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TURBULENT DISPERSION OF DYNAMIC PARTICLES

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ABSTRACT

This paper discusses turbulent dispersion of particles whose size and inertia are such that it cannot be assumed that they will follow exactly the fluctuations of the fluid elements with which they are associated. The turbulent field is assumed to be stationary, homogeneous, and isotropic. Stationary, extraneous force fields for the particles may exist. The method of generalized harmonic analysis (Wiener) is used to determine the statistical particle dispersion parameters in terms of the spectrum density of the turbulent field and the physical characteristics of the particles. The discussion is restricted to one-dimensional problems; it includes consideration of the relation between turbulent dispersion and Brownian motion of the particles due to molecular impacts.

An illustrative example of dispersion calculation with a given turbulence spectrum is presented, and aspects of practical applications are discussed briefly.

OBJECT

The object of the research is to study both theoretically and experimentally (1) the dispersion of airborne particulates and (2) the penetration of these particulates into structures as a function of atmospheric turbulence and wind velocity and direction.

1. INTRODUCTION

The problem of the dispersion of material particles in a turbulent fluid concerns meteorologists, chemists, engineers, and others in many ways. The fact that particle dispersion constitutes a direct and striking manifestation of the mechanism of fluid turbulence makes it an interesting aerodynamic problem.

Turbulent dispersion of particles is a random process. The velocity of a particle can be considered as the response of a dynamic system which has turbulence impacts as its input. Since turbulent velocity is random in nature, so must the particle velocity and displacement also be random. It is the statistically defined properties of the particle motion which are of primary interest.

In contrast to Taylor's treatment of turbulent diffusion (1921), which identifies the transportable property with certain fluid elements, the present study considers the particles as dynamic bodies which may have amplitudes of fluctuations different from those of the fluid elements with which they are associated. A distinction is thus being made between turbulent diffusion and turbulent dispersion. Turbulent diffusion depends on the character of turbulence only while turbulent dispersion of particles depends on both the character of turbulence and the physical properties of the particles. When the particle size decreases below a certain threshold value, depending on the frequency spectrum of the fluctuations, kinematic viscosity, etc., the particles will drift to and fro in full accord with their associated fluid elements. Then the particle dispersion becomes identified with the fluid diffusion.

It should be noted that the results hereby obtained are valid only when the particles are so sparsely distributed that the dynamic effect of the embedded particles on the fluid can be ignored. For instance, turbulent dispersion of atmospheric aerosols, such as smoke, fog, etc., would satisfy this requirement.

In reviewing the literature, we have found only a treatise by Tchen (1947) in which the problem of particle dispersion in a turbulent fluid is treated. Tchen's method of approach is different from ours, although some of the results are complementary.

2. SIGNIFICANT FORCES ACTING ON A PARTICLE

An important step in the study of the motion of material particles which are suspended in a turbulent fluid (either liquid or gas) is the recognition of all the significant forces experienced by the rapidly fluctuating particles. Some of these forces that are relatively insignificant in normal experience may become important when they are scaled to the mass and size of a minute particle which is of the order of 5×10^{-11} g for a one-micron (10^{-4} cm) particle with specific gravity of unity.

In order to analyze the problem, we assume that the suspension of material particles is so dilute that every particle, spherical in shape, behaves as if it were alone in a fluid medium of infinite extent. Furthermore, we restrict this study to the one-dimensional case. This should not be a serious drawback for the purpose of exploring the fundamental mechanism of particle dispersion in an isotropic turbulent fluid.

The forces acting on a particle in a turbulent fluid can be classified as follows:

Extraneous Forces $\sum F$.—These include forces due to gravity, buoyancy, and the thermal field and electrostatic field, if such fields exist. We are not concerned here with the detail analysis of these forces which have been discussed elsewhere (Green, 1953).

Fluid Resistance.—Stoke's formula is adopted for the fluid resistance to a spherical particle

$$D_1 = 6\pi a\mu(V - U) \tag{2.1}$$

where a is the radius of the particle, and μ is the coefficient of fluid viscosity; V and U are velocities for particle and fluid, respectively. It should be noted that Stoke's formula holds provided $(V - U)a \ll \nu$, where $\nu = \mu\rho_0^{-1}$ is the kinematic viscosity of the fluid, ρ_0 being the density of the fluid. This constitutes one of the governing factors concerning the particle size range for which the present analysis is valid.

