ERRATA SHEET

Page		Original	Corrected
1	bottom	physics	Physics
5	bottom	(1959)	(1949)
7	Eq. (3.4)	$n(p)$	$n(P)$
7	Eq. (3.4)	$\exp-(\underline{c},-\underline{s})^2$	$\exp-(\underline{c}-\underline{s})^2$
8	middle	near-continouus	near-continuum
9	top	near-continuous	near-continuum
10	Eq. (4.9)	$(h^2\ \underline{\delta}-\underline{h}^2)$	$(h^2\underline{\delta}-\underline{h}^2)^m$
10	Eq. (4.9)	$\frac{g_1 f}{2^{n-1} m!}$	$\frac{g_1 f}{2^{m-1} m!}$
11	Eq. (4.10)	$\int \underline{c}^h c^\ell f\ d\underline{c}$	$\int \underline{c}^n c^\ell f\ d\underline{c}$
11	Eq. (4.13)	$g_{1a}+g_{1b}+g_2+g_3$	$g_{1a}+g_{1b}+g_2-g_3$
11	Eq. (4.14)	${}^\ell g_{1a}^n+{}^\ell g_{1b}^n-{}^\ell g_2^n+{}^\ell g_3^n$	${}^\ell g_{1a}^n+{}^\ell g_{1b}^n+{}^\ell g_2^n-{}^\ell g_3^n$
14	Eq. (4.23)	$\Gamma(m+\frac{1}{2})\sum_{j=0}^{n}\sum_{m=0}^{N(j/2)}$	$\sum_{j=0}^{n}\sum_{m=0}^{N(j/2)}\Gamma(m+\frac{1}{2})$
15	bottom	j and n	j and m
16	top	$-{}^0g^1\ {}^0g^1$	$-\frac{1}{2}\ {}^0g^1\ {}^0g^1$
16		${}^0g^2\ {}^2f^0$	${}^0g^0\ {}^2f^0$
16		${}^2g^0\ {}^2g^0$	${}^2g^0\ {}^0g^0$
17	Eq. (5.4)	$e(\underline{q},-c\underline{q}_0)\cos\tau$	$E(\underline{q},-c\underline{q}_0)\underline{q}_0\ \cos\tau$
17	bottom	*drag	*Drag

ERATTA SHEET (concluded)

Page		Original	Corrected
20	middle	the collisional distribution and	(delete the 4 words)
21	middle	(o.g. $< \frac{\lambda}{a} <$ 10)	(0.5 $< \frac{\lambda}{d} <$ 10)
22	bottom	this equation	his equation
23	bottom	coutte	couette
25		ℓ	b
26		x	χ
27		x	χ

THE UNIVERSITY OF MICHIGAN

COLLEGE OF ENGINEERING
Department of Aeronautical and Astronautical Engineering

Final Report

STUDY OF AIR DRAG AT HIGH ALTITUDE

Prepared for the project by

V. C. Liu

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PREFACE

This final report is submitted on the work done under Subcontract with the University of Utah for the Prime Contract No. AF 19(604)-6658 with the Air Force Cambridge Research Laboratories for the period from July 15, 1962, to June 30, 1964.

THE UNIVERSITY OF MICHIGAN PROJECT PERSONNEL

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GENERAL DISCUSSION

A. OBJECTIVE OF THE PRESENT STUDY

A rational theory or accurate calibration of sphere drag corresponding to the free flight conditions of a falling sphere in the upper atmosphere is of utmost interest in the current program of upper atmosphere measurement. Unfortunately neither has been available for the entire experimental range of interest.

The experimental data obtained from laboratory calibrations of sphere drag in the rarefied flows have been scanty, lacking especially for the flow regime intermediate between the free-molecule and the continuum flows. Recently there have been some experimental sphere drag studies at hypersonic speeds for the purpose of simulating re-entry conditions. These results turn out to be not very useful to the falling sphere experiment because of the extremely high temperature for the skins of the sphere; furthermore these skin temperatures are unknown to a considerable degree. The fact that the sphere drag in a highly rarefied atmosphere depends critically on the skin temperature makes the experimental sphere drag data for re-entry even less favorable for extrapolation to the falling sphere experiments.

In view of the unsatisfactory status of sphere drag theory, one tends to use various forms of semi-empirical approach which consists of extrapolating sphere drag on the basis of available experimental data with the guidance of physically plausible functions to represent the sphere drag coefficient. This approach would be doomed to failure if the empirical formula is used beyond its limited range of validity. In general, one uses a limited number of empirical constants to represent some physical quantities which vary with many parameters.

It is felt that the present situation of the sphere drag problem pertaining to upper air measurements calls for an examination of sphere drag from the viewpoint of the Boltzmann equation in kinetic theory.

B. METHOD OF APPROACH TO THE PROBLEM OF SPHERE DRAG IN TRANSITIONAL FLOWS

It has been generally agreed that aerodynamics of semi-rarefied gas of neutral particles can be adequately represented by the Boltzmann equation in kinetic theory. It is also well known that Boltzmann equation has been notoriously resistant to solution except in its application to the relatively simple cases such as the transport phenomena and free-molecule flows. Consequently, numerous kinetic models for the evaluation of molecular collisions have appeared in the literature as approximations to Boltzmann's collision integral. Such efforts are certainly worthwhile in view of the significance of the problem provided the consequences of both the shortcomings and the advantages of such approximations are carefully clarified. Unfortunately this is usually not the case.

A common procedure in reporting such approximate analysis in the scientific literatures starts with some drastic simplification of the binary collision integral without careful discussion of its consequences, then the simplified kinetic equation, thus obtained, is applied to some special problems usually the internal flow problems such as Couette flows. The analysis proceeds with the solution for the simplified equation and compares its results with some other models. The most damaging consequence of such analyses is the claiming of broad implications of the agreement of the calculated results on the basis of some applications to the simple special problems. Much of the existing confusion and conflicting impression of the kinetic theory of the transition flows result from these unjustifiable claims.

In the present approach which is described in the scientific report, authored by V. C. Liu, S. C. Pan, and H. Jew, attached at the end of this report, we obtain the first-order iteration starting with the free-molecule solution of Boltzmann equation as its zero-order approximation. The exact binary collision integral is expanded in terms of Hermite polynomials for molecules interacting under the Maxwellian potential.

C. RESULTS AND DISCUSSION

Calculations of sphere drag in transition flows based on the present solution of first-order iteration are made for spheres with skin temperature equal to the free stream temperature.

It is found that the present results of sphere drag coefficient depends in a very simplified manner on the speed ratios (or molecular Mach numbers as they are sometimes called). For detail functional dependance, see the quoted report attached at the end of this final report.

On the Sphere Drag in Flows of Almost-Free Molecules*

by

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Abstract

A kinetic theory of sphere drag in the transition flows is presented. The kinetic method starts with the Boltzmann equation for the Maxwellian molecules and having the exact binary collision integral in first order Knudsen iteration. The collision integral is expanded as a function of Hermite polynomials in molecular velocity. The Hermite coefficients are taken as functions of the space vector. The sphere drag analysis is thus reduced to a problem of integration involving tensor algebra.

Although the general method of approach to the problem is valid for the flow of any speed provided it is in the regime near the free molecules, the calculated results, based on Hermite expansion truncated beyond the second order, are estimated to be accurate for speed ratio (free stream velocity/the most probable molecular speed of the free stream) less than unity. The calculations are for spheres at the free stream temperature and having perfect diffuse reflection for the incident molecules at the solid surface. The computed results agree with Millikan's measurements for the extremely slowly moving spheres within a few percent over a wide range of Knudsen numbers ($0.5 < \lambda/d < 10$).

A result of possible great significance is that the sphere drag of the almost-free molecular flows expressed in units of the free molecular drag at the corresponding speed ratio is found essentially independent of the speed ratio for

*This work was supported in parts by the upper Atmospheric physics Laboratory, Geophysics Research Directorate, AFCRL and the Phoenix Memorial Research Grant of the University of Michigan.

the range of speed ratios the present calculation is supposed to be valid.

(I) Introduction

The recent interest on sphere drag in a rarefied gas has been stimulated, partly at least, by the advent of the earth satellite experiments and the upper atmosphere measurements. The theory of sphere drag in the extremely rarefied medium such that the state of free molecules[1] exists, is well known provided the reflection mechanisms of the incident molecules on the solid surface are adequately represented by accommodation coefficients of Maxwell and Knudsen.

Numerous attempts have been made recently in the determination of sphere drag for the state slightly less rarefied than free molecules. This has been called the near-free molecular flows (or almost-free molecular flows). Many of these attempts are limited either to the flows of extremely high speeds[2,3] or to the extremely low speeds[4]. These restrictions have been introduced to make the analyses tractable. In the present study, an effort is made to remove the limitations imposed in previous analyses on the speed of the sphere. The purpose of the present work is two-fold: (1) to obtain the sphere drag in a rarefied atmosphere which is of utmost interest in upper atmosphere measurements, (2) to understand the fundamental nature of the transition flows with a model which is realistic enough such that accurate drag measurements, either free flight or laboratory, can be made in the near future in order to make meaningful comparisons.

It is a matter of simple dimensional analysis to show that the flows around geometrically similar bodies without the influence of external force are dynamically similar, provided the flows have equal speed ratio (s) and Knudsen number (Kn) respectively.

Experimental results of sphere drags corresponding to the Knudsen numbers of the near-free molecular flows rarely exist except for the excellent measurements by Millikan[5] which covers a wide range of Knudsen number ($.01 < K_n < 10$) at however extremely low speed ratios ($s < 10^{-5}$). Millikan's data

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[1] G. N. Patterson, Molecular Flow of Gases, John Wiley, 1956

[2] D. R. Willis, "A Study of Near Free Molecule Flow" Rand Report R-339 (1959)

[3] R. M. L. Baker and A. F. Charwat, Phys. Fluids 1, 73 (1958)

[4] Z. Szymanski, Arch. Mech. Stos. (Warsaw) 8, 449 (1956); 9, 35 (1957)

[5] R.A. Millikan, Phy. Rev. 22, 1 (1923)

for which the experimental conditions were clearly defined may serve as important check points for theory of sphere drag at asymptotically low speed ratios.

The elementary kinetic theory of the transition flows based on collision statistics[3,6] can only give a gross quantitative answer to the problem of interest as was painstakingly cautioned[6]. A rigorous theory must start with the Boltzmann equation. The present study on sphere drag gives the first order (Kn^{-1}) iteration of the Boltzmann equation for the Maxwellian molecules[7]. The basic approach is similar to that of Szymanski[4] except that we preserve the nonlinear terms of speed ratio in the Boltzmann collision integral which is essential in the treatment of high speed flows. The inclusion of these nonlinear terms considerably complicates the collision integral analysis.

