# UNIT-PRICE PROCUREMENT AUCTIONS: STRUCTURAL RESULTS AND COMPARISONS

by

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# CHAPTER 1

# Overview

This dissertation studies unit-price procurement auctions, focusing on the structural impacts of suppliers' capacity and of system uncertainties. My interest in this research area was directly motivated by my two-year involvement with a major energy company at Michigan. I was part of several projects including capacity investment in storage and transmission systems, bidding in procurement auctions, and integrating decisions related to natural gas inventory and financial hedging.

The energy industry can be characterized by complex market mechanism, intense competition, and also voluminous data, resulting in significant number of interesting questions. Analysis of such industry naturally leads to a broad range of methodologies. I specifically embrace stochastic optimization, game theory, and econometrics, when attempting to answer the questions I outline below. The managerial insights, while derived in the context of energy industry, can be generalized and applied in other related fields, including supply chain management, procurement management, and auction design.

Chapter 2 is motivated by the high price volatility in wholesale electricity markets and by the different formats of auctions observed in electricity markets, emphasized by the ongoing debate about their advantages and disadvantages. Our objectives are to interpret the high price volatility in the wholesale electricity auctions and to compare two prevailing market designs, discriminatory and uniform ones, used in the United Kingdom and the United States respectively. Two interrelated questions are examined: (a) how price volatility is related to industry capacity structure and demand uncertainty; (b) how to compare the performance of a uniform auction (charging all suppliers a uniform market-clearing price) and a discriminatory auction (charging each supplier a price equal to her/his bid), in terms of both average price paid by the auctioneer and price volatilities.

We consider a basic model with multiple symmetric bidders as well as a model with two asymmetric bidders. We show that, in the absence of exogenous sources of uncertainty such as asymmetric information and random demand, price may not be constant and price dispersion may stem endogenously from electricity producers' randomized bidding due to the prevalence of a mixed-strategy equilibrium. The bid distribution and consequent price variance are mainly determined by system utilization. Introduction of demand uncertainty increases the likelihood of price dispersion (due to a wider range of system utilizations that lead to mixed-strategy equilibria), but not necessarily the magnitude of price variability. Numerical studies further illustrate that demand uncertainty has a secondary contribution to price dispersion, compared with system utilization. Based on the model of conditional price dispersion that we propose, the empirical study on the New England Power Pool qualitatively supports our theoretical predictions.

The comparison between the uniform and discriminatory auctions indicates that, at symmetric equilibria, they yield the same average price but the discriminatory auction results in lower price volatility. This lesson continues to hold for the auctions with uncertain demand and well describes the cases with two asymmetric bidders. In contrast to the two schools of auction theorists who argue one auction's efficiency superiority over the other, our results suggest that the two auctions have the similar efficiency but the discriminatory auction has lower volatility. These insights are also of practical value for the procurement managers in other industries who face competitive suppliers with capacity constraints.

In Chapter 3, we continue to study the effect of capacity constraints, but in more realistic, and thus more complicated settings. We focus on discriminatory auction and extend Chapter 2 by considering multiple asymmetric bidders. The contributions are both methodological and in providing economic insights.

The game we consider is equivalent to the Bertrand-Edgeworth competition (capacitated suppliers competing in price) with inelastic demand. We derive the equilibrium structure, which can be characterized as follows. Pure-strategy equilibrium is achieved only under restricted system conditions. Otherwise, a mixed-strategy equilibrium prevails. In these cases, we show that (a) all active suppliers randomize their bids within their price intervals, (b) for suppliers with the same cost, their bidding intervals have a nested structure, (c) there exists a market leader whose bidding interval covers those of the other players and she is usually the one with the highest capacity, (d) when demand increases, the lower bound of market leader increases continuously, while the upper bound jumps from one discrete cost level to another; thus, the price range expands and contracts alternately.

Based on the structural results, we propose a numerical algorithm to compute the bid distribution at equilibrium. Our numerical tests provide new insight about a result in Kreps & Scheinkman (1983) that lower capacity yields more aggressive pricing behavior (as indicated by stochastically lower prices). We demonstrate that, while the result always holds in two-player games, in multiple-player settings, both costs and capacities influence the bidding strategy. If the capacities are identical, a player with lower cost will indeed bid more aggressively. In general, however, the equilibrium has a nested structure in which a low-capacity player prices within the range chosen by a high-capacity player, given that their costs are the same. Finally, we prove that the structural properties of equilibrium are robust and can be extended to games with price-elastic demand. Since the properties above are also new in price-elastic setting, this chapter extends the analysis of Bertrand-Edgeworth oligopoly competition.

The fourth chapter extends the basic model by including asymmetric information about suppliers' costs and capacities. It also compares the efficiency of discriminatory and uniform auctions under this more general setting. The setup of the game is similar to the previous chapters. Multiple suppliers, with their individual costs and capacities, submit price bids. We continue to assume that each supplier submits a single unit price, which allows for analytical tractability and new economic and managerial insights. We analyze two models: one where production cost is private information, and another where capacity is private information.

When each supplier's cost is privately known, we show that the equilibrium bidding strategy is jointly controlled by (a) the marginal contribution of a supplier's capacity in satisfying demand, and (b) the probability distribution of uncertain costs. We establish an efficiency equivalence outcome between discriminatory and uniform auctions, which generalizes revenue equivalence in auction literature (Vickrey 1961) and payment equivalence in unit-price auction with complete information (Chapter 2). Numerical study indicates that, in a discriminatory auction, a supplier's bid is increasing in its cost and system utilization; while in a uniform auction, the monotonicity in utilization rates holds in two regions, but is disrupted at the transition point.

When a supplier's capacity is private information, we separately analyze discriminatory and uniform auctions. For discriminatory auctions, monotone equilibrium always exists, where supplier's bids decrease in capacity and the revenues increase in capacity. For uniform auctions, monotone equilibrium always exists for two-bidder case, but may not exist for three or more suppliers. If it exists, the behavior is similar to the discriminatory auction (decreasing bids and increasing revenues), but revenue equivalence does not hold. An interesting implication of revenue monotonicity is that it is incentive compatible for firms to bid full capacity (in contrast to capacity withholding). Numerical study indicates that, if monotone equilibrium exists for a certain demand level, it also exists for all higher demands.

The next three chapters consist of the three essays.

# CHAPTER 2

# Price Dispersion in Electricity Auctions

## 2.1 Introduction

Since the year 2000, high price volatility, including occasional extreme price shocks, has been the most prominent characteristic of wholesale electricity markets. Figure 2.1 displays hourly spot electricity prices in the New England market (NEPOOL) over the period January 2004 through June 2006, with prices ranging from \$0/MWh to large price spikes above \$200/MWh. The price behavior observed in NEPOOL is far from unique. For the same time period, the prices in other major electricity trading hubs (PJM, NYISO, and MISO) range from -\$10/MWh[1] to above \$250/MWh, including NYISO, the New York City hub, where price has a record low of -\$279/MWh and a record high of 1894/MWh.

In this paper we focus on wholesale markets, where retailers are facing price insensitive demand and suppliers have constant costs and capacities. While many examples have some features of such markets, its extreme case is deregulated wholesale electricity market, which both motivated this study and is our focus throughout the paper.

Major electricity buyers are the energy distributors who procure power from wholesale markets and transmit it to their customers (residential, commercial, and a portion of industrial users). The retail prices paid by electricity end-consumers are regulated and, thus, very stable in short-to-medium term, with instantaneous demand almost inelastic to the wholesale prices. On the supply side, each electricity

[1] Negative electricity prices may result from suppliers' incentive to maintain power generation in order to overcome high start-up costs.

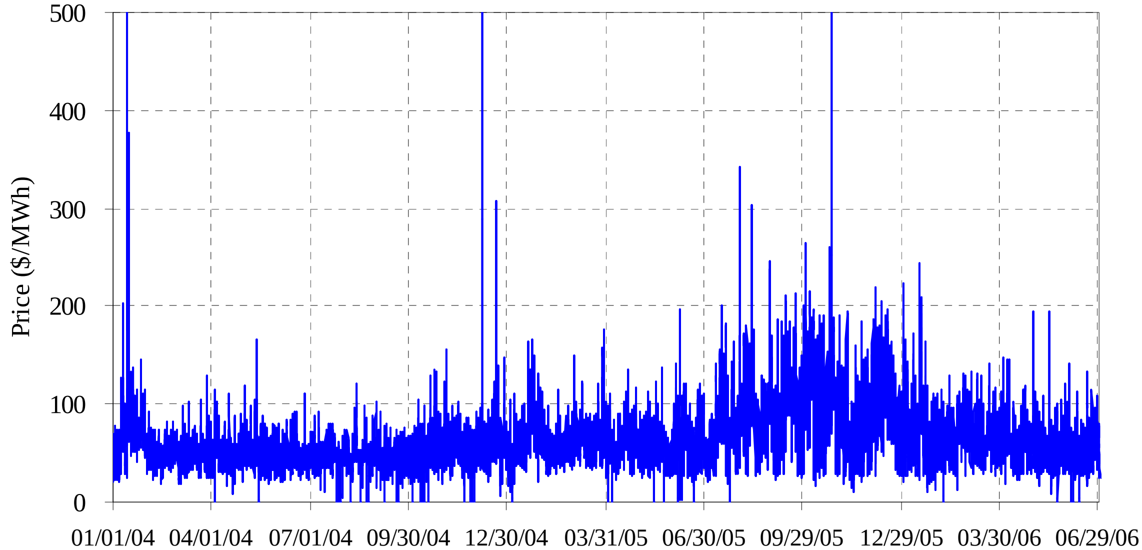


**Figure 2.1.** NEPOOL Hourly Electricity Spot Prices (Jan.2004-Jun.2006)

generation unit with specific technology has a fixed capacity associated with it and a fairly stable variable cost.

Our first objective is to characterize the drivers and structural reasons behind price variability. In order to understand their significance, we examine effects of individual factors such as capacity, end-customer demand variability, and suppliers' asymmetry. Our approach is to answer these questions through a theoretical model that captures the critical inter-dependencies of energy market. While previous models assumed a competitive equilibrium outcome, where all suppliers are pricing at their marginal costs, we relax this assumption and examine whether price dispersion is driven by structural factors and whether it might exist even without demand uncertainty (critical for the existing models). Later we use data from wholesale energy markets to evaluate how consistent they are with our theoretical model.

Our second objective, that builds on the structural form of pricing strategies, is to evaluate and compare two dominant forms of auctions, uniform and discriminatory, in order to provide justification which of them is more appropriate for electricity markets, with emphasis both on efficiency (average prices) as well as price dispersion (price volatility). This is motivated by recent policy changes and the debates surrounding energy market. Specifically, in March 2001, seeking a better market performance, the British government implemented a radical reform in the electricity trading arrangements, and replaced uniform auction (UA) with a discriminatory (or pay-as-bid) auction (DA). Also, in November 2000 during the California crisis, the California Power Exchange appointed a panel of significant auction theorists to investigate a similar proposal, which suggested that UA action is preferred. This view is not, however, uniformly shared in literature.

To answer these two questions we model a procurement auction where demand is price-independent and suppliers have fixed capacities and constant marginal production costs. All information about generators' costs and capacities is public. Each supplier submits a bid, which is the price for operating her/his generation unit. If bid is accepted, full capacity or any portion of it can be dispatched.[2] Both types of auctions, uniform and discriminatory, have the same allocation scheme: system operator admits the suppliers one by one, according to increasing bid prices, until either demand is satisfied or all suppliers are dispatched. In case of a price-tie, each supplier has the same probability to be selected first.

The above model simplifies real electricity auctions for analytical tractability. First, we assume symmetric information about suppliers' costs and capacities among auction bidders.[3] Second, we assume constant marginal production costs. Since one firm may own multiple plants with different generating technologies, this assumption implies that the generating plants are the actual auction bidders and they operate as separate profit units. We also omit the startup costs. Arguably, the supplier's unit variable cost may be viewed as a startup-adjusted average cost. Since the demand pattern has strong intraday and weekly patterns and weather forecast is publicly available, the short-term load profile is fairly predictable. Upon bidding, a supplier has a good estimate of how long her generation unit will be used, if the bid is admitted.

[2]In other words, we purposefully rule out the possibility of strategic withholding, which has been identified as a possible measure for suppliers to exercise their market power. It is a relevant issue, but outsides the scope of this paper. See Hogan (2001) for some related discussion. This setup allows us to concentrate on the effects of capacity structure on price dynamics.

[3]Consequently, we do not study information asymmetry, which is central to significant portion of research in auction theory.

Finally, we restrict supplier's bid to be only one price and impose that the total capacity must be fully committed.

The main contributions of our paper are as follows. From technical point of view, the paper completely characterizes the equilibrium structure for N-bidder symmetric auctions. The extensions include random demand and two-bidder asymmetric cases. Our discussion, under both auction schemes, focuses on probabilistic properties of the unit prices paid by the electricity buyers and on comparison of DA and UA. With respect to our first objective, interpreting and characterizing price dispersion, we show that price dispersion may stem from suppliers' strategically randomized bidding. The factors directly influencing it are: capacity structure, cost structure, average capacity utilization, and demand uncertainty. Higher capacity utilization yields higher expected price, while price variance is maximized at intermediate levels of capacity. Introduction of demand uncertainty increases the chance of price dispersion (i.e., manifesting itself through mixed-strategy equilibrium), but not necessarily the magnitude of price variance. In order to test the robustness of the above lessons in asymmetric settings, we consider the asymmetric two-bidder case as an extension. The numerical studies indicate that, for given system utilization, increasing capacity asymmetry leads to a higher expected price and an initial increasing and possibly an eventual drop of the price variance.

The second area of our interest, comparison of the performance of DA and UA, also leads to interesting findings. (Our paper is the first analytical paper to compare the price volatilities resulting from the two auctions.) Most importantly, for N-bidder symmetric auctions with deterministic demand, the unique symmetric equilibria for DA and UA correspond to the same expected price, while DA results in lower price variance. While with no uncertainty other equilibria may exist, reasonably levels of demand uncertainty imply uniqueness of equilibrium for a UA and this unique equilibrium is symmetric. Our conclusions about average prices and their volatility are robust throughout the whole range of possible uncertainty as we focus on symmetric equilibria. Through a numerical study we examine asymmetric two-bidders cases with random demand and show that buyers, on average, pay similar prices under both market designs but UA, in majority of cases we observed, yields higher price variance than DA.

Since the original motivation came from highly volatile electricity market, we attempt to illustrate our theoretical findings using empirical data. As we do not have access to DA for U.K. market, we look at the nature of price dispersion for the US market. Since Quantile-Regression (Q-R) model allows to characterize price dispersion more comprehensively, compared to conditional-moment models, we implement this tool to demonstrate the price dispersion conditional on system demand. The empirical observation seem to be consistent with our structural results for price prediction. While we do not have access to any data that would allow comparison of two auction formats, we review a closely-related field – experimental economics, and point that it provides direct support for our conclusions.

The remainder of the paper is organized as follows. Next section describes the relevant literature, Section 2.3 describes the model. In Section 2.4 we derive solution for auctions with symmetric bidders and formally establish that the average prices are the same for DA and UA while variability is smaller for DA. Section 2.5 investigates two extensions – impacts of random demand and asymmetric bidders. Section 2.6 presents the preliminary empirical tests of price dispersion, and Section 2.7 concludes the paper and discusses its practical and policy implications.

## 2.2 Literature Review

Our focus is on two research questions (a) existence of price dispersion and (b) a comparison of two auction formats. Each of these two questions has its own stream of research associated with it. Two substreams, dealing with price dispersion in wholesale electricity markets, are within financial engineering and economics literature, respectively.

The financial engineering literature directly models the electricity price dynamics as continuous-time diffusion processes, and calibrates the models by fitting actual price data. For modeling the price process, mean-reverting model with jumps has been a popular choice (Kaminski 1997 and Deng 2000). For estimating and forecasting price volatilities, conditional autoregressive heteroskedasticity (ARCH) model and its variations (GARCH, EGARCH, etc.) are widely used (Duffie et al., 1998, and Goto and Karolyi, 2003). For quantifying the probability of extreme events in the electricity markets, extreme-value theory (EVT) is introduced in Bystrom (2005). The above models have a common objective – to capture the probabilistic properties of electricity price dynamics. While they are very popular in firms dealing with risk management, as pointed out by Duffie et al. (1998), changes in volatility are not generated by a mathematical model, but rather by real-world events that have significance which may at first only be apparent to engineers, geologists, economists or geopolitical analyst. Our paper differs with these papers in that we intend to identify the structural reasons for electricity price dispersion rather than to statistically describe the phenomena.

Within economics literature, there are two groups of relevant papers. Those that consider price dispersion based on competitive equilibrium and those where price dispersion is based on and explained within the framework of mixed strategy. Our paper belongs to the second group.

Within the first group, while not concentrating on price dispersion itself, several papers on energy market economics provide insights into the possible reasons for price dispersion. The inelasticity of both electricity demand and supply is identified as a key driver of the volatile prices (Borenstein, 2002 and Wilson 2002). The argument assumes, however, a competitive equilibrium outcome, where all suppliers are pricing at their marginal costs. Switching on new generation units clearly leads to kinks in marginal cost curve, and the changing demand drives price volatility, as different marginal cost of unit called into operation. Our paper does not use competitive outcome (does not assume price is equal marginal cost) and price dispersion exists even with deterministic demand.

Papers in the second group allow suppliers to price strategically and typically consider mixed-strategy equilibria. The primary mechanism behind their argument is that suppliers may randomize their bids (as a result of mixed-strategy equilibrium) above their costs. In other words, suppliers exercise their market power in the competition. [4]

The first relevant papers are Varian (1980) and Burdett and Judd (1983). They are first to directly use mixed-strategy equilibrium to interpret (spatial) price dispersion in retail markets. In Varian (1980), the existence of uninformed customers provides an incentive to randomize prices. In similar spirit, in Burdett and Judd (1983) customers observe a limited number of price quotes, which leads to price dispersion. The critical element, costly search (or customers not observing all prices) does not take place in our paper. Instead, limited capacity plays a pivotal role.

The possibility that capacitated firms may play mixed pricing strategy is first documented in the literature of Bertrand-Edgeworth game – see Vives (2000) for a comprehensive review of the related papers. While the focus of their paper is on deriving equivalence of two pricing games, Kreps and Scheinkman (1987, KS) was the first that presented a complete duopoly solution with asymmetric capacities. FFH (2006) considers both discriminatory and uniform auction, both for two firms. Their results for DA are similar to B-E solution in KS. Similarity is expected as the discriminatory unit-price auction model can be viewed as a B-E game with inelastic demand (Chapter 3). We extend the solution for DA and for UA to symmetric oligopoly. Chapter 3 analyze asymmetric oligopoly for DA.

FFH and Chapter 3 are clearly the closest papers to ours. Chapter 3 considers DA for asymmetric oligopoly with deterministic demand and provides several properties of equilibrium outcomes. These generalize the results for DA listed in this paper. Chapter 3, however, does not study the impacts of demand uncertainty, does not consider UA, and does not compare auction formats. We compare our paper with FFH in more detail below, since it is also relevant to our second research objective.

The paper's second objective is to compare the performance of two prevailing market designs for trading energy within wholesale market, discriminatory auctions (DA) and uniform auctions (UA).

The importance of this question is emphasized not only by volume of relevant

[4]Market power is defined as the capability of a firm to raise the market price above the industrial marginal cost level.

papers, but also by a public debate. In the process of global energy deregulation, uniform auction has become dominantly adopted electricity procurement mechanism and, as mentioned in the introduction, in March 2001, to improve market efficiency, the British government replaced uniform auction (UA) with a discriminatory auction (DA). However, during the California crisis, the panel of significant auction theorists (Kahn, Cramton, Porter, and Tabors, 2001) (KCPT) rejected discriminatory auction by predicting that the change would introduce new inefficiencies. The panel's prediction was not, however, based on any specific model of interactions. Similar message is expressed in Wolfram 1999. Other theoretical literature (e.g., Febra et al 2006) predicts that DA auctions are more efficient. The question thus remains open.

The two auction formats have been studied in multi-unit and shared auction literature, motivated by the treasury auctions, where both auction formats have been implemented. The theoretical analysis focuses primarily on revenue of auctioneer and allocation efficiency (i.e., whether the goods are awarded to the buyers with highest valuations, or in procurement setting the suppliers with the lowest costs). Binmore and Swierzbinski (2000) and Ausubel and Cramton (2002) both point out that the efficiency and revenue ranking of the two auctions is ambiguous and may be influenced by equilibrium selection, bidders' valuation structure, and asymmetry of the system. Krishna and Perry (1998) establishes revenue equivalence in multiunit auctions. The key distinction of these papers from ours is that they focus on the impacts of asymmetric information. The dominance of auctions is primarily driven by the information rents. Our paper assumes complete information, so the issue of information rents does not exist.

Wilson (1979) and Wang and Zender (2002) consider these two auctions under the setting of perfectly divisible goods. Wang and Zender show that in symmetric-bidders setting, there always exist equilibria of uniform auction with lower expected revenue for the auctioneer, implying superiority of discriminatory auction. As illustrated by Wilson (1979), when allowed to submit continuous demand schedule, bidders can reduce the intensity of their competition and therefore reduce the revenue of the auctioneer. Simply, continuous schedule reduces the benefits (to the auctioneer) of undercutting. In electricity auctions, since a supplier's bidding decision is restricted to limited number of price-quantity pairs, the nature of the competition is significantly different from the Wilson and Wang and Zender's models, which limit the application of their results.

Our paper formally models the two auction formats and confirms Wilfram (1999) and KCPT (2001)'s argument that bidders will *bid* (stochastically) higher in DA than in UA. However, this does not imply that the prices *paid* will be higher. Our analytical results for symmetric settings (and numerical study for asymmetric ones) suggest that the average prices in both auctions are the same (very close to each other).

Since FFH is very close to our paper, we describe it in more detail. Most importantly, FFH argues that uniform auctions result in higher average prices than discriminatory auctions. Our paper does suggest that DA is a better market design, but not due to average price but due to price variability, i.e., both auctions have the same average price but DA yields lower price variances. The difference is driven by different objectives, slightly different setting, and significantly different equilibrium selection criterium. Specifically:

- FFH concentrate on market efficiency and compares average prices, while we compare the stochastic performance of both auctions.

- FFH focuses on duopoly case. Since most deregulated electricity markets are largely decentralized, with no suppliers (or hardly any) dominating the pricing competition, we focus on oligopoly settings, which is more realistic and highlights the role of relative influence of suppliers on pricing policy.

- For UA we select a different equilibrium. FFH's choice requires some form of pre-game communication, while our criterium is based on independent bidding assumption. This leads to important differences.

Extending on the last point: in UA, outside of perfectly competitive solutions, FFH select pure-strategy equilibria, where one player prices at the market cap (chosen by regulator), while the other players set their prices very low. Demand is cleared at price cap and all players are paid at the price equal to market cap. While the predictions of FFH is that market price is always a cost or price cap, in practice a range of market prices is observed. Also, with some randomness, the pure-strategy equilibrium does not exist. Our selection of symmetric equilibrium allows for a consistent choice of equilibrium for cases of no randomness, small randomness, and high randomness of demand. The range of predicted prices is also more consistent with reality (allowing many prices between cost and price cap). Importantly, we disagree with FFH's equilibrium selection, since such a price outcome is a violation of independent bidding assumption in auction setting – the pure-strategy equilibrium needs pregame communication among all suppliers, which is clearly prohibited by anti-trust law.[5]

The empirical literature on the treasury auctions is summarized in Binmore and Swierzbinski (2000) and it seems to be inconclusive with respect to ranking of the two auction formats. For example, Simon (1994) estimates that switching from DA to UA in 1970's resulted in large loss of revenue for the US Treasury; Nyborg and Sundaresan (1996) estimate the effect between a small loss and moderate gains; while Malvey and Archibald (1998) claim small gains. Empirical comparison of the two auction formats in electricity markets could be conducted only in UK and the findings (Evans and Green 2002, Newbery 2003 and Fabra and Toro 2003) are also controversial, due to major structural changes of the markets taking place during the switch from uniform auction to discriminatory auctions.

The comparison of two auctions formats was also studied in the laboratory settings. Motivated by the electricity auctions, Mount et al. (2002) reports that "both uniform auction and discriminatory auction produce average prices fifty percent above the competitive levels. However, the prices for the uniform price auction are more volatile with many price spikes." Similar test was conducted by Rassenti et al. (RSW, 2001), and their experiments indicate that (a) a DA consistently generates lower price volatility; (b) the average prices of the two auctions have no significant difference for

[5]While relative benefits are not modeled neither in our paper nor in FFH, the asymmetric equilibrium analyzed in FFH puts the price setter at a relative disadvantage due to not using whole capacity.

high demand, but (c) DA yields higher average price for low demand. [6] Our paper is the first analytical paper to compare the price volatilities resulting from the two auctions. Our analytical results provide direct support to the cited above experimental papers dealing with electricity auction design.

## 2.3 The Model

*Game Description.* Consider wholesale electricity procurement auction with $N$ potential suppliers. Supplier $i$ is assumed to have unit variable cost $c_i \geq 0$ and production capacity $k_i > 0$, for $i = 1, 2, \ldots, N$, which are common knowledge. The random electricity demand $\xi$ is generated by price-insensitive consumers and the auctioneer procures electricity from suppliers to satisfy the demand as much as possible. The suppliers compete to serve demand by submitting unit prices.

The sequence of events is as follows. Distribution of demand $\xi$ is known to all suppliers. During the auction, suppliers *independently* submit (sealed) prices $\{p_i\}_{i=1}^{N}$ to the auctioneer. It is assumed that supplier $i$'s bid price $p_i$ is bounded by price cap $B$, imposed by the regulator, i.e., $p_i \in [0, B]$. After demand is realized and aggregated, the auctioneer calls suppliers into operation based on their bids. The lowest-bid supplier is admitted first. If her capacity cannot cover the demand, the auctioneer moves to the next lowest-bid supplier, and so on, until the demand is filled or no capacity is left. We assume ties are broken by first granting orders to the efficient suppliers (those with lower production costs). If suppliers with the same costs form a tie, each supplier gets a demand share proportional to her capacity.[7]

The following notation is used throughout the paper. Let $\mathbf{p}_{-i} \equiv (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$ and supplier $i$'s realized sales as $z_i(p_i, \mathbf{p}_{-i}) = k_i r_i(p_i, \mathbf{p}_{-i})$, where $r_i$ is the fraction of her bid quantity accepted by the auctioneer. The above assumptions

[6] Here both "low" and "high" demand sustain pure-strategy equilibria with competitive price level, so they can be both viewed as low-demand state.

