.

Lemma 5.2. [20, p. 39] Let $z \in P_2$ on a disk $B(P,R)$ with center P and radius R. Then there exist functions $a_0(r)$, $a_n(r)$, $b_n(r)$, $n = 1, 2, \ldots,$ of class P_2'' (and therefore of class P_2') on each interval (r_0, R) with $0 < r_0 < R$, such that the series

$$\text{a} \quad \frac{a_0(r)}{2} + \sum_{n=1}^{\infty} [a_n(r) \cos n\theta + b_n(r) \sin n\theta]$$

converges in $L_2(0, 2\pi)$ to a function $\overline{z}(r, \theta)$ for each r, the convergence being absolute and uniform in θ for almost every fixed r. Moreover, the function $\overline{z}(r, \theta)$ is equivalent to z, and

$$D(z, B(P,R)) = \pi \int_0^R r[\frac{a_0'^2}{2} + \sum_{j=1}^{\infty} \{a_n'^2 + b_n'^2 + \frac{n^2(a_n^2 + b_n^2)}{r^2}\}]dr \;.$$

Lemma 5.3. [30, pp. 103-123] Let the functions ϕ_{00}, ϕ_{10}, ϕ_{01} be given on the boundary of the disk $B(P,R)$, and suppose that there is a function $\zeta \in H^2(B(P,R))$ such that $\zeta = \phi_{00}$, $\frac{\partial \zeta}{\partial u} = \phi_{10}$, $\frac{\partial \zeta}{\partial v} = \phi_{01}$ on the boundary. Then there exists a unique function $z \in H^2(B(P,R))$

satisfying these boundary conditions and minimizing the integral

$$J(\zeta, B(P,R)) = \int_{B(P,R)} (|\zeta_{uu}|^2 + 2|\zeta_{uv}|^2 + |\zeta_{vv}|^2) du\, dv$$

among all such functions. The function z has continuous derivatives of all orders on the domain $B^o(P,R) : \{ w : |w - P| < R\}$ and satisfies the biharmonic equation

$$\Delta^2 z = \frac{\partial^4 z}{\partial u^4} + 2 \frac{\partial^4 z}{\partial u^2 \partial v^2} + \frac{\partial^4 z}{\partial v^2} = 0$$

on $B^o(P,R)$. Furthermore, z is the only function biharmonic on $B^o(P,R)$ and satisfying the given boundary conditions.

<u>Theorem 5.3</u>. Denote by Q' the (open) region $Q^o - \sigma$. For each K_n, z_{K_n} satisfies the oondition

(5.1)

$$D[z_{K_n}, B(w_0, R)] \leq D[z_{K_n}, B(w_0, a)] \; (R/a)^{\lambda} \quad ,$$

$$\lambda = \frac{m}{2M} \quad , \qquad 0 \leq R \leq a \quad ,$$

for every disk $B(w_0, a) \subset Q'$. Thus the vectors z_{K_n} are equicontinuous on every closed subdomain of Q'.

<u>Proof</u>. To prove equicontinuity assuming that relation (5.1) is true, we note that since $I(z_{K_n}, Q) \to \delta$ (Theorem 5.2), there is a constant C such that $I(z_{K_n}, Q) \leq C\, m/2$. Thus

$$D[z_{K_n}, B(w_0, a)] \leq D[z_{K_n}, Q] \leq \frac{2}{m} I[z_{K_n}, Q] \leq C$$

for all K_n, $n = 1, 2, \ldots$. Then by relation (5.1)

$$D[z_{K_n}, B(w_0, R)] \leq D[z_{K_n}, B(w_0, a)] \ (R/a)^{\lambda} \leq C(R/a)^{\lambda} ,$$

and so by Lemma (5.1) we have

$$(5.2) \qquad |z_{K_n}(w_1) - z_{K_n}(w_2)| \leq N \left(\frac{|w_1 - w_2|}{a}\right)^{\lambda/2} ,$$

independently of K_n. But this says that the vectors z_{K_n} are equicontinuous on every closed subdomain of Q'.