There is an additional fluid resistance which is attributed to the relative particle acceleration $[\dot{V}(t) - \dot{U}(t)]$ (Bassat, 1910; Tchen, 1947).

$$D_2 = 6\pi a\mu \left[\frac{a}{(\pi\nu)^{1/2}} \int_{-\infty}^{t} \frac{\dot{V}(\xi) - \dot{U}(\xi)}{(t - \xi)^{1/2}} d\xi \right] . \tag{2.2}$$

For a sinusoidal motion, i.e., $(V - U) \sim \sin \omega t$, where ω is the angular velocity of fluctuation. $D_2D_1^{-1}$ is of the order $a\omega^{1/2}\nu^{-1/2}$, which is in general negligibly small unless $\omega\nu^{-1}$ happens to be very large, such as $\omega\nu^{-1} \approx a^{-2}$.

Fluid Inertia.—The pressure gradient, which is induced by the acceleration of fluid, causes an apparent mass addition, $(2/3)\pi a^3\rho_0$, to the particle in addition to a pressure force of magnitude $2\pi a^3\rho_0\dot{U}$ (Tchen, 1947).

The random molecular impacts which produce Brownian motion of the particles are not considered until Section 8.

3. EQUATION OF PARTICLE MOTION

In view of the discussions in Section 2, we can write the equation of motion of a dynamic particle with density ρ_1 which is suspended in a turbulent fluid in the form

$$\frac{4}{3}\pi a^3(\rho_1 + \frac{\rho_0}{2})\dot{V} + 6\pi a\mu \left[V - U + \frac{a}{(\pi\nu)^{1/2}} \int_{-\infty}^{t} \frac{\dot{V}(\xi) - \dot{U}(\xi)}{(t - \xi)^{1/2}} d\xi\right] - 2\pi a^3\rho_0\dot{U} - \sum F = 0. \quad (3.1)$$

To simplify (3.1), we introduce the transformation

$$\begin{aligned} U &= \bar{U} + u \\ V &= \bar{V} + v \end{aligned} \quad (3.2)$$

where $\bar{U}$ and $\bar{V}$ represent the mean values of U and V, respectively; u and v are the fluctuating values such that

$$\begin{aligned} \bar{u} &= 0 \\ \bar{v} &= 0 . \end{aligned} \quad (3.3)$$

Furthermore,

$$\bar{V} = \bar{U} + V_d \quad (3.4)$$

where $V_d = (6\,\pi a\mu)^{-1} \sum F$ is equal to the particle drifting velocity due to the stationary extraneous forces $\sum F$. This is obvious in view of condition (3.3).

Equation (3.1) thus becomes

$$(2\rho_1 + \rho_0)\,\dot{v} + \frac{9\nu\rho_0}{a^2}\left[v - u + \frac{a}{(\pi\nu)^{1/2}} \int_{-\infty}^{t} \frac{\dot{v}(\xi) - \dot{u}(\xi)}{(t - \xi)^{1/2}} d\xi\right] - 3\rho_0\dot{u} = 0. \quad (3.5)$$

The equation of particle motion (3.5), which involves random function u(t) which in turn is prescribed only by its statistically defined properties, is called a stochastic differential equation. The ultimate aim of solving a stochastic differential equation is to seek a probability distribution such as $W(v, t, v_0)$ which governs the probability of occurrence of the random velocity v at time t given that $v = v_0$ at $t = 0$.

Because of the measurements made by Simmons and Salter (1938) and Townsend (1947), u(t) of homogeneous and isotropic turbulence is known to have a Gaussian distribution within experimental error. It can be shown that v(t), which is linearly related to u(t), will also have a distribution of Gaussian type.

To prescribe a probability distribution of a Gaussian process, we need to know only the mean value and the variance of the random variable in question (Cramer, 1946). The former vanishes in view of (3.3). The variance, $\overline{v^2(t)}$, alone is required to prescribe the probability distribution of v(t).

It is further assumed that u(t) is stationary or statistically homogeneous with respect to time. This is true only when the decay of turbulence is negligible.