(II) Formal Iteration of the Boltzmann Equation

Consider a gas with Maxwellian molecules[7] of mass m, number density n_∞ and with the most probable velocity w_∞. To describe the molecular distribution surrounding a spherical body of diameter d placed in a free stream of velocity $\underline{u}_\infty$, we introduce a molecular distribution function $F(\underline{c}, \underline{r})$ for the molecules at point $\underline{r}$ with velocity $\underline{c}$ in dimensionless form with displacement and velocity expressed in units of $d/2$ and w_∞ respectively. If the differential collision cross section for the molecular collisions can be expressed in units of the momentum cross section B_1 for Maxwellian molecules[8], we can write the steady state Boltzmann equation for Maxwellian molecules in dimensionless form as follows:

$$\underline{c} \cdot \frac{\partial F}{\partial \underline{r}} = \frac{d n_\infty B_1}{2 w_\infty} \int d\underline{c}_1 \int_0^{2\pi} d\epsilon \int_0^{\pi/2} d\theta \, B(\theta) \; (F' F_1' - F F_1) \tag{2.1}$$

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[6] V.C. Liu, J. Aero. Sci. 25, 779 (1958); also J. Fluid Mech. 5, 481 (1959)
[7] S. Chapman and T.G. Cowling, Mathematical Theory of Non-uniform Gases, (Cambridge 1951)
[8] H. Grad, Comm. Pure and App. Math. 2, 325 (1959) (Note: the definition of $B(\theta)$ is different from Grad's by a factor of B_1.)

where

$$B_1 = 0.343 \left(\frac{2K}{m}\right)^{\frac{1}{2}} \qquad \text{(inter-molecular force} = K/\eta^5\text{)}$$

$$B(\theta)\, d\theta\, d\epsilon = b\, db\, d\epsilon / 0.343 \quad \text{(b = impact parameter, See Fig. 1)}$$

A physical interpretation can be given to the dimensionless constant in front of the integrals in Equation (2. 1). If we define a mean free path λ based on the momentum cross section B_1 and use the definition of viscosity μ from transport theory for Maxwellian molecules, namely*

$$\mu = kT/6B_1 = 0.491(4/\pi)^{\frac{1}{2}}\, mn_\infty w_\infty \lambda \tag{2. 2}$$

we can rewrite (2. 1) as

$$\underline{c} \cdot \frac{\partial F}{\partial \underline{r}} = 0.075\,(d/\lambda) \int d\underline{c}_1 \int_0^{2\pi} d\epsilon \int_0^{\pi/2} d\theta\, B(\theta)(F'F_1' - FF_1) \tag{2. 3}$$

in other words, this dimensionless constant of interest is inversely proportional to the Knudsen number (λ/d).

For the studies of flows at high Knudsen numbers, it is appropriate to expand the distribution function F into a power series in d/λ, the inverse of Knudsen number[9],

$$F = \overset{0}{F} + (d/\lambda)\overset{1}{F} + (d/\lambda)^2\overset{2}{F} + \cdots \tag{2. 4}$$

Equation (2. 3), after the substitution of the expansion (2. 4) and a re-arrangement of terms in the powers of d/λ results in the sequence of equations:

$$\underline{c} \cdot \frac{\partial \overset{0}{F}}{\partial \underline{r}} = 0 \tag{2. 5}$$

$$\underline{c} \cdot \frac{\partial \overset{1}{F}}{\partial \underline{r}} = 0.075(d/\lambda) \int d\underline{c}_1 \int_0^{2\pi} d\epsilon \int_0^{\pi/2} d\theta\, B(\theta)(\overset{0}{F}'\overset{0}{F}_1' - \overset{0}{F}\overset{0}{F}_1) \tag{2. 6}$$

. . . .

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It is observed that the left hand side of Equation (2. 5) is the derivative of $\overset{0}{F}$ in the direction of the vector $\underline{c}$ in the molecular phase space. At each

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[9]G. Jaffe, Ann. d. Phys. 6, 195 (1930); See also C. S. Wang Chang and G. E. Uhlenbeck, Univ. of Mich. ERI Rept. M999 (1953)

*For the Maxwellian molecules $B_1 = B_1^{(n)}$ following notations used in Reference 8.

point in this space ($\underline{c}, \underline{r}$), this vector points in the direction of the molecular trajectory through that point, which is also the characteristic curve of the equation. Hence if q denotes arc length along a trajectory, Equation (2. 5) and (2. 6) become respectively

$$c \frac{d\overset{0}{F}}{dq} = 0 \qquad (2.7)$$

$$c \frac{d\overset{1}{F}}{dq} = 0.075(d/\lambda)E(\underline{r}, \underline{c}) \qquad (2.8)$$

where $E(\underline{r}, \underline{c})$ denotes the multiple integral in Equation (2. 6), the physical significance of which will be discussed later.

Since Equation (2. 7) states that the function $\overset{0}{F}$ is constant along a trajectory, it is the distribution function for the free molecular, or collisionless, flows when appropriate boundary conditions of interest have been satisfied. Equation (2. 8), which contains a collision term expressed in terms of the free molecular distribution $\overset{0}{F}$, may be considered as the kinetic equation of flow of the almost free molecules[10]. It may be noted that the previous analyses made on the flows of the almost-free molecules[3,6] are, in essence, some macroscopic moments of Equation (2. 8) for mass and momentum fluxes etc., after drastic simplifications have been made to the collision integral E.

It is further noted that in view of the mathematical structure of Equations (2. 7) and (2. 8), the analysis of the rarefied gas invariably involves integrations along the characteristic curves when the flow fields need to be mapped. In the case of the free molecular flows, this is quite similar to the problem of geometrical optics. In fact, the line of sight principle will be adopted in mapping the distribution $\overset{0}{F}(\underline{r}, \underline{c})$.

(III) Zeroth Order Approximation to the Molecular Distribution

As a prelude to the analysis of flows of the almost-free molecules, we must first obtain the solution $\overset{0}{F}$ to the collisionless Equation (2. 7) with boundary conditions prescribed on the sphere and upstream. This constitutes our zeroth

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[10] V. C. Liu, Univ. of Mich. ORA Rept. 02885-11-F (1962)

order approximation to the problem. It might be of interest to note here that in the aerodynamics of free molecules[1] where the primary interest is usually the total momentum and energy transfer between the molecules and the solid body, the molecular distribution for the entire flow field surrounding the body, namely $\overset{0}{F}(\underline{r}, \underline{c})$, is not of interest. In the present study, however, $\overset{0}{F}(\underline{r}, \underline{c})$ must be obtained before one can start on the first order iteration by the use of Equation (2. 8).

It is noted that the aggregation of molecules at any point $(\underline{r})$ in a free molecular flow field must come from either of two sources: the free stream and the reflected stream from the solid surface. It is possible to express $\overset{0}{F}$ as the sum of two component functions each from one of two mentioned sources with weighting factors; the weighting factor contains a unit vector which represents the boundary of the cone of sight at a point and is a discontinuous function[10]. The expression so obtained for $\overset{0}{F}$ containing a closed form weighting factor is more elegant but not effective in the eventual integrations for the collision integral E. Thus we simply express $\overset{0}{F}$ in terms of functions $f(\underline{r}, \underline{c})$ and $g(\underline{r}, \underline{c})$ as follows:

$$\overset{0}{F} = f + g \tag{3.1}$$

where

$$f = \pi^{-\frac{3}{2}} \exp - (\underline{c} - \underline{s})^2 \qquad \text{for all regions} \tag{3.2}$$

and

$$g = \begin{cases} 0 & \text{for region I (See Fig. 2)} \quad (3.3) \\ \pi^{-\frac{3}{2}}\, n(p) \exp - c^2 - \pi^{-\frac{3}{2}} \exp - (\underline{c}, - \underline{s})^2 & \text{for region II} \quad (3.4) \end{cases}$$

where $\underline{s}$ denotes the free stream velocity in units of w_∞, i. e., $\underline{s} = \underline{u}_\infty / w_\infty$ and the equivalent number density n(P) of reflected molecules at point P on the sphere (See Fig. 2) is given by[1]

$$n(P) = \exp - (\underline{s} \cdot \underline{n}_o)^2 - \sqrt{\pi}\, \underline{s} \cdot \underline{n}_o + \sqrt{\pi}\, \underline{s} \cdot \underline{n}_o \operatorname{erf} (\underline{s} \cdot \underline{n}_o) \tag{3.5}$$

In (3. 5) $\underline{n}_o$ denotes a unit vector as shown in Fig. 2. The subscript 'o' always designates a unit vector in the direction of the vector it subscripts, e. g., $\underline{s}_o = \underline{s} / |\underline{s}|$, etc.

It can be shown, by referring to Fig. 2 that

$$\underline{n}_o \cdot \underline{s}_o = \underline{s}_o \cdot \underline{r} - (\underline{s}_o \cdot \underline{c}_o)\left[(\underline{r} \cdot \underline{c}_o) - (1 - r^2 \sin^2 \chi)^{\frac{1}{2}}\right] \tag{3.6}$$

In the present study the function n(P) is approximated by a linear function with s-dependent coefficients $\alpha(s)$ and $\beta(s)$ as follows:

$$n(P) = \alpha s x + \beta \tag{3.7}$$

where $x = \underline{s}_o \cdot \underline{n}_o$ and the parameters α, β are to be determined by considering the conservation of particles for the upstream and downstream semi-spherical surfaces respectively (See Appendix 1).

(IV) A Mathematical Representation of the First Order Collision Effect

In the interest of aerodynamic drag, we shall be concerned only with a few lower moments of the distribution function $\overset{1}{F}$ rather than the function itself. Accordingly, the present approach will dwell primarily with the determination of moments for $\overset{1}{F}$ from the use of Equation (2.8). The fact, however, that the distribution $\overset{0}{F}$ appearing in the integrand of the collision integral E is a discontinuous function—an inherent feature of the free molecular distribution—makes it unfruitful to use the moment-generation technique of Grad[8] which appears effective for the near-continuous flows only[11].