To prove relation (5.1) we use the minimizing property of z_{K_n} and the relation (4.5) to see that

$$(5.3) \qquad \tfrac{1}{2} m D[z_{K_n}, B(w_0, R)] \leq I[z_{K_n}, B(w_0, R)]$$

$$\leq I[\zeta, B(w_0, R)] \leq \tfrac{1}{2} M D[\zeta, B(w_0, R)] ,$$

where ζ is the biharmonic function on $B^{o}(w_0, R)$ having boundary values $\zeta(R, \theta) = z_{K_n}(R, \theta)$, $\frac{\partial \zeta}{\partial u}(R, \theta) = \frac{\partial z_{K_n}}{\partial u}(R, \theta)$, $\frac{\partial \zeta}{\partial v}(R, \theta) = \frac{\partial z_{K_n}}{\partial v}(R, \theta)$ (Lemma 5.3).

Let (r, θ) be polar coordinates with pole at w_0. By Lemma 5.2 we have

$$z_{K_n}(r,\theta) = \frac{a_0(r)}{2} + \sum_{n=1}^{\infty} [a_n(r)\cos n\theta + b_n(r)\sin n\theta] , \quad 0 \le r \le a ,$$

$$\zeta(r,\theta) = \frac{A_0(r)}{2} + \sum_{n=1}^{\infty} [A_n(r)\cos n\theta + B_n(r)\sin n\theta] , \quad 0 \le r \le R .$$

Since ζ is biharmonic and has the same Dirichlet data as z_{K_n} on $B^*(w_0, R)$, one may easily obtain

$$A_n(r) = c_n(r/R)^n + d_n(r/R)^{n+2} , \quad B_n(r) = e_n(r/R)^n + f_n(r/R)^{n+2} ,$$

where c_n, d_n, e_n, f_n are constants defined by

$$2c_n = (n+2)\alpha_n - \beta_n , \qquad 2d_n = \beta_n - n\alpha_n ,$$

$$\alpha_n = a_n(R) , \qquad \beta_n = Ra_n'(R) , \qquad n \ge 0 ,$$

and similar formulas for e_n and f_n for $n > 0$. Thus if we set

$$\psi(R) = D[z_{K_n}, B(w_0, R)]$$

we obtain

$$\psi(R) \le CD[\zeta, B(w_0, R)]$$

$$= C\pi \int_0^R r\left\{ \frac{A_0'^2}{2} + \sum_{n=1}^{\infty} [A_n'^2 + B_n'^2 + r^{-2}n^2(A_n^2 + B_n^2)] \right\} dr$$

$$\le 2C'R\psi'(R) , \qquad C' = \frac{M}{m}\pi .$$

The last member of the inequality follows by termwise comparison.

Thus we have

$$\psi(R) \leq AR\psi'(R) \quad , \qquad A > 1 \quad ,$$

or $0 \leq AR\psi'(R) - \psi(R)$, which yields

$$\frac{d}{dR}(R^{-\frac{1}{A}}\psi(R)) = R^{-\frac{1}{A}}\psi'(R) - \frac{1}{A} R^{-\frac{1}{A}-1}\psi(R)$$

$$= R^{-\frac{1}{A}}(\psi'(R) - \frac{1}{A}R^{-1}\psi(R))$$

$$\geq 0 \quad .$$

Since $\psi(0) = 0$, we have

$$R^{-\frac{1}{A}}\psi(R) \leq a^{-\frac{1}{A}}\psi(a) \quad ,$$

or

$$D[z_{K_n}, B(w_0, R)] \leq (\frac{R}{a})^{1/A} D[z_{K_n}, B(w_0, a)]$$

$$= (\frac{R}{a})^{\lambda} D[z_{K_n}, B(w_0, a)] \quad .$$

Note: This proof may be found in [21, p. 573].

From Theorem 5.3 and the fact that the numbers $D[z_{K_n}, Q]$ are uniformly bounded, as well as $L_2[z_{K_n}, Q]$, we obtain the following result.

Theorem 5.4. There exists a vector $z \in P_2(Q)$ such that a subsequence $\{z_p\}$ of $\{z_{K_n}\}$ converges weakly in $P_2(Q)$ to z and uniformly on every closed subdomain of Q' to z. Moreover, $z(Q^*)$ lies on the torus $\mathcal{T}$, $z(Q^*)$ links H, and $I_0(z, Q) \leq \delta$.