[7] The mixed-strategy equilibrium solutions are independent of such rules, since any forms of rationing will eliminates the chance of price-tie at equilibrium. For Bertrand-like pure-strategy equilibrium to sustain, the more efficient supplier must possess higher priority. See Deneckere and Kovenock (1996) for a discussion of rationing rules in Bertrand-Edgeworth games with *asymmetric* unit costs.

lead to

$$r_i(p_i, \mathbf{p}_{-i}) = 1 \wedge \frac{[\xi - \sum_{n \neq i} k_n \delta_{(p_n < p_i)} - \sum_{n \neq i} k_n \delta_{(p_n = p_i, c_n < c_i)}]^+}{k_i + \sum_{n \neq i} k_n \delta_{(p_n = p_i, c_n = c_i)}},$$

where $\delta_{(A)} = 1$ if $A$ is true, 0 otherwise. To investigate a supplier's sales at the proximity of a certain price, we define $r_i^-$ and $r_i^+$, and simplify[8] them as

$$\text{(a) } r_i^-(p_i, \mathbf{p}_{-i}) \equiv \lim_{p \uparrow p_i} r_i(p, \mathbf{p}_{-i}) = 1 \wedge \frac{[\xi - \sum_{n \neq i} k_n \delta_{(p_n < p_i)}]^+}{k_i}, \tag{2.1}$$

$$\text{(b) } r_i^+(p_i, \mathbf{p}_{-i}) \equiv \lim_{p \downarrow p_i} r_i(p, \mathbf{p}_{-i}) = 1 \wedge \frac{[\xi - \sum_{n \neq i} k_n \delta_{(p_n \leq p_i)}]^+}{k_i}.$$

It is easy to verify that $r_i^-(p_i, \mathbf{p}_{-i}) \geq r_i(p_i, \mathbf{p}_{-i}) \geq r_i^+(p_i, \mathbf{p}_{-i})$.

Two auction types are considered in our paper. In a discriminatory auction (DA), an admitted supplier is paid at her bid price, while in a uniform auction (UA), all of the selected suppliers are paid at a uniform price equal to the highest bid admitted (i.e., the highest price among all admitted suppliers). We use superscripts (or subscripts when convenient) $d$ and $u$ to denote the two auction formats. Under each of the two auction formats, supplier $i$ maximizes his expected payoff, where

$$\text{(a) } R_i^d(p_i, \mathbf{p}_{-i}) = (p_i - c_i) k_i r_i(p_i, \mathbf{p}_{-i}),$$

$$\text{(b) } R_i^u(p_i, \mathbf{p}_{-i}) = (\max_n \{p_n : r_n(p_n, \mathbf{p}_{-n}) > 0\} - c_i) k_i r_i(p_i, \mathbf{p}_{-i}).$$

Note that $r_i^-(p_i, \mathbf{p}_{-i}) \neq r_i(p_i, \mathbf{p}_{-i})$ (or $r_i(p_i, \mathbf{p}_{-i}) \neq r_i^+(p_i, \mathbf{p}_{-i})$) only if $p_i = \max\{\mathbf{p}_{-i} : r_n(p_n, \mathbf{p}_{-n}) > 0\}$. Hence, we have the following useful observation, for both DA and UA,

$$R_i^-(p_i, \mathbf{p}_{-i}) - R_i(p_i, \mathbf{p}_{-i}) = (p_i - c_i) k_i [r_i^-(p_i, \mathbf{p}_{-i}) - r_i(p_i, \mathbf{p}_{-i})] \tag{2.2}$$

$$R_i(p_i, \mathbf{p}_{-i}) - R_i^+(p_i, \mathbf{p}_{-i}) = (p_i - c_i) k_i [r_i(p_i, \mathbf{p}_{-i}) - r_i^+(p_i, \mathbf{p}_{-i})]$$

where $R_i^-(p_i, \mathbf{p}_{-i}) \equiv \lim_{p \uparrow p_i} R_i(p, \mathbf{p}_{-i})$ and $R_i^+(p_i, \mathbf{p}_{-i}) \equiv \lim_{p \downarrow p_i} R_i(p, \mathbf{p}_{-i})$.[9]

As we show in Section 2.4, pure strategy equilibria exist only under restricted conditions. In general, suppliers are forced to play mixed strategies. Mixed strategies

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[8]Equations (2.1) follow $\lim_{p \uparrow p_i} \delta_{(p_k < p)} = \delta_{(p_k < p_i)}$, $\lim_{p \downarrow p_i} \delta_{(p_k < p)} = \delta_{(p_k \leq p_i)}$, and $\lim_{p \to p_i} \delta_{(p_k = p)} = 0$.

[9]By (2.2), the existence of $R_i^-(\mathbf{p})$ and $R_i^+(\mathbf{p})$ for both UA and DA follows (i) boundedness and monotonicity of $r_i(\mathbf{p})$ in $p_i$, (ii) continuity of $p_i - c_i$, and (iii) continuity of $max\, \{p_n : r_n(\mathbf{p}) > 0\} - c_i$ in $p_i$.

also fit the practitioner's opinion about the situation they face, which is discussed in Section 2.7.

*Mixed Strategies.* The corresponding notation related to mixed-strategy equilibrium analysis is introduced here. Supplier $i$'s mixed-strategy is denoted by $\sigma_i$, a random variable with support $[0, B]$. Define $F_i(p; \sigma_i) \equiv \Pr\{\sigma_i \leq p\}$ as the cumulative distribution function for $\sigma_i$ and $m_i(p; \sigma_i) \equiv \Pr\{\sigma_i = p\}$ the probability mass at price $p$. Denote $\overline{p}_i(\sigma_i) \equiv \inf\{p : F_i(p; \sigma_i) = 1\}$ and $\underline{p}_i(\sigma_i) \equiv \sup\{p : F_i(p; \sigma_i) = 0\}$ as the upper and lower pricing bounds for $\sigma_i$. Given the opponents' mixed-strategy $\sigma_{-i} \equiv (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n)$, supplier $i$ has random sales and payoff when choosing price $p$. Let $\hat{z}_i(p, \sigma_{-i}) \equiv \mathsf{E}_{\sigma_{-i}}[z_i(p, \sigma_{-i})]$, $\hat{r}_i(p, \sigma_{-i}) \equiv \mathsf{E}_{\sigma_{-i}}[r_i(p, \sigma_{-i})]$, and $\hat{R}_i(p, \sigma_{-i}) \equiv \mathsf{E}_{\sigma_{-i}}[R_i(p, \sigma_{-i})]$ represent her expected sales, expected sales fraction, and expected payoff, respectively.

Denote $\sigma^* \equiv (\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*)$ as a mixed-strategy equilibrium and $ER_i(\sigma^*) \equiv \hat{R}_i(\sigma_i^*, \sigma_{-i}^*)$ supplier $i$'s expected equilibrium payoff. For simplicity, we suppress the equilibrium-associated notation by omitting $\sigma^*$. For example, $F_i(p_i) = F_i(p_i; \sigma_i = \sigma_i^*)$ and $\overline{p}_i = \overline{p}_i(\sigma_i^*)$. Similarly, we use shorthand notation $\hat{z}_i(p_i) = \hat{z}_i(p_i, \sigma_{-i}^*)$, $\hat{r}_i(p_i) = \hat{r}_i(p_i, \sigma_{-i}^*)$, $\hat{R}_i(p_i) = \hat{R}_i(p_i, \sigma_{-i}^*)$, and $ER_i = ER_i(\sigma^*)$. Corresponding to (2.1), we also define $\hat{r}_i^-(p_i) \equiv \lim_{p \uparrow p_i} \hat{r}_i(p)$ and $\hat{r}_i^+(p_i) \equiv \lim_{p \downarrow p_i} \hat{r}_i(p)$. $\hat{R}_i^-(p)$ and $\hat{R}_i^+(p)$ are defined similarly. As $r_i(p) \in [0, 1]$ for all $p$, applying *bounded convergence theorem*, we have $\hat{r}_i^-(p) = \mathsf{E}_{\sigma_{-i}^*}[r_i^-(p)]$ and $\hat{r}_i^+(p) = \mathsf{E}_{\sigma_{-i}^*}[r_i^+(p)]$. The following observation is very useful: for any mixed-strategy equilibrium,[10]

$$\text{(a) } m_i(p) > 0 \text{ implies } \hat{R}_i(p) = ER_i; \tag{2.3}$$

$$\text{(b) } F_i(p) > F_i(p') \text{ for all } p' < p \text{ implies } \hat{R}_i^-(p) \equiv \lim_{p' \uparrow p} \hat{R}_i(p') = ER_i;$$

$$\text{(c) } F_i(p) < F_i(p') \text{ for all } p' > p \text{ implies } \hat{R}_i^+(p) \equiv \lim_{p' \downarrow p} \hat{R}_i(p') = ER_i.$$

[10]Part (a) is obvious. For part (b), the monotonicity of $F_i$ at $p$'s left neighborhood implies $\Pr\{\sigma_i^* \in [p - \Delta, p)\} > 0$ for any $\Delta > 0$, and consequently, $\tilde{p} \in [p - \Delta, p)$ exists such that $\hat{R}_i(\tilde{p}) = ER_i$. [Otherwise, if $\hat{R}_i(p') < ER_i$ for all $p' \in [p - \Delta, p)$, we must have $\Pr\{\sigma_i^* \in [p - \Delta, p)\} = 0$, which contradicts to the initial assumption.] As $\Delta$ converges to zero, we have $\hat{R}_i^-(p) = ER_i$ where existence of $\hat{R}_i^-(p)$ follows Footnote 9. Similarly we can show part (c).

## 2.4 Symmetric Auctions with Deterministic Demand

This section derives both key results of the paper in symmetric oligopoly case. We characterize when suppliers have incentive to randomized prices for each auction type. We show that price dispersion may stem endogenously from suppliers' strategic bidding behaviors even in a deterministic economic system. We derive the distribution of prices and show how it depends on capacities and costs. Understanding pricing strategies for both types of auctions allow us to compare them from point of view of prices that buyers pay. We show that they result in the same expected price, but the same variance. The equilibrium structures identified in this section and the following insights are robust and will be extended later to more general settings with asymmetric bidders and random demand.

Our basic model assumes symmetric bidders and deterministic (or perfectly foreseeable) demand. These assumptions make the analysis tractable and yield closed-form equilibrium solutions. Specifically, here we assume $k_i = k$ and $c_i = c$ for all $i = 1, 2, \ldots, N$ and seek Nash equilibria for both types of auctions. The analysis for symmetric DA and UA relies heavily on the order statistics of bids. We introduce the common notation here. Denote $b_m^{(N)}$ the $m$-th *lowest* bid among $N$ bids and $G_m^{(N)}$ its c.d.f. Reversely, denote $b_{(m)}^{(N)}$ the $m$-th *highest* bid and $G_{(m)}^{(N)}$ its c.d.f. Clearly, $b_m^{(N)} = b_{(N+1-m)}^{(N)}$ and $G_m^{(N)} = G_{(N+1-m)}^{(N)}$. Superscript $(N)$ is omitted for simplicity when context is clear and especially when other superscripts are needed. In the following subsection, we separately analyze DA and UA, which allows us later compare their performances. We omit those analytical derivations that are well established in theory of order statistics.

### 2.4.1 Equilibrium Analysis

We first analyze discriminatory auction and show general structure of equilibrium (symmetry and uniqueness), then we derive the exact analytical form of equilibrium. Later we follow with the analysis of uniform auction.

*Discriminatory Auction.* Pure-strategy equilibrium can be achieved only under restricted market conditions. For high demand $\xi \geq Nk$, all suppliers price at the cap $B$. Thus, they have the maximal possible market power. On the other hand, for low demand $\xi \leq (N-1)k$, the auction becomes very competitive and the demand is cleared at price equal to cost $c$. If a supplier prices $c$, the remaining suppliers can still satisfy the whole demand, so the deviation cannot bring any profit. Clearly, $\left\{p_i^{d*} = c\right\}_{i=1}^{N}$ is the only symmetric equilibrium.[11]

For the intermediate demand $(N-1)k < \xi < Nk$, pure-strategy equilibrium does not exist. This results from three incompatible tensions in the competition. First, if different prices are chosen, supplier $i$ who prices lower than the highest bid will have an incentive to raise price as close as possible just below the highest bid, because $r_i(p_i) = 1$ for all $p_i < \max\{\mathbf{p}_{-i}\}$. Second, if all suppliersprices are extremely close, the highest-price supplier has an incentive to set her own bid slightly lower than next highest price to achieve $r^{-}(p) = 1 > r(p)$, which collectively causes the highest price to drop. Third, if a uniform price $p^* = c$ is shared by all suppliers or everybody prices very close to the cost $c$, everyone obtains a zero payoff, so supplier $i$ will be better off choosing $p_i = B$, because $R_i(B) = (B-c)[\xi - (N-1)k] > 0$. The above three tensions lead to the formation of a mixed-strategy equilibrium. The most important implication is that price dispersion may happen in a DA without exogenous randomness (like demand uncertainty or information asymmetry). The fundamental driver is the imperfect competition among oligopolistic suppliers. While mixed strategies were analyzed in oligopolistic competition (e.g., KS), we use the concept in auction setting and show that it explains price variability when demand and supply are price insensitive. The randomization of the bidding price is consistent with conversation we heard from traders. The following proposition describes full structure of the equilibrium.

**Proposition 1** *A symmetric DA has a unique symmetric Nash equilibrium. (a) For $\xi \geq Nk$, it is a pure-strategy equilibrium with $\{p_i^{d*} = B\}_{i=1}^{N}$; (b) For $\xi \leq (N-1)k$, it is a pure-strategy equilibrium with $\{p_i^{d*} = c\}_{i=1}^{N}$; (c) For $(N-1)k < \xi < Nk$, it is a*

[11]Note that when demand $\xi$ is less than $(N-2)k$, the equilibrium is sustained even if certain player chooses a price higher than $c$. It only requires $\lceil \xi/k \rceil + 1$ bidders choosing $c$ where $\lceil x \rceil \equiv \min\{i \in \mathcal{Z} : i \geq x\}$. Hence, multiple equilibria may exist, but they are all payoff-equivalent (zero profit for everyone).

*mixed-strategy equilibrium, with equilibrium distribution function*

$$F^d(p) = (\frac{k}{Nk-\xi}\frac{p-\underline{p}^d}{p-c})^{\frac{1}{N-1}} \quad \textit{for } p \in [\underline{p}^d, B], \tag{2.4}$$

*where* $\underline{p}^d = c + \frac{(B-c)[\xi-(N-1)k]}{k}$.

Below we outline the critical element of the proof (simple derivations and algebraic steps are omitted). According to Theorem 6 in Dasgupta and Maskin (1986)[?], a symmetric mixed-strategy equilibrium exists. To derive the equilibrium distribution $F^d$, we first establish that, for any symmetric mixed-strategy equilibrium $\{\sigma_i^* = \sigma_d^*\}_{i=1}^N$, the equilibrium distribution function $F^d$ must be continuous and strictly increasing (Lemma A1). It implies that, according to (2.3-b,c), any $p \in [\underline{p}^d, \bar{p}^d] \cap (c, B]$ yields the expected equilibrium payoff $\hat{R}(p) = ER^d = \hat{R}(\bar{p}^d) = (\bar{p}^d - c)[\xi - (N-1)k]$. The optimality of $ER^d$ requires $\bar{p}^d = B$ and $ER^d = (B-c)[\xi - (N-1)k] > 0$. By (2.3-c), we have $\hat{R}^+(\underline{p}^d) = ER^d > 0$, implying $\underline{p}^d > c$ and therefore $m(\underline{p}^d) = 0$. It follows that $\hat{r}^+(\underline{p}^d) = \hat{r}(\underline{p}^d) = 1$, and $\underline{p}^d$ can be derived from $ER^d = \hat{R}^+(\underline{p}^d) = (\underline{p}^d - c)k$. For $p \in [\underline{p}^d, B]$, $F^d(p)$ must satisfy

$$\hat{R}(p) = (p-c)\sum_{n=0}^{N-1} \binom{N-1}{n} F^d(p)^n \bar{F}^d(p)^{N-1-n} z_n = ER^d \quad \text{with } z_n = k \wedge [\xi - nk]^+. \tag{2.5}$$

It is possible to show that the unique solution to (2.5) is equation (2.4) above, which completes the proof of the proposition. Note that cdf of price $F$ does not have mass point at the price cap $B$ (by the same logic as for lack of mass points within the price range).

Notice that, the equilibrium outcomes presented in Proposition 1 are determined by five factors $\{k, \xi, N, c, B\}$. Define $\rho \equiv \frac{\xi}{Nk}$ as the *aggregated utilization*, and the determinant factors become $\{\rho, N, c, B\}$. The three cases in Proposition 1 correspond to (a) $\rho \geq 1$, (b) $\rho \leq \frac{N-1}{N}$, and (c) $\frac{N-1}{N} < \rho < 1$, respectively. For case (c), the equilibrium distribution function can be expressed as

$$F^d(p) = [\frac{1}{N(1-\rho)}\frac{p-\underline{p}^d}{p-c}]^{\frac{1}{N-1}} = [\frac{B-c}{p-c} - \frac{B-p}{N(1-\rho)(p-c)}]^{\frac{1}{N-1}} \text{ for } p \in [\underline{p}^d, B]. \tag{2.6}$$

Since all marginal costs are $c$, any price above $c$ is a manifestation of pricing power. From equation (2.6), the equilibrium bids are stochastically increasing in

$\rho$ and stochastically decreasing in $N$. It suggests that suppliers' pricing power is primarily determined by system utilization, represented by $\rho$, and market decentralization (delegated by $N$). Higher utilization gives higher pricing power to suppliers, while increasing number of suppliers dilutes it. While Proposition 1 considers only symmetric solution, it is possible that asymmetric equilibrium can exist. Proposition 11 in Appendix A.2, however, excludes such a possibility. We now are ready to move to analysis of uniform auction. As opposite to discriminatory auction, asymmetric equilibrium can exist and our analysis need to be more detailed.

*Uniform Auction.* The existence of a symmetric mixed-strategy equilibrium for a UA follows again from Dasgupta and Maskin (1986) Theorem 6. For high demand $\xi \geq Nk$ or low demand $\xi \leq (N-1)k$, the symmetric solution reduces to the same pure-strategy equilibria achieved in a DA, with all suppliers pricing at $B$ or $c$. For intermediate demand $(N-1)d < \xi < Nk$, there is no symmetric pure-strategy equilibrium due to the same incentive to undercut other suppliers as in a DA.

Despite the above similarities, the competitive nature of DA and UA is different. In a UA, those suppliers who price below the highest dispatched bidder have no incentive to raise price, enjoying the benefits of a " free ride" as price takers; while in a DA, a low-price bidder always has an incentive to increase price as long as she can maintain a sales fraction of 1. For $(N-1)k < \xi < Nk$, this difference matters — unlike DA, a UA has multiple asymmetric pure-strategy equilibria. Consider the following bidding outcome. One bidder chooses $B$, while the rest of the bidders price at cost $c$. The high bidder's payoff is $R(B) = (B-c)[\xi - (N-1)k] > 0$. If she deviates to any price $p \in (c, B)$, her sales is still the same, but the profit margin $(p-c)$ is strictly decreased. A price cut to $c$ or below increases her market share but reduces the price margin to zero. Hence pricing at $B$ is her optimal strategy. For other bidders, given that the demand is cleared at price $B$, the profit equals to monopoly-equivalent payoff $(B-c)k$. Thus, a pure-strategy equilibrium is sustained. Note that the price takers may raise price above $c$, as long as it is *not* high enough to trigger the price maker's defecting from $B$. A comprehensive characterization of the asymmetric equilibria is presented in the following lemma.

**Lemma 1** *For a symmetric UA with* $(N-1)k < \xi < Nk$*, a pure-strategy equilibrium must satisfy* $b^u_{(1)} = B$ *and* $b^u_{(2)} \leq c + \frac{(B-c)[\xi-(N-1)k]}{k}$*.*

Note that the upper bound for $b^u_{(2)}$ is equal to $\underline{p}^d$, the lower pricing bound of the mixed-strategy equilibrium in a DA, because it is the lowest price securing the highest bidder's profit at price $B$.

Similarly to DA, also for UA we can describe the structure of symmetric Nash equilibrium. The impacts of market power can be characterized by $\rho$ and $N$.

**Proposition 2** *A symmetric UA has a unique symmetric Nash equilibrium. (a) For* $\xi \geq Nk$*, it is a pure-strategy equilibrium with* $\{p^{u*}_i = B\}^N_{i=1}$*; (b) For* $\xi \leq (N-1)k$*, it is a pure-strategy equilibrium with* $\{p^{u*}_i = c\}^N_{i=1}$*; (c) For* $(N-1)k < \xi < Nk$*, it is a mixed-strategy equilibrium, with equilibrium distribution function*

$$F^u(p) = (\frac{p-c}{B-c})^{\frac{\xi-(N-1)k}{(N-1)(Nk-\xi)}} = (\frac{p-c}{B-c})^{\frac{N\rho-N+1}{N(N-1)(1-\rho)}} \quad \text{for } p \in [c, B]. \tag{2.7}$$

Let us sketch the justification of Proposition 2. Clearly, from Lemma 1, all pure-strategy equilibria are asymmetric for $\xi \in ((N-1)k, Nk)$. Thus, the existing symmetric solution must be a mixed-strategy one. Similarly to DA, we can show that a symmetric mixed-strategy equilibrium must have $F^u(p)$ continuous and strictly increasing in $p \in [\underline{p}^u, \bar{p}^u] \cap (c, B]$. The continuity and monotonicity of $F^u$ implies $\hat{R}(p) = ER^u$ for all $p \in [\underline{p}^u, \bar{p}^u]$. As $m(\bar{p}^u) = 0$, or equivalently, $\Pr\{\sigma^*_u < \bar{p}^u\} = 1$, pricing at $\bar{p}^u$ results in market clearing price $\bar{p}^u$ and sales of $\xi-(N-1)k$ with probability 1. The optimality of $\bar{p}^u$ implies $\bar{p}^u = B$ and $ER^u = \hat{R}(\bar{p}^u) = (B-c)[\xi-(N-1)k]$. For any $p \in [\underline{p}^u, B]$, by monotonicity of $F^u$, supplier $i$'s expected payoff satisfies,

$$\hat{R}_i(p) = G^{(-i)}_{(1)}(p) \cdot (p-c)[\xi-(N-1)k] + \int_p^B (v-c)k dG^{(-i)}_{(1)}(v) = ER^u. \tag{2.8}$$

Since equation (2.8) cannot be solved explicitly, we consider its first order condition, $\frac{d\hat{R}_i(p)}{dp} = 0$, which yields an ordinary differential equation (2.9) of $G^{(-i)}_{(1)}(p)$:

$$\dot{G}^{(-i)}_{(1)}(p) \cdot (p-c)(\xi-Nk) + G^{(-i)}_{(1)}(p) \cdot [\xi-(N-1)k] = 0. \tag{2.9}$$

The general solution to the ODE is $G^{(-i)}_{(1)}(p) = C_0(p-c)^{\frac{\xi-(N-1)k}{Nk-\xi}}$ where $C_0$ is a constant. From the boundary conditions $G^{(-i)}_{(1)}(B) = F^u(B)^{N-1} = 1$ and $G^{(-i)}_{(1)}(\underline{p}^u) =$

$F^u(\underline{p}^u)^{N-1} = 0$, we have $C_0 = (B-c)^{-\frac{\xi-(N-1)k}{Nk-\xi}}$ and $\underline{p}^u = c$. Also, from $G^{(-i)}_{(1)}(p) = F^u(p)^{N-1}$, we obtain the equilibrium distribution function described in (2.7) above, which complete the outline of the proof.

Note that the price range is always $[c, B]$. That is, for all mixed-strategy equilibria (but pure-strategy ones), any price between $c$ and $B$ can be observed in a UA, while in a DA the lower bound $\underline{p}$ increases in utilization $\rho$ and decreases in the number of suppliers $N$. The equilibrium solution in a UA stochastically increases in $\rho$ and decreases in $N$, resulting in similar to DA economic interpretation of how utilization and decentralization influence the suppliers' market power, manifesting itself by randomized bidding above the cost.

While for a DA, with $\frac{N-1}{N} < \rho < 1$, the symmetric mixed-strategy equilibrium is the unique Nash solution; for a UA, we have identified both symmetric mixed-strategy equilibrium and asymmetric pure-strategy equilibrium. In any pure-strategy equilibrium, the task of price making is assigned to one supplier; while in symmetric mixed-strategy equilibrium, this responsibility is equally and randomly shared among all bidders. Obviously a continuum of equilibria exist, morphing between the symmetric mixed-strategy equilibrium and any pure-strategy equilibrium. As these equilibria differ in multiple dimensions, it is difficult to provide a general solution form. We exclude mixed-strategy solutions that can be reduced to payoff-equivalent pure-strategy equilibria such as $\{p^{u*}_i = B, \sigma^{u*}_j < B$ almost surely for all $j \neq i\}$. Formally, we define a mixed-strategy equilibrium as *irreducible* if $\Pr\{\sigma^{u*}_i < B\} > 0$ for all $i$. Proposition 2 (Appendix A.3) presents several structural properties of irreducible equilibria with continuous and monotone distribution functions. Based on these properties in Proposition 3, we construct a family of mixed-strategy equilibria that clearly illustrate increasing asymmetry between the price maker and the remaining suppliers. (Note that, there exist other solutions outside of this family.)

**Proposition 3** *For symmetric UA with* $(N-1)k < \xi < Nk$*, any* $(h, m^u_B) \in \{1, 2, \ldots, N\} \times [0, 1)$ *defines an irreducible mixed-strategy equilibrium with distribution* *functions,*

$$(a)\ F_h^u(p) = (1 - m_B^u)(\frac{p-c}{B-c})^{\frac{\xi-(N-1)k}{(N-1)(Nk-\xi)}} \quad for\ p \in [c, B)\ and\ m_h^u(B) = m_B^u, \quad (2.10)$$

$$(b)\ F_i^u(p) = (\frac{p-c}{B-c})^{\frac{\xi-(N-1)k}{(N-1)(Nk-\xi)}} \quad for\ p \in [c, B]\ and\ i \neq h.$$

*The equilibrium payoff is* $ER_i^u = (B-c)[\xi - (N-1)k + \delta_{(i\neq h)} m_B^u (Nk - \xi)]$, *for all* $i$.