It is observed that the right hand side of Equation (2.8) can be interpreted as the distribution function of molecules having collisions in the neighborhood of the point $(\underline{r})$. In more precise statement, it can be said that $0.075\,(d/\lambda)E(\underline{r}, \underline{c})\,d\underline{r}d\underline{c}$ represents the net rate of change of the number of molecules situated in the region $\underline{r}$, $\underline{r} + d\underline{r}$ and having velocities between $\underline{c}$ and $\underline{c} + d\underline{c}$. Note that collisions have the tendency to randomize the molecular arrangement hence to smooth out the discontinuity in molecular arrangement in the molecular phase space. It is therefore postulated that for the almost-free molecules the collisional distribution $E(\underline{r}, \underline{c})$ should be better suited for representation, compared to F for instance, by a series of three dimensional Hermite polynomials $H^{(n)}_{ijk\ldots}(\underline{c})$ which has been used by Grad[8]

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[11] R. Goldberg, "Slow Flow of a Rarefied Gas Past a Spherical Obstacle", Thesis, New York Univ. (1954)

for representing the distribution $F(\underline{r}, \underline{c})$ in the near-continuous flows. This must be considered as the fundamental hypothesis of the present approach.

(IV-1) Series Expansion for distribution $E(\underline{r}, \underline{c})$

It is assumed that

$$E(\underline{r}, \underline{c}) = (2\pi)^{-\frac{3}{2}} \exp -(c^2/2) \sum_{n=0}^{\infty} a^{(n)}_{ijk\ldots} \; H^{(n)}_{ijk\ldots} \tag{4.1}$$

(i, j, k ranges from 1 to 3)

where the coefficients $a^{(n)}_{ijk\ldots}$, symmetrical tensors of the nth order, are functions of $\underline{r}$, and $H^{(n)}_{ijk\ldots}$, the three dimensional Hermite polynomials of the nth order, are functions of $\underline{c}$. The coefficients are, in turn, expressible as

$$a^{(n)}_{ijk\ldots} = \frac{1}{n!} \int d\underline{c} \; H^{(n)}_{ijk\ldots} \; E(\underline{r}, \underline{c}) \tag{4.2}$$

Equation (4.2), after the substitution of $\overset{0}{F} = f + g$ from (3.1) and $E(\underline{r}, \underline{c})$ from (2.6) and (2.8), also the use of symmetry relations for the binary collision integral[7], becomes

$$a^{(n)} = \frac{1}{n!} \int\int d\underline{c} d\underline{c}_1 \int_0^{2\pi} d\epsilon \int_0^{\pi/2} d\theta \, B(\theta) \, [H^{(n)}] (g_1 f + \tfrac{1}{2} g_1 g) \tag{4.3}$$

where*

$$[H^{(n)}] = H^{(n)}(\underline{c}') + H^{(n)}(\underline{c}_1') - H^{(n)}(\underline{c}) - H^{(n)}(\underline{c}_1) \tag{4.4}$$

From conservations of momentum and energy in elastic collisions of molecules, we obtain the velocities after collision (See Fig. 1)

$$\begin{aligned} \underline{c}_1' &= \underline{c}_1 + h \cos\theta \, \underline{k}_o \\ \underline{c}' &= \underline{c} - h \cos\theta \, \underline{k}_o \end{aligned} \tag{4.5}$$

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*Subscripts ijk... have been dropped since the order of tensor is designated by subscript (n).

where h is the velocity of approach in the center of mass system; $\underline{k}_o$, the unit vector pointing to the direction of interaction force that changes the vector $\underline{h}$ to $\underline{h}'$. The substitution of (4.5) in (4.4) and the use of the identity:

$$H^{(n)}(\underline{a} + \underline{b}) = \sum_{j=0}^{n} H^{(n-j)}(\underline{a})\underline{b}^j$$

leads to

$$[H^{(n)}] = \sum_{j=1}^{n} \{H^{(n-j)}(\underline{c}_1) + (-1)^j H^{(n-j)}(\underline{c})\} \cos^j \theta \, h^j \, \underline{k}_o^j \tag{4.6}$$

Further development of (4.3) involves integrals of the following type:

$$\int_0^{2\pi} d\epsilon \, k_o^j = \sum_{m=0}^{N(j/2)} \underline{h}_o^{(j-2m)} (\underline{\delta} - \underline{h}_o^2)^m \frac{\pi(-1)^j \cos^{j-2m} \theta \sin^{2m} \theta}{2^{m-1} \quad m!} \tag{4.7}$$

where N(j/2) denotes the highest integer not greater than j/2; $\underline{\delta}$, the unit tensor or Kronecker delta δ_{ij} with i, j omitted[8]. (For the derivation of (4.7) see appendix 2).

$$\text{Define} \quad B_{jm} = \pi \int_0^{\pi/2} \sin^{2m}\theta \cos^{2j-2m}\theta \, B(\theta) \, d\theta \tag{4.8}$$

The expression (4.3), after the use of (4.6), (4.7) and (4.8), becomes

$$a^{(n)} = \frac{1}{n!} \sum_{j=1}^{n} \sum_{m=0}^{N(j/2)} \iint d\underline{c} d\underline{c}_1 \{H^{(n-j)}(\underline{c}) +$$

$$+ (-1)^j H^{(n-j)}(\underline{c}_1)\} \underline{h}^{j-2m} (h^2 \delta - h^2) B_{jm} \left(\frac{g_1 f}{2^{n-1} m!} + \frac{g_1 g}{2^m m!} \right) \tag{4.9}$$

So far, the expansion has been given to the nth order. In the evaluation for collision effects, we shall truncate the series beyond n = 2; the mathematical effort of higher order terms becomes prohibitively heavy and is probably not rewarding because, as it will be shown, the present approximation appears satisfactory.

(IV - 2) The Evaluation of the Coefficient $a^{(2)}$

A careful study of (4.8) will show that the final evaluation of the coefficient $a^{(2)}$ amounts to the calculations of the following types of moment integrals:

(1) $$^{\ell}f^{n} = \int \underline{c}^{h} c^{\ell} f d\underline{c} \tag{4.10}$$

over the whole velocity space, which can be integrated without much trouble. The results of the first few moment integrals ($n \leq 2$) are as follows:

$$^{0}f^{0} = 1;\ ^{0}f^{1} = \underline{s};\ ^{0}f^{2} = \underline{s}^{2} + \tfrac{1}{2}\underline{\delta}\ ;\ ^{2}f^{0} = s^{2} + \tfrac{3}{2} \tag{4.11}$$

(2) $$^{\ell}g^{n} = \int \underline{c}^{n} c^{\ell} g d\underline{c} \tag{4.12}$$

over a conical domain subtended at the point $\underline{r}$ by the sphere of diameter d. (See Fig. 2) To accomplish the integration we divide the g-function into four parts:

$$g = g_{1a} + g_{1b} + g_{2} + g_{3} \tag{4.13}$$

where

$$g_{1a} = \pi^{-\frac{3}{2}} \alpha s\, (\underline{s}_{o} \cdot \underline{r}) \exp -c^{2}$$

$$g_{1b} = -\pi^{-\frac{3}{2}} \alpha s\, (\underline{s}_{o} \cdot \underline{c}_{o}) \{\underline{r} \cdot \underline{c}_{o} -(1 - r^{2} \sin^{2}\chi)^{\frac{1}{2}}\} \exp -c^{2}$$

$$g_{2} = \pi^{-\frac{3}{2}} \beta \exp - c^{2}$$

$$g_{3} = \pi^{-\frac{3}{2}} \exp -(\underline{c} - \underline{s})^{2}$$

and $$^{\ell}g^{n} = {}^{\ell}g_{1a}^{n} + {}^{\ell}g_{1b}^{n} - {}^{\ell}g_{2}^{n} + {}^{\ell}g_{3}^{n} \tag{4.14}$$

These moment integrals have been evaluated and the results are as follows:

$$^{\ell}g_{1a}^{n} + {}^{\ell}g_{2}^{n} = (\alpha\, \underline{s} \cdot \underline{r} + \beta)\pi^{-\frac{3}{2}}\, \Gamma\left(\frac{n + \ell + 3}{2}\right) \sum_{m=0}^{N(j/2)} \underline{r}_{o}^{n-2m} (\underline{\delta} - \underline{r}_{o}^{2})^{m} \alpha_{nm} \tag{4.15}$$

where

$$\alpha_{nm} = \pi \int_0^{\sin^{-1}\frac{1}{r}} \frac{\cos^{n-2m}\chi \, \sin^{2m+1}\chi \, d\chi}{2^m \, m!} \tag{4.16}$$

and

$$ {}^{\ell}g^{n}_{1b} = -\pi^{-\frac{3}{2}}\,\Gamma\left(\frac{n+\ell+3}{2}\right)\,\alpha\underline{s}\cdot \sum_{m=0}^{N\left(\frac{n+1}{2}\right)} \underline{r}_o^{\,n+1-2m}(\underline{\delta}-\underline{r}_o^{2})^{m}\beta_{nm} \tag{4.17}$$

$$\beta_{nm} = \pi \int_0^{\sin^{-1}\frac{1}{r}} \frac{r\cos\chi - (1-r^2\sin^2\chi)^{\frac{1}{2}}\sin^{2m+1}\chi\,\cos^{n+1-2m}\chi\,d\chi}{2^m\,m!} \tag{4.18}$$

The coefficients α_{nm} and β_{nm} are functions of r^{-1} and are expanded up to the power r^{-5}. These coefficients have been evaluated and the results are as follows $(n \leq 2)$:

$$\begin{aligned} &\alpha_{00} = (\pi/8)(4r^{-2} + r^{-4} + \ldots) \;; \quad \alpha_{10} = (\pi/2)r^{-2} \\ &\alpha_{20} = (\pi/8)(4r^{-2} - r^{-4} + \ldots) \;; \quad \alpha_{21} = (\pi/8)(r^{-4} + \ldots) \end{aligned} \tag{4.19}$$

$$\begin{aligned} &\beta_{00} = (\pi/48)(24r^{-1} - 16r^{-2} - 6r^{-3} - r^{5} + \ldots)\;; \\ &\qquad\qquad \beta_{10} = (\pi/60)(30r^{-1} - 20r^{-2} - 15r^{-3} + 4r^{-4} + \ldots) \\ &\beta_{11} = (\pi/120)(15r^{-3} - 8r^{-4} + \ldots)\;; \\ &\qquad\qquad \beta_{21} = (\pi/120)(15r^{-3} - 8r^{-4} - 5r^{-5} + \ldots) \end{aligned} \tag{4.20}$$

with these coefficients we obtain the moment integrals ${}^{\ell}g^{n}_{1a}$, ${}^{\ell}g^{n}_{1b}$ and ${}^{\ell}g^{n}_{2}$ with $n \leq 2$.