Proof. The statements about convergence follow from the above remarks. Thus z is continuous on every closed subdomain of Q', and we may take $z \in P_2'(Q)$. By the lower semicontinuity of I_0, if $\overline{R} \subset Q'$ is any closed subdomain,

$$I_0(z, R) \leq \varliminf_{p \to \infty} I_0(z_p, R) \leq \lim_{p \to \infty} I(z_p, Q) = \delta \quad .$$

The fact that $z(Q^*)$ lies on $\mathcal{T}$ (in the sense previously defined) follows from Corollary 2.2.2. Thus it remains to show that $z(Q^*)$ links H.

Let Q_η be any subrectangle of Q whose boundary Q_η^* contains the segment σ. By [23, p. 99, Theorem 2.11(b)], $z_p \to z$ weakly on Q_η^o, and by [23, p. 100, Theorem 2.12], $z_p \to z$ strongly in L_2 on Q_η^*. Thus there is a subsequence, again called $\{z_p\}$, such that $z_p \to z$ a.e. (1-dimensional measure) on Q_η^*. Since $z_p(\sigma)$ is contained in a closed set $\overline{F}$ (bounded away from $\mathcal{T}$), for almost all $w \in \sigma$ we have $z(w) \in \overline{F}$. On the other hand, $z \in P_2'(Q)$. Therefore, there is a point $w_0 \in \sigma$ and a line parallel to the u-coordinate axis, passing through w_0, such that z is continuous along this line. Let us say $w_0 = (u_0, v_0)$, and the line $L_0 : v = v_0$ is the one in question. Since $z(w_0) \in \overline{F}$, the distance from $z(w_0)$ to $\mathcal{T}$ is > 0. Thus there is a point near w_0, say, (u_1, v_0), such that $z(u_1, v_0)$ is at a distance > 0 from $\mathcal{T}$. But z is continuous at (u_1, v_0), so there is a neighborhood, say a square $R : [u_2 \leq u \leq u_3, v_2 \leq v \leq v_3]$, of (u_1, v_0) such that the distance of $z(w)$ from $\mathcal{T}$ is $> \tau > 0$ for all $w \in R$. But as shown above, there is a strip S near Q^* such that $z(w)$ is within, say, $\tau/4$ of $\mathcal{T}$ for all w in this strip. Thus,

$$0 < \frac{3\tau}{4} < |z(u', v) - z(u'', v)|$$

$$\leq \left| \int_{u''}^{u'} z_u(u, v) du \right| \leq \int_{u''}^{u'} |z_u(u, v)| du \quad ,$$

$u'' < u'$, $(u'', v) \in R$, $(u', v) \in S$. Since this holds for all v with $v_2 \leq v \leq v_3$,

$$0 < \int_{v_2}^{v_3} \frac{3\tau}{4} dv \leq \int_{v_2}^{v_3} \int_{u''}^{u'} |z_u(u, v)| du\, dv \leq \int_Q |z_u(u, v)| du\, dv$$

$$\leq \left(\int_Q |z_u(u, v)|^2 du\, dv \right)^{\frac{1}{2}} \leq (D(z, Q))^{\frac{1}{2}}$$

Therefore,

$$0 < \frac{m}{2} D(z, Q) \leq I(z, Q) \quad .$$

We now assume for the moment that all our conditions are satisfied on the unit disk B, and we shall prove that z is linked. Then by a conformal mapping the result will hold for Q.