Various statistical properties of the randomly fluctuating quantities to be discussed in the following are averages taken over the fluctuations of an ensemble of macroscopically identical systems (ensemble average). In physical measurements, the time average of the quantity taken over the fluctuations of a single system usually is the only feasible experimental average. It has been shown by Birkhoff (Kampé de Fériet, 1950) that, for a stationary process, the time average of a fluctuating quantity, if taken over a sufficiently long time, will agree with the ensemble average.

4. MATHEMATICAL REPRESENTATION OF THE TURBULENT VELOCITY

Consider a truncated stationary function u(t) such that

$$u_T(t) = 0 \qquad \text{when } |t| > T$$

$$u_T(t) = u_t(t) \qquad \text{when } |t| \le T$$

where T is long compared to the scale of turbulence (Lagrangian). The quantity $u_T(t)$, being stationary and continuous in mean squares, possesses a Fourier expansion

$$u_T(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\omega t} A(\omega) d\omega \tag{4.1}$$

where

$$A(\omega) = \frac{1}{(2\pi)^{1/2}} \int_{-T}^{T} e^{-i\omega t} u_T(t) dt \; . \tag{4.2}$$

To describe the sequential behavior of a random process, the spectrum density and autocorrelation function are generally used.

In view of the Parseval theorem (Wiener, 1933),

$$\int_{-\infty}^{\infty} u_T^2(t) dt = \int_{-T}^{T} u_T^2(t) dt = \int_{-\infty}^{\infty} |A(\omega)|^2 d\omega \; . \tag{4.3}$$

Since $|A(\omega)|^2$ is an even function of ω,

$$\overline{u^2(t)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-\infty}^{+\infty} |A(\omega)|^2 d\omega = \int_0^{\infty} \frac{|A(\omega)|^2}{T} d\omega \; . \tag{4.4}$$

For a stationary random function u(t), $\overline{u^2(t)}$ has a constant value. The quantity $|A(\omega)|^2 T^{-1}$, therefore, tends to a definite limit and is defined as the spectrum density $p_u(\omega)$ of the random function u(t).

From the physical point of view, Fourier analysis of the turbulent velocity field amounts to resolving the motion into components of different linear sizes which make additive contributions to the total energy,

$$\overline{u^2(t)} = \int_0^\infty p_u(\omega)d\omega \; . \tag{4.5}$$

Consider next the autocorrelation function $\phi_u(\tau)$;

$$\phi_u(\tau) = \overline{u(t)u(t+\tau)} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} u(t)u(t+\tau)dt \tag{4.6}$$

which has the following properties:

$$\phi_u(0) = \overline{u^2(t)} \; , \tag{4.7}$$

$$\lim_{\tau\to\infty} \phi_u(\tau) = 0, \quad \text{and} \tag{4.8}$$

$$\phi_u(\tau) = \phi_u(-\tau) \; . \tag{4.9}$$

The following relations between $\phi_u(\tau)$ and $p_u(\omega)$ are known as the Wiener-Khintchine theorem (Wiener, 1933):

$$\phi_u(\tau) = \int_0^\infty p_u(\omega)\cos \omega\tau \; d\omega \tag{4.10}$$

and

$$p_u(\omega) = \frac{2}{\pi} \int_0^\infty \phi_u(\tau) \cos \omega\tau \; d\tau \tag{4.11}$$

from which it can be shown that

$$\frac{\pi}{2} p_u(\omega)_{\omega=0} = \int_0^\infty \phi_u(\tau)d\tau \; . \tag{4.12}$$

Successive differentiation of $\phi_u(\tau)$, both (4.9) and (4.10), gives

$$\left(\frac{\partial^2 \phi_u}{\partial t^2}\right)_{\tau=0} = - \int_0^{\infty} \omega^2 p_u(\omega) \, d\omega \qquad (4.13)$$

which relates the derivative of the autocorrelation function at $\tau = o$ to the second moment of the spectrum density.

The above discussion represents a brief resume of some pertinent results of the theory of random functions which will be needed in the following discussion.

5. SOLUTION OF THE PARTICLE EQUATION

The fact that u(t) can be adequately represented by a Fourier integral and that Equation (3.5) prescribes a stable linear system suggest the use of the generalized harmonic analysis approach (Wiener, 1933; Lin, 1943).