The proposition can be easily proved by verifying the expected payoffs for all suppliers and we omit the details. When multiple equilibria exist, equilibrium selection is critical for evaluating different auction formats. Within the whole equilibrium set, the *focal* points are the pure-strategy equilibria(most asymmetric) and the symmetric mixed-strategy equilibrium(most symmetric). The pure-strategy solutions described in Lemma 1 are "attractive" because the bidders' behavior is deterministic and their total payoff is maximized.[12] Price maker has the same profit as in symmetric mixed-strategy equilibrium, while all other players are strictly better off. However, in sealed-bid auction context, these pure-strategy equilibria have major shortcomings. First, from a behaviorial perspective, they rely heavily on pre-game communication — the $N-1$ price takers needs to achieve an agreement to collectively impose a "threat" to the price maker and the price maker must be informed that she is under such threat. Suppliers bidding at $c$ have no profit if nobody ends up bidding high. Since the price maker does not get higher payoff than in a mixed-strategy equilibrium, it is not clear if anyone would agree to become a price maker and play the pure-strategy equilibrium. Clearly, the communication needed to achieve a pure-strategy equilibrium violates the original assumption of independent bidding and would be considered as violation of antitrust law. Compared with the pure-strategy equilibria, the symmetric mixed-strategy equilibrium is also more informative because its stochastic features are arguably shared by other non-symmetric mixed-strategy equilibria(e.g., the ones described in Proposition 3). Due to the above considerations, we will focus in the rest of the paper primarily on the symmetric equilibrium for UA and then discuss

[12]Based on such equilibrium selection, FFH (Proposition 5) omits all mixed-strategy solutions for symmetric oligopoly model and argues that UA yields higher price (deterministic price $B$) than DA (random price with $B$ as upper bound). Also note that FFH's equilibrium selection and efficiency argument imply that there is no price dispersion in uniform auctions.

the relationship with asymmetric pure-strategy ones.

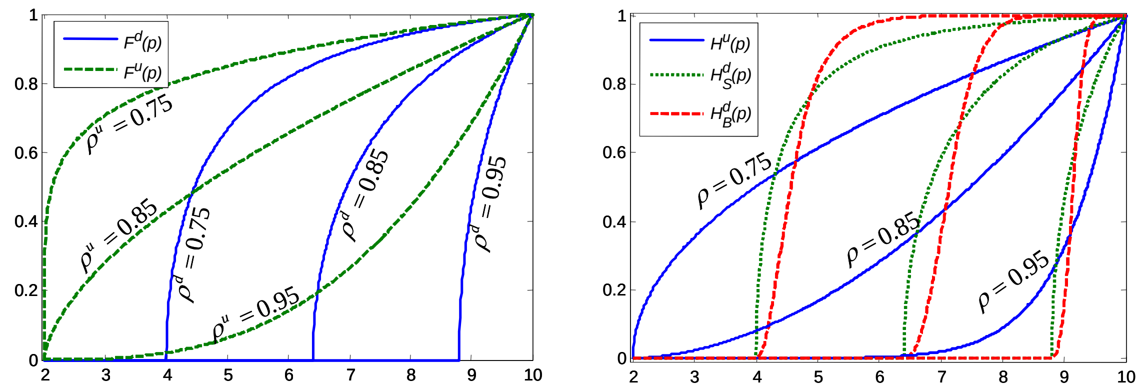
### 2.4.2 Comparison of Symmetric DA and UA

One of the main objectives of this paper is to compare the stochastic performances of DA and UA. For $\xi \leq (N-1)k$ or $\xi \geq Nk$, both auctions yield the same pure-strategy equilibria involving uniform pricing outcome ($c$ or $B$). The interesting case is for demand $(N-1)k < \xi < Nk$. Since the payment schemes in DA and UA are different, instead of comparing the distributions of supplier's equilibrium bids, $F_i^d(p)$ and $F_i^u(p)$, we need to compare the average price for the capacities called into operation, which we label as *transaction price* $P$. For UA, $P^u$ is the market clearing price. For DA, the admitted suppliers are compensated at different rates (equal to their bid prices) but all electricity buyers pay one single price that covers the total revenue of the suppliers. In order to compare prices paid by customers in two auctions, for DA, we consider the randomly sampled *seller price* $P_S^d$. In seller price, the probability of being certain price is equal to the probability that the corresponding capacity is admitted. Denote by $P_B^d$ average price, or the *buyer price*, which is a sales-weighted average of the realized bids. Formally, for an auction with $(N-1)k < \xi < Nk$, the above prices can be expressed as

$$
\begin{aligned}
&\text{(a) } P_S^d \equiv \left\{ b_{(1)}^d, \frac{\xi-(N-1)k}{\xi} \right\} \oplus \left\{ b_{(2)}^d, \frac{k}{\xi} \right\} \oplus \ldots\ldots \oplus \left\{ b_{(N)}^d, \frac{k}{\xi} \right\} \\
&\text{(b) } P_B^d \equiv \frac{\xi-(N-1)k}{\xi} \cdot b_{(1)}^d + \frac{k}{\xi} \cdot b_{(2)}^d + \ldots\ldots + \frac{k}{\xi} \cdot b_{(N)}^d \\
&\text{(c) } P^u \equiv b_{(1)}^u,
\end{aligned}
\tag{2.11}
$$

where $\tilde{y} = \oplus_{n=1}^N \{\tilde{x}_n, f_n\}$ denotes a random variable $\tilde{y}$, being a mixture of $N$ random variables, such that $\tilde{x}_n$ is chosen with probability $f_n$.

Denote $H_S^d(\cdot)$, $H_B^d(\cdot)$, and $H^u(\cdot)$ as the c.d.f. of $P_S^d$, $P_B^d$ and $P^u$ and they are computed in Appendix A.4. Figure 2.2 illustrates the behavior of equilibrium bids and transaction prices in both DA and UA. Obviously, suppliers bid stochastically lower in a UA than in a DA, see Figure 2.2(a). However, since everyone is paid at the highest bid in a UA, the transaction prices illustrated in Figure 2.2(b) are significantly increased compared to an individual supplier's bids in Figure 2.2(a);




(a) Distributions of Equilibrium Bids (b) Distributions of Transaction Prices

**Figure 2.2.** Equilibrium Bids and Prices in Discriminatory and Uniform Auctions
Auction Setting: $(N = 3, c = 2, B = 10)$

while for a DA, the differences between $H_S^d(\cdot)$ and $F^d(\cdot)$ are less significant. Note that $H_S^d$ is stochastically smaller than $F^d$, because among the $N$ i.i.d. bids, the highest realized value is selected with a smaller chance $f_{(1)} = \frac{\xi-(N-1)k}{\xi}$ than other ones $f_{(n)} = \frac{k}{\xi}$ for $n > 1$. Note also that when utilization $\rho$ is increased, bids and transaction prices in both auctions increase stochastically, but in a different fashion. For a DA, a higher $\rho$ implies a higher pricing bound $\underline{p}^d$, resulting in condensing of bidding range towards the price cap $B$; while for $F^u$ and $H^u$ in a UA, increasing utilization leads to the preponderance of probability mass shifting from $\underline{p}^d$ to $B$, with pricing range remaining $[c, B]$. Finally, Figure 2.2(b) illustrates comparisons among $P_S^d$, $P_B^d$, and $P^u$ for given $\rho$. First note that, given any realized bid vector $\mathbf{b}$, $P_S^d$ is a random variable with discrete values while $P_B^d$ is a constant. Therefore, $P_S^d$ must be a mean-preserving spread of $P_B^d$. The more interesting comparison is between the two auction formats. We observe that, for each $\rho$, there is a single cross between $H^u$ and $H_S^d$. It suggests that $H^u$ may be a mean-preserving spread of $H_S^d$. This conjecture is formally established in the following proposition. See Appendix A.5 for its proof.

**Theorem 1** *Consider symmetric auctions with $(N-1)k < \xi < Nk$. The transaction prices $P_S^d$ and $P_B^d$ in a discriminatory auction and $P^u$ in a uniform auction satisfy:*
*(a) Equal Expected Price:* $\mathsf{E}[P^u] = \mathsf{E}[P_S^d] = \mathsf{E}[P_B^d] = (B-c)\frac{N[\xi-(N-1)k]}{\xi} + c$;
*(b) Variability Ordering: $P^u$ is stochastically more variable than $P_S^d$ and $P_S^d$ is*

*stochastically more variable than* $P_B^d$.

The equality of expected prices can be easily justified. At a symmetric mixed-strategy equilibrium $\sigma_d^*$ or $\sigma_u^*$, a bidder expects to earn the equilibrium payoff when pricing at $B$. Despite the different payment schemes, a bid equal to $B$ must be the highest one, so a bidder obtains the same expected payoff $ER^d = ER^u = (B - c)[\xi - (N-1)k]$. The intuition behind variability ordering can be established by revisiting the competitive natures of the two auctions. As we explained, one of the key incentives in a DA is missing in a UA — low-price parties' benefit by approaching the highest bidder. In other words, there is stronger force in a DA to contract the bid range, which is consistent with prices in DA being less dispersed.

Proposition 1 has some surprising consequences. First, equality (a) contradicts FFH's claim[13] that uniform auctions yield higher prices. The difference is due to the equilibrium selection for uniform auctions. FFH assume that, in UA, pure-strategy equilibria described in Lemma 1 is chosen; we argue that the asymmetric pure-strategy solution is not consistent with the independent bidding assumption and that the required communication makes the comparison with DA "unfair." However, when symmetric bidding is assumed, DA and UA result in the same expected price. Second, our analysis of the symmetric equilibria predicts that discriminatory auction outperforms uniform auctions by generating lower price volatility.

## 2.5 Extensions

This section extends the previous section in two dimensions, demand uncertainty and asymmetry of bidders. We investigate the robustness of the main takeaways from the previous section and also obtain new insights about the influence of the other system factors.

[13]FFH(2006) Proposition 5 (indirectly) suggests $\mathsf{E}[P^d] = (B-c)\frac{\xi-(N-1)k}{k} + c < B = \mathsf{E}[P^u]$.

### 2.5.1 Random Demand

Sections 2.4 assumes demand to be known (perfectly foreseeable). In practice, some leadtimes are involved. For instance, in electricity day-ahead (hour-ahead) markets, suppliers submit bids one day (one hour) before the actual dispatch. Therefore, demand uncertainty is an element of decision process, which is the focus of this section. For simplicity, the analysis of this section focuses mainly on symmetric auctions. Selected results for asymmetric auctions are reported as extensions.

*Equilibrium Analysis of Symmetric Auctions.* Suppose all bidders share a common belief about the demand distribution with cdf $\Phi(\cdot)$. Define the lowest demand level $\underline{\xi} \equiv \inf\{\xi : \Phi(\xi) > 0\}$ and the highest one $\bar{\xi} \equiv \sup\{\xi : \Phi(\xi) < 1\}$. For the ease of exposition, we assume $\Pr\{\xi = \underline{\xi}\} = \Pr\{\xi = \bar{\xi}\} = 0$ and $\Phi(\xi)$ is strictly increasing in $\xi \in [\underline{\xi}, \bar{\xi}]$. We first characterize the equilibrium structure. With the intuition established in Section 2.4, pure-strategy equilibrium can be easily derived for both DA and UA. For low demand with $\bar{\xi} \leq (N-1)k$, the highest bidder obtains zero sales with probability 1. Both auctions are competitive and sustain pure-strategy equilibrium $\{p_i^* = c\}_{i=1}^N$. Similarly, for high demand with $\underline{\xi} \geq Nk$, all bidders choose $\{p_i^* = B\}_{i=1}^N$. Note that these two cases probably cover a relatively small range of possible demand realizations. The interesting case is when $\frac{\underline{\xi}}{N} < k < \frac{\bar{\xi}}{N-1}$ and no symmetric pure-strategy equilibrium exists. Symmetric equilibria are still our primary focus.

- *Discriminatory Auctions* It is useful to define the (ex ante) expected sales $Z_n$ for the $n$-th lowest bidder,

$$Z_n \equiv \int_{(n-1)k}^{nk} [\xi - (n-1)k] d\Phi(\xi) + k\bar{\Phi}(nk) = \int_{(n-1)k}^{nk} \bar{\Phi}(\xi) d\xi.$$

Notice that $\{Z_n\}_{n=1}^N$ are constant values for given demand distribution $\Phi(\cdot)$. Monotonicity of $\bar{\Phi}(\xi)$ and $\frac{\underline{\xi}}{N} < k < \frac{\bar{\xi}}{N-1}$ implies $Z_1 \geq Z_2 \geq \ldots \geq Z_n$ and $Z_{N-1} > Z_n$. We also define the total expected sales $X \equiv \mathsf{E}[\min\{\xi, Nk\}] = \sum_{n=1}^N Z_n$. Following the logic of Lemma A.1, we can show the continuity and strict monotonicity of $F^d(\cdot)$ in $[\underline{p}^d, B]$, and

$$ER^d = (B-c)Z_n,\ \bar{p}^d = B,\ \underline{p}^d = \frac{ER^d}{Z_1} + c.$$

By (2.3b, c) we also have, for any $p \in [\underline{p}^d, B]$,

$$\hat{R}^d(p) = (p-c)\sum_{n=0}^{N-1}\binom{N-1}{n}F^d(p)^n\bar{F}^d(p)^{N-n-1}Z_n = ER^d. \qquad (2.12)$$

Unlike the deterministic case, there is no closed-form solution to equation (2.12) and we need to solve the equation numerically.

- *Uniform Auctions* The derivation of the solutions to UA is more complicated. We first compute supplier $i$'s expected payoff $\hat{R}_i^u(p)$, given that other players follow $F^u$. Define function $\delta_n(\xi) \equiv \begin{cases} \delta_{(nk-k<\xi\leq nk)} & \text{for } n<N \\ \delta_{(\xi>Nk-k)} & \text{for } n=N \end{cases}$, indicating the number of active suppliers associated with realized demand $\xi$. To simplify the notation, denote $b_n \equiv b_n^{(-i)}$ and $G_n(p) \equiv G_n^{(N-1)}(p)$. We also denote $b_0 < c$ and $b_n > B$, and correspondingly, $G_0(p) = 1$ and $G_n(p) = 0$ for all $p \in [c, B]$. The continuity and strict monotonicity of $F^u$ can be established, implying that possibility of price tie can be ignored. Now we have

$$R_i^u(p|\xi, \mathbf{b}^{-i}) = \sum_{n=1}^{N}\delta_n(\xi)\delta_{(b_{n-1}<p<b_n)}(p-c)z_n(\xi) + \sum_{n=1}^{N-1}\delta_{n+1}(\xi)\delta_{(p<b_n)}(b_n-c)k,$$

where $z_n = \min\{k, [\xi-(n-1)k]^+\}$. Taking expectation over $\xi$ and $\mathbf{b}^{-i}$, we have

$$\hat{R}_i^u(p) = (p-c)\sum_{n=1}^{N}[G_{n-1}(p)-G_n(p)]Y_n + k\sum_{n=1}^{N-1}\Phi_{n+1}\int_p^B(v-c)dG_n(v), \qquad (2.13)$$

where $Y_n \equiv \begin{cases} Z_n - k\bar{\Phi}(nk) & \text{for } n<N \\ Z_n & \text{for } n=N \end{cases}$ and $\Phi_n \equiv \begin{cases} \Phi(nk)-\Phi(nk-k) & \text{for } n<N \\ \bar{\Phi}(Nk-k) & \text{for } n=N \end{cases}$

The first order condition leads to the following ODE of $F^u$

$$\sum_{n=0}^{N-1}\tbinom{N-1}{n}(F^u)^n(\bar{F}^u)^{N-(n+1)}Y_{n+1} \qquad (2.14)$$
$$= (p-c)\dot{F}^u\sum_{n=1}^{N-1}(N-n)\tbinom{N-1}{n-1}(F^u)^{n-1}(\bar{F}^u)^{N-n-1}(Z_n-Z_{n+1}),$$

with boundary condition $F^u(B)=1$. [See Appendix A.6 for the derivation of (2.13) and (2.14).] Similarly to DA, (2.14) can only solved numerically. The equilibrium solutions are summarized in Proposition 2. (See Appendix A.7 for its proof.)

**Theorem 2** *Both symmetric DA and symmetric UA have unique symmetric Nash equilibria. (a) For* $\underline{\xi} \geq Nk$*, they are identical pure-strategy equilibria with* $\{p_i^* = B\}_{i=1}^N$*; (b) For* $\bar{\xi} \leq (N-1)k$*, they are pure-strategy equilibria with* $\{p_i^* = c\}_{i=1}^N$*; (c) For* $\frac{\underline{\xi}}{N} < k < \frac{\bar{\xi}}{N-1}$*, they are mixed-strategy equilibria. The equilibrium distribution* $F^d(p)$ *solves equation (2.12) for a DA; and* $F^u(p)$ *solves ODE (2.14) for a UA.*

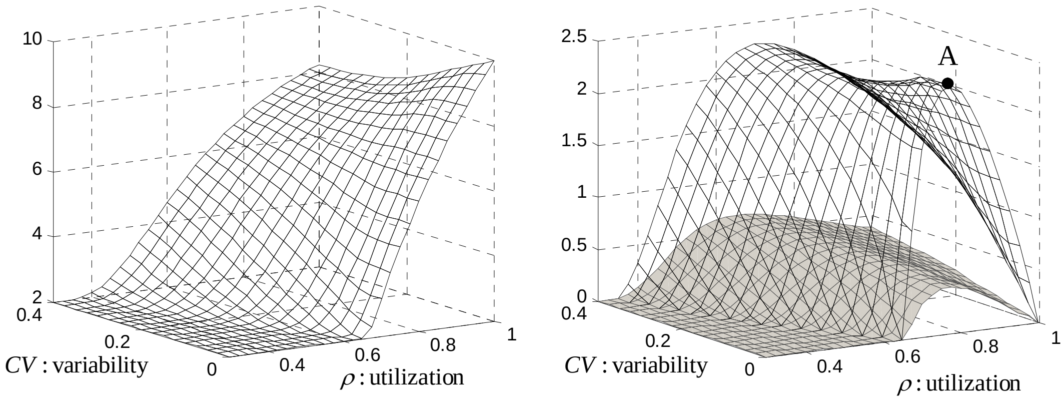
It is easy to see that the demand uncertainty may eliminate asymmetric equilibrium for uniform auctions. Recall that multiple asymmetric pure-strategy equilibria exist for UA with deterministic demand $(N-1)k < \xi < Nk$ (Lemma 1). They all involve a complete separation between price maker and price takers. However, when demand is random with $\underline{\xi} < (N-1)k$, a pure-strategy profile $\{p_1 = B, p_i = c \text{ for } i \neq 1\}$ cannot be sustained, because with probability $\Pr\{\xi < (N-1)k\} > 0$ the price is set other suppliers but the highest one. Now bidding a low price is not a best response for any possible price-setting bidder, who will either raise bid (for better expected margin) or reduce bid slightly to undercut other suppliers (for larger expected sales). It leads to non-existence of pure-strategy solution. In other words, a random demand with $\underline{\xi} < (N-1)k < \bar{\xi}$ guarantees the prevalence of mixed-strategy equilibrium (not pure ones). This serves as additional justification for choosing to analyze the symmetric mixed-strategy equilibria.

*Impacts of Demand Uncertainty and Auction Comparison.* To compare the performance of the two auctions, we again consider buyer prices $P^u$ and $P^d$ with distributions of $H^d(p)$ and $H^u(p)$. Appendix A.8 explains numerical derivation of both $H^d(p)$ and $H^u(p)$. It is easy to see that Expected Price Equality described in Proposition 1(a) still holds under stochastic demand.

$$\mathsf{E}[P^d] = \mathsf{E}[P^u] = (B-c)\frac{NZ_n}{X} + c \tag{2.15}$$

We are interested in whether the variability ordering for UA and DA described in Proposition 1 also holds in random demand case. Since there is no closed-form solution for both $H^d(\cdot)$ and $H^u(\cdot)$, we use numerical experiments to answer this question.

The numerical test is set as follows. Consider 3-player symmetric DA and UA, with the total capacity normalized to $Nk = 1$ (i.e., $k = 1/3$), and therefore, $\rho = \frac{\mathsf{E}[\xi]}{Nk} = \mathsf{E}[\xi]$.




(a) Expectations of $P^u$ and $P^d_B$ (b) Standard Deviations of $P^u$ and $P^d_B$

**Figure 2.3.** Impacts of Demand Uncertainty on Transaction Prices
Auction Setting: $(N = 3, B = 10, c = 2)$
Note: In the right graph, values associated with DA are meshed on the grey surfaces.

Let price cap be $B = 10$ and cost $c = 2$. The demand is $\xi = \rho + \epsilon$, where the expected demand $\rho \in [0, 1]$ and random shock $\epsilon$ has symmetric triangle distribution $\mathsf{Tr}(-r, r)$. This triangle distribution has a standard deviation of $\frac{r}{\sqrt{6}}$ and coefficient of variation $CV = \frac{r}{\sqrt{6}\rho}$, which can be easily controlled by adjusting $r$ and $\rho$. Triangle distribution is easy to analyze and has a unimodal pdf, which is typical for commodity demand. To enforce $\underline{\xi} \geq 0$, $r$ is controlled to be less than or equal to $\rho$. [14]

For any given $(\rho, CV)$, the equilibrium solution $F^d$ and $F^u$ to DA and UA can be numerically solved based on (2.12) and (2.14), as explained in Appendix A.8. The experiment outcomes are presented in Figure 2.5.1. Deterministic demand corresponds to $CV = 0$, and serves as a benchmark for evaluating the impact of demand variability. The following qualitative behaviors are observed.

(a) *Buyer price in DA is stochastically less variable than in UA.* The numerical tests confirm our conjecture. $P^d$ is much more variable than $P^u$. At intermediate utilizations, the standard deviation of prices in DA is about one third of that in UA.

(b) *Demand variability increases the chance of price dispersion, but not necessarily its magnitude.* In Figure 2.5.1(b) the vertical axis shows price variance and zero

[14]We performed limited experiments for other distributions and found that the qualitative performances identified below are fairly robust.

price variance corresponds to pure-strategy pricing outcomes. Intuitively, we expect price to start to vary for bigger demand variability ($CV$). As $CV$ increases, for both $DA$ and $UA$, the range of utilizations with zero price variance is indeed shrinking and eventually disappears.

Our intuition would also suggest, that the bigger demand variability, the bigger price dispersion. This, however, is often not the case. In many cases, price dispersion is nearly constant across various levels of demand variability. For intermediate utilizations, introduction of demand variability may even decrease the price variance in both auctions. This phenomenon appears more significant in UA, see Figure 2.5.1(b). This could be referred to as "the mixing role" of demand variability. In a deterministic case, the price variance peaks at an intermediate utilization (point A in Figure 2.5.1(b)). With random demand shocks, the price variance becomes an equivalent of weighted average of price variances. (Obviously, the underlying behavior is more complicated as the sellers' bidding strategies change.)

One of the questions we faced in the initial phases of interaction with a major northwest energy trading company was whether the price variance is "primarily" driven by demand variance. Our analytical solutions and numerical illustrations suggest that it is not the case. Even with very small (or none) demand variability price dispersion (traders' gambling) is significant. With bigger demand uncertainty, the range of utilizations where price dispersion (gambling) takes place quickly extends, but the size of price dispersion is not significantly influenced, suggesting that the structure of the interactions (use of auction) are primarily driving price variance in the energy markets.

### 2.5.2 Asymmetric Bidders with Random Demand

Analyzing asymmetric auctions with multiple bidders is technically challenging. see Chapter 3. We summarize known properties and, for the purpose of illustrating the typical behavior, we use numerical examples.

To keep analytical tractability, we only consider the case of two bidders. Without loss of generality, we assume $c_1 \leq c_2$ throughout this section. For given costs $(c_1, c_2)$

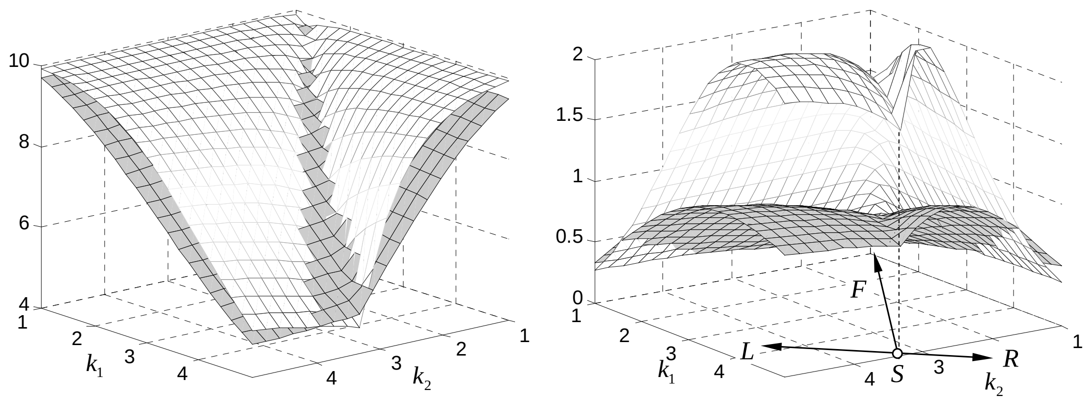
and price cap $B$, the resulting equilibrium depends on the combination of capacities $(k_1, k_2)$ and random demand $\xi$. The derivation is similar to KS. The complete solution in presented in Appendix A.9. Since FFH contains a subset of these results, this section focuses only on the results not discussed in FFH and highlights the new insights not discussed in the previous sections.

Similarly to the symmetric case, DA always has a unique equilibrium, either pure-strategy or mixed-strategy one. There are, however, some differences. (i) When a mixed-strategy equilibrium prevails, one supplier may, with a positive probability, choose a bid equal to the price cap $B$. With identical costs, low capacity supplier bids more aggressively (with stochastically lower bids) than high-capacity one.[15] That is, the high-capacity supplier $(h)$ has $m_h^d(B) > 0$ and prices stochastically higher than the low-capacity party $(l)$. The intuition is similar to KS who study a duopoly capacitated pricing game. Supplier with higher price gets residual demand and, thus, only partially utilizes her capacity. This hurts more low-capacity supplier who, therefore, tends to price more aggressively.

For UA, similar to the symmetric case, we may have a continuum of equilibria, which happens outside of two extremes (competitive equilibrium $\{c_2, c_2\}$ and monopoly-like equilibrium $\{B, B\}$). Each of them can be qualitatively viewed as a mixture of a pure-strategy equilibrium and a " pure" mixed-strategy equilibrium (a solution with $m_1^u(B) = m_2^u(B) = 0$). Our "pure" mixed strategy is structurally similar to the symmetric equilibrium for UA.

We acknowledge that in a duopoly setting, FFH's selection of pure-strategy equilibrium based on payoff-dominance has a better standing than in the oligopoly case. This is because it does not require pre-game communication among the price takers (there is only one). However, as each supplier prefers to be a price-taker and there are two asymmetric equilibria, certain " agreement" is still needed for a specific equilibrium to be played, as in the symmetric case. Also, FFH's selection again suggests that there is no price dispersion between the two possible market clearing prices $c_2$

[15]Numerical examples suggest that this remains the case with asymmetric costs, unless low capacity supplier has much higher cost.