Thus, since $z(B^*)$ lies on $\mathcal{T}$, there is a strip $S_0 : [r_0 \leq r \leq 1;\ 0 \leq \theta \leq 2\pi]$ such that z maps every closed curve in S_0 onto a curve whose distance from $\mathcal{T}$ is less than d/4 (d = dist[H, $\mathcal{T}$]). Hence for $w \in S_0$ it must be that z(w) does not lie on H; in fact, the distance from z(w) to H is $> \frac{d}{2}$. Let $\varepsilon > 0$ be given, $\varepsilon < d/4$. Then there is a number r_ε, $0 < r_\varepsilon < 1$, such that the circle $r = r_\varepsilon$ encloses all points w which map onto H under z, and such that the curve $M_\varepsilon : z = z(r_\varepsilon, \theta),\ 0 \leq \theta \leq 2\pi$, has its greatest distance from $\mathcal{T}$ less than $\varepsilon/2$. By the uniform convergence of z_p toward z on each closed subdomain of B' (B' corresponds to Q', i.e., a circular arc is omitted), we may choose p so large that

$$|z_p(r_\varepsilon, \theta) - z(r_\varepsilon, \theta)| < \varepsilon/2 \quad , \quad 0 \leq \theta \leq 2\pi \quad .$$

Thus the curve $M_{p\varepsilon} : z = z_p(r_\varepsilon, \theta)$ is at a distance $\leq \varepsilon$ from $\mathcal{T}$.

Now suppose that the curve $M_\varepsilon : z = z(r_\varepsilon, \theta),\ 0 \leq \theta \leq 2\pi$, does not link H. Then neither does the curve $M_{p\varepsilon} : z = z_p(r_\varepsilon, \theta),\ 0 \leq \theta \leq 2\pi$ link H (since they can be continuously deformed into each other without touching H). However, the curve $M_p : z = z_p(1, \theta),\ 0 \leq \theta \leq 2\pi$, does link H, so the intersection number corresponding to the image under z_p of the ring $r_\varepsilon \leq r \leq 1$ and the circle H is different from 0.

Since $I(z, B) > 0$, there is a subdomain, say,

$B_0 : [r \leq b;\ 0 \leq \theta \leq 2\pi]$, $0 < b < r_\varepsilon$, such that $I(z, B_0) > 0$.

Therefore, we have for some $\beta > 0$

$$0 < \beta \leq I(z, B_0) < \underline{\lim}_{p \to \infty} I(z_p, B_0) \quad ,$$

so after extracting a subsequence we rename it $\{z_p\}$ and we may assume that $0 < \beta \leq I(z_p, B_0)$ for all p.

Next, set $F_p(\theta) = \int_b^1 |z_{p_r}(r, \theta)|^2 r\, dr$. Then $F_p(\theta) \in L_1(0, 2\pi)$, and

$$\int_0^{2\pi} F_p(\theta) d\theta = \int_0^{2\pi} \int_b^1 |z_{p_r}(r, \theta)|^2 r\, dr\, d\theta \leq D(z_p, B) \leq A,$$

A = constant .

Now if $F_p(\theta) > \frac{A}{b\pi}$ a.e. on $(0, 2\pi)$, then we would have $\int_0^{2\pi} F_p(\theta)\, d\theta > A$, a contradiction. Therefore, there is a set of positive measure for which $F_p(\theta) \leq \frac{A}{b\pi}$; we let θ_p be in that set. Then

$$\int_b^1 |z_{p_r}(r, \theta_p)|^2 r\, dr \leq \frac{A}{b\pi} .$$

Hence for almost all r, $b \leq r \leq 1$,

$$b|z_p(r,\theta_p) - z_p(1,\theta_p)|^2 = b|\int_r^1 z_p(r,\theta_p)dr|^2$$

$$\leq b(\int_r^1 |z_p(r,\theta_p)|dr)^2$$

$$\leq b(1-r)\int_r^1 |z_{p_r}(r,\theta_p)|^2 dr$$

$$\leq (1-r)\int_b^1 |z_{p_r}(r,\theta_p)|^2 r\,dr$$

$$\leq (1-r)\frac{A}{b\pi} \quad .$$

Hence for almost all r, $b \leq r \leq 1$,

$$|z_p(r,\theta_p) - z_p(1,\theta_p)|^2 \leq (1-r)\frac{A}{\pi} \quad ,$$

and since z_p is continuous, this relation holds for all r in the interval.