It is obvious that the response v(t), will have fluctuations similar to those of the input function u(t). We resolve u(t) and v(t) into Fourier components

$$u(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\omega t} A(\omega) d\omega \qquad (5.1)$$

and

$$v(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\omega t} B(\omega) \, d\omega \quad . \qquad (5.2)$$

Substitute (5.1) and (5.2) into Equation (3.5), leading to

$$\psi(\omega) = \frac{B(\omega)}{A(\omega)} = 1 - \frac{1}{1 - \frac{9}{4}\frac{i\rho_o}{\rho_1 - \rho_o}\left[\frac{2i}{3} + \frac{2\nu}{\omega a^2} + 2\left(\frac{i\nu}{\omega a^2}\right)^{1/2}\right]} \qquad (5.3)$$

The function $\psi(\omega)$ denotes the ratio of the sinusoidal component velocity of the spherical particle to the corresponding component velocity of the fluid element at the position of its center when the spherical particle is absent.

So far the analysis has been carried for an arbitrary fluid medium, liquid or gas. The function $\psi(\omega)$ may be simplified for some special cases.

Nonviscous Fluid.—When the viscous effect of the fluid is negligible compared to its inertia effect, $\psi(\omega)$ becomes

$$\psi = \frac{3\rho_0}{2\rho_1+\rho_0} . \tag{5.4}$$

The function ψ is, therefore, less or greater than unity according as ρ_1 is greater or less than ρ_0.

When $a\omega^{1/2}\nu^{-1/2} \ll 1$.—In most problems concerning turbulent dispersion of material particles, it becomes valid to assume $a\omega^{1/2}\nu^{-1/2} \ll 1$. Hence,

$$\psi(\omega) = 1 - \frac{1}{\left[1 - \frac{9}{2}\frac{i\rho_0\nu}{(\rho_1 - \rho_0)\omega a^2}\right]} . \tag{5.5}$$

If $\omega a^2\nu^{-1}$, though itself small, is large compared with $\rho_0(\rho_1 - \rho_0)^{-1}$, $\psi(\omega)$ may become very small. This means that the spherical particles remain nearly at rest as the turbulent flow beats upon them.

When the radius of a particle decreases, the inertia diminishes as a^3, whereas the surface on which viscosity acts diminishes as a^2 only. It is therefore to be expected that a stage will be reached when $\psi(\omega)$ approximates to unity, in which case the particle drifts to and fro with its corresponding fluid element in an identical manner. In other words, the particle dispersion can be completely identified by the turbulent fluid diffusion. This condition is satisfied provided

$$a^2 \ll \frac{9}{2} \frac{\rho_0}{\rho_1 - \rho_0} \frac{\nu}{\omega} \quad . \tag{5.6}$$

For a water droplet in air fluctuating with a maximum frequency of sinusoidal component air motion of one cycle per second, the limiting radius of the droplet should be much less than .01 cm.

Now, from (5.2) and (5.3), we have

$$v(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{i\omega t} \psi(\omega) A(\omega) d\omega . \tag{5.7}$$

In view of the Parseval theorem [see also (4.4)],

$$\begin{aligned} \overline{v^2} &= \int_0^{\infty} \frac{|\psi(\omega)|^2 |A(\omega)|^2}{T} d\omega \\ &= \int_0^{\infty} |\psi(\omega)|^2 p_u(\omega) d\omega . \end{aligned} \tag{5.8}$$

Hence,

$$p_v(\omega) = |\psi(\omega)|^2 p_u(\omega) \quad . \tag{5.9}$$

Since $\psi(\omega)$ generally vanishes for $\omega \to \infty$, $p_v(\omega)$ the spectrum density of $v(t)$, will go to zero for $\omega \to \infty$ faster than $p_u(\omega)$ will. This has the effect of "smoothing" the particle velocity function $v(t)$ as compared to $u(t)$.