$$ {}^{0}g^{0}_{1a} = (\alpha \underline{s} \cdot \underline{r}_o/32)\,(8r^{-1} + 2r^{-3} + r^{-5} + \ldots) $$

$$ {}^{2}g^{0}_{1a} = (\alpha \underline{s} \cdot \underline{r}_o/64)\,(24r^{-1} + 6r^{-3} + 3r^{-5} + \ldots) $$

$$ {}^{0}g^{1}_{1a} = \alpha \underline{s} \cdot \underline{r}_o\,(\sqrt{\pi}\;2r)^{-1}\underline{r}_o $$

$$ {}^{0}g^{2}_{1a} = \alpha \underline{s} \cdot \underline{r}_o\,[(24r^{-1} - 12r^{-3} - 3r^{-5} + \ldots)\underline{r}_o^{2} + (6r^{-3} + 2r^{-5} + \ldots)\underline{\delta}]/64 $$

$$ {}^{0}g^{0}_{1b} = -\alpha \underline{s} \cdot \underline{r}_o\,(24r^{-1} - 16r^{-2} - 6r^{-3} - r^{-5} + \ldots)/96 \tag{4.21} $$

$$ {}^{0}g^{1}_{1b} = -[\alpha \underline{s} \cdot \underline{r}_o(60r^{-1} - 40r^{-2} - 45r^{-3} + 16r^{-4} + \ldots)\underline{r}_o - \alpha\, s(15r^{-3} - 8r^{-4} + \ldots)]/120\sqrt{\pi} $$

$$ {}^{2}g^{0}_{1b} = -\alpha \underline{s} \cdot \underline{r}_o\,(24r^{-1} - 16r^{-2} - 6r^{-3} - r^{-5} + \ldots)/64 $$

$$ {}^{0}g^{2}_{1b} = -\alpha \underline{s} \cdot \underline{r}_o[(120r^{-1} - 80r^{-2} - 150r^{-3} + 64r^{-4} + 35r^{-5})\underline{r}_o^{2}/320 $$

$$ + (30r^{-3} - 16r^{-4} - 10r^{-5})\underline{\delta}/320] - \alpha \underline{s} \cdot \underline{r}_o(30r^{-3} - 16r^{-4} - 10r^{-5})/320 $$

$$ {}^{0}g^{0}_{2} = \beta(4r^{-2} + r^{-4} + \ldots)/16 $$

$$ {}^{2}g^{0}_{2} = \beta(12r^{-2} + 3r^{-4} + \ldots)/32 \tag{4.22} $$

$$ {}^{0}g^{2}_{2} = \beta[(12r^{-2} - 6r^{-4} + \ldots)\underline{r}_o^{2} + (3r^{-4} + \ldots)\underline{\delta}]/32 $$

The remaining moment integral ${}^{\ell}g^{n}_{3}$ turns out to be the most difficult one to cope with. To facilitate the integration we introduce the Cartesian coordinates $(\underline{r}_o, \underline{t}_o, \underline{t}'_o)$ where $\underline{r}_o$ is the unit vector pointing from center of the sphere to the point of interest; $\underline{t}_o$ is the unit vector, perpendicular to $\underline{r}_o$, lying in the plane containing $\underline{r}_o$ and $\underline{s}_o$; $\underline{t}'_o$ is the unit vector perpendicular to both $\underline{r}_o$ and $\underline{t}_o$ (see Fig. 3). In terms of the new vectors, we obtain, after a lengthy algebraic manipulation, the ${}^{\ell}g_3^n$ in the following form:

$$
\begin{aligned}
{}^{\ell}g_3^n = {} & 2\pi^{-\frac{3}{2}}\Gamma(\tfrac{1}{2})\Gamma(m+\tfrac{1}{2}) \sum_{j=0}^{n} \sum_{m=0}^{N(j/2)} e^{-s^2}\, \underline{r}_o^{\,n-j}\, \underline{t}_o^{\,j-2m}\, \underline{t}'^{\,2m}_o \int_0^\infty e^{-c^2} c^{n+\ell+2}\, dc \;\times \\
& \times \int_0^{\sin^{-1}(\frac{1}{r})} d\chi\, e^{-2sc\cos\theta\cos\chi} \sin^{j+1}\chi \cos^{n-j}\chi \;\times \\
& \times \left(\frac{d^{j-2m}}{dz^{j-2m}} \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{k!\,(k+m)!} \right)_{z=2sc\sin\chi\sin\Theta}
\end{aligned}
\qquad (4.23)
$$

It turns out that the manner in which we split the tensor into spherical components $(\underline{r}_o, \underline{t}_o, \underline{t}'_o)$ makes possible the further advance in the effective evaluation of ${}^{\ell}g_3^n$. We shall illustrate this by showing the terms associated with the powers of $\underline{r}_o$, $\underline{t}_o$ and $\underline{t}'_o$ in the expanded (r^{-1}) power expression for ${}^{\ell}g_3^n$:

$$\underline{r}_o^{\,n} \ldots (\pi)^{-\frac{1}{2}} e^{-s^2}\, \underline{r}_o^{\,n} \left[r^{-2} f_{n+\ell+2} - (\tfrac{1}{4}) r^{-4} \left\{ (n-1) f_{n+\ell+2} - 2s\cos\Theta f_{n+\ell+3} - 2s^2 \sin^2\Theta f_{n+\ell+4} \right\} + \ldots \right]$$

$(j = 0,\ m = 0)$

$$\underline{r}_o^{\,n-1}\underline{t}_o \ldots (\tfrac{1}{2})(\pi)^{-\frac{1}{2}} e^{-s^2}\, s\sin\Theta\, \underline{r}_o^{\,n-1}\underline{t}_o\, r^{-4} f_{n+\ell+3} + \ldots \qquad (j=1,\ m=0,\ k=1)$$

$$\underline{r}_o^{\,n-2}\underline{t}_o^{\,2} \ldots (\tfrac{1}{4})(\pi)^{-\frac{1}{2}} e^{-s^2}\, \underline{r}_o^{\,n-2}\underline{t}_o^{\,2}\, r^{-4} f_{n+\ell+2} + \ldots \qquad (j=2,\ m=0,\ k=1) \qquad (4.24)$$

$$\underline{r}_o^{\,n-2}\underline{t}'^{\,2}_o \ldots (\tfrac{1}{4})(\pi)^{-\frac{1}{2}} e^{-s^2}\, \underline{r}_o^{\,n-2}\underline{t}'^{\,2}_o\, r^{-4} f_{n+\ell+2} + \ldots \qquad (j=2,\ m=1,\ k=0)$$

where $f_n(s\cos\Theta) = \int_0^\infty e^{-c^2} c^n e^{-2sc\cos\Theta} dc = \sum_m \frac{(-1)^m 2^{m-1}(s\cos\Theta)^m}{m!} \Gamma(\frac{n+m+1}{2})$

It is significant to note that in the r^{-1} power expansion for ${}^{\ell}g_3^n$ the orders of $\underline{t}_o'$ appear only in even powers which means that there is no rotational component of molecular flow around $\underline{s}_o$; also as functions of Θ, s, and r^{-1}, the lower order of $\underline{t}_o$ and $\underline{t}_o'$ always accompany with the lower orders of r^{-1} as shown in (4. 23). Hence for any order of n, there will be only a few terms of the low order $\underline{t}_o$ and $\underline{t}_o'$ assuming importance in the expansion. It implies that the moment integrals ${}^{\ell}g_3^n$ are predominantly r-dependent functions.

Again the moments of ${}^{\ell}g_3^n$ up to n = 2 and the expansion in power series of r^{-1} to the 5th order are evaluated and presented as follows:

$$
\begin{aligned}
{}^0g_3^0 &= (\pi)^{-\frac{1}{2}} e^{-s^2} [4r^{-2} f_2 + r^{-4}(f_2 + 2s\cos\Theta f_3 + 2s^2\sin^2\Theta f_4)]/4 \\
{}^2g_3^0 &= (\pi)^{-\frac{1}{2}} e^{-s^2} [4r^{-2} f_4 + r^{-4}(f_4 + 2s\cos\Theta f_5 + 2s^2\sin^2\Theta f_6)]/4 \\
{}^0g_3^1 &= (\pi)^{-\frac{1}{2}} e^{-s^2} [4r^{-2} f_3 + r^{-4}(2s\cos\Theta f_4 + 2s^2\sin^2\Theta f_5)]\underline{r}_o/4 + \\
&\quad + (\pi)^{-\frac{1}{2}} e^{-s^2}\ s\sin\Theta\, \underline{t}_o\, r^{-4} f_4/2 \qquad\qquad (4.25) \\
{}^0g_3^2 &= (\pi)^{-\frac{1}{2}} e^{-s^2} [4r^{-2} f_4 + r^{-4}(-f_4 + 2s\cos\Theta f_5 + 2s^2\sin^2\Theta f_6)]\, \underline{r}_o^2/4 + \\
&\quad + (\pi)^{-\frac{1}{2}} e^{-s^2}\, s\sin\Theta r^{-4} f_5\, \underline{t}_o\underline{r}_o/2 + (\pi)^{-\frac{1}{2}} e^{-s^2} r^{-4} f_4(\underline{t}_o^2 + \underline{t}_o'^2)/4
\end{aligned}
$$

(IV - 3) General Expression for the Collisional Distribution $E(\underline{r}, \underline{c})$

Finally the expression for the collisional distribution $E(\underline{r}, \underline{c})$ can be given in computable form. It is noted that the first non-zero term in the expansion for $E(\underline{r}, \underline{c})$ is the second order term because both $[H^{(0)}]$ and $[H^{(1)}]$ vanish. For n = 2 the possible j and n are as follows:

$$(j, m) = (1, 0),\ (2, 0) \text{ and } (2, 1).$$

With these values of j, m we obtain

$$a^{(2)}_{ij}(\underline{r}) = [\,{}^0g^2\,{}^0f^0 - {}^0g^1\,{}^0f^1 + {}^0g^0\,{}^0f^2 + {}^0g^2\,{}^0g^0 - {}^0g^1\,{}^0g^1\,](2B_{20} - B_{10} - B_{21}) +$$

$$+[\,{}^2g^0\,{}^0f^0 - 2\,{}^0g^1\cdot{}^0f^1 + {}^0g^2\,{}^2f^0 + {}^2g^0\,{}^2g^0 - {}^0g^1\cdot{}^0g^1\,]\,\underline{\delta}$$

