STRING THEORY EFFECTS ON BLACK HOLE PHYSICS

by

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CHAPTER I

Introduction

The thermodynamic nature of black holes presents one of the major challenges in theoretical physics. A complete understanding requires a consistent theory that unifies gravity and quantum mechanics. String theory successfully provides a microscopic description of gravity, but more generally a significant amount of the recent progress is due to the holographic description of black holes.

In this thesis we present recent developments on black hole physics. We investigate string theory effects on classical solutions, and discuss aspects of the their dual description. In the remainder of this section we discuss the motivations that inspired our work, and give an overview the key features of our results.

1.1 Black Hole Physics

In the context of astrophysics, a black hole represents the final stage of gravitational collapse of a massive object such as a star. When the internal pressure of the star is insufficient to resist its own gravity, the material is squeezed to high densities. Once it collapses to within the Schwarzschild radius [1] the object forms a region from which nothing can escape. The boundary of this region defines an event horizon. In the classical theory of general relativity, a black hole is a solution to Einstein's equations that contains

[1] A characteristic radius associated with every mass. If that mass could be compressed to fit within that radius, no force could stop it from continuing to collapse into a gravitational singularity.

an event horizon. The size and shape of this horizon is uniquely determined by the mass, rotation and electric (or magnetic) charge of the black hole.

Almost forty years ago, the analysis presented in [1, 2, 3] showed that a stationary black hole can exchange energy with its surroundings. A remarkable result was the close analogy between the laws of black hole mechanics and thermodynamical principles. Basically, the mass of the black hole behaves as the energy of a thermodynamic system, the surface gravity of the event horizon as the temperature, and the area of the event horizon would follow the entropy.

Classically the physical relationship of these quantities breaks down since the black hole cannot radiate, *i.e.* the temperature of a classical black hole is zero. However, in the context of quantum field theory in a curved background, a stationary black hole emits thermal radiation –Hawking's Radiation– with a characteristic temperature determined by the surface gravity of the event horizon [4]. Therefore one identifies the black hole parameters (mass, area and surface gravity) with thermodynamics quantities (energy, entropy and temperature).

The Bekenstein-Hawking area law for the black hole entropy gives rise to several questions. In thermodynamics, the state of a system at equilibrium can be described by intensive quantities (temperature) and extensive quantities (energy, entropy). The statistical approach is to derive these quantities from the dynamics of the microscopic constituents and their interactions. From a macroscopic perspective, the entropy is a state function that measures how far from equalization the system is with respect to its surrounding, and can only be known empirically. While in statistical mechanics the entropy is a function of the distribution of the system on its microscopic states, *e.g.* the number of possible configurations for a fixed number of particles at a given temperature. It is natural to ask if the entropy of the black hole can be understood as a collection of microstates characterized by the mass, rotation and electric charge that define the macroscopic system.

The nature of these microstates for the black hole is related to one of the fundamental problems of physics: a consistent description of the dynamics of space-time and quantum physics. For a certain class of black holes, string theory provides a precise microscopic description that involves D-branes, fundamental strings and other objects carrying the same set of charges as the black hole. The statistical entropy is given by the degeneracy of states determined by the dynamics of the string excitations. The microscopic (D-branes and strings) and macroscopic (black hole) calculations of the entropy are performed independently, and for large values of charge both quantities exactly agree [5].

The great majority of black holes that have a microscopic description share an important property: extremality. This means that the solution has zero temperature, hence do not Hawking radiate and are stable. In some cases an extremal black hole is also invariant under certain supersymmetry transformations, the so-called BPS black holes. The stability properties and supersymmetry gives some control over the microscopic dynamics, and justifies the comparison between the statistical entropy and the black hole entropy.

However, there are still several unanswered questions concerning the quantum nature of black holes. The Bekenstein-Hawking entropy applies when the gravitational action is just the standard Einstein-Hilbert term. This classical action is a low-energy effective theory; therefore it will receive corrections from higher dimension operators. The area law for the entropy is only a leading order result, valid when the black hole is much larger than the Planck scale. In the microscopic theory, the derivation of the statistical entropy is usually done for large values of charges in the system. For certain configurations the spectrum of the theory is highly constrained by the symmetries of the system. This gives access to powerful counting formulas which are valid everywhere and so it is possible to compute subleading corrections to the microscopic entropy. Matching the corrections to the area law and the statistical entropy will give a more detailed understanding of black holes.

In four dimensional (4D) supergravity the program of including higher derivatives has received much attention. The effects on black holes became tractable due to the off-shell formalism of $N = 2$ supergravity, where the corrections are packaged in the prepotential [6, 7, 8, 9]. Including the subleading corrections, their entropies successfully matched the statistical entropy [10, 11].[2] The OSV conjecture [16], which proposes a connection of topological strings with the entropy of 4D black holes, has also stimulated recent work [17, 18, 19, 20, 21, 22, 23].

Despite the exact agreement, this result seems quite mysterious since only a selected subset of terms in the supergravity action are being used. The full effective action of string theory is expected to contain an infinite series of higher derivative terms, and generically it also includes non-local contributions. There is at present no understanding why additional 4D terms can be neglected which *a priori* are of the same order. By contrast, in five dimensional (5D) supergravity there is greater control on the corrections. The enhanced symmetries inherent to the 5D near horizon geometry allow one to systematically compute the entropy. In fact, one can prove a non-renormalization theorem [24, 25] stating that the entropy gets no corrections from terms with more than four-derivatives. All higher derivative terms either vanish on the solution or can be removed by a field redefinition [26].

For the above reasons, we confine our attention to four-derivative corrections to five dimensional supergravity. The relevant terms are determined by gauge and gravitational anomalies, and correspond to the supersymmetric completion of the mixed Chern-Simons term. With the full corrected BPS black hole solutions in hand, we compute the corrections to the macroscopic entropy. In the course of our discussion we will compare with the appropriate microscopic quantities, and emphasize the distinctions and similarities between

[2] In [12] a different combination of higher derivative terms was studied –Gauss-Bonet interaction– and it was found to also correctly capture the subleading entropy [13, 14, 15].

4D and 5D black holes.

1.1.1 Extremization principles

As we emphasized above, a systematic quantitative study of higher derivative corrections will give us a better understanding of the quantum aspects of black holes. In addition it addresses the question: is the thermodynamic nature of Einstein theory just a peculiar accident or a robust feature of all covariant theories of gravity?

In [27, 28, 29] the question was resolved and the authors gave a generalized expression of the black hole entropy, known as Wald's entropy formula. The idea was to construct the first law of black hole mechanics valid for any theory which is invariant under diffeomorphisms. In this construction, the black hole entropy is related to the Noether charge of diffeomorphisms under the Killing vector field which generates the event horizon. Further, the entropy can always be expressed as a local geometric density integrated over the horizon. In practice it is rather complicated to evaluate Wald's entropy directly but there is a short-cut that applies to extremal black holes.

An extremal black hole has the remarkable property of exhibiting an attractor behavior. Basically, the equations of motion have a fixed basin of attraction at the horizon where the scalar moduli fields are fixed in terms of charges carried by the black hole. A priori, the moduli are continuous parameters which can be freely specified at infinity raising the possibility that the entropy might depend on their values. Since the entropy is a surface integral over the horizon, it depends only on the values of the moduli at the horizon, and these turn out to be insensitive to the values at infinity. Therefore the black hole entropy is a function purely of conserved charges, *i.e.* mass, electric/magnetic charge and angular momentum.

One of the advantages of the attractor mechanism, is that the limiting value of the geometry decouples from the asymptotically flat solution and it constitutes a solution in its own right. In addition the solution has enhanced symmetry, and for BPS black holes it corresponds to maximal supersymmetry. As nicely constructed in [24] and [11] the problem of finding an attractor solution to a general higher derivative action can be reduced to the problem of extremizing a single function of the scale sizes and moduli. Because of the enhanced symmetry of the background, the extremization procedure just involves solving algebraic equations. Further, the entropy reduces to evaluating the appropriate functional at its extremum. These extremization principles greatly simplifies the evaluation of Wald's entropy. This is the procedure we will use here to compute the black hole entropy in the presence of higher derivative terms.

1.1.2 Resolution of singularities

Until now we have only discussed the thermodynamic nature of black holes. Another interesting phenomena is the resolution of singularities in the classical theory. There are different solutions in general relativity that unavoidably contain a singularity. For example, in most cosmological models at $t = 0$ matter is crushed to high densities creating a time-like singularity, *i.e.* the Big-Bang. Therefore, in the early universe general relativity is no longer valid and string theory should provide a description at this stage. There has been a significant amount of work to address this in string theory, *e.g.* see [30, 31, 32, 33, 34] and references within.

In the context of black holes, one encounters solutions that contain a space-like naked singularity in the two-derivative theory. Resolving this singularity is another interesting application of higher-derivative corrections. One class of singular solution is the so-called *small black holes*. They have a zero size horizon classically and hence the singularity is exposed. These solutions have vanishing Bekenstein-Hawking entropy at the leading order, but still possess a non-zero entropy in the microscopic theory. As discussed in [35, 36, 37, 38], the effect of higher-order corrections is to provide a string scale horizon that covers the singularity and the geometry becomes qualitatively similar to a "regular" black hole. Further it is also possible to compute the corrected Wald entropy which agrees with the statistical degeneracy.

In five dimensions we can also construct extended magnetic sources corresponding to black strings. These sources allow for another class of *small* solutions with zero size horizon denoted as *small black strings*. In contrast with the small black hole, the singularity is extended and not point-like as above. Also the configuration carries no entropy even in the microscopic theory. Therefore instead of shielding the singularity with a finite size horizon, the singularity is smoothed out entirely [39, 40].

The simplest example of a small black string corresponds to a fundamental heterotic string without any momentum excitations. For this system we find a completely regular solution at the four-derivative level, which smoothly interpolates between the near horizon geometry and asymptotically flat space. Unfortunately, for these configurations there is no small parameter suppressing higher order terms. The analysis only suggest that the solution remains smooth after additional corrections are taken into account, but the precise numerical values of the fields will likely be modified.

1.2 Holographic Description of Black Holes

The holographic description of Anti-de Sitter space has played a crucial role in understanding the quantum nature of black holes. The basic idea behind the AdS/CFT correspondence is the identification of the symmetries of a gravitational theory on Anti-de Sitter space (AdS) and a dual conformal field theory (CFT) located at the boundary [41, 42]. The correspondence states an equivalence between the partition function of the gravitational theory on AdS and the dual CFT. The entropy of the black hole is then governed by the degeneracy of states of the CFT.

In [41] the correspondence was realized by studying the low-energy description of D-branes in string theory. An alternative derivation of the duality relies only on general properties of a diffeomorphism-invariant theory [43] and does not utilize string theory. Here we will briefly explain the arguments that lead to the duality. For a complete discussion see [44] and references within.

1.2.1 Holography

The concept of holography is quiet natural in gravity. An indication of such a description is the Bekenstein bound [45] which implies that the number of degrees of freedom of a gravitational system grows as the area and not the volume of a given region. A standard quantum field theory does not behave in such a way. This suggests that in quantum gravity the bulk dynamics is completely described by a boundary theory with one degree of freedom less, *i.e.* a holographic principle.

For the case of three dimensional gravity it possible to explicitly realize the boundary theory. The observation made in [43] is that the asymptotic symmetry group of AdS_3 is generated by two copies of the Virasoro algebra. Therefore a 3D theory of gravity on AdS is dual to a 2D CFT. To reach this conclusion, first we identify all allowed diffeomorphisms that preserve the boundary conditions of AdS_3. By analyzing the transformation properties of the gravitational stress tensor under these diffeomorphisms, one finds that it has the properties of a stress tensor in a quantum CFT. In particular it transforms like a tensor plus a Schwarzian derivative under the conformal group. The enhancement of the asymptotic $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ conformal symmetry to a Virasoro algebra gives a non-zero central charge. The central charge of the conformal theory is the quantity that governs the entropy of the black hole via Cardy's formula [46].

This derivation rests on a key assumption: a 3D quantum theory of gravity must exist. The above analysis only deals with the classical gravitational theory and will only capture limited features of the dual quantum theory. For example one can only determine the central charge of the theory which is sufficient to compute the entropy, but the exact conformal theory and degeneracies at every mass level or other dynamical aspects are out of reach. String theory is a quantum theory of gravity, hence it gives a framework where the duality can be further explored and generalized to other dimensions.

1.2.2 String theory and AdS/CFT

We now turn to the derivation of the correspondence as done in [41]. String theory on a Dp-brane background contains open and closed string modes.[3] Open strings correspond to excitations of the D-branes, and closed strings are the excitations of empty space. The system has two perturbative descriptions governed by the massless modes of open or closed strings. When the energy of the system is small compared to the inverse string length l_s, the low-energy effective theory of open strings on N Dp-branes is a $U(N)$ conformal field theory in $(p+1)$-dimensions. For small loop parameter the perturbative analysis of the Yang-Mills theory can be trusted. This requires $g_s N \ll 1$, where g_s is the string coupling.

The closed massless string modes give a gravity multiplet in ten dimensions, therefore the effective theory is type II supergravity in 10D. Dp-branes are massive charged objects which act as sources for various fields that support the black hole. The low-energy limit is achieved by introducing certain scaling limits which decouples the near horizon geometry from the asymptotic flat space. The resulting geometry is a gravitational theory on AdS$_{p+2}$. The validity of supergravity requires that the radius of curvature is large compared to the string length l_s. This implies that the theory is strongly coupled, *i.e.* $g_s N \gg 1$ and $N \gg 1$.

We presented two low-energy theories for the Dp-brane system: a supergravity solution on AdS$_{p+2}$, and a conformal field theory on $(p+1)$-dimensions. Each perturbative description is valid for different values of the coupling, making extremely hard a direct comparison between them. However, there are aspects of these theories that do not de-

[3] A Dp-brane is a $(p+1)$-dimensional hyperplane in spacetime where an open string can end. Due to the open-closed string duality, this means that a D-brane is also a source of closed strings.

pend on the coupling. For example the global symmetries, spectrum of chiral operators and moduli space among others remain unchanged in the strong and weakly coupled regime. The exact matching of these quantities is what motivated the conjecture in [41].

The AdS/CFT correspondence is a strong/weak coupling duality. In the large N limit, it relates the region of weak coupling in the field theory to the high curvature string theory, and vice versa. For this reason it is called a duality, since perturbatively the two descriptions look different.

For a large class of black holes the accounting of the microscopic entropy is due to the appearance of an AdS_3 factor in the near horizon geometry [47, 48]. The two standard constructions of these geometries in string theory are: D1-D5 system [5, 44, 49] yielding a 5D black hole with near horizon geometry $\mathrm{AdS}_3 \times S^3$, and wrapped M5-branes [10] that give rise to a 5D black string (or 4D black hole) with near horizon geometry $\mathrm{AdS}_3 \times S^2$. In each case, in the weakly coupled regime the systems are governed by a specific supersymmetric gauge theory that flows to a nontrivial two dimensional CFT.

Throughout our work we will use AdS/CFT to explore two different aspect of black holes. First the AdS_3 constructions, namely the D1-D5 system and M5-branes, will provide a precise framework where we can compare the sub-leading effects of both the macroscopic and microscopic theories. This provides a more stringent test of string theory beyond the leading order.

Currently the black holes that are well understood contain an AdS_3 factor. For black holes with an universal AdS_2 factor in the near horizon geometry there have been several attempts to construct a consistent microscopic description. This motivates our second application of AdS/CFT. We develop holographic renormalization for AdS_2 backgrounds in a conventional manner. As we will derive, the conformal symmetry group is enhanced to a Virasoro algebra which provides a setting to discuss the entropy of black holes with an AdS_2 factor.

1.3 Overview

The thesis is organized as follows. In Chapter II we construct asymptotically flat five dimensional black holes and strings in the presence of higher derivatives terms. The four derivative terms are the supersymetric completion of the mixed gauge-gravitational Chern-Simons term, which are obtained using the off-shell version of Poincaré supergravity. The solutions naturally divide in two depending on whether the Killing vector is null or timelike. Null solutions correspond to 5D black strings, and timelike backgrounds correspond to 5D black holes and black rings.

Chapter III focuses on the corrections to the black hole entropy. We start the discussion by constructing the attractor solutions and observe that they describe the near horizon geometries of the solutions found in Chapter II. With the attractor geometry fixed, we then turn to evaluating the black hole entropy via extremization principles. For black strings with an AdS_3 factor finding the entropy is reduced to computing the central charge using c-extremization. For black holes with an AdS_2 factor, we apply the entropy function and compute the subleading corrections to the black hole entropy.

To complete the comparison of the macroscopic and microscopic corrections to the black hole entropy, in Chapter IV we analyze the statistical corrections to the D1-D5 black hole in five dimensions. The counting problem for D1-D5 system on $K3$ is well under control [50, 51, 14], and it will allow us to calculate subleading corrections in the same regime of validity as the Wald entropy in R^2 corrected supergravity. We will use this result to clarify the relation between 4D and 5D black holes.

Finally, in Chapter V we discuss holography for two-dimensional Maxwell-dilaton gravity. Our goal is to develop the holographic description of AdS_2 for the gravitational theory systematically, following the procedures that are well-known from the AdS/CFT correspondence in higher dimensions. From the transformation properties of the energy-momentum tensor we determine the central charge. In addition, we verify that our result for the central charge agrees with the Brown–Henneaux central charge for AdS_3 spacetimes [43].

The discussion presented in Chapter II and III is based on [52, 39, 53, 40] which was done in collaboration with Joshua L. Davis, Per Kraus and Finn Larsen. The contents in chapter IV are based on [52]. and [54] which was done in collaboration with Sameer Murthy. The results in chapter V were done in collaboration with Daniel Grumiller, Finn Larsen and Robert McNees, and have been published in [55].

CHAPTER II

Black Solutions with Higher Derivative Terms

In this chapter we construct stationary solutions to five dimensional supergravity in the presence of four derivative terms. The solutions we obtain preserve a fraction of the supersymmetry and are asymptotically flat. The space of solutions is characterized by the charges the *black* object supports, and the isometries preserved by the geometry. First, there are supersymmetric solutions carrying magnetic charge; such objects are *black strings* extended in one spatial direction [56, 57]. Second, there are supersymmetric, electrically charged, rotating (BMPV) *black holes* [5, 58, 59]. Finally, *black rings* are objects that can carry both electric and magnetic charges, and rotate in two independent planes [60, 61].

Five dimensional supergravity can be thought as arising from the dimensional reduction of M-theory on a Calabi-Yau threefold (see Appendix C.2). The theory contains some number of vector multiplets, determined by the Hodge numbers of the Calabi-Yau. It then turns out that the action up to four derivatives is completely determined in terms of two additional pieces of topological data, namely the triple intersection numbers and second Chern class of the Calabi-Yau. The task of finding solutions to this theory is greatly simplified by working in a fully off-shell formalism [62, 63, 64, 65]. This means that enough auxiliary fields are introduced so that the supersymmetry transformations are independent of the action.[1] This is a great advantage, because the supersymmetry transformation

[1] A familiar example of this is $N = 1$ supersymmetric field theory in four dimensions, where the

laws are very simple, while the explicit action is quite complicated. In looking for BPS solutions we first exhaust the conditions implied by unbroken supersymmetry; in the off-shell formalism it follows that this part of the analysis proceeds the same whether one considers the two, four, or even higher derivative solutions. Much of the solution thereby can be determined without great effort. Only at the very end do we need to consider some of the equations of motion in order to complete the solutions. In general, we find that the full solution can be expressed algebraically in terms of a single function, which obeys a nonlinear ordinary differential equation.

The chapter is organized as follows. We begin with a review of superconformal gravity and the inclusion of higher derivative terms using the off-shell formalism. We sketch the construction of the off-shell multiplets, and outline the gauge fixing procedure that gives $D = 5$ R^2-corrected Poincaré supergravity. Our discussion of asymptotically flat solutions divides into two parts. The half BPS solutions have a distinguished Killing vector, which can be null or timelike. The Null case corresponds to black strings, and the timelike case includes rotating black holes and black rings. We show how to systematically construct these solutions, starting by applying the conditions of unbroken supersymmetry, and then imposing the equations of motion for the Maxwell and auxiliary fields. The discussion is based on the results found in [52, 39, 53, 40].

2.1 Conformal Supergravity

The low energy limit of a supersymmetric compactification of string theory is a supergravity theory. While the Lagrangians of these theories can in principle be extracted from string S-matrix computations, in practice a more efficient method is to work directly in field theory, by demanding invariance under local supersymmetry. This approach typically uses the so-called Noether method. In this procedure one starts with an action invariant

superspace construction ensures that the supersymmetry algebra closes without having to use the equations of motion.

under global supersymmetry, and then attempts to incorporate local invariance iteratively. For a two-derivative Lagrangian the possible matter couplings are usually known, and the process of constructing the action and transformation rules for the fields only involves a finite number of steps. The incorporation of higher derivative terms increases enormously the possible terms in the action and transformation rules. For example, one might start by including a specific four-derivative interaction and using the two-derivative transformation rules. This will generate additional four-derivative terms that will necessitate modifications to the supersymmetry transformations. Now, these modified transformations will generate six-derivative terms in the Lagrangian and so forth. In general, it may take many steps, if not an infinite number, for this iterative procedure to terminate, making the construction extremely difficult and tedious.

A more systematic approach to obtaining an invariant action is by constructing off-shell representations of the supersymmetry algebra. The advantage of this formalism is that the construction of invariants is well-defined since the transformation rules are fixed. Now, the theory we are aiming for is an off-shell version of Poincaré supergravity. However, it turns out that the construction of off-shell multiplets is greatly simplified by first considering a theory with a larger gauge invariance, and then at the end gauge fixing down to Poincaré supergravity. In five dimensions, it turns out that extending conformal supergravity to a gauge theory described by the superalgebra $F(4)$ gives an irreducible off-shell realization of the gravity and matter multiplets. The cost of this procedure is the inclusion of additional symmetries and compensating fields, which have no physical degrees of freedom. The construction of a supergravity theory from the gauge theory is first done by imposing constraints that identify the gauge theory as a gravity theory. Then, gauge fixing appropriately the values of certain compensating fields, one reduces the superconformal theory to Poincaré supergravity. This has been extensively studied for $d \leqslant 6$ superconformal theories; for more details we refer the reader to [66, 67, 68, 62, 63, 69, 64, 65].

One of the fruitful applications of this formalism is the construction of higher-derivative Lagrangians. Specifically, we will consider the four-derivative corrections to $N=2$ supergravity which arise from string theory. In five dimensional theories, there is a special mixed gauge-gravitational Chern-Simons term given by

$$\mathcal{L}_{\rm CS} = \frac{c_{2I}}{24\cdot 16}\epsilon_{abcde}A^{Ia}R^{bcfg}R^{de}{}_{fg}\ . \tag{2.1}$$

The coefficient of this term is precisely determined in string/M-theory by $M5$-brane anomaly cancellation via anomaly inflow [70]. The constants c_{2I} are understood as the expansion coefficients of the second Chern class of the Calabi-Yau threefold on which the eleven-dimensional M-theory is compactified. In [65] all terms related by supersymmetry to (2.1) were derived using the superconformal formalism.

Our present goal is to simply present the main concepts of the superconformal formalism and the specific results we will use to study black holes and other such objects in the next sections. In the following subsections we outline the field content and transformation rules for the gravity and matter multiplets and, after gauge fixing the superconformal theory, we present the R^2 supersymmetric completion of $N=2$ supergravity.

2.1.1 Off-shell supersymmetry multiplets

In this section we will discuss the field content of the relevant five dimensional supersymmetry multiplets. Before describing each multiplet, let us briefly describe some aspects of the superconformal formalism. The five dimensional theory is obtained by first constructing a gauge theory with gauge symmetry given by the supergroup $F(4)$. The generators $\mathbf{X}_A$ and corresponding gauge fields h_μ^A for this theory are

$$\begin{aligned}\mathbf{X}_A :&\ \mathbf{P}_a\ ,\quad \mathbf{M}_{ab}\ ,\quad \mathbf{D}\ ,\ \mathbf{K}_a\ ,\ \mathbf{U}_{ij}\ ,\ \mathbf{Q}_i\ ,\ \mathbf{S}^i\\ h_\mu^A :&\ e_\mu^a\ ,\quad \omega_\mu^{ab}\ ,\quad b_\mu\ ,\ f_\mu^a\ ,\ V_\mu^{ij}\ ,\ \psi_\mu^i\ ,\ \phi_\mu^i\end{aligned} \tag{2.2}$$

where $a, b = 0, \ldots, 4$ are tangent space indices, $\mu, \nu = 0, \ldots, 4$ are (curved) spacetime indices and $i, j = 1, 2$ are $SU(2)$ indices. The generators of the Poincaré algebra are translations $\mathbf{P}_a$ and Lorentz transformations $\mathbf{M}_{ab}$. The special conformal transformations and dilatations are generated by $\mathbf{D}$ and $\mathbf{K}_a$, respectively. $\mathbf{U}_{ij}$ is the generator of $SU(2)$ and the fermionic generators for supersymmetry and conformal supersymmetry are the symplectic Majorana spinors $\mathbf{Q}_i$ and $\mathbf{S}^i$.

The next step is to construct from the superconformal gauge theory a conformal supergravity theory, *i.e.* our symmetries have to be realized as space-time symmetries rather than internal symmetries. This procedure is well known [71] and it is achieved by applying torsion-less constraints[2] over the curvatures, which are

$$\hat{R}^a_{\mu\nu}(P) = 0\ , \quad \gamma^\mu \hat{R}^i_{\mu\nu}(Q) = 0\ , \quad \hat{R}_\mu{}^a(M) = 0\ . \tag{2.3}$$

Here the curvatures are defined as commutators of the conformal supercovariant derivatives, that is

$$[\hat{\mathcal{D}}_\mu, \hat{\mathcal{D}}_\nu] = -\hat{R}^A_{\mu\nu}\mathbf{X}_A\ , \tag{2.4}$$

with

$$\hat{\mathcal{D}}_\mu = \partial_\mu - h^A_\mu \mathbf{X}_A\ , \tag{2.5}$$

where we are summing over $\mathbf{X}_A = \left\{\mathbf{M}_{ab},\ \mathbf{D},\ \mathbf{K}_a,\ \mathbf{U}^{ij},\ \mathbf{Q}_i,\ \mathbf{S}^i\right\}$. By solving (2.3), some of the gauge fields will become dependent fields. Assuming that the vielbein e^a_μ is invertible, the first constraint will determine the connection ω^{ab}_μ. The second and third constraints fix ϕ^i_μ and f^a_μ, respectively, making them dependent fields as well [62, 63].

The final step in constructing the off-shell gravity multiplet is adding auxiliary fields. In order to understand this, it is useful to track the number of independent bosonic and fermionic components. Before imposing (2.3) the gauge fields are composed of 96 bosonic

[2] In the literature, (2.3) are often called the *conventional constraints.*

and 64 fermionic gauge fields. The curvature constraints fix the connections ω_μ^{ab}, ϕ_μ^i and f_μ^a, eliminating their degrees of freedom. The new number of degrees of freedom is then the total number of components of the remaining gauge fields minus the total number of the generators $\mathbf{X}_A$. This counting results in 21+24 degrees of freedom. Adding auxiliary fields, which will include extra transformation rules and modifications to the supersymmetry algebra, solves this final mismatch in the number of bosonic and fermionic degrees of freedom. The procedure has been outlined in [66].

The construction sketched above gives the irreducible **Weyl multiplet**, which describes $32 + 32$ degrees of freedom and contains the following fields,

$$e_\mu^a \ , \ V_\mu^{ij} \ , \ b_\mu \ , \ v_{ab} \ , \ D \ , \ \psi_\mu^i \ , \ \chi^i \ . \tag{2.6}$$

As we mentioned before, in order to have a closed algebra it is necessary to include compensators, *i.e.* auxiliary fields. For the Weyl multiplet, the non-propagating fields are an antisymmetric two-form tensor v_{ab}, a scalar field D and an $SU(2)$ Majorana spinor χ^i.

The two matter multiplets relevant for our purposes are the vector multiplet and hypermultiplet. The off-shell components of the **vector multiplet** are,

$$M^I \ , \ A_\mu^I \ , \ Y_{ij}^I \ , \ \Omega_i^I \ .$$

M^I are scalar fields and A_μ^I gauge fields. The multiplet also contains a $SU(2)$ triplet auxiliary field Y_{ij}^I and the $SU(2)$ Majorana spinor Ω_i^I. The index I labels the generators of the gauge group G. For brevity, we consider G as $n_V + 1$ copies of $U(1)$; the generalization to non-Abelian gauge groups is discussed in [62] and [63]. The field strength is given by

$$\hat{F}_{\mu\nu}^I = 2\partial_{[\mu} A_{\nu]}^I + 4i\bar{\psi}_{[\mu}\gamma_{\nu]}\Omega^I - 2i\bar{\psi}_\mu\psi_\nu M^I \ .$$

The components of the **hypermultiplet**, are

$$\mathcal{A}_i^\alpha \ , \ \zeta^\alpha \ , \ \mathcal{F}_\alpha^i \ ,$$

where the index $\alpha = 1 \cdots 2r$ represents $USp(2r)$. The scalars $\mathcal{A}_i^\alpha$ are anti-hermitian, ζ^α is a Majorana spinor and $\mathcal{F}_\alpha^i$ are auxiliary fields. For our discussion, the relevant supersymmetry transformation is given by

$$\delta\zeta^\alpha = \gamma^a \hat{\mathcal{D}}_a \mathcal{A}_j^\alpha \epsilon^j - \gamma \cdot v \mathcal{A}_j^\alpha \epsilon^j + 3\mathcal{A}_j^\alpha \eta^j \ . \tag{2.7}$$

As we will discuss shortly, (2.7) will allow us to consistently preserve the Poincaré gauge by performing a compensating $\mathbf{S}^i$ transformation, *i.e.* fix η^i in terms of ϵ^i.

2.1.2 Off-shell Poincaré supergravity

Our main interest is five dimensional Poincaré supergravity. Starting from the superconformal theory, it is possible to gauge fix the additional conformal symmetries and consistently obtain an off-shell representation of $N = 2$ supergravity. This requires choosing the *vevs* of certain fields associated with the conformal group and the R-symmetry, which spontaneously breaks the superconformal symmetry. The procedure does not make use of the equations of motion, and the number of symmetries and degrees of freedom eliminated is balanced. This makes the process reversible and therefore, the conformal theory is gauge equivalent to Poincaré supergravity [72].

We will start by considering the Weyl multiplet coupled to $n_V + 1$ vector fields and one hypermultiplet.[3] To illustrate the gauge fixing procedure, consider the two-derivative Lagrangian describing the bosonic sector of the conformal theory

$$\begin{aligned} \mathcal{L}_B =& 2\mathcal{D}^a \mathcal{A}_i^\alpha \mathcal{D}_a \mathcal{A}_\alpha^i + \mathcal{A}^2 \left(\frac{1}{4}D + \frac{3}{8}R - \frac{1}{2}v^2\right) \\ &+ \mathcal{N}\left(\frac{1}{2}D - \frac{1}{4}R + 3v^2\right) + 2\mathcal{N}_I v^{ab} F_{ab}^I + \frac{1}{4}\mathcal{N}_{IJ} F_{ab}^I F^{Jab} \\ &- \mathcal{N}_{IJ}\left(\frac{1}{2}\mathcal{D}^a M^I \mathcal{D}_a M^J + Y_{ij}^I Y^{Jij}\right) + \frac{1}{24e} c_{IJK} A_a^I F_{bc}^J F_{de}^K \epsilon^{abcde} \ , \end{aligned} \tag{2.8}$$

[3] One could include additional hyper multiplets (or other matter fields not discussed here), which would require the inclusion of non-dynamical multiplets in order to consistently eliminate the extra gauge symmetries, obscuring the procedure.

with

$$\mathcal{N} = \frac{1}{6} c_{IJK} M^I M^J M^K \ , \tag{2.9}$$

where c_{IJK} are constants, and $\mathcal{N}_I$ and $\mathcal{N}_{IJ}$ are derivatives of $\mathcal{N}$ with respect to M^I

$$\mathcal{N}_I \equiv \frac{\partial \mathcal{N}}{\partial M^I} = \frac{1}{2} c_{IJK} M^J M^K \ , \ \mathcal{N}_{IJ} \equiv \frac{\partial^2 \mathcal{N}}{\partial M^I \partial M^J} = c_{IJK} M^K \ . \tag{2.10}$$

For the detailed derivation of (2.8) we refer the reader to [65]. The first step towards gauge fixing the theory is to notice that the dilatational field b_μ only appears in (2.8) through the covariant derivatives of the matter fields. This allows us to fix special conformal transformations by choosing the gauge $b_\mu = 0$.

In order to have the canonical normalization for the Ricci scalar in (2.8), our gauge choice for the dilatational group is $\mathcal{A}^2 = -2$. Notice that in the two-derivative theory this gauge choice, combined with the equations of motion of the auxiliary field D, gives the very special geometry constraint $\mathcal{N} = 1$ (see Appendix A.1).

The $SU(2)$ symmetry is fixed by identifying the indices in the hypermultiplet scalar, *i.e.* $\mathcal{A}^i_\alpha = \delta^i_\alpha$. Finally, since we restricted the discussion to an Abelian gauge group for the vector multiplet, the auxiliary fields V^{ij}_μ and Y^I_{ij} will only appear quadratically in (2.8). Therefore, it is appropriate for the ungauged theory to set both Y^I_{ij} and V^{ij}_μ to zero.