Therefore, the oscillation on the radial segment $N_p : [\theta = \theta_p;\ r_\varepsilon \leq r \leq 1]$ is less than $\varepsilon/2$ if r_ε is chosen close enough to 1. The point $z_p(1,\theta_p)$ is on a manifold $\mathfrak{M}_p$, the greatest distance from $\mathfrak{M}_p$ to $\mathcal{T}$ goes to 0 as $p \to \infty$. Therefore, the values of z_p on N_p are at a distance less than ε from $\mathcal{T}$ if p is large enough. But by the inequality above, r_ε is independent of p, so we may rotate and obtain $N_p = N$ for all sufficiently large p.

We now cut the ring $R_\varepsilon : [r_\varepsilon < r < 1,\ 0 \leq \theta \leq 2\pi]$ along N to obtain a simply connected domain R'_ε, whose boundary is mapped by z_p onto a continuous curve at a (greatest) distance less than ε from $\mathcal{T}$, and this curve is linked with H since the intersection number of $z_p(R_\varepsilon)$ with H is not zero. (There can be no point $w \in N$ such that $z_p(w)$ lies on H, since the distance of $z_p(w)$ from $\mathcal{T}$ is $< \varepsilon$, while the distance from H to $\mathcal{T}$ is $> \varepsilon$.)

We now have

$$I(z_p, R'_\varepsilon) = I(z_p, B) - I(z_p, B - R'_\varepsilon) \quad ,$$

and

$$I(z_p, B - R'_\varepsilon) \geq I(z_p, B_0) \geq \beta > 0,$$

so that

$$I(z_p, R'_\varepsilon) < I(z_p, B) - \beta \quad .$$

Letting $p \to \infty$ we have

$$\varlimsup_{p\to\infty} I(z_p, R'_\varepsilon) \leq \delta - \beta \quad .$$

By a conformal mapping we can transform R'_ε onto B and z_p into a vector ζ_p defined on B with $I(z_p, R'_\varepsilon) = I(\zeta_p, B)$.

The new sequence $\{\zeta_p\}$ is an admissible sequence.

Hence

$$\delta \leq \varlimsup_{p\to\infty} I(\zeta_p, B) = \varlimsup_{p\to\infty} I(z_p, R'_\varepsilon) \leq \delta - \beta \ ,$$

contradiction. Therefore, we conclude that z must link H.

Now by a conformal mapping, the result holds on the square Q.

Note: The proof of linking is a variant of that in [11, pp. 216, 217].

<u>Theorem 5.5.</u> $I_0(z, Q) = \delta$.

<u>Proof.</u> The vector z is not yet known to be admissible since it is not necessarily continuous on the segment σ. Let $\varepsilon > 0$ be given. We choose a rectangle A containing σ in its interior such that $\overline{A} \subset Q^o$, and such that

$$\frac{M}{2} D(z, A) < \varepsilon \quad .$$

This, of course, may always be done, by the absolute continuity of the integral. The boundary values of z on A^* are continuous, and therefore there is a harmonic function h whose boundary values agree with those of z on A^*, and moreover, h is continuous on $\overline{A}$. We define a new vector

$$z'(w) = \begin{Bmatrix} h(w) , & w \in A \\ z(w) , & w \in Q - A \end{Bmatrix} \quad .$$

which is now continuous on Q^o, linked with H (preceding theorem), and whose boundary lies on $\mathcal{T}$. That is, z' is admissible. Furthermore,

$$D(z', A) = D(h, A) \leq D(z, A) \quad ,$$

and all these remarks hold for any such rectangle contained in A. Therefore,

$$\delta \leq I_0(z', Q) = I_0(z, Q-A) + I_0(z', A)$$

$$\leq I_0(z, Q-A) + I(z', A)$$

$$\leq I_0(z, Q-A) + \frac{M}{2} D(z', A) < I_0(z, Q-A) + \varepsilon \quad .$$

Thus,

$$\delta < I_0(z, Q-A) + \varepsilon$$

$$\leq I_0(z, Q) + \varepsilon \quad .$$

But $\varepsilon > 0$ was arbitrary, and so it must be that $\delta \leq I_0(z, Q)$. By the preceding theorem, $\delta \geq I_0(z, Q)$. Therefore, $I_0(z, Q) = \delta$.