The evaluation of B_{20}, B_{10}, B_{21} for the Maxwellian molecules gives[12]

$$2B_{20} - B_{10} - B_{21} = -1.0562$$

V. Contribution of Momentum Flux by the Collided Molecules

(V - 1) Formulation of Sphere Drag in an Almost-Collisionless Flow

From the use of the collisional distribution $E(\underline{r}, \underline{c})$ we can evaluate the momentum flux to the sphere contributed by the first order collisional effect. This constitutes, of course, an additional term to the sphere drag due to the free molecules. Note that the rate of change of the molecular distribution in the neighborhood, $\underline{q}$, $\underline{q} + d\underline{q}$ (See Fig. 4) with velocities lying between $\underline{c}$, $\underline{c} + d\underline{c}$ is

$$0.075\,(d/\lambda)\;E(\underline{q}, \underline{c})\,d\underline{q}\,d\underline{c}$$

where $\underline{q}$ denotes the position of point Q; the origin of the vector $\underline{q}$ is at point P on the spherical surface (See Fig. 4). The direction of the velocity vector $\underline{c}$ at point Q is specified such that it extends from Q to intersect the surface of the sphere at P; hence

$$\underline{c} = -c\,\frac{\underline{PQ}}{|\underline{PQ}|} = -c\underline{q}_0 \tag{5.1}$$

Furthermore, the molecules with velocity $\underline{c}$ originating from Q and intersecting a surface element $d\sigma$ at P, lie in

$$d\underline{c} = c^2\,dc\,\cos\tau\;d\theta/q^2 \tag{5.2}$$

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[12] J.C. Maxwell, collected works; (Dover).

where τ denotes the angle between $\underline{q}$ and the normal $\underline{n}_o$ to the surface at P. From the use of Equations (5.1) and (5.2) we obtain the change of the incident molecular flux and momentum flux to a unit area at P,

$$N(P) = 0.075\,(d/\lambda) \int_R d\underline{q} \int_0^\infty dc\, c^2\, E(\underline{q}, -c\underline{q}_o) \cos\tau / q^2 \tag{5.3}$$

and

$$M_i(P) = -0.075m(d/\lambda) \int_R d\underline{q} \int_0^\infty dc\, c^3\, E(\underline{q}, -c\underline{q}_o) \cos\tau / q^2 \tag{5.4}$$

respectively. The domain R refers to the Semi-infinite region bounded by the plane tangent to the surface at P (See Fig. 4). The momentum flux taken from the body by the reflected molecules of the amount N(P) is, assuming diffuse reflection,

$$M_r(P) = \frac{1}{2}\pi^{\frac{1}{2}} m\, N(P)\, \underline{n}_o \tag{5.5}$$

The net change of momentum flux for an area $d\sigma$ around the point P is

$$[M_i(P) - M_r(P)]d\sigma \tag{5.6}$$

and the sphere drag contributed by the first order collisional effect, in dimensionless drag coefficient*

$$\Delta C_D = 0.6\,(d/\lambda)(\pi d^2 s)^{-1} \int_{\text{sphere}} d\sigma[M_i(P) - \frac{1}{2}\pi^{\frac{1}{2}} mN(P)\underline{n}_o] \tag{5.7}$$

(V - 2) The Computation

In order to facilitate the integration processes in the formulations of the sphere drag, we must introduce a coordinate transformation from the $(\underline{r}_o, \underline{t}_o, \underline{t}'_o)$-system, on which the moment integrals ${}^{\ell}g^{n}$ have been prescribed, to a new (q, τ, σ)-system in terms of which the sphere drag will be conveniently expressed. Referring to Fig. 4, $\underline{k}$ is in the same direction as $\underline{n}_o$. $\underline{i}$ and $\underline{k}$ are coplanar with $\underline{s}_o$ and $\underline{n}_o$ while $\underline{j}$ is normal to them. In terms of these unit

---

*drag coefficient is defined as the ratio of the drag force in question and the total dynamic pressure based on the sphere cross sectional area $(\pi d^2/4)$ and the free stream density and velocity.

vectors we express the following quantities:

$$\underline{q} = q \sin\tau \cos\sigma\, \underline{i} + q \sin\tau \sin\sigma\, \underline{j} + q\cos\tau\, \underline{k}$$

$$\underline{r} = q \sin\tau \cos\sigma\, \underline{i} + q \sin\tau \sin\sigma\, \underline{j} + (1 + q\cos\tau)\, \underline{k} \tag{5.8}$$

$$\underline{s}_o = -\sin\Phi\, \underline{i} - \cos\Phi\, \underline{k}$$

Integrals (5.3) and (5.4) become respectively:

$$N(\Phi) = \int_0^\infty dq \int_0^{\pi/2} d\tau \sin\tau \cos\tau \int_0^{2\pi} d\sigma \int_0^\infty dc\, c^2\, a_{ij}^{(2)}\, H_{ij}^{(2)} \tag{5.9}$$

$$M_i(\Phi) = \int_0^\infty dq \int_0^{\pi/2} d\tau \sin\tau \cos\tau \int_0^{2\pi} d\sigma \int_0^\infty dc\, c^2\, m\, (\underline{s}_o \cdot \underline{c})\, a_{ij}^{(2)}\, H_{ij}^{(2)} \tag{5.10}$$

To evaluate the integrals (5.9) and (5.10) we must begin with the contraction of the second order tensors $a^{(2)}$ and $H^{(2)}$. Since

$$H_{ij}^{(2)} = c_i c_j - \delta_{ij}$$

and $a_{ij}^{(2)}$ given in Section (IV - 3) contains terms: $\underline{r}_o^2$, $\underline{\delta}$, $\underline{s}_o\, \underline{r}_o$ etc., as a preliminary step, we need contractions of the following terms:

$$H^{(2)} \underline{r}_o^2 = (\underline{c}_o \cdot \underline{r}_o)^2 c^2 - 1$$

$$H^{(2)} \underline{\delta} = c^2 - 3$$

$$H^{(2)} \underline{s}_o \cdot \underline{r}_o = (\underline{c}_o \cdot \underline{s}_o)(\underline{c}_o \cdot \underline{r}_o) c^2 - \underline{s}_o \cdot \underline{r}$$

$$H^{(2)} \underline{r}_o \underline{t}_o \sin\Theta = (\underline{r}_o \cdot \underline{c}_o)[(\underline{s}_o \cdot \underline{c}_o) - (\underline{s}_o \cdot \underline{r}_o)(\underline{r}_o \cdot \underline{c}_o)] c^2 \tag{5.11}$$

$$H^{(2)}(\underline{t}_o^2 + \underline{t}'^2_o) = [1 - (\underline{c}_o \cdot \underline{r}_o)^2] c^2 - 2$$

$$H^{(2)} s_o^2 = c^2 (s_o \cdot c_o)^2 - 1$$

$$H^{(2)} s_o t_o \sin\Theta = (c_o \cdot s_o)[(s_o \cdot c_o) - (s_o \cdot r_o)(r_o \cdot c_o)]c^2 - 1 + (s_o \cdot r_o)^2$$

The following terms which appear in the above contracted results are now expressed with the new spherical system.

$$(s_o \cdot r_o) = -(q \sin\tau \cos\sigma \sin\Phi + \cos\Phi + q\cos\tau\cos\Phi)/r$$

$$(s_o \cdot c_o) = (\sin\tau\cos\sigma\sin\Phi + \cos\tau\cos\Phi) \tag{5.12}$$

$$(c_o \cdot r_o) = -(q + \cos\tau)/r$$

After the substitution of the contracted result of $H^{(2)}a^{(2)}$ in (5.9) and (5.10), the general term of which appear of the following type:

$$q^k \cos^\ell \tau \sin^m \tau \cos^\alpha \Phi \sin^\beta \Phi \cos^\gamma \sigma / r^p \tag{5.13}$$

Let

$$I^{k\ell m}_{p\alpha\beta\gamma} = \int_0^\infty dq \int_0^{\pi/2} d\tau \int_0^\pi d\Phi \int_0^{2\pi} d\sigma\, q^k \cos^\ell\tau \sin^m\tau \cos^\alpha\Phi \sin^\beta\Phi \cos^\gamma\sigma / r^p \tag{5.14}$$

and note that $I^{k\ell m}_{p\alpha\beta\gamma}$ vanishes after integration with respect to σ with odd γ; with even γ and next integration for Φ, $I^{k\ell m}_{p\alpha\beta\gamma}$ again vanishes with odd α.

In addition to the functions of (5.14), we need some more building blocks to facilitate the computation of sphere drag which are defined as follows:

Let $\omega = s_o \cdot r_o$; $y = s_o \cdot c_o$; $z = c_o \cdot r_o$, we define

$$A_{ij} = \int_0^\infty dq \int_0^{\pi/2} d\tau \int_0^\pi d\Phi \int_0^{2\pi} d\sigma\, y\omega^i r^{-j} \sin\Phi \sin\tau\cos\tau \qquad (i,\ \text{odd}) \tag{5.15}$$

$$A^*_{ij} = \int_0^\infty dq \int_0^{\pi/2} d\tau \int_0^\pi d\Phi \int_0^{2\pi} d\sigma\, \omega^i r^{-j} \sin\Phi\cos\Phi\sin\tau\cos\tau \quad (i,\ \text{odd}) \tag{5.16}$$

Similarly we may define B_{ij}, D_{ij} and G_{ij} by replacing y in (5.15) with zy^2 (with i, even), z^2y (with i, odd) and y^3 (with i, odd) respectively; also define B^*_{ij}, D^*_{ij}, and G^*_{ij} by replacing ω^j in (5.16) with $zy\omega^i$ (with i, even), $z^2\omega^i$ (with i, odd) respectively.

(V - 3) Procedure for Numerical Analysis

The double integrals

$$I_p^{k\ell m} = \int_0^\infty dq \int_0^{\pi/2} d\tau \, q^k \cos^\ell \tau \, \sin^m \tau / r^p \tag{5.17}$$

obtained from (5.14), after elementary integrations, are numerically integrated on the IBM 7090 Computer for integer indices in the ranges: $0 \le k \le 5$, $1 \le \ell \le 8$, $1 \le m \le 7$, $3 \le p \le 10$ with $k + 1 < p$.

The basic integrals A_{ij}, A^*_{ij}, B_{kj}, B^*_{kj}, D_{ij}, D^*_{ij}, G_{ij} and G^*_{ij} with integer indices in ranges $i = 1, 3$, $j = 2$ to 5 and $k = 0, 2, 4$ are computed. These basic integrals are the building blocks which enable us to calculate the collisional distribution and the sphere drag. The computed functions are tabulated in Table I.