Summarizing, our gauge choice is given by:

$$\begin{gathered} \mathcal{A}^i_\alpha = \delta^i_\alpha \ , \ \mathcal{A}^2 = -2 \ , \\ b_\mu = 0 \ , \ V^{ij}_\mu = 0 \ , \ Y^I_{ij} = 0 \ . \end{gathered} \tag{2.11}$$

Substituting (2.11) into (2.8) gives rise to the two-derivative Lagrangian

$$\begin{aligned} \mathcal{L}_0 = & -\frac{1}{2} D - \frac{3}{4} R + v^2 + \mathcal{N} \left(\frac{1}{2} D - \frac{1}{4} R + 3 v^2 \right) + 2 \mathcal{N}_I v^{ab} F^I_{ab} \\ & + \mathcal{N}_{IJ} \left(\frac{1}{4} F^I_{ab} F^{Jab} + \frac{1}{2} \partial_a M^I \partial^a M^J \right) + \frac{1}{24} c_{IJK} A^I_a F^J_{bc} F^K_{de} \epsilon^{abcde} \ . \end{aligned} \tag{2.12}$$

The gauge choice (2.11) remains valid for the higher-derivative Lagrangian constructed in [65], and it reads

$$
\begin{aligned}
\mathcal{L}_1 = \frac{c_{2I}}{24}\Big(&\frac{1}{16}\epsilon_{abcde}A^{Ia}R^{bcfg}R^{de}{}_{fg} + \frac{1}{8}M^I C^{abcd}C_{abcd} + \frac{1}{12}M^I D^2 + \frac{1}{6}F^{Iab}v_{ab}D \\
&+ \frac{1}{3}M^I C_{abcd}v^{ab}v^{cd} + \frac{1}{2}F^{Iab}C_{abcd}v^{cd} + \frac{8}{3}M^I v_{ab}\mathcal{D}^b\mathcal{D}_c v^{ac} \\
&- \frac{16}{9}M^I v^{ac}v_{cb}R_a{}^b - \frac{2}{9}M^I v^2 R + \frac{4}{3}M^I\mathcal{D}^a v^{bc}\mathcal{D}_a v_{bc} + \frac{4}{3}M^I\mathcal{D}^a v^{bc}\mathcal{D}_b v_{ca} \\
&- \frac{2}{3}M^I\epsilon_{abcde}v^{ab}v^{cd}\mathcal{D}_f v^{ef} + \frac{2}{3}F^{Iab}\epsilon_{abcde}v^{cf}\mathcal{D}_f v^{de} + F^{Iab}\epsilon_{abcde}v^c{}_f\mathcal{D}^d v^{ef} \\
&- \frac{4}{3}F^{Iab}v_{ac}v^{cd}v_{db} - \frac{1}{3}F^{Iab}v_{ab}v^2 + 4M^I v_{ab}v^{bc}v_{cd}v^{da} - M^I(v^2)^2\Big) ,
\end{aligned}
\tag{2.13}
$$

where the overall coefficient in $\mathcal{L}_1$ is fixed by the anomaly cancelation condition (2.1). The symbol $\mathcal{D}_a$ now refers to the usual covariant derivative of general relativity and should not be confused with the conformal covariant derivatives of the previous sections. Indeed, the presence of the auxiliary fields D and v_{ab} are the only remnants of the superconformal formalism.

The supersymmetry transformations are also affected by the gauge fixing. In particular the parameter η^i associated to **S**-supersymmetry is fixed. The BPS condition for the hypermultiplet fermion follows from (2.7)

$$
\gamma^a\hat{\mathcal{D}}_a\mathcal{A}_j^\alpha\epsilon^j - \gamma\cdot v\mathcal{A}_j^\alpha\epsilon^j + 3\mathcal{A}_j^\alpha\eta^j = 0 \ . \tag{2.14}
$$

For the field configuration (2.11), we can solve (2.14) for η^i,

$$
\eta^i = \frac{1}{3}\gamma\cdot v\epsilon^i \ . \tag{2.15}
$$

Replacing (2.15) in the transformation rules for the remaining fermionic fields, one obtains the following the residual supersymmetry transformations[4]

[4]We now leave the i indices implicit since they play very little role in what follows. See [73] for a discussion of this point.

$$\delta\psi_\mu = \left(\mathcal{D}_\mu + \frac{1}{2}v^{ab}\gamma_{\mu ab} - \frac{1}{3}\gamma_\mu\gamma\cdot v\right)\epsilon\ , \tag{2.16a}$$

$$\delta\Omega^I = \left(-\frac{1}{4}\gamma\cdot F^I - \frac{1}{2}\gamma^a\partial_a M^I - \frac{1}{3}M^I\gamma\cdot v\right)\epsilon\ , \tag{2.16b}$$

$$\delta\chi = \left(D - 2\gamma^c\gamma^{ab}\mathcal{D}_a v_{bc} - 2\gamma^a\epsilon_{abcde}v^{bc}v^{de} + \frac{4}{3}(\gamma\cdot v)^2\right)\epsilon\ . \tag{2.16c}$$

It is the vanishing of these transformations which constitute the BPS conditions in the off-shell Poincaré supergravity.

Summarizing, we provided an off-shell Lagrangian and appropriate supersymmetry transformations for five-dimensional supergravity with R^2 terms. This forms the starting point for our detailed analysis of corrections to black holes and similar objects. The theory is described by the action

$$S = \frac{1}{4\pi^2}\int d^5x\sqrt{g}\left(\mathcal{L}_0 + \mathcal{L}_1\right)\ , \tag{2.17}$$

where the bosonic part of the leading (two-derivative) Lagrangian is (2.12) and the bosonic higher derivative corrections are described by (2.13). The supersymmetry variations of the fermionic fields around bosonic backgrounds are given by (2.16).

Note that the four-derivative Lagrangian (2.13) is proportional to the constants c_{2I}, which can be thought of as the effective expansion parameters of the theory. Furthermore, the expansion coefficients c_{2I} make no appearance in the supersymmetry transformations (2.16) for the supersymmetry algebra is completely off-shell, *i.e.* independent of the action of the theory.

2.1.3 Integrating out the auxiliary fields

We have termed the fields v_{ab}, D, and χ as auxiliary fields. This nomenclature is clear from the viewpoint of the superconformal symmetry of Section 2.1, where these fields were added to compensate for the mismatch between the number of bosonic and fermionic degrees of freedom. However, focusing on the bosonic fields, from the point of view of the leading-order action (2.12) the fields v_{ab} and D are also auxiliary variables in the sense of possessing algebraic equations of motion. It can be easily seen that substituting the equations of motion for v_{ab} and D into (2.12) leads to the on-shell two-derivative supergravity Lagrangian (A.15)

$$\mathcal{L} = -R - G_{IJ}\partial_a M^I \partial^a M^J - \frac{1}{2} G_{IJ} F^I_{ab} F^{Jab} + \frac{1}{24} c_{IJK} A^I_a F^J_{bc} F^K_{de} \epsilon^{abcde} \ , \tag{2.18}$$

with G_{IJ} metric on the scalar moduli space [74]

$$G_{IJ} = -\frac{1}{2}\, \partial_I \partial_J (\ln \mathcal{N})|_{\mathcal{N}=1} = \frac{1}{2} \left(\mathcal{N}_I \mathcal{N}_J - \mathcal{N}_{IJ} \right) \ . \tag{2.19}$$

When the higher-derivative corrections encapsulated in (2.13) are taken into account, the two-form v_{ab} no longer has an algebraic equation of motion. It seems fair to now ask in what sense it is still an auxiliary field. To sensibly interpret this, we must recall that the Lagrangian including stringy corrections should be understood as an effective Lagrangian, *i.e.* part of a derivative expansion suppressed by powers of the five-dimensional Planck scale. Thus, it is only sensible to integrate out the auxiliary fields iteratively, in an expansion in inverse powers of the Planck mass or, equivalently, in powers of the constants c_{2I}.

2.1.4 Comments on field redefinitions

In higher-derivative theories of gravity, the precise form of the Lagrangian is ambiguous due to possible field redefinitions. For example, one may consider

$$g_{\mu\nu} \to g_{\mu\nu} + aRg_{\mu\nu} + bR_{\mu\nu} + \dots \ , \tag{2.20}$$

for some dimensionful constants a and b, or generalizations involving the matter fields. Field redefinitions leave the leading order Einstein-Hilbert action invariant, but can change the coefficients and form of the R^2 terms. Since they mix terms of different orders in derivatives it is generally ambiguous to label certain terms as “two-derivative” or “higher-derivative”.

One of the advantages of the off-shell formalism we employ is that it addresses these ambiguities. The reason is that the off-shell supersymmetry transformations are independent of the action, yet they do not mix different orders in derivatives (if we assign the auxiliary fields v_{ab} and D derivative orders of one and two, respectively). General field redefinitions of the form (2.20) would modify the supersymmetry algebra and mix orders of derivatives. Thus if we restrict to variables where the supersymmetry transformations take their off-shell forms, *e.g.* in (2.16), then most of the field redefinition ambiguity is fixed. In our formalism it is therefore meaningful to label terms by their order in derivatives [65].

2.1.5 Modified very special geometry

In the on-shell theory, there is a constraint (A.13) imposed by hand

$$\mathcal{N} \equiv \frac{1}{6} c_{IJK} M^I M^J M^K = 1 \ . \tag{2.21}$$

This is known as the *very special geometry* constraint and indicates that not all of the Kähler moduli are independent fields. Interestingly, the off-shell formalism does not require this to be imposed externally. Rather, the equation of motion for D following from the two-derivative Lagrangian (2.12) is precisely this condition. This immediately implies that (2.21) does not hold in the presence of higher-derivative corrections since D also appears in the four-derivative Lagrangian (2.13). Indeed, the D equation of motion following from the full Lagrangian $\mathcal{L}_0 + \mathcal{L}_1$ is

$$\mathcal{N} = 1 - \frac{c_{2I}}{72} \left[F^I_{ab} v^{ab} + M^I D \right] \ . \tag{2.22}$$

Very special geometry is an interesting mathematical structure in its own right and the modified very special geometry is also likely to be an interesting structure, but we will have little to say about that here. Indeed, it would be of much interest to explore this topic further. For the present purposes, we use (2.22) as just another equation in specifying our solutions.

2.1.6 Isometries and projections on Killing spinors

In the following we will be investigating supersymmetric solutions to the theory described above. While we will consider maximally supersymmetric solutions, for which the supersymmetry parameter ϵ in the BPS conditions is understood to be unconstrained, we will also discuss asymptotically flat solutions such as black holes and black strings. These asymptotically flat solutions break some fraction of supersymmetry and so ϵ is expected to satisfy some sort of projective constraint(s). We can derive this projection in the following way (analogous to that of [75] in the on-shell formalism) which is generally applicable. Assume the existence of some spinor ϵ satisfying the BPS condition from the gravitino variation

$$\left[\mathcal{D}_\mu + \frac{1}{6}v^{ab}e_\mu^{\ c}\left(\gamma_{abc} - 4\eta_{ac}\gamma_b\right)\right]\epsilon = 0 \ . \tag{2.23}$$

Now define the vector, $V_\mu = -\bar{\epsilon}\gamma_\mu\epsilon$ and use (2.23) to compute its covariant derivative

$$\begin{aligned} \mathcal{D}_\mu V_\nu &= -\frac{1}{6}v^{ab}e_\mu^{\ c}e_\nu^{\ d}\bar{\epsilon}\left([\gamma_{abc},\gamma_d] + 4\eta_{ac}\{\gamma_b,\gamma_d\}\right)\epsilon \ , \\ &= -\frac{1}{6}v^{ab}e_\mu^{\ c}e_\nu^{\ d}\bar{\epsilon}\left(\gamma_{abcd} + 8\eta_{ac}\eta_{bd}\right)\epsilon \ . \end{aligned} \tag{2.24}$$

The right-hand side in the second line is anti-symmetric under exchange of μ and ν, thus V_μ is a Killing vector. One can now use various Fierz identities [75] to derive a projection obeyed by the Killing spinor

$$V^\mu\gamma_\mu\epsilon = -f\epsilon \ , \tag{2.25}$$

where $f = \sqrt{V^\mu V_\mu}$. Since there is only one condition on ϵ, this argument leads to solutions which preserve half of the supersymmetries.

The details of the supersymmetry analysis are qualitatively different for solutions with a null isometry ($f^2 = 0$) and those with a timelike isometry ($f^2 > 0$). We will study these two cases in turn in subsequent sections. The analysis proceeds as follows. One introduces a metric *ansatz* with an isometry that we identify with $V_\mu = -\bar{\epsilon}\gamma_\mu\epsilon$. This determines a projection obeyed by the Killing spinor via (2.25). One then uses the BPS conditions to obtain as much information as possible about the undetermined functions of the *ansatz*. In the off-shell formalism, the results of this analysis are completely independent of the action. Equations of motion, which do of course depend on the precise form of the action, are then imposed as needed to completely specify the solution.

2.2 Black Strings and Null Supersymmetry

We now begin investigating asymptotically flat solutions which preserve only a fraction of the supersymmetry of the theory. We begin with black string solutions, which were first discussed in [56]. In particular, we will study corrections to the Calabi-Yau black strings studied in [57]. These solutions each have at least one null isometry so we will determine the off-shell supersymmetry conditions for any such spacetime. The conditions from supersymmetry do not completely specify the solution and we will require more conditions on the functions in our *ansatz*, including equations of motion from the full higher-derivative Lagrangian. We will specialize to purely magnetically charged strings which carry no momentum along their length; this is precisely the case studied in [39]. For a discussion on the more general case see [40]. Under these assumptions we will only need to use the equation of motion for D and the Bianchi identity for F^I to completely specify the solution.

2.2.1 Ansatz: magnetic background

We are interested in supersymmetric black string solutions carrying magnetic charges p^I. We assume translation invariance along the string, and spherical symmetry in the transverse directions. To make these symmetries explicit, we write our *ansatz* as

$$ds^2 = e^{2U_1(r)}\left(dt^2 - dx_4^2\right) - e^{-4U_2(r)}dx^i dx^i \ , \quad dx^i dx^i = dr^2 + r^2 d\Omega_2^2 \ , \tag{2.26}$$

where $i = 1, 2, 3$. The two-forms F^I and v will be proportional to the volume form on S^2. We chose the vielbein as

$$e^{\hat{a}} = e^{U_1} dx^a \ , \qquad a = 0, 4 \ ,$$
$$e^{\hat{i}} = e^{-2U_2} dx^i \ , \quad i = 1, 2, 3 \ . \tag{2.27}$$

The non-trivial spin connections are

$$\omega_a{}^{\hat{a}\hat{i}} = -e^{U_1+2U_2}\partial_i U_1 \ , \ \omega_k{}^{\hat{i}\hat{j}} = 2\delta^i_k \partial_j U_2 - 2\delta^j_k \partial_i U_2 \ . \tag{2.28}$$

2.2.2 Supersymmetry conditions

We start the construction of our solutions by requiring that the supersymmetry conditions (2.16) vanish in the background (2.26). The supersymmmetry parameter ϵ is constant along the string and obeys

$$\gamma_{\hat{t}\hat{4}}\epsilon = -\epsilon \ . \tag{2.29}$$

Gravitino variation. We first analyze the gravitino variation (2.16a) set equal to zero

$$\delta\psi_\mu = \left(\mathcal{D}_\mu + \frac{1}{2}v^{ab}\gamma_{\mu ab} - \frac{1}{3}\gamma_\mu \gamma \cdot v\right)\epsilon = 0 \ . \tag{2.30}$$

For the background (2.26), the covariant derivative is

$$\mathcal{D}_a = \partial_a - \frac{1}{2}e^{U_1+2U_2}\partial_i U_1 \gamma_{\hat{a}\hat{i}} \ ,$$
$$\mathcal{D}_i = \partial_i + \partial_j U_2 \gamma_{\hat{i}\hat{j}} \ . \tag{2.31}$$

Along the string, equation (2.30) simplifies to

$$\left[-\frac{1}{2}e^{U_1+2U_2}\partial_i U_1 \gamma_{\hat{a}\hat{i}} + \frac{1}{6}e^{U_1} v_{\hat{i}\hat{j}}\gamma_{\hat{a}\hat{i}\hat{j}}\right]\epsilon = 0 \ . \tag{2.32}$$

It is convenient to use the projection (2.29) in the form

$$\gamma_{\hat{i}\hat{j}\hat{k}}\epsilon = -\varepsilon_{ijk}\epsilon \ , \tag{2.33}$$

where $\varepsilon_{123} = 1$. Then

$$\gamma_{\hat{i}\hat{j}}\epsilon = \gamma^{\hat{k}}\gamma_{\hat{i}\hat{j}\hat{k}}\epsilon = \varepsilon_{ijk}\gamma_{\hat{k}}\epsilon \ . \tag{2.34}$$

Now (2.32) becomes

$$\left[-\frac{1}{2}e^{U_1+2U_2}\partial_k U_1 + \frac{1}{6}e^{U_1}v_{\hat{i}\hat{j}}\varepsilon_{ijk}\right]\gamma_{\hat{a}\hat{k}}\epsilon = 0 \ , \tag{2.35}$$

from which we can solve for the auxiliary field,

$$v_{\hat{i}\hat{j}} = \frac{3}{2}e^{2U_2}\varepsilon_{ijk}\partial_k U_1 \ , \tag{2.36}$$

or in coordinate frame

$$v_{ij} = \frac{3}{2}e^{-2U_2}\varepsilon_{ijk}\partial_k U_1 \ . \tag{2.37}$$

Consider now the components of the gravitino variation along x^i,

$$\left[\partial_i + \partial_j U_2 \gamma_{\hat{i}\hat{j}} + \frac{1}{2}v_{\hat{j}\hat{k}}\left(\gamma_{i\hat{j}\hat{k}} - \frac{2}{3}\gamma_i\gamma_{\hat{j}\hat{k}}\right)\right]\epsilon = 0 \ . \tag{2.38}$$

The $v_{\hat{j}\hat{k}}$ terms split into a "radial" part where either j, k is equal to i, and an "angular" part where $i \neq j \neq k$. Thus we have two conditions

$$0 = \left(\partial_i - \frac{1}{6}\varepsilon_{ijk}e_i{}^{\hat{i}}v_{\hat{j}\hat{k}}\right)\epsilon \ , \tag{2.39a}$$

$$0 = \left(\partial_j U_2 \varepsilon_{ijk} + \frac{2}{3}v_{\hat{i}\hat{k}}e_i^{\hat{i}}\right)\gamma_{\hat{k}}\epsilon \ , \tag{2.39b}$$

where there is no summation over i. Equation (2.39b) leads to $U_2 = U_1$, so we will drop the subscripts on U from now on. Solving (2.39a) then leads to

$$\left(\partial_i - \frac{1}{2}\partial_i U\right)\epsilon = 0 \ , \tag{2.40}$$

and so the Killing spinor takes the form

$$\epsilon = e^{U/2}\epsilon_0 \ , \tag{2.41}$$

where ϵ_0 is some constant spinor.

It will be convenient to use cylindrical coordinates from now on. The metric takes the form

$$ds^2 = e^{2U}\left(dt^2 - dx_4^2\right) - e^{-4U}(dr^2 + r^2 d\Omega_2^2) \ , \tag{2.42}$$

in terms of a single function $U(r)$. The coordinate frame expression (2.37) is a tensor statement on the 3-dimensional base space, where ε_{abc} is a completely anti-symmetric tensor with components $\pm\sqrt{g}$. So in cylindrical coordinates the auxiliary two-form is

$$v_{\theta\phi} = \frac{3}{2} e^{-2U} r^2 \sin\theta \partial_r U \ , \quad v_{\hat{\theta}\hat{\phi}} = \frac{3}{2} e^{2U} \partial_r U \ , \tag{2.43}$$

with other components vanishing due to spherical symmetry in the transverse space. The projection (2.29) in cylindrical coordinates can be written as

$$\gamma_{\hat{r}\hat{\theta}\hat{\phi}} \epsilon = -\epsilon \ . \tag{2.44}$$

Gaugino variation. Evaluated on the magnetic background, the gaugino variation $\delta\Omega^I = 0$ in (2.16b) gives

$$\left(\gamma_{\hat{\theta}\hat{\phi}} F^{I\hat{\theta}\hat{\phi}} + \gamma^{\hat{r}} e^r_{\hat{r}} \partial_r M^I + \frac{4}{3} M^I \gamma_{\hat{\theta}\hat{\phi}} v^{\hat{\theta}\hat{\phi}} \right) \epsilon = 0 \ . \tag{2.45}$$

Using (2.44) and solving for the field strength we get

$$\begin{aligned} F^{I\hat{\theta}\hat{\phi}} &= e^{2U} \partial_r M^I - \frac{4}{3} M^I v^{\hat{\theta}\hat{\phi}} \\ &= \partial_r (M^I e^{-2U}) e^{4U} \ . \end{aligned} \tag{2.46}$$

In coordinate frame, (2.46) becomes

$$F^I_{\theta\phi} = \partial_r (M^I e^{-2U}) r^2 \sin\theta \ . \tag{2.47}$$

This equation is the first hint of the expected attractor behavior: the flow of the scalars M^I is completely determined by the magnetic field F^I.

Auxiliary fermion variation. The last supersymmetry variation to solve is $\delta\chi = 0$ in (2.16c). Neglecting the ϵ-terms since we look for parity invariant solutions, this condition is

$$\left(D - 2\gamma^c \gamma^{ab} \mathcal{D}_a v_{bc} + \frac{4}{3} (\gamma \cdot v)^2 \right) \epsilon = 0 \ . \tag{2.48}$$

The relevant components of the covariant derivative of v for the contraction in (2.48) are

$$\mathcal{D}_\theta v_{r\phi} = \mathcal{D}_\phi v_{\theta r} = -\Gamma^\theta_{\theta r} v_{\theta\phi} \ , \ \mathcal{D}_r v_{\theta\phi} = \partial_r v_{\theta\phi} - 2\Gamma^\theta_{\theta r} v_{\theta\phi} \ , \tag{2.49}$$

with

$$\Gamma^\theta_{\theta r} = \Gamma^\phi_{\phi r} = -2\partial_r U + \frac{1}{r} \ . \tag{2.50}$$

Then, the second term in (2.48) becomes

$$\begin{aligned} \gamma^c\gamma^{ab}\mathcal{D}_a v_{bc} &= e^r_{\hat{r}} e^\theta_{\hat{\theta}} e^\phi_{\hat{\phi}} \left(-4\mathcal{D}_\theta v_{r\phi} + 2\mathcal{D}_r v_{\theta\phi}\right) \gamma^{\hat{r}\hat{\theta}\hat{\phi}} \\ &= 2\frac{e^{6U}}{r^2 \sin\theta} \partial_r v_{\theta\phi} \gamma^{\hat{r}\hat{\theta}\hat{\phi}} \ , \\ &= \frac{3}{2} e^{6U} \nabla^2 (e^{-2U}) \gamma_{\hat{r}\hat{\theta}\hat{\phi}} \ , \end{aligned} \tag{2.51}$$

with $\nabla^2 = \partial_i\partial_i = r^{-2}\partial_r(r^2\partial_r)$ due to spherical symmetry. Inserting (2.51) in (2.48) we have

$$\left(D - 3e^{6U}\nabla^2(e^{-2U})\gamma_{\hat{r}\hat{\theta}\hat{\phi}} - \frac{16}{3}(v_{\hat{\theta}\hat{\phi}})^2\right)\epsilon = 0 \ , \tag{2.52}$$

where we used

$$(\gamma \cdot v)^2 = -4(v_{\hat{\theta}\hat{\phi}})^2 \ . \tag{2.53}$$

Using the projection (2.44) and substituting the auxiliary field (2.43) into (2.52) we find

$$\begin{aligned} D &= -3e^{6U}\nabla^2(e^{-2U}) + \frac{16}{3}(v_{\hat{\theta}\hat{\phi}})^2 \\ &= 3e^{6U}\left(-\nabla^2(e^{-2U}) + 4e^{-2U}(\nabla U)^2\right) \\ &= 6e^{4U}\nabla^2 U \ . \end{aligned} \tag{2.54}$$

What we have found so far is that supersymmetry demands a metric of the form (2.42), an auxiliary two tensor of the form (2.43), the gauge field strengths (2.47), and the auxiliary D-field (2.54). All told the entire solution is now specified in terms of the functions M^I and U which are not fixed by supersymmetry alone.

2.2.3 Equations of motion

Having exhausted the implications of unbroken supersymmetry, we now need to use information from the equations of motion.

Maxwell's equations. Any specific string solution is parametrized by the values of the magnetic charges as measured by surface integrals at infinity. These in turn determine the gauge fields in the interior via the Maxwell equations.

We first consider the equation of motion

$$\partial_\theta \left(\sqrt{g} \frac{\partial \mathcal{L}}{\partial F^I_{\theta\phi}} \right) = 0 \ . \tag{2.55}$$

Spherical symmetry implies that the expression in parenthesis is a function of r only, hence (2.55) is satisfied identically for any field strength $F^I_{\theta\phi} = \mathcal{F}^I(r)\sin\theta$. Thus we get no new information from this equation of motion.

In the magnetic case the nontrivial condition arises from the Bianchi identity $dF^I = 0$. The point is that the expression (2.46) for F^I determined from supersymmetry is not automatically a closed form. Therefore, the Bianchi identity

$$\partial_r F^I_{\theta\phi} = \partial_r \left(r^2 \partial_r (M^I e^{-2U}) \right) \sin\theta = 0 \ , \tag{2.56}$$

is nontrivial. Physically, this is because supersymmetry is consistent with any extended distribution of magnetic charges, while here we are demanding the absence of charge away from the origin. The equation (2.56) integrates to

$$r^2 \partial_r (M^I e^{-2U}) = -\frac{p^I}{2} \ , \tag{2.57}$$

where p^I is the quantized magnetic charge carried by F^I. We note that the field strength

$$F^I = -\frac{p^I}{2} \epsilon_2 \ , \tag{2.58}$$

with ϵ_2 the volume form on the unit S^2. Note that F^I does not get modified after including higher derivatives, since the magnetic charge is topological.

The solutions to (2.57) are harmonic functions on the three-dimensional base space. We are just interested in the simplest solution

$$M^I e^{-2U} = H^I = M^I_\infty + \frac{p^I}{2r} \ , \tag{2.59}$$

with M^I_∞ the value of M^I in the asymptotically flat region where $U = 0$.

D equation. So far, by imposing the conditions for supersymmetry and integrating the Bianchi identity, we have been able to write our solution in terms of one unknown function $U(r)$. To determine this remaining function we use the equation of motion for the auxiliary field D. Inspecting (2.12) and (2.13) we see that the only D-dependent terms in the Lagrangian are

$$\mathcal{L}_D = \frac{1}{2}(\mathcal{N} - 1)D + \frac{c_{2I}}{24}\left(\frac{1}{12}M^I D^2 + \frac{1}{6}F^{Iab}v_{ab}D\right) \ . \tag{2.60}$$

Therefore, the equation of motion for D is

$$\mathcal{N} = 1 - \frac{c_{2I}}{72}\left(F^I_{ab}v^{ab} + M^I D\right) \ . \tag{2.61}$$

Inserting the gauge-field (2.47), the auxiliary field (2.43), and the D-field (2.54) gives

$$e^{-6U} = \frac{1}{6}c_{IJK}H^I H^J H^K + \frac{c_{2I}}{24}\left(\nabla H^I \nabla U + 2H^I \nabla^2 U\right) \ . \tag{2.62}$$

Here H^I are the harmonic functions defined in (2.59) and we used

$$\mathcal{N} = \frac{1}{6}c_{IJK}H^I H^J H^K e^{6U} \ . \tag{2.63}$$

The D constraint (2.62) is now an ordinary differential equation that determines $U(r)$. Its solution specifies the entire geometry and all the matter fields.

We can solve (2.62) exactly in the near horizon region. This case corresponds to vanishing integration constants in (2.59) so that

$$H^I = \frac{p^I}{2r} \ . \tag{2.64}$$

Then (2.62) gives

$$e^{-6U} = \frac{1}{8r^3}\left(p^3 + \frac{1}{12}c_2 \cdot p\right) = \frac{\ell_S^3}{r^3} \ , \tag{2.65}$$

where $p^3 = \frac{1}{6}c_{IJK}p^Ip^Jp^K$. The geometry in this case is $\mathrm{AdS}_3 \times S^2$ with the scale ℓ_S.

The asymptotically flat solutions to (2.62) cannot in general be found in closed form. In the following two subsections we discuss an approximate solution and an example of numerical integration.

2.2.4 Corrected geometry for large black strings

One way to find solutions to (2.62) is by perturbation theory. This strategy captures the correct physics when the solution is regular already in the leading order theory, *i.e.* for large black strings. Accordingly, the starting point is the familiar solution

$$e^{-6U_0} = \frac{1}{6}c_{IJK}H^IH^JH^K \ , \tag{2.66}$$

to the two-derivative theory. This solves (2.62) with $c_{2I} = 0$.

Although c_{2I} is not small it will be multiplied by terms that are of higher order in the derivative expansion. It is therefore meaningful to expand the full solution to (2.62) in the form

$$e^{-6U} = e^{-6U_0} + c_{2I}\varepsilon^I + \frac{1}{2}c_{2I}c_{2J}\varepsilon^{IJ} + \dots \ , \tag{2.67}$$

where $\varepsilon^I(r), \varepsilon^{IJ}(r), \dots$ determine the corrected geometry with increasing precision.

Inserting (2.67) in (2.62) and keeping only the terms linear in c_{2I} we find the first order correction[5]

$$\varepsilon^I = \frac{1}{24}(\nabla H^I \cdot \nabla U_0 + 2H^I\nabla^2 U_0) \ . \tag{2.68}$$

[5] It is understood that the correction ε^I is only defined in the combination $c_{2I}\varepsilon^I$.

Iterating, we find the second order correction

$$\varepsilon^{IJ} = -\frac{1}{72}\left(\nabla H^I \cdot \nabla(e^{6U_0}\varepsilon^J) + 2H^I\nabla^2(e^{6U_0}\varepsilon^J)\right) \ , \tag{2.69}$$

where the first order correction ε^I is given by (2.68). Higher orders can be computed similarly. In summary, we find that starting from a smooth solution to the two-derivative theory we can systematically and explicitly compute the higher order corrections. The series is expected to be uniformly convergent.

In the near horizon limit (2.64), the full solution (2.65) is recovered exactly when taking the leading correction (2.68) into account. As indicated by (2.65), the effect of the higher derivative corrections is to expand the sphere by a specific amount (which is small for large charges). The perturbative solution gives approximate expressions for the corrections also in the bulk of the solution. Numerical analysis indicates that the corrections remain positive so at any value of the isotropic coordinate r the corresponding sphere is expanded by a specific amount.

In this section we have focused on large black strings, that is, those which are non-singular in the leading supergravity description. In the next section we turn to *small strings*, particularly the important case of *fundamental strings*.

2.2.5 Small Black Holes and Strings

One of the main motivations for studying higher derivative corrections is their potential to regularize geometries that are singular in the lowest order supergravity approximation [24, 36, 37, 38, 76]. One version of this phenomenon occurs for black holes possessing a nonzero entropy, where the effect of the higher derivative terms is not to remove the black hole singularity, but rather to shield it with an event horizon. The resulting spacetime is then qualitatively similar to that of an ordinary "large" black hole. Examples of this occur for both four and five dimensional black holes in string theory. A second, and in many ways more striking, example pertains to the case in which the solution has a vanishing entropy.

In this case the singularity, instead of being shielded by a finite size event horizon, is smoothed out entirely. Our five dimensional string solutions provide an explicit realization of this.

To realize the latter type of solutions, we consider magnetic string solutions whose charge configurations satisfy $p^3 = \frac{1}{6}c_{IJK}p^I p^J p^K = 0$. We refer to these as *small strings*. From (2.65) (see also Section 3.1.2) our string solutions had a near horizon $\mathrm{AdS}_3 \times S^2$ geometry with AdS scale size given by

$$\ell_A^3 = p^3 + \frac{1}{12}c_2 \cdot p \ . \tag{2.70}$$

For small strings the geometry is singular in the two derivative approximation, since $\ell_A = 0$. Conversely, $\ell_A \neq 0$ when the correction proportional to c_{2I} is taken into account. Thus it appears that a spacetime singularity has been resolved. To understand the causal structure we can note that our metric is a particular example of the general class of geometries studied in [56]. The resulting Penrose diagram is like that of the M5-brane in eleven dimensions. In particular, the geometry is completely smooth, and there is no finite entropy event horizon.

We should, however, close one potential loophole. In principle, it could be that the actual near string geometry realized in the full asymptotically flat solution is not the regular solution that is consistent with the charges, but instead a deformed but still singular geometry. In order to exclude this possibility we must construct the complete solution that smoothly interpolates between the regular near horizon geometry and asymptotically flat space. In this section we present such an interpolating solution, thereby confirming that the singularity is indeed smoothed out.

Since the near string geometry after corrections are taken into account has an AdS_3 factor, it is natural to ask whether the AdS/CFT correspondence applies, and to determine what special features the holography might exhibit. This question has attracted significant attention recently and remains an active area of inquiry [77, 78, 73, 79].

A particularly important example of a small string is obtained when the Calabi-Yau is $K3 \times T^2$ (see footnote (9) for some caveats on these compactifications), and the only magnetic charge that is turned on is that corresponding to an $M5$-brane wrapping the $K3$. The resulting 5D string is then dual, via IIA-heterotic duality, to the fundamental heterotic string [35, 80]. In the following we will construct the explicit solution for this particular example.

Let M^1 be the single modulus on the torus and M^i be the moduli of $K3$ where $i = 2, \ldots, 23$. The charge configuration of interest specifies the harmonic functions as

$$\begin{aligned} H^1 &= M^1_\infty + \frac{p^1}{2r} \ , \\ H^i &= M^i_\infty \ , \ \ i = 2, \ldots, 23 \ . \end{aligned} \tag{2.71}$$

The only non-vanishing intersection numbers are $c_{1ij} = c_{ij}$ where c_{ij} is the intersection matrix for $K3$. To simplify, we choose M^i_∞ consistent with $\frac{1}{2} c_{ij} M^i_\infty M^j_\infty = 1$, so that (2.63) becomes

$$\mathcal{N} e^{-6U} = \frac{1}{6} c_{IJK} H^I H^J H^K = H^1 \ . \tag{2.72}$$

The master equation (2.62) now reads

$$H^1 = e^{-6U} - \left[\partial_r H^1 \partial_r U + 2H^1 \ \frac{1}{r^2} \partial_r (r^2 \partial_r U) \right] \ , \tag{2.73}$$

where we used $c_2(K3) = 24$ and $c_{2i} = 0$. We can write this more explicitly as

$$1 + \frac{p^1}{2r} = e^{-6U} - 2(1 + \frac{p^1}{2r}) U'' - \frac{4}{r} \left(1 + \frac{3p^1}{8r} \right) U' \ , \tag{2.74}$$

where primes denote derivatives with respect to r. Note that we set $M^1_\infty = 1$; a general value can be restored by a rescaling of p^1 and a shift of U.

In our units distance r is measured in units of the 5D Planck length. The parameter p^1 is a pure number counting the fundamental strings. For a given p^1, it is straightforward to integrate (2.74) numerically. Instead, to gain some analytical insight we will take $p^1 \gg 1$

so as to have an expansion parameter. We will analyze the problem one region at a time.