(a) Expectations of $P^u$ and $P^d_B$

(b) Standard Deviations of $P^u$ and $P^d_B$

**Figure 2.4.** Expectations and Variances of Transaction Prices
In both graphs, values associated with DA are represented by the grey surfaces, cost vector $c = [1, 3]$, price cap $B = 10$, demand has (truncated) normal distribution $N(4, 2)$.

and $B$ in UA.

With some demand randomness (usually moderate), nonuniqueness of equilibrium for UA disappears. Assuming that price cap is not very small, it is sufficient that there is a chance of demand realizations straddle the smaller of the capacities $\underline{\xi} < \min\{k_1, k_2\} < \bar{\xi}$. The reason for it is as follows. The only possible pure-strategy equilibria are when the price is equal cost or the cap, $B$. A chance of demand exceeding lower capacity, $\bar{\xi} > \min\{k_1, k_2\}$, removes possibility of selling at $\max\{c_1, c_2\}$. If demand may be smaller than lower capacity, $\bar{\xi} > \min\{k_1, k_2\}$, there is a chance for each supplier to be the price setter (supplier' bid becomes the price) and none of the suppliers is wiling to set the price at its cost. Obviously in these cases both suppliers pricing at price cap $B$ is not sustainable, eliminating possibility of pure strategy equilibria. Thus, demand uncertainty has two impacts on the price variability – on one hand, it reduces the system uncertainty by eliminating multiplicity of equilibria; on the other hand, it enforces the price variability by making a mixed-strategy equilibrium the unique solution to the game.

*A. Average price and price variability.* With unique solution for a UA, any ambiguity in comparing the two auctions disappears. Figure 2.4 shows means and standard deviations for both auction formats, as a function of combinations of capacities $(k_1, k_2)$.

Our tests indicate that (a) the expected prices associated with DA and UA are quite similar; the ranking of their values is ambiguous and the differences are small; (b) the standard deviation is significantly higher in a UA than in a DA for most of $\mathbf{k}$. It suggests that, in asymmetric settings, the same lessons hold as in symmetric ones – UA yields nearly the same expected price as DA but higher price volatility.

*B. Capacity Asymmetry* We are also interested in how capacity structure influences the price dispersion. Since the qualitative characteristics of two auctions appear to be similar, we only discuss the case of DA. In Figure 2.4(b), we illustrate the effect of total capacity and the effect of capacity asymmetry. A reduction of total capacity (an increase of the system utilization) is illustrated by moving from $S$ to $F$. The expected price increases, as expected, while the variance increases first and then decreases. The price variance is maximized at intermediate utilization, implying robustness of the same observation for symmetric case.

Capacity asymmetry is illustrated by considering capacity combinations between $L$ and $R$. In the nearly symmetric setting, corresponding to point $S$, both expected buyer prices and their variances have local minima.[16] The set of nearly symmetric capacity combinations $SF$ corresponds to the boundary separating sets $\Omega_3^d$ and $\Omega_4^d$ in the description of DA's solution structure (see Appendix A.9). For points on this boundary (curve $SF$), the unique mixed-strategy equilibrium has no mass points, $m_1^d(B) = m_2^d(B) = 0$. Thus, if total capacity is fairly evenly[17] allocated between the two suppliers, the auction is more competitive and we observe lower expected price.

On line $LR$ the total capacity and thus system utilization are constant. If the two suppliers' capacities become unbalanced (i.e., $S$ moves towards either $L$ or $R$), the supplier with increasing portion of system capacity will bid, with increasing probability, at price cap $B$, i.e., gains dominant pricing power. Consequently, the average price increases and so does the price standard deviation.When capacity allocation

[16]The trajectories for UA and DA are not necessarily identical, but they are very close to each other as illustrated in the Figure 2.4.

[17]Cost asymmetry influences the boundary: according to Appendix A.9, suppliers' capacities are proportional to their maximum profit margins $(B - c_1) : (B - c_2)$. With fairly high price caps, these capacities are nearly symmetric.

becomes very unbalanced, $\mathsf{Std}[P^d]$ may decrease, since $\mathsf{E}[P^d]$ is approaching $B$ and does not vary so much. Thus, capacity asymmetry reduces the market competitiveness and leads to an increase of expected buyer price and increase and decrease of the price variability.

## 2.6 Preliminary Empirical Study

Our paper focuses on two related problems, the rationale for price dispersion and the comparison between DA and UA. Since we do not have access to data for the U.K. electricity market (the only marketplace that adopted DA), we investigate the uniform auction used in the U.S., and therefore, our partial tests refer only to the first problem, the structure of price dispersion. New England Power Pool (NEPOOL) is chosen due to its geographical integrity and availability of data.

*Research Design and Data Description.* Our objective is to see whether our theoretical predictions are consistent with the observed data. Our theoretical analysis shows that bidding strategies and corresponding price dispersion are linked to the demand level, decentralization of capacity, and asymmetry of production technology (cost and capacity). Since in reality the capacity profile and costs are fairly stable, while demand changes, we will observe the effect of single primary factor, demand level, on price dispersion. Our purpose is to use empirical data to provide *qualitative illustration* of derived dynamics rather than to formally test the model.

In general, there are two approaches to describe pricing policies. One is to directly study the bidding decisions of individual generating units. It is used in Wolfram (1998), which analyzes the daily electricity auction in U.K. The other approach is to estimate the aggregated price mark-up, adopted in Wolfram (1999, on U.K. market) and Borenstein, et al. (2002, on California market). In principle, as a preliminary test, this section takes the second approach. Our main distinction from the above papers is that we focus primarily on the stochastic properties of the price (price dispersion), rather than the average price level.

Since we are interested in describing price dispersion as a function of demand, a natural choice would be to estimate the different moments of price, such as mean, variance, and skewness, conditional on the demand. A more comprehensive characterization is to directly estimate the conditional distribution of the price. This can be done through the Quantile Regression (QR) model introduced by Koenker and Bassett (1978). Due to its advantage over the classic conditional-mean models, QR model is gaining popularity in many areas of applied econometrics. See Koenker & Hallock (2001) for a good introduction and Koenker (2005) for detailed coverage. While not used so far in analyzing market power and auction data, we find QR very appropriate in the settings we study, to estimate a family of price quantile curves as functions of the actual demand.

Among possible concerns, the critical ones are potential endogeneity and influence of factors not captured in the data. The possible endogeneity between demand and price is marginal in our setting because electricity demand is price independent and primarily driven by the weather (Engle, Mustafa, and Rice 1992). Among multiple factors that may influence price dispersion, the main ones are fuel prices and generation outage. As the fossil-fuel (for example, natural gas) generating plants are often the infra-marginal units at NEPOOL, the change of fuel price moves the cost curve on a daily basis, and subsequently the realized electricity price. The generation outage influences the price dispersion in two ways – 1) it directly modifies the marginal cost curve; 2) creates information asymmetry, as some of the suppliers may be aware of the outage of other units, but not necessarily all. In the analysis below, we attempt to control the impacts of both factors.

Next we describe the data we selected for the study. We examine the hourly demand and real-time price for NEPOOL. The available data covers the period March 2003 through June 2006.[18] Since the new Standard Market Design (SMD) was initiated at NEPOOL in March 2003, we omit the first year to eliminate (or at lest lessen) the effect of any potential learning. As the daily outage data for NEPOOL and daily natural gas price for a major pricing point in New England, Algonquin Cityhub are

[18]Spot price and load data are obtained from website http://www.iso-ne.com, where additional information about the NEPOOL market can be found.

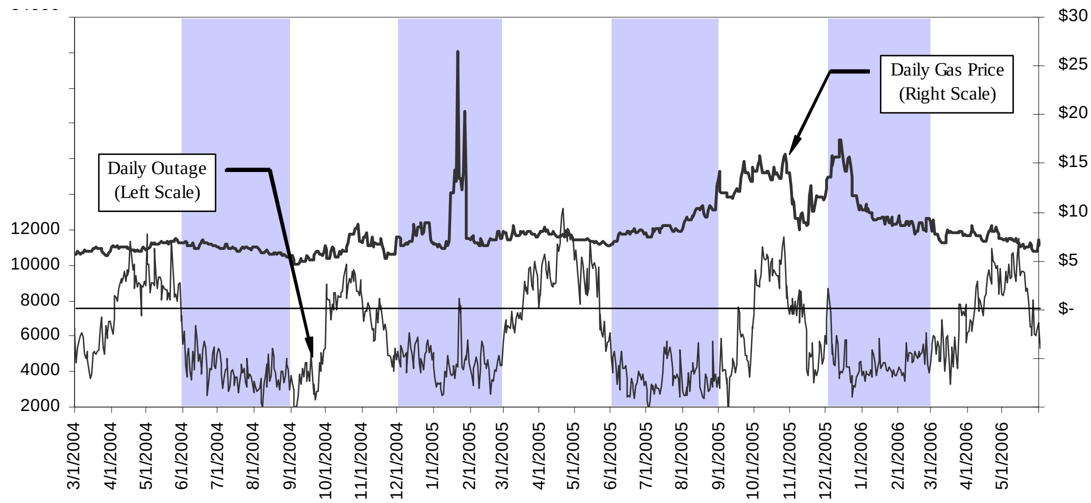



**Figure 2.5.** Natural Gas Prices at Algonquin Cityhub and Electricity Outage at NEPOOL

available, we control for the impact of outage and of fuel price. Figure 2.5 displays two time series, daily prices and daily outages. It is standard to classify the data as heating season, cooling season, and shoulder months. (The data presented in Figure 2.5 includes spring-shoulder months (Mar-May 2004), cooling season (Jun-Aug 2004), fall-shoulder months (Sep-Nov 2004), heating season (Dec 2004 - Feb 2005), and then another cycle.) Since outages are an additional source of variability and since outages are much higher during the shoulder months than in the heating and cooling seasons, in our empirical analysis we omit all shoulder months and concentrate on the first cooling season (Jun-Aug 2004). We do not control for weather – influence of the changing weather manifests itself in demand changing over time.

*Marginal Cost Curve and Qualitative Hypotheses.* In this subsection we first discuss the price dispersion that we expect to observe based on our theoretical analysis and structure of the cost-capacity profile at NEPOOL. A direct implication of our analysis is that price should exhibit heteroscedasticity conditional on different demand levels. A more interesting and relevant question is how price variance behaves as a function of the aggregated marginal cost curve. The discussion in Sections 2.4 and 2.5.1 suggests that, given symmetric suppliers, the price tends to be more dispersed for intermediate demands between 0 and capacity of all suppliers, as illustrated in Figure 2.6 (left). For the symmetric uniform auction, we compute and plot nine quantile curves (for

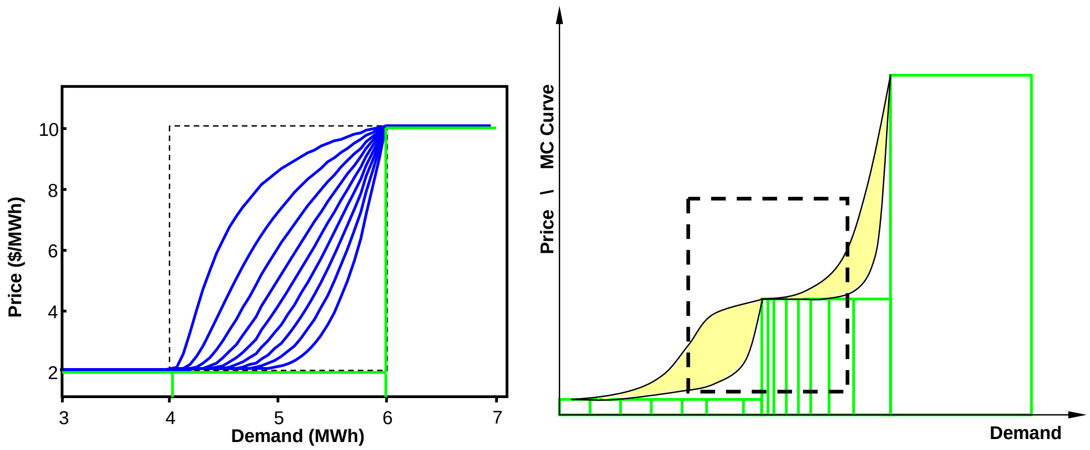



**Figure 2.6.** Numerical Examples Illustrating The Qualitative Hypotheses

10%, 20%, ... and 90%), illustrating the trend of price dispersion as demand increases.

Note, however, that in practice suppliers' marginal costs are not equal. In practice, it is convenient to express marginal cost as a function of cumulative capacity, where all units are ordered from lowest to highest marginal cost. This is referred to as marginal cost (MC) curve. MC curve is critical in our empirical illustration. A major northwest energy trading company is building such curves for NEPOOL on daily basis and provides several historical MC snapshots within our sample period. Since we focus on the 2004 cooling season, we use the MC curve for the median date (July 15th 2004), see Figure 2.7.[19] Note that, for the time period we consider (summer 2004), the influence of natural gas is less severe, because the price of natural gas, a major electricity generating fuel, is quite stable as illustrated in Figure 2.5. As marked on Figure 2.7, we have four groups of suppliers.we also plot the the histogram of hourly demands (the lower graph in Figure 2.7). Thus, we conceptually treat the MC curve as consisting of two cost levels, first being groups I and II, and second being group III, with nearly constant cost in each group, as illustrated in Figure 2.6(b).

While our model assumes identical cost for many suppliers and we do not provide a general solution for two levels of cost, we numerically verify that the solution for two costs can be represented as two "stacked up" solutions, each for a constant marginal

[19]The curve-providing company adopts a method similar to Borenstein et al. (2002) for MC estimation.

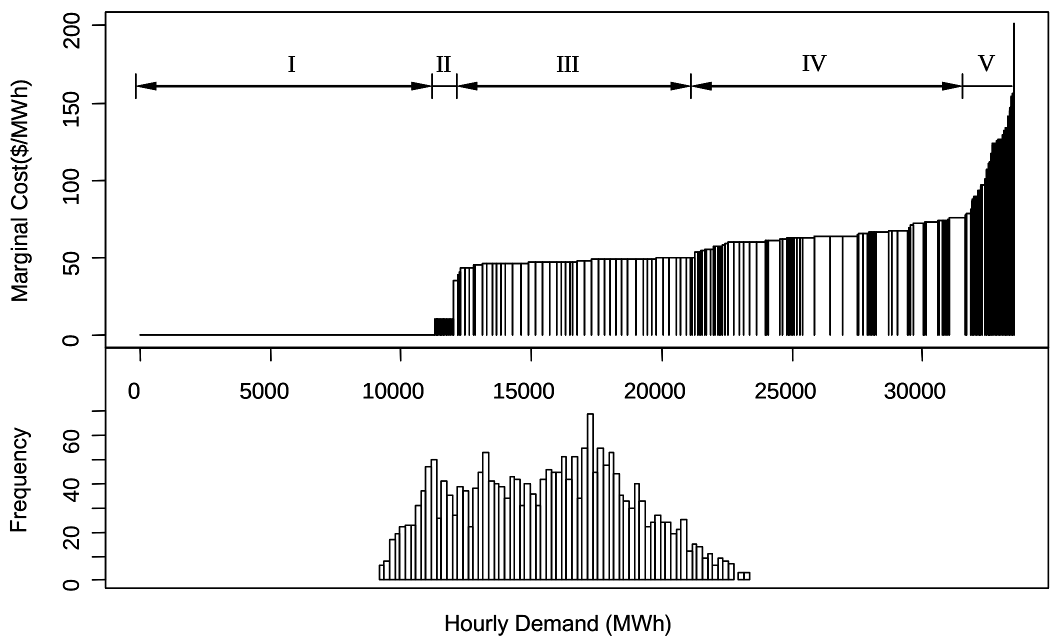



**Figure 2.7.** Marginal Cost Curve (07-15-2004) and Histogram of Hourly Loads (Jun-Aug 2004)

cost, see Figure 2.6(b). The solution has an intuitive structure. When demand does not activate the high-cost suppliers or only activates a small portion of them, they are under intensive competition and, therefore, expected to price at their costs or very close to it. For the lower-cost suppliers, pricing above the high cost group is a dominated strategy, so the higher cost is effectively a price cap for the low-cost suppliers playing the same role as $B$ in our symmetric models. When demand activate high-cost suppliers, the low cost supplier can secure their position of being a price taker by pricing low (such as at the higher cost or just below). The game is now played effectively only among the high-cost suppliers.

The 9 quantile curves illustrate the solution structure of above analysis and the grey box indicates the area where we expect to observe the trend of price dispersion, based on the histogram of demand distribution for the sample period (Figure 2.7). This numerical example suggests that, when demand moves towards the next stack of suppliers, the price range becomes narrower, and with demand continuing to increase it eventually expands. Furthermore, for demands in left of the shaded area, price is left skewed (with more heavier weight close to the cost level); while in the right of the shaded area, the opposite is the case. Our objective is to investigate how the actual

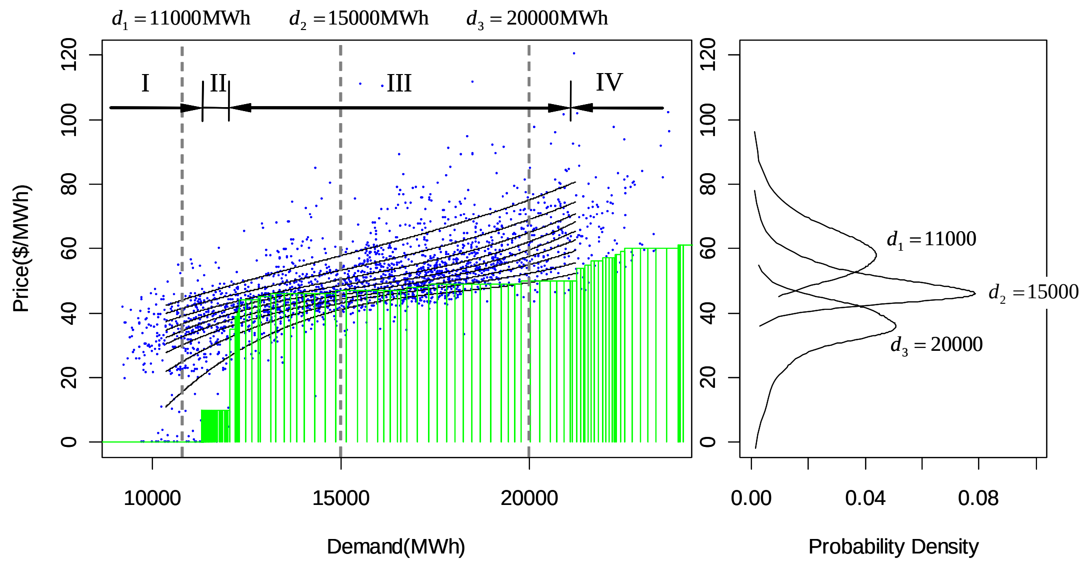



**Figure 2.8.** Quantile Regression Curves and Conditional Price Dispersion

price distribution behaves and whether the above prediction is supported.

*One-Variable Conditional Price Dispersion.* In this section we explain our methodology in more detail and then present the main results. Next subsection asks about robustness of the observations.

The individual points in Figure 2.8(a) are a scatterplot of hourly demand and price data. Clearly, as demand increases, the price tends to increase but exhibits wide dispersion. It also illustrates significant nonlinearity between demand and price. Figure 2.8(a) also superimposes 9 estimated quantile regression curves (for 10, 20, . . . , and 90 percent) on the scatterplot. Each curve is specified as a cubic function of the demand to allow for the intuitive shapes of quantile curves.[20] Each curves is specified as follows: for $\tau = 10\%, 20\%, \ldots, 90\%$,

$$P(\tau|d) = \alpha(\tau) + \beta_1(\tau)d + \beta_2(\tau)d^2 + \beta_3(\tau)d^3. \tag{2.16}$$

The curves in Figure 2.8 show the conditional heteroscedasticity of electricity prices and imply that demand not only determines the average price level, but also significantly influences the price variance and skewness. We can see that the realized

[20]The trend of demand-price in Figure 2.8(a) has a concave-convex curvature, which can well be captured by a cubic function. We have tried other nonlinear and nonparametric expressions, and the estimated Q-R curves appear to be robust.

prices are less dispersed for middle-range demand and more dispersed at the two ends. This is exactly what our theoretical model suggested. To show this more explicitly, we estimate a family of quantile functions for every 1% and compute the conditional density function of the market price. Figure 2.8(b) illustrates three probability density curves conditional on demand levels of 11000, 15000, and 20000 MWh.[21] The price distribution for demand $d_2 = 15000$ has narrower domain than the ones $d_1 = 11000$ and $d_3 = 20000$. Furthermore, price for $d_1$ is left-skewed and price for $d_3$ is right-skewed, as predicted by our numerical example.

*Robustness: Two-Stage Estimation of Conditional Price Dispersion.* The discussion in the previous section suggests that, for given MC curve at NEPOOL, the prices tend to be less dispersed for intermediate demand and more dispersed for both low and high demand. Clearly several uncontrolled factors may contribute to the price dispersion, such as fuel price, outage, and load shape (with strong daily periodicity). Also, the observation is based on one cooling season (summer 2004) and it is not clear whether it would hold for other seasons. The purpose of this subsection is to control the impacts of various system factors and examine other time periods.

We design the following two-stage test. In the first stage, we consider a single-equation model, where average price is a function of demand, gas price, outage, and time-of-the-day indicator, as shown below. In the second stage, we run quantile regression on the residuals of first-stage equation. Similar to model (2.16), we focus on the residuals' distribution conditional on the demand, highlighting the influences of market power. The model is as follows,

$$\text{(a) } P = \alpha + \beta_1 d + \beta_2 d^2 + \beta_3 d^3 + \gamma c_{GAS} + \delta T_{out} + \sum_{i=1}^{23} \theta_i D_i + \varepsilon \qquad (2.17)$$

$$\text{(b) } \varepsilon(\tau|d) = \alpha'(\tau) + \beta_1'(\tau)d + \beta_2'(\tau)d^2 + \beta_3'(\tau)d^3 \quad \text{for } \tau = 10\%, 20\%, \ldots, 90\%.$$

where $c_{GAS}$ is the daily gas price at Algonquin for the trading day, $T_{out}$ is the daily outage at NEPOOL, and $D_i$ denote the hourly dummy.

[21] We note that Koenker (2005) estimated conditional distributions using Q-R in an analysis of the serial correlation of daily temperatures.

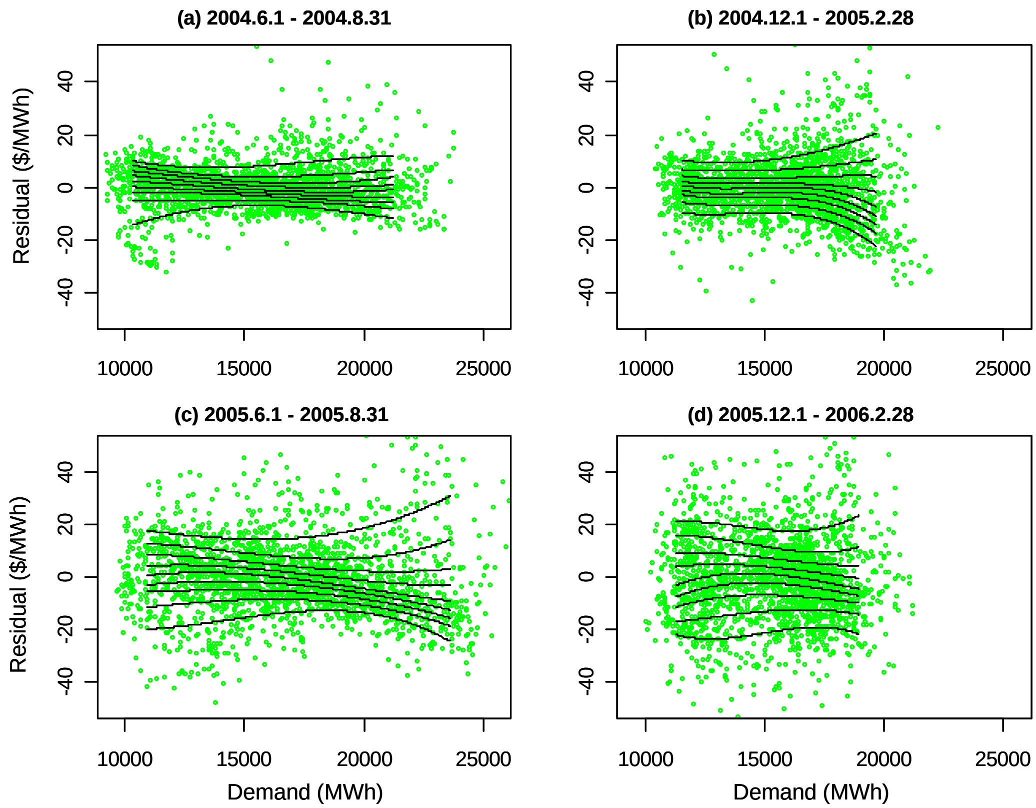



**Figure 2.9.** Quantile Curves for Residuals of Conditional Mean Models

In Figure 2.9, we illustrate the four families of estimated quantile curves, corresponding to the four electricity peak seasons. They show dispersion of the price around the mean. The first one uses the same sample set as in Figure 2.8. In Figure 2.9(a) we observe the same behavior of price dispersion as in Figure 2.8(a). It suggests that after controlling the impacts of the additional factors, price dispersion has the same characteristics as a function of demand level (as a proxy of market power). The same pattern can be observed in the other three graphs, suggesting consistency of results among the datasets we considered.

## 2.7 Discussion and Concluding Remarks

Motivated by the widely observed price dispersion in electricity markets and the ongoing debate about design of wholesale electricity market, this paper investigates the sources of price dispersion and compares the stochastic performance of two prevailing market designs, DA and UA. We model the critical elements of the auction as a game among energy suppliers. The analysis of the game allows us to characterize the equilibrium solutions and evaluate their sensitivity to the underlying assumptions. Specifically, it enables us to investigate the probabilistic properties of prices paid by electricity buyers and to compare their performances under DA and UA. To test the empirical implications of our analysis, we introduce Quantile Regression model. By linking hourly-spot prices for the cooling season in 2004 at NEPOOL with NEPOOL's marginal cost curve, we illustrate the consistency between these observations and behavior predicted by our model.

Our paper has a number of economic implications:

*Interpretation of Price Dispersion.* Our model indicates that one of important sources of price dispersion is intentional randomization, manifesting itself as a mixed strategy. While settings and mechanism are noticeably different, Varian (1980) uses mixed-strategy equilibrium to explain the empirical failure of "law of one price" in consumer markets. In order to price discriminate between informed and uninformed customers, stores may randomly choose sales, which disables the uninformed consumers from learning about future prices. The two critical elements of that paper, market segmentation and information asymmetry, do not play any role in our auction settings. Instead, limited capacity causes price randomization. (At a high level of abstraction, both stores' randomized sales and electricity suppliers' randomized bidding can be treated as intentional strategies to achieve profitability.)

