ENGINEERING RESEARCH INSTITUTE
THE UNIVERSITY OF MICHIGAN
ANN ARBOR

ON THE MOTION OF A PROJECTILE IN THE ATMOSPHERE

Vi-Cheng Liu

ABSTRACT

The equation of rectilinear motion of a projectile which moves in an atmosphere, of which the density decreases exponentially with the altitude, is solved. It is found that the velocity of the projectile can be expressed explicitly in terms of confluent hypergeometric functions. This theory is applied to treat two specific problems: (1) the flight analysis of a sounding rocket during the free-flight period and (2) the calculation of ambient temperature from the trajectory of a spherical projectile.

1. INTRODUCTION

The theory of flight of a ballistic projectile in a resisting medium dates back to Newton. In the meantime, various approximations for the flight parameters which are involved in the equation of motion have been used.[1] The most important ones are the aerodynamic-drag coefficient of the projectile and the ambient-air density. The former is usually considered as functions of Mach number, the latter as functions of the altitude.

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[1]Superscript numbers indicate similarly numbered references at the end of this note.

The theory of flight of a sounding rocket during the free-flight period deals essentially with the same problem as that of a ballistic projectile. The only difference is that the air-density factor is more critical because of the extreme altitude attained by the former; hence, a more realistic approximation to the ambient-air density is necessary. In view of the multiplicity of parameters involved, the equation of motion of the projectile (or rocket) is generally treated either by a laborious method of step-wise integration or by the analog computer technique.

A recent experiment,[2] in which a falling sphere is used as an ambient-temperature probe in the upper-atmosphere measurements, stimulates new interest to the trajectory problem. Measured trajectory data of a spherical projectile are used to determine the ambient-air density, from which the ambient-air temperature can be derived.

In this note, new approximations to the aerodynamic-drag coefficient and ambient-air density, which are believed to be more accurate than previous representations,[1] are introduced. Particle drag, which is caused by the presence of aerosols in the atmosphere, is allowed in the present analysis. This, of course, warrants consideration only in some special cases.

It is found that the projectile velocity is expressible explicitly in terms of confluent hypergeometric functions. The significance of this result, which is believed to be new, is twofold: (1) In treating "direct" problems, for instance, like the trajectory analysis of a sounding rocket, the present theory

can be used to save the labor of step-wise integration if tabulation of the particular confluent hypergeometric function is available. (2) In the case of an "indirect" problem such as the computation of the ambient temperature from a falling-sphere trajectory, it can be applied with advantage because of the elimination of the numerical differentiation process which is involved in the original theory.[2]

2. SIGNIFICANT FORCES

The word "projectile" is used here referring to any axially symmetric body in free flight. We restrict our discussion to the case with negligible spin and yaw.

The force system acting on a projectile includes: (1) the gravity mg; (2) the bouyancy ρgV, where V represents the volume displaced by the projectile, and ρ, the ambient-air density; (3) the aerodynamic drag D; (4) particle drag d, which is caused by the presence of aerosols in the atmosphere; and (5) the inertial force $(m + \Delta m)dv/dt$, where Δm, Munk's apparent additional mass, is equal to one half the mass of air displaced by a spherical projectile; and dv/dt, the projectile acceleration.

The bouyancy and the inertial force of the apparent additional mass are significant only for projectiles of the balloon type. Though, under sea-level conditions, the particle drag due to water droplets, for instance, may be insignificant when compared with the aerodynamic drag, it is conceivable, however, that in the cumulus clouds at high altitudes, the drag due to water droplets may

become significant. While the water-droplet content depends primarily on the ambient temperature, the ambient-air density decreases with the altitude exponentially.

It is not intended here for us to become involved in detailed computation of the particle drag. Rather, we shall investigate its effect on the projectile motion since the particle-drag term in the equation of motion follows a functional dependence upon the altitude, which is, in general, different from that of the aerodynamic drag. As a first approximation, the particle drag is prescribed as $d = C_d \sigma A v^2 / 2$, according to the Newtonian concept of drag. The particles are assumed to be mass points at rest; hence, C_d should depend only on the position and shape of the projectile and be independent of its size A and velocity v. The particle concentration is represented by σ.