The $\mathbf{AdS}_3 \times S^2$-region. This is the leading order behavior close to the string. According to our magnetic attractor solution in the form (2.65) we expect the precise asymptotics

$$e^{-6U} \to \frac{\ell_S^3}{r^3} \ , \ r \to 0 \ , \tag{2.75}$$

where the S^2-radius is given by

$$\ell_S = \left(\frac{p^1}{4}\right)^{1/3} \ . \tag{2.76}$$

For $p^1 \gg 1$ this is much smaller than the scale size of a large string, which from (2.70) has scale $\sim p$. However, it is nevertheless much larger than the 5D Planck scale. The modulus describing the volume of the internal T^2 is

$$M^1 = \frac{p^1}{2\ell_S} = 2^{-1/3}(p^1)^{2/3} \ , \tag{2.77}$$

which also corresponds to the length scale $(p^1)^{1/3}$.

The near-string region. We next seek a solution in the entire range $r \ll p^1$ which includes the scale (2.76) but reaches further out. In fact, it may be taken to be all of space in a scaling limit where $p^1 \to \infty$.

In the near string region (2.74) reduces to

$$\frac{p^1}{2r} = e^{-6U} - \frac{p^1}{r}U'' - \frac{3p^1}{2r^2}U' \ . \tag{2.78}$$

We can scale out the string number p^1 by substituting

$$e^{-6U(r)} = \frac{p^1}{4r^3}e^{-6\Delta(r)} \ , \tag{2.79}$$

which amounts to

$$U(r) = \frac{1}{2}\ln\frac{r}{\ell_S} + \Delta(r) \ . \tag{2.80}$$

This gives

$$\Delta'' + \frac{3}{2r}\Delta' + \frac{1}{4r^2}(1 - e^{-6\Delta}) + \frac{1}{2} = 0 \ , \tag{2.81}$$

which describes the geometry in the entire region $r \ll p^1$. The asymptotic behavior at small r is

$$\Delta(r) = -\frac{1}{13}r^2 + \frac{3}{(13)^3}r^4 + \frac{20}{9(13)^4}r^6 + \cdots \ . \tag{2.82}$$

Since $\Delta(r) \to 0$ smoothly as $r \to 0$ we have an analytical description of the approach to the $\mathrm{AdS}_3 \times S^2$ region.

The asymptotic behavior for large r is also smooth. Expanding in $u = \frac{1}{r}$ we find

$$\Delta(r) = -\frac{1}{6}\ln(2r^2) - \frac{1}{36}\frac{1}{r^2} + \frac{13}{12 \cdot 36}\frac{1}{r^4} + \cdots \ . \tag{2.83}$$

It is straightforward to solve (2.81) numerically. Figure 2.1 shows the curve that interpolates between the asymptotic forms (2.82) and (2.83). The oscillatory behavior in the intermediate region is characteristic of higher derivative theories. We comment in more detail below.

In the original variable $U(r)$ the approximation (2.83) gives

$$e^{-6U} = \frac{p^1}{2r}\left(1 + \frac{1}{6r^2} - \frac{1}{6r^4} + \ldots\right) \ , \tag{2.84}$$

for large r. The leading behavior, $e^{-6U} = H^1 \sim \frac{p^1}{2r}$, agrees with the near string behavior in two-derivative supergravity. In the full theory this singular region is replaced by a smooth geometry.

The approach to asymptotically flat space. We still need to analyze the region where r is large, meaning $r \sim p^1$ or larger. Here we encounter some subtleties in matching the solution on to the asymptotically flat region.

In the asymptotic region the full equation (2.73) simplifies to

$$1+\frac{p^1}{2r}=e^{-6U}-2(1+\frac{p^1}{2r})U'' \ . \tag{2.85}$$

Terms with explicit factors of $1/r$ were neglected, but we kept derivatives with respect to r so as to allow for Planck scale structure, even though $r \sim p^1 \gg 1$. Changing variables as

$$e^{-6U}=(1+\frac{p^1}{2r})e^{-6W} \ , \tag{2.86}$$

we find

$$W''=\frac{1}{2}(e^{-6W}-1)\simeq -3W \ . \tag{2.87}$$

The expansion for small W is justified because (2.84) imposes the boundary condition $W \to 0$ for $r \ll p^1$.

The solution $W=0$ expected from two-derivative supergravity is in fact a solution to (2.87), but there are also more general solutions of the form

$$W=A\sin(\sqrt{3}r+\delta) \ . \tag{2.88}$$

The amplitude of this solution is undamped, so it is not really an intrinsic feature of the localized string solution we consider. Instead it is a property of fluctuations about flat space, albeit an unphysical one. The existence of such spurious solutions is a well-known feature of theories with higher derivatives, and is related to the possibility of field redefinitions [36, 37, 38, 81]. In the present context the issue is that the oscillatory solutions can be mapped to zero by a new choice of variables, such as $\tilde{W}=(\nabla^2-3)W$.

To summarize, modulo the one subtlety associated with field redefinitions, we have found a smooth solution interpolating between the near horizon $\mathrm{AdS}_3 \times S^2$ attractor and asymptotically flat space. The solution is completely regular, the causal structure being the same as that of an M5-brane in eleven dimensions. While our result is highly suggestive of the existence of a smooth solution of the full theory with all higher derivative corrections included, we cannot establish with certainty that this is the case. The reason is that for small strings there is no small parameter suppressing even higher derivative terms. Indeed, it is easy to check that in the near horizon region terms in the action with more than four derivatives contribute at the same order as those included in the present analysis. As a result, the precise numerical results for the attractor moduli and scale size are expected to receive corrections of order unity. On the other hand, it seems highly plausible that the solution will remain smooth even after these additional corrections have been taken into account.

2.3 Timelike Supersymmetry – Black Holes and Rings

We now turn to the case in which the Killing vector V^μ is timelike over some region of the solution. This class of solutions includes 5D black holes and black rings. The analysis that follows is mainly taken from [39, 53, 40].

2.3.1 Metric ansatz

We start with a general metric *ansatz* with timelike Killing vector $\frac{\partial}{\partial t}$,

$$ds^2 = e^{4U_1(x)}(dt+\omega)^2 - e^{-2U_2(x)}ds_B^2 \ . \tag{2.89}$$

Here ω is a 1-form on the 4D base B with coordinates x^i with $i = 1, \ldots, 4$. We choose vielbeins

$$e^{\hat{t}} = e^{2U_1}(dt+\omega) \ , \quad e^{\hat{i}} = e^{-U_2}\tilde{e}^{\hat{i}} \ , \tag{2.90}$$

where $\tilde{e}^{\hat{i}}$ are vielbeins for ds_B^2. The corresponding spin connection is

$$\begin{aligned} \omega^{\hat{t}}{}_{\hat{i}} &= 2e^{U_2}\tilde{\nabla}_{\hat{i}}U_1 e^{\hat{t}} + \frac{1}{2}e^{2U_1+U_2}d\omega_{\hat{i}\hat{j}}\tilde{e}^{\hat{j}} \ , \\ \omega^{\hat{i}}{}_{\hat{j}} &= \tilde{\omega}^{\hat{i}}{}_{\hat{j}} + \frac{1}{2}e^{2U_1+2U_2}d\omega_{\hat{i}\hat{j}}e^{\hat{t}} + e^{U_2}\tilde{\nabla}_{\hat{i}}U_2 e^{\hat{j}} - e^{U_2}\tilde{\nabla}_{\hat{j}}U_2 e^{\hat{i}} \ . \end{aligned} \tag{2.91}$$

We will adopt the following convention for hatted indices. Hatted indices of five dimensional tensors are orthonormal with respect to the full 5D metric, whereas those of tensors defined on the base space are orthonormal with respect to ds_B^2. For example, $d\omega$ is defined to live on the base, and so obeys $d\omega_{ij} = \tilde{e}_i^{\hat{k}}\tilde{e}_j^{\hat{l}}d\omega_{\hat{k}\hat{l}}$. Furthermore, the tilde on $\tilde{\nabla}_{\hat{i}}$ indicates that the $\hat{i}$ index is orthonormal with respect to the base metric. To avoid confusion, we comment below when two different types of hatted indices are used in a single equation.

The Hodge dual on the base space is defined as

$$\star_4\alpha_{\hat{i}\hat{j}} = \frac{1}{2}\epsilon_{\hat{i}\hat{j}\hat{k}\hat{l}}\alpha^{\hat{k}\hat{l}} \ , \tag{2.92}$$

with $\epsilon_{\hat{1}\hat{2}\hat{3}\hat{4}} = 1$. A 2-form on the base space can be decomposed into self-dual and anti-self-dual forms,

$$\alpha = \alpha^+ + \alpha^- \ , \tag{2.93}$$

where $\star_4\alpha^\pm = \pm\alpha^\pm$.

Equation (2.25) tells us to look for supersymmetric solutions with a Killing spinor obeying the projection

$$\gamma^{\hat{t}}\epsilon = -\epsilon \ , \tag{2.94}$$

with a useful alternative form being

$$\alpha^{-\hat{i}\hat{j}}\gamma_{\hat{i}\hat{j}}\epsilon = 0 \ , \tag{2.95}$$

where $\alpha^{-\hat{i}\hat{j}}$ is any two-form that is anti-self-dual on the 4D base space. The strategy we employ is the same as for the null projection discussed in the previous section: we first exhaust the conditions implied by unbroken supersymmetry, and then impose some of the equations of motion or other constraints.

2.3.2 Supersymmetry conditions

There are three supersymmetry conditions we need to solve. Following the same procedure as in the previous section we first impose a vanishing gravitino variation,

$$\delta\psi_\mu = \left[\mathcal{D}_\mu + \frac{1}{2}v^{ab}\gamma_{\mu ab} - \frac{1}{3}\gamma_\mu\gamma\cdot v\right]\epsilon = 0 \ . \tag{2.96}$$

Evaluated in our background, the time component of equation (2.96) reads

$$\left[\partial_t - e^{2U_1+U_2}\partial_i U_1\gamma_{\hat{i}} - \frac{2}{3}e^{2U_1}v^{\hat{t}\hat{i}}\gamma_{\hat{i}} - \frac{1}{4}e^{4U_1+2U_2}d\omega_{ij}\gamma^{\hat{i}\hat{j}} - \frac{1}{6}e^{2U_1}v_{\hat{i}\hat{j}}\gamma^{\hat{i}\hat{j}}\right]\epsilon = 0 \ , \tag{2.97}$$

where we used the projection (2.94). The terms proportional to $\gamma_{\hat{i}}$ and $\gamma_{\hat{i}\hat{j}}$ give the conditions

$$\begin{aligned} v_{\hat{t}\hat{i}} &= \frac{3}{2}e^{U_2}\tilde{\nabla}_{\hat{i}}U_1 \ , \\ v^+_{\hat{i}\hat{j}} &= -\frac{3}{4}e^{2U_1}d\omega^+_{\hat{i}\hat{j}} \ . \end{aligned} \tag{2.98}$$

The spatial component of the gravitino variation (2.96) simplifies to

$$\left[\tilde{\nabla}_i + \frac{1}{2}\partial_j U_2\gamma_{\hat{i}\hat{j}} + v^{\hat{t}\hat{k}}e_i{}^{\hat{j}}\left(\gamma_{\hat{j}\hat{k}} - \frac{2}{3}\gamma_{\hat{j}}\gamma_{\hat{k}}\right) - e_i{}^{\hat{k}}\left(v^-_{\hat{k}\hat{j}} + \frac{1}{4}e^{2U_1}d\omega^-_{\hat{k}\hat{j}}\right)\gamma^{\hat{j}}\right]\epsilon = 0 \ , \tag{2.99}$$

where we used the results from (2.98). The last term in (2.99) relates the anti-self-dual pieces of v and $d\omega$,

$$v^-_{\hat{i}\hat{j}} = -\frac{1}{4}e^{2U_1}d\omega^-_{\hat{i}\hat{j}} \ . \tag{2.100}$$

To forestall confusion, we note that in equations (2.98) and (2.100) the indices on v are orthonormal with respect to the full 5D metric, while those on $d\omega$ are orthonormal with respect to the base metric.

The remaining components of (2.99) impose equality of the two metric functions $U_1 = U_2 \equiv U$ and determine the Killing spinor as

$$\epsilon = e^{U(x)}\epsilon_0 \ , \tag{2.101}$$

with ϵ_0 a covariantly constant spinor on the base, $\tilde{\nabla}_i\epsilon_0 = 0$. This implies that the base space is hyperKähler.[6]

The gaugino variation is given by

$$\delta\Omega^I = \left[-\frac{1}{4}\gamma\cdot F^I - \frac{1}{2}\gamma^a\partial_a M^I - \frac{1}{3}M^I\gamma\cdot v\right]\epsilon = 0 \ . \tag{2.102}$$

[6]Recall that there is an implicit $SU(2)$ index on the spinor ϵ. One can then construct three distinct two-forms, $\Phi^{ij}_{ab} = \bar{\epsilon}^i\gamma_{ab}\epsilon^j$, which enjoy an $SU(2)$ algebra. This algebra defines the hyperKähler structure of the base space B; see [75] for details.

This condition determines the electric and self-dual pieces of F^I_{ab},

(2.103)
$$F^{I\hat{t}\hat{i}} = e^{-U}\tilde{\nabla}_{\hat{i}}(e^{2U}M^I) \ ,$$
$$F^{I+} = -\frac{4}{3}M^I v^+ \ .$$

Defining the anti-self-dual form

(2.104)
$$\Theta^I = -e^{2U}M^I d\omega^- + F^{I-} \ ,$$

the field strength can be written as

(2.105)
$$F^I = d(M^I e^{\hat{t}}) + \Theta^I \ .$$

The Bianchi identity implies that Θ^I is closed. We emphasize that Θ^I, or more precisely F^{I-}, is undetermined by supersymmetry. These anti-self-dual components are important for black ring geometries, but vanish for rotating black holes.

Finally, the variation of the auxiliary fermion is

(2.106)
$$\delta\chi = \left[D - 2\gamma^c\gamma^{ab}\mathcal{D}_a v_{bc} - 2\gamma^a\epsilon_{abcde}v^{bc}v^{de} + \frac{4}{3}(\gamma\cdot v)^2\right]\epsilon = 0 \ .$$

Using equations (2.98) and (2.100), the terms proportional to one or two gamma matrices cancel identically. The terms independent of $\gamma_{\hat{i}}$ give an equation for D, which reads

(2.107)
$$D = 3e^{2U}(\tilde{\nabla}^2 U - 6(\tilde{\nabla}U)^2) + \frac{1}{2}e^{8U}(3d\omega^+_{\hat{i}\hat{j}}d\omega^{+\hat{i}\hat{j}} + d\omega^-_{\hat{i}\hat{j}}d\omega^{-\hat{i}\hat{j}}) \ .$$

2.3.3 Maxwell equations

The part of the action containing the gauge fields is

(2.108)
$$S^{(A)} = \frac{1}{4\pi^2}\int d^5x\sqrt{g}\left(\mathcal{L}_0^{(A)} + \mathcal{L}_1^{(A)}\right) \ ,$$

where the two-derivative terms are

(2.109)
$$\mathcal{L}_0^{(A)} = 2\mathcal{N}_I v^{ab}F^I_{ab} + \frac{1}{4}\mathcal{N}_{IJ}F^I_{ab}F^{Jab} + \frac{1}{24}c_{IJK}A^I_a F^J_{bc}F^K_{de}\epsilon^{abcde} \ ,$$

and the four-derivative contributions are

$$\mathcal{L}_1^{(A)} = \frac{c_{2I}}{24}\Big(\frac{1}{16}\epsilon^{abcde}A_a^I R_{bc}{}^{fg}R_{defg} + \frac{2}{3}\epsilon_{abcde}F^{Iab}v^{cf}\mathcal{D}_f v^{de} + \epsilon_{abcde}F^{Iab}v^c{}_f\mathcal{D}^d v^{ef} \\ + \frac{1}{6}F^{Iab}v_{ab}D + \frac{1}{2}F^{Iab}C_{abcd}v^{cd} - \frac{4}{3}F^{Iab}v_{ac}v^{cd}v_{db} - \frac{1}{3}F^{Iab}v_{ab}v^2\Big) \ . \tag{2.110}$$

Variation of (2.108) with respect to A_μ^I gives,

$$\nabla_\mu\left(4\mathcal{N}_I v^{\mu\nu} + \mathcal{N}_{IJ}F^{J\mu\nu} + 2\frac{\delta\mathcal{L}_1}{\delta F_{\mu\nu}^I}\right) \\ = \frac{1}{8}c_{IJK}F_{\alpha\beta}^J F_{\sigma\rho}^K\epsilon^{\nu\alpha\beta\sigma\rho} + \frac{c_{2I}}{24\cdot 16}\epsilon^{\nu\alpha\beta\sigma\rho}R_{\alpha\beta\mu\gamma}R_{\sigma\rho}{}^{\mu\gamma} \ , \tag{2.111}$$

with

$$2\frac{\delta\mathcal{L}_1}{\delta F^{Iab}} = \frac{c_{2I}}{24}\Big(\frac{1}{3}v_{ab}D - \frac{8}{3}v_{ac}v^{cd}v_{db} - \frac{2}{3}v_{ab}v^2 + C_{abcd}v^{cd} \\ + \frac{4}{3}\epsilon_{abcde}v^{cf}\mathcal{D}_f v^{de} + 2\epsilon_{abcde}v^c{}_f\mathcal{D}^d v^{ef}\Big) \ , \tag{2.112}$$

and

$$\frac{\delta\mathcal{L}_1}{\delta F_{\mu\nu}^I} = e_a{}^\mu e_b{}^\nu\frac{\delta\mathcal{L}_1}{\delta F_{ab}^I} \ . \tag{2.113}$$

A lengthy computation is now required in order to expand and simplify (2.111). After making heavy use of the conditions derived from supersymmetry, we find that the spatial components of (2.111) are satisfied identically, while the time component reduces to

$$\tilde\nabla^2\left[M_I e^{-2U} - \frac{c_{2I}}{24}\left(3(\tilde\nabla U)^2 - \frac{1}{4}e^{6U}d\omega^{+\hat i\hat j}d\omega^+_{\hat i\hat j} - \frac{1}{12}e^{6U}d\omega^{-\hat i\hat j}d\omega^-_{\hat i\hat j}\right)\right] \\ = \frac{1}{2}c_{IJK}\Theta^J\cdot\Theta^K + \frac{c_{2I}}{24}\frac{1}{8}\tilde R^{\hat i\hat j\hat k\hat l}\tilde R_{\hat i\hat j\hat k\hat l} \ , \tag{2.114}$$

where $\Theta^I\cdot\Theta^J = \Theta^{I\hat i\hat j}\Theta^J_{\hat i\hat j}$ and $\tilde R_{\hat i\hat j\hat k\hat l}$ is the Riemann tensor of the metric on the base. Note also that the indices on $\Theta^I_{\hat i\hat j}$ are defined to be orthonormal with respect to the metric on the base.

2.3.4 D equation

The equation of motion for the auxiliary field D was given in (2.22). In the present case it becomes

$$
\begin{aligned}
\mathcal{N} = 1 - \frac{c_{2I}}{24} e^{2U} \Bigg[& M^I \left(\tilde{\nabla}^2 U - 4(\tilde{\nabla} U)^2 \right) + \tilde{\nabla}_{\hat{i}} M^I \tilde{\nabla}_{\hat{i}} U \\
& + \frac{1}{4} e^{6U} M^I \left(d\omega^{+\hat{i}\hat{j}} d\omega^{+}_{\hat{i}\hat{j}} + \frac{1}{3} d\omega^{-\hat{i}\hat{j}} d\omega^{-}_{\hat{i}\hat{j}} \right) - \frac{1}{12} e^{4U} \Theta^I_{\hat{i}\hat{j}} d\omega^{-\hat{i}\hat{j}} \Bigg] .
\end{aligned} \tag{2.115}
$$

2.3.5 v equation

The final ingredient needed to completely determine the general solution is the v equation of motion. In fact, for the explicit solutions considered here, namely the spinning black holes, this information is not needed. It is however needed to determine the black ring solution, and so we display the result. The full v equation of motion is rather forbidding, and so we simplify by considering just a flat base space. Furthermore, simplifications result upon contracting the v equation with $d\omega$. It turns out that the v equation contracted with $d\omega^+$ is automatically satisfied given our prior results, and so, after a lengthy calculation, we are left with

$$
\begin{aligned}
\frac{1}{4} d\omega^{-\hat{i}\hat{j}} d\omega^{-}_{\hat{i}\hat{j}} + \frac{1}{8} e^{-2U} M_I \Theta^{I\hat{i}\hat{j}} d\omega^{-}_{\hat{i}\hat{j}} = -\frac{c_{2I}}{16 \cdot 24} d\omega^{-\hat{i}\hat{j}} \Big[& -\frac{1}{6} e^{-6U} \tilde{\nabla}^2 (e^{6U} \Theta^I_{\hat{i}\hat{j}}) \\
& + 4 \tilde{\nabla}_{\hat{j}} \tilde{\nabla}_{\hat{k}} (e^{2U} M^I d\omega^{+}_{\hat{i}\hat{k}}) + \frac{1}{3} \tilde{\nabla}^2 (e^{2U} M^I d\omega^{-}_{\hat{i}\hat{j}}) \\
& + \frac{1}{6} e^{6U} \Theta^I_{\hat{i}\hat{j}} \left((3 d\omega^{+\hat{k}\hat{l}} d\omega^{+}_{\hat{k}\hat{l}} + d\omega^{-\hat{k}\hat{l}} d\omega^{-}_{\hat{k}\hat{l}} \right) \Big] .
\end{aligned} \tag{2.116}
$$

2.3.6 Spinning black holes on Gibbons-Hawking space

We now focus on 5D electrically charged spinning black hole solutions. The main simplification here is that we take[7]

$$
d\omega^- = \Theta^I = 0 . \tag{2.117}
$$

Then, to determine the full solution the relevant equations (2.114) and (2.115) become

$$
\tilde{\nabla}^2 \left[M_I e^{-2U} - \frac{c_{2I}}{24} \left(3(\tilde{\nabla} U)^2 - \frac{1}{4} e^{6U} d\omega^{+\hat{i}\hat{j}} d\omega^{+}_{\hat{i}\hat{j}} \right) \right] = \frac{c_{2I}}{24 \cdot 8} \tilde{R}^{\hat{i}\hat{j}\hat{k}\hat{l}} \tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} , \tag{2.118}
$$

[7]The two-derivative BMPV solution [5, 58, 59] enjoys such a property. We include (2.117) as part of our *ansatz* to investigate higher-derivative corrections to this solution.

and

$$\mathcal{N} = 1 - \frac{c_{2I}}{24} e^{2U} \Big[M^I \left(\tilde{\nabla}^2 U - 4(\tilde{\nabla} U)^2 \right) + \tilde{\nabla}_{\hat{i}} M^I \tilde{\nabla}_{\hat{i}} U + \frac{1}{4} e^{6U} M^I d\omega^{+\hat{i}\hat{j}} d\omega^{+}_{\hat{i}\hat{j}} \Big] \ . \tag{2.119}$$

The base space is now taken to be a Gibbons-Hawking space with metric

$$ds_B^2 = (H^0)^{-1}(dx^5 + \vec{\chi} \cdot d\vec{x})^2 + H^0 d\vec{x}^2 \ , \tag{2.120}$$

and H^0 and $\vec{\chi}$ satisfying[8]

$$\vec{\nabla} H^0 = \vec{\nabla} \times \vec{\chi} \ , \tag{2.121}$$

which in turn implies that H^0 is harmonic on $\mathbb{R}^3$, up to isolated singularities. The x^5 direction is taken to be compact, $x^5 \cong x^5 + 4\pi$, and an isometry direction for the entire solution. We note a few special cases. Setting $H^0 = 1/|\vec{x}|$ yields the flat metric on $\mathbb{R}^4$ in Gibbons-Hawking coordinates. Taking $H^0 = 1$ yields a flat metric on $\mathbb{R}^3 \times S^1$. A more interesting choice is the charge p^0 Taub-NUT space with

$$H^0 = H^0_\infty + \frac{p^0}{|\vec{x}|} \ . \tag{2.122}$$

For general p^0 the geometry has a conical singularity but the $p^0 = 1$ case is non-singular.

With this choice of base space we find

$$\tilde{R}^{\hat{i}\hat{j}\hat{k}\hat{l}} \tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} = 2\tilde{\nabla}^2 \left(\frac{(\tilde{\nabla} H^0)^2}{(H^0)^2} \right) + \dots \ , \tag{2.123}$$

where the dots represent δ-functions due to possible isolated singularities in H^0, such as in (2.122). Thus we can write $\tilde{R}^{\hat{i}\hat{j}\hat{k}\hat{l}} \tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}}$ as a total Laplacian

$$\tilde{R}^{\hat{i}\hat{j}\hat{k}\hat{l}} \tilde{R}_{\hat{i}\hat{j}\hat{k}\hat{l}} = \tilde{\nabla}^2 \Phi \equiv \tilde{\nabla}^2 \left(2 \frac{(\tilde{\nabla} H^0)^2}{(H^0)^2} + \sum_i \frac{a_i}{|\vec{x} - \vec{x}_i|} \right) \ , \tag{2.124}$$

for some coefficients a_i. We can now solve (2.118) as

$$M_I e^{-2U} - \frac{c_{2I}}{24} \left[3(\tilde{\nabla} U)^2 - \frac{1}{4} e^{6U} d\omega^{+\hat{i}\hat{j}} d\omega^{+}_{\hat{i}\hat{j}} \right] - \frac{c_{2I}}{24 \cdot 8} \Phi = H_I \ , \tag{2.125}$$

[8] Take care not confuse $\vec{\nabla}$, the gradient on $\mathbb{R}^3$, with $\tilde{\nabla}$, the covariant derivative on the four-dimensional Gibbons-Hawking space.

where $\tilde{\nabla}^2 H_I = 0$. The choice

$$H_I = M_I^\infty + \frac{q_I}{4\rho}\ , \quad \rho = |\vec{x}|\ , \tag{2.126}$$

identifies the constants of integration M_I^∞ with the value of the moduli at infinity, and q_I as the conserved 5D electric charges in the case that the base is $\mathbb{R}^4$. In Section 4.3.1 we will closely study the case of a Taub-NUT base space and see that there are modifications to the asymptotics controlled by the coefficients a_i in (2.124).

The rotation, as encoded in $d\omega^+$, is determined uniquely from closure and self-duality to be

$$d\omega^+ = -\frac{J}{8\rho^2} x^m \left(\tilde{e}^{\hat{5}} \wedge \tilde{e}^{\hat{m}} + \frac{1}{2}\epsilon_{\hat{m}\hat{n}\hat{p}} \tilde{e}^{\hat{n}} \wedge \tilde{e}^{\hat{p}} \right)\ , \tag{2.127}$$

where $\tilde{e}^a$ are the obvious vielbeins for the Gibbons-Hawking metric (2.120) and the orientation is $\epsilon_{\hat{5}\hat{m}\hat{n}\hat{p}} = 1$. With this normalization, J is the angular momentum of the 5D spinning black hole (note that for the supersymmetric black hole the two independent angular momenta in 5D must be equal.)

Now that $d\omega^+$ has been specified the full solution can be found as follows. After using (2.125) to find M_I we determine M^I by solving $M_I = \frac{1}{2} c_{IJK} M^J M^K$ (this can be done explicitly only for special choices of c_{IJK}). We then insert M^I into (2.119) to obtain a nonlinear, second order, differential equation for $U = U(\rho)$. This last equation typically can be solved only by numerical integration. However, the near horizon limit of the solution can be computed analytically as we do in Section 3.1.4.

2.3.7 Example: $K3 \times T^2$ compactifications

We can find more explicit results in the special case of $K3 \times T^2$ compactifications.[9] In this case $c_{1ij} = c_{ij}$, $i, j = 2, \ldots 23$ are the only nontrivial intersection numbers and $c_{2,i} = 0$,

[9] The formalism used in this review only describes the gauge fields which lie in five-dimensional $N = 2$ vector multiplets. These arise by wrapping the eleven-dimensional three-form on two-cycles dual to elements of $H^2(CY_3)$, and are the only lower-dimensional vectors for a generic Calabi-Yau compactification. On the other hand, $K3 \times T^2$ compactifications preserve $N = 4$ supersymmetry. When decomposed into $N = 2$ language, the $N = 4$ gravity multiplet gives rise to two $N = 2$

$c_{2,1} = 24$ are the 2nd Chern-class coefficients. We define c^{ij} to be the inverse of the $K3$ intersection matrix c_{ij}.

We first derive the attractor solution. Our procedure instructs us to first find the hatted variables in terms of conserved charges by inverting (3.32). In the present case we find

$$\hat{M}^1 = \sqrt{\frac{\frac{1}{2}c^{ij}q_i q_j + \frac{4J^2}{(q_1+1)^2}}{q_1+3}} \ , \qquad \hat{M}^i = \sqrt{\frac{q_1+3}{\frac{1}{2}c^{ij}q_i q_j + \frac{4J^2}{(q_1+1)^2}}}\, c^{ij} q_j \ , \tag{2.128}$$

and

$$\hat{J} = \sqrt{\frac{q_1+3}{\frac{1}{2}c^{ij}q_i q_j + \frac{4J^2}{(q_1+1)^2}}}\, \frac{J}{q_1+1} \ . \tag{2.129}$$

All quantities of interest are given in terms of these variables. For completeness, we display the entropy of this solution here, although it will be derived later in Section 3.2.2 for an arbitrary Calabi-Yau compactification. For a spinning black hole the entropy is given by (3.56) which, after substitution of (2.128) and (2.129) and the intersection numbers and Chern class coefficients for $K3 \times T^2$, becomes

$$S \ = \ 2\pi \sqrt{\frac{1}{2}c^{ij}q_i q_j (q_1+3) - \frac{(q_1-1)(q_1+3)}{(q_1+1)^2} J^2} \ . \tag{2.130}$$

In the case of $K3 \times T^2$ the charge q_1 corresponding to M2-branes wrapping T^2 is apparently special; the higher order corrections to the entropy are encoded entirely in the modified functional dependence on q_1.

We now turn to the full asymptotically flat solution in the static case $J = 0$. The full solution can be expressed explicitly in terms of the function U, which obeys a nonlinear ODE requiring a numerical treatment. We first invert $M_I = \frac{1}{2}c_{IJK}M^J M^K$ as

$$M^1 = \sqrt{\frac{c^{ij}M_i M_j}{2M_1}} \ , \quad M^i = c^{ij} M_j \sqrt{\frac{2M_1}{c^{kl}M_k M_l}} \ . \tag{2.131}$$

gravitino multiplets which contain a total of four extra gauge fields. From the M-theory perspective, these fields arise from Kaluza-Klein compactification along one-cycles in the T^2 factor; two gauge fields arise from compactifying the metric and two two-forms (which, in five dimensions, are dual to vectors) arise from the eleven-dimensional three-form. Our description of $K3 \times T^2$ compactifications is valid only for solutions which are not charged under these additional fields.

Substituting into (2.125) gives

$$M^1 = \left(\frac{e^{2U}c^{ij}H_iH_j}{2H_1+6U'^2}\right)^{1/2} \quad , \quad M^i = \left(\frac{e^{2U}c^{ij}H_iH_j}{2H_1+6U'^2}\right)^{-1/2} e^{2U}c^{ij}H_j \ , \tag{2.132}$$

where $'$ denotes differentiation with respect to $r = 2\sqrt{\rho}$ (in terms of r the base space metric becomes $dr^2 + r^2 d\Omega_3^2$). The special geometry constraint (2.119) is

$$\frac{1}{2}c_{ij}M^iM^jM^1 - 1 + e^{2U}\left[(U'' + \frac{3}{r}U' - 4U'^2)M^1 + U'M^{1'}\right] = 0 \ . \tag{2.133}$$

The problem is now to insert (2.132) into (2.133) and solve for $U(r)$.

This is straightforward to solve numerically given specific choices of charges. Consider a small black hole, $q_1 = 0$ with $q_2 = q_3 = 1,\ c^{23} = 1$. We also assume $H = H_2 = H_3 = 1 + \frac{1}{r^2}$ are the only harmonic functions not equal to unity. Then (2.133) becomes

$$HU'' + (1+3U'^2)\left[\left(\frac{3}{r} + \frac{1}{r^3}\right)U' + H\right] - e^{-3U}(1+3U'^2)^{3/2} = 0 \ . \tag{2.134}$$

The boundary conditions are fixed by matching to the small r behavior

$$e^{-2U} \sim \frac{\ell_S^2}{r^2} \ , \tag{2.135}$$

with $\ell_S = 3^{-1/6}$. The result of the numerical solution for $U(r)$ is shown in Figure 2.2.

2.3.8 Comments on black rings

Black rings [60, 61] incorporate nonzero Θ^I and $d\omega^-$. After choosing the base space, the two-form Θ^I can be determined by the requirements of closure and anti-self duality. In the two-derivative limit $d\omega^-$ can be determined from the v equation of motion according to

$$d\omega^- = \frac{1}{2}e^{-2U}M_I\Theta^I \ . \tag{2.136}$$

In the higher derivative case there is instead equation (2.116), which has not yet been solved. The full black ring solution is therefore not available at present. We can, however, find the near horizon geometry of the black ring and an expression for its entropy. This question will be revisited in the Section 3.2.3.

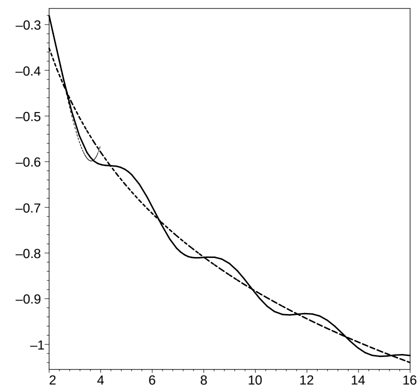

Figure 2.1: Analytical and numerical results for $\Delta(r)$ in the near string region. The solid curve describes the numerical solution of (2.81). The dotted curve represent the analytical solution for small values of r given by (2.82), and the dashed curved is the approximate solution for large values of r (2.83).

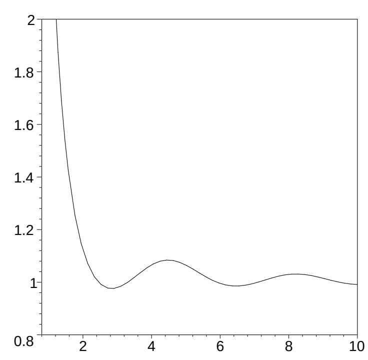

Figure 2.2: Numerical solution of equation (2.134); the curve represents $e^{-2U(r)}$ for small values of r. The oscillatory behavior is characteristic of higher derivative theories and we discussed in Section 2.2.5.