By the Wiener-Khintchine theorem [see (4.10) and (4.11)],

$$\phi_v(\tau) = \overline{v(t)\, v(t + \tau)} = \int_0^{\infty} |\psi(\omega)|^2 p_u(\omega) \cos \omega\tau \, d\omega \tag{5.10}$$

and

$$p_v(\omega) = \frac{2}{\pi} \int_0^{\infty} \phi_v(\tau) \cos \omega\tau \, d\tau \quad . \tag{5.11}$$

For $\omega = 0$,

$$\frac{\pi}{2}\, p_v(\omega)_{\omega=0} = \int_0^\infty \phi_v(\tau)\, d\tau \quad . \tag{5.12}$$

It can be shown that

$$p_v(\omega)_{\omega=0} = p_u(\omega)_{\omega=0}$$

in view of the result: $\psi(\omega)_{\omega=0} = 1$ [see (5.3)]. Hence,

$$\int_0^\infty \phi_u(\tau) d\tau = \int_0^\infty \phi_v(\tau) d\tau \ . \tag{5.13}$$

It is well known (Dryden, 1939; Batchelor, 1949) that when the diffusion time t is very large, the turbulent diffusion of a given initial distribution of marked fluid in a homogeneous isotropic turbulent field can be closely represented by a Fickian-type diffusion equation with the diffusion coefficient proportional to the integral on the left-hand side of (5.13). The conclusion as represented by (5.13) therefore justifies the use of the same diffusion equation to prescribe the dispersion of the particles as well as the marked fluid, provided that t is very large (see Section 6).

It should be noted that a result similar to (5.13) has been previously obtained by Tchen (1947) by a different method.

6. PARTICLE DISPERSION

The variance of the displacement y(t) has been commonly used (Taylor, 1921) as a statistical property of turbulent diffusion. The corresponding parameter for particle dispersion can be similarly derived (Kampé de Fériet, 1939; Frenkiel, 1953):

$$\overline{y^2(t)} = 2 \int_0^t (t - \tau)\, \overline{v(t)v(t + \tau)}\, d\tau \ . \tag{6.1}$$

Substituting (5.10) in (6.1), we find

$$\overline{y^2(t)} = 2\int_0^t (t - \tau)\int_0^\infty |\psi(\omega)|^2 p_u(\omega) \cos \omega\tau \, d\omega \qquad (6.2)$$

Interchanging the limits of (6.2) and integrating it by parts, we get

$$\overline{y^2(t)} = \int_0^\infty |\psi(\omega)|^2 p_u(\omega) \frac{2(1 - \cos \omega t)}{\omega} d\omega \ . \qquad (6.3)$$

In comparison with the result for turbulent diffusion (Kampé de Fériet, 1939) which applies only when condition (5.6) is satisfied, the dispersion parameter, as expressed by (6.3), for larger particles has a weighing factor $|\psi(\omega)|^2$ in the integrand. As $\psi(\omega)$ generally decreases monotonously and vanishes when $\omega \to \infty$. The properties of the dispersion associated with eddies of different sizes will be presented in the qualitative sense. For instance, eddies of low frequencies contribute most to particle dispersion when t is large.

The properties of $\overline{y^2(t)}$ at

$$\text{(i)} \quad t \gg \int_0^\infty \phi_v(\tau) d\tau$$

and

$$\text{(ii)} \quad t \ll \int_0^\infty \phi_v(\tau) d\tau$$

can be investigated by rearranging Equation (6.1).

For case (i) write

$$\overline{y^2(t)} = 2t\int_0^\infty \phi_v(\tau) d\tau - 2\int_0^\infty \tau\phi_v(\tau) d\tau \ . \qquad (6.4)$$

The second term on the right-hand side of (6.4) represents the first moment of the correlation function and is a constant. Its significance, as compared to the first term, diminishes as t increases. When t is very large

$$\overline{y^2(t)} \approx 2t \int_0^\infty \phi_v(\tau)d\tau = 2t \int_0^\infty \phi_u(\tau)d\tau \tag{6.5}$$

in view of (5.13). The quantity $(2t)^{-1}\overline{y^2}$ can be shown (Kenard, 1938) to be equal to the dispersion coefficient of a Fickian-type equation when t is very large.

For case (ii), expand $\phi_v(\tau)$ into a power series in τ, and note that $\phi_v(\tau)_{t=0} = \overline{v^2}$. Therefore, we obtain from (6.1), after substituting (4.13) and (5.8),

$$\overline{y^2(t)} \approx t^2 \int_0^\infty |\psi(\omega)|^2 p_u(\omega)d\omega - \frac{t^4}{12} \int_0^\infty \omega^2 |\psi(\omega)|^2 p_u(\omega)d\omega \, . \tag{6.6}$$

The second term on the right-hand side of (6.6) becomes very small, compared to the first term, when t is very small. Hence,

$$\overline{y^2(t)} \approx t^2 \int_0^\infty |\psi(\omega)|^2 p_u(\omega)d\omega \, . \tag{6.7}$$

When particles are dispersed from a fixed source, the variance $\overline{y^2(t)}$, or rather its square root value, indicates the size of the cloud of dispersion at time t.