In economics literature, intentional randomization, as an interpretation of mixed strategy, has been criticized for its lack of behavioral applicability. [22] Obviously, work of Harsanyi (1973) may provide an alternative interpretation. Harsanyi establishes a direct connection between pure-strategy Bayesian equilibrium and mixed-strategy solution. The Harsanyi's argument, adapted for electricity market, would be that an electricity supplier in real world, instead of randomizing the bids, may price deterministically according to certain privately observed signal(s). It could be her actual

[22] For example, Cachon and Netessine (2003), argue that " mixed strategies have not been applied in SCM, in part because it is not clear how a manager would actually implement a mixed strategy. . . . It seems unreasonable to suggest that a manager should 'flip a coin' when choosing capacity."

cost, capacity outage, or even personal view of the system uncertainty (such as electricity demand forecast) that give rise to some price changes. With such information asymmetry, a supplier's mixed strategy at equilibrium is just her competitors' appropriate[23] belief about her possible actions. Thus, price dispersion may well be interpreted by the prevalence of mixed-strategy equilibrium.

While both interpretations (purposeful randomization and reaction to private signals) are possible, our conversations with traders of a major electricity trading company indicated that the traders do use purposeful randomization when deciding their bids. (We expect that use of private information also plays a role in some situations.)

*Implication to Energy Risk Management.* Central task in any risk management applications is to estimate and forecast price volatilities. The popular models used in practice (ARCH and its variations) estimate the price volatility based on the historical prices. To our best knowledge, there is no risk management literature which incorporates information about other system factors. A practical reason is that availability of such data is more limited compared to price data. Importantly, we point to an additional critical predictor of price volatility. As shown in Section 2.6, the level of electricity price dispersion is heavily influenced by the system capacity utilization (or demand level). Since, in electricity markets, data for electricity demand is as easily available as the price data, including demand information seems as an appropriate adjustment.

*Implication to Procurement Auction Design.* The ongoing debate about electricity auction designs has focused on the efficiency of discriminatory and uniform auctions and, particularly, on the resulting price levels. Two schools formed among several leading auction theorists (represented by KCPT and FFH respectively) point in opposite directions. Our analysis and numerical tests suggest, similarly to FFH, that DA is a "better" market design. However, we do not endorse FFH's argument that DA yields lower prices compared to UA. Our paper shows the payment equivalence under symmetric setting and illustrates that such relation holds approximately under

[23]Harsanyi shows that a mixed-strategy equilibrium is the limit of a sequence of pure-strategy Bayesian equilibria corresponding to diminishing uncertainties of private signals.

asymmetric scenarios. Thus, we do differ with both of the two schools in terms of market efficiency. We prefer, however, DA over UA because electricity buyers will experience lower price volatility under DA. Considering the high price volatility in the U.S. market (where UA is adopted) and significant attention paid to it, this message has a potential to influence policy decisions.

Independent qualitative confirmation of our results comes from the area of experimental economics, as recent developments of experimental economics provide an alternative approach to compare the two market designs. The laboratory observations have supported neither schools' view (both claim one market design is more efficient than another). Mount et al. (2002) reports that " both uniform auction and discriminatory auction produce average prices fifty percent above the competitive levels. However, the prices for the uniform price auction are more volatile with many price spikes." It is a direct support of our theoretical finding. Similar test was conducted by Rassenti et al. (RSW, 2001), and their experiments indicate that (a) a DA consistently generates lower price volatility; (b) the average prices of the two auctions have no difference for high demand, but (c) DA yields higher average price for low demand.[24] Our theory is consistent with (a) and (b), but fails to explain (c). Despite lack of perfect consistency, compared to other theoretical papers we are aware off, our paper provides predictions closest to experimental observations.

[24]Here both " low" and " high" demand sustain pure-strategy equilibria with competitive price level, so they can be both viewed as low-demand state as we mentioned in the extended abstract.

# CHAPTER 3

# Bertrand-Edgeworth Auction with Multiple Asymmetric Bidders

## 3.1 Introduction

*Bertrand-Edgeworth Auction.* Bertrand-Edgeworth game (i.e., capacitated firms competing in price) is one of the most important building blocks for competitive models. It allows to study how capacity constraints influence firms' pricing decision in competitive environment and links the two classic forms of oligopoly competitions, in price and in quantity. In two recent papers about electricity auction designs, Fabra et al. (2006) and Chapter 2 revisit the Bertrand-Edgeworth game, because of its resemblance to the discriminatory auction implemented in the England and Wales electricity markets in 2001. Their point of departure from existing Bertrand-Edgeworth analysis is to assume inelastic demand[1] and fixed pricing cap. Demand inelasticity has always been identified as a key driver for the high-price volatility in electricity markets and selection of pricing cap is of high interest for regulatory economists after the California energy crisis in 2000-2001. As these features are more common in procurement auctions than in oligopoly pricing competitions, we label the model as a Bertrand-Edgeworth auction.

*Contributions.* The paper contributes in terms of both methodology and economic insights. From methodological point of view, we structurally characterize the solution to B-E auctions. We show that a deterministic (i.e., pure-strategy) equilibrium outcome is achieved only under restricted system conditions. Otherwise, a

[1] As an extension, Fabra et al. consider the duopoly case with downward-sloping demand function, which is solved in Kreps and Scheinkman (1983).

mixed-strategy equilibrium prevails with each player randomizing prices over a certain price interval. In mixed-strategy case, we show that (a) there exists a market leader whose bidding interval covers those of all other players and she is usually the one with relatively high capacity; (b) when demand increases, the lower bound of the pricing range is a piece-wise linear function that increases continuously, while the upper bound jumps from one discrete cost level to another. Consequently, the price range expands and contracts alternately. Applying these analytical results, we derive closed-form solution for a family of B-E auctions, where all active suppliers have similar size of capacity. The solution unifies the existing B-E solutions (asymmetric duopoly in Fabra et al. (2006) and symmetric oligopoly in Chapter 2. As these structural properties can be extended to games with elastic demand, the two existing B-E solutions in Kreps and Scheinkman (1983) and Vives (1986) are also special cases of our solution.

For more general asymmetric oligopoly settings, we investigate the impacts of capacity using numerical simulations. Our tests illustrate the limitation of one important implication in Kreps and Scheinkman (1983) that, for the same costs, lower capacity yields more aggressive pricing behavior, as indicated by stochastically lower bids. Such conclusion holds only in two-player games; in multiple-player settings, the equilibrium has instead a nested structure, where a low-capacity player prices within the range chosen by a high-capacity player, given that their costs are the same.

*Literature.* Three groups of papers are related to our work. The first stream of research directly studies B-E games. The fundamental papers analyzing B-E games include Kreps and Scheinkman (1983), Vives (1986), Osborne and Pitchik (1986), and Vives (1999). Vives (1999) contains a comprehensive review of the research in B-E area. Our analytical results are a generalization of the solution to asymmetric oligopoly settings. Similar solution technique is also used in Burdett and Judd (1983) and Varian (1988), both of which study the price dispersion in retail markets and interpret it as a mixed-strategy equilibrium. The primary difference between their models and B-E games that we consider, is that the mixed-strategy in these papers is not driven by capacity constraints, but by information asymmetry or costly consumer search. Allen and Hellwig (1986) also studies oligopoly B-E competitions, but their results focus on the aggregated impact of market decentralization. For example, they show that when the number of suppliers increases, the Nash equilibria of B-E games converge to a competitive outcome. While the insight itself is confirmed in Chapter 2 such analysis of the limit cases is not our focus. The second group of papers analyzes discontinuous games. As indicated in this paper, B-E games usually result in mixed-strategy equilibria, due to discontinuities of the payoff functions. The theoretical foundations of this field are laid out by Dasgupta and Maskin (1986), Simon (1986) and Reny (1999), who focus on the general conditions for mixed-strategy equilibrium to exist. Our paper instead concentrates on the equilibrium structure and resulting economic insights. The third group of papers includes the literature on shared auctions is also related to our work. Besides the two directly related papers (Fabra et al. (2006) and Chapter 2), other important papers include Wilson (1979), Wang and Zender (2002), and Klemperer and Meyer (1989). The main difference is that they assume continuous supply functions submitted by bidders, while we assume only unit-price and supply quantity can be submitted. Such difference leads to different bidders' behavior and quite different insights.

Due to technical nature of most proofs, after presenting the model in Section 3.2, in Section 3.3 we outline the analytical results and briefly explain the intuition. Based on analytical results, in Section 3.4 we derive closed-form solution to quasi-symmetric B-E auctions and numerically study the impact of capacity. In Section 3.5, we outline an extension of our model by considering demand elasticity. Appendix A is a stand-alone document and it contains all technical derivations we refer to. Appendix B contains description of the computational algorithm that numerically finds the equilibrium.

## 3.2 Model

*Game description.* Consider a non-strategic auctioneer and $N$ competing (strategic) suppliers. Supplier $i$, for $i = 1, 2, \dots N$, submits a bid of quantity-price pair $(q_i, p_i)$, while the auctioneer collects the $N$ bids and chooses a portfolio of suppliers that satisfies deterministic demand $d$ at a minimal cost. The auctioneer imposes a price cap $b$ so that only prices $p_i \leq b$ are accepted. Each supplier has a linear production cost, with per-unit cost of $c_i$, and fixed production capacity, $x_i$. The auctioneer can ask each supplier to provide any quantity, $q_i$, up to her bid limit, $x_i$, and pays the bid price, $p_i$, per unit requested. Given any vector of bid offers $(\mathbf{p}, \mathbf{q}) = (\{p_i\}_{i=1}^N, \{q_i\}_{i=1}^N)$, the auctioneer's problem is easily solved. The auctioneer will start at the lowest bid price and accept $\min\{d, q_i\}$, then move to the next lowest bid price, until demand is filled. The suppliers are aware of auctioneer's strategy and simultaneously submit quantity-price bids. We seek a Nash equilibrium among these competing suppliers.

It is assumed that $d > 0$, $x_i > 0$, $c_i \in [0, b)$ for $i = 1, 2, \ldots, N$ and $b > 0$. All problem parameters are common knowledge among the players. Suppliers are indexed according to increasing costs, $0 \leq c_1 \leq c_2 \leq \ldots \leq c_N < b$. We assume that the auctioneer will satisfy his demand if possible, even if he has to pay $b$ per unit to do it. For ease of exposition, we denote $c_{N+1} := b$, which may be interpreted as the auctioneer's reserve cost to meet the demand (in the form of penalties paid to end users). Since any bid price less than a supplier's cost yields a strictly negative payoff for any positive number of units supplied, all suppliers who bid any capacity into the market will bid $p_i \in [c_i, b]$.

*Allocation rule.* The auctioneer minimizes his cost of satisfying demand, given the quantities and prices bid into the market. We assume ties are broken by first granting orders to the efficient suppliers (those with lower production costs). If suppliers with the same cost form a tie, each supplier gets a demand share proportional to her capacity. These assumptions lead to supplier $i$ receiving demand for fraction $r_i(\mathbf{p}, \mathbf{q})$ of her bid capacity $q_i$:

$$r_i(\mathbf{p}, \mathbf{q}) = \min\left\{1, \frac{[d - \sum_{n \neq i} q_k \delta_{(p_k < p_i)} - \sum_{k \neq i} q_k \delta_{(p_k = p_i, c_k < c_i)}]^+}{q_i + \sum_{k \neq i} q_k \delta_{(p_k = p_i, c_k = c_i)}}\right\}. \tag{3.1}$$

where $\delta_{(.)}$ denotes an indicator function. Supplier $i$'s realized sales is denoted by $z_i = q_i r_i$. We use $\mathbf{y}_{-i}$ to denote vector $(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_N)$, where $y$ may refer to price, quantity, or resulting policy.

**Lemma 2** ***Full-Capacity Bidding:*** *For all suppliers, it is (weakly) optimal to bid all their capacities into the auction, that is $q_i = x_i$ for all $i$.*

**Proof.** Proof of Lemma 16. As $R_i(\mathbf{p},\mathbf{q}) = (p_i - c_i)z_i(\mathbf{p},\mathbf{q})$ and $p_i \geq c_i$, it is sufficient to show that $z_i(\mathbf{p},\mathbf{q})$ is nondecreasing in $q_i$. By (3.1), we have

$$z_i(\mathbf{p},\mathbf{q}) = q_i r_i(\mathbf{p},\mathbf{q}) = \min\left\{q_i, \frac{[d - \sum_{k\neq i} q_k \delta_{(p_k<p_i)} - \sum_{k\neq i} q_k \delta_{(p_k=p_i, c_k<c_i)}]^+}{1 + q_i^{-1} \cdot \sum_{k\neq i} q_k \delta_{(p_k=p_i, c_k=c_i)}}\right\}.$$

As $[d - \sum_{k\neq i} q_k \delta_{(p_k<p_i)} - \sum_{k\neq i} q_k \delta_{(p_k=p_i,c_k<c_i)}]^+$ and $\sum_{k\neq i} q_k \delta_{(p_k=p_i,c_k=c_i)}$ are non-negative constants for fixed $\mathbf{p}$ and $\mathbf{q}_{-i}$, $z_i$ is nondecreasing in $q_i$. ■

Henceforth, without loss of generality we assume that $q_i = x_i$ for all players $i$ and the bidding game is reduced to an oligopoly pricing game. Accordingly, in what follows we use the condensed notation $r_i(\mathbf{p}) = r_i(\mathbf{p},\mathbf{x})$ and $z_i(\mathbf{p}) = z_i(\mathbf{p},\mathbf{x})$. Clearly, bidder $i$'s problem is,

$$\max_{p_i \in [c_i, b]} \; R_i(p_i, \mathbf{p}_{-i}) = (p_i - c_i)x_i r_i(p_i, \mathbf{p}_{-i}).$$

*Mixed strategies.* It is well known that a B-E game may result in mixed-strategy equilibrium and we introduce the corresponding notation here. Supplier $i$'s mixed-strategy is denoted by $\sigma_i$, a random variable with support $[c_i, b]$. Define $F_i(p;\sigma_i)$ as the cumulative distribution function for $\sigma_i$ and $m_i(p;\sigma_i)$ the probability mass at price $p$. Denote $\overline{p}_i(\sigma_i) := \inf\{p : F_i(p;\sigma_i) = 1\}$ and $\underline{p}_i(\sigma_i) := \sup\{p : F_i(p;\sigma_i) = 0\}$ as the upper and lower bounds for bidding strategy $\sigma_i$. Given the opponents' mixed-strategy $\sigma_{-i}$, supplier $i$ has random sales and payoff, when choosing price $p$. Define $\bar{z}(p,\sigma_{-i}) := \mathsf{E}_{\sigma_{-i}}[z_i(p,\sigma_{-i})]$ as her expected sales, and similarly we define her expected sales ratio $\bar{r}_i(p,\sigma_{-i})$ and her expected payoff $\bar{R}_i(p,\sigma_{-i})$.

Denote $\sigma^*$ as a mixed-strategy equilibrium and $ER_i(\sigma^*) := \bar{R}_i(\sigma_i^*, \sigma_{-i}^*)$ supplier $i$'s expected equilibrium payoff. For simplicity, we suppress the equilibrium-associated notation by omitting $\sigma^*$. For example, $F_i(p_i) = F_i(p_i;\sigma_i = \sigma_i^*)$ and $\overline{p}_i = \overline{p}_i(\sigma_i^*)$. Similarly, we use shorthand notation $\bar{z}_i(p_i) = \bar{z}_i(p_i,\sigma_{-i}^*)$, $\bar{r}_i(p_i) = \bar{r}_i(p_i,\sigma_{-i}^*)$, $\bar{R}_i(p_i) = \bar{R}_i(p_i,\sigma_{-i}^*)$, and $ER_i = ER_i(\sigma^*)$.

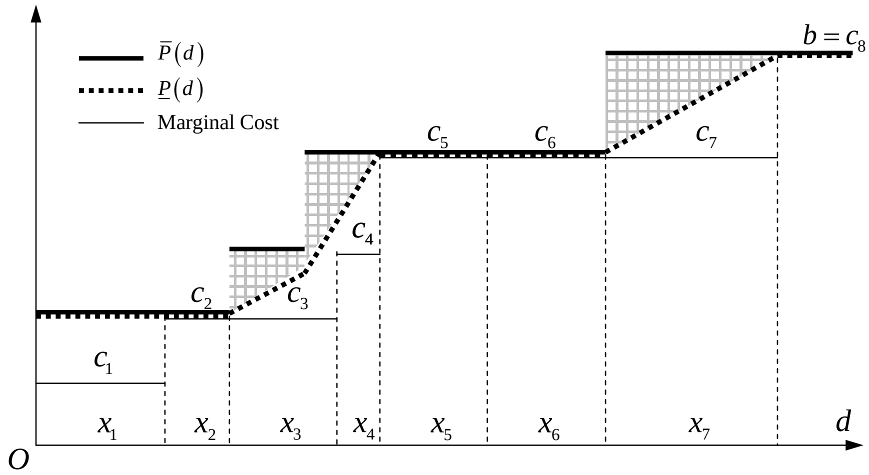



**Figure 3.1.** Intervals of Effective Equilibrium Bids Corresponding to Different Demands

## 3.3 Main Equilibrium Results

*An illustrative example.* We start with a numerical example that illustrates the structure of the equilibrium solution. Consider seven suppliers, who differ in production costs, capacities, or both. By stocking their capacities one by one according to increasing production costs, we construct the aggregated marginal cost curve for the industry, see Figure 3.1. Given the marginal-cost curve, the equilibrium outcome depends on the level of demand $d$. We refer to the suppliers whose bids are accepted (with positive production) as *active* or *activated* ones, depending on the context. Formally, $A = \{i : r_i(p_i^*, \mathbf{p}_{-i}^*) > 0\}$, is the set of active suppliers. When demand is low, not all suppliers are activated and the competition is among the more efficient suppliers (those with lower generation costs). To avoid activating more suppliers into the game, their bids are bounded from above, so that the inefficient suppliers (those with costs higher than this bound) are effectively bid out of the auction. More specifically, for any given demand level and corresponding Nash equilibrium, we denote $\underline{P}(d) = \min\{\underline{p}_i | ER_i > 0\}$ and $\overline{P}(d) = \max\{\overline{p}_i | ER_i > 0\}$ as the lower and upper bounds of the bids placed by profitable suppliers. By increasing the demand level from zero to above the total capacities $\sum_{i=1}^{N} x_i$, we construct a continuum of B-E auctions and plot the trajectories of $\underline{P}(d)$ and $\overline{P}(d)$ in Figure 3.1 as functions of demand $d$.

When $\underline{P}(d) = \overline{P}(d)$, a pure-strategy equilibrium is reached, where all profitable suppliers bid uniformly the same price. When $\underline{P}(d) < \overline{P}(d)$, the auction ends with a mixed-strategy equilibrium, where profitable suppliers randomize the bids within the interval $[\underline{P}(d), \overline{P}(d)]$. $\underline{P}(d)$ is a continuous and piecewise linear function of demand $d$, while $\overline{P}(d)$ is a non-decreasing step function, with values equal to certain cost levels. As demand increases, the effective bidding range shifts upward and it may expand and contract alternatively. To explain the above qualitative characteristics, we below describe the Nash equilibrium.

*Equilibrium normalization.* We first impose an assumption that simplifies the analysis without any loss of mathematical generality and economic insights. When $\overline{P}(d) < b$ the suppliers with cost higher than $\overline{P}(d)$ are never activated and have zero payoffs. Thus, multiple equilibria may exist since these inactive suppliers are indifferent among all prices between their cost and the price cap. Of course, their bids have no impact on the economic transfers among agents. We label such equilibria as *payoff equivalent*. We choose one among these, which we call a *normalized* equilibrium, where nonprofitable suppliers bid their costs ($R_i(\mathbf{p}) = 0$ implies $p_i^* = c_i$). We show that for any Nash equilibrium (pure-strategy or mixed-strategy), there exists a normalized, payoff equivalent equilibrium (Lemmas 20 and 23) and that applies to all costs, both greater and equal to $\overline{P}(d)$. The proof for mixed strategy is delayed until preliminary results for mixed strategy are described. In what follows, we analyze the normalized equilibria.

*Pure-strategy equilibrium.* In a pure-strategy equilibrium all active suppliers bid the same price. If there was any price difference among them, any active supplier who bids below the highest bid must sell all of her capacity and, by raising her bid towards the highest active bid, her capacity remains fully utilized, while the payoff is strictly increased. Lemma 17 formally shows that in pure-strategy equilibrium no price separation exists among the active bidders. The remaining question is what price will be chosen. We show that it must be either the upper limit $b$ or one of the costs (otherwise all active bidders would raise the price). The lowest cost that may play this role is the cost of the first firm to put total bid capacity above demand (a minor modification is required if demand is less than the quantity bid by the lowest-cost firm). Formally (Proposition 17), the following is the bid price

$$P^* = \begin{cases} c_2 & \text{for } d < x_1 & \text{(a)} \\ c_j & \text{for } \sum_{k=1}^{j-1} x_k \leq d < \sum_{k=1}^{j} x_k \text{ and } j \geq 2 & \text{(b)} \\ b & \text{for } d \geq \sum_k x_k & \text{(c)} \end{cases} \tag{3.2}$$

Any price corresponding to a cost lower than $P^*$ cannot activate enough capacity to cover demand. Also, any price above $P^*$ activates a total capacity strictly higher than demand and triggers a demand rationing. As at least one supplier $i$ with positive profitable margin ($P^* - c_i > 0$) does not fully utilize her capacity, she has an incentive to decrease the price and increase her capacity utilization.

Since we consider only normalized equilibria, the above argument suggests the pure-strategy equilibrium, if exists, is uniquely determined as $p_i^* = \min\{P^*, c_i\}$ for all $i$ (Proposition 18). If, at this price vector no supplier is tempted to defect, then that price vector is an equilibrium. Analyzing the benefits of defecting is easy because by (C.7) $R_i(P^*) = (P^* - c_i)(x_i \wedge d)$ for $i$ with $c_i < P^*$ and $R_i(p^*) = 0$ otherwise. Formally, we define the payoff of supplier $i$

$$S_i(p) := R_i(p, \mathbf{p}_{-i} = \mathbf{p}^*_{-i}) = (p - c_i)[d - \sum_{k \neq i} x_k \delta_{(c_k < p)}]^+ \quad \text{for } p \in (c_i, b]. \tag{3.3}$$

Strategy profile $p_i^* = \min\{P^*, c_i\}$ sustains a pure-strategy equilibrium if and only if $R_i(P^*) \geq S_i(c_j)$ for all $i$ with $c_i \leq P^*$ and $c_j > P^*$. Conversely, if there is any supplier $i$ having $R_i(P^*) < S_i(c_j)$ for certain $c_j > P^*$, there is no pure-strategy equilibrium and we turn to seek mixed-strategy equilibrium.

In next subsections, we describe the structural properties of the Nash equilibrium. The accompanying lemmas and propositions are presented in the stand-alone appendix of this chapter (Appendix B).

*Preliminary properties of mixed-strategy equilibrium.* We first establish several properties that any mixed-strategy equilibrium must satisfy. The first one relates to the upper bound on the pricing range, the next two to the lower bound on the pricing range, while the fourth one describes ordering of $P^*$ and $\underline{P}$.

First preliminary property: Given that there exists supplier(s) obtaining positive market share by bidding up to $\overline{P}$, any supplier with cost $c_i < \overline{P}$ also expects to obtain a positive sales (and therefore a positive payoff) by bidding between $c_i$ and $\overline{P}$. Formally, the upper bound of effective bids $\overline{P}$ separates the profitable bidders from the nonprofitable ones: (Lemma 22) $c_i < \overline{P} \Leftrightarrow ER_i > 0$.

Second preliminary property: Lemma 24(a) proves that a supplier with $\underline{p}_i = \underline{P}$ has an expected payoff of $ER_i = (\underline{P}-c_i) \min\{d, x_i\}$. Otherwise, if $ER_i < (\underline{P}-c_i) \min\{d, x_i\}$, supplier $i$ would have an incentive to defect to $\underline{P}$'s left neighborhood, which provides a bigger payoff $(\underline{P} - c_i) \min\{d, x_i\} > ER_i$.

Third preliminary property: Among the profitable suppliers, for some of them, their withdrawal would translate into not satisfying the market demand at a certain price $p$. In general, supplier $i$ is considered *critical at price* $p$ if its cost $c_i$ is smaller than price $p$ and its capacity is necessary to meet the demand, that is $\sum_{k \neq i} x_k \delta_{(c_k<p)} \leq d$. In Lemma 24b, we show that, for those suppliers who are critical at price $\underline{P}$, we must have $\underline{p}_i = \underline{P}$. If this is not the case, (i.e., $\underline{P} < \underline{p}_i$) the suppliers with $\underline{p}_j \in [\underline{P}, \underline{p}_i)$ have an incentive to raise their lower bidding bound, without losing any sales.

Fourth preliminary property: Another observation in Figure 3.1 can also be formally justified; the lower bidding bound in a mixed-strategy equilibrium is always above (strictly above for mixed strategy) the price $P^*$ determined by (C.7). Since pure-strategy equilibrium does not exists, we must have at least one supplier $i$ with $c_i < P^*$ and $(c_j - c_i)[d - \sum_{k \neq i} x_k \delta_{(c_k<c_j)}] > (P^* - c_i) \min\{d, x_i\}$ for certain $c_j > P^*$. It suggests that $\sum_{k \neq i} x_k \delta_{(c_k<P^*)} < d$, and therefore, $\underline{P} \leq P^*$ is impossible.

*Anchoring supplier.* We show (Lemma 25) that at most one profitable supplier has positive probability mass $m_i(\overline{P}) > 0$ at the upper bound $\overline{P}$ (implying $R_i(\overline{P}) = ER_i$). (Otherwise, if there was more than one supplier doing so, there would be a positive chance of a price tie at $\overline{P}$. From $\overline{P} > \underline{P} > P^*$, we have that $\sum_k x_k \delta_{(c_k<\overline{P})} > d$, implying the rationing rule (C.1) applies on that occasion. Consequently, the least efficient supplier among them (with the highest cost) would prefer to shift the probability mass towards $\overline{P}$'s left neighborhood, where a higher expected sales can be achieved. Thus, more than one supplier choosing $m_i(\overline{P}) > 0$ cannot be an equilibrium outcome.