CHAPTER III

Black Hole Entropy and Extremization Principles

Any candidate theory of quantum gravity should elucidate the microphysics of black holes. One of the great successes of string theory is to provide an accounting of the entropy of many black holes in terms of a microscopic counting of states

$$S_{BH} = \frac{A}{4G} = \log \Omega_{string} \ . \tag{3.1}$$

The leading order entropy is given by the Bekenstein-Hawking area law, derived from the classical Einstein-Hilbert action. Any theory of quantum gravity will in general contain higher dimension operators in the low energy effective Lagrangian, *i.e.* terms which contain more than two derivatives of the fundamental fields. The area law for the black hole entropy is therefore only valid in the limit that the black hole is much larger than the Planck and string scales. Analogously, the explicit counting of states is usually done in the limit of large mass and charge, where powerful formulas for the asymptotic degeneracies are available. Thus one expects corrections on both sides of (3.1), and matching these corrections leads to an even more detailed understanding of string theory and black holes. An ambitious long term goal is to verify (3.1) *exactly*, as this would surely signal that we have achieved a fundamental understanding of quantum gravity.

In the following we compute the leading order corrections to the entropy of 5D black holes and strings constructed in the previous chapter. On the gravity side, the corrections to the entropy can be computed using Bekenstein-Hawking-Wald entropy [27, 28, 29]. This is a generalization of Bekenstein-Hawking area law formula, which is valid for any diffeomorphism and gauge invariant local effective action, and so can be applied to effective Lagrangians arising in string theory or otherwise. The Wald formula greatly simplifies for an extremal black hole with a near horizon AdS_2 or AdS_3 factor. For AdS_2 the entropy function formalism [11] is appropriate, while for AdS_3 c-extremization [24] is most efficient. Both methods are extremization principles that rely on the enhanced symmetry of the horizon geometry and the attractor mechanism. These two ingredients then result on the entropy depending only on the near horizon data, *i.e.* black holes have no-hair.

All of the solutions constructed in the previous chapter have a near horizon region with enhanced supersymmetry. This fact implies that the near horizon geometries are much simpler to obtain than the full asymptotically flat solutions, since some of the equations of motion can be traded for the simpler conditions following from enhanced supersymmetry. Therefore, we show how to obtain the near horizon solutions directly. We also exhibit the higher derivative version of the attractor mechanism, which fixes the moduli in terms of the electric and magnetic fluxes in the near horizon region.

After constructing the near horizon regions we turn to evaluating the black hole entropy. This is not completely straightforward, since Wald's formula does not directly apply, as the five dimensional action contains non-gauge invariant Chern-Simons terms. In the AdS_3 case, finding the black hole entropy can be reduced to finding the generalized Brown-Henneaux central charges [43] of the underlying Virasoro algebras.[1] For a general Lagrangian, an efficient c-extremization formula is available [24], which reduces the computation of the central charges to solving a set of algebraic equations. In the supersymmetric

[1]To be precise, the central charges take into account the contributions from all *local* terms in the effective action. Additional nonlocal terms are also present due to the fact that the black hole has a different topology than Minkowski space. These contributions come from the worldlines of particles winding around the horizon [82, 83, 84].

context the procedure is even simpler, since the central charges can be read off from the coefficients of the Chern-Simons terms [85, 24, 25]. We carry out both procedures and show that they agree. For AdS_2, there is a similar extremization recipe based on the so-called entropy function [11]. Applying the entropy function here requires a bit of extra work, since the Chern-Simons terms need to be rewritten in a gauge invariant form in order for Wald's formula to apply. We carry this out and obtain the explicit entropy formulas for our black hole solutions. The results turn out to be remarkably simple. The discussion is based on the results found in [52, 39, 53, 40].

3.1 Attractor Solutions

An important property of extremal black holes is attractor behavior. The literature on the attractor mechanism is extensive but the original works which first explored the phenomenon include [86, 87, 88, 89, 90]. Furthermore, useful reviews which approach the subject from different viewpoints include [91, 15, 92, 93].

In general, there exist BPS and non-BPS extremal black holes, and both display attractor behavior. The non-BPS branch is quite interesting, but it will not be discussed here (see [24, 94, 95], and [96, 97, 98, 99, 100] for discussion).

We would like to reconsider BPS attractors within the higher derivative setting developed in this review. First, recall that attractor behavior involves two related aspects:

- **Attractor mechanism:** Within a fixed basin of attraction, the scalar fields flow to constants at the black hole horizon which depend on the black hole charges alone. In particular the endpoint of the attractor flow is independent of the initial conditions, *i.e.* the values of the asymptotic moduli.

- **Attractor solution:** The limiting value of the geometry (and the associated matter fields) near the black hole horizon constitutes a solution in its own right, independently of the flow. One remarkable feature is that the attractor solution has enhanced, in fact maximal, supersymmetry. This property is highly constraining, and so the solution can be analyzed in much detail.

In the previous chapter we constructed the full asymptotically flat solutions, from which attractor solutions are extracted by taking appropriate near horizon limits. But since this method obscures the intrinsic simplicity of the attractor solutions, it is instructive to construct the attractor solutions directly. This is what we do in this section.

3.1.1 Maximal supersymmetry in the off-shell formalism

As we have emphasized, the BPS attractor solution has maximal supersymmetry. Thus we consider the vanishing of the supersymmetry variations (2.16), which we repeat for ease of reference

$$0 = \left(\mathcal{D}_\mu + \frac{1}{2}v^{ab}\gamma_{\mu ab} - \frac{1}{3}\gamma_\mu \gamma \cdot v\right)\epsilon \ , \tag{3.2a}$$

$$0 = \left(-\frac{1}{4}\gamma \cdot F^I - \frac{1}{2}\gamma^a \partial_a M^I - \frac{1}{3}M^I \gamma \cdot v\right)\epsilon \ , \tag{3.2b}$$

$$0 = \left(D - 2\gamma^c\gamma^{ab}\mathcal{D}_a v_{bc} - 2\gamma^a \epsilon_{abcde}v^{bc}v^{de} + \frac{4}{3}(\gamma \cdot v)^2\right)\epsilon \ . \tag{3.2c}$$

The supersymmetry parameter ϵ should be subject to no projection conditions if the solution is to preserve maximal supersymmetry. Therefore, terms with different structures of γ-matrices cannot cancel each other on the attractor solution. In gaugino variation (3.2b) there are two independent γ-matrix structures: γ^a and γ^{ab}. Requiring that each term vanishes demands

$$M^I = \text{constant} \ , \tag{3.3}$$

and

$$F^I = -\frac{4}{3}M^I v \ . \tag{3.4}$$

Constancy of the scalar fields is a familiar feature of attractors in two-derivative gravity. The values of the constants will be determined below. The second result (3.4) is special to the off-shell formalism in that it identifies the auxiliary two-form v with the graviphoton field strength.

We next extract the information from (3.2c). We can write it as

$$\left[(D - \frac{8}{3}v^2) - 2\gamma^{abc}\mathcal{D}_a v_{bc} + 2\gamma^a(\mathcal{D}^b v_{ba} - \frac{1}{3}\epsilon_{abcde}v^{bc}v^{de})\right]\epsilon = 0 \ , \tag{3.5}$$

by using the algebraic identities

$$\begin{aligned}
&\gamma^{ab}\gamma^{cd} = -(\eta^{ac}\eta^{bd} - \eta^{ad}\eta^{bc}) - (\gamma^{ac}\eta^{bd} - \gamma^{bc}\eta^{ad} + \gamma^{bd}\eta^{ac} - \gamma^{ad}\eta^{bc}) + \gamma^{abcd} \ , \\
&\gamma^a\gamma^{bc} = \eta^{ab}\gamma^c - \eta^{ac}\gamma^b + \gamma^{abc} \ , \\
&\gamma_{abcde} = \epsilon_{abcde} \ .
\end{aligned} \tag{3.6}$$

Again, maximal supersymmetry precludes any cancelation between different tensor structures of the γ-matrices. We therefore determine the value of the D-field as

$$D = \frac{8}{3}v^2 \ , \tag{3.7}$$

and we find that the auxiliary two-form v must satisfy

$$\begin{aligned}
\epsilon^{abcde}\mathcal{D}_a v_{bc} &= 0 \ , \\
\mathcal{D}^b v_{ba} - \frac{1}{3}\epsilon_{abcde}v^{bc}v^{de} &= 0 \ .
\end{aligned} \tag{3.8}$$

Both equations support the identification of the auxiliary field v with the graviphoton field strength in minimal supergravity. The first equation in (3.8) is analogous to the Bianchi identity, and the second is analogous to the two-derivative equation of motion for a gauge field with Chern-Simons coupling. Note that v satisfies the equations of motion of two-derivative minimal supergravity even though, here, we have not assumed an action yet.

The final piece of information from maximal supersymmetry is the vanishing of the gravitino variation, corresponding to (3.2a). This equation is identical to the gravitino variation of minimal supergravity, with the auxiliary two-form v taking the role of the graviphoton field strength. The solutions to minimal supergravity have been classified completely [75]. Adapted to our notation, the solutions with maximal supersymmetry are:

- Flat space.
- A certain class of pp-waves.
- Generalized Gödel space-times.
- $\mathrm{AdS}_3 \times S^2$ with geometry

  $$ds^2 = \ell_A^2 ds^2_{AdS} - \ell_S^2 d\Omega_2^2 \ , \quad \text{with} \quad \ell_A = 2\ell_S \ . \tag{3.9}$$

  Note that supersymmetry relates the two radii. Additionally, v is proportional to the volume form on S^2

  $$v = \frac{3}{4}\ell_S \epsilon_{S^2} \ . \tag{3.10}$$

- $\mathrm{AdS}_2 \times S^3$ with geometry

  $$ds^2 = \ell_A^2 ds^2_{AdS} - \ell_S^2 d\Omega_3^2 \ , \quad \text{with} \quad \ell_A = \frac{1}{2}\ell_S \ , \tag{3.11}$$

  Again, supersymmetry relates the two radii. In this case v is proportional to the volume form on AdS_2

  $$v = \frac{3}{4}\ell_A \epsilon_{AdS_2} \ . \tag{3.12}$$

- The near horizon BMPV solution or the *rotating attractor*.

The computations leading to the above classification of solutions to minimal supergravity use the equations of motion and the Bianchi identity for the field strength, as well as the on-shell supersymmetry transformations. Presently, we analyze the consequences of maximal supersymmetry in the off-shell formalism, but do not wish to apply the equations of motion yet, because they depend on the action. Fortunately, we found in (3.8) that supersymmetry imposes the standard two-derivative equation of motion and Bianchi identity for the auxiliary two-tensor v, which in turn can be identified with the graviphoton in minimal supergravity. In our context, the classification therefore gives precisely the conditions for maximal supersymmetry, with no actual equations of motion imposed.

We will not repeat the general classification of [75] but just explain why the possibilities are so limited and derive the quantitative results given above. First recall that there exists an integrability condition obtained from the commutator of covariant derivatives acting on a spinor

$$[\mathcal{D}_\mu, \mathcal{D}_\nu]\,\epsilon = \frac{1}{4} R_{\mu\nu ab}\gamma^{ab}\epsilon\ . \tag{3.13}$$

We can evaluate the left-hand side of (3.13) by differentiating and then antisymmetrizing the BPS condition resulting from the gravitino variation, *i.e.* the first equation in (3.2). The resulting equations are rather unwieldy, but they can be simplified to purely algebraic conditions by using reorderings akin to (3.6) along with the supersymmetry conditions (3.8). The terms proportional to the tensor structure γ^{ab} give the Riemann tensor

$$\begin{aligned} R_{\mu\nu\rho\sigma} = &-\frac{16}{9} v_{\mu\nu}v_{\rho\sigma} - \frac{4}{3}(v_{\mu\rho}v_{\nu\sigma} - v_{\mu\sigma}v_{\nu\rho}) + \frac{2}{9}(g_{\mu\rho}g_{\nu\sigma} - g_{\nu\rho}g_{\mu\sigma})v^2 \\ &- \frac{4}{9}\left(g_{\mu\sigma}v_{\rho\tau}v^{\tau}{}_{\nu} - g_{\mu\rho}v_{\sigma\tau}v^{\tau}{}_{\nu} - g_{\nu\sigma}v_{\rho\tau}v^{\tau}{}_{\mu} + g_{\nu\rho}v_{\sigma\tau}v^{\tau}{}_{\mu}\right)\ . \end{aligned} \tag{3.14}$$

Conceptually, we might want to start with a v that solves equations (3.8), since then the geometry is completely determined by (3.14). But the two sets of equations are of course entangled. Also, one must further check that the gravitino variation does in fact vanish, and not just its commutator.

The most basic solutions for the study of black holes and strings are the $\mathrm{AdS}_3 \times S^2$ and $\mathrm{AdS}_2 \times S^3$ geometries. For these, the solutions to the supersymmetry conditions (3.8) are given by magnetic and electric fluxes, as in (3.10) and (3.12). In each case we can insert in (3.14) and verify that the geometry is in fact maximally symmetric and that the scales ℓ_A, ℓ_S are related to those of v in the manner indicated.

3.1.2 The magnetic attractor solution

So far we have just analyzed the consequences of supersymmetry. In order to determine the solutions completely we also need information from the equations of motion. We next show how this works in the case of the simplest nontrivial attractor solution, the $\mathrm{AdS}_3 \times S^2$ that is interpreted as the near horizon geometry of a magnetic string.

The key ingredient beyond maximal supersymmetry is the modified very special geometry constraint

$$\begin{aligned}\frac{1}{6}c_{IJK}M^I M^J M^K &= 1 - \frac{c_{2I}}{72}\left(F^I \cdot v + M^I D\right) \ , \\ &= 1 - \frac{c_{2I}}{54}M^I v^2 \ , \\ &= 1 - \frac{c_{2I}}{12}\frac{M^I}{\ell_A^2} \ .\end{aligned} \tag{3.15}$$

We first used the D equation of motion (2.22) and then simplified using (3.4) and (3.7). In the last line we used

$$v^2 = \frac{9}{8\ell_S^2} = \frac{9}{2\ell_A^2} \ , \tag{3.16}$$

from (3.10) and (3.9).

In $\mathrm{AdS}_3 \times S^2$ the field strengths (3.4) become

$$F^I = -\frac{4}{3}M^I v = -\frac{1}{2}M^I \ell_A \epsilon_{S^2} \ . \tag{3.17}$$

In our normalization the magnetic fluxes are fixed as

$$F^I = -\frac{p^I}{2}\epsilon_{S^2} \ , \tag{3.18}$$

so we determine the scalar fields as

$$M^I = \frac{p^I}{\ell_A} \ . \tag{3.19}$$

Inserting this into the modified very special geometry constraint (3.15) we finally determine the precise scale of the geometry

$$\ell_A^3 = \frac{1}{6}c_{IJK}p^I p^J p^K + \frac{1}{12}c_{2I}p^I \equiv p^3 + \frac{1}{12}c_2 \cdot p \ . \tag{3.20}$$

The preceding three equations specify the attractor solution completely in terms of magnetic charges p^I.

3.1.3 The electric attractor solution

The electric attractor solution is the $\mathrm{AdS}_2 \times S^3$ near horizon geometry of a non-rotating ($J = 0$) 5D black hole. The scales of the geometry are

$$\ell \equiv \ell_A = \frac{1}{2}\ell_S \ , \tag{3.21}$$

and the auxiliary two-form v in (3.12) gives

$$v^2 = -\frac{9}{8\ell^2} \ , \tag{3.22}$$

so that the modified very special geometry constraint (2.22) becomes

$$\frac{1}{6}c_{IJK}M^I M^J M^K = 1 - \frac{c_{2I}M^I}{54}v^2 = 1 + \frac{c_{2I}M^I}{48\ell^2} \ . \tag{3.23}$$

We can write this in a more convenient way by introducing the re-scaled moduli

$$\hat{M}^I = 2\ell M^I \ , \tag{3.24}$$

so that

$$\ell^3 = \frac{1}{8}\left[\frac{1}{6}c_{IJK}\hat{M}^I \hat{M}^J \hat{M}^K - \frac{c_{2I}}{12}\hat{M}^I\right] \ . \tag{3.25}$$

This equation gives the scale of the geometry in terms of the rescaled moduli.

We would often like to specify the solution in terms its electric charges, rather than the re-scaled moduli. Electric charges may be defined as integration constants in Gauss' law, a step that depends on the detailed action of the theory. We carried out this analysis in Section 2.3.6, and after solving (2.125) for this background we find

$$q_I = \frac{1}{2} c_{IJK} \hat{M}^J \hat{M}^K - \frac{c_{2I}}{8} \ , \tag{3.26}$$

where we dropped the constants in (2.126). If the q_I are given, this relation determines the rescaled moduli $\hat{M}^I$ and so, through (3.25), the scale ℓ.

3.1.4 The rotating attractor

Just as the non-rotating 5D black hole can be generalized to include rotation, the electric attractor just studied is a special case of a more general rotating attractor. For simplicity we take the base to be flat $\mathbb{R}^4$, deferring the Taub-NUT case to the next chapter.

The attractor solution corresponds to dropping the constant in the harmonic functions (2.122), (2.126) and considering a metric factor of the form

$$e^{2U} = \frac{\rho}{\ell^2} \ . \tag{3.27}$$

To display the near horizon solution it is useful to define the re-scaled quantities

$$\hat{M}^I = 2\ell M^I \ , \quad \hat{J} = \frac{1}{8\ell^3} J \ . \tag{3.28}$$

The resulting geometry takes the form

$$\begin{aligned} ds^2 &= w^2 \left[(1 + (e^0)^2)(\rho^2 d\tau^2 - \frac{d\rho^2}{\rho^2} - d\theta^2 - \sin^2\theta d\phi^2) - (dy + \cos\theta d\phi)^2 \right] \ , \\ v &= -\frac{3}{4} w (d\tau \wedge d\rho - e^0 \sin\theta d\theta \wedge d\phi) \ . \end{aligned} \tag{3.29}$$

The geometry describes a spatial circle nontrivially fibered over $\mathrm{AdS}_2 \times S^2$. The parameters w and e^0 specify the scale sizes and angular momentum of the solution

$$e^0 = -\frac{\hat{J}}{\sqrt{1 - \hat{J}^2}} \ , \quad w = \ell \sqrt{1 - \hat{J}^2} \ , \tag{3.30}$$

where ℓ can be identified with the radii of the AdS_2 and S^2 factors in string frame. The two-form v is a solution to the supersymmetry conditions (3.8) written in the convenient form

$$d \star v + \frac{4}{3} v \wedge v = 0 \ . \tag{3.31}$$

The modified special geometry constraint follows by inserting v in the second line of (3.15).

The background (3.29) was derived by taking the near horizon limit of the full solution constructed in Section 2.3.6. Inserting the various functions into the modified very special geometry constraint (2.119) and the relation expressing flux conservation (2.125) we verify that this *ansatz* gives an exact maximally supersymmetric solution.

Still the near horizon geometry is not completely fixed: we need to relate asymptotic charges (J, q_I) to re-scaled variables $(\hat{J}, \hat{M}^I)$. We proceed by solving the equations (2.119), (2.125) written in the form

$$\begin{aligned} J &= \left(\frac{1}{3!} c_{IJK} \hat{M}^I \hat{M}^J \hat{M}^K - \frac{c_{2I} \hat{M}^I}{12} (1 - 2\hat{J}^2) \right) \hat{J} \ , \\ q_I &= \frac{1}{2} c_{IJK} \hat{M}^J \hat{M}^K - \frac{c_{2I}}{8} \left(1 - \frac{4}{3} \hat{J}^2 \right) \ . \end{aligned} \tag{3.32}$$

With the solution in hand we compute

$$\begin{aligned} \ell^3 &= \frac{1}{8} \left(\frac{1}{3!} c_{IJK} \hat{M}^I \hat{M}^J \hat{M}^K - \frac{c_{2I} \hat{M}^I}{12} (1 - 2\hat{J}^2) \right) \ , \\ M^I &= \frac{1}{2\ell} \hat{M}^I \ , \end{aligned} \tag{3.33}$$

to find the values for the physical scale of the solution ℓ and the physical moduli M^I, written as functions of (J, q_I). A novel feature of the higher derivative attractor mechanism is that the fixed values of the moduli depend on the angular momentum as well as the electric charges. From (3.32) it is clear that the J dependence only appears through the higher derivative terms.

In general it is of course rather difficult to invert (3.32) explicitly. This is the situation also before higher derivative corrections have been taken into account and/or if angular momentum is neglected. However, in the large charge regime we can make the dependence on the higher derivative corrections manifest in an inverse charge expansion. Let us define the dual charges q^I through

$$q_I = \frac{1}{2} c_{IJK} q^J q^K \ . \tag{3.34}$$

We also define[2]

$$Q^{3/2} = \frac{1}{3!} c_{IJK} q^I q^J q^K \ , \tag{3.35a}$$

$$C_{IJ} = c_{IJK} q^K \ . \tag{3.35b}$$

Each of these quantities depend on charges and Calabi-Yau data but not on moduli.

With the definitions (3.34) and (3.35) we can invert (3.32) for large charges (*i.e.* expand to first order in c_{2I}) and find

$$\begin{aligned} \hat{M}^I &= q^I + \frac{1}{8}\left(1 - \frac{4}{3}\frac{J^2}{Q^3}\right) C^{IJ} c_{2J} + \dots \ , \\ \hat{J} &= \frac{J}{Q^{3/2}}\left(1 + \frac{c_2 \cdot q}{48 Q^{3/2}}\left[1 - 4\frac{J^2}{Q^3}\right]\right) + \dots \ , \end{aligned} \tag{3.36}$$

where C^{IJ} is the inverse of the matrix C_{IJ} defined in (3.35b). Then (3.33) gives the physical scale of the geometry and the physical moduli as

$$\begin{aligned} \ell &= \frac{1}{2} Q^{1/2}\left(1 - \frac{c_2 \cdot q}{144 Q^{3/2}}\left[1 - 4\frac{J^2}{Q^3}\right]\right) + \dots \ , \\ M^I &= \frac{q^I}{Q^{1/2}}\left(1 + \frac{c_2 \cdot q}{144 Q^{3/2}}\left[1 - 4\frac{J^2}{Q^3}\right]\right) + \frac{1}{8 Q^{1/2}}\left(1 - \frac{4}{3}\frac{J^2}{Q^3}\right) C^{IJ} c_{2J} + \dots \ . \end{aligned} \tag{3.37}$$

3.2 Extremization Principles

An important application of the solutions we construct is to the study of gravitational thermodynamics. The higher derivative corrections to the supergravity solutions are interesting for this purpose because they are sensitive to details of the microscopic statistical description.

The black hole entropy is famously given by the Bekenstein-Hawking area law

$$S = \frac{1}{4 G_D} A_{D-2} \ . \tag{3.38}$$

[2]The $\frac{3}{2}$ power is introduced so that Q has the same dimension as the physical charges q_I.

This expression applies only when the gravitational action is just the standard Einstein-Hilbert term. In general, one must use instead the Wald entropy formula[3]

(3.39)
$$S = -\frac{1}{8G_D}\int_{\text{hor}} d^{D-2}x\sqrt{h}\frac{\delta\mathcal{L}_D}{\delta R_{\mu\nu\rho\sigma}}\epsilon^{\mu\nu}\epsilon^{\rho\sigma} \ .$$

This reduces to (3.38) for the two-derivative action, but generally the density one must integrate over the event horizon is more complicated than the canonical volume form. In practice, it is in fact rather cumbersome to evaluate (3.39) and evaluate the requisite integral but there is a short-cut that applies to black holes with near horizon geometry presented as a fibration over $\text{AdS}_2 \times S^2$. Then the Wald entropy (3.39) is the Legendre transform of the on-shell action[4] up to an overall numerical factor. This general procedure is known as the entropy function formalism [11]. In Section 3.2.2 we apply the entropy function formalism to our five dimensional black hole solutions with $\text{AdS}_2 \times S^3$ near horizon geometry.

Although we analyze a theory in five dimensions, we can discuss four dimensional black holes by adding excitations to black strings with $\text{AdS}_3 \times S^2$ near string geometry. For large excitation energy the black hole entropy is given by Cardy's formula

(3.40)
$$S = 2\pi\left[\sqrt{\frac{c_L}{6}\left(h_L - \frac{c_L}{24}\right)} + \sqrt{\frac{c_R}{6}\left(h_R - \frac{c_R}{24}\right)}\ \right] \ ,$$

where h_L, h_R are eigenvalues of the AdS_3 energy generators $L_0, \bar{L}_0$. Since Cardy's formula can be justified in both the gravitational description[5] and also in the dual CFT, the central charge becomes a proxy for the entropy in the $\text{AdS}_3 \times S^2$ setting. It is therefore the central charge that we want to compute for our solutions. The central charge is convenient to compute because it is just the on-shell action, up to an overall numerical factor. This

[3]Theories with gravitational Chern-Simons terms may violate diffeomorphism invariance. Then Wald's formula does not apply and one must use a further generalization due to Tachikawa [101].

[4]This refers to the usual notion of "on-shell action", *i.e.* the action evaluated on a solution to all of the equations of motion. This is not to be confused with the sense of "on-shell" that we have been using throughout this review, *i.e.* with only the auxiliary field equations of motion imposed.

[5]In fact Cardy's formula (3.40) agrees with the Wald entropy whenever diffeomorphism invariance applies ($c_L = c_R$) [24, 102], or with Tachikawa's generalization [101] when $c_L \neq c_R$.

methodology is known as c-extremization [24]. In Section 3.2.1 we apply c-extremization to our five dimensional black string solutions with $\mathrm{AdS}_3 \times S^2$ near horizon geometry.

The entropy function formalism and the c-extremization procedure can be carried out while keeping arbitrary the scales of the AdS and sphere geometries, as well as matter fields consistent with the symmetries. These parameters are then determined by the extremization procedure in a manner independent of supersymmetry. The computations therefore constitute an important consistency check on the explicit Lagrangian and other parts of the framework.

3.2.1 Black strings and c-extremization

The general c-extremization procedure considers a $\mathrm{AdS}_3 \times S^{D-3}$ solution to a theory with action of the form

$$S = \frac{1}{16\pi G_D} \int d^D x \sqrt{g} \mathcal{L} + S_{CS} + S_{\text{bndy}} \ . \tag{3.41}$$

The Chern-Simons terms (if any) are collected in the term S_{CS}, and S_{bndy} are the terms regulating the infrared divergences at the boundary of AdS_3. The total central charge

$$c = \frac{1}{2}(c_L + c_R) \ , \tag{3.42}$$

is essentially the trace anomaly of the CFT, which in turn is encoded in the on-shell action of the theory. The precise relation is

$$c = -\frac{3\Omega_{D-3}}{8G_D} \ell_A^3 \ell_S^{D-3} \mathcal{L}_{\text{ext}} \ , \tag{3.43}$$

with the understanding that the action must be extremized over all parameters, with magnetic charges through S^{D-3} kept fixed.

We want to apply this formalism to the black string attractor solution found in Section 3.1.2. The isometries of the near horizon region determines the form of the solution as

$$
\begin{aligned}
ds^2 &= \ell_A^2 ds^2_{AdS} - \ell_S^2 d\Omega_2^2 \ , \\
F^I &= -\frac{p^I}{2}\epsilon_2 \ , \\
v &= V\epsilon_2 \ , \\
M^I &= mp^I \ .
\end{aligned} \tag{3.44}
$$

In Section 3.1.2 we used maximal supersymmetry and the modified very special geometry constraint to determine the parameters ℓ_A, ℓ_S, V, m and the auxiliary scalar D in terms of the magnetic charges p^I as

$$
\begin{aligned}
V &= \frac{3}{8}\ell_A \ , \ D = \frac{12}{\ell_A^2} \ , \ m = \frac{1}{\ell_A} \ , \\
\ell_S &= \frac{1}{2}\ell_A \ , \ \ell_A^3 = p^3 + \frac{1}{12}c_2 \cdot p \ ,
\end{aligned} \tag{3.45}
$$

where

$$
p^3 \equiv \frac{1}{6}c_{IJK}p^I p^J p^K \ . \tag{3.46}
$$

However, it is instructive to use just the *ansatz* (3.44) for now. Inserting this *ansatz* into the leading order Lagrangian (2.12) we find

$$
\begin{aligned}
\mathcal{L}_0 = 2\bigg(&\frac{1}{4}(p^3m^3 - 1)D - \frac{1}{4}(p^3m^3 + 3)\left(\frac{3}{\ell_A^2} - \frac{1}{\ell_S^2}\right) \\
&+ \frac{1}{\ell_S^4}\left((3p^3m^3 + 1)V^2 + 3p^3m^2V\right) + \frac{3p^3}{\ell_S^4}\frac{m}{8}\bigg) \ ,
\end{aligned} \tag{3.47}
$$

and the four derivative Lagrangian (2.13) yields

$$
\begin{aligned}
\mathcal{L}_1 = \frac{c_2 \cdot p}{24}\bigg[&\frac{m}{4}\left(\frac{1}{\ell_A^2} - \frac{1}{\ell_S^2}\right)^2 + \frac{2}{3}\frac{V^3}{\ell_S^8} + 4m\frac{V^4}{\ell_S^8} + m\frac{D^2}{12} + \frac{D}{6}\frac{V}{\ell_S^4} \\
&- \frac{2}{3}m\frac{V^2}{\ell_S^4}\left(\frac{3}{\ell_A^2} + \frac{5}{\ell_S^2}\right) + \frac{1}{2}\frac{V}{\ell_S^4}\left(\frac{1}{\ell_A^2} - \frac{1}{\ell_S^2}\right)\bigg] \ .
\end{aligned} \tag{3.48}
$$

According to c-extremization we now need to extremize the c-function

$$
c(\ell_A, \ell_S, V, D, m) = -6\ell_A^3\ell_S^2(\mathcal{L}_0 + \mathcal{L}_1) \ , \tag{3.49}
$$

with respect to all variables. The resulting extremization conditions are quite involved. For example, the variation of (3.49) with respect to m gives

$$\begin{aligned}\frac{3p^3m^2}{4}\left(D-\frac{3}{\ell_A^2}+\frac{1}{\ell_S^2}\right)+\frac{3p^3}{\ell_S^4}\left(3m^2V^2+2mV+\frac{1}{8}\right)\\+\frac{c_2\cdot p}{48}\left[\frac{1}{4}\left(\frac{1}{\ell_A^2}-\frac{1}{\ell_S^2}\right)^2+4\frac{V^4}{\ell_S^8}+\frac{D^2}{12}-\frac{2}{3}\frac{V^2}{\ell_S^4}\left(\frac{3}{\ell_A^2}+\frac{5}{\ell_S^2}\right)\right]=0\ .\end{aligned} \tag{3.50}$$

It would be very difficult to solve equations with such complexity without any guidance. Fortunately we already determined the attractor solution (3.45) and it is straightforward to verify that it does indeed satisfy (3.50). We can similarly vary the c-function (3.49) with respect to ℓ_A, ℓ_S, V, D and show that the resulting equations are satisfied by the attractor solution (3.45). Thus the attractor solution extremizes the c-function (3.49) as it should.

Since we have proceeded indirectly we have not excluded the possibility that c-extremization could have other solutions with the same charge configuration. Such solutions would not be supersymmetric. This possibility further imposes the point that c-extremization is logically independent from the considerations using maximal supersymmetry that determined the attractor solution in the first place. The success of c-extremization therefore constitutes a valuable consistency check on the entire framework.

At this point we have verified that the c-function is extremized on the attractor solution (3.45). The central charge is now simply the value of the (3.49) on that solution. The computation gives

$$c=6p^3+\frac{3}{4}c_2\cdot p\ . \tag{3.51}$$

In order to put this result in perspective, let us recall the microscopic interpretation of these black strings [10]. We can interpret $N=2$ supergravity in five dimensions as the low energy limit of M-theory compactified on some Calabi-Yau threefold CY_3. The black string in five dimensions corresponds to a M5-brane wrapping a 4-cycle in CY_3 that has component p^I along the basis four-cycle ω_I. The central charges of the effective string CFT are known to be [10, 85]

$$c_L = c_{IJK}p^I p^J p^K + \frac{1}{2}c_2 \cdot p \ , \quad c_R = c_{IJK}p^I p^J p^K + c_2 \cdot p \ , \tag{3.52}$$

where c_{IJK} are the triple intersection numbers of the CY_3, and c_{2I} are the expansion coefficients of the second Chern class. Computing the total central charge (3.42) from (3.52) we find precise agreement with our result (3.51) found by c-extremization.

It is worth noting that the simple form of the central charge comes about in a rather nontrivial way in the c-extremization procedure. The radius of curvature ℓ_A from the last line of (3.45) introduces powers of $(p^3 + \frac{1}{12}c_2 \cdot p)^{1/3}$ in the denominator of the Lagrangian (3.47)-(3.48). It is only due to intricate cancelations that the final result (3.51) becomes a polynomial in the charges p^I.

3.2.2 Black hole entropy

We want to compute the entropy of black hole solutions with $\mathrm{AdS}_2 \times S^3$ near horizon geometry. As mentioned in the introduction to this section the most efficient method to find the entropy is by use of the entropy function [11], which amounts to computing the Legendre transform of the Lagrangian density evaluated on the near horizon solution. Some care is needed because the 5D action contains non-gauge invariant Chern-Simons terms while the entropy function method applies to gauge invariant actions.

We first review the general procedure for determining the entropy from the near horizon solution, mainly following [103, 53]. The general setup is valid for spinning black holes as well as black rings.

The near horizon geometries of interest take the form of a circle fibered over an $\mathrm{AdS}_2 \times S^2$ base:

$$ds^2 = w^{-1}\Big[v_1\Big(\rho^2 d\tau^2 - \frac{d\rho^2}{\rho^2}\Big) - v_2(d\theta^2 + \sin^2\theta d\phi^2)\Big] - w^2\Big(dx^5 + e^0\rho d\tau + p^0\cos\theta d\phi\Big)^2 \ ,$$

$$A^I = e^I\rho d\tau + p^I\cos\theta + a^I\Big(dx^5 + e^0\rho d\tau + p^0\cos\theta d\phi\Big) \ ,$$

$$v = -\frac{1}{4\mathcal{N}}M_I F^I \ .$$

The parameters w, $v_{1,2}$, a^I and all scalar fields are assumed to be constant. Kaluza-Klein reduction along x^5 yields a 4D theory on $\mathrm{AdS}_2 \times S^2$. The solution carries the magnetic charges p^I, while e^I denote electric potentials.[6]

Omitting the Chern-Simons terms for the moment, let the action be

$$I = \frac{1}{4\pi^2}\int d^5x\sqrt{g}\mathcal{L} \ . \tag{3.53}$$

Define

$$f = \frac{1}{4\pi^2}\int d\theta d\phi dx^5\sqrt{g}\mathcal{L} \ . \tag{3.54}$$

Then the black hole entropy is

$$S = 2\pi\Big(e^0\frac{\partial f}{\partial e^0} + e^I\frac{\partial f}{\partial e^I} - f\Big) \ . \tag{3.55}$$

Here w, $v_{1,2}$ etc. take their on-shell values. One way to find these values is to extremize f while holding fixed the magnetic charges and electric potentials. The general extremization problem would be quite complicated given the complexity of our four-derivative action. Fortunately, in the cases of interest we already know the values of all fields from the explicit solutions.