7. PROBABILITY DISTRIBUTIONS

Following the discussion of Section 3 we can write the probability density for v(t) (Cramer, 1946):

$$W(v) = \frac{1}{(2\pi\overline{v^2})^{1/2}} \exp\left(- \frac{v^2}{2\overline{v^2}}\right) \, . \tag{7.1}$$

Introduce (3.2) and (3.4), so that

$$W(V) \quad = \quad \frac{1}{(2\pi\overline{v^2})^{1/2}} \exp\left[-\frac{(V - \bar{U} - V_d)^2}{2\overline{v^2}}\right] . \tag{7.2}$$

A similar probability density for y(t) can be written if it is known to be a Gaussian type. It is expected that when t is sufficiently large, the random variable y(t) should have a Gaussian distribution in view of the central limit theorem (Batchelor, 1949), in which case, according to (3.4),

$$W(y,t) \quad = \quad \frac{1}{(2\pi\overline{y^2})^{1/2}} \exp\left[\frac{(y - \bar{U}t - V_d t)^2}{2\overline{y^2}}\right] . \tag{7.3}$$

The probability density $W(y,t)$ represents the spatial (one-dimension) distribution of the mean concentration of the particles. The practical applications of the probability density $W(v)$ will be found in the treatment of coagulation and impaction problems.

8. RELATION BETWEEN TURBULENT DISPERSION AND BROWNIAN MOTION

Brownian motion of particles is attributed to the direct molecular impacts whose frequency, in a liquid, is in the order of 10^{21} per second. From the very definition of fluid elements as comprising a continuous medium, one can justifiably assume that the random forces due to molecular impacts and those due to turbulent impacts are statistically independent sets of forces insofar as the dynamics of particles is concerned.

It is known in the theory of random functions that the spectrum density of the sum of two or more independent random functions is the sum of the spectrum densities of the random functions separately. Hence, the total motion of particles can be obtained by using the total spectrum density.

It can also be shown that the particle displacements due to Brownian forces and those due to turbulent impacts are statistically independent. Therefore, the variance of the total particle displacement is equal to the sum of the variances of the partial displacements due to Brownian forces and turbulence, respectively.

9. ILLUSTRATIVE EXAMPLE

In the discussion so far, the question of the actual shape of the turbulence spectrum density function has been left open. The only restriction is that the integral, which involves the spectrum density function, must converge. This indeed simplifies the discussion of the general problem. The final answer, however, can be determined only after the turbulence spectrum density is prescribed.

Consider the spectrum density

$$p_u(\omega) = \frac{2}{\pi} \frac{\omega_o}{\omega_o^2 + \omega^2} \overline{u^2} \tag{9.1}$$

where ω_o is the characteristic frequency of fluctuation whose corresponding autocorrelation function $\phi_u(\tau)$ can be shown to be of the form

$$\phi_u(\tau) \sim \exp\left(-\frac{|\tau|}{L}\right)$$

where L is the Lagrangian scale of turbulence.

We do not wish to stress the importance of this spectrum, though measurements (Dryden, 1938; Liepmann, et al., 1951) indicate that (9.1) represents a close approximation of the spectrum density of some turbulence fields.

Let

$$\beta = \frac{2}{9} \frac{\rho_1 - \rho_o}{\rho_o} \frac{a^2}{\nu} \tag{9.2}$$

From (5.5) we obtain

$$|\psi(\omega)|^2 = \frac{1}{1 + \beta^2\omega^2} \tag{9.3}$$

Substituting (9.1) and (9.3) into (5.8)

$$\overline{v^2} = \frac{2\overline{u^2}}{\pi} \int_0^\infty \frac{1}{1 + \beta^2\omega^2} \frac{\omega_0}{\omega_0{}^2 + \omega^2} d\omega \quad . \tag{9.4}$$

Equation (9.4) becomes, after integration,

$$\overline{v^2} = \frac{\overline{u^2}}{1 + \beta\omega_0} \tag{9.5}$$

Similarly, from (6.3), we get

$$\overline{y^2(t)} = \frac{4\overline{u^2}}{\pi} \int_0^\infty \frac{1}{1 + \beta^2\omega^2} \frac{\omega_0}{\omega_0{}^2 + \omega^2} \frac{1 - \cos \omega t}{\omega^2} d\omega \quad . \tag{9.6}$$