Suppose there exists a supplier with $m_i(\overline{P}) > 0$. Since $\bar{R}_i(\overline{P}) = ER_i > 0$ and $m_j(\overline{P}) = 0$ for all other profitable suppliers, we have that $ER_i = S_i(\overline{P}) > 0$, implying $d - \sum_{k \neq i} x_k \delta_{(c_k < \overline{P})} > 0$. According to the second preliminary property (Lemma 24a), supplier $i$ must choose $\underline{p}_i = \underline{P}$. When there is no supplier with $m_i(\overline{P}) > 0$, it can be similarly established that any supplier $i$ with $\overline{p}_i = \overline{P}$ satisfies the above two properties ($ER_i = S_i(\overline{P})$ and $\underline{p}_i = \underline{P}$). Formally, we label supplier $i_A$ as an *anchoring supplier* if she satisfies (a) $c_{i_A} < \overline{P}$, (b) $\bar{p}_{i_A} = \overline{P}$, and (c) $m_{i_A}(\overline{P}) \geq m_j(\overline{P})$ for all $j$ with $c_j < \overline{P}$. This observation allows us to determine the bidding interval $[\underline{P}, \overline{P}]$, if only we can identify the supplier with these characteristics. We can further show (Lemma 27) that $\bar{p}_{i_A} = \overline{P} = \arg\max\{S_{i_A}(c_j) : c_j > P^*\}$. It highlights supplier $i_A$'s role in forming the mixed-strategy equilibrium, because her incentive to defect from $P^*$ to $\overline{P}$ is the primary reason that a pure-strategy outcome $p_i^* = \max\{c_i, P^*\}$ cannot be an equilibrium.

*Computing price bounds.* The task of deriving $\underline{P}$ and $\overline{P}$ boils down to identifying supplier $i_A$. As the procedure is analytically complicated (Lemmas 27-29 and Proposition 19), we only outline the critical elements. First, we can narrow down the search for $i_A$ to the suppliers who are critical at price $P^*$, i.e., $i_A \in \Omega := \{i \in \mathcal{N} : c_i \leq P^* \text{ and } \sum_{k \neq i} x_k \delta_{(c_k \leq P^*)} < d\}$. Second, there may exist multiple $c_j$'s that maximize $S_{i_A}(c_j)$. We show that, under such circumstance, $\overline{P}$ has to be the smallest one among the multiple maximizers. Therefore, we define for all $i \in \Omega$ the *trial* value for the upper bidding bound $\overline{P}_i^T := \min\{\arg\max\{S_i(c_j) | c_j > P^*\}\}$. Clearly, $\overline{P} \in \{\overline{P}_i^T | i \in \Omega\}$. Thirdly, $ER_{i_A} = S_{i_A}(\overline{P}) = \bar{R}_{i_A}(\underline{P}) = (\underline{P} - c_{i_A}) \min\{x_{i_A}, d\}$ implies $\underline{p}_{i_A} = \underline{P} = \frac{S_{i_A}(\overline{P})}{\min\{d, x_{i_A}\}} + c_{i_A}$. Thus, we define for $i \in \Omega$ the *trial* value of lower bidding bound $\underline{P}_i^T := \frac{\max\{S_i(c_j) | c_j > P^*\}}{\min\{d, x_i\}} + c_i$ and $\underline{P} \in \{\underline{P}_i^T | i \in \Omega\}$ must hold. We can actually identify $\underline{P}$ by showing that the anchoring supplier must have the largest trial value of lower bound among all candidates, i.e., $\underline{P} = \underline{P}_{i_A}^T = \max_{i \in \Omega}\{\underline{P}_i^T\}$. If there exists one supplier $i$ satisfying $\underline{P}_i^T > \underline{P}_j^T$ for all $j \in \Omega \setminus \{i\}$, she must be the anchoring supplier and her trial value of the upper bound $\overline{P}_i^T$ must be $\overline{P}$. Some added complexity results from the possibility that multiple suppliers in $\Omega$ attain $\underline{P}_k = \max_{i \in \Omega}\{\underline{P}_i^T\} = \underline{P}$, which impedes the determination of $\overline{P}$. We resolve this last ambiguity by showing that, if multiple suppliers have $\underline{P}_k = \underline{P}$, the anchoring supplier must have the smallest trial value of the upper bound among these suppliers, i.e., $\overline{P} = \min\{\overline{P}_i^T | i \in \{k | \underline{P}_k^T = \underline{P}\}\}$. If multiple suppliers have both $\underline{P}_k = \underline{P}$ and $\overline{P}_k = \overline{P}$, they are all anchoring suppliers. The process above completely determines the anchoring supplier $i_A$ and the two pricing bounds $\{\underline{P}, \overline{P}\}$.

*Distributional properties.* Having uniquely determined the price interval for effective bids, a further characterization of the mixed-strategy equilibrium is based on investigation of the distribution function $F_i$. Unfortunately, closed-form solutions exist only for some special cases – we present these in the next section. Outside those cases, mixed-strategy equilibrium can only be numerically constructed. Both the analytical and numerical solutions rely on one useful result that we describe next. We prove (Proposition 16) that the equilibrium distribution function is continuous for any critical supplier at price $\overline{P}$, i.e., there is no probability mass associated with any price within $(\underline{P}, \overline{P})$. The intuition is as follows. If supplier $i$ arranges a probability mass at price $\tilde{p}$, it deters all other players from pricing at $\tilde{p}$ because of the possibility of sharing demand with agent $i$ and this disadvantage also extends to $\tilde{p}$'s right neighborhood. As no other supplier will price at $\tilde{p}$ or its right neighborhood, it is not optimal for supplier $i$ to arrange a positive probability mass at $\tilde{p}$.

Summarizing the properties above we have the following theorem:

**Theorem 3** *(1)* ***Pure-Strategy Equilibrium:*** *A normalized pure-strategy equilibrium exists and is unique if and only if, for each supplier $i$ with cost $c_i \leq P^*$ and for all $c_j > P^*$, $(P^* - c_i)\min\{d, x_i\} \geq S_i(c_j)$. The pure-strategy equilibrium is $p_i^* = \max\{P^*, c_i\}$ for all $i$.*

*(2)* ***Mixed-Strategy Equilibrium:*** *A normalized mixed-strategy equilibrium satisfies the following. (a) There exists an anchoring supplier $i_A$ such that $c_{i_A} \leq P^*$, $\underline{p}_{i_A} = \underline{P} = \frac{ER_{i_A}}{\min\{d, x_{i_A}\}} + c_{i_A}$, $\bar{p}_{i_A} = \overline{P} = \arg\max_{c_j > P^*} S_{i_A}(c_j)$, and $ER_{i_A} = S_{i_A}(\overline{P})$; (b) The bidding interval $[\underline{P}, \overline{P}]$ can be uniquely determined by $\underline{P} = \max_{i\in\Omega}\{\underline{P}_i^T := \frac{\max_{c_j>P^*} S_i(c_j)}{\min\{x_i, d\}} + c_i\}$ and $\overline{P} = \min_{i\in\{k|\underline{P}_k^T = \underline{P}\}}\{\overline{P}_i^T = \min\{\arg\max_{c_j>P^*} S_i(c_j)\}\}$; (c) For any critical supplier $i$, $m_i(p) = 0$ for all $p \in (\min\{c_i, \underline{P}\}, \overline{P})$.*

We close this section by revisiting and explaining further the example shown in

Figure 3.1. Consistent with derivations above, $\overline{P}(d)$ in Figure 3.1 is a step function and takes values equal to costs of some inefficient supplier(s). More interestingly, $\underline{P}(d)$ is piece-wise linear in $d$. It is because $\underline{P}(d) = \frac{S_i(\overline{P})}{d \wedge x_{i_A}} + c_{i_A}$ is linear in $d$ (by (C.18) and $x_{i_A} < d$). When $\overline{P}(d)$ jumps to a higher cost level, there is a change of the slope of $\underline{P}(d)$, as illustrated by the kink of $\underline{P}(d)$ for $\sum_{k=1}^{2} x_k < d < \sum_{k=1}^{3} x_k$. Moreover, when multiple suppliers have the same cost, assume for a moment that the suppliers with the same cost are ordered from smallest to largest capacity. As Figure 3.1 illustrates, pure-strategy equilibrium usually collapses when demand activates the supplier with the highest capacity (such as $d > \sum_{k=1}^{2} x_k$ and $d > \sum_{k=1}^{6} x_k$). Note that last supplier within the group becomes capable to defect from a competitive pricing level. Correspondingly, in the resulting mixed-strategy equilibrium, she becomes the anchoring supplier.

## 3.4 Applications

*Analytical solution to quasi-symmetric B-E auctions.* Applying the above results, we can compute the equilibrium distribution functions $\{F_i(p)\}_{i=1}^{N}$ for some cases analytically and the remaining ones numerically. In this subsection, we derive the closed-form solution for B-E auctions when each supplier is critical at price $\overline{P}$, where $\underline{P}$ and $\overline{P}$ are derived by Theorem 3(2-b). We label such an auction as *quasi-symmetric.*

According to the third preliminary property, we have $\underline{p}_i = \underline{P}$ for all $i$ with $c_i < \overline{P}$. For $p$ in $\underline{P}$'s right neighborhood, we can compute the equilibrium solution $\{F_i(p)\}_{i \in \mathbf{I}_p}$ where the index set is defined as the set of players for whom $p$ is within its support, $\mathbf{I}_p := \{i : F_i(p) \in (0,1)\}$ and the number of players $N_p := ||\mathbf{I}_p||$. In the below derivation, we only compute distribution functions for players in $\mathbf{I}_p$.

According to Theorem 3.(2c), $F_i(p)$ is *continuous*, and guided by the same intuition,[2] we can show that $F_i(p)$ is also *strictly increasing* in $p \in (\underline{P}, \overline{p}_i)$. Continu-

[2]Under the assumption of quasi-symmetry, if the probability distribution $F_i$ has a hole $[\alpha, \beta]$ within $p \in (\underline{P}, \overline{p}_i)$, the rest of suppliers will find that $\beta$ generates strictly higher profit than any price in $[\alpha, \beta)$. Therefore, rest of the supplier should also arrange zero probability within $[\alpha, \beta)$. This in turn implies that, for supplier $i$, $\bar{R}_i(\alpha) < \bar{R}_i(\beta)$, and therefore, this hole shall further expand leftwards, which contradicts the optimality of $F_i$.

ity and monotonicity imply $\bar{R}_i(p) = ER_i$.[3] According to the assumption of quasi-symmetric auction, supplier $i$'s expected payoff at $p$ is $\bar{R}_i(p) = (p - c_i)[\prod_{k \neq i} F_k \cdot (d - \sum_{k \neq i} x_k) + (1 - \prod_{k \neq i} F_i) \cdot x_i]$. Applying the second preliminary property, we must have $ER_i = (\underline{P} - c_i)x_i$ for all $i$. From $\bar{R}_i(p) = ER_i$, we have $\prod_{k \neq i} F_k = \frac{x_i}{\sum_k x_k - d} \frac{p - \underline{P}}{p - c_i}$. Multiplying the equations across all $i$, we have expression of $(\prod_k F_k)^{N_p - 1}$. Taking $(N_p - 1)$th root of the expression and then dividing it by $\prod_{k \neq i} F_k$, we obtain

$$F_i(p) = \left[ \frac{p - \underline{P}}{\sum_{k=1}^{N} x_k \delta_{\{c_k < \overline{P}\}} - d} \cdot \prod_{k \in \mathbf{I}_p} (\frac{x_k}{p - c_k}) \right]^{\frac{1}{N_p - 1}} \frac{p - c_i}{x_i} \text{ for } i \in \mathbf{I}_p. \qquad (3.4)$$

Figure 3.2 shows an example of the equilibrium distribution functions for a 3-bidder B-E auction. By applying Theorem 3, we identify supplier 3 as the anchoring supplier and derive $\underline{P}$ and $\overline{P} = b$. It is easy to verify the capacity profile satisfies the condition of quasi-symmetry, so equation (3.4) can be applied directly at $\underline{P}$'s right neighborhood. Note that, when $p$ moves upward, $F_1(p)$ reaches value of 1, at price $\overline{p}_1$. Since $F_1(p) = 1$ for all $p > \overline{p}_1$, within interval $(\overline{p}_1, \overline{P})$, we solve the problem for suppliers 2 and 3 only. The same procedure is still valid and the solution has the form of (3.4), with the only modification of $\mathbf{I}'_p = \{2, 3\}$ and $N'_p = \left\|\mathbf{I}'_p\right\| = 2$. For any quasi-symmetric B-E auction, we repeatedly compute the distribution functions based on equation (3.4) for the modified set $\mathbf{I}_p$. Note that, the above derivation process has suggested, that the solution for a quasi-symmetric B-E auction is unique.

The N-bidder symmetric solution in Chapter 2 is a special case of (3.4). Similarly, the 2-bidder asymmetric solution in Fabra et al. (2006) is also a special case of this procedure. (Since for any price, at least two players are in the $\mathbf{I}_p$ set, in duopoly, the price ranges of two players must be identical. Thus, the quasi-symmetric capacity condition is not needed and we need to replace $x_k$ by $\hat{x}_k = \min\{d, x_k\}$ for $k = 1, 2$.)

*Numerical study for general B-E auctions.* Outside the quasi-symmetric auctions, closed-form solutions usually do not exist. However, inspired by the above procedure, we propose a numerical scheme (defined in detail in Appendix B) to numerically

[3] $\bar{R}_i(p) = ER_i$ results from the following observations established in Chapter 2: (a) $m_i(p) > 0$ implies $\bar{R}_i(p) = ER_i$; (b) $F_i(p) > F_i(p')$ for all $p' < p$ implies $\bar{R}_i^-(p) \equiv \lim_{p' \uparrow p} \bar{R}_i(p') = ER_i$; (c) $F_i(p) < F_i(p')$ for all $p' > p$ implies $\bar{R}_i^+(p) \equiv \lim_{p' \downarrow p} \bar{R}_i(p') = ER_i$.

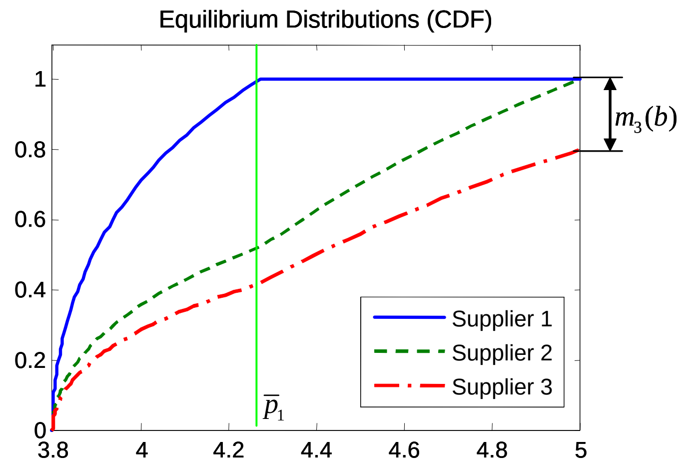



**Figure 3.2.** Solution for 3-Bidder B-E Auction

Example Parameters: $\mathbf{c} = [1, 2, 2]\,, \mathbf{x} = [3, 4, 5]\,, d = 10, b = 5$

compute the equilibrium solutions for general B-E auctions. The algorithm relies on four assumptions

(A1) $F_i(p)$ is continuous in $p \in [\underline{P}, \overline{P})$; partially established in Theorem 3.(2c) and

(A2) $F_i(p)$ is strictly increasing in $(\underline{p}_i, \overline{p}_i)$.

(A3) $\bar{R}_i(p)$ is strictly increasing in $p \in [c_i, \underline{p}_i)$ for $i \in \Pi$.

(A4) There is unique mixed-strategy equilibrium.

These four assumptions hold for any quasi-symmetric case. While we are not aware of any counter-examples, we have not been able to establish that they hold in general. Hence, they are used as assumptions in our numerical procedure.

In Figure 3.3, we first present the solution to a four-player asymmetric bidding game, illustrating the effects of cost and capacity. The equilibrium has the following features: (a) the anchoring supplier 4 has probability mass at $b = 10$; (b) supplier 3's bids also covers the whole (anchoring) interval, but her bids are stochastically smaller than supplier 4's; (c) supplier 2 chooses $\underline{P}$ as her lower bound, but $\overline{p}_2$ is smaller than $\overline{P}$; (d) supplier 4 randomizes over a subset of supplier 2's strategy set with $\underline{p}_2 < \underline{p}_1 < \overline{p}_1 < \overline{p}_2$. Suppliers 1, 2, and 3 have the same cost $c_1 = c_2 = c_3$, but their capacities are ordered $x_1 < x_2 < x_3$, and supplier 4 has $c_4 > c_3$ and $x_3 = x_4$.

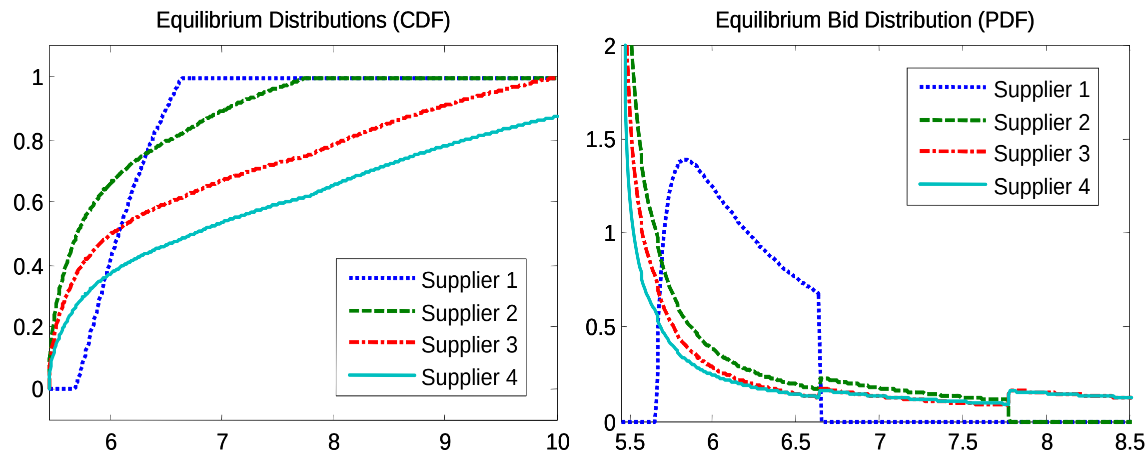



**Figure 3.3.** Solution for 4-Bidder B-E auction

Example Parameters:$\mathbf{c} = [2, 2, 2, 3]\,, b = 10, \mathbf{x} = [0.6, 3, 4, 4]\,, d = 9$

Our example illustrates the impacts of capacity on price formation – when costs are similar, capacities determine the ranges of price randomization. Particularly, if $c_k = c_l$ and $x_k < x_l$, then $[\underline{p}_k, \overline{p}_k]$ is *nested* within $[\underline{p}_l, \overline{p}_l]$. For example, suppliers 1, 2, and 3 have identical costs and ordered capacities, so their bidding intervals satisfy $\underline{p}_3 \leq \underline{p}_2 < \underline{p}_1 < \overline{p}_1 < \overline{p}_2 < \overline{p}_3$. This observation is different from an observation reported in Kreps and Scheinkman (1983). In their analysis of B-E competition of duopoly with the same cost, the player with lower capacity chooses a *more aggressive* pricing strategy by bidding stochastically lower. Their interpretation is that the low-capacity party has a higher risk to be outbid, because serving the residual demand means a much smaller market share than being called first. By bidding lower, the low-capacity player reduces the chance to be outbid. Note that the game in Kreps and Scheinkman (1983) assumes elastic demand $d(p)$, but the same behavior exist and the same lessons apply when demand is inelastic, see Fabra et al. (2006). However, as indicated in Figure 3.3, the claim does not hold when there are more than two bidders. In general, only nested pricing structure can be claimed. In the special case of two players, due to choosing the same pricing interval, and the high-capacity supplier possibly arranging a probability mass at the price cap $b$, her bids are stochastically higher.

The intuition in Kreps and Scheinkman (1983) about low-capacity supplier's vulnerability to be outbid remains valid, but this behavioral description is a part of a bigger picture. When several players have different capacities, the high-capacity parties anchor the competition by setting the price interval. A low-capacity supplier has a smaller stake in the game and behaves like a price taker, selecting the optimal price range that maximizes her payoff. In fact, our numerical algorithm reflects such intuition, when locating the lower pricing bound $\underline{p}_i$ if $\underline{p}_i > \underline{P}$. For example, for B-E auction illustrated in Figure 3.3, given $\{F_2, F_3, F_4\}$ determined in $\underline{P}$'s right neighborhood, $\underline{p}_1 = \arg\max_p \bar{R}_1(p, F_2(p), F_3(p), F_4(p))$. Of course, player 1's active bidding influences high-capacity players' bidding strategy in $[\underline{p}_1, \overline{p}_1]$ at equilibrium. These are reflected in Figure 3.3(b), where pdf $\{f_2, f_3, f_4\}$ lose continuity at $\underline{p}_1$ and $\overline{p}_1$.

## 3.5 An Extension

*Demand Elasticity.* In the preceding sections, we have characterized the equilibrium structure for games with deterministic inelastic demand. We investigate here whether the same properties hold for more general demand. The following theorem summarizes the equilibrium structure for general demand functions.

**Theorem 4** *Suppose demand function $d(p)$ is non-increasing and concave. The following hold:*

*(i) (**Pure-Strategy Equilibrium**) Define*

$$P^* := \begin{cases} \underset{p\in[c_1,c_2]}{\arg\max}\{(p-c_1)\,[d(p)\wedge x_1]\} & \text{if } d(c_2) < x_1 \\ c_j & \text{if } \sum_{k=1}^{j-1} x_k < d(c_j) < \sum_{k=1}^{j} x_k \text{ and } j \geq 2 \\ p(\sum_{k=1}^{j} x_k) & \text{if } d(c_{j+1}) \leq \sum_{k=1}^{j} x_k \leq d(c_j) \\ b & \text{if } d(b) \geq \sum_k x_k \end{cases} \tag{3.5}$$

*where $p(\cdot)$ is the inverse demand function. A unique normalized pure-strategy equilibrium exists with $p_i^* = P^* \vee c_i$, if and only if $P^* = \arg\max_{p\geq c_i}\{S_i(p) := (p-c_i)[d(p) - \sum_{k\neq i} x_k \delta_{(c_k<p)}]^+\}$ for all $i$ with $c_i \leq P^*$.*

*(ii) (**Properties of Mixed-Strategy Equilibrium**) A mixed-strategy equilibrium satisfy (ii-a) $i \in \Pi := \{k : ER_k > 0\}$ if and only if $c_i < \overline{P}$; (ii-b) $\underline{p}_i = \underline{P}$ implies*

*$ER_i = (\underline{P} - c_i)[x_i \wedge d(\underline{P})]$; (ii-c) $i \in \Pi$ and $\sum_{k \in \Pi \setminus \{i\}} x_k \le d(\underline{P})$ implies $\underline{p}_i = \underline{P}$; (ii-d) $\underline{P} > P^*$; (ii-e) at most one profitable supplier has $m_i(\overline{P}) > 0$.*

*(iii) (**Anchoring Supplier**) At a mixed-strategy equilibrium, these exists a profitable supplier $i_A$ such that $\overline{p}_{i_A} = \overline{P}$ and $m_{i_A}(\overline{P}) \ge m_j(\overline{P})$ for all $j \in \Pi \setminus \{i_A\}$. Supplier $i_A$ satisfies (iii-a) $ER_{i_A} = S_{i_A}(\overline{P})$; (iii-b) $c_{i_A} \le P^*$; (iii-c) $\underline{p}_{i_A} = \underline{P} = \frac{ER_{i_A}}{d(\underline{P}) \wedge x_{i_A}} + c_{i_A}$; (iii-d) $\overline{P} = \arg\max\{S_{i_A}(p) : p \in (P^*, b]\}$.*

*(iii) (**Pricing Range**) The pricing bounds of a mixed-strategy equilibrium can be determined as follows. For all $i \in \Omega := \{i : c_i \le P^*$ and $\sum_{k \ne i} x_k \delta_{(c_k < P^*)} < d(P^*)\}$, define $\bar{R}_i^T := \max\{S_i(p) : p > P^*\}$, $\overline{P}_i^T := \min\{\arg\max\{S_i(p) : p \in (P^*, b]\}\}$, and $\underline{P}_i^T \in [P^*, \overline{P}_i^T]$ as the unique solution $p \in [P^*, \overline{P}_i^T]$ to $(p - c_i)[x_i \wedge d(p)] = \bar{R}_i^T$. The pricing bounds are $\underline{P} = \max_{i \in \Omega}\left\{\underline{P}_i^T\right\}$ and $\overline{P} = \min_{i \in \Omega}\{\overline{P}_i^T : \underline{P}_i^T = \underline{P}\}$.*

Note that, the regularity imposed on the demand function implies that $d(p)$ is either a constant or strictly decreasing in $p$, which implies existence of a reverse function $p(d)$. As the analysis of decreasing-demand case is very similar to our previous results, we only explain the equation (3.5). As illustrated in Figure 3.4, vertical lines for demand are replaced by decreasing curves $p(d)$. $P^*$ corresponds to the point where reverse demand function $p(d)$ crosses the marginal cost curve, except when $c_1 < c_2$ and $d(c_2) < x_1$. For this exceptional case, supplier 1 will not choose any price above $c_2$ to avoid activating more capacities than the demand. For price below $c_2$, supplier 1 prices like a monopoly so $p_1^* = P^*$ must be a profit maximizer within $[c_1, c_2]$. The proof of Theorem 4 for elastic demand-case largely repeats the arguments for inelastic-demand case, with $d$ replaced by $d(\cdot)$. Technical difficulties are added because downward-price elasticity makes it less obvious how price adjustment changes one's payoff and concavity of the demand function is necessary for the proof to hold. A complete proof is presented in Appendix C.

Theorem 4 can be applied to derive the mixed-strategy equilibrium. Similar to the inelastic demand case, if no supplier chooses a probability mass at any price $p \in [\underline{P}, \overline{P})$ and all active players have strictly increasing $F_k$ in $p \in [\underline{p}_k, \overline{p}_k]$, then the equilibrium distributions can be computed progressively from $\underline{P}$ to $\overline{P}$. For special cases, like duopoly and symmetric oligopoly, this approach yields closed-form solutions.