To prepare for the final contraction of the Hermite polynomial $H^{(2)}$ and its coefficient, $a^{(2)}$, we first evaluate its component functions associated with tensors: $\underline{s}^2$, $\underline{r}^2$, $(\underline{s}\,\underline{r})$, $\underline{\delta}$, $\underline{t}^2 + \underline{t}'^2$, $(\underline{s}\;\underline{t})$ and $(\underline{r}\;\underline{t})$. The collection of the moment integrals ${}^{\ell}f^n$ and ${}^{\ell}g^n$ (for $\ell = 0, 2$ and $n = 0, 1, 2$) grouped in separate terms of scalar, vectorial and tensorial species for $a^{(2)}$ are readied to contract with $H^{(2)}$. The final integration of the moments of the collisional distribution $E(\underline{r}, \underline{c})$ over the molecular phase space are made to obtain the collisional momentum transfer to the sphere and its associated dimensionless drag coefficient ΔC_D, see eq.(5.7) which is a function of the speed ratio s and inversely proportional to the Knudsen number (λ/d). The computations for ΔC_D have been made for $s = 0.1, 0.2 \ldots 1.5$ and also for $s = 10^{-5}$.

(V - 4) Results of Computations

The results of computations of the aerodynamic drag for a sphere in an almost collisionless flow are shown in Fig. 5. The drag components contributed by the incident species (M_i) and the reflected species (N) respectively are also shown in Fig. 6. All the calculations are based upon the assumed condition that the sphere temperature is equal to the free stream temperature, and perfect diffuse reflection prevails. Although it is difficult to prove the convergence of the expansion, we are able to establish the rapid decrease of the absolute values of the succeeding terms in all the expansions when the speed ratio s is not much larger than 1.

It is significant to note that the theoretical result calculated for a very small value of s ($s = 10^{-5}$) agrees with Millikan's measured values* at corresponding s over a wide range of Knudsen numbers ($0.g < \frac{\lambda}{a} < 10$). It is also noted that over a wide range of speed ratios ($s < 1.0$) the drag coefficient ratio $C_D/C_{D f.m.}$ ($C_{D f.m.}$ denotes the drag coefficient of the sphere in which the free molecules flow at the same speed ratio) depends only on the Knudsen number (λ/d) in this first order approximation (See Fig. 7). Even for $1 < s < 1.5$ the computed results for $C_D/C_{D f.m.}$ show only slight dependence on s. We feel however that to vindicate this conclusion with respect to the range $s > 1$, higher order terms of s in the expansions must be included.

VI Discussions and Conclusions

The classification of rarefied flows, e.g., the continuum, transition, free molecular, has been traditionally based on values of the Knudsen number, a ratio of the mean free path (λ) and a characteristic dimension of the body (d). In view of the asymptotic solutions of the Boltzmann equation we can give a different viewpoint to the flow regimes.

Consider the flow field around a body. Within a distance much less than

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*In this comparison the same expression for the mean free path is used in calculating Knudsen numbers.

a mean free path from the body the molecular distribution would exhibit the features of quasi-free molecular flow since the collisions between the streams incident on and reflected from the surface is dominated by the collisions between the incident stream and the surface. On the other hand, at distances of many free paths away from the body the molecular distribution is almost locally Maxwellian, provided that sufficiently large volumes are used for sampling; hence, it can be treated as quasi-continuum. At the intermediate distance from the body, the molecular distribution will deviate from both the asymptotic solutions mentioned above. The significance of each of the three flow regimes pertaining to a given flow must be determined by the Knudsen number in question.

Although there is little doubt that the transition flow structure in a monatomic gas is contained in the Boltzmann equation, we cannot treat the transition flow as an entity because the contemporary solutions to this flow regime are valid only either near continuum or near-free molecular flow. In the former class there is Goldberg's solution to the thirteen moment equations which are linearized for the problem of a slowly moving sphere. Contrary to Grad's remark[13], Goldberg's sphere drag solution cannot be expected to cover the whole range of mean-free-paths because it would not be a meaningful approximation to the free or near-free molecular flows (see Fig. 5).

In the latter class there is a valuable solution by Szymanski[4] which is a first order Knudsen iteration of the Boltzmann equation. Like Goldberg, Szymanski also limits his discussion to the slowly moving sphere such that linearization of the disturbance effect can be used. Unfortunately, we found two gross errors* in Szymanski's development and hence have serious reservation about the validity of his results. In any event, his theory is developed for the flows of extremely low speeds only.

At the other extreme of speed ratios ($s >> 1$) there are theories of sphere drag based on either Boltzmann equation with simplified collision integral[2] or

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[13] H. Grad: Principles of the Kinetic Theory of Gases, Handbuch der Phys., vol. 12, p. 292, Springer, 1958.

*One in this equation (7.10); another, equation (7.14) in Reference 4.

collision statistics[3]. In both cases many gross approximations have been introduced. Since the answer obtained often is a small difference of two large quantities, serious doubt is cast upon the validity of the results. On the experimental side there is hardly any data available for such high Knudsen numbers. Besides, much of the sphere drag measured at intermediate Knudsen numbers is made with unknown surface temperature on which the sphere drag strongly depends.

The present theory based on the exact* Boltzmann collision integral for Maxwellian molecules is formulated for high Knudsen number ($Kn \geq 1$) and a speed range $0 < s < 1$; the upper limit for s is not of the cut-off nature. In fact, it can be extended to higher values of s with more terms in the expansions. The effect on the sphere drag due to the use of an artificial molecular model, such as the Maxwellian molecules is difficult to estimate. It would be, of course, very desirable to treat the problem with a more realistic model such as the elastic spheres; the mathematics thus involved will be quite formidable.

The lack of sphere drag measurements at corresponding Knudsen number and surface temperature makes it impossible to ascertain the accuracy of the present theory; nevertheless its close agreement (within 5 per cent) with Millikan's experimental results at the low speeds in a wide range of Knudsen numbers does appear encouraging. A composite plot of sphere drag against Knudsen numbers including the continuum range is given in Fig. 5 to lend some support to the present theory.

Much of the contemporary studies of flows, at moderately high Knudsen numbers, with exact Boltzmann equation approach are limited to simple internal flows such as Coutte flows, etc. Should the present results be considered favorable, it could suggest that the method of the present approach might have opened a new effective avenue to the mysterious regime of transition flows.

*The only approximation introduced is at representation of the molecular flux of the reflected molecules.

Table I

$A_{12} = -2.09439$	$A_{13} = -1.04719$	$A_{14} = -0.69813$	$A_{15} = -0.52358$
$A_{32} = -1.25658$	$A_{34} = -0.41880$	$A_{35} = -0.31408$	
$B_{02} = -2.09439$	$B_{04} = -0.69813$	$B_{05} = -0.52358$	$B_{22} = -1.11696$
$B_{24} = -0.33504$	$B_{42} = -0.77781$	$B_{44} = -0.22733$	$B_{45} = -0.16451$
$D_{12} = -1.74524$	$D_{13} = -0.78530$	$D_{14} = -0.48859$	$D_{15} = -0.34896$
$D_{32} = -1.04704$	$D_{34} = -0.29309$	$D_{35} = -0.20931$	
$G_{12} = -1.25662$	$G_{13} = -0.62832$	$G_{14} = -0.41885$	$G_{15} = -0.31415$
$G_{32} = -0.83773$	$G_{34} = -0.26324$		

$A^*_{12} = -2.09436$	$A^*_{13} = -1.19542$	$A^*_{14} = -0.85682$	$A^*_{15} = -0.67410$
$A^*_{32} = -1.25655$	$A^*_{34} = -0.51400$	$A^*_{35} = -0.40436$	
$B^*_{02} = -1.39626$	$B^*_{04} = -0.46541$	$B^*_{05} = -0.34905$	$B^*_{22} = -0.93080$
$B^*_{24} = -0.33347$	$B^*_{42} = -0.67809$	$B^*_{44} = -0.22621$	$B^*_{45} = -0.16737$
$D^*_{12} = -1.62889$	$D^*_{13} = -0.82237$	$D^*_{14} = -0.54361$	$D^*_{15} = -0.40314$
$D^*_{32} = -0.97723$	$D^*_{34} = -0.33440$	$D^*_{35} = -0.24179$	
$G^*_{12} = -0.97738$	$G^*_{13} = -0.51834$	$G^*_{14} = -0.35753$	$G^*_{15} = -0.27446$
$G^*_{32} = -0.69811$	$G^*_{34} = -0.24640$		

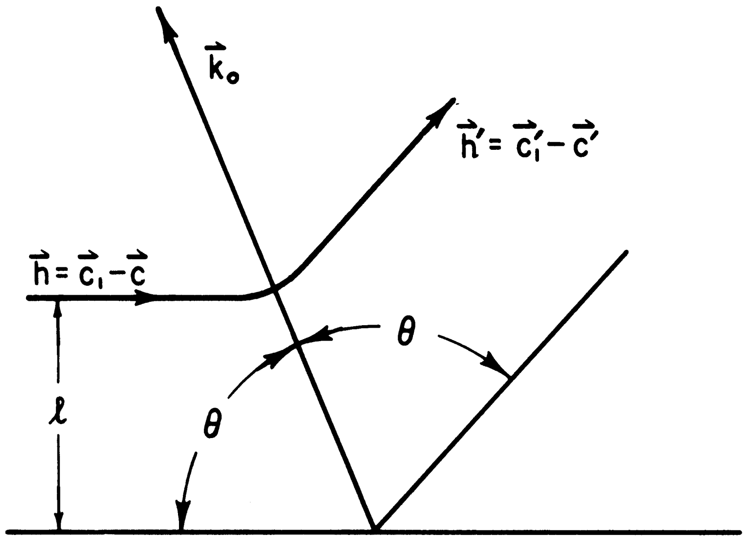

Fig. 1 Kinematics of Collision.