Now it remains only to show that z is continuous on the whole of Q^o. We shall prove that z is continuous on a closed subdomain of Q^o which contains σ. This is all that is necessary. We may take for such a subdomain a large (concentric) inscribed disk B.

Lemma 5.4. [19, p. 42]. Let B be a region bounded by a finite number of circles. Let z be a vector of class $P_2(B)$, whose boundary values are continuous as functions on B^*. Suppose also that there exists a number $K \geq 1$ such that

$$D(z, G) \leq K \; D(H(z, G), G)$$

for every subregion G, of B, which is of class K (see [19, p. 5]),

$H(z, G)$ being the function harmonic on G and coinciding with z on G^*. Then z is equivalent to a function $\tilde{z}$ which is continuous on $\bar{B}$ and takes on these boundary values continuously.

<u>Lemma 5.5</u>. If G is any Jordan region such that $\bar{G} \subset B^o$ and measure $(G^*) = 0$, then

$$I_0(z, G) = \lim_{p\to\infty} I(z_p, G) \quad .$$

<u>Proof</u>. By Theorems 5.2 and 5.5 we have

$$\lim_{p\to\infty} I(z_p, Q) = \delta = I_0(z, Q) \quad . \tag{5.4}$$

We shall prove the lemma for every $\bar{G} \subset Q^o$, and then it will be true for every $\bar{G} \subset B^o$ of the prescribed type. Furthermore, if G is a region such that there is a sequence of closed subregions $\bar{R} \subset G^o$ invading G, and on each $\bar{R}$, $z_p \to z$ uniformly, each z_p being continuous on $\bar{R}$, we have (by the lower semicontinuity of I_0)

$$I_0(z, R) \leq \underline{\lim}_{p\to\infty} I_0(z_p, R) \leq \underline{\lim}_{p\to\infty} I_0(z_p, G) \quad .$$

Letting R invade G, this yields

$$I_0(z, G) \leq \underline{\lim}_{p\to\infty} I_0(z_p, G) \tag{5.5}$$

Let $\varepsilon > 0$ be given. By relation (5.4) above, there is a number N_1 such that $p \geq N_1$ implies

(5.6) $$-\frac{\varepsilon}{3} < I(z_p, Q) - I_0(z, Q) < \varepsilon/3 \quad .$$

In view of relation (5.5), there is a number N_2 such that $p \geq N_2$ implies

$$I_0(z, G) < I_0(z_p, G) + \varepsilon \quad ,$$

or since $I_0(z_p, G) \leq I(z_p, G)$,

(5.7) $$-\varepsilon < I(z_p, G) - I_0(z, G) \quad .$$

Let $G' = Q - \overline{G}$. Since meas $G^* = 0$, $I(z,Q) = I(z,G) + I(z,G')$, and similarly for $I_0(z,Q)$, $I(z_p,Q)$, $I_0(z_p,Q)$. We now construct a region G_η defined by

$$G_\eta = G'_\eta \cap Q_{\eta'} \quad ,$$

$$G'_\eta = \{ w \in G; : \text{dist } w, G^* \geq \eta \} \quad ,$$

$$Q_{\eta'} = \{ w \in Q : \text{dist } w, Q^* \geq \eta \} \quad ,$$

where $\eta > 0$ is chosen so small that

(5.8) $$I_0(z, G' - G_\eta) < \varepsilon/3 \quad .$$

Now G_η is a compact set, meas $G^* = 0$, and G_η is at a positive distance from $\overline{G}$ and from Q^*. Therefore, we may cover G_η by a finite grid of closed squares $\{ \overline{Q}_i \}_{i=1}^{M}$ whose sides are parallel to the coordinate axes, each square having the same side-length, and whose diagonals are so small that if $w \in \overline{Q}_i \cap G_\eta$, then $\overline{Q}_i \cap \overline{G} = \emptyset$ and $\overline{Q}_i \cap Q^* = \emptyset$, $i = 1, 2, \ldots, M$.