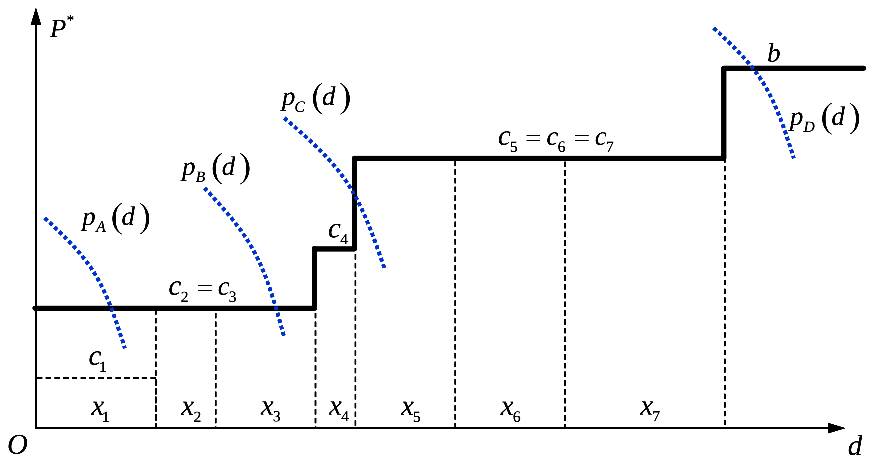



**Figure 3.4.** Derivation of $P^*$ under Different Demand Functions

The duopoly solution is presented in Kreps and Scheinkman (1983) and symmetric solution in Vives (1986). In fact, with $d = d(p)$, where $d(p)$ is concave, (3.4), when applied to specific settings, describes the closed-form solutions in the following four papers Kreps and Scheinkman (1983), Vives (1986), Fabra et al. (2006), and Chapter 2. In this spirit, it unifies (generalizes) the previously known results. Due to its significant generality, we refer to equation (3.4) as the *canonical* solution to B-E auctions/games.

# CHAPTER 4

# Unit-Price Procurement Auctions with Asymmetric Information about Costs and Capacities

## 4.1 Introduction

*Motivation.* Share auctions are widely used in public and private sectors. Their behavior is typically analyzed using continuous bidding functions. We follow an alternative approach, using unit-price bids. Within unit-price auctions we are able to provide structural description of the resulting equilibria.

Share auctions assume that the good is perfectly divisible and that the bidders are not forced to bid one unit each (as in single-item auctions) or a multiple of that unit (as in multi-unit auctions). Two significant applications of share auctions are electricity/procurement auctions and treasury auctions. Electricity auctions have all elements that are critical in our model: asymmetric information, stable, but uncertain costs, and limited capacity. Treasury auctions have some characteristics of our model. (They are, however, heavily studied in literature.)

- *Electricity (Reverse/Procurement) Auctions* Resulting from the deregulation of energy industry in Europe and the United States, major wholesale-electricity markets were re-organized and are currently structured around procurement auctions. Two different formats are used in United Kingdom and United States, discriminatory and uniform, respectively. The most important features of the electricity auctions are (a) stable marginal production costs and (b) rigid capacity for a specific generation unit, both cost structure and available capacity are, however, not known to other bidders until much later time (actual bids are published with a delay, usually 6 months, and they provide information about capacity and partial information about costs).

- *Treasury Auctions* They are probably the most influential application of share auctions, due to their great scale and significant impact on the economy (influencing the whole bond market). While electricity auctions are reverse one, treasury auctions are normal and thus the role of cost is played by individual valuation (which is uncertain). Also, individual budgets in treasury auctions act similarly to capacity in reverse auctions.

*Analytical Challenge.* Two modeling strategies are adopted to analyze share auctions, continuous-bidding schedule and unit-price bids. Each of these has certain strengths, but also certain limitations due to the need for analytical tractability, while preserving the important characteristics of share auctions.

The first strategy assumes that the bidding schedule is a continuous function of quantity and price (Wilson, 1979, Back & Zender 1993, and Wang & Zender, 2002). The major drawback is that in practice the number of bids is limited and even with limited number of bids, bidders usually do not place the maximum number. Empirical evidence in treasury auctions (Bikhchandani & Huang, 1993) and electricity auctions Chapter 2 indicates that most auction participants use very small number of price-quantity pairs (mostly one or two). Another limitation of this modeling strategy is that each supplier's is assumed to be able to satisfy the total demand, that is, there is no capacity limits for suppliers' bids. It is particularly a poor fit for electricity auctions, where capacity constraint seems to be an important driver of bidding strategy.

While continuous bidding strategy assumes that bidders can place (unrealistically) many bids, the alternative modeling strategy (FFH 2007 and Chapter 2) moves in the opposite direction. It assumes that each supplier submits only one bid consisting of single production price for its capacity. This approach is suitable to highlight the influences of suppliers' capacities on the market outcomes. While capacity is a natural part of the papers in this literature stream, this work is the first one to study the impacts of asymmetric information (capacity and cost) in unit-price auctions.

*Main Contributions.* This paper extends the unit-price auctions with symmetric bidders (Chapter 2) by allowing asymmetric information about suppliers' production costs and capacities. It is well known that analyzing a game with asymmetric information in multiple dimensions is technically challenging. Consequently, we present two separate models, one where cost and one where capacity is privately known, while the other component has a deterministic value. Also, as typically done in auction literature, we restrict our solution search to separating Bayesian equilibria with symmetric monotone bidding strategies.

For auctions with asymmetric information about suppliers' costs, we derive the close-form bidding strategies for discriminatory and uniform auctions. The solutions are governed by two independent levers, characterizing each type of the suppliers. We label the first one as market power. In this paper, market power is interpreted as the marginal contribution of a supplier's capacity in satisfying the demand. The influence of market power is thoroughly studied in symmetric information setting by Chapter 2 and can be represented by system utilization (demand/total capacity) and number of suppliers. The second lever is uncertainty about the supplier's cost, which is private information. In a monotone separating equilibrium, a supplier with lower production cost obtains an information rent from the auctioneer. The impacts of information asymmetry are well studied in classic single-item auctions with private valuations.

Market power describes the bidding range. When the utilization (demand divided by sum of all capacities is above a critical threshold of $(N-1)/N$, the range reaches the price cap. For all other utilizations, the range spans up to the highest possible cost (same for all utilizations). Within each of these two cases, the bidding strategy changes as a function of both realized cost value (the type of a supplier) and market power. The bids in discriminatory auctions are increasing in the system utilization and so does the lower bound of equilibrium bids. For uniform auctions, the supplier with lowest cost type always bids true value, and equilibrium bidding strategy is increasing in system utilization in regions below and above the critical threshold $(N-1)/N$, but the monotonicity fails at that point. In this more general situation (limited capacity and uncertain cost), we show that despite the presence of joint impacts of market power and information rent, a generalized revenue equivalence between the discriminatory and uniform auctions continues to hold. This unifies several well-known equivalence outcomes.

Our paper is first to model asymmetric information about suppliers' capacities in auction literature. Compared with auctions with publicly unknown costs, the analysis is considerably more complicated. The paper provides complete analysis for two-bidder cases and derives the monotone equilibrium solutions for N-bidder cases. For discriminatory auctions, monotone equilibrium always exists, where the equilibrium bid is decreasing in a supplier's capacity, while its expected revenue is increasing. For uniform auctions, monotone equilibrium does not always exist. Without extending the analysis to non-monotone bidding strategies or mixed strategies, we restrict our focus on the cases where monotone separating equilibrium prevails. Similarly to the discriminatory counterpart, the equilibrium has decreasing bidding strategies and increasing expected payoffs, but the two auctions does not yield the same expected payment for the auctioneer. Numerical studies indicate that uniform auctions usually yields higher payoffs for suppliers, but the magnitude is not significant.

One concern for unit-price auction models is about the assumption that suppliers always fully commit their capacities. This is not an issue for discriminatory auctions since a supplier's payoff increases in its bidding quantity, in other words, bidding full capacity is incentive compatible. However, for uniform auctions, a supplier's capacity reduction may result in higher market clearing price and possibly increases its payoff. The analytical discussion about capacity withholding is very limited and whether the above scenario could actually happen remains unknown. Interestingly, our finding that a uniform auction with monotone equilibrium bids has increasing expected payoffs in a supplier's actual capacity, implies that capacity withholding may not be incentive compatible for uniform auctions. In other words, the close-form bidding strategies, if solving a uniform auction, are also an equilibrium solution when decision space is expanded to two dimensions of price and quantity. This is an important milestone in understanding the nature of capacity withholding. The discussion about uniform auctions is bounded by the fact that monotone separating equilibrium may not exist when demand is relatively low.

The remainder of the paper is organized as follows. Next section reviews the related literature and Section 3 presents the model of unit-price auctions. In Section 4 we study the cases where suppliers' costs are private information and establish the generalized revenue (payment) equivalence outcome. Section 4 discusses the models when the asymmetric information is about capacity, 2-bidder cases are discussed in great details and N-bidder game is heuristically solved. Numerical studies are conducted to investigate important issues such as revenue comparison and capacity withholding. Section 5 concludes the paper and discusses its practical and policy implications.

## 4.2 Literature Review

This paper relates to several groups of works in economics and management science literature. We summarize them below and highlight the primary difference with our paper.

*Unit-Price Procurement Auction with Complete Information.* The most closely related works to this paper are FFH(2007) and Chapter 2, which first use the unit-price procurement auction to model wholesale electricity markets. FFH focuses on the case of two asymmetric bidders where uniform auction have multiple equilibria including both pure-strategy and mixed-strategy ones, but discriminatory auction has unique mixed-strategy solution. FFH argues that uniform auctions yield higher expected payment for the auctioneer and is then an inferior design, based on their selection of the pure-strategy equilibria in uniform auctions. Chapter 2 focuses on oligopoly scenario where the unique symmetric mixed-strategy equilibrium is the natural focal point among the equilibrium family for uniform auctions. Chapter 2 shows that symmetric equilibrium has the same expected payment as the unique equilibrium in a discriminatory auction, but yields higher price volatility. Therefore, Chapter 2 agrees with FFH's auction selection but differs in that the dominance of discriminatory auctions is only in second order. Both FFH and Chapter 2 assume symmetric information, stripping off the cost and capacity uncertainty among the suppliers. This paper can be viewed as an extension of Chapter 2's oligopoly model by exploring the impacts of information asymmetry among bidders.

*Single-item and Multi-unit Auctions with Private Values.* When asymmetric information is about cost, our unit-price share auction model is closely related to the classic private-value auction literature (sees Krishna 2002 for a comprehensive summary of the auction theories). In fact, when demand is equal to individual capacity times a positive integer, our model reduces to multi-unit auctions with single-unit bids. Furthermore, when demand is less than a supplier's capacity, the discriminatory and uniform auctions both reduces to single-item first-price auction. Therefore, information rent has a similar bite in auctioneer's profit as as in private-value auctions. A stark distinction takes place when the system utilization crosses the critical threshold (N-1)/N identified in Chapter 2. The equilibrium is jointly controlled by supplier's sizable power and information advantage, where the first impact cannot be seen in classic auction literature. Our generalized revenue(payment) equivalence marries the famous revenue equivalence outcome established by Vickrey (1961) and Chapter 2's equivalence outcome due to market power. Bidding capacity (demand size in normal auctions) being uncertain as in our second model, is only possible in share auctions, and therefore not discussed within this body of literature. Pekeč and Tsetlin (2008) address this issue by considering uncertain number of bidders in a multi-unit single-bid auction. It is shown that, when the magnitude of this uncertainty is significant, the discriminatory auction yields higher revenues to the auctioneer and therefore dominates the uniform auction. This dominance result is consistent to the numerical observations in our unit-price auctions with uncertain capacity. However, in their model, bidders have symmetric information among uncertainty, while bidder's capacity is private information in our model.

*Share Auction with Continuous Bidding Curve.* The literature of share auctions with continuous bidding functions differs from classic unit-price auction models in expanding bidders' decision space. Wilson (1979) pioneers to revisit the classic single-item auction under the continuous bidding assumption. It is shown that differential bids largely enhance the bidders' pricing power and result in inferior outcomes to the auctioneer. This observation holds for both discriminatory and uniform designs and the revenue of the two auction formats can be equivalent under specific equilibrium selection. Back and Zender (1993) extends Wilson (1979) by demonstrating how a uniform auction can yield continuum of equilibria, most of which leads to lower expected revenue the auctioneer than the discriminatory auctions. A more complete characterization of equilibrium bidding strategy under continuous bidding assumption is provided later in Wang and Zender (2002).

In similar spirit to Wilson (1979), Klemperer and Meyer (KM) (1989) presents a model named as "supply function equilibrium" to model to quantity/price competition of oligopoly pricing game. The model can be viewed as special uniform auction with price elastic demand and it is shown that continuous bidding curve can facilitate ex post optimality for bidders as an uncertainty scrutinizer. In contrast to these models, our unit-price model restricts the bidder's decision space and therefore limits their pricing power. In reality, most share auctions allow limited number of bids (equivalent to step-wise function with limited number of jumps) and empirically, most bidders choose to use one or two bidding pairs (Bikhchandani and Huang, 1993 and Chapter 2). Therefore, unit-price model can be an interesting complement to continuous bidding model in studying share auctions, with former restricting bidders' action space while the latter relaxes it. A recent paper by Holmberg (2007) uses KM's supply function equilibrium to model uniform auctions in electricity markets by considering the impacts of capacity constraints and constant marginal costs. The setting of their game is virtually identical to the uniform auctions in FFH (2007) and Chapter 2 while the solution differs drastically. This paper differs from Holmberg further in considering the impacts of asymmetric information on costs and capacities.

*Procurement Auction in Private Sector.* The increasing usage of auctions in public and business procurement generates many interesting research questions. Ewarhart and Fieseler (2003) studies procurement auctions of unit-price contracts in the construction tendering context. The auctioneer looks for one contractor among many bidders who compete in offering a bundle of unit-prices for several construction input factors. As all bidders have sufficient capacities, the primary attention is about how an efficient scoring rule is designed to rank the unit-price bundles. Chen (2007) studies the procurement auction design for a retailer (modeled as a news vendor) who faces random demand and a pool of suppliers with private information about their costs. The optimal design is to offer a quantity-payment menu and ask the suppliers to bid the whole contract. The primary distinction of our model is to search for a set of capacitated suppliers, while capacity is not an issue for their settings.

*Capacity Withholding in Uniform Auctions.* One concern about uniform auctions is that suppliers may withhold some of their capacity so that market clearing price is increased. If the increasing of profit margin dominates the adverse effect of reduced sale quantity, a supplier will choose to do so. Ausubel and Cramton (2002) first establish this result in multi-unit auction setting, showing that a bidder will shade the bids for second and latter units, but bids truthfully for the first unit (as in second price auction). However, this result may not be a theoretical support for existence of capacity withholding, since shading prices from the true value may also take place in our unit-price auction models, and it remains unclear whether it is in a bidder's interest to cut off certain portion of its capacity. Lave and Perekhodtsev (2001) establish an equilibrium where a bidder does not bid in the whole cost curve. However, this phenomenon is largely driven by their assumption that bidders have to price every unit at its actual marginal cost and the only decision is to whether the whole curve shall be submitted. Dechnenaux and Kovenock (2007) use a similar model as FFH (2007) and HLK (2008). The distinction is that demand is price elastic and they consider the environment of repeated games. They show that a colluded equilibrium exists where all supplier withhold certain portion of their capacities under the threat of being punished in a poor equilibrium if they defects from the collusion. Withholding will not happen in one-shot game. Our model of unit-price auction with privately known capacity information provides some insights about whether capacity withholding is in a supplier's own interest, particularly when his capacity is only privately known.

*Empirical Studies* The debate about discriminatory and uniform auctions is largely motivated by the treasury and electricity auctions where both auction formats are implemented. The empirical studies about the treasury auctions are summarized in Binmore and Swierzbinski (2000). We note that the empirical evidence have also presented a foggy picture about the ranking of the two auctions. For examples, Simon (1994) estimates that the switch from DA to UA in 1970's resulted in large loss in the revenue for the US Treasury; Nyborg & Sundaresan (1996) estimate the revenue changes due to the switch may range from small losses to moderate gains; and Malvey and Archibald (1998) find the switch produce small gains of revenue for the Treasury. Empirical comparison of the two auction formats in electricity markets can be conducted only in UK and the findings (Evans & Green 2002, and Fabra & Toro 2003) are too controversial, since there are other major structural changes of the markets that happen concurrently with the switch from uniform auction to discriminatory auctions.

## 4.3 The Model

*System Setup* Consider a procurement auction with $N$ potential suppliers. The auctioneer faces exogenously determined demand $\xi$ and suppliers compete in offering their capacities through submitting unit prices. Denote supplier $i$'s cost-capacity profile as $\{c_i, k_i\}$ and his bid price as $p_i$ which cannot be higher than the price cap $B$ set by the auctioneer, i.e., $p_i \in [0, B]$ for $i = 1, 2, \ldots, N$.

*Sequence of Events and Allocation Scheme* During the auction, all suppliers independently submit (sealed) bids $\{p_i\}_{i=1}^N$ to the auctioneer. The lowest-bid supplier is admitted first. If her capacity cannot cover the demand, the auctioneer moves to the next lowest-bid supplier, and so on, until the demand is filled or no capacity is left. We assume ties are broken by first granting orders to the efficient suppliers (those with lower production costs), and among iso-cost suppliers, a supplier with higher capacity is given higher allocation priority. When the price-tie is formed among multiple suppliers with the same cost and the same capacity, the "pie" is split evenly.[1] Denote the vector of supplier $i$'s competing bids as $\mathbf{p}_{-i} \equiv (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N)$ and supplier $i$ 's realized sales as $z_i(p_i, \mathbf{p}_{-i}) = k_i r_i(p_i, \mathbf{p}_{-i})$ where $r_i$ is the fraction of her bid quantity accepted by the auctioneer. The above assumptions imply

$$z_i(p_i, \mathbf{p}_{-i}) = \min\{k_i, [\xi - \frac{\sum_{n \neq i} k_n(\delta_{(p_n<p_i)} + \delta_{(p_n=p_i, c_n<c_i)} + \delta_{(p_n=p_i, c_n=c_i, k_n>k_i)})}{1 + \sum_{n \neq i} \delta_{(p_n=p_i, c_n=c_i, k_n=k_i)}}]^+\}, \tag{4.1}$$

where $\delta_{(A)} = 1$ if $A$ is true and 0 otherwise.

*Payment Schemes and Value Representation* Two payment schemes are considered. In a discriminatory auction (DA, hereafter), an admitted supplier is paid at her bid price, while in a uniform auction (UA, hereafter), all of the selected suppliers are paid at the same uniform price equal to the highest bid selected. We use superscripts (or subscripts when convenient) $d$ and $u$ to denote the two auction formats. Under these two schemes, supplier $i$'s profit is written as

$$\begin{aligned} \text{(a)} R_i^d(p_i, \mathbf{p}_{-i}) &= (p_i - c_i) z_i(p_i, \mathbf{p}_{-i}), \qquad (4.2) \\ \text{(b)} R_i^u(p_i, \mathbf{p}_{-i}) &= (\max_n \{p_n : z_n(p_n, \mathbf{p}_{-n}) > 0\} - c_i) z_i(p_i, \mathbf{p}_{-i}). \end{aligned}$$

*Information Structure* Each supplier's cost-capacity profile $\{c_i, k_i\}$ is private information and all suppliers share a common prior distribution about other suppliers' cost-capacity profiles, denoted by independent identical probability distributions $\mathcal{P}(c_n, k_n)$ for $n = 1, 2, \ldots, N$. It is well known that, even for single item auctions, when asymmetric information has two independent dimensions, auction models become analytically intractable. Therefore, we will analyze the impacts of two types of asymmetric information (cost and capacity) separately. In other words, we will consider two families of auctions, the ones with uncertain cost and the ones with uncertain capacities.

*Model Label* We focus on games with symmetric bidders. For ease of exposition, we use the following four-letter labels to denote different models (D/U)A(C/K)-$n$,

[1] The tie-breaking rule is specified for analytical simplicity, so that discussion of $\varepsilon$-equilibrium is avoided.

where (D/U) denotes discriminatory/uniform, (C/K) denotes uncertain cost/capacity, and $n$ the number of bidders. For examples, DAC-N denotes N-bidder discriminatory auction with uncertain costs, and UAK-2 denotes 2-bidder (duopoly) uniform auction with uncertain capacities.

## 4.4 Auction with Uncertain Cost

*Additional Notation* In this section, we suppose all suppliers have constant capacity $k > 0$ and their costs $\{c_i\}_{i=1}^N$ have independent identical distributions, with *continuous and strictly increasing* c.d.f. $G(c)$ in $c = [\underline{c}, \bar{c}]$ . The information about demand $\xi$, capacity $k$, and cost distribution $G(c)$ is a common knowledge. Our analysis focuses on symmetric Perfect Bayesian Equilibrium (PBE) $p = \beta(c)$. One assumption that we use in our heuristic derivation of the PBE is that $\beta(c)$ is *strictly increasing and differentiable* in $c \in [\underline{c}, \bar{c}]$. Denote $\beta^{-1}(\cdot)$ as the inverse function of $\beta$.

The derivation relies heavily on order statistics. Particularly, we analyze bidder $i$'s decision rule, given that his $N-1$ competitors follow a symmetric bidding strategy that we are looking for. For the $N-1$ i.i.d. competing bids $\mathbf{p}_{-i}$, denote $b_m$ as the $m$-th lowest one. Accordingly, denote $c_m$ as $m$-th lowest cost among the $\mathbf{c}_{-i}$ and $L_m$ as its cdf,

$$\begin{aligned} L_m(c) &\equiv \Pr\{c_m < c\} = \sum_{j=m}^{N-1} \Pr\{c_m < c \le c_{m+1}\} \qquad (4.3) \\ &= \sum_{j=m}^{N-1} \binom{N-1}{j} G(c)^j \bar{G}(c)^{N-1-j} \quad \text{for } m = 1, 2, \ldots, N-1. \end{aligned}$$

### 4.4.1 Discriminatory Auction (DAC-N)

*Derivation.* In this section, we first derive the symmetric bidding strategy $p = \beta(c)$ and then show it is a PBE.

*Case for $\xi \ge Nk$.* As supplier $i$'s payoff $(p - c_i)k$ is independent of other players' bids, it is easy to see that $\beta(c) = B$ is the solution.

*Case for $0 < \xi < Nk$.* Consider the decision of supplier $i$ whose cost is $c$. Suppose supplier $m \neq i$ follows $\beta$ which is *strictly increasing and differentiable* with choice of price $p$, supplier $i'$s expected payoff becomes

$$R(p) = (p-c)z(G(\beta^{-1}(p))) \text{ where} z(G) = \sum_{n=0}^{N-1} \binom{N-1}{n} G^n \bar{G}^{N-1-n} \left[k \wedge (\xi - nk)^+\right]. \tag{4.4}$$

The first order condition is

$$\frac{dR(p)}{dp} = z(G(\beta^{-1}(p))) + (p-c)\frac{dz(G)}{dG}\frac{dG}{dc}(\beta^{-1}(p))\frac{1}{\beta'(\beta(p))} = 0.$$

At a symmetric equilibrium, $p = \beta(c)$, and equivalently, $c = \beta^{-1}(p)$. So the FOC reads

$$z(G(c)) + (\beta(c) - c)\frac{dz(G(c))}{dc}\frac{1}{\beta'(c)} = 0, \text{ i.e.,} \frac{d}{dc}\left[(\beta(c) - c)z(G(c))\right] = -z(G(c)).$$

It leads to the following solution,

$$(\beta(c) - c)z(G(c)) = C_D + \int_c^{\bar{c}} z(G(\theta))d\theta, \tag{4.5}$$

where $C_D = [\beta(\bar{c}) - \bar{c}]\, z(G(\bar{c})) = [\beta(\bar{c}) - \bar{c}]\, z(1) = [\beta(\bar{c}) - \bar{c}]\, [k \wedge (\xi - Nk + k)^+]$. If $\xi \leq (N-1)k$, $C_D = 0$. If $\xi > (N-1)k$ and supplier $i$'s type is $\bar{c}$, then she is the last to be selected by the system. Therefore, the optimal choice is to price at $B$. Thus, $\beta(\bar{c}) = B$ and $C_D = (B - \bar{c})\,[k \wedge (\xi - Nk + k)]$. While above we show the necessary conditions for symmetric equilibrium, we now show that indeed it is an equilibrium.

**Proposition 4** *The symmetric equilibrium bidding strategy for model DAC-N is*

$$\begin{aligned} \beta_D(c) &= c + \frac{C_D + \int_c^{\bar{c}} z(G(\theta))d\theta}{z(G(c))} \\ \textit{where } C_D &= \begin{cases} 0 & \textit{for } \xi \leq (N-1)k \\ (B - \bar{c})\,[k \wedge (\xi - Nk + k)] & \textit{for } \xi > (N-1)k \end{cases} \end{aligned} \tag{4.6}$$

**Proof.** When $\xi \geq Nk$, we have $z(G(c)) = k$ and $C_P = (B - c)k$, so $\beta(c) = B$. It is clearly the optimal decision.

When $\xi < Nk$, we first show that $\beta(c)$ is an increasing function. From (4.6), we have

$$\frac{d\beta(c)}{dc} = -\frac{C_P + \int_c^{\bar{c}} z(G(\theta))d\theta}{z^2(G(c))} \cdot \frac{dG(c)}{dc} \cdot \frac{dz(G(c))}{dG}.$$

As the first two terms are positive, in order to show $\frac{d\beta(c)}{dc} \geq 0$, we only need to prove $\frac{dz(G)}{dG} \leq 0$. Denoting $z_n = \min\{k, (\xi - nk)^+\}$ we have $z(G) = \sum_{n=0}^{N-1} \binom{N-1}{n} G^n \bar{G}^{N-1-n} z_n$ and

$$\begin{aligned}
\frac{dz(G)}{dG} &= \sum_{n=1}^{N-1} \frac{(N-1)!}{(n-1)!(N-n-1)!} G^{n-1} \bar{G}^{N-n-1} z_n - \sum_{n=0}^{N-2} \frac{(N-1)!}{n!(N-n-2)!} G^n \bar{G}^{N-n-2} z_n \\
&= \sum_{n=1}^{N-1} \frac{(N-1)!}{(n-1)!(N-n-1)!} G^{n-1} \bar{G}^{N-n-1} (z_n - z_{n-1}). \\
[\text{by } n' &\equiv n+1 \text{ in the 2nd term.}]
\end{aligned}$$

Clearly the difference $z_n - z_{n-1} \leq 0$ for all $n$. Since $\sum_{n=1}^{N-1}(z_n - z_{n-1}) = z_{N-1} - z_0 = [\xi - (N-1)k]^+ - (k \wedge \xi) < 0$, at least one difference is negative, implying $\frac{dz(G)}{dG} < 0$.