The Chern-Simons term is handled by first reducing the action along x^5 and then adding a total derivative to $\mathcal{L}$ to restore gauge invariance in the resulting 4D action [53] (it is of course not possible to restore gauge invariance in 5D).

[6] An important point, discussed at length in Section 4.3, is that e^I are conjugate to 4D electric charges, which differ from the 5D charges.

The result of this computation is the entropy formula

$$S = 2\pi\sqrt{1-\hat{J}^2}\left(\frac{1}{6}c_{IJK}\hat{M}^I\hat{M}^J\hat{M}^K + \frac{1}{6}\hat{J}^2 c_{2I}\hat{M}^I\right) , \tag{3.56}$$

where the rescaled moduli are evaluated at their attractor values (3.32).

We can also express the entropy in terms of the conserved charges. We first use (3.33) to find an expression in terms of geometrical variables

$$S = 2\pi\sqrt{(2\ell)^6 - J^2}\Big(1 + \frac{c_{2I}M^I}{48\ell^2}\Big) , \tag{3.57}$$

and then expand to first order in c_{2I} using (3.37) to find

$$S = 2\pi\sqrt{Q^3 - J^2}\Big(1 + \frac{c_2 \cdot q}{16}\frac{Q^{3/2}}{(Q^3-J^2)} + \cdots\Big) . \tag{3.58}$$

From the standpoint of our 5D supergravity action (3.56) is an exact expression for the entropy. But as a statement about black hole entropy in string theory it is only valid to first order in c_{2I}, since we have only kept terms in the effective action up to four derivatives. The situation here is to be contrasted with that for 5D black strings, where anomaly arguments imply that the entropy is uncorrected by terms beyond four derivatives. The anomaly argument relies on the presence of an AdS_3 factor [24, 25], which is absent for the 5D black holes considered in this section.

3.2.3 Black ring entropy

Although we have not yet determined the complete black ring solution we can compute its entropy by applying the entropy function formalism to the black ring attractor.

For the black ring the near horizon solution is

$$\begin{aligned} ds^2 &= w^{-1}v_3\Big[\Big(\rho^2 d\tau^2 - \frac{d\rho^2}{\rho^2}\Big) - d\Omega^2\Big] - w^2\Big(dx^5 + e^0\rho d\tau\Big)^2 , \\ A^I &= -\frac{1}{2}p^I\cos\theta d\phi - \frac{e^I}{e^0}dx^5 . \end{aligned} \tag{3.59}$$

Further details of the solution follow from the fact that the near horizon geometry is a magnetic attractor. The near horizon geometry is a product of a BTZ black hole and an

S^2, and there is enhanced supersymmetry. These conditions imply

$$
\begin{aligned}
M^I &= \frac{p^I}{2we^0} \ , \\
v_3 &= w^3(e^0)^2 \ , \\
D &= \frac{3}{w^2(e^0)^2} \ , \\
v &= -\frac{3}{4}we^0 \sin\theta d\theta \wedge d\phi \ .
\end{aligned}
\tag{3.60}
$$

The computation of the entropy in terms of the entropy function proceeds as in the case of the spinning black hole. The result is

$$
S = \frac{2\pi}{e^0}\left(\frac{1}{6}c_{IJK}p^Ip^Jp^K + \frac{1}{6}c_{2I}p^I\right) \ . \tag{3.61}
$$

The entropy is expressed above in terms of magnetic charges p^I and the potential e^0, but the preferred form of the entropy would be a function of the conserved asymptotic charges. To get a formula purely in terms of the charges (p^I, q_I) and the angular momenta we need to trade away e^0. But for this one needs knowledge of more than just the near horizon geometry, which, as we noted above, is not available at present.

Let us finally note that the entropy can be expressed in geometric variables as

$$
S = (2-\mathcal{N})\frac{A}{\pi} = (2-\mathcal{N})\frac{A}{4G_5} \ , \tag{3.62}
$$

where A is the area of the event horizon. In the two-derivative limit we have $\mathcal{N} = 1$ and we recover the Bekenstein-Hawking entropy.

CHAPTER IV

The Statistical Entropy of Five Dimensional Black Holes

One important question in string theory is to understand the black hole entropy from a statistical point of view, *i.e.* as the logarithm of the number of quantum states. For a certain class of extremal black holes that preserve a fraction of supersymmetry, this question has been extensively studied. The microscopic configuration involves D-branes, fundamental strings and other objects carrying the same set of charges as the black hole. At weak coupling the gravitational backreaction can be ignored, and due to stability properties of the system the dynamics of the microscopic theory is under control. Supersymmetry then allows to extend the results to strong coupling where gravity is dominant and the system becomes a black hole. The statistical entropy is obtained by performing an inverse Fourier transform of the partition function at weak coupling, and in the large charge limit this quantity exactly agrees with the Bekenstein-Hawking area law [5, 50, 10, 15]. As emphasized in Chapter III, corrections beyond the large charge limit to both macroscopic and microscopic entropy should exactly agree. In the following we compute the corrections to the statistical entropy in the overlapping regime of validity with Wald entropy in R^2 corrected supergravity.

For four dimensional black holes, the exact counting of microstates beyond the large charge estimate has been achieved in $N = 4$ string theory using the construction of the partition function for 1/4 BPS dyons[1] in terms of an auxilliary mathematical function called the Igusa cusp form: Φ_{10}, the unique weight 10 modular form of $Sp(2,\mathbb{Z})$. This counting formula was originally conjectured in [50] and then derived in [51, 14] using a D-brane-monopole setup, and generalized to the counting of all dyons in [104]. Using the modular transformation properties of Φ_{10}, one can systematically deduce the sub-leading corrections to the 4D black hole entropy [13, 14, 15].

The derivation of the counting formula uses the relation of the 4D black holes in question to a three charge spinning black hole in 5D which has come to be known as the 4D-5D connection [105]. The 4D black holes carry one extra charge which corresponds to a unit Kaluza-Klein (KK) monopole at the center of which the 5D black hole is placed. By making the modulus of the KK circle small or large, the authors of [51] then argue that the entropy of the 4D and 5D black holes are related. More precisely the microstates of the 4D system can be counted by putting together the microstates of the 5D system, and the states which are bound to the KK monopole itself [14].

For the 5D black hole, the microscopic analysis in [5] used the related two dimensional superconformal field theory (SCFT) with target space $Sym^{Q_1Q_5+1}(K3)$, and energy eigenvalues equal to the momentum n. In this 2D SCFT, one can apply the Cardy formula to estimate the density of states at high energies. The Cardy formula is valid for energies much larger than the central charge, *i.e.* $n \gg Q_1Q_5$. There is a systematic procedure to compute corrections to the Cardy formula [82, 106] in the parameter $Q_1Q_5 \ll n$.

In the gravity theory when the Schwarzschild radius is much larger than the string length, the configuration looks like a big 5D black hole carrying electric charge (Q_1, Q_5, n). In the type II theory on $K3$ this radius is given by $R^2_{Sch} = Q_1Q_5/n$, in string units. One can now look at finer structures and probe higher derivative corrections to the black

[1]In four dimensions, a dyon is a BPS states which carries both electric (Q) and magnetic (P) charge.

hole entropy; these sigma model corrections to supergravity will be governed by the small parameter n/Q_1Q_5. This is exactly the opposite regime to the one above where one can compute corrections to the Cardy formula. Therefore one cannot naively compare the macroscopic corrections with the microscopic corrections in the Cardy limit.

We need a new tool to compute the sub-leading expansions of the statistical entropy in the non-Cardy regime.[2] Such a tool can be found by using the above 4D-5D connection in reverse – we can rewrite the 5D partition function in terms of the 4D partition function plus some corrections which physically have to do with the *stripping off* of the modes stuck to the KK monopole. Mathematically this is expressed as a precise relation between the 5D and the 4D partition functions. Having done this, we can use the powerful mathematical properties of the function Φ_{10} to deduce systematically the corrections to the 5D entropy.

Finally, we use our technique to clear up a slightly confusing point in the literature having to do with the 4D-5D connection, which gives a relation between the entropies of black holes in four and five dimensions [105]. This connection involves 5D solutions whose base metric is a Taub-NUT. The Taub-NUT geometry interpolates between $\mathbb{R}^4$ at the origin and $\mathbb{R}^3 \times S^1$ at infinity, and the size of the S^1 is freely adjustable. By placing a black hole at the origin and dialing the S^1 radius we can interpolate between black holes with 4D and 5D asymptotics. Since the attractor mechanism implies that the BPS entropy is independent of moduli, it is expected that the 4D and 5D black hole entropy formulas are closely related. Higher derivative corrections turn out to introduce an interesting twist to this story [53]. The relation between the 4D and 5D black hole charges is not the naive one expected from the lowest order solutions. It turns out that there is a subtle shift in

[2] This would not be necessary if one can map the counting problem to that of finding the density of states in the Cardy regime of a different CFT. Indeed, as was observed in [107, 108], the entropy of the 5D black hole which we consider can be expressed to subleading order as a Cardy formula of a putative dual SCFT with $L_0 = Q_1$ and $c = 6Q_5(n+3)$. It would be very interesting to understand the microscopic origin of such a SCFT with these values of charges. In the remainder of this chapter, the phrase "away from the Cardy limit" should be taken to mean "away from the Cardy limit of any currently understood microscopic SCFT, and in particular the D1-D5-p SCFT".

the definition of charge in the 5D theory having to do with the curvature of the Taub-NUT space, which changes the entropy expressed in terms of the 4D charges. As we shall show, this small change agrees precisely with the different sub-leading contributions in the 4D and 5D microscopic formulas.

The discussion presented in this chapter is based on the articles [52, 54].

4.1 Black Holes on $K3\times\mathbf{T}^2$

Our case of interest are corrections to the entropy of 1/4 BPS black holes in $N=4$ with internal manifold $\mathrm{CY}_3{=}K3\times\mathrm{T}^2$. In the following, we will apply our findings in Section 3.1.4 and 3.2.2 to compute the corrections to the entropy of these black holes.

In the eleven dimensional language, the electric charges q^I correspond to M2-branes wrapping two-cycles. In type IIB string theory the M2-branes wrapping two-cycles of $K3$ correspond to D1-D5 branes where the D5 wrapping on $K3$ and the D1 wrapped along S^1, and adding momentum n along S^1 corresponds to M2-branes wrapping T^2. For $K3\times\mathrm{T}^2$, $\hat{M}^1$ denotes the modulus on the torus and $\hat{M}^i$ the moduli on K3, with $i=2,\dots 23$. The non-trivial intersection numbers and second Chern class are

$$c_{1ij}=c_{ij}\ ,\quad c_{2,1}=c_2(K3)=24\ . \tag{4.1}$$

For this specific manifold, equations (3.32) are invertible allowing to write $(\hat{M}^I,\hat{J})$ in terms of (q_I,J)

$$\hat{M}^1=\sqrt{\frac{1}{2}q_iq_jc^{ij}+\frac{4J^2}{(q_1+\frac{c_2}{24})^2}}(q_1+\frac{c_2}{8})\ , \tag{4.2a}$$

$$\hat{M}^i=c^{ij}q_j\sqrt{\frac{(q_1+\frac{c_2}{8})}{\frac{1}{2}q_iq_jc^{ij}+\frac{4J^2}{(q_1+\frac{c_2}{24})^2}}}\ , \tag{4.2b}$$

$$\hat{J}=\frac{J}{q_1+\frac{c_2}{24}}\sqrt{\frac{(q_1+\frac{c_2}{8})}{\frac{1}{2}q_iq_jc^{ij}+\frac{4J^2}{(q_1+\frac{c_2}{24})^2}}}\ , \tag{4.2c}$$

where we define c^{ij} as the inverse of c_{ij}. Inserting (4.2) in (3.56), the entropy as function of charges becomes

$$S = 2\pi\sqrt{\frac{1}{2}q_i q_j c^{ij}\left(q_1 + \frac{c_2}{8}\right) - \frac{\left(q_1 - \frac{c_2}{24}\right)\left(q_1 + \frac{c_2}{8}\right)}{\left(q_1 + \frac{c_2}{24}\right)^2} J^2} \ . \tag{4.3}$$

Expanding to first order in c_2 gives

$$S = 2\pi\sqrt{Q^3 - J^2}\left(1 + \frac{3}{2}\frac{Q_1 Q_5}{Q^3 - J^2} + \dots\right) \ , \tag{4.4}$$

where we identified the IIB charges as

$$Q_1 Q_5 = \frac{1}{2}c^{ij}q_i q_j \ , \quad n = q_1 \ , \quad Q^3 - J^2 = Q_1 Q_5 n - J^2 \ . \tag{4.5}$$

If all the charges (J, q_i) scale equally, the expression to sub-leading order is:

$$S = 2\pi\sqrt{Q_1 Q_5 n}\left(1 + \frac{3}{2n} - \frac{J^2}{2Q_1 Q_5 n} + \dots\right) \ , \tag{4.6}$$

where the sub-leading dependence on angular momentum is due to leading supergravity result. The higher derivatives terms give rise to corrections proportional to J as displayed in (4.4), but are not important at this order.

Summarizing, we have an expression for the sub-leading corrections to the entropy (4.6) for rotating five dimensional black holes. These corrections come from the supersymmetric completion of $c_{2I}A^I \wedge \mathrm{Tr}(R^2)$, which are believed to be the complete four derivative gauge invariant terms in five dimensions. The entropy (4.6) is what we would like to compare with the microscopic counting formula.

4.2 The Microscopic Degeneracy Formula

The 5D counting problem of the D1-D5 system on $K3$ is captured by a $(4,4)$ two-dimensional superconformal field theory (SCFT) along the worldvolume $\mathbb{R}\times S^1$ with target space $\mathrm{Sym}^{Q_1Q_5+1}(K3)$ [109]. We denote this sigma model SCFT by

$$X^{5D} = \sigma(\mathrm{Sym}^{Q_1Q_5+1}(K3)) \ . \tag{4.7}$$

Two of the charges Q_1, Q_5 that the black hole carries appear in the definition of the sigma model. The third charge momentum n and the angular momentum l appear as the eigenvalues of the hamiltonian L_0 and R-charge $J_0/2$ of the sigma model. The charges are related to the number of D-branes in the following fashion

$$Q_5 = N_5 \ , \quad Q_1 = N_1 - N_5 \ , \tag{4.8}$$

because there is an effective negative unit one-brane charge generated by the five-brane wrapped on the $K3$. The relevant object which captures the degeneracy of BPS states is the elliptic genus

$$\chi(X^{5D}; q, y) \equiv \mathrm{Tr}_{RR}^{X^{5D}} (-1)^{J_0 - \widetilde{J}_0} q^{L_0} \widetilde{q}^{\bar{L}_0} y^{J_0} \equiv \sum_{n,l} c^{5D}(Q_1 Q_5, n, l) q^n y^l \ . \tag{4.9}$$

To estimate the growth of the coefficients of this SCFT, we can use Cardy's formula and spectral flow in the SCFT [5, 82, 106]

$$\Omega \sim \exp\left(\sqrt{\frac{c}{6} L_0 - J^2}\right) + \dots \ , \tag{4.10}$$

with c the central charge of the theory. For the SCFT (4.7) we have

$$c = 6 Q_1 Q_5 \ , \quad L_0 = n \ , \quad J^2 = \frac{l^2}{4} \ . \tag{4.11}$$

Plugging in these values to (4.10), we get

$$\Omega(Q_1, Q_5, n, l) \sim \exp(2\pi \sqrt{Q_1 Q_5 n - l^2/4}) + \dots \ , \tag{4.12}$$

The approximation (4.12) is valid at high values of L_0 compared to c, *i.e.* $n \gg Q_1 Q_5$. One can actually systematically compute corrections to this result using an exact formula which determines the fourier coefficients of the elliptic genus of a symmetric product SCFT in terms of the fourier coefficients of the original SCFT (in this case $K3$) [110]. The formula relies on the modular transformation properties of the elliptic genus under $SL(2, \mathbb{Z})$ and uses the Jacobi-Rademacher expansion [82, 106]. By its nature, it is expressed as a series of corrections to the Cardy formula and can be used as above when $n \gg Q_1 Q_5$.

On the other hand, the black hole entropy function is valid for large values of charges when all the charges scale equally, *i.e.* $Q_1Q_5 \gg n \gg 1$. In order to meaningfully compare the two expressions, we would need to re-sum the Farey tail expansion in Q_1Q_5/n and reexpress it as an expansion in n/Q_1Q_5, which *a priori* seems to be a difficult problem.

However, we can make progress using the relation of the elliptic genus of the symmetric product to the Siegel modular form Φ_{10}. This is known as the Igusa cusp form and is the unique weight 10 modular form of $Sp(2,\mathbb{Z})$. Using the more powerful modular transformation properties and a saddle point approximation, we can compute the expansion of the above elliptic genus for any regime of charges, in particular $n/Q_1Q_5 \ll 1$. Physically, this is related to the 4D-5D lift which we shall discuss in a following section. In this section, we shall simply use this relation to our calculational advantage.

The generating function of the elliptic genus of the symmetric product is given by [110]

$$Z(\rho,\sigma,v) \equiv \sum_{k=0}^{\infty} p^k \chi(Sym^k(X);q,y) = \prod_{n>0,m\geq 0,l} \frac{1}{(1-p^n q^m y^l)^{c(nm,l)}} \ , \tag{4.13}$$

where we have set

$$q = e^{2\pi i\rho} \ , \quad p = e^{2\pi i\sigma} \ , \quad y = e^{2\pi i v} \ , \tag{4.14}$$

and the coefficients $c(n,l)$ are defined through

$$\chi(X;q,y) = \sum_{n,l} c(n,l) q^n y^l \ . \tag{4.15}$$

For $X = K3$, this generating function is related to the Igusa cusp form Φ_{10} as [50],

$$Z(\rho,\sigma,v) = \frac{f^{KK}(\rho,\sigma,v)}{\Phi_{10}(\rho,\sigma,v)}, \tag{4.16}$$

where

$$\begin{aligned} f^{KK}(\rho,\sigma,v) &= p\,q\,y\,(1-y^{-1})^2 \prod_{m=1}^{\infty}(1-q^m)^{20}(1-q^m y)^2(1-q^m y^{-1})^2 \\ &= p\,\eta^{18}(\rho)\,\vartheta_1^2(v,\rho) \ . \end{aligned} \tag{4.17}$$

We are interested in the microscopic degeneracy of the system with charges (Q_1, Q_5, n, l), which is given by the coefficient $c(n,l)$ of the sigma model (4.7). This can be expressed as an inverse Fourier transform of the generating function $Z(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$

$$\Omega^{5D}(Q_1, Q_5, n, l) = \oint_{\mathcal{C}} d\widetilde{\rho} d\widetilde{\sigma} d\widetilde{v}\, e^{-2i\pi(\widetilde{\rho}n+\widetilde{\sigma}(Q_1Q_5+1)+l\widetilde{v})}\, Z(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \ . \tag{4.18}$$

The contour $\mathcal{C}$ in the above integral is presented in Appendix B.1. In the 4D theory, the choice of contour was important for the analysis of BPS decays and the associated walls of marginal stability [111, 112]. The decays happened precisely when the contour crossed a pole related to the decay. These effects did not affect the power series expansion for the entropy, but were exponentially small corrections in the degeneracy formula.

In five dimensions, it is expected from a supergravity analysis that there are no such decays corresponding to real codimension one walls [113]. Note in this context that the purely v dependent factors in the function f^{KK} which have a zero at $v = 0$. These poles therefore do not exist in the 5D partition function. It would be interesting to analyze in more detail all the poles of the partition function in the 5D theory. However, for the purpose of computing power law corrections to the entropy our analysis is sufficient.

4.2.1 Saddle point approximation

We can solve the integral (4.18) in two steps as in [13, 14, 15]. First, we notice that the dominant pole of the expression $1/\Phi_{10}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$ is *not* factored out by the function $f^{KK}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v})$. We can therefore do a contour integral around this pole and the residue is an integral over two remaining coordinates. This can be approximated by the saddle point method to give an asymptotic expansion. We follow the method of [13, 14] of which we present some relevant details in Appendix B.1. The actual evaluation only relies on the fact that the charges n, Q_1Q_5 are large and not on the relative magnitude of the two charges.[3]

[3]This fact was also used for computing the four dimensional black hole entropy in a region where Q^2, P^2 are large, and one was much larger than the other [14].

We find the following expression which is to be evaluated at its extremum,

$$S^{5D}_{stat} = S_0 + S_1 \ , \tag{4.19}$$

where identify the classical (S_0) and first correction to the large charge limit (S_1) as

$$S_0 = -2\pi i\widetilde{\rho}n - 2\pi i\widetilde{\sigma}(Q_1Q_5+1) + 2\pi i(\frac{1}{2} - \tilde{v})l \ , \tag{4.20a}$$

$$S_1 = 12\ln\widetilde{\sigma} - \ln\eta^{24}(\rho) - \ln\eta^{24}(\sigma) + \ln f^{KK}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) \ , \tag{4.20b}$$

with

$$\widetilde{\rho} = \frac{\rho\sigma}{\rho+\sigma}, \quad \widetilde{\sigma} = -\frac{1}{\rho+\sigma}, \quad \widetilde{v} = \frac{1}{2} - \sqrt{\frac{1}{4} + \widetilde{\rho}\widetilde{\sigma}} \ . \tag{4.21}$$

Since we are interested in the answer to only the first order beyond the large charge limit, we can extremize only the classical part S_0 and evaluate the full expression (4.19) at those values. By minimizing the classical functional S_0 we obtain

$$\begin{aligned} \widetilde{\rho} &= \frac{i}{2}\frac{Q_1Q_5+1}{\sqrt{Q^3-J^2}} \ , \\ \widetilde{\sigma} &= \frac{i}{2}\frac{n}{\sqrt{Q^3-J^2}} \ , \end{aligned} \tag{4.22}$$

where[4]

$$Q^3 - J^2 \equiv (Q_1Q_5+1)n - l^2/4 \ . \tag{4.23}$$

Plugging (4.22) in (4.20) gives

$$S_0(Q_1,Q_5,n) = 2\pi\sqrt{Q^3-J^2} \tag{4.24}$$

and

$$\begin{aligned} S_1(Q_1,Q_5,n) = &-\pi\frac{n}{\sqrt{Q^3-J^2}} - 24\ln\eta\left(\frac{l+i2\sqrt{Q^3-J^2}}{2n}\right) - 24\ln\eta\left(\frac{-l+i2\sqrt{Q^3-J^2}}{2n}\right) \\ &+ 18\ln\eta\left(\frac{iQ_1Q_5}{2\sqrt{Q^3-J^2}}\right) + 2\ln\vartheta_1\left(\frac{1}{2} - \frac{il}{4\sqrt{Q^3-J^2}}, \frac{iQ_1Q_5}{2\sqrt{Q^3-J^2}}\right) + \dots \ . \end{aligned} \tag{4.25}$$

[4]The shift of one in Q_1Q_5 is not important to sub-leading order in the black hole regime $Q_1Q_5 >> n$, note the difference with (4.5). This shift will be important in the Cardy regime which we discuss below.

4.2.2 Supergravity limit

In the limit where all the charges (n, Q_1, Q_5, l) are large and scale uniformly, we can use the expansion of the functions $\eta(\tau)$, $\vartheta_1(v,\tau)$ (Appendix B.2) and after dropping higher terms we get

$$
\begin{aligned}
S_1(Q_1,Q_5,n) &= 4\pi\frac{\sqrt{Q^3-J^2}}{n} - \pi\frac{Q_1Q_5}{\sqrt{Q^3-J^2}} + \dots \\
&= 3\pi\sqrt{\frac{Q_1Q_5}{n}} + \dots
\end{aligned} \tag{4.26}
$$

Combining (4.24) and (4.26), the full entropy formula reads

$$
S^{5D}(Q_1,Q_5,n) = 2\pi\sqrt{Q_1Q_5n}\left(1 + \frac{3}{2n} - \frac{l^2}{8Q_1Q_5n}\right) + \dots \tag{4.27}
$$

We see that this agrees with the macroscopic result (4.6) in the same regime of large charges.

4.2.3 Cardy limit

In the opposite limit, when $n \gg Q_1Q_5$, and $Q^3 - J^2 \gg 1$ we can also expand the result (4.25) to sub-leading order. In order to do that, we first need to use the modular transformation properties of the various functions (Appendix B.2)

$$
\begin{aligned}
S_1(Q_1,Q_5,n) = &-\pi\frac{n}{\sqrt{Q^3-J^2}} - 24\ln\eta\left(\frac{l+i2\sqrt{Q^3-J^2}}{2Q_1Q_5}\right) - 24\ln\eta\left(\frac{-l+i2\sqrt{Q^3-J^2}}{2Q_1Q_5}\right) \\
&+ 18\ln\eta\left(i\frac{2\sqrt{Q^3-J^2}}{Q_1Q_5}\right) + 2\ln\vartheta_1\left(-i\frac{2\sqrt{Q^3-J^2}}{Q_1Q_5}\left[\frac{1}{2} - \frac{il}{4\sqrt{Q^3-J^2}}\right], i\frac{2\sqrt{Q^3-J^2}}{Q_1Q_5}\right) \\
&+ 2\pi\frac{2n}{\sqrt{Q^3-J^2}}\left[\frac{1}{2} - \frac{l}{i4\sqrt{Q^3-J^2}}\right]^2 + \dots
\end{aligned} \tag{4.28}
$$

Dropping terms of higher order in Q_1Q_5/n in (4.28) we get

$$
\begin{aligned}
S_1(Q_1,Q_5,n) = &\, 4\pi\frac{n}{\sqrt{Q^3-J^2}} - 3\pi\frac{\sqrt{Q^3-J^2}}{Q_1Q_5} \\
&+ \pi\frac{n}{\sqrt{Q^3-J^2}} - \pi\frac{n}{\sqrt{Q^3-J^2}} - \pi\frac{n}{\sqrt{Q^3-J^2}} + \dots \\
= &\, 0 + \mathcal{O}\left(\frac{1}{\sqrt{Q_1Q_5n - l^2/4}}\right) + \dots,
\end{aligned} \tag{4.29}
$$

which finally allows us to write the entropy as

$$S^{5D}(Q_1, Q_5, n) = 2\pi\sqrt{(Q_1Q_5+1)n - l^2/4}\left(1 + \mathcal{O}\left(\frac{1}{Q_1Q_5n - l^2/4}\right)\right) + \dots \tag{4.30}$$

Note that unlike in the other limit, all the terms suppressed by $1/Q_1Q_5$ have dropped away, and the first sub-leading term is suppressed by $1/(Q^3 - J^2)$. This is exactly in agreement with the more familiar Jacobi-Rademacher expansion to the same order which we have sketched in Appendix B.3.

4.3 Quantum/String Corrections to the 4D-5D Connection

The 4D-5D connection [105, 114, 115, 116, 117] is a relation between a black hole in five dimensions carrying three gauge charges plus angular momentum, and a black hole in four dimensions carrying the above charges and in addition, a unit[5] Taub-NUT charge. The angular momentum in the five dimensions becomes mometum along the Taub-NUT circle at infinity in four dimensions. On application to a rotating BMPV black hole preserving 1/4 supersymmetry, the 5D black hole can be related to a four dimensional 1/4 dyonic black hole. This relation can be used to derive an exact counting formula for 1/4 BPS dyons in $N = 4$ string theory [51, 14].

As a consequence of the attractor mechanism, the entropy of extremal black holes is independent of asymptotic value of moduli. By tuning one of these moduli, one can make the curvature of the Taub-NUT space large or small. Therefore it seems reasonable to relate the entropy of 4D dyonic black holes with 5D black holes and the leading order prescription [51] is

$$S^{4D}(Q_1Q_5+1, n, l) = S^{5D}(Q_1Q_5, n, l)\ , \tag{4.31}$$

This equation however, will receive corrections[6] at sub-leading order

$$S^{4D}(Q_1, Q_5, n, l) = S^{5D}(Q_1, Q_5, n, l)\left(1 + \frac{c_1}{Q^2} + \dots\right)\ . \tag{4.32}$$

[5]This has been extended recently to the case when there are multiple KK monopoles [104].
[6]These corrections are not related to the shift in the charges in (4.31).

The computation of these corrections boils down to computing the sub-leading corrections to the 5D black hole entropy and comparing with the known sub-leading corrections to the 4D black hole entropy. The results of the previous sections fill in this gap, and we can now explain the origin of the small difference in the 4D and the 5D black hole entropy both from the microscopic and macroscopic viewpoints.

In the regime of charges that all the charges are large and scaled equally, the 5D entropy is (4.4), (4.27)

$$S^{5D}(Q_1, Q_5, n, l) = 2\pi\sqrt{Q_1 Q_5 n}\left(1 + \frac{3}{2n} - \frac{l^2}{8Q_1 Q_5 n}\right) + \ldots \tag{4.33}$$

In the same limit, the corresponding 4D black hole with one additional Taub-NUT charge is (see the review [15] and references therein)

$$S^{4D}(Q_1, Q_5, n, l) = 2\pi\sqrt{Q_1 Q_5 n}\left(1 + \frac{2}{n} - \frac{l^2}{8Q_1 Q_5 n}\right) + \ldots \tag{4.34}$$

The discrepancy between the two expressions is essentially accounted for by the Taub-NUT space whose small effects remain at all values of the moduli. The interesting fact is that the actual micro and macro mechanisms are different. As we explain below, in the microscopic theory, the Taub-NUT space gives rise to additional bound states, which changes the degeneracy function, whereas in the macroscopic formalism, the Taub-NUT space changes the final value of entropy because of a Chern-Simon coupling in the effective action. It is a non-trivial reflection of the consistency of string theory that the two mechanisms in different regimes of parameter space account quantitatively for the same effect.

4.3.1 Macroscopic mechanism

There is a rich web of interconnections between supergravity theories in diverse dimensions, and it is illuminating to consider the relations between solutions to these different theories. A solution with a spacelike isometry can be converted to a lower dimensional one by Kaluza-Klein reduction along the isometry direction. Conversely, a solution can be up-lifted to one higher dimension by interpreting a gauge field as the off-diagonal components of a higher dimensional metric.

The BPS equations governing general 4D supersymmetric solutions are well established, including the contributions from a class of four-derivative corrections [9]. On the other hand, we have obtained the corresponding 5D BPS equations. At the two-derivative level, the authors in [117] showed that the 4D BPS equations can be mapped to a special case of the 5D BPS equations. That is to say, the general 4D BPS solution can be interpreted as the rewriting of a 5D solution. The generalization of this correspondence with four-derivative corrections unfortunately fails, apparently no simple relation between the two sets of solutions [40].

We now turn to the relation between the entropies of four and five dimensional black holes. To illustrate the salient issues consider the simplest case of electrically charged, non-rotating, 5D black holes, and their 4D analogues. At the two-derivative level the following relation holds [105]

$$S^{5D}(q_I) = S^{4D}(q_I, p^0 = 1) \ . \tag{4.35}$$

This formula is motivated by placing the 5D black hole at the tip of Taub-NUT. Since Taub-NUT is a unit charge Kaluza-Klein monopole, this yields a 4D black hole carrying magnetic charge $p^0 = 1$. On the other hand, suppose that we sit at a fixed distance from the black hole and then expand the size of the Taub-NUT circle to infinity. Since Taub-NUT looks like $\mathbb{R}^4$ near the origin it is clear that this limiting process gives back the original 5D black hole. Finally, the moduli independence of the entropy yields (4.35).

The preceding argument contains a hidden assumption, namely that the act of placing the black hole in Taub-NUT does not change its electric charge. But why should this be so? In fact it is not, as was first noticed [39] and further studied in [53]. The reason is that higher derivative terms induce a delocalized charge density on the Taub-NUT, so that the charge carried by the 4D black hole is actually that of the 5D black hole plus that of the Taub-NUT.

Expanding on this point, let us return to the general solutions of Section 2.3.6, *i.e.* spinning black holes on a Gibbons-Hawking base. The Maxwell equations led to (2.118) which demonstrates that the curvature on the base space provides a delocalized source for the gauge field. This effect should be expected simply from the fact that we deal with an action with a $\int A^I \wedge R \wedge R$ Chern-Simons term.

To make explicit the relation between the charges, consider a general action of gauge fields in the language of forms

$$S = \frac{1}{4\pi^2} \int_{\mathcal{M}_5} \star_5 \mathcal{L}\left(A^I, F^I\right) \ . \tag{4.36}$$

The Euler-Lagrange equations of motion are

$$d \star_5 \frac{\partial \mathcal{L}}{\partial F^I} = \star_5 \frac{\partial \mathcal{L}}{\partial A^I} \ . \tag{4.37}$$

Since the left side is exact, we see that this identifies a divergenceless current

$$j_I = \frac{\partial \mathcal{L}}{\partial A^I} \ . \tag{4.38}$$

The conserved charge is obtained by integrating $\star_5 j_I$ over a spacelike slice Σ, suitably normalized. Through the equations of motion and Stoke's theorem this can be expressed as an integral over the asymptotic boundary of Σ

$$Q_I = -\frac{1}{4\pi^2} \int_{\partial\Sigma} \star_5 \frac{\partial \mathcal{L}}{\partial F^I} \ , \tag{4.39}$$

which clearly reproduces the conventional $Q \sim \int \star F$ for the Maxwell action.

For the present case, we consider solutions where the gauge fields fall off sufficiently fast that only the two-derivative terms in the Lagrangian lead to non-zero contributions to the surface integral in (4.39). Our charge formula is then

$$Q_I = -\frac{1}{2\pi^2}\int_{\partial\Sigma}\left(\frac{1}{2}\mathcal{N}_{IJ}\star_5 F^J + 2M_I \star_5 v\right) . \tag{4.40}$$

For our solutions with timelike supersymmetry, we can identify Σ with the hyper Kähler base space.