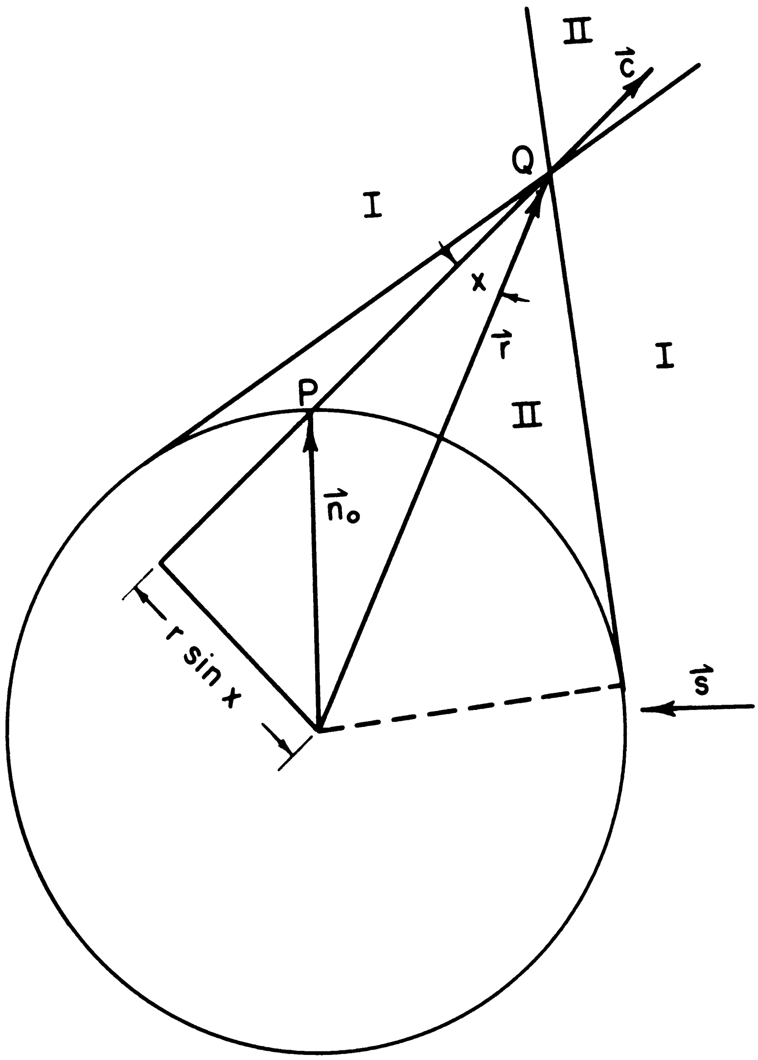

Fig. 2 Sphere-cone Geometry.

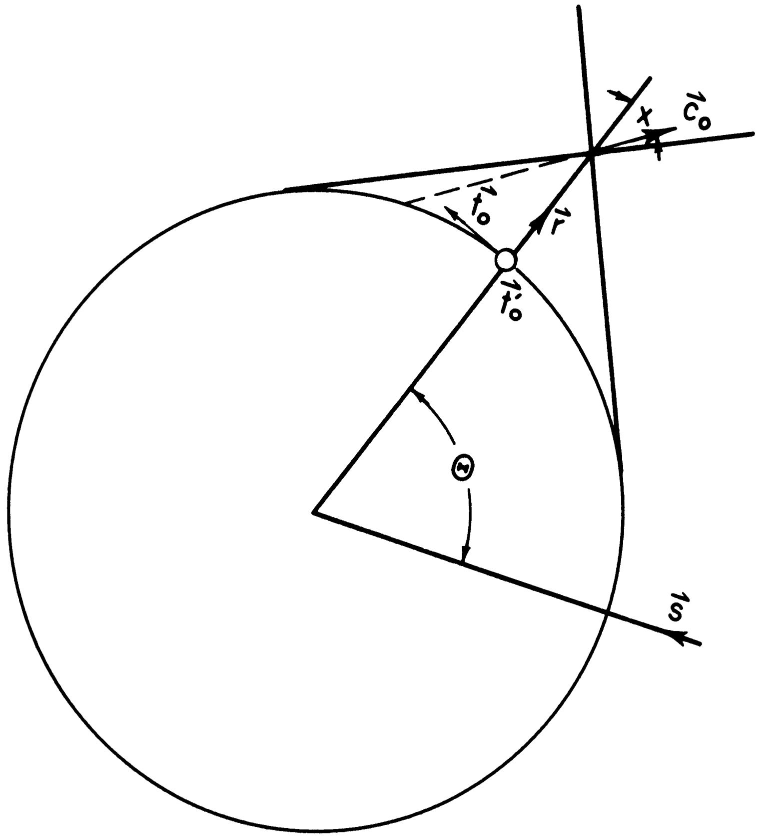

Fig. 3 Spherical Tensor Geometry.

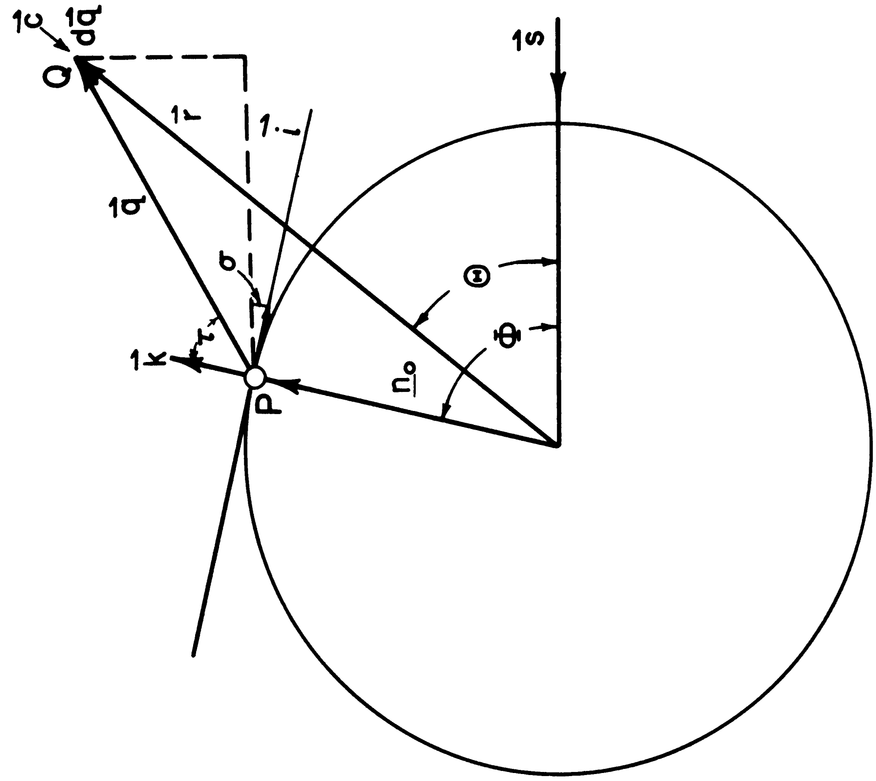

Fig. 4 Field Coordinates.

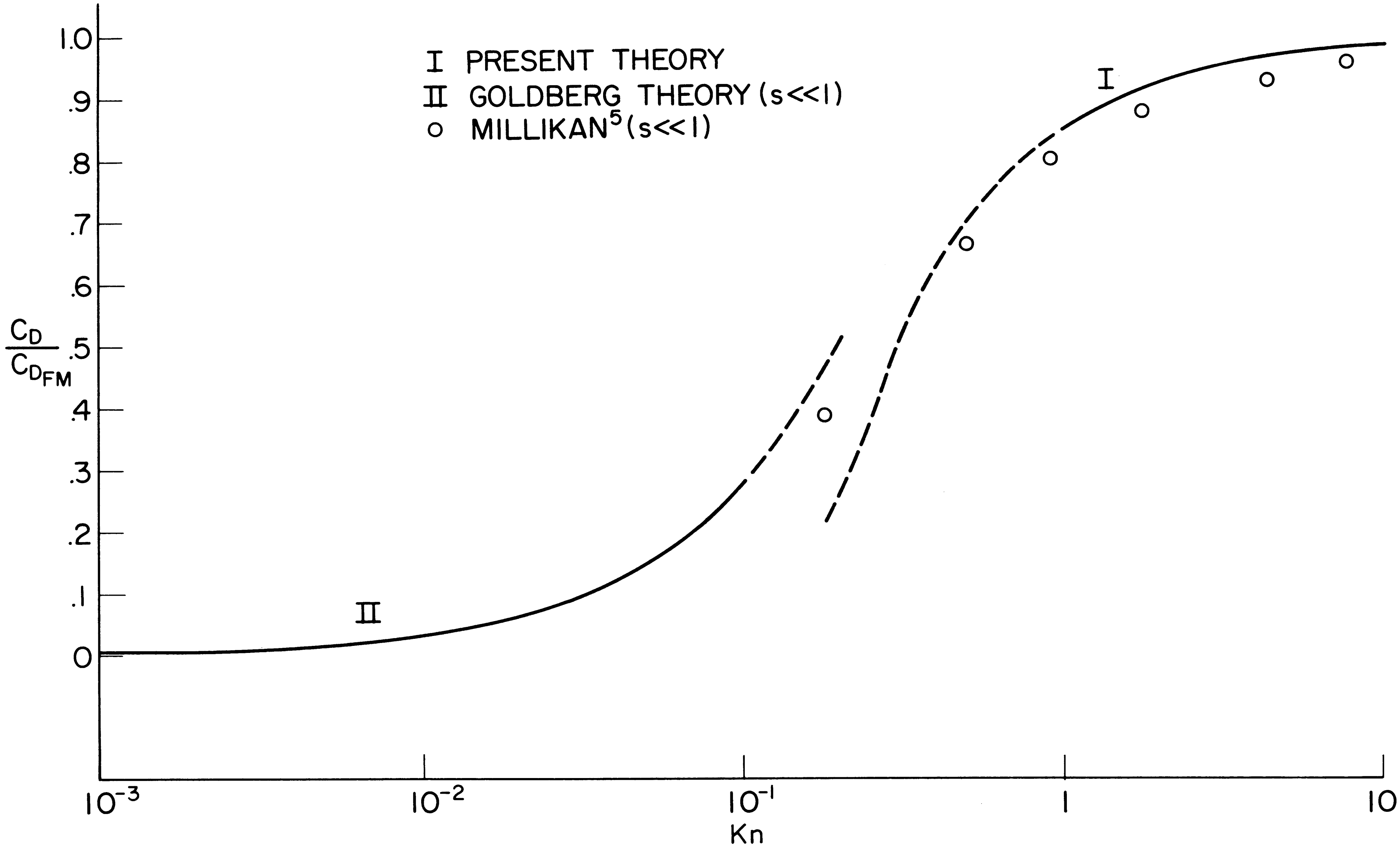
I PRESENT THEORY
II GOLDBERG THEORY (s<<1)
o MILLIKAN5 (s<<1)
$\frac{C_D}{C_{D_{FM}}}$
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
I
II
10^{-3}
10^{-2}
10^{-1}
1
10
Kn