3. APPROXIMATE REPRESENTATION OF THE AERODYNAMIC-DRAG COEFFICIENT

It can be demonstrated with a dimensional analysis that the aerodynamic-drag coefficient, defined as $C_D = D/(\rho V^2 A / 2)$, for geometrically similar projectiles depends primarily upon the Mach number and Reynolds number. For projectiles moving at high speeds, the dependence of C_D on the Reynolds number is much less significant than that on the Mach number except in cases for which the Reynolds numbers are low, such as those at the high altitudes. In those cases, however, the aerodynamic-drag force is usually negligible as compared to the gravity.

For a specific projectile, the variation of C_D with Mach number M, in general, shows distinctly different characteristics in the three Mach number ranges: (1) subsonic, (2) transonic, and (3) supersonic. Qualitatively speaking, $C_D(M)$ starts with roughly constant values (for different projectiles) then increases gradually as transonic range is approached. During the transonic range it increases until the supersonic range begins. $C_D(M)$ in the supersonic range decreases with a gradient which is very large at first and gradually diminishes to a small value.

Reliable values of C_D still rely mostly on measurements from model tests. Theoretical consideration may, however, serve as a guide to the choice of emperical functions for prescribing C_D.

Consider separately the compressibility effect in the subsonic flows for slender and round bodies. In the former case, to account for the compressibility effect on the pressure coefficient, either the Prandtl-Glauert factor $(1 - M^2)^{-1/2}$ or a modified version of it (for instance, the Karman-Tsien formula or Goethert's generalized Prandtl-Glauert formula) can be used. These factors, when expanded in a power series of M, can be expressed in the form

$$C_o + C_2 M^2 \tag{3.1}$$

after neglecting higher-order terms. In the latter case, as an index of the compressibility on the drag coefficient of a round body, the impact-pressure coefficient at the stagnation point can be shown to be

$$C_p = 1 + \frac{M^2}{4} + \dots \qquad (3.2)$$

It has also been shown by measurements that the compressibility effect on the skin-friction coefficient and the coefficient of drag due to interference between body and fins can be considered as proportional to the powers of the Prandtl-Glauert factor.

A large fraction of the drag force on a projectile in the supersonic flows are attributed to the frontal shock wave and the base pressure. For a typical body like a cylinder with a nose cone, the drag coefficient can be shown to be approximately of the form

$$C_o + C_3 M^{-2} \qquad (3.3)$$

The validity of the above expression is expected to extend to the hypersonic range.[3] Relatively less is known concerning the theory of drag in a transonic flow, which is a mixture of subsonic and supersonic flows. It is therefore suggested to use

$$C_D = C_o + C_2 M^2 + C_3 M^{-2} \qquad (3.4)$$

as a general expression for approximating the drag coefficient of a projectile in the following analysis. C_o, C_2, and C_3 in (3.4) are empirical constants to be prescribed for each Mach-number range.

4. EQUATION OF RECTILINEAR MOTION OF A PROJECTILE

Consider a nonspinning projectile which ascends at zero angle of yaw in a stationary atmosphere. By collecting terms of forces which have been discussed in Section 2, we obtain the equation governing the rectilinear motion of a projectile along the vertical axis y.

$$\frac{1}{2}(C_0 + C_2 M^2 + C_3 M^{-2})A\rho v^2 + \frac{1}{2}C_d A\sigma v^2 - \rho g V + mg + (m + \triangle m)v\frac{dv}{dy} = 0 \qquad (4.1)$$

For an isothermal atmosphere with temperature T in altitude interval $(y - y_0)$, the ambient-air density can be expressed as $\rho = \rho_0 \exp\left[-(y - y_0)/H\right]$ where H, the scale height, is defined as $kT/m_1 g$-- k = Boltzmann's constant; m_1 = mean molecular mass of air .