Equation (9.6) can be reduced to an integrable form by resolving its integrand into partial fractions. Thus,

$$\frac{\overline{y^2(t)}}{\overline{u^2}} = \frac{2}{\omega_0}t - \frac{2}{(1 - \beta^2\omega_0^2)\omega_0{}^2} [1 - \exp(-\omega_0 t)] + \frac{2\beta^3\omega_0}{1 - \beta^2\omega_0{}^2} \left[1 - \exp(-\frac{t}{\beta})\right] \tag{9.7}$$

The quantity $\overline{y^2(t)}$ approaches $2\overline{u^2}t(\omega_0)^{-1}$ when t approaches ∞. This checks with the general conclusion given in (6.5) because

$$2t \int_0^\infty \phi_u(\tau)d\tau = \pi t p_u(\omega)_{\omega=0} = \frac{2\overline{u^2}}{\omega_0} t \quad .$$

When t is very small, $\overline{y^2(t)}$ becomes approximately

$$\overline{y^2(t)} \longrightarrow \frac{\overline{u^2}}{1 + \beta\omega_0} t^2 = \overline{v^2}\, t^2$$

which again checks with the general conclusion given in (6.7).

More discussion concerning the properties of the spectrum density (9.1) can be found in works on diffusion (Kampé de Fériet, 1939; Frankiel, 1953) and on Brownian motion (Wang and Uhlenbeck, 1945).

10. PRACTICAL APPLICATIONS

Physical problems of turbulent dispersion generally involve motion in three dimensions. To generalize the simplified (one-dimensional) analysis to the general case (three-dimensional) would not present any difficulty if one can be certain that the random particle velocity components are statistically independent. This condition can be guaranteed when the turbulent velocity components are statistically independent [see Equation (3.1)]. The latter has been generally assumed for the case of homogeneous isotropic turbulence (Frenkiel, 1953).

The present theory can be applied to problems of two general types. The first is the calculation of the dispersion of particles emitted from a fixed or moving source in a turbulent atmosphere, the spectrum of which is known. The dispersion of pollens can be considered as a typical example. Secondly, by measuring the dispersion of a cluster of floating "particles" one can calculate the turbulence spectrum of the atmosphere in which the particles are embedded. The "particles" here act as "filters" to component motions of turbulence. The latter can be conceived in a method of measuring clear air turbulence by tracking the dispersion of floating balloons or other artificial "particles."

The present study also establishes the basis of an analysis of more complicated problems such as the coagulation of particles and their impaction on surfaces which will be treated in another paper.

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TABLE OF SYMBOLS

U	=	fluid velocity
$\bar{U}$	=	mean fluid velocity
$u(t)$	=	turbulent fluid velocity fluctuation
V	=	particle velocity
$\bar{V}$	=	mean particle velocity
$V_d = \frac{\Sigma F}{6\pi\mu a}$	=	particle drifting velocity due to stationary extraneous forces ΣF
$v(t)$	=	particle velocity fluctuation
$y(t)$	=	particle displacement fluctuation
$\phi_u(\tau)$	=	$\frac{1}{2T}\int_{-T}^{T} u(t)u(t + \tau)d\tau$
$\phi_v(\tau)$	=	$\frac{1}{2T}\int_{-T}^{T} v(t)v(t + \tau)d\tau$
$p_u(\omega)$	=	spectrum density of $u(t)$
$p_v(\omega)$	=	spectrum density of $v(t)$
ω	=	angular frequency of fluctuation
a	=	particle radius
ρ_1	=	particle density
ρ_o	=	fluid density
μ	=	coefficient of fluid viscosity
$\nu = \mu/\rho_o$	=	kinematic viscosity
β	=	$\frac{2}{9}\frac{a^2}{\nu}\frac{\rho_1 - \rho_o}{\rho_o}$
L	=	$\frac{\int_0^\infty \phi_u(\tau)d\tau}{[\overline{u^2(t)}]^{1/2}\,[\overline{u^2(t + \tau)}]^{1/2}}$ = Lagrangian scale of turbulence

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