Fig. 5

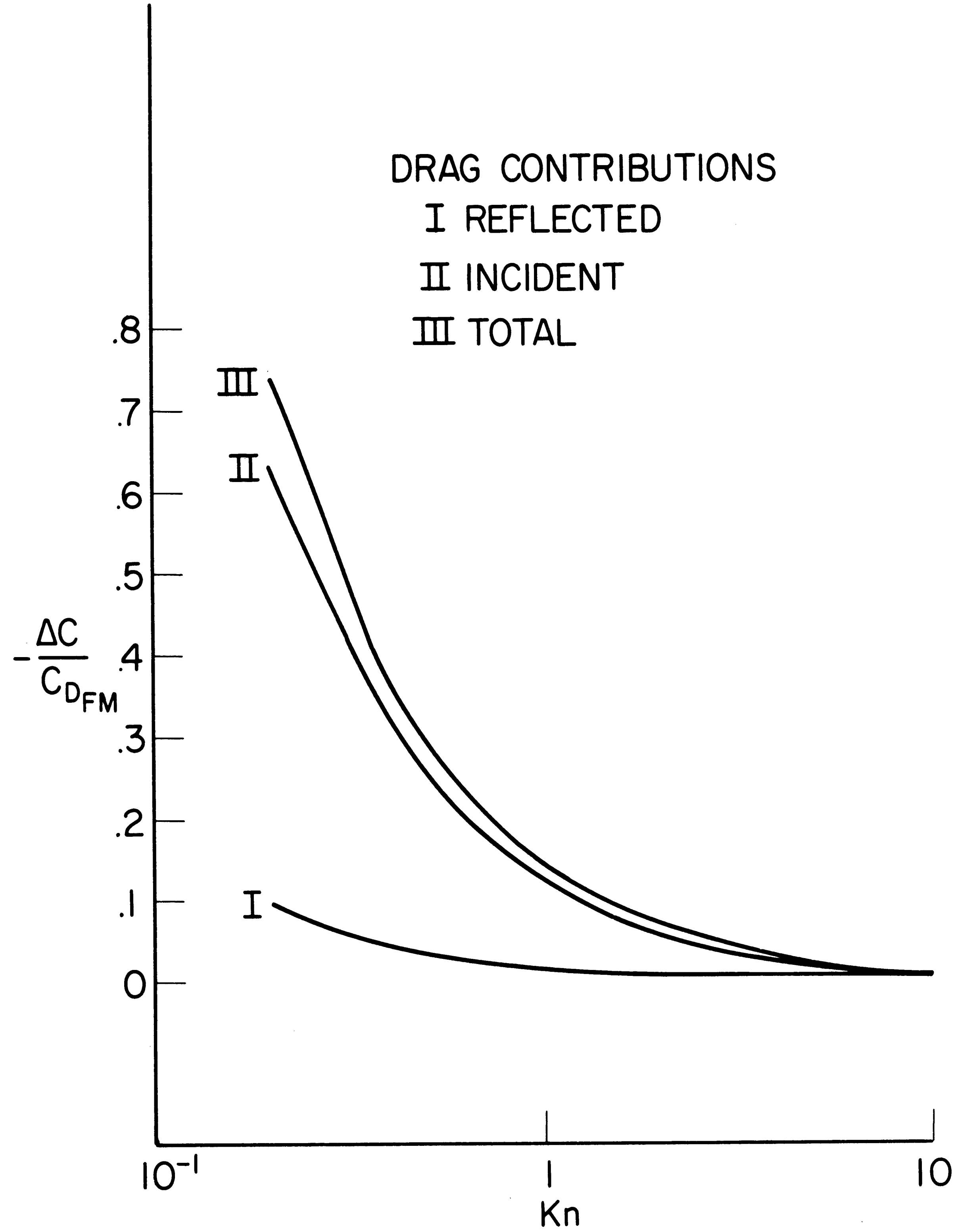
DRAG CONTRIBUTIONS
I REFLECTED
II INCIDENT
III TOTAL
.8
.7
.6
.5
.4
.3
.2
.1
0
$-\frac{\Delta C}{C_{D_{FM}}}$
III
II
I
10^{-1}
1
10
Kn

Fig. 6

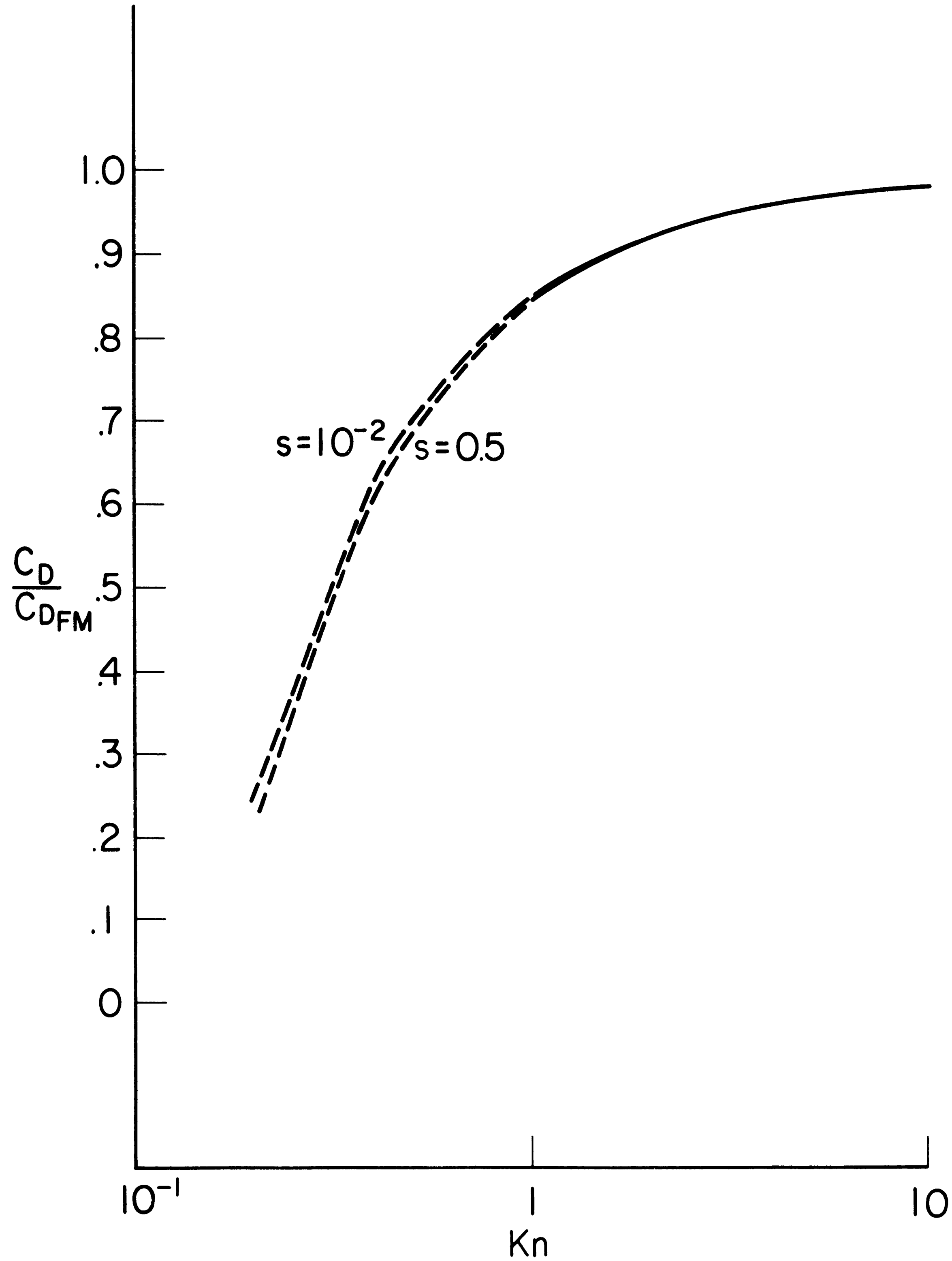

Fig. 7

Appendix 1 Determination of $\alpha(s)$ and $\beta(s)$

The equivalent number density of the reflected molecules at point P on the surface (See Eq. 3.2)

$$n(P) = \exp -(sx)^2 - \sqrt{\pi}\ sx + \sqrt{\pi}\ sx\ \mathrm{erf}(sx) \equiv n_1 \tag{1}$$

where $x = \underline{n}_o \cdot \underline{s}_o$ is approximated by the linear function:

$$n(P) = \alpha s x + \beta \equiv n_2 \tag{2}$$

such that the total number of molecules reflected from the semi-spherical sphere on the upstream side based on (1) is equal to that based on (2); similarly for the reflected molecules for the downstream semi-spherical surface, i.e.,

$$\int_0^{-1} n_1\, dx = \int_0^{-1} n_2\, dx$$

$$\int_0^{1} n_1\, dx = \int_0^{1} n_2\, dx$$

Performing the integrations and solving for α and β we obtain

$$\alpha = -\pi^{\frac{1}{2}}$$

$$\beta = \pi^{\frac{1}{2}}[(1 + 2s^2)\ \mathrm{erf}\, s + 2s\,\pi^{-\frac{1}{2}} \exp - s^2]/4s$$

Appendix 2. Derivation of formula for $\int_0^{2\pi} d\epsilon \; \underline{k}_o^j$

This tensor integral of rank j appears in the present theory and is worth consideration. Figure 1 of the text gives a picture of the relationship between vectors $\underline{k}_o$ and $\underline{h}_o$. From Figure 1 and by vector addition theory (let $\underline{h}_o \cdot \underline{k}_o = -\cos\psi$)

$$\underline{k}_o = -\underline{h}_o \cos\psi + \underline{p}_o \sin\psi \tag{1}$$

and

$$\underline{k}_o^j = (-\underline{h}_o \cos\psi + \underline{p}_o \sin\psi)^j$$

$$= \sum_{m=0}^{j} (-)^{j-m} \, \underline{h}_o^{j-m} \cos^{j-m}\psi \, \sin^m\psi \, \underline{p}_o^m \tag{2}$$

Now with $\underline{h}_o$ and $\underline{p}_o$ orthogonal and $\underline{p}_o = (0, \cos\epsilon, \sin\epsilon)$

we have

$$\int_0^{2\pi} \underline{p}_o \, d\epsilon = 0 \tag{3}$$

and

$$\int_0^{2\pi} \underline{p}_o^2 \, d\epsilon = \pi(\underline{\delta} - \underline{h}_o^2) \tag{4}$$

Generalizing these integrals, we find

$$\int_0^{2\pi} \underline{p}_o^i \, d\epsilon = \begin{cases} 0 & , \text{ i odd} \\ \pi \dfrac{(\underline{\delta} - \underline{h}_o^2)^{i/2}}{2^{\frac{i}{2}-1} \left(\frac{i}{2}\right)!} & , \text{ i even} \end{cases} \tag{5}$$

From (2) and (5), it follows that

$$\int_0^{2\pi} d\epsilon \; \underline{k}_o^j = \pi \sum_{m=0}^{N(\frac{j}{2})} (-)^j \underline{h}_o^{j-2m} \cos^{j-2m}\psi \, \sin^{2m}\psi \; \frac{(\underline{\delta} - \underline{h}_o^2)^m}{2^{m-1} m!} \quad .$$