To simplify Eq. (4.1), we introduce

$$Z = M^2 = (\gamma gH)^{-1} v^2, \quad x = \exp\left[-(y - y_0)/H\right], \text{ and } \rho = \rho_0 x \qquad (4.2)$$

where γ= ratio of specific heats of air, and obtain

$$\frac{dZ}{dx} = C_2 \alpha Z^2 + (C_0 \alpha + \frac{\beta}{x})Z + (C_3 \alpha - \delta + \frac{\epsilon}{x}) \qquad (4.3)$$

where

$$\alpha = A\rho_0 H(m + \triangle m)^{-1}, \quad \beta = C_d A\sigma H(m + \triangle m)^{-1}, \quad \delta = 2\rho_0 V\gamma^{-1}(m + \triangle m)^{-1}, \text{ and}$$

$$\epsilon = 2m\gamma^{-1}(m + \triangle m)^{-1} \qquad (4.4)$$

Equation (4.3) is nonlinear and of the Riccati type. It can be transformed through use of the substitution

$$u = \exp\left[-\int \alpha C_2 Z dx\right] \qquad (4.5)$$

into a linear equation

$$\frac{d^2u}{dx^2} - (\alpha C_o + \frac{\beta}{x}) \frac{du}{dx} + \alpha C_2(\alpha C_3 - \delta + \frac{\epsilon}{x})u = 0 \tag{4.6}$$

Equation (4.6), which has a regular singularity at $x = 0$ and an

irregular singularity at $x = \infty$, can be converted into the canonical form

$$\xi \frac{d^2\phi}{d\xi^2} + (b - \xi) \frac{d\phi}{d\xi} - a\phi = 0 \tag{4.7}$$

with

$$\xi = 2\Omega x, \quad \Omega^2 = \frac{\alpha^2 C_o^2}{4} + \alpha C_2(\delta - \alpha C_3), \quad \phi = u \exp\left[\Omega - \frac{1}{2}\alpha C_o\right]x, \quad b = -\beta, \quad \text{and}$$

$$a = \frac{1}{2}(\frac{1}{2}\alpha C_o\beta - \alpha C_2\epsilon - \beta\Omega)\Omega^{-1} \tag{4.8}$$

One of the fundamental solutions of (4.7) is the confluent hyper-

geometric function[4]

$$\phi(a, b, \xi) = {}_1F_1(a, b, \xi) = \sum_{n=0}^{\infty} \frac{(a)_n \xi^n}{(b)_n \, n!} \tag{4.9}$$

where

$$(a)_n = a(a + 1) \ldots (a + n - 1) \qquad n \geq 1$$

$$(a)_o = 1$$

The other one is

$$\xi^{1-b} \, {}_1F_1(1 + a - b, \ 2 - b; \xi) \tag{4.10}$$

provided b is not an integer. The general solution of (4.7) with nonintegral b

may be written as

$$\Phi = B \, {}_1F_1(a, \ b; \xi) + C \xi^{1-b} \, {}_1F_1(1 + a - b, \ 2 - b; \xi) \tag{4.11}$$

where B and C are arbitrary constants. Confluent hypergeometric functions have

been fairly extensively tabuleted.[5,6,7]

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In view of (4.2), (4.5), (4.8), and (4.11), we obtain

$$v^2 = \frac{2\gamma g H \Lambda}{\alpha C_2} \left[\frac{1}{2} - \frac{\alpha C_0}{4\Lambda} - \frac{1}{\Phi} \frac{d\Phi}{d\xi} \right] \tag{4.12}$$

Because of the term $(d\Phi/d\xi)/\Phi$, one of the arbitrary constants of Φ as given in Eq. (4.11) will be eliminated in Eq. (4.12) for v^2. Since Eq. (4.3) is of the first order, there is only one constant of integration. This is consistent with the physical aspect of the problem. The constant of integration is to be determined by the use of the boundary condition of the projectile trajectory.

Consider the restriction on b in (4.11). Since $b = -\beta$, where β is defined as $C_d A \sigma H (m + \Delta m)^{-1}$, which, in general, is less than unity, the limitation on the values of b is not significant as far as the present analysis is concerned. The only notable exception is the case $b = 0$, which corresponds to the problem when the particle drag is null.