Now let us compare the charge computations for two distinct solutions, one with a flat $\mathbb{R}^4$ base space and another on Taub-NUT, considering just the non-rotating black hole for simplicity. As mentioned previously, both $\mathbb{R}^4$ and Taub-NUT can be written as

$$ds^2 = (H^0)^{-1}(dx^5 + \vec{\chi}\cdot d\vec{x})^2 + H^0\left(d\rho^2 + \rho^2 d\Omega_2\right) , \tag{4.41}$$

where

$$H^0 = \eta + \frac{1}{\rho} , \tag{4.42}$$

with $\eta = 0$ for $\mathbb{R}^4$ and $\eta = 1$ for Taub-NUT. The coordinate x^5 is compact with period 4π, and we choose the orientation $\epsilon_{\hat\rho\hat\theta\hat\phi\hat5} = 1$. Using the formulas from Section 2.3 for the gauge fields and auxiliary field, we see that in Gibbons-Hawking coordinates

$$Q_I = \lim_{\rho\to\infty}\left[-4\rho^2\partial_\rho\left(e^{-2U}M_I\right)\right] , \tag{4.43}$$

which is independent of the Gibbons-Hawking function H^0.

Now recall the result from the higher-derivative Maxwell equation (2.125)

$$M_I e^{-2U} - \frac{c_{2I}}{8}(\tilde\nabla U)^2 - \frac{c_{2I}}{24\cdot 8}\Phi = 1 + \frac{q_I}{4\rho} , \tag{4.44}$$

where Φ is defined in (2.124) as

$$\tilde R^2_{\hat i\hat j\hat k\hat l} = \tilde\nabla^2\Phi \equiv \tilde\nabla^2\left(2\frac{(\tilde\nabla H^0)^2}{(H^0)^2} + \sum_i \frac{a_i}{|\vec{x}-\vec{x}_i|}\right) . \tag{4.45}$$

Both $\mathbb{R}^4$ and $p^0 = 1$ Taub-NUT are completely smooth geometries and so there are no singularities in the corresponding $\tilde R^2$. On the other hand, (4.45) has manifestly singular terms unless the a_i are chosen to cancel singularities in the H^0 dependent term. Comparing with (4.42), we see that smoothness is assured when the a_i are chosen such that

$$\tilde{R}^2_{\hat{i}\hat{j}\hat{k}\hat{l}} = \tilde{\nabla}^2 \left(2\frac{(\tilde{\nabla}H^0)^2}{(H^0)^2} - \frac{2}{\rho} \right) . \tag{4.46}$$

The solution (4.44) is now fully specified as

$$M_I e^{-2U} - \frac{c_{2I}}{8}(\tilde{\nabla}U)^2 - \frac{c_{2I}}{24\cdot 4}\left(\frac{(\tilde{\nabla}H^0)^2}{(H^0)^2} - \frac{1}{\rho} \right) = 1 + \frac{q_I}{4\rho} . \tag{4.47}$$

In the absence of stringy corrections, *i.e.* for $c_{2I} = 0$, we have $M_I e^{-2U} = 1 + \frac{q_I}{4\rho}$ which gives $Q_I = q_I$ independent of the base space geometry. However, including these corrections we see that aymptotically[7]

$$M_I e^{-2U} = 1 + \frac{1}{4}\left(q_I - \eta\frac{c_{2I}}{24} \right) \rho^{-1} + O(\rho^{-2}) , \tag{4.48}$$

yielding the asymptotic charge

$$Q_I = q_I - \eta\frac{c_{2I}}{24} . \tag{4.49}$$

The preceding computation tells us that formula (4.35) gets modified to

$$S^{5D}(q_I) = S^{4D}(q_I - \frac{c_{2I}}{24}, p^0 = 1) . \tag{4.50}$$

The shift in q_I exactly accounts for the different coefficients for the subleading terms in (4.33) and (4.34). An analogous, and more complicated, relation holds in the case of rotating black holes; see [53] for the details.

4.3.2 Microscopic mechanism

The microscopic setup in type IIB string theory on $K3$ has a D1-D5-p system with the D5 branes wrapping the $K3$, and the effective D1-D5 string with momentum p wrapping a circle S^1. The rest of the five dimensions is a KK monopole (Taub-NUT geometry)

[7]We are ignoring the $\frac{c_{2I}}{8}(\tilde{\nabla}U)^2$ term since one can check that it falls off too rapidly as $\rho \to \infty$ to contribute to Q_I.

which asymptotes to $\mathbb{R}^{3,1} \times \widetilde{S}^1$. The branes sit at the center of the Taub-NUT space where spacetime looks like $\mathbb{R}^{4,1}$. The counting of 1/4 BPS dyons is done by looking at low energy excitations of this system. The counting problem effectively becomes a product of three decoupled systems [14] which we can paraphrase as computing the modified elliptic genus of the following 2D SCFT:

$$X^{4D} = X^{5D} \times \sigma(TN_1) \times \sigma_L(KK - P) \tag{4.51a}$$

$$X^{5D} = \sigma(Sym^{Q_1Q_5+1}(K3)) \tag{4.51b}$$

The first factor which is a symmetric product theory which controls the 5D BPS counting problem of the D1-D5 system. The piece $\sigma(TN_1)$ describes the bound states of the center of mass of the $D1$-$D5$ with the KK monopole. The piece σ_L(KK-P) describes the bound states of the KK monopole and momentum and is a conformal field theory of 24 left-moving bosons of the heterotic string, which can be deduced from the duality between the Type-IIB KK-P system and the heterotic F1-P system. The presence of the second and third factor is crucial for establishing S-duality and the wall-crossing phenomena in 4D.

The degeneracy of the BPS states of the theory X^{4D} is given in terms of the partition function which is the inverse of the Igusa cusp form, the unique weight 10 modular form of $Sp(2, \mathbb{Z})$.

$$\Omega^{4D}(Q_1, Q_5, n, l) = \oint_{\mathcal{C}} d\rho d\sigma dv \, e^{-2i\pi(\rho n + \sigma Q_1 Q_5 + lv)} \, \frac{1}{\Phi_{10}(\rho, \sigma, v)}. \tag{4.52}$$

This partition function is understood by separately counting the three decoupled pieces in the formula (4.51a) above. The degeneracies of the theory X^{5D} is given by a similar inverse fourier transform (4.18) with a partition function $Z(\rho, \sigma, v)$ which differs sightly from that of the 4D theory.

The discrepancy between the two partition functions (4.16) is due to the factors $\sigma(TN_1)\times$ $\sigma_L(KK - P)$ which completes the 5D system into the 4D system. The BPS partition function of the extra piece related to the KK monopole is precisely $f^{KK}(\rho, \sigma, v)$ (4.16), (4.17). Physically, most of the entropy of the dyonic black hole comes from the first factor in (4.51a) which governs the 5D black hole, but a small fraction of the entropy of the 4D black hole comes from the bound states of momentum and center of mass with the KK monopole itself. This small fraction precisely accounts for the sub-leading corrections to the 4D-5D connection formula[8].

In addition this new tool allows us to understand certain features of 5D black holes and contrast them against 4D black holes. The first such feature is spacetime duality. The 4D duality group is bigger than the 5D one, and in particular it contains the 4D electric-magnetic duality which is absent in 5D. The manifestation of this duality which exchanges $n \equiv Q^2 \leftrightarrow P^2 \equiv Q_1 Q_5$ appears through the pre-potential in the 4D gravity theory, to which worldsheet/membrane instantons (depending on the duality frame) wrapping the T^2 contribute in a crucial way. These instanton contributions complete the classical linear prepotential into a transcendental function related to the Jacobi η function which is S-duality invariant. The entropy function which depends on the prepotential is thus also duality invariant.

In five dimensions, one of the circles which these worldsheets/branes wrap becomes large and the five dimensional supergravity does not see their effects, and only the contributions $P^2 \gg Q^2$ are retained. The entropy function only contains the residue of the leading linear piece which is *not* duality invariant as expected since 5D supergravity admits no such S-duality.

[8] In the limit of large charges in which we evaluate the integral, the contribution from the KK monopole piece comes purely from the ground state, and one can explicitly see the equivalence to the macroscopic mechanism already at this level of the calculation.

CHAPTER V

Holographic Description of AdS_2 Black Holes

In different contexts within string theory and black holes physics, one encounters configurations that contain a two dimensional AdS factor. For example, AdS_2 appears in 2D quantum gravity as an invariant $SL(2,\mathbb{R})$ ground state for Liouville theory [118, 119]. More interestingly, AdS_2 is an attractor solution that universally governs the near horizon geometry of all extremal black holes [120] (see Chapter IV for more details). It is therefore natural to study quantum black holes by applying the AdS/CFT correspondence to the AdS_2 factor. There have been several interesting attempts at implementing this strategy [121, 122, 123, 124, 125, 126, 127, 128, 129, 130, 131] but AdS_2 holography remains enigmatic, at least compared with the much more straightforward case of AdS_3 holography.

The approach presented here was proposed initially by Hartman and Strominger [132], in the context of Maxwell-dilaton gravity with bulk action

$$I_{\text{bulk}} = \frac{\alpha}{2\pi}\int_{\mathcal{M}} d^2x\sqrt{-g}\,\left[e^{-2\phi}\left(R+\frac{8}{L^2}\right)-\frac{L^2}{4}F^2\right]\ . \tag{5.1}$$

These authors pointed out that, for this theory, the usual conformal diffeomorphisms must be accompanied by gauge transformations, in order to maintain boundary conditions. They found that the combined transformations satisfy a Virasoro algebra with a specific central charge. These results suggest a close relation to the AdS_3 theory.

In the following we develop the holographic description of AdS_2 for the theory (5.1) systematically, following the procedures that are well-known from the AdS/CFT correspondence in higher dimensions.

Specifically, we apply the standard holographic renormalization procedure [133, 134, 135, 136] to asymptotically AdS_2 spacetimes. The basic idea is to impose precise boundary conditions and determine the boundary counterterms needed for a consistent variational principle. These counterterms encode the infrared divergences of the bulk theory. Surprisingly, consistency of the theory requires a boundary term that takes the form of a mass term for the gauge field that appears to violate gauge invariance. However, we demonstrate that the new counterterm is invariant with respect to all gauge variations that preserve the boundary conditions.

The divergences removed by the counterterms will also render a finite conserved quantities at the boundary, such as the energy-momentum stress tensor. According to AdS/CFT, the stress tensor should exhibit the properties of the a stress tensor in a quantum CFT. In particular it should transform like a tensor plus a Schwarzian derivative under the conformal group. By analyzing the transformation properties of the current under diffeomorphism accompanied by a gauge transformations, we verify the enhancement of the asymptotic $SL(2,\mathbb{R})$ conformal symmetry of the theory to a Virasoro algebra, and obtain a non-zero central charge.

Since holography is better understood in three dimensional gravity, it is useful to show that our results in two dimensions are consistent with dimensional reduction of standard results in three dimensions. In particular, we verify that our result for the central charge agrees with the Brown–Henneaux central charge for AdS_3 spacetimes [43]. Still there are some important issues relate to the details of the KK-reduction. In our embedding of asymptotically AdS_2 into AdS_3 we maintain Lorentzian signature and reduce along a direction that is light-like in the boundary theory, but space-like in the bulk. A satisfying feature of the set-up is that the null reduction on the boundary manifestly freezes the holomorphic sector of the boundary theory in its ground state, as it must since the global symmetry is reduced from $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ to $SL(2,\mathbb{R})$. The corollary is that the boundary theory dual to asymptotically AdS_2 necessarily becomes the chiral part of a CFT and such a theory is not generally consistent by itself [137, 138]. The study of the ensuing microscopic questions is beyond the scope presented here.

Finally, we use our results to discuss the entropy of black holes in AdS_2. To be more precise, we use general principles to determine enough features of the microscopic theory that we can determine its entropy, but we do not discuss detailed implementations in string theory. This is in the spirit of the well-known microscopic derivation of the entropy of the BTZ black hole in 3D [47], and also previous related results in AdS_2 [122, 139, 140, 141].

The main lesson we draw from our results is that, even for AdS_2, the AdS/CFT correspondence can be implemented in a rather conventional manner. The discussion is based on the results found in [55].

5.1 Boundary Counterterms in Maxwell-Dilaton AdS Gravity

In this section we study a charged version of a specific 2D dilaton gravity. We construct a well-defined variational principle for this model by adding boundary terms to the standard action, including a novel boundary mass term for the $U(1)$ gauge field.

5.1.1 Bulk action and equations of motion

There exist many 2D dilaton gravity models that admit an AdS ground state (see [142, 143] and references therein). For the sake of specificity we pick a simple example — the Jackiw–Teitelboim model [144] — and add a minimally coupled $U(1)$ gauge field. The bulk action

$$I_{\text{bulk}} = \frac{\alpha}{2\pi} \int_{\mathcal{M}} d^2x \sqrt{-g} \left[e^{-2\phi} \left(R + \frac{8}{L^2} \right) - \frac{L^2}{4} F^2 \right] , \tag{5.2}$$

is normalized by the dimensionless constant α which is left unspecified for the time being. For constant dilaton backgrounds we eventually employ the relation

$$\alpha = -\frac{1}{8G_2} e^{2\phi} \tag{5.3}$$

between the 2D Newton constant G_2 and α. While the factors in (5.3) are the usual ones (see Appendix C.1), the sign will be justified in later sections by computing various physical quantities.

The variation of the action with respect to the fields takes the form

$$\delta\, I_{\text{bulk}} = \frac{\alpha}{2\pi} \int_{\mathcal{M}} d^2x\sqrt{-g}\, \Big[\mathcal{E}^{\mu\nu}\, \delta g_{\mu\nu} + \mathcal{E}_\phi\, \delta\phi + \mathcal{E}^\mu\, \delta A_\mu\Big] + \text{boundary terms} \ , \tag{5.4}$$

with

$$\mathcal{E}_{\mu\nu} = \nabla_\mu \nabla_\nu e^{-2\phi} - g_{\mu\nu}\, \nabla^2 e^{-2\phi} + \frac{4}{L^2}\, e^{-2\phi}\, g_{\mu\nu} + \frac{L^2}{2}\, F_\mu{}^\lambda\, F_{\nu\lambda} - \frac{L^2}{8}\, g_{\mu\nu}\, F^2 \ , \tag{5.5a}$$

$$\mathcal{E}_\phi = -2\, e^{-2\phi} \left(R + \frac{8}{L^2}\right) \ , \tag{5.5b}$$

$$\mathcal{E}_\mu = L^2\, \nabla^\nu F_{\nu\mu} \ . \tag{5.5c}$$

Setting each of these equal to zero yields the equations of motion for the theory. The boundary terms will be discussed in Section 5.1.2 below.

All classical solutions to (5.5) can be found in closed form [142, 143]. Some aspects of generic solutions with non-constant dilaton will be discussed in Section 5.5, below. Until then we focus on solutions with constant dilaton, since those exhibit an interesting enhanced symmetry. This can be seen by noting that the dilaton equation $\mathcal{E}_\phi = 0$ implies that all classical solutions must be spacetimes of constant (negative) curvature. Such a space is maximally symmetric and exhibits three Killing vectors, *i.e.* it is locally (and asymptotically) AdS$_2$. A non-constant dilaton breaks the $SL(2,\mathbb{R})$ algebra generated by these Killing vectors to $U(1)$, but a constant dilaton respects the full AdS$_2$ algebra.

With constant dilaton the equations of motion reduce to

$$R + \frac{8}{L^2} = 0 \;, \qquad \nabla^\nu F_{\nu\mu} = 0 \;, \qquad e^{-2\phi} = -\frac{L^4}{32}\, F^2 \;. \tag{5.6}$$

The middle equation in (5.6) is satisfied by a covariantly constant field strength

$$F_{\mu\nu} = 2E\, \epsilon_{\mu\nu}, \tag{5.7}$$

where E is a constant of motion determining the strength of the electric field. The last equation in (5.6) determines the dilaton in terms of the electric field,

$$e^{-2\phi} = \frac{L^4}{4}\, E^2 \,. \tag{5.8}$$

Expressing the electric field in terms of the dilaton, we can rewrite (5.7) as $F_{\mu\nu} = \frac{4}{L^2}\, e^{-\phi}\, \epsilon_{\mu\nu}$. Without loss of generality, we have chosen the sign of E to be positive. The first equation in (5.6) requires the scalar curvature to be constant and negative. Working in a coordinate and $U(1)$ gauge where the metric and gauge field take the form

$$ds^2 = d\eta^2 + g_{tt}\, dt^2 = d\eta^2 + h_{tt}\, dt^2 \;, \qquad A_\mu dx^\mu = A_t(\eta, t)\, dt \;, \tag{5.9}$$

the curvature condition simplifies to the linear differential equation

$$\frac{\partial^2}{\partial \eta^2} \sqrt{-g} = \frac{4}{L^2}\, \sqrt{-g}\,, \tag{5.10}$$

which is solved by $\sqrt{-g} = \big(h_0(t)\, e^{2\eta/L} + h_1(t)\, e^{-2\eta/L}\big)/2$. Therefore, a general solution to (5.6) is given by

$$g_{\mu\nu} dx^\mu dx^\nu = d\eta^2 - \frac{1}{4}\, \Big(h_0(t)\, e^{2\eta/L} + h_1(t)\, e^{-2\eta/L}\Big)^2\, dt^2 \;, \tag{5.11a}$$

$$A_\mu dx^\mu = \frac{1}{L}\, e^{-\phi}\, \Big(h_0(t)\, e^{2\eta/L} - h_1(t)\, e^{-2\eta/L} + a(t)\Big)\, dt \;, \tag{5.11b}$$

$$\phi = \text{constant} \;, \tag{5.11c}$$

where h_0, h_1, and a are arbitrary functions of t. This solution can be further simplified by fixing the residual gauge freedom in (5.9). In particular, the $U(1)$ transformation

$A_\mu \to A_\mu + \partial_\mu \Lambda(t)$ preserves the condition $A_\eta = 0$, and a redefinition $h_0(t)dt \to dt$ of the time coordinate preserves the conditions $g_{\eta\eta} = 1$ and $g_{\eta t} = 0$. This remaining freedom is fixed by requiring $a(t) = 0$ and $h_0(t) = 1$. Thus, the general gauge-fixed solution of the equations of motion depends on the constant ϕ, specified by the boundary conditions, and an arbitrary function $h_1(t)$.

Following the standard implementation of the AdS/CFT correspondence in higher dimensions, we describe asymptotically AdS_2 field configurations by (5.9) with the Fefferman-Graham expansions:

$$h_{tt} = e^{4\eta/L}\, g_{tt}^{(0)} + g_{tt}^{(1)} + e^{-4\eta/L}\, g_{tt}^{(2)} + \dots\ , \tag{5.12a}$$

$$A_t = e^{2\eta/L}\, A_t^{(0)} + A_t^{(1)} + e^{-2\eta/L}\, A_t^{(2)} + \dots\ , \tag{5.12b}$$

$$\phi = \phi^{(0)} + e^{-2\eta/L}\, \phi^{(1)} + \dots\ . \tag{5.12c}$$

Our explicit solutions (5.11) take this form with asymptotic values

$$g_{tt}^{(0)} = -\frac{1}{4}\,, \qquad A_t^{(0)} = \frac{1}{L}\, e^{-\phi^{(0)}}\,, \qquad \phi^{(0)} = \text{constant}\,, \tag{5.13}$$

and specific values for the remaining expansion coefficients in (5.12). The variational principle considers general off-shell field configurations with (5.13) imposed as boundary conditions, but the remaining expansion coefficients are free to vary from their on-shell values.

5.1.2 Boundary terms

An action principle based on (5.2) requires a number of boundary terms:

$$I = I_{\text{bulk}} + I_{\text{GHY}} + I_{\text{counter}} = I_{\text{bulk}} + I_{\text{boundary}}\ . \tag{5.14}$$

The boundary action I_{GHY} is the dilaton gravity analog of the Gibbons–Hawking–York (GHY) term [145, 146], and it is given by

$$I_{\text{GHY}} = \frac{\alpha}{\pi} \int_{\partial\mathcal{M}} dx \sqrt{-h}\, e^{-2\phi}\, K\,, \tag{5.15}$$

where h is the determinant of the induced metric on $\partial\mathcal{M}$, and K the trace of the extrinsic curvature (our conventions are summarized in Appendix C.1). This term is necessary for the action to have a well-defined boundary value problem for fields satisfying Dirichlet conditions at $\partial\mathcal{M}$. However, on spacetimes with non-compact spatial sections this is not sufficient for a consistent variational principle. We must include in (5.14) a set of 'boundary counterterms' so that the action is extremized by asymptotically AdS_2 solutions of the equations of motion. In order to preserve the boundary value problem these counterterms can only depend on quantities intrinsic to the boundary. Requiring diffeomorphism invariance along the boundary leads to the generic *ansatz*

$$I_{\text{counter}} = \int_{\partial\mathcal{M}} dx\sqrt{-h}\,\mathcal{L}_{\text{counter}}(A^a A_a, \phi)\,. \tag{5.16}$$

In the special case of vanishing gauge field the counterterm must reduce to $\mathcal{L}_{\text{counter}} \propto e^{-2\phi}$, cf. e.g. [147]. In the presence of a gauge field the bulk action contains a term that scales quadratically with the field strength. Therefore, the counterterm may contain an additional contribution that scales quadratically with the gauge field. This lets us refine the *ansatz* (5.16) to

$$I_{\text{counter}} = \frac{\alpha}{\pi}\int_{\partial\mathcal{M}} dx\sqrt{-h}\,\Big[\lambda\, e^{-2\phi} + m\, A^a A_a\Big]\,. \tag{5.17}$$

The coefficients λ, m of the boundary counterterms will be determined in the following.

With these preliminaries the variation of the action (5.14) takes the form

$$\delta I = \int_{\partial\mathcal{M}} dx\sqrt{-h}\,\Big[(\pi^{ab}+p^{ab})\,\delta h_{ab} + (\pi_\phi + p_\phi)\,\delta\phi + (\pi^a + p^a)\,\delta A_a\Big] + \text{bulk terms}\,, \tag{5.18}$$

where the bulk terms were considered already in the variation of the bulk action (5.4). The boundary contributions are given by

$$\pi^{tt} + p^{tt} \;= \frac{\alpha}{2\pi}\left(h^{tt}\, n^\mu\partial_\mu e^{-2\phi} + \lambda\, h^{tt}\, e^{-2\phi} + m\, h^{tt}\, A^t A_t - 2\, m\, A^t A^t\right)\,, \tag{5.19a}$$

$$\pi^t + p^t \;= \frac{\alpha}{2\pi}\left(-L^2\, n_\mu\, F^{\mu t} + 4\, m\, A^t\right)\,, \tag{5.19b}$$

$$\pi_\phi + p_\phi = -2\frac{\alpha}{\pi}\, e^{-2\phi}\left(K + \lambda\right)\,. \tag{5.19c}$$

In our notation 'π' is the part of the momentum that comes from the variation of the bulk action and the GHY term, and 'p' represents the contribution from the boundary counterterms.

For the action to be extremized the terms in (5.18) must vanish for generic variations of the fields that preserve the boundary conditions (5.13). If we consider field configurations admitting an asymptotic expansion of the form (5.12), then the boundary terms should vanish for arbitrary variations of the fields whose leading asymptotic behavior is:

$$\delta h_{tt} = \delta g_{tt}^{(1)} = \text{finite} \tag{5.20a}$$

$$\delta A_t = \delta A_t^{(1)} = \text{finite} \tag{5.20b}$$

$$\delta \phi = e^{-2\eta/L}\, \delta\phi^{(1)} \to 0 \tag{5.20c}$$

We refer to variations of the form (5.20) as "variations that preserve the boundary conditions".

Inserting the asymptotic behavior (5.12) in (5.19a)-(5.19c), the boundary terms in (5.18) become

$$\begin{aligned} \delta I \Big|_{\text{EOM}} = \frac{\alpha}{\pi} \int_{\partial \mathcal{M}} dt\, \Big[&- e^{-2\phi} \left(\lambda + \frac{4}{L^2} m\right) e^{-2\eta/L}\, \delta h_{tt} - e^{-2\phi} \left(\frac{2}{L} + \lambda\right) e^{2\eta/L}\, \delta\phi \\ &+ 2e^{-\phi} \left(1 - \frac{2}{L} m\right) \delta A_t + \ldots \Big] \, , \end{aligned} \tag{5.21}$$

where '...' indicates terms that vanish at spatial infinity for any field variations that preserve the boundary conditions. The leading terms in (5.21) vanish for λ and m given by

$$\lambda = -\frac{2}{L} \, , \qquad m = \frac{L}{2} \, . \tag{5.22}$$

As a consistency check we note that these *two* values cancel *three* terms in (5.21). Also, the value of λ, which is present for dilaton gravity with no Maxwell term, agrees with previous computations [147]. [1] With the values (5.22) the variational principle is well-defined because the variation of the on-shell action vanishes for all variations that preserve the boundary conditions.

In summary, the full action

$$I=\frac{\alpha}{2\pi}\int_{\mathcal{M}}d^2x\sqrt{-g}\left[e^{-2\phi}\left(R+\frac{8}{L^2}\right)-\frac{L^2}{4}F^{\mu\nu}F_{\mu\nu}\right]$$
$$+\frac{\alpha}{\pi}\int_{\partial\mathcal{M}}dx\,\sqrt{-h}\left[e^{-2\phi}\left(K-\frac{2}{L}\right)+\frac{L}{2}A^aA_a\right]\,,$$

has a well-defined boundary value problem, a well-defined variational principle, and is extremized by asymptotically AdS_2 solutions of the form (5.12).

5.1.3 Boundary mass term and gauge invariance

The boundary term

$$I_{\text{new}}=\frac{\alpha L}{2\pi}\int_{\partial\mathcal{M}}dx\sqrt{-h}\,A^aA_a \tag{5.23}$$

is novel and requires some attention, because it would seem to spoil invariance under the gauge transformations

$$A_\mu\to A_\mu+\partial_\mu\Lambda\ . \tag{5.24}$$

The purpose of this section is to show that the mass term (5.23) is in fact invariant under gauge transformations that preserve the gauge condition $A_\eta=0$ and the boundary condition specified in (5.20b).

The gauge parameter Λ must have the asymptotic form

$$\Lambda=\Lambda^{(0)}(t)+\Lambda^{(1)}(t)\,e^{-2\eta/L}+\mathcal{O}\left(e^{-4\eta/L}\right) \tag{5.25}$$

[1]We also comment on the only previous example of A^2 boundary terms that we are aware of [148]. That work employs the Einstein frame, which is not accessible in 2D, and many of the expressions appearing in that paper indeed diverge when applied to 2D. An exception is their equation (92), which determines the numerical factor N_0 in the boundary mass term (90) for the gauge field B_i. Equation (92) has two solutions, and the authors of [148] exclusively consider the trivial one $N_0=0$, *i.e.* there is no boundary mass term. However, the other solution leads to a non-vanishing boundary mass term for the gauge field. Translating their notations to ours ($d=1$, $N_0=2m\alpha/\pi$, $K_0=\alpha L^2/(2\pi)$, $\ell=L/2$) we find perfect agreement between the non-trivial solution $N_0=K_0/\ell$ of their equation (90) and our result (5.22).

in order that the asymptotic behavior

$$A_t = A_t^{(0)}\, e^{2\eta/L} + \mathcal{O}(1) \ , \tag{5.26a}$$

$$A_\eta = \mathcal{O}\left(e^{-2\eta/L}\right) \ , \tag{5.26b}$$

of the gauge field is preserved. Indeed, allowing some positive power of $e^{2\eta/L}$ in the expansion (5.25) of Λ would spoil this property.

Having established the most general gauge transformation consistent with our boundary conditions we can investigate whether the counterterm (5.23) is gauge invariant. Acting with the gauge transformation (5.24) and taking the asymptotic expansions (5.25) and (5.26) into account yields

$$\delta_\Lambda I_{\text{new}} = \frac{\alpha L}{\pi} \lim_{\eta\to\infty} \int_{\partial\mathcal{M}} dt \sqrt{-h}\, h^{tt} A_t\, \delta_\Lambda A_t = -\frac{2\alpha L}{\pi}\, A_t^{(0)} \int_{\partial\mathcal{M}} dt\, \partial_t \Lambda^{(0)} \ . \tag{5.27}$$

The same result holds for the full action (5.23), because all other terms in I are manifestly gauge invariant. The integral in (5.27) vanishes for continuous gauge transformations if $\Lambda^{(0)}$ takes the same value at the initial and final times. In those cases the counterterm (5.23) and the full action (5.23) are both gauge invariant with respect to gauge transformations that asymptote to (5.25).

The "large" gauge transformations that do not automatically leave the action invariant are also interesting. As an example, we consider the discontinuous gauge transformation

$$\Lambda^{(0)}(t) = 2\pi\, q_m\, \theta(t - t_0) \ , \tag{5.28}$$

where q_m is the dimensionless magnetic monopole charge with a convenient normalization. We assume that t_0 is contained in $\partial\mathcal{M}$, so that the delta function obtained from $\partial_t\Lambda^{(0)}$ is supported. Inserting the discontinuous gauge transformation (5.28) into the gauge variation of the action (5.27) gives

$$\delta_\Lambda I = \delta_\Lambda I_{\text{new}} = -2\alpha\, L^2 E\, q_m \ , \tag{5.29}$$

which tells us that the full action is shifted by a constant. We investigate now under which conditions this constant is an integer multiple of 2π.

The 2D Gauss law relates the electric field E to the dimensionless electric charge q_e:

$$E = -\frac{\pi\, q_e}{\alpha\, L^2}\ . \tag{5.30}$$

Again we have chosen a convenient normalization.[2] The Gauss law (5.30) allows to rewrite the gauge shift of the action (5.29) in a suggestive way:

$$\delta_\Lambda I = \delta_\Lambda I_{\text{new}} = 2\pi\, q_e\, q_m\ . \tag{5.31}$$

Thus, as long as magnetic and electric charge obey the Dirac quantization condition

$$q_e\, q_m \in \mathbb{Z}\ , \tag{5.32}$$

the action just shifts by multiples of 2π. We shall assume that this is the case. Then I_{new} and I are gauge invariant modulo 2π despite of the apparent gauge non-invariance of the boundary mass term $m\, A^a A_a$.

In conclusion, the full action (5.23) is gauge invariant with respect to all gauge variations (5.25) that preserve the boundary conditions (5.12) provided the integral in (5.27) vanishes (modulo 2π). This is the case if the Dirac quantization condition (5.32) holds.

5.2 Boundary Stress Tensor and Central Charge

The behavior of the on-shell action is characterized by the linear response functions of the boundary theory[3]

$$T^{ab} = \frac{2}{\sqrt{-h}}\,\frac{\delta I}{\delta h_{ab}}\ , \qquad J^a = \frac{1}{\sqrt{-h}}\,\frac{\delta I}{\delta A_a}\ . \tag{5.33}$$

[2]If we set $\alpha\, L^2/(2\pi) = 1$ then the action (5.1) has a Maxwell-term with standard normalization. In that case our Gauss law (5.30) simplifies to $2E = -q_e$. The factor 2 appears here because in our conventions the relation between field strength and electric field contains such a factor, $F_{\mu\nu} = 2E\,\epsilon_{\mu\nu}$. Thus, apart from the sign, the normalization in (5.30) leads to the standard normalization of electric charge in 2D. The sign is a consequence of our desire to have positive E for positive q_e in the case of negative α.

[3]These are the same conventions as in [133]. The boundary current and stress tensor used here is related to the definitions in [132] by $J^a = \frac{1}{2\pi} J^a_{\text{HS}}$ and $T^{ab} = \frac{1}{2\pi} T^{ab}_{\text{HS}}$.

The response function for the dilaton, which is not relevant for the present considerations, is discussed in [147]. The general expressions (5.19a) and (5.19b) give

$$T_{tt} = \frac{\alpha}{\pi}\big(- \frac{2}{L}\, h_{tt}\, e^{-2\phi} - \frac{L}{2}\, A_t A_t\big) \ , \tag{5.34a}$$

$$J^t = \frac{\alpha}{2\pi}\big(- L^2 n_\mu F^{\mu t} + 2LA^t\big) \ . \tag{5.34b}$$

We want to find the transformation properties of these functions under the asymptotic symmetries of the theory; *i.e.* under the combination of bulk diffeomorphisms and $U(1)$ gauge transformations that act non-trivially at $\partial\mathcal{M}$, while preserving the boundary conditions and the choice of gauge.

A diffeomorphism $x^\mu \to x^\mu + \epsilon^\mu(x)$ transforms the fields as

$$\delta_\epsilon g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu \ , \qquad \delta_\epsilon A_\mu = \epsilon^\nu\, \nabla_\nu A_\mu + A_\nu \nabla_\mu \epsilon^\nu \ . \tag{5.35}$$

The background geometry is specified by the gauge conditions $g_{\eta\eta} = 1$, $g_{\eta t} = 0$, and the boundary condition that fixes the leading term $g^{(0)}_{tt}$ in the asymptotic expansion (5.12a) of h_{tt}. These conditions are preserved by the diffeomorphisms

$$\epsilon^\eta = -\frac{L}{2}\,\partial_t \xi(t) \ , \qquad \epsilon^t = \xi(t) + \frac{L^2}{2}\left(e^{4\eta/L} + h_1(t)\right)^{-1}\, \partial_t^2 \xi(t) \ , \tag{5.36}$$

where ξ is an arbitrary function of the coordinate t. Under (5.36), the boundary metric transforms according to

$$\delta_\epsilon h_{tt} = -\left(1 + e^{-4\eta/L} h_1(t)\right)\left(h_1(t)\,\partial_t \xi(t) + \frac{1}{2}\,\xi(t)\partial_t h_1(t) + \frac{L^2}{4}\,\partial_t^3 \xi(t)\right) \ . \tag{5.37}$$

Turning to the gauge field, the change in A_η due to the diffeomorphism (5.36) is

$$\delta_\epsilon A_\eta = -2\, e^{-\phi}\, \frac{\left(e^{2\eta/L} - h_1(t)\, e^{-2\eta/L}\right)}{\left(e^{2\eta/L} + h_1(t)\, e^{-2\eta/L}\right)^2}\, \partial_t^2 \xi(t) \ . \tag{5.38}$$

Thus, diffeomorphisms with $\partial_t^2 \xi \neq 0$ do not preserve the $U(1)$ gauge condition $A_\eta = 0$. The gauge is restored by the compensating gauge transformation $A_\mu \to A_\mu + \partial_\mu \Lambda$, with Λ given by

$$\Lambda = -L\, e^{-\phi}\left(e^{2\eta/L} + h_1(t)\, e^{-2\eta/L}\right)^{-1}\, \partial_t^2 \xi(t) \ . \tag{5.39}$$

The effect of the combined diffeomorphism and $U(1)$ gauge transformation on A_t is

$$(\delta_\epsilon + \delta_\Lambda)\, A_t = -e^{-2\eta/L}\, e^{-\phi} \left(\frac{1}{L}\xi(t)\partial_t h_1(t) + \frac{2}{L}\, h_1(t)\, \partial_t \xi(t) + \frac{L}{2}\, \partial_t^3 \xi(t) \right) \; . \tag{5.40}$$

This transformation preserves the boundary condition (5.12b) for A_t, as well as the condition $A_t^{(1)} = 0$ that was used to fix the residual $U(1)$ gauge freedom. Thus, the asymptotic symmetries of the theory are generated by a diffeomorphism (5.36) accompanied by the $U(1)$ gauge transformation (5.39). Under such transformations the metric and gauge field behave as (5.37) and (5.40), respectively.