Then (by relation (5.5), changing the name G to Q_i) I_0 is lower semicontinuous on each Q_i, and thus also on the closed set $G_Q = \bigcup_{i=1}^{M} \overline{Q}_i \subset G'$. Therefore, there is a number N_3 such that $\rho \geq N_3$ implies that

$$I_0(z, G_Q) < I_0(z_p, G_Q) + \varepsilon/3 \leq I(z_p, G_Q) + \varepsilon/3 \quad ,$$

or

$$(5.9) \qquad I_0(z, G_Q) - I(z_p, G_Q) < \varepsilon/3 \quad .$$

Also from (5.8) we have

$$(5.10) \qquad I_0(z, G' - G_Q) \leq I_0(z, G' - G_\eta) < \varepsilon/3 \quad .$$

Now let $N = \max\{N_1, N_2, N_3\}$. Then for $p \geq N$, using relations (5.6)—(5.10), we have

$$\begin{aligned} -\varepsilon < I(z_p, G) - I_0(z, G) &= I(z_p, Q) - I(z_p, G') - I_0(z, Q) + I_0(z, G') \\ &= (I(z_p, Q) - I_0(z, Q)) + (I_0(z, G') - I(z_p, G')) \\ &< \varepsilon/3 + (I_0(z, G_Q) - I(z_p, G_Q)) + I_0(z, G' - G_Q) - I(z_p, G' - G_Q) \\ &< \varepsilon/3 < \varepsilon/3 + \varepsilon/3 - I(z_p, G' - G_Q) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \quad . \end{aligned}$$

Therefore, $\lim_{p \to \infty} I(z_p, G) = I_0(z, G)$, where G is a subdomain of Q of the proper type. Therefore, this also holds for subdomains of B.

Now if G is any subregion of the type of Lemma 5.4, we have by Lemma 5.5 and the lower semicontinuity of $D(z, G)$ (with respect to weak convergence in class P_2)

$$(5.11) \qquad \frac{m}{2} D(z, G) \leq \underline{\lim}_{p\to\infty} \frac{m}{2} D(z_p, G) \leq \lim_{p\to\infty} I(z_p, G) = I_0(z, G) \quad .$$

However, from the minimizing property of z (Theorem 5.5) we obtain (using relation (5.11))

$$\frac{m}{2} D(z, G) \leq I_0(z, G) \leq I(z, G) \leq I(H(z, G), G) \leq \frac{M}{2} D(H(z, G), G),$$

or

$$(5.12) \qquad D(z, G) \leq KD(H(z, G), G) \quad ,$$

where $H(z, G)$ is the harmonic function coinciding with z on G^*, and $K = \frac{M}{m} \geq 1$. However, by Lemma 5.4, relation (5.12) implies z is continuous on B. Therefore, z is continuous on Q^o, z links H, and the boundary of z lies on $\mathcal{T}$. Thus z is an admissible vector and $I_0(z, Q) = \delta \leq L$. But then since z is admissible, $I_0(z, Q) \geq L$. Thus $I_0(z, Q) = L$, and the existence theorem is proved.

Remark. It can be seen now that the existence theorem holds for much more general fixed manifolds. For example, we could take any manifold whose complement in E^3 is homeomorphic to the complement of a simple closed curve (or a torus). Thus a capstan surface and a cylinder fall into this category.

More generally, we could take a piece of a plane (or deformation of a plane) and attach an arc to the plane at the endpoints of the arc. The resulting manifold would be our fixed manifold. Moreover, the arc need not even be simple; the position of an admissible surface is described only by the intersection number of the surface with a fixed simple closed curve. However, the existence theorem gives no information as to the structure of the trace of the minimizing surface on the fixed manifold.

BIBLIOGRAPHY

1. P. Alexandroff and H. Hopf, Topologie I, Julius Springer, Berlin, 1935.

2. J. W. Calkin, "Functions of several variables and absolute continuity, I," Duke Math. J. 6 (1940), 170-186.

3. C. Carathéodory, "Untersuchungen über die Konformen Abbildungen von festen und veränderlichen Gebieten," Math. Ann. 72 (1912), 107-144.

4. L. Cesari, "Una uguaglianza fondamentale per l'area delle superficie," Atti Della Reale Accademia D'Italia 14 (1944), 891-950.