Equation (4.7) with $b = 0$ has been studied by Bateman.[8] The solution of (4.7) for this particular case is known as Bateman's k-function. The general solution of Eq. (4.7), when $b = 0$, is bounded in the neighborhood of $\xi = 0$.*

It is to be noted that when $C_2 = 0$, Eq. (4.3) becomes a linear equation of the first order, which can be integrated immediately in terms of elementary functions.[9]

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*This is pointed out to the author by Professor Robert C. F. Bartels.

It is also to be noted that the equation of motion of a projectile with

the assumption $C_D \sim M^{-1}$ (a satisfactory approximation for the wave drag of

airfoil sections in hypersonic flows) and $\beta = 0$ is again a linear equation of the

first order if v and t are used as a dependent and an independent variable, re-

spectively. [10]

It is interesting to note that the function ϕ in Eq. (4.7) can be ex-

pressed in terms of Bessel functions provided that the condition $2C_2 \epsilon = C_o \beta$ is

fulfilled. Other special cases of the confluent hypergeometric function will be

obtained for the solution ϕ when values of the parameters b and a in Eq. (4.7)

fulfill appropriate restrictions in each individual case.

5. ENGINEERING APPLICATION

The result of the present analysis may be applied to the following

problems:

5.1 Flight Analysis of a Sounding Rocket.

A rocket which carries sounding instruments to high altitudes for

measuring the upper atmosphere is called a sounding rocket. It is usually

launched vertically upward to gain the utmost peak altitude. In view of the

simplicity of the designed trajectory, it is generally equipped with the least

guidance. As a first approximation, it is plausible to assume that a sounding

rocket with normal performance moves rectilinearly at zero angle of yaw.

Hence, the present analysis can be applied in the free-flight period.

For a solid-propellant rocket, which usually has large jet-propulsive thrust of a short duration, we can estimate the burnout velocity and altitude by neglecting the aerodynamic drag in comparison to the forward thrust. The classical method is to integrate the equation of motion of a rocket with mass m as a function of time t:

$$C \frac{dm}{dt} + m(\frac{dv}{dt} + g) = 0 \qquad (5.1)$$

where C is the constant exhaust velocity. This leads to

$$v_b = - C \ln\frac{m_b}{m_o} - g(t_b - t_o) \qquad (5.2)$$

$$y_b = - C \int_{t_o}^{t_b} (\ln \frac{m_b}{m_o})dt - \frac{1}{2} g(t_b - t_o)^2 \qquad (5.3)$$

where subscripts o and b denote initial and burnout condition, respectively. The above calculations can be improved through an iterative process in regard to the aerodynamic-drag correction.

5.2 Ambient Temperature Computation from Trajectory Data of a Falling Sphere.

The idea of using a free-falling sphere, of which the trajectory can be accurately measured, to probe the upper atmosphere for the purpose of measuring the ambient temperature was originally conceived by Jones.[2] This was carried out by incorporating a miniature Doppler receiver-transmitter in an inflated sphere, which was released from a sounding rocket near its peak altitude. The sphere trajectory is determined by a DOVAP system,[11] which consists of a transmitter and an array of receivers on the ground.

In an alternative scheme, as proposed by the author, the ground stations of transmitters and receivers are replaced by a mobile station which has a transmitter and a receiver just like those used in the DOVAP system except in miniature sizes. The mobile station is incorporated in a body, with negligible drag-to-weight ratio, which is to be released simultaneously with the sphere from the rocket. With this scheme, the velocity of the sphere, relative to an apparently null-drag trajectory or pseudo-vacuum trajectory, is obtained from Doppler cycles which are to be telemetered from the mobile station and recorded along a time scale. It is assumed that both the mobile station and the sphere fall vertically with no relative tumble.

One serious drawback of the falling-sphere method is the need of numerical differentiation of the velocity data in the process of computing ambient-air density according to the original theory.[2] As it is known, numerical differentiation is a notoriously inaccurate process because it exaggerates the irregularities of the numerical function whose derivatives are to be determined. This process can be avoided if the present theory of projectile trajectory is applied in step-wise calculation of the scale height H, which is proportional to the ambient-air temperature T.

6. ACKNOWLEDGMENT

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