We can now return to our goal of computing the transformation of the linear response functions (5.34) under the asymptotic symmetries of the theory. The change in the stress tensor (5.34a) due to the combined diffeomorphism (5.36) and $U(1)$ gauge transformation (5.39) takes the form

$$(\delta_\epsilon + \delta_\Lambda) T_{tt} = 2\, T_{tt}\, \partial_t \xi + \xi\, \partial_t T_{tt} - \frac{c}{24\pi}\, L\, \partial_t^3 \xi(t) \; . \tag{5.41}$$

The first two terms are the usual tensor transformation due to the diffeomorphism. In addition, there is an anomalous term generated by the $U(1)$ component of the asymptotic symmetry. We included a factor L in the anomalous term in (5.41) in order to make the central charge c dimensionless. Using the expressions (5.37) and (5.40) for the transformation of the fields we verify the general form (5.41) and determine the central charge

$$c = -24\, \alpha e^{-2\phi} \; . \tag{5.42}$$

The relation (5.3) allows us to rewrite (5.42) in the more aesthetically pleasing form

$$c = \frac{3}{G_2} \; . \tag{5.43}$$

The requirement that the central charge should be positive determines $\alpha < 0$ as the physically correct sign. We shall see the same (unusual) sign appearing as the physically correct one in later sections.

Another suggestive expression for the central charge is

$$c = 3\,\mathrm{Vol}_L\,\mathcal{L}_{2\mathrm{D}}\ , \tag{5.44}$$

where the volume element $\mathrm{Vol}_L = 2\pi L^2$ and Lagrangian density $\mathcal{L}_{2\mathrm{D}} = \frac{4\alpha}{\pi L^2} e^{-2\phi}$ is related to the on-shell bulk action (5.1) by

$$I_{\mathrm{bulk}}\Big|_{\mathrm{EOM}} = -\int_{\mathcal{M}} d^2x\sqrt{-g}\,\mathcal{L}_{2\mathrm{D}}\ . \tag{5.45}$$

The central charge (5.44) is the natural starting point for computation of higher derivative corrections to the central charge, in the spirit of [24, 11, 128, 130].

So far we considered just the transformation property of the energy momentum tensor (5.34a). We should also consider the response of the boundary current (5.34b) to a gauge transformation. Generally, we write the transformation of a current as

$$\delta_\Lambda J_t = -\frac{k}{4\pi}\,L\,\partial_t\Lambda\ , \tag{5.46}$$

where the level k parametrizes the gauge anomaly. The only term in (5.34b) that changes under a gauge transformation is the term proportional to A_t. The resulting variation of the boundary current takes the form (5.46) with the level

$$k = -4\alpha = \frac{1}{2G_2}e^{2\phi}\ . \tag{5.47}$$

Our definitions of central charge (5.41) and level in (5.46) are similar to the corresponding definitions in 2D CFT. However, they differ by the introduction of the AdS scale L, needed to keep these quantities dimensionless. We could have introduced another length scale instead, and the anomalies would then be rescaled correspondingly as a result. Since c and k would change the same way under such a rescaling we may want to express the central charge (5.43) in terms of the level (5.47) as

$$c = 6\,k\,e^{-2\phi}\,. \tag{5.48}$$

This result is insensitive to the length scale introduced in the definitions of the anomalies, as long as the same scale is used in the two definitions.

Expressing the dilaton (5.8) in terms of the electric field we find yet another form of the central charge

$$c = \frac{3}{2}\, kE^2 L^4 \ . \tag{5.49}$$

As it stands, this result is twice as large as the result found in [132]. However, there the anomaly is attributed to two contributions, from T_{++} and T_{--} related to the two boundaries of global AdS_2. We introduce a single energy-momentum tensor T_{tt}, as seems appropriate when the boundary theory has just one spacetime dimension. In general spacetimes, T_{tt} would be a density but in one spacetime dimension there are no spatial dimensions, and so the "density" is the same as the energy. Such an energy-momentum tensor cannot be divided into left- and right-moving parts. Thus our computation agree with [132] even though our interpretations differ.

5.3 3D Reduction and Connection with 2D

Asymptotically AdS_2 backgrounds have a non-trivial $SL(2,\mathbb{R})$ group acting on the boundary that can be interpreted as one of the two $SL(2,\mathbb{R})$ groups associated to AdS_3. To do so, we compactify pure gravity in 3D with a negative cosmological constant on a circle and find the map to the Maxwell-dilaton gravity (5.1). This dimensional reduction also shows that the AdS_2 boundary stress tensor and central charge found here are consistent with the corresponding quantities in AdS_3.

5.3.1 Three dimensional gravity

Our starting point is pure three dimensional gravity described by an action

$$I = \frac{1}{16\pi G_3}\int d^3x\sqrt{-g}\left(\mathcal{R}+\frac{2}{\ell^2}\right) + \frac{1}{8\pi G_3}\int d^2y\sqrt{-\gamma}\left(\mathcal{K}-\frac{1}{\ell}\right) \ , \tag{5.50}$$

that is a sum over bulk and boundary actions like in the schematic equation (5.14). The 3D stress-tensor defined as

$$\delta I = \frac{1}{2}\int d^2y\sqrt{-\gamma}\, T^{ab}_{\text{3D}}\,\delta\gamma_{ab}\ , \tag{5.51}$$

becomes [133]

$$T^{\text{3D}}_{ab} = -\frac{1}{8\pi G_3}\Big(\mathcal{K}_{ab} - \mathcal{K}\gamma_{ab} + \frac{1}{\ell}\gamma_{ab}\Big)\ . \tag{5.52}$$

For asymptotically AdS_3 spaces we can always choose Fefferman-Graham coordinates, where the bulk metric takes the form

$$ds^2 = d\eta^2 + \gamma_{ab}\,dy^a dy^b\ ,\quad \gamma_{ab} = e^{2\eta/\ell}\gamma^{(0)}_{ab} + \gamma^{(2)}_{ab} + \dots\ . \tag{5.53}$$

The functions $\gamma^{(i)}_{ab}$ depend only on the boundary coordinate y^a with $a,b = 1,2$. The boundary is located at $\eta\to\infty$, and $\gamma^{(0)}_{ab}$ is the 2D boundary metric defined up to conformal transformations. The energy momentum tensor (5.52) evaluated in the coordinates (5.53) is

$$T^{\text{3D}}_{ab} = \frac{1}{8\pi G_3\ell}\Big(\gamma^{(2)}_{ab} - \gamma^{cd}_{(0)}\gamma^{(2)}_{cd}\gamma^{(0)}_{ab}\Big)\ . \tag{5.54}$$

In the case of pure gravity (5.50) we can be more explicit and write the exact solution as [149]

$$ds^2 = d\eta^2 + \Big(\frac{\ell^2}{4}e^{2\eta/\ell} + 4g_+g_-e^{-2\eta/\ell}\Big)dx^+dx^- + \ell\Big(g_+(dx^+)^2 + g_-(dx^-)^2\Big)\ . \tag{5.55}$$

We assumed a flat boundary metric $\gamma^{(0)}_{ab}$ parameterized by light-cone coordinates $x^{\pm}$. The function g_+ (g_-) depends exclusively on x^+ (x^-). For this family of solutions the energy-momentum tensor (5.54) becomes

$$T^{\text{3D}}_{++} = \frac{1}{8\pi G_3}\,g_+\ ,\qquad T^{\text{3D}}_{--} = \frac{1}{8\pi G_3}\,g_-\ . \tag{5.56}$$

5.3.2 Kaluza-Klein reduction

Dimensional reduction is implemented by writing the 3D metric as

$$ds^2 = e^{-2\psi}\ell^2(dz + \tilde{A}_\mu dx^\mu)^2 + \tilde{g}_{\mu\nu}dx^\mu dx^\nu \ . \tag{5.57}$$

The 2D metric $\tilde{g}_{\mu\nu}$, the scalar field ψ, and the gauge field $\tilde{A}_\mu$ all depend only on x^μ ($\mu = 1, 2$). The coordinate z has period 2π. The 3D Ricci scalar expressed in terms of 2D fields reads

$$\mathcal{R} = \tilde{R} - 2e^{\psi}\tilde{\nabla}^2 e^{-\psi} - \frac{\ell^2}{4}e^{-2\psi}\tilde{F}^2 \ . \tag{5.58}$$

The 2D scalar curvature $\tilde{R}$, and the covariant derivatives $\tilde{\nabla}_\mu$, are constructed from $\tilde{g}_{\mu\nu}$. Inserting (5.58) in the 3D bulk action in (5.50) gives the 2D bulk action

$$\tilde{I}_{\text{bulk}} = \frac{\ell}{8G_3}\int d^2x\sqrt{-\tilde{g}}e^{-\psi}\left(\tilde{R} + \frac{2}{\ell^2} - \frac{\ell^2}{4}e^{-2\psi}\tilde{F}^2\right) \ . \tag{5.59}$$

The action (5.59) is on-shell equivalent to the action (5.23) for the constant dilaton solutions (5.11). To find the precise dictionary we first compare the equations of motion. Variation of the action (5.59) with respect to the scalar ψ and metric $\tilde{g}_{\mu\nu}$ gives

$$\tilde{R} + \frac{2}{\ell^2} - \frac{3\ell^2}{4}\,e^{-2\psi}\tilde{F}^2 = 0 \ , \tag{5.60a}$$

$$\tilde{g}_{\mu\nu}\left(\frac{1}{\ell^2} - \frac{\ell^2}{8}\,e^{-2\psi}\tilde{F}^2\right) + \frac{\ell^2}{2}\,e^{-2\psi}\tilde{F}_{\mu\alpha}\tilde{F}_\nu{}^\alpha = 0 \ , \tag{5.60b}$$

which implies[4]

$$e^{-2\psi}\tilde{F}^2 = -\frac{8}{\ell^4} \ , \tag{5.61a}$$

$$\tilde{R} = -\frac{8}{\ell^2} \ . \tag{5.61b}$$

The analogous equations derived from the 2D action (5.6) take the same form, but with

[4]A check on the algebra: inserting (5.61) into the formula (5.58) for the 3D Ricci scalar yields $\mathcal{R} = -6/\ell^2$, concurrent with our definition of the 3D AdS radius.

the identifications

(5.62a) $$\tilde{g}_{\mu\nu} = a^2 g_{\mu\nu} \ ,$$

(5.62b) $$\ell = aL \ ,$$

(5.62c) $$e^{-\psi}\tilde{F}_{\mu\nu} = \frac{1}{2}e^{\phi}F_{\mu\nu} \ ,$$

with a an arbitrary constant.

In order to match the overall normalization on-shell we evaluate the bulk action (5.59) using the on-shell relations (5.61a) and (5.61b)

(5.63) $$\tilde{I}_{\text{bulk}} = -\frac{\ell}{2G_3}\int d^2x \frac{\sqrt{-\tilde{g}}}{\ell^2}e^{-\psi} \ .$$

and compare with the analogous expression

(5.64) $$I_{\text{bulk}} = \frac{4\alpha}{\pi}\int d^2x \frac{\sqrt{-g}}{L^2}e^{-2\phi} \ .$$

computed directly from the 2D action (5.23). Equating the on-shell actions $I_{\text{bulk}} = \tilde{I}_{\text{bulk}}$ and simplifying using (5.62a), (5.62b) we find

(5.65) $$\alpha = -\frac{\pi\,\ell}{8G_3}e^{2\phi-\psi} \, .$$

We see again that the unusual sign $\alpha < 0$ is the physically correct one. According to (5.3) we can write the 3D/2D identification as

(5.66) $$\frac{1}{G_2} = \frac{\pi\ell e^{-\psi}}{G_3} \ .$$

So far we determined the 3D/2D on-shell dictionary by comparing equations of motions and the bulk action. In Appendix C.2 we verify that the same identification (5.65) also guarantees that the boundary actions agree. Additionally, we show that the 3D/2D dictionary identifies the 3D solutions (5.55) with the general 2D solutions (5.11). These checks give confidence in our 3D/2D map.

In summary, our final result for the dictionary between the 2D theory and the KK reduction of the 3D theory is given by the identifications (5.62) and the relation (5.65) between normalization constants. We emphasize that the map is on-shell; it is between solutions and their properties. The full off-shell theories do not agree, as is evident from the sign in (5.65). The restriction to on-shell configurations will not play any role here but it may be important in other applications.

5.3.3 Conserved currents and central charge

Applying the 3D/2D dictionary from the previous subsection (and elaborations in Appendix C.2), we now compare the linear response functions and the central charge computed by reduction from 3D to those computed directly in 2D.

The starting point is the 3D energy momentum tensor (5.51). The KK-reduction formula (C.7) decomposes the variation of the boundary metric γ_{ab} as

$$\delta\gamma_{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \delta\tilde{h}_{tt} + \ell^2 e^{-2\psi} \begin{pmatrix} 2\tilde{A}_t & 1 \\ 1 & 0 \end{pmatrix} \delta\tilde{A}_t \ , \tag{5.67}$$

and the determinant $\sqrt{-\gamma} = \ell e^{-\psi}\sqrt{-\tilde{h}}$ so that the 3D stress tensor (5.51) becomes

$$\begin{aligned} \delta I &= \int dx\sqrt{-\tilde{h}}\Big[\frac{1}{2}\big(2\pi\ell e^{-\psi}\big)T^{tt}_{3d}\ \delta\tilde{h}_{tt} + \big(\pi\ell^3 e^{-3\psi}\big)\big(T^{tt}_{3d}\tilde{A}_t + T^{zt}_{3d}\big)]\ 2\delta\tilde{A}_t\Big] \\ &= \int dx\sqrt{-h}(h^{tt})^2\Big[\frac{1}{2}\big(2\pi L e^{-\psi}T^{3d}_{tt}\big)\delta h_{tt} + \big(\frac{\pi}{2}L^3 e^{-\psi+2\phi}A_t T^{3d}_{tt}\big)\delta A_t\Big] \ , \end{aligned} \tag{5.68}$$

where we used the 3D-2D dictionary (5.62) and wrote the variation of the boundary fields as

$$\delta\tilde{h}_{tt} = a^2\delta h_{tt} \ , \quad \delta\tilde{A}_t = \frac{1}{2}e^{\psi+\phi}\delta A_t \ . \tag{5.69}$$

Indices of the stress tensor in (5.68) are lowered and raised with $\tilde{h}_{tt}$ and $\tilde{h}^{tt}$, respectively. Comparing (5.68) with the 2D definition of stress tensor and current (5.33) we find

$$T^{\text{2D}}_{tt} = 2\pi L e^{-\psi}T^{\text{3D}}_{tt} \ , \tag{5.70a}$$

$$J_t = \frac{\pi}{2}L^3 h^{tt} e^{-\psi+2\phi}A_t T^{\text{3D}}_{tt} \ , \tag{5.70b}$$

for the relation between 3D and 2D quantities.

The next step is to rewrite the 3D energy momentum tensor (5.56) in a notation more appropriate for comparison with 2D. We first rescale coordinates according to (C.14) and then transform into 2D variables using (C.16b), (C.17a). The result is

$$T_{tt}^{\text{3D}} = -\frac{1}{8\pi G_3 \ell} h_1 \ , \qquad T_{zz}^{\text{3D}} = \frac{1}{8\pi G_3} \ell e^{-2\psi} \ . \tag{5.71}$$

Inserting these expressions in (5.70), along with the asymptotic values of the background fields in the solution (5.11), we find

$$T_{tt}^{\text{2D}} = \frac{2\alpha}{L\pi} e^{-2\phi} h_1 \ , \tag{5.72a}$$

$$J_t = -\frac{2\alpha}{\pi} e^{-\phi} e^{-2\eta/L} h_1 \ , \tag{5.72b}$$

after simplifications using our 3D-2D dictionary (5.62) and the rescaling mentioned just before (C.17). The current (5.72b) vanishes on the boundary $\eta \to \infty$ but the subleading term given here is significant for some applications. The expressions (5.72) are our results for the 2D linear response functions, computed by reduction from 3D. They should be compared with the analogous functions (5.34) defined directly in 2D, with those latter expressions evaluated on the solution (5.11). These results agree precisely.

Using the relations between the conserved currents, we now proceed to compare the central charges in 2D and 3D. Under the diffeomorphisms which preserve the three dimensional boundary, the 3D stress tensor transforms as [133]

$$\delta T_{tt}^{\text{3D}} = 2T_{tt}^{\text{3D}} \partial_t \xi(t) + \xi(t) \partial_t T_{tt}^{\text{3D}} - \frac{c}{24\pi} \partial_t^3 \xi(t) \ , \tag{5.73}$$

with the central term given by the standard Brown–Henneaux central charge

$$c_{\text{3D}} = \frac{3\ell}{2G_3} \ . \tag{5.74}$$

From the relation (5.70a) between 2D and 3D stress tensor, and by comparing the transformations (5.41) and (5.73), the central charges are related as

$$c_{\text{2D}} = 2\pi e^{-\psi} c_{\text{3D}} \ . \tag{5.75}$$

Inserting (5.74) and using (5.65) we find

$$c_{\text{2D}} = 2\pi e^{-\psi}\left(\frac{3\ell}{2G_3}\right) = -24\alpha e^{-2\phi} \ . \tag{5.76}$$

This is the result for the 2D central charge, obtained by reduction from 3D. It agrees precisely with the central charge (5.42) obtained directly in 2D.

In summary, in this section we have given an explicit map between 3D and 2D. We have shown that it correctly maps the equations of motion and the on-shell actions, it maps 3D solutions to those found directly in 2D, it maps the linear response functions correctly between the two pictures, and it maps the central charge correctly.

5.4 Black Hole Thermodynamics

In this section we apply our results to discuss the entropy of 2D black holes. We start by computing the temperature and mass of the black hole and the relation of these quantities to the 2D stress tensor. By using the renormalized on-shell action and the first law of thermodynamics, we obtain the Bekenstein-Hawking entropy. Finally, we discuss the identification of the black hole entropy with the ground state entropy of the dual CFT.

5.4.1 Stress tensor for $\mathbf{AdS_2}$ black holes

For $h_0 = 1$ and constant h_1, the solution (5.11) becomes

$$ds^2 = d\eta^2 - \frac{1}{4}e^{4\eta/L}\left(1 + h_1 e^{-4\eta/L}\right)^2 dt^2 \ , \tag{5.77a}$$

$$A_t = \frac{1}{L}e^{-\phi}e^{2\eta/L}\left(1 - h_1 e^{-4\eta/L}\right) \ . \tag{5.77b}$$

Solutions with positive h_1 correspond to global AdS_2 with radius $\ell_A = L/2$, while solutions with negative h_1 describe black hole geometries.

An AdS_2 black hole with horizon at $\eta = \eta_0$ corresponds to h_1 given by

$$h_1 = -e^{4\eta_0/L}. \tag{5.78}$$

Regularity of the Euclidean metrics near the horizon determines the imaginary periodicity $t \sim t + i\beta$ as

$$\beta = \pi L e^{-2\eta_0/L} \ . \tag{5.79}$$

We identify the temperature of the 2D black hole as $T = \beta^{-1}$.

Our general AdS_2 stress tensor (5.34), (5.72a) is

$$T_{tt} = \frac{2\alpha}{\pi L} e^{-2\phi} h_1 = -\frac{h_1}{4\pi G_2 L} \ . \tag{5.80}$$

The stress tensor for global AdS_2 ($h_1 > 0$) is negative. This is reasonable, because the Casimir energy of AdS_3 is negative as well. Importantly, the black hole solutions ($h_1 < 0$) are assigned positive energy, as they should be. The assignment $\alpha < 0$ is needed in (5.80) to reach this result, giving further confidence in our determination of that sign.

We can rewrite the stress tensor (5.80) as

$$T_{tt} = \frac{\pi L T^2}{4G_2} = c\,\frac{\pi L T^2}{12} \ , \tag{5.81}$$

where we used the central charge (5.43). We interpret this form of the energy as a remnant of the 3D origin of the theory, as the right movers of a 2D CFT.

The mass is generally identified as the local charge of the current generated by the Killing vector ∂_t. This amounts to the prescription

$$M = \sqrt{-g^{tt}}\, T_{tt} = 2e^{-2\eta/L}\, T_{tt} \to 0 \ , \tag{5.82}$$

for the mass measured asymptotically as $\eta \to \infty$. The solutions (5.77) are therefore all assigned vanishing mass, due to the redshift as the boundary is approached. We will see in the following that this result is needed to uphold the Bekenstein-Hawking area law.

5.4.2 On-shell action and Bekenstein-Hawking entropy

The boundary terms in (5.23) were constructed so that the variational principle is well-defined, but they are also supposed to cancel divergences and render the on-shell action finite. It is instructive to compute its value.

The on-shell bulk action (A.14) becomes

$$
\begin{aligned}
I_{\text{bulk}} &= \frac{2\alpha}{\pi L^2} e^{-2\phi} \int dt d\eta \left(e^{2\eta/L} + h_1 e^{-2\eta/L} \right) \\
&= \frac{\alpha\beta}{\pi L} e^{-2\phi} \left(e^{2\eta/L} - h_1 e^{-2\eta/L} \right)\Big|_{\eta_0}^{\infty} \ , \qquad (5.83)
\end{aligned}
$$

for the 2D black hole (5.77). The boundary terms in (5.23) were evaluated in (C.12) with the result

$$
I_{\text{boundary}} = -\frac{2\alpha\beta}{\pi L} e^{-2\phi} \sqrt{-h_{tt}} = -\frac{\alpha\beta}{\pi L} e^{-2\phi} e^{2\eta/L} \ . \qquad (5.84)
$$

The divergence at the boundary $\eta \to \infty$ cancels the corresponding divergence in the bulk action (5.83). The renormalized on-shell action becomes finite with the value

$$
I = I_{\text{bulk}} + I_{\text{boundary}} = -\frac{2\alpha\beta}{\pi L} e^{-2\phi} e^{2\eta_0/L} = -2\alpha e^{-2\phi} = \frac{1}{4G_2} \ . \qquad (5.85)
$$

The third equality used (5.79) and the last one used (5.3).

We computed the on-shell action in Lorentzian signature to conform with the conventions elsewhere in the chapter. The Euclidean action has the opposite sign $I_E = -I$, and it is that action which is related to the free energy in the standard manner

$$
\beta F = I_E = \beta M - S \ , \qquad (5.86)
$$

when we consider the canonical ensemble.[5] We found vanishing M in (5.82) and so the black hole entropy becomes

$$
S = -I_E = I = \frac{1}{4G_2} \ . \qquad (5.87)
$$

This is the standard Bekenstein-Hawking result.

[5] Strictly, the on-shell action is related to a thermodynamic potential that is a function of the temperature T and the electrostatic potential Φ. However, the boundary term for the gauge field leads to a net charge $Q = 0$, and so the thermodynamic potential reduces to the standard Helmholtz free energy.

5.4.3 Black hole entropy from Cardy's formula

One of the motivations for determining the central charge of AdS_2 is that it may provide a short-cut to the black hole entropy. We will just make preliminary comments on this application.

A 2D chiral CFT with c_0 degrees of freedom living on a circle with radius R has entropy given by the Cardy formula

$$S = 2\pi\sqrt{\frac{c_0}{6}(2\pi RH)}\ . \tag{5.88}$$

Here H denotes the energy of the system. This formula generally applies when $2\pi RH \gg \frac{c_0}{24}$ but, for the CFTs relevant for black holes, we expect it to hold also for $2\pi RH \sim \frac{c_0}{24}$ [82]. Since the Casimir energy for such a theory is $2\pi RH = \frac{c_0}{24}$ we recover the universal ground state entropy

$$S = 2\pi \cdot \frac{c_0}{12}\ . \tag{5.89}$$

Relating the number of degrees of freedom c_0 to our result for the central charge (5.43) as $c_0 = c/(2\pi)$ we find the ground state entropy

$$S = \frac{c}{12} = \frac{1}{4G_2}\ , \tag{5.90}$$

in agreement with the Bekenstein-Hawking entropy.

The relation $c_0 = c/(2\pi)$ is not self-evident. We have defined the central charge by the transformation property (5.41) with stress tensor normalized as in (5.33). This gives the same normalization of central charge as in [132]. As we have already emphasized, the length scale L introduced in (5.41) to render the central charge dimensionless is rather arbitrary. We could have introduced $2\pi L$ instead, corresponding to

$$(\delta_\epsilon + \delta_\Lambda)T_{tt} = 2\,T_{tt}\,\partial_t\xi + \xi\,\partial_t T_{tt} - \frac{c_0}{12}\,L\,\partial_t^3\xi(t)\ . \tag{5.91}$$

It is apparently this definition that leads to c_0, the measure of degrees of freedom.

The situation is illuminated by our 3D-2D dictionary. We can implement this by using the 3D origin of the 2D coordinate t (C.14) or, simpler, the 3D origin of the 2D central charge (5.75). In 3D the dimensionless central charge that counts the degrees of freedom is introduced without need of an arbitrary scale. The relation (5.75) to the 2D central charge therefore motivates the factor 2π in $c_0 = c/(2\pi)$. Furthermore, there is a conformal rescaling of the central charge due to an induced dilaton $e^{-\psi}$. To get a feel for this consider the canonical 4D BPS black holes [150, 151], supported by four mutually BPS charges n_1, n_2, n_3, n_0 of which n_0 is the KK-momentum along the circle. The conformal rescaling brings the 3D central charge $c_{3D} = 6n_1n_2n_3$ into the more symmetrical value

$$c_0 = 6\sqrt{n_1n_2n_3n_0} \tag{5.92}$$

It would be interesting to understand this value directly from the 2D point of view.

It is natural to consider a more general problem. The 2D black holes (5.77) are lifted by our 2D/3D map in Section 5.3 to the general BTZ black holes in three dimensions. The BTZ black holes are dual to a 2D CFT, with both right and left movers. The 2D description keeps only one chirality and so it is challenging to understand how the general entropy can be accounted for directly in 2D. Our result equating the ground state entropy of the chiral 2D CFT with the Bekenstein-Hawking entropy of any AdS_2 black hole indicates that this is in fact possible, but the details remain puzzling.

5.5 Backgrounds with Non-constant Dilaton

In this section we generalize our considerations to backgrounds with non-constant dilaton. We find that the counterterms determined for constant dilaton give a well-defined variational principle also in the case of a non-constant dilaton. We discuss some properties of the general solutions. In particular we identify an extremal solution that reduces to the constant dilaton solution (5.11) in a near horizon limit. For recent work on non-constant dilaton solutions in 2D Maxwell-Dilaton gravity see [152].

5.5.1 General solution with non-constant dilaton

We start by finding the solutions to the equations of motion. The spacetime and gauge curvature are determined by solving $\mathcal{E}_\phi = 0$ and $\mathcal{E}_\mu = 0$ in (5.5), which gives

$$R = -\frac{8}{L^2}\,, \qquad F_{\mu\nu} = 2E\,\epsilon_{\mu\nu}\;. \tag{5.93}$$

In the case of non-constant dilaton we may use the dilaton as one of the coordinates

$$e^{-2\phi} = \frac{r}{L}\;. \tag{5.94a}$$

This statement is true everywhere except on bifurcation points of bifurcate Killing horizons. We do not exhaustively discuss global issues here and therefore disregard this subtlety. For dimensional reasons we have included a factor $1/L$ on the right hand side of the definition (5.94a). Using the residual gauge freedom we employ again a gauge where the line element is diagonal and the gauge field has only a time component,

$$ds^2 = g_{rr}\,dr^2 + g_{tt}\,dt^2\,, \qquad A_\mu dx^\mu = A_t\,dt\,. \tag{5.94b}$$

Solving $\mathcal{E}_\phi = 0$ yields $g_{tt} = -1/g_{rr}$, and the last equations of motion $\mathcal{E}_{\mu\nu} = 0$ gives

$$g_{tt} = -\frac{4r^2}{L^2} + 2L^3E^2\,r + 4M\;, \tag{5.94c}$$

and

$$A_t = 2Er\,. \tag{5.94d}$$

The electric field E and ‘mass’ M are constants of motion. The former has dimension of inverse length squared, the latter is dimensionless in our notation.

There is a Killing vector $k = \partial_t$ that leaves the metric, gauge field and dilaton invariant. There are two other Killing vectors that leave invariant the metric, but not the dilaton. This is the breaking of $SL(2,\mathbb{R})$ to $U(1)$ mentioned before (5.6).

The Killing horizons are determined by the zeroes of the Killing norm. The norm squared is given by $k^\mu k^\nu g_{\mu\nu} = g_{tt}$, and therefore by solving $g_{tt} = 0$ the horizons are located at

$$r_h = L\left[\frac{E^2L^4}{4} \pm \sqrt{\left(\frac{E^2L^4}{4}\right)^2 + M}\right] . \tag{5.95}$$

For positive M, there is exactly one positive solution to (5.95). If M is negative two Killing horizons exist, provided the inequality $E^2 > 4\sqrt{-M}/L^4$ holds. If the inequality is saturated,

$$M_{\text{ext}} = -\frac{L^8E^4}{16} , \tag{5.96}$$

then the Killing horizon becomes extremal and the value of the dilaton (5.94a) on the extremal horizon, $r_h/L = L^4E^2/4$, coincides with the constant dilaton result (5.8). This is consistent with the universality of extremal black hole spacetimes [120].

The geometric properties of the solution (5.94) are developed further in [142, 143] and references therein. The thermodynamic properties are a special case of those discussed in [147].[6]

5.5.2 Asymptotic geometry and counterterms

In order to compare the asymptotic geometry with our previous results we introduce

$$e^{2\eta/L} = \frac{4r}{L} , \tag{5.97}$$

and write the solutions (5.94) in the form

$$g_{\mu\nu}dx^\mu dx^\nu = d\eta^2 - \frac{1}{4}e^{4\eta/L}dt^2 + \dots , \tag{5.98a}$$

$$A_\mu dx^\mu = \frac{1}{2}LEe^{2\eta/L}dt , \tag{5.98b}$$

$$\phi = -\frac{\eta}{L} . \tag{5.98c}$$

[6] The solution (5.94) is the special case $U(X) = 0$, $V(X) = -\frac{4X}{L^2} + L^2E^2$ where the functions U, V are introduced in the definition of the action (1.1) of [147] and $X = e^{-2\phi}$ is the dilaton field.

At this order the solutions agree with the constant dilaton background (5.11), except the dilaton diverges linearly with η rather than approaching a constant. Therefore, the solution (5.94) may be called 'linear dilaton solution'.

Asymptotically linear dilaton solutions have Fefferman-Graham like expressions analogous to (5.12), except that we must allow a logarithmic modification in the expansion of the dilaton

$$h_{tt} = e^{4\eta/L}\, h_{tt}^{(0)} + h_{tt}^{(1)} + \dots\ , \tag{5.99a}$$

$$A_t = e^{2\eta/L}\, A_t^{(0)} + A_t^{(1)} + \dots\ , \tag{5.99b}$$

$$\phi = \eta\, \phi^{(\log)} + \phi^{(0)} + \dots\ . \tag{5.99c}$$

As for the full solution, the asymptotic geometry and field strength of the linear dilaton solutions respect the asymptotic $SL(2,\mathbb{R})$ symmetry, but the asymptotic dilaton respects only the Killing vector ∂_t. The explicit linear dilaton solution (5.94) is obviously of the asymptotically linear dilaton form form (5.99). Its boundary values are

$$h_{tt}^{(0)} = -\frac{1}{4}\,, \qquad A_t^{(0)} = \frac{1}{2}LE\,, \qquad \phi^{(\log)} = -\frac{1}{L}\,. \tag{5.100}$$

We want to set up a consistent boundary value problem and variational principle, as in Section 5.1. There we wrote down the most general local counter terms and determined their coefficients, essentially by demanding the vanishing of the momenta (5.19) at the boundary. A non-constant dilaton could give rise to some additional local boundary terms, but all candidate terms vanish too rapidly to affect the variational principle — they are irrelevant terms in the boundary theory. Since there are no new counter terms and the coefficients of the existing ones are fixed by considering constant dilaton configurations, it must be that the same counter terms suffice also in the more general case.

We can verify this argument by explicit computation. For our choice of counter terms, the momenta (5.19b), (5.19c) vanish no matter the dilaton profile. For consistency we have verified that (5.19a) also does not lead to new conditions. We conclude that the logarithmic modification in the dilaton sector inherent to (5.100) does not destroy the consistency of the full action (5.23). Moreover, the discussion of gauge invariance in Section 5.1.3 also carries through. This is so, because the new boundary term (5.23) does not depend on the dilaton field.

In summary, we find that the full action (5.23) encompasses the boundary value problems (5.13) and (5.100), has a well-defined variational principle, and is extremized by the constant dilaton solutions (5.11) as well as by the linear dilaton solutions (5.94). Our discussion therefore exhausts all solutions to the equations of motion (5.5).

5.6 Discussion

We conclude this chapter with a few comments on questions that are left for future work:

- **Universal Central Charge:** Our result for the central charge (5.43) can be written as

  $$c = \frac{3}{G_2}\ . \tag{5.101}$$

  This form of the central charge does not depend on the detailed matter in the theory, *i.e.* the Maxwell field and the charge of the solution under that field. This raises the possibility that the central charge (5.101) could be universal, *i.e.* independent of the matter in the theory. It would therefore be interesting to study more general theories and establish in which cases (5.101) applies.