5. ________, "La nozione di integrale sopra una superficie in forma parametrica," Ann. Scuola Norm. Sup. Pisa 13 (1947), 77-117.

6. ________, "Condizioni sufficienti per la semicontinuita degli integrali sopra una superficie in forma parametrica," Ann. Scuola Norm. Sup. Pisa 14 (1948), 47-79.

7. ________, "Su un particolare processo di retrazione per superficie," Riv. Mat. Univ. Parma 3(1952), 25-42.

8. ________, "An existence theorem for integrals on parametric surfaces," Amer. J. Math. 74 (1952), 265-295.

9. ________, Surface Area, Princeton University Press, Princeton, New Jersey, 1956.

10. R. Courant, "On a generalized form of Plateau's problem," Trans. Amer. Math. Soc. 50 (1941), 40-47.

11. ________, Dirichlet's Principle, Interscience Publishers, New York, 1950.

12. R. Courant and N. Davids, "Minimal surfaces spanning closed manifolds," Proc. Nat. Acad. Sci., U. S. A., 26 (1940), 194-199.

13. J. M. Danskin, "On the existence of minimizing surfaces in parametric double integral problems in the calculus of variations," Riv. Mat. Univ. Parma 3 (1952), 43-63.

14. N. Davids, "Minimal surfaces spanning closed manifolds and having prescribed topological position," Amer. J. Math. 64 (1942), 348-362.

15. F. W. Gehring, "The Carathéodory convergence theorem for quasiconformal mappings in space," Ann. Acad. Sci. Fenn. A. I. 336 (1964), 1-21.

16. G. M. Goluzin, Geometrische Funktionentheorie, Deutscher Verlag der Wissenschaften, Berlin, 1957.

17. H. Lewy, "On minimal surfaces with partially free boundary," Comm. Pure Appl. Math. 4 (1951), 1-13.

18. C. B. Morrey, "An analytic characterization of surfaces of finite Lebesgue area.I," Amer. J. Math. 57 (1935), 692-702.

19. ________, "Functions of several variables and absolute continuity. II," Duke. Math. J. 6 (1940), 187-215.

20. ________, "Multiple integral problems in the calculus of variations and related topics," Univ. Cal. Publ. Math., N. S. Vol. 1, 1943, 1-130.

21. ________, "The parametric variational problem for double integrals," Comm. Pure Appl. Math. 14 (1961), 569-575.

22. ________, "Multiple integrals in the calculus of variations," Colloq. Lec., Amer. Math. Soc., August, 1964.

23. C. B. Morrey and J. Eells, "A variational method in the theory of harmonic integrals. I," Ann. of Math. 63 (1956), 91-128.

24. E. A. Nordhaus, "The problem of Bolza for double integrals in the calculus of variations," Contributions to the Calculus of Variations, 1938-1941, University of Chicago Press, Chicago, 1942.

25. Yu. G. Reshetnyak, "A new proof of a theorem concerning the existence of an absolute minimum for two dimensional problems in the calculus of variations in parametric form," Sibirsk. Mat. Z. 3 (1962), 744-768. (In Russian)

26. J. Serrin, "On the definition and properties of certain variational integrals," Trans. Amer. Math. Soc. 101 (1961), 139-167.

27. A. G. Sigalov, "Two dimensional problems of the calculus of variations," Uspehi Matem. Nauk. N. S., 6 (1951), 16-101. (In Russian)

28. H. A. Simmons, "The first and second variations of a double integral for the case of variable limits," Trans. Amer. Math. Soc. 28 (1926), 235-251.

29. ________, "The first and second variations of an n-tuple integral in the case of variable limits," Trans. Amer. Math. Soc. 36 (1934), 29-43.

30. S. L. Sobolev, Applications of Functional Analysis in Mathematical Physics, Translations of Mathematical Monographs, Vol. 7, American Mathematical Society, Providence, 1963.

31. L. Tonelli, "L'estremo assoluto degli integrali doppi," Ann. Schola Norm. Sup. Pisa 2 (1933), 89-130.

32. L. H. Turner, "On the direct method in the calculus of variations," Ph.D. Thesis, Purdue University, Lafayette, Indiana, 1957.