- **Mass Terms for Gauge Fields:** One of the subtleties we encountered here was the presence of the mass term

  $$I_{\text{new}} \sim \int_{\partial\mathcal{M}} m\, A^2\ , \tag{5.102}$$

for the boundary gauge field. Related boundary terms are known from Chern-Simons theory in three dimensions [153, 84], but apparently not in higher dimensions. It would be interesting to find situations where such boundary terms do appear in higher dimensions, after all. A challenge is that typical boundary conditions have the gauge field falling off so fast at infinity that these boundary terms are not relevant, but there may be settings with gauge fields that fall off more slowly.

- **Unitarity:** Our computations have several unusual signs. The most prominent one is that the overall constant in the action (5.1) must be negative

$$\alpha < 0 \ . \tag{5.103}$$

With this assignment the various terms in the action would appear to have the "wrong" sign, raising concerns about the unitarity. The sign we use is required to get positive central charge, positive 3D Newton constant, positive energy of the 2D black holes, and positive black hole entropy. This suggests that $\alpha < 0$ is in fact the physical sign. Nevertheless, a more direct understanding of unitarity would be desirable.

CHAPTER VI

Conclusion

We conclude this thesis by summarizing our findings. We highlight the implications of our results and suggest future directions.

Higher derivative corrections

The novelty of our approach is that we utilize the off-shell formalism of supergravity. This framework has several advantages. First, the construction of invariant Lagrangians is greatly simplified when dealing with higher derivative terms. The supersymmetry transformations do not depend on the Lagrangian, and therefore are not affected by higher order terms. Also the formalism assures that the truncation is consistent, *i.e.* there are no field redefinition ambiguities that could mix orders in derivatives. Most importantly, the construction of the full asymptotically flat solution is surprisingly simple. The bulk analysis of the BPS conditions is completely parallel to the two-derivative case. Higher derivative corrections only manifest themselves towards the end of the construction, where they yield corrections to the special geometry conditions and Maxwell equation.

An important application of these constructions is to resolve the singularities of small black strings. These objects correspond to classical solutions with a naked singularity and vanishing entropy. Once the stringy corrections are included, we obtained completely smooth geometries with the correct asymptotic behavior at the four-derivative level. Since there is no small parameter suppressing higher-order terms, our analysis can only suggest that the solution will remain smooth.

After the corrections are taken into account, this solution has an AdS_3 factor in the near-string region. The question of whether the small black string allows a dual description has attracted significant attention [77, 78, 73, 79]. Identifying precisely the dual conformal theory has been challenging due to the appearance of non-linear algebras among other difficulties. Still it remains an interesting direction of future research.

Black hole entropy

The microscopic corrections to the black hole entropy that we found agreed with the macroscopic supergravity theory with R^2 corrections. The higher order corrections used in five dimensions are known to be a complete set of terms at the four derivative level for supergravity coupled to vector multiplets. The agreement with the microscopic calculation to this order in α' is a more stringent test of string theory than in the four dimensional case.

The two gravitational backgrounds we considered are: magnetic string solutions, with near horizon geometry $\mathrm{AdS}_3 \times S^2$; and electric particle solutions, with near horizon geometry $\mathrm{AdS}_2 \times S^3$. The microscopic description corresponds to wrapped M5-branes in M-theory and the D1-D5 system in type IIB string theory, respectively. In the following we summarize our findings for each case.

The extra symmetries inherent in the near horizon $\mathrm{AdS}_3 \times S^2$ implies that the corrected entropy formula is governed by the coefficients of the Chern-Simons terms in the supergravity action. The key observation is that the entropy formula is controlled by the values of the left and right moving central charges of the associated 2D CFT, and due to supersymmetry these are completely determined by gauge and gravitational anomalies.

To verify this picture explicitly, we found the corrected near horizon $\mathrm{AdS}_3 \times S^2$ geometry. We showed that the result is in precise agreement with the values inferred from the supersymmetry/anomaly based argument, and thereby verified that the entropy is indeed controlled by the Chern-Simons terms. More generally, this same logic leads to the conclusion that there are no further corrections to the central charges from additional higher derivative terms (*i.e.* more than four derivatives), since we have already taken into account the full set of terms related by supersymmetry to the Chern-Simons terms.

We also studied the effect of the Taub-NUT geometry on the sub-leading corrections to the entropy for the D1-D5 black hole. This space contains a contractible circle that allows one to lift a 4D black hole to a 5D black hole by tuning the size of the circle. In the presence of the gauge-gravitational Chern-Simons term, the curvature of Taub-NUT acts as a source of electric charge that leads to different sub-leading corrections to the black hole entropy in 5D versus 4D. In the microscopic theory, due to the presence of Taub-NUT, the spectrum of states acquires additional modes from the center of mass motion of the system and momentum along the circle. These additional states exactly account for the shift between 5D and 4D corrections to the entropy.

The difference between the five and four dimensional entropy raises an interesting puzzle. The 4D and 5D black holes have the same near horizon geometry. Assuming that all of the dynamics that accounts for the entropy is stored in the horizon, the attractor mechanism then implies that both black holes should carry the same entropy. What we found here contradicts this statement and seems to imply that the black hole has hair, *i.e.* there are degrees of freedom living outside the horizon. In [154, 155] the authors where able to precisely identify the additional modes in the microscopic theory in four and five dimensions.

In the gravitational side this question is more subtle. The extremization principles are constructed based on the assumption that the black hole has no hair, therefore they will only capture the entropy associated to the horizon. It still remains a challenge to determine in the gravitational theory the appropriate generalization of the entropy function that will describe delocalized effects of the mixed gauge-gravitational Chern-Simons term and reproduce the microscopic results.

Holography of AdS_2

We applied the standard holographic renormalization procedure to asymptotically AdS_2 spacetimes. The basic idea is to impose precise boundary conditions and determine the boundary counterterms needed for a consistent variational principle. The divergences removed by the counterterms rendered a finite energy-momentum tensor at the boundary. By analyzing the transformation properties of the current under diffeomorphism accompanied by a gauge transformations, we verified the enhancement of the asymptotic $SL(2,\mathbb{R})$ conformal symmetry of the theory to a Virasoro algebra, and obtained a non-zero central charge. The main lesson is that for AdS_2 the AdS/CFT correspondence can be implemented in a conventional manner.

Our result for the central charge does not depend on the detailed matter in the theory, *i.e.* the Maxwell field and the charge of the solution under that field. This raises the possibility that the central charge could be independent of the matter in the theory. It would therefore be interesting to study more general theories and establish in which cases our expression applies.

One of the difficulties of applying AdS/CFT to two dimensional spaces is due to the two disconnected boundaries of AdS_2. A possible way to understand the connection (or disconnection) of the dual theories at each boundary is from a higher dimensional point of view. AdS_3 can be obtained as a circle fibration over AdS_2. It would be interesting to clearly understand geometrically the location of two boundaries of AdS_2 in the AdS_3, and also precisely relate the dual theories of each AdS space. This should allow us to better identify the appropriate microscopic operators and states that account for the statistical entropy in AdS_2.

APPENDICES

APPENDIX A

Five Dimensional On-Shell Supergravity

In this appendix we briefly review the two-derivative supergravity theory which we are interested, $N = 2$ supergravity in five dimensions coupled to an arbitrary number of vector multiplets. We review how this theory is embedded in string theory as a dimensional reduction of eleven-dimensional supergravity on a Calabi-Yau threefold [156, 157].

A.1 M-theory on a Calabi-Yau threefold

We begin with the action

$$S_{11} = -\frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-G} \left(R + \frac{1}{2} |\mathcal{F}_4|^2 \right) + \frac{1}{12\kappa_{11}^2} \int \mathcal{A}_3 \wedge \mathcal{F}_4 \wedge \mathcal{F}_4 \ , \tag{A.1}$$

which is the bosonic part of the low energy eleven-dimensional supergravity theory. The perturbative spectrum contains the graviton G_{MN}, the gravitino Ψ_M, and the three-form potential $\mathcal{A}_3$ with field strength $\mathcal{F}_4 = d\mathcal{A}_3$. This theory is maximally supersymmetric, possessing 32 independent supersymmetries. There are spatially extended half-BPS solitons, the $M2$ and $M5$-branes, which carry the electric and magnetic charges, respectively, of the flux $\mathcal{F}_4$.

The five-dimensional theory of interest is obtained via compactification on a Calabi-Yau threefold CY_3 and depends only on topological data of the compactification manifold. Let J_I be a basis of closed $(1,1)$-forms spanning the Dolbeault cohomology group $H^{(1,1)}(CY_3)$

and let $h^{(1,1)} = dim\left(H^{(1,1)}\left(CY_3\right)\right)$. We can then expand the Kähler form J on CY_3 as

$$J = M^I J_I \ , \qquad I = 1 \ldots h^{(1,1)} \ . \tag{A.2}$$

By de Rham's theorem[1] we can choose a basis of two-cycles ω^K for the homology group $H_2\left(CY_3\right)$ such that

$$\int_{\omega^K} J_I = \delta_I^K \ . \tag{A.3}$$

Thus the real-valued expansion coefficients M^I can be understood as the volumes of the two-cycles ω^I

$$M^I = \int_{\omega^I} J \ . \tag{A.4}$$

The M^I are known as *Kähler moduli* and they act as scalar fields in the effective five-dimensional theory. We will often refer to the M^I simply as the *moduli* since the other Calabi-Yau moduli, the *complex structure moduli*, lie in $D = 5$ hypermultiplets and are decoupled for the purposes of investigating stationary solutions.[2]We will therefore largely ignore the hypermultiplets in the following.

The eleven-dimensional three-form potential can be decomposed after compactification as

$$\mathcal{A}_3 = A^I \wedge J_I \ , \tag{A.5}$$

where A^I is a one-form living in $D = 5$. This results in $h^{(1,1)}$ vector fields in the five-dimensional effective theory. Since the J_I are closed, the field strengths are given by

$$\mathcal{F}_4 = F^I \wedge J_I \ , \tag{A.6}$$

[1]This theorem asserts the duality between the homology group $H_2\left(\mathcal{M}\right)$ and the *de Rham* cohomology $H^2\left(\mathcal{M}\right)$ for a manifold $\mathcal{M}$. For a Calabi-Yau threefold there are no $(0,2)$ or $(2,0)$ forms in the cohomology so we have a duality between H_2 and the *Dolbeault* cohomology $H^{(1,1)}$.

[2]By *decoupled*, we mean that they can be set to constant values in a way consistent with the BPS conditions and equations of motion of the theory.

where $F^I = dA^I$. The eleven-dimensional Chern-Simons term reduces to

$$\int_{\mathcal{M}_{11}} \mathcal{A}_3 \wedge \mathcal{F}_4 \wedge \mathcal{F}_4 = \int_{CY_3} J_I \wedge J_J \wedge J_K \int_{\mathcal{M}_5} A^I \wedge F^J \wedge F^K \tag{A.7}$$

$$= c_{IJK} \int_{\mathcal{M}_5} A^I \wedge F^J \wedge F^K \ , \tag{A.8}$$

where in the last line we have used the definition of the Calabi-Yau manifold *intersection numbers*

$$c_{IJK} = \int_{CY_3} J_I \wedge J_J \wedge J_K \ . \tag{A.9}$$

The nomenclature arises since c_{IJK} can be regarded as counting the number of triple intersections of the four-cycles ω_I, ω_J, and ω_K, which are basis elements of the homology group $H_4\left(CY_3\right)$. This basis has been chosen to be dual to the previously introduced basis of $H_2\left(CY_3\right)$, *i.e.* with normalized inner product

$$\left(\omega^I, \omega_J\right) = \delta^I_J \ , \tag{A.10}$$

where this inner product counts the number of intersections of the cycles ω^I and ω_J.

The above is almost sufficient to write down the $D = 5$ Lagrangian, but there is an important constraint that must be considered separately. To fill up the five-dimensional supersymmetry multiplets one linear combination of the aforementioned vectors must reside in the gravity multiplet. This vector is called the *graviphoton* and is given by

$$A^{grav}_\mu = M_I A^I_\mu \ , \tag{A.11}$$

where the M_I are the volumes of the basis four-cycles ω_I

$$M_I = \frac{1}{2}\int_{\omega_I} J \wedge J = \frac{1}{2}\int_{CY_3} J \wedge J \wedge J_I = \frac{1}{2} c_{IJK} M^J M^K \ . \tag{A.12}$$

Since one combination of the vectors arising from compactification does not live in a vector multiplet, the same must be true of the scalars. It turns out that the total Calabi-Yau volume, which we call $\mathcal{N}$, sits in a hypermultiplet. Due to the decoupling of hypermultiplets we can simply fix the value of the volume, and so we arrive at the *very special geometry* constraint

$$\mathcal{N} \equiv \frac{1}{3!}\int_{CY_3} J\wedge J\wedge J = \frac{1}{6}c_{IJK}M^I M^J M^K = 1 \ . \tag{A.13}$$

Due to the above considerations the index I runs over $1\ldots(n_V+1)$, where n_V is the number of independent vector multiplets in the effective theory.

Choosing units[3] such that $\kappa_{11}^2 = \kappa_5^2\mathcal{N} = 2\pi^2$, the action for the theory outlined above is

$$S = \frac{1}{4\pi^2}\int_{\mathcal{M}_5} d^5x\sqrt{|g|}\mathcal{L} \ , \tag{A.14}$$

with Lagrangian

$$\mathcal{L} = -R - G_{IJ}\partial_a M^I\partial^a M^J - \frac{1}{2}G_{IJ}F^I_{ab}F^{Jab} + \frac{1}{24}c_{IJK}A^I_a F^J_{bc}F^K_{de}\epsilon^{abcde} \ . \tag{A.15}$$

The metric on the scalar moduli space is [74]

$$G_{IJ} = \frac{1}{2}\int_{CY_3} J_I\wedge *J_J = -\frac{1}{2}\,\partial_I\partial_J(\ln\mathcal{N})|_{\mathcal{N}=1} = \frac{1}{2}\left(\mathcal{N}_I\mathcal{N}_J - \mathcal{N}_{IJ}\right) \ , \tag{A.16}$$

where the $*$ denotes Hodge duality within the Calabi-Yau and $\mathcal{N}_I$ and $\mathcal{N}_{IJ}$ denote derivatives of $\mathcal{N}$ with respect to the moduli

$$\mathcal{N}_I \equiv \partial_I\mathcal{N} = \frac{1}{2}c_{IJK}M^J M^K = M_I \ , \quad \mathcal{N}_{IJ} \equiv \partial_I\partial_J\mathcal{N} = c_{IJK}M^K \ . \tag{A.17}$$

As previously stated, the eleven-dimensional theory is maximally supersymmetric with 32 independent supersymmetries. A generic Calabi-Yau manifold has $SU(3)$ holonomy, reduced from $SU(4)(\cong SO(6))$ for a generic six-dimensional manifold; thus it preserves 1/4 supersymmetry, or eight independent supersymmetries. More precisely, this is the number of explicit supersymmetries for general c_{IJK}; for special values of the c_{IJK} there are more supersymmetries which are implicit in our formalism. These values correspond

[3]Equivalently, our units are such that the five-dimensional Newton's constant is $G_5 = \frac{\pi}{4}$.

to compactification on a manifold $\mathcal{M}$ of further restricted holonomy; for $\mathcal{M} = T^2 \times K3$ there are 16 supersymmetries, while for $\mathcal{M} = T^6$ there are 32. For these cases with more than eight supersymmetries, there will be additional gauge fields in the five-dimensional spectrum. The formalism used in this review can describe solutions in these theories which are uncharged under the additional gauge fields.

A.2 M-branes and $D = 5$ solutions

Eleven-dimensional supergravity has asymptotically flat solutions with non-trivial four-form flux. In the full quantum description, these solutions are understood to be sourced by certain solitonic objects, the M-branes. Specifically, the $M2$-*brane* is an extended object with a $(2+1)$-dimensional worldvolume which carries the unit electric charge associated with $\mathcal{A}_3$. Conversely, the $M5$-*brane* carries the unit magnetic charge of $\mathcal{A}_3$ and has a $(5+1)$-dimensional worldvolume. The worldvolumes of these objects can be wrapped around various cycles in a Calabi-Yau and so lead to sources in the effective five-dimensional theory.

The five-dimensional theory has a wealth of interesting supersymmetric solutions including black holes, black strings and black rings. As indicated above, these each can be embedded into M-theory as a bound state of M-branes. In particular, wrapping an $M2$-brane around one of the basis two-cycles ω^I leads to a five-dimensional solution carrying electric charges

$$q_I \equiv -\int_{S^3} \frac{\delta S}{\delta F^I} = \frac{1}{2\pi^2} \int_{S^3} G_{IJ} \star F^J \ , \tag{A.18}$$

where the integral is taken over the asymptotic three-sphere surrounding the black hole. Wrapping an $M5$ brane around one of the basis four-cycles ω_I gives an infinitely extended one-dimensional string,[4] carrying the magnetic charges

$$p^I = -\frac{1}{2\pi}\int_{S^2} F^I \ , \tag{A.19}$$

where one integrates over the asymptotic two-sphere surrounding the string. Further, there are dyonic solutions constructed from both $M2$ and $M5$-branes. These can take the form of either infinite strings with an extended electric charge density along their volume, or a black ring, where the $M5$-branes contribute non-conserved magnetic dipole moments.

[4]This is not to be confused with a fundamental string, although special configurations are dual to an infinite heterotic string.

APPENDIX B

Counting Formulas

This appendix contains some technical details used in Chapter IV. We briefly sketch the evaluation of the contour and saddle point integral discussed in Section 4.2, some relevant properties of the Jacobi functions and some details of the Jacobi-Rademacher expansion.

B.1 Some details of the evaluation of the contour and saddle point integral

In this appendix, we shall sketch some relevant details about the evaluation of the integral (4.18) which we recall here. Consider

$$\Omega^{5D}(Q_1, Q_5, n, l) = \oint_{\mathcal{C}} d\widetilde{\rho} d\widetilde{\sigma} d\widetilde{v}\, e^{-2i\pi(\widetilde{\rho}n + \widetilde{\sigma}(Q_1 Q_5 + 1) + l\widetilde{v})}\, Z(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \ . \tag{B.1}$$

The integral above is over the contour

$$\begin{aligned} &0 < \Re(\widetilde{\rho}) \leq 1 \ , \quad 0 < \Re(\widetilde{\sigma}) \leq 1 \ , \quad 0 < \Re(\widetilde{v}) \leq 1 \ , \\ &\Im(\widetilde{\rho}) \gg 1 \ , \quad \Im(\widetilde{\sigma}) \gg 1 \ , \quad \Im(\widetilde{v}) \gg 1 \ , \end{aligned} \tag{B.2}$$

over the three coordinates, where $\Re$ and $\Im$ denote the real and imaginary parts. This defines the integration curve $\mathcal{C}$ as a 3-torus in the Siegel upper half-plane. The imaginary parts are taken to be large to guarentee convergence. As we shall see below, the dominant pole in the function is not affected, and we can therefore perform the contour integral around that pole. This gives a prescription for the contour. As mentioned in the text, it is expected that there is no dependence on the moduli in the 5D theory, and therefore there are no other poles where wall-crossing behavior occurs in the 5D integral. A precise analysis of the contour as was done in 4D [112] remains to be done.

We mostly follow [13] in our notation and conventions. First we need to do a contour integral in the $\widetilde{v}$ coordinate, which picks up the residue at various poles. These poles occur at zeros of the function Φ_{10} and the poles of the function f^{KK}. For large charges, the dominant contribution when the exponent takes its largest value at its saddle point. This was analyzed in [50]. When f^{KK} is not present, this dominant divisor is

$$\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v} = 0 \ . \tag{B.3}$$

We can check that the function (4.17)

$$f^{KK}(\rho, \sigma, v) = p\, \eta^{18}(\rho)\, \vartheta_1^2(v, \rho) \ , \tag{B.4}$$

does not take away this pole, and does not alter the dominance of this pole. We can now carry out the contour integration in the variable $\widetilde{v}$ around the zero of the above divisor

$$\widetilde{v}_{\pm} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \widetilde{\rho}\widetilde{\sigma}} \ . \tag{B.5}$$

In the contour integration, the variables $\widetilde{\rho}$ and $\widetilde{\sigma}$ are held fixed and we choose the negative value of the square root $\widetilde{v}_{-}$.

The modular properties of the function Φ_{10} under $Sp(2, \mathbb{Z})$ allow us to factorize it around the value $\widetilde{v} = \widetilde{v}_{-}$. The integrand in (4.18) behaves like:

(B.6)

$$C \exp\left(-2\pi i(\widetilde{\rho}n + \widetilde{s}(Q_1Q_5 + 1) + 2\widetilde{v}l)\right) \widetilde{\sigma}^{12}(\widetilde{v} - \widetilde{v}_{+})^{-2}(\widetilde{v} - \widetilde{v}_{-})^{-2}\eta^{-24}(\rho)\eta^{-24}(\sigma) f^{KK}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \ .$$

Using this factorization, we can evaluate the contour integral, and then perform a saddle point analysis of the remaining integral over $(\widetilde{\rho}, \widetilde{\sigma})$. This relatively straightforward procedure gives the expression (4.19) to be evaluated at its extremum.

Note that the function f^{KK} does not have any poles in the interior of the region we are considering, but has many zeroes. These zeroes do not include the divisor (B.3). Therefore the dominant pole of Φ_{10}^{-1} remains the dominant pole of the 5D integrand Z. Note however that f^{KK} does have a zero at $\widetilde{v} = 0$ which takes away the pole at the same value of the function Φ_{10}^{-1}. This means that there is no wall crossing behavior in the five dimensional theory due to this pole. For the evaluation of the integral, these observations mean that the presence of the function f^{KK} changes the analysis only through its appearance in the entropy function (4.19) to be extremized.

B.2 The Jacobi η and ϑ functions and their properties

We define

$$q = e^{2\pi i\tau} \ , \quad y = e^{2\pi i v} \ . \tag{B.7}$$

The Jacobi eta function is defined as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \ . \tag{B.8}$$

The odd Jacobi theta function is

$$\vartheta_1(v, \tau) = -2q^{\frac{1}{8}} \sin(\pi v) \prod_{m=1}^{\infty} (1 - q^m)(1 - q^m y)(1 - q^m y^{-1}) \ . \tag{B.9}$$

For large imaginary values of $\tau = it$, $t \to \infty$, we have $q \to 0$ most of the terms in the product become unity and these functions admit an expansion of the form

$$\eta(\tau) = -\frac{\pi}{12} t + \ldots \tag{B.10}$$

These functions satisfy the modular properties:

$$\begin{aligned} \eta(-\frac{1}{\tau}) &= \sqrt{-i\tau}\,\eta(\tau) \\ \vartheta_1(\frac{v}{\tau}, -\frac{1}{\tau}) &= i\sqrt{-i\tau}\, e^{i\pi v^2/\tau} \vartheta_1(v, \tau) \ . \end{aligned} \tag{B.11}$$

For the ϑ function, the expansion depends on the value of v compared to τ, but similar expansions are possible.

B.3 The Jacobi-Rademacher expansion

The Jacobi-Rademacher expansion [82, 106] is a very powerful (exact) expansion containing both power law and exponential corrections to the Cardy estimate. Here, we are only interested in the first power law correction, which can be estimated by using a Jacobi modular transformation and a saddle point expansion.

The counting of 1/4 BPS states of the D1-D5 system on $K3$ is summarized by the elliptic genus of the 2D SCFT $Sym^k(K3)$ with $k = Q_1Q_5 + 1$. This elliptic genus can be expanded in a theta function decomposition

$$\begin{aligned} \chi(Sym^k(K3);\tau,z) &= -\sum_{l=-k+1}^{k}\sum_{n\in\mathbb{Z}} c(n,\mu)\, q^{n-l^2/4k}\,\theta_{l,k}(z,\tau) \\ &\equiv -\sum_{l=-k+1}^{k} h_l(\tau)\,\theta_{l,k}(z,\tau)\ . \end{aligned} \tag{B.12}$$

We write

$$h_l(\tau) = \sum_{m=0}^{\infty} H_l(m)\, q^{m-\frac{l^2}{4k}}\ . \tag{B.13}$$

We can estimate the value of the coefficients $H_l(n)$ when $n \gg k$ using the Cardy's formula after doing a modular transformation on the elliptic genus and performing a saddle point expansion

$$H_l(n) = (const)\, e^{\pi i l}\frac{k}{(4nk-l^2)^{\frac{1}{2}}}\, I_{3/2}(2\pi\sqrt{nk-l^2/4}) + \dots\ , \tag{B.14}$$

where the dots denote terms which are exponentially suppressed. There is actually an exact formula which captures all the exponentially sub-leading terms [82, 106] which we don't need here.

Here $I_{3/2}$ is the modified Bessel function of the first type. The index 3/2 appears because the weight of the vector valued modular form $H_\mu(z)$ is $w = -\frac{1}{2}$. Note that by definition, the elliptic genus has weight zero, but the θ functions have weight $+\frac{1}{2}$, so the functions H_μ

have weight $-\frac{1}{2}$. This function in fact has an expression in terms of elementary functions

$$I_{3/2}(z) = \sqrt{\frac{2}{\pi z}} \left(\cosh(z) - \frac{\sinh(z)}{z} \right) . \tag{B.15}$$

The entropy is the logarithm of the degeneracy $H_\mu(n)$. With $k = Q_1Q_5 + 1$, we have $z = 2\pi\sqrt{(Q_1Q_5+1)n - \ell^2/4}$. The entropy is equal to

$$\begin{aligned} S^{5D} &= \log\left(e^z \left[1 - \frac{1}{z} \right] \right) + \dots \\ &= 2\pi\sqrt{(Q_1Q_5+1)n - l^2/4} \left(1 + \frac{1}{4\pi^2(Q_1Q_5 n - l^2/4)} + \dots \right) , \end{aligned} \tag{B.16}$$

which is in agreement with (4.30).

APPENDIX C

Two dimensional Maxwell-Dilaton Gravity

In this appendix we gather the conventions and notations used in Chapter V. Also, some calculations concerning the dictionary between 3D and 2D gravity are discussed below.

C.1 Conventions and Notations

The 2D Newton's constant is determined by requiring that the normalization of the gravitational action is given by

$$I = -\frac{1}{16\pi G_2}\int_{\mathcal{M}} d^2x\,\sqrt{-g}\,R + \ldots, \tag{C.1}$$

where the unusual minus sign comes from requiring positivity of several physical quantities, as explained in the body of the paper. Comparing (5.2) with (C.1) gives the relation (5.3) for constant dilaton backgrounds.

For sake of compatibility with other literature we choose a somewhat unusual normalization of the AdS radius L so that in 2D $R_{AdS} = -8/L^2$, and for electric field E we use $F_{\mu\nu} = 2E\,\epsilon_{\mu\nu}$. For the same reason our gauge field has inverse length dimension. As usual, the quantity $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength for the gauge field A_μ, its square is defined as $F^2 = F^{\mu\nu}F_{\mu\nu} = -8E^2$, and the dilaton field ϕ is defined by its coupling to the Ricci scalar of the form $e^{-2\phi}\,R$.

Minkowskian signature $-,+,\ldots$ is used throughout this paper. Curvature is defined such that the Ricci-scalar is negative for AdS. The symbol $\mathcal{M}$ denotes a 2D manifold with coordinates x^μ, whereas $\partial\mathcal{M}$ denotes its timelike boundary with coordinate x^a and induced metric h_{ab}. We denote the 2D epsilon-tensor by

$$\epsilon_{\mu\nu} = \sqrt{-g}\,\varepsilon_{\mu\nu} \ , \tag{C.2}$$

and fix the sign of the epsilon-symbol as $\varepsilon^{t\eta} = -\varepsilon_{t\eta} = 1$.

In our 2D study we use exclusively the Fefferman-Graham type of coordinate system

$$ds^2 = d\eta^2 + g_{tt}\,dt^2 \ , \tag{C.3}$$

in which the single component of the induced metric on $\partial\mathcal{M}$ is given by $h_{tt} = g_{tt}$ with 'determinant' $h = h_{tt}$. In the same coordinate system the outward pointing unit vector normal to $\partial\mathcal{M}$ is given by $n^\mu = \delta^\mu_\eta$, and the trace of the extrinsic curvature is given by

$$K = \frac{1}{2}h^{tt}\,\partial_\eta h_{tt} \ . \tag{C.4}$$

Our conventions in 3D are as follows. Again we use exclusively the Fefferman-Graham type of coordinate system

$$ds^2 = d\eta^2 + \gamma_{ab}\,dx^a dx^b \ , \tag{C.5}$$

in which the induced metric on the boundary is given by the 2D metric γ_{ab}. In the same coordinate system the extrinsic curvature is given by

$$\mathcal{K}_{ab} = \frac{1}{2}\partial_\eta \gamma_{ab} \ , \tag{C.6}$$

with trace $\mathcal{K} = \gamma^{ab}\mathcal{K}_{ab}$. The 3D AdS radius ℓ is normalized in a standard way, $\mathcal{R}_{\text{AdS}} = -6/\ell^2$. Without loss of generality we assume that the AdS radii are positive: $L, \ell > 0$.

C.2 Dictionary between 2D and 3D theories

We have derived in Section 5.3.2 the relation (5.65) between the normalization constants in 2D and 3D. As a consistency check on our 3D interpretation of the 2D theory we show in

Section C.2.1 that the boundary terms also reduce correctly. Also, since the 2D Maxwell-dilaton theory is on-shell equivalent to the KK-reduction of 3D gravity, the 3D solutions respecting the appropriate isometry must agree with a 2D solution. We construct the explicit map in Section C.2.2.

C.2.1 Kaluza-Klein reduction: the boundary terms

Applying the KK-reduction (5.57) to a 3D metric in the Fefferman-Graham form (5.53) we can write

$$ds^2 = e^{-2\psi}\ell^2(dz + \tilde{A}_t dt)^2 + \tilde{h}_{tt}dt^2 + d\eta^2 \ . \tag{C.7}$$

Here we identify $\tilde{h}_{tt}$ as the metric of the 1D boundary of the 2D metric $\tilde{g}_{\mu\nu}dx^\mu dx^\mu$. Surfaces of (infinite) constant η define the boundary in both 3D and in 2D, and so we can use η as the radial coordinate in both cases. The 3D trace of extrinsic curvature becomes

$$\mathcal{K} = \tilde{K} - \partial_\eta \psi \ , \tag{C.8}$$

with $\tilde{K}$ the extrinsic curvature of the one dimensional boundary $\tilde{h}_{tt}$. The boundary term of the 3D theory in (5.50) therefore reduces to the boundary term

$$\tilde{I}_{\text{boundary}} = \frac{\ell}{4G_3}\int dt\sqrt{-\tilde{h}}\, e^{-\psi}\left(\tilde{K} - \frac{1}{\ell}\right) \ , \tag{C.9}$$

of the 2D theory. The term proportional to the gradient in ψ canceled an identical term arising when integrating the bulk term (5.58) by parts.

In order to show that our 2D theory is equivalent on-shell to the KK-reduction of the 3D theory we must match (C.9) with the boundary term

$$I_{\text{boundary}} = \frac{\alpha}{\pi}\int dt\sqrt{-h}\, e^{-2\phi}\left(K - \frac{2}{L} + \frac{L}{2}e^{2\phi}A^a A_a\right) \tag{C.10}$$

determined directly in 2D. Evaluating (C.10) on the asymptotic AdS_2 backgrounds (5.11) we have

$$K = \frac{2}{L} \ , \qquad h^{ab}A_a A_b = -e^{-2\phi}\frac{4}{L^2} \ . \tag{C.11}$$

and so

$$I_{\text{boundary}} = -\frac{\alpha}{\pi}\int dt\sqrt{-h}e^{-2\phi}\frac{2}{L}$$
$$= \frac{\ell}{4G_3}\int dt\sqrt{-\tilde{h}}e^{-\psi}\frac{1}{\ell}\ , \tag{C.12}$$

where in the last line we used our 3D-2D dictionary (5.62a), (5.65).

Asymptotically AdS_2 solutions of the theory defined by the bulk action (5.59) have extrinsic curvature $\tilde{K} = \frac{2}{\ell}$ and therefore the on-shell value of the KK reduced boundary action (C.9) exactly agrees with the on-shell value of the 2D boundary action (C.12), i.e., $\tilde{I}_{\text{boundary}} = I_{\text{boundary}}$. This is what we wanted to show.

C.2.2 Asymptotically AdS solutions

Our starting point is the general 3D solution (5.55). For compactifications along $z = x^+$ we consider $g_+ =$ constant, and rewrite the solution in the form (C.7) as

$$\begin{aligned} ds^2 \; &= \; \Big(\frac{g_+}{\ell}\Big)\ell^2\Big[dx^+ + \frac{1}{2\ell}e^{2\eta/\ell}\sqrt{\frac{\ell}{g_+}}\big(1+\frac{16}{\ell^2}g_+g_-(t)e^{-4\eta/\ell}\big)dt\Big]^2 \\ &\quad -\frac{1}{4}e^{4\eta/\ell}\big(1-\frac{16}{\ell^2}g_+g_-(t)e^{-4\eta/\ell}\big)^2dt^2 + d\eta^2\ , \end{aligned} \tag{C.13}$$

with

$$t = \frac{\ell}{4}\sqrt{\frac{\ell}{g_+}}\,x^-\ . \tag{C.14}$$

Comparing with the Ansatz (5.57) we read off the 2D metric

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu = -\frac{1}{4}e^{4\eta/\ell}\Big(1-\frac{16}{\ell^2}g_+g_-(t)e^{-4\eta/\ell}\Big)^2dt^2 + d\eta^2\ , \tag{C.15}$$

and the matter fields

$$\tilde{A} = \frac{1}{2\ell}e^{2\eta/\ell}e^{\psi}\Big(1+\frac{16}{\ell^2}g_+g_-(t)e^{-4\eta/\ell}\Big)dt\ , \tag{C.16a}$$

$$e^{-2\psi} = \frac{g_+}{\ell}\ . \tag{C.16b}$$

The solution (C.15)-(C.16) should be equivalent to the asymptotically AdS_2 solutions (5.11) found directly in 2D. After the coordinate transformation $(\eta,\, t) \to \frac{1}{a}\,(\eta,\, t)$ in (5.11) this expectation is correct, and we use the dictionary (5.62), (5.65) to find the relations $h_0(t) = 1$, $a(t) = 0$ and

$$h_1 = -\frac{16}{\ell^2} g_+ g_-(t) \ , \tag{C.17a}$$

$$\alpha = -\frac{\pi\ell}{8G_3} e^{2\phi} \sqrt{\frac{g_+}{\ell}} \ , \tag{C.17b}$$

between the parameters of the solutions.

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