Next, suppose that all suppliers other than $i$ follow strategy $\beta$, given in (4.6). We show that it is optimal for supplier $i$ to follow $\beta$. First notice that it is not optimal for supplier $i$ to bid lower than $\beta(\underline{c})$ because all $p \leq \beta(\underline{c})$ yield an expected sales ratio of 1 almost surely. It is also easy to see that supplier $i$ will not bid above $\beta(\bar{c})$: when $\xi > (N-1)k$, $\beta(\bar{c}) = B$ and higher price is not feasible; when $\xi \leq (N-1)k$, higher price than $\beta(\bar{c}) = \bar{c}$ yields a sales ratio of zero almost surely. Denote $s = \beta^{-1}(p)$. If supplier $i$ defects from price $\beta(c)$ to $p$, her expected profit is

$$R_D(s, c) = (\beta(s) - c)z(G(s)) = -cz(G(s)) + \beta(s)z(G(s)) = R(s, s) + (s - c)z(G(s)).$$

By (4.5), we have $R_D(c, c) = (\beta(c) - c)z(G(c)) = C_P + \int_c^{\bar{c}} z(G(\theta))d\theta$. It then follows

$$\begin{aligned}
R_D(c, c) - R_D(s, c) &= -(s - c)z(G(s)) + R(c, c) - R(s, s) \\
&= (c - s)z(G(s)) + \int_c^{\bar{c}} z(G(\theta))d\theta - \int_s^{\bar{c}} z(G(\theta))d\theta \\
&= \begin{cases} \int_s^c [z(G(s)) - z(G(\theta))]\, d\theta & \text{for } s < c \\ \int_c^s [z(G(\theta)) - z(G(s))]\, d\theta & \text{for } s > c \end{cases}.
\end{aligned}$$

As $G(c)$ is strictly increasing in $c \in [\underline{c}, \bar{c}]$ and $Z(\cdot)$ is strictly decreasing in $G$, we have $R(c, c) > R(s, c)$ for any $s \neq c$. That is, any deviation from $p = \beta(c)$ causes a profit loss for supplier $i$. ■

We delay to show the numerical examples of the equilibrium bidding strategy after driving the solutions for uniform auctions.

### 4.4.2 Uniform Auction (UAC-N)

*Derivation***.** Similarly to the previous section, we first heuristically derive the equilibrium bidding strategy.

If $\xi \geq Nk$, the equilibrium bidding strategy is $\beta(c) = B$.

If $\xi < Nk$, suppose there exists a symmetric monotone equilibrium bidding strategy $\beta(c)$, which is *strictly increasing* and *differentiable* in $c$. For supplier $i$ with cost $c$, when bidding price $p$, her payoff conditional on $\{\mathbf{b}_{-i}, \xi\}$ is

$$\sum_{m=1}^{N} \delta_{(\xi \in ((m-1)k, mk])} \delta_{(b_{m-1} < p \leq b_m)} (p - c) \left[\xi - (m-1)k\right] + \sum_{n=1}^{N-1} \delta_{(\xi \in (nk, (n+1)k])} (b_n - c) k \delta_{(p < b_n)}.$$

For $\xi \in ((m-1)k, mk]$, we have that the $m$-th lowest bidder makes the price. Given that $\beta(\cdot)$ is played by all of supplier $i$'s competitors, his profit, by bidding price $p$, is

$$R(\mathbf{c}^{-i}, p) = (p - c) \delta_{(c_{m-1} < \beta^{(-1)}(p) \leq c_m)} \left[\xi - (m-1)k\right] + (\beta(c_{m-1}) - c) k \delta_{(p \leq \beta(c_{m-1}))}.$$

Taking expectation over all competitors' types, we have

$$\begin{aligned} \bar{R}(p) &= \mathsf{E}_{\mathbf{c}^{-i}} \left[R(\mathbf{c}^{-i}, p)\right] \qquad (4.7) \\ &= (p - c) \Pr\left\{c_{m-1} < \beta^{(-1)}(p) \leq c_m\right\} \left[\xi - (m-1)k\right] \\ &\quad + \mathsf{E}_{c_{m-1}} \left[(\beta(c_{m-1}) - c) k \delta_{(p \leq \beta(c_{m-1}))}\right] \\ &= (p - c) \left[\xi - (m-1)k\right] \left[L_{m-1}(\beta^{(-1)}(p)) - L_m(\beta^{(-1)}(p))\right] \\ &\quad + \int_p^B (v - c) k dL_{m-1}(\beta^{(-1)}(v)), \end{aligned}$$

where the last expression follows (4.3).

Now we have

$$\begin{aligned} \frac{d\bar{R}(p)}{dp} &= (p - c) \left[\xi - (m-1)k\right] \left[\frac{d}{dp} L_{m-1}(\beta^{(-1)}(p)) - \frac{d}{dp} L_m(\beta^{(-1)}(p))\right] \\ &\quad + \left[L_{m-1}(\beta^{(-1)}(p)) - L_m(\beta^{(-1)}(p))\right] \left[\xi - (m-1)k\right] - (p - c) k \frac{d}{dp} L_{m-1}(\beta^{(-1)}(p)) \end{aligned}$$

FOC $\frac{d\bar{R}(p)}{dp} = 0$ reads,

$$\begin{aligned} & L_{m-1}(\beta^{(-1)}(p)) - L_m(\beta^{(-1)}(p)) \\ &= (p - c) \left[\frac{d}{dp} L_m(\beta^{(-1)}(p)) + \frac{mk - \xi}{\xi - (m-1)k} \frac{d}{dp} L_{m-1}(\beta^{(-1)}(p))\right] \\ &= \frac{(p - c) \left[\frac{d}{dc} L_m(\beta^{(-1)}(p)) + \frac{mk - \xi}{\xi - (m-1)k} \frac{d}{dc} L_{m-1}(\beta^{(-1)}(p))\right]}{\beta'(\beta^{(-1)}(p))} \end{aligned}$$

At equilibrium, we must have $\beta^{(-1)}(p) = c$ and $p = \beta(c)$, so we have ODE,

$$\beta'(c)\left[L_{m-1}(c) - L_m(c)\right] = \left[\beta(c) - c\right]\left[\frac{mk-\xi}{\xi-(m-1)k}\frac{d}{dc}L_{m-1}(c) + \frac{d}{dc}L_m(c)\right]. \quad (4.8)$$

Define $\alpha(c) \equiv \beta(c) - c$ and $T(c) \equiv \frac{\frac{mk-\xi}{\xi-(m-1)k}\frac{d}{dc}L_{m-1}(c)+\frac{d}{dc}L_m(c)}{L_{m-1}(c)-L_m(c)}$. From (4.3), we have

$$\begin{aligned}
\frac{d}{dc}L_m(c) &= \frac{d}{dc}\left[\sum_{j=m}^{N-1}\binom{N-1}{j}G(c)^j\bar{G}(c)^{N-1-j}\right] \\
&= g(c)\sum_{j=m}^{N-1}\binom{N-1}{j}\left[jG^{j-1}\bar{G}^{N-1-j} - (N-1-j)G^j\bar{G}^{N-2-j}\right] \\
&= g(c)\left[\sum_{j=m}^{N-1}\frac{(N-1)!G^{j-1}\bar{G}^{N-1-j}}{(j-1)!(N-1-j)!} - \sum_{j=m}^{N-2}\frac{(N-1)!G^j\bar{G}^{N-2-j}}{j!(N-2-j)!}\right] \\
&= (N-1)g(c)\left[\sum_{j=m}^{N-1}\frac{(N-2)!\cdot G^{j-1}\bar{G}^{N-1-j}}{(j-1)!(N-1-j)!} - \sum_{j=m}^{N-2}\frac{(N-2)!\cdot G^j\bar{G}^{N-2-j}}{j!(N-2-j)!}\right] \\
&= (N-1)g(c)\left[\sum_{k=m-1}^{N-2}\frac{(N-2)!}{k!(N-2-k)!}G^k\bar{G}^{N-2-k} - \sum_{j=m}^{N-2}\frac{(N-2)!}{j!(N-2-j)!}G^j\bar{G}^{N-2-j}\right] \\
&= (N-1)\frac{(N-2)!}{(m-1)!(N-1-m)!}G^{m-1}\bar{G}^{N-1-m}\cdot g(c) \\
&= m\binom{N-1}{m}G^{m-1}\bar{G}^{N-1-m}\cdot g(c).
\end{aligned}$$

It follows

$$\begin{aligned}
T(c) &= \frac{\frac{mk-\xi}{\xi-(m-1)k}(m-1)\binom{N-1}{m-1}G^{m-2}(c)\bar{G}^{N-m}(c) + \binom{N-1}{m}G^{m-1}(c)\bar{G}^{N-m-1}(c)}{\binom{N-1}{m-1}G(c)^{m-1}\bar{G}(c)^{N-m}}g(c) \\
&= \frac{(mk-\xi)(m-1)}{\xi-(m-1)k}\frac{g(c)}{G(c)} + (N-m)\frac{g(c)}{\bar{G}(c)}.
\end{aligned}$$

The ODE(4.8) of $\beta(c)$ becomes an ODE of $\alpha(c)$,

$$\alpha'(c) + 1 = T(c)\alpha(c). \quad (4.9)$$

The solution to the first-order homogeneous equation $\alpha'(c) = T(c)\alpha(c)$ is

$$\alpha(c) = \exp(-\int_c^{\bar{c}} T(s)ds).$$

The solution of ODE (4.9) has the form of

$$\alpha(c) = \gamma(c)\exp(-\int_c^{\bar{c}} T(s)ds) = \frac{\gamma(c)}{\exp\int_c^{\bar{c}} T(s)ds}.$$

Substitute it into (4.9), we have

$$\gamma'(c) = -\exp \int_c^{\bar{c}} T(s)ds,$$

then

$$\gamma(c) = \int_c^{\bar{c}} (\exp \int_v^{\bar{c}} T(s)ds)dv + C_U,$$

and

$$\begin{aligned}\alpha(c) &= \int_c^{\bar{c}} \frac{\exp \int_v^{\bar{c}} T(s)ds}{\exp \int_c^{\bar{c}} T(s)ds} dv + \frac{C_U}{\exp \int_c^{\bar{c}} T(s)ds} \\ &= \int_c^{\bar{c}} \frac{dv}{\exp \int_c^v T(s)ds} + \frac{C_U}{\exp \int_c^{\bar{c}} T(s)ds}.\end{aligned}$$

*Boundary condition.* We study the individual rationality condition for type $c = \bar{c}$. The solution above implies $\beta(\bar{c}) = \bar{c} + C_U$. Similar discussion as in the case for DA leads to $C_U = 0$ for $\xi \le (N-1)k$ and $C_U = (B - \bar{c})$ for $\xi > (N-1)k$. Therefore the solution of ( 4.8) is

$$\beta(c) = c + \int_c^{\bar{c}} \frac{dv}{\exp \int_c^v T(s)ds} + \frac{C_U}{\exp \int_c^{\bar{c}} T(v)dv}.$$

**Proposition 5** *The symmetric bidding strategy for UAC-N is,* $\beta(c) = B$ *for* $\xi \ge Nk$*; for* $0 < \xi < Nk$ *and* $m = \left\lceil \frac{\xi}{k} \right\rceil$*,*

$$\begin{aligned}\beta_U(c) &= c + \int_c^{\bar{c}} \frac{dv}{\exp \int_c^v T(s)ds} + \frac{C_U}{\exp \int_c^{\bar{c}} T(v)dv} \qquad (4.10) \\ \textit{where } C_U &= \begin{cases} 0 & \textit{for } m \le N-1 \\ B - \bar{c} & \textit{for } m = N \end{cases} \\ T(c) &= \frac{(mk - \xi)(m-1)}{\xi - (m-1)k} \frac{g(c)}{G(c)} + (N - m)\frac{g(c)}{\bar{G}(c)}\end{aligned}$$

**Proof.** We only discuss the case for $0 < \xi < Nk$.

1. Monotonicity of $\beta(c)$. It is easy to see from (4.8)

$$\beta'(c) = T(c)(\beta(c) - c) = T(c)\alpha(c) = T(c)\left[\int_c^{\bar{c}} \frac{dv}{\exp \int_c^v T(s)ds} + \frac{C_U}{\exp \int_c^{\bar{c}} T(v)dv}\right] > 0.$$

2. Optimality of $\beta(c)$. When bidding strategy (4.10) is played by all bidders, the expected payoff for supplier with cost $c$ is, by (4.7),

$$\begin{aligned}R_U(c) &= (\beta(c) - c)\left[\xi - (m-1)k\right]\left[L_{m-1}(c) - L_m(c)\right] + \int_c^B (\beta(v) - c)kdL_{m-1}(v) \\ R'_U(c) &= \left[\xi - (m-1)k\right]\left\{(\beta'(c) - 1)\left[L_{m-1}(c) - L_m(c)\right] + \alpha(c)(L'_{m-1}(c) - L'_m(c))\right\} \\ &\quad -\alpha(c)kL'_{m-1}(c) - k(1 - L_{m-1}(c)).\end{aligned}$$

It follows

$$
\begin{aligned}
& R'_U(c) + k(1 - L_{m-1}(c)) \\
= \; & [\xi - (m-1)k] \left[ (T\alpha - 1)(L_{m-1} - L_m) + \alpha(L'_{m-1} - L'_m) \right] - \alpha k L'_{m-1} \\
= \; & [\xi - (m-1)k] \left[ \left( \frac{\frac{mk-\xi}{\xi-(m-1)k} L'_{m-1} + L'_m}{L_{m-1} - L_m} \alpha - 1 \right)(L_{m-1} - L_m) + \alpha(L'_{m-1} - L'_m) \right] \\
& - \alpha k L'_{m-1} \\
= \; & [\xi - (m-1)k] \left[ \alpha \frac{mk - \xi}{\xi - (m-1)k} L'_{m-1} + \alpha L'_m - (L_{m-1} - L_m) + \alpha(L'_{m-1} - L'_m) \right] \\
& - \alpha k L'_{m-1} \\
= \; & \alpha(mk - \xi) L'_{m-1} + [\xi - (m-1)k] \left[ L_m - L_{m-1} + \alpha L'_{m-1} \right] - \alpha k L'_{m-1} \\
= \; & \alpha L'_{m-1} [mk - \xi + \xi - (m-1)k - k] + [\xi - (m-1)k] [L_m - L_{m-1}] \\
= \; & [\xi - (m-1)k] [L_m - L_{m-1}] .
\end{aligned}
$$

Consequently,

$$
R'_U(c) = -[\xi - (m-1)k] \binom{N-1}{m-1} G^{m-1} \bar{G}^{N-m} - k \sum_{j=0}^{m-2} \binom{N-1}{j} G^j \bar{G}^{N-1-j}.
$$

Now we consider the proof of Proposition 4, we have

$$
R'_D(c) = \frac{dR_D(c,c)}{dc} = -z(G(c)) = -\sum_{n=0}^{N-1} \binom{N-1}{n} G^n \bar{G}^{N-1-n} z_n = R'_U(c).
$$

Since $R_U(\bar{c}) = R_D(\bar{c}) = (B - \bar{c}) [\xi - (N-1)k]^+ = C_D$, we have

$$
R_U(c) = R_D(c) = C_D + \int_c^{\bar{c}} z(G(\theta)) d\theta.
$$

Similarly to the proof for DA, we define

$$
\begin{aligned}
R_U(s,c) \; = \; & (\beta(s) - c) [\xi - (m-1)k] [L_{m-1}(s) - L_m(s)] + \int_s^{\bar{c}} (\beta(v) - c) k dL_{m-1}(v) \\
= \; & (\beta(s) - c) [\xi - (m-1)k] [L_{m-1}(s) - L_m(s)] + \int_s^{\bar{c}} (\beta(v) - s) k dL_{m-1}(v) \\
& + (s - c) k (1 - L_{m-1}(s)) \\
= \; & R_U(s,s) + (s - c) [(\xi - (m-1)k)(L_{m-1}(s) - L_m(s)) + k(1 - L_{m-1}(s))] \\
= \; & R_U(s,s) + (s - c) z(G(s)) = R_D(s,c).
\end{aligned}
$$

According the proof of Proposition 4, we have $R_U(c,c) - R_U(s,c) = R_D(c,c) - R_D(s,c) < 0$ for all $s \neq c$. ■

The proofs of Propositions 4 and 5 also suggest the following facts.

**Theorem 5** *(Payment Equivalence and Rent Separation) We have*

$$R_U(c) = R_D(c) = (B - \bar{c})\,[\xi - (N-1)k]^+ + \int_c^{\bar{c}} z(G(\theta))d\theta. \tag{4.11}$$

*Interpretation.* Theorem 5 has interesting economic interpretation. The first term denotes the market power rent shared by all types. Chapter 2 establishes the equivalence outcome under symmetric information and both auctions yields the same revenue, equal to the first term. It denotes a supplier's largest payment if he is the last supplier to be dispatched. Since such a rent exists even when the production cost is certain, now it must be shared by all cost types. In a procurement auction, $\bar{c}$ is the most inferior cost type, everyone get at least $(B - \bar{c})\,[\xi - (N-1)k]^+$. The second term of (4.11) denotes the information rent charged by a supplier with type $c$. Auction is by and large an adverse selection mechanism. In a type-revealing equilibrium (here through monotone bidding), the principal has to pay a higher rent to a more favorable type so that it won't be mimicked by a less favorable type. In this case, type $\bar{c}$ collects no information rent and the marginal rent collected by type $c$ is its expected sales $Z(G(c))$.

*Numerical Studies.* Our main contribution on the unit-price auction models with uncertain cost is to explicitly characterize the bidding strategies. Next we present several examples demonstrating how system parameters shape the bidding strategy. In Figure 4.1, we have two panels (columns) of graphs. In panel (a), the bidders' costs are uniformly distributed in interval $[0,1]$, while panel (b) uses a beta distribution with parameters $a = b = 2$. Note that uniform distribution is also a beta distribution with $a = b = 1$. Graphs (a-1) and (b-1) plot the density and cumulative distribution functions. Beta (2,2) is chosen for its bell shape, which is common in most real situations. All auctions tested here also share the following parameters: $N = 3$, $B = 2$, $k = 1$, $\underline{c} = 0$, and $\bar{c} = 1$.

Graphs (a-2) and (b-2) display the equilibrium bidding strategies for discrimi-

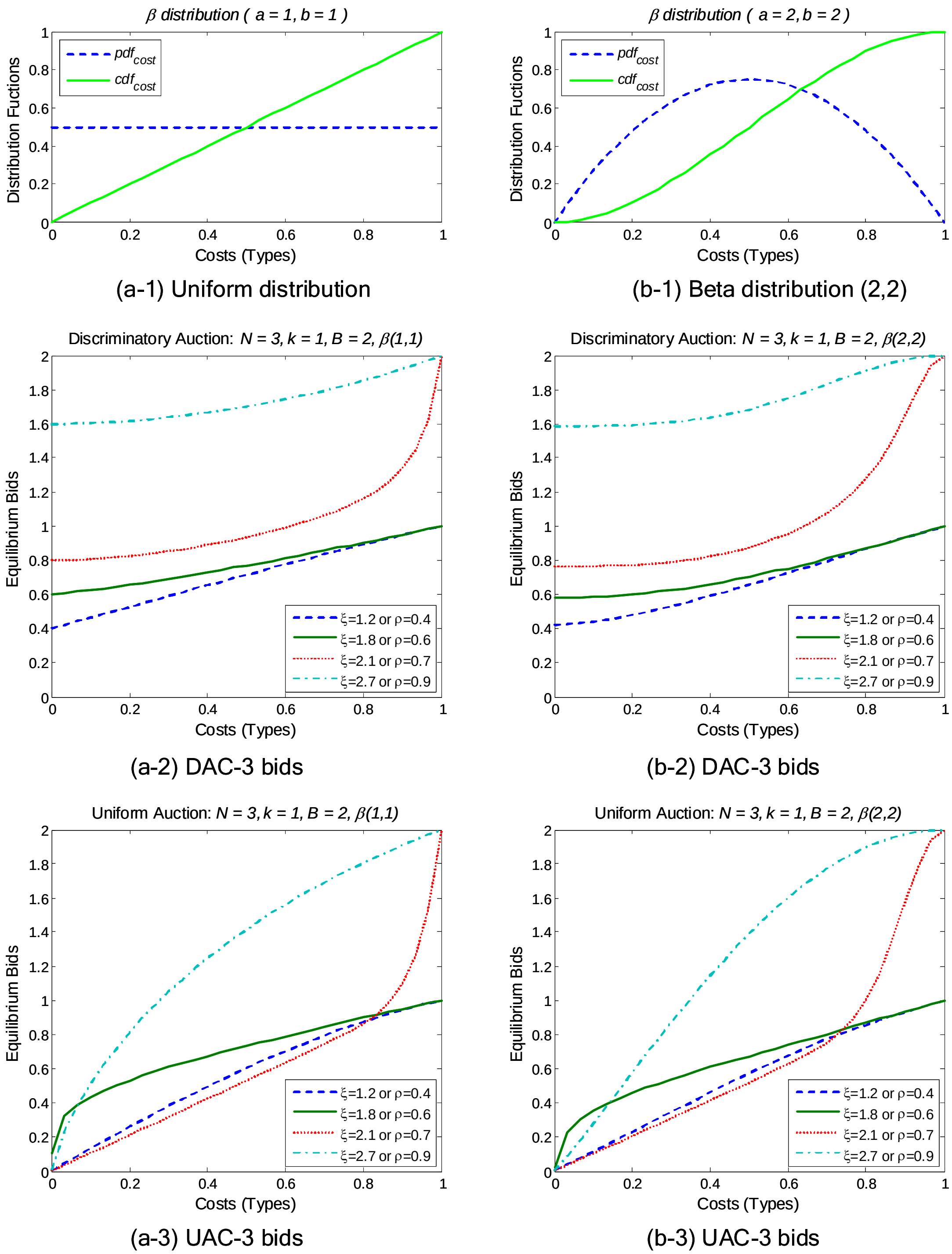



**Figure 4.1.** Equilibrium Bidding Functions under Different Distributions and Utilizations

natory auctions. We consider four demand values of $\xi$ (or equivalently utilizations $\rho = \frac{\xi}{Nk}$). The first observation is that the equilibrium bids are increasing in demand (utilization) for all types. The upper bound of the price range $\beta(\bar{c})$ is either $\bar{c}$ or $B$, depending on whether demand $\xi$ is higher than $(N-1)k$, or $\xi$ higher than $\frac{N-1}{N}$. The lower bound $\beta(\underline{c})$ also increases in demand, resulting in a shrinking price range unless demand crosses the critical threshold. Comparing graph (a-2) and (b-2), it appears that the shape of cost distribution function has minor impacts on the structure bidding strategy.

Graph (a-3) and (b-3) plot the bidding curves for uniform auctions, where the same four demand levels are used as in the discriminatory case. The bidding curve differs from the ones of discriminatory auctions in the following aspects. First, the low pricing bound $\beta(\underline{c})$ is fixed to be zero. Notice that the lowest bid guarantees a full-capacity admission and charges the market clearing price. As cost $\underline{c}$ is the most advantageous type, and therefore, it will choose the lowest price possible, $\beta(\underline{c}) = \underline{c} = 0$. The upper price bound $\beta(\bar{c})$ has the same property as in the discriminatory case, but bidding strategy $\beta(c)$ is not necessarily monotone in demand or utilization when it cross the critical threshold. For example, for the low types close to $\underline{c}$, the bid price for demand $\xi = 2.7$ $\rho = 0.7$), is lower than the price associated with demand $\xi = 1.8$ ($\rho = 0.6$).

## 4.5 Auction with Uncertain Capacity

*Additional Notation* In this section, we study unit-price auctions where bidders' capacities is private information. This is a major departure from the existing auction literature which has focused primarily on the bidders' private valuations of the goods. This section gives a detailed analysis for models with two bidders. For the general case with $N$ bidders, we heuristically derive the equilibrium bidding strategies without proving their optimality (necessary conditions for PBE).

Suppose all suppliers' costs are deterministic and identical. For simplicity, we normalize it as $c = 0$. Suppliers' capacities $\{k_i\}_{i=1}^{N}$ have independent identical distributions, with continuous and strictly increasing c.d.f. $H(k)$ in $k \in \left[\underline{k}, \bar{k}\right]$ where

$0 \le \underline{k} < \bar{k} < \xi$. [2] Denote its density function as $h(k)$. Similar to the previous section, we focus on symmetric PBE $p = \gamma(k)$.

### 4.5.1 Discriminatory Auction (DAK-2)

*Derivation.* In this section, we first derive the PBE heuristically and then prove that obtained solution satisfies the equilibrium conditions. We first consider two simple cases.

(1) For $\xi \le \underline{k}$, each supplier can cover the demand for sure. The game reduces to a simple two-player Bertrand Game. The equilibrium bidding strategy is $\gamma(k) = 0$ for all $k$. The interesting case is,

(2) For demand with $\underline{k} < \xi < \overline{k}$. Suppose there exists a symmetric PBE where a bidder with capacity $k$ chooses the following bidding strategy $p = \gamma(k)$. We further assume $\gamma(\cdot)$ is a *strictly decreasing and differentiable* function and $\gamma^{-1}(\cdot)$ denote its inverse function. (Appendix A.1 demonstrates that equilibrium with increasing bidding strategy does not exist.)

Consider supplier 1's bidding decision $p$ when his capacity is $k$. Suppose that supplier 2 follows the equilibrium strategy $\gamma$. We first compute supplier 1's expected payoff when choosing price $p$. Notice that, the decreasing bidding strategy implies he is outbidding bidder 2 with a type

$$R(p) = p\left\{kH(\gamma^{-1}(p)) + \int_{\gamma^{-1}(p)}^{\overline{k}} [k \wedge (\xi - x)]\, h(x)dx\right\}.$$

The above expression results from the strictly decreasing bidding strategy $\gamma$: (a) $p < \gamma(k_2)$ for all $k < \gamma^{-1}(p)$ and vice versa. The first order derivative is

$$\begin{aligned}\frac{dR(p)}{dp} &= kH(\gamma^{-1}(p)) + \int_{\gamma^{-1}(p)}^{\overline{k}} [k \wedge (\xi - x)]\, h(x)dx \\ &\quad + pkh(\gamma^{-1}(p))\frac{d\gamma^{-1}(p)}{dp} - \left[k \wedge (\xi - \gamma^{-1}(p))\right] h(\gamma^{-1}(p))\frac{d\gamma^{-1}(p)}{dp}\end{aligned}$$

[2]The assumption $\bar{k} < \xi$ is not important for discriminatory auctions, but essential for uniform auctions. The symmetric PBE will collapse if $\bar{k} > \xi$ and we may need to expand our search to mixed strategies.

$$= p\left\{k - \left[k \wedge (\xi - \gamma^{-1}(p))\right]\right\} \frac{h(\gamma^{-1}(p))}{\gamma'(\gamma^{-1}(p))}$$
$$+kH(\gamma^{-1}(p)) + \int_{\gamma^{-1}(p)}^{\bar{k}} [k \wedge (\xi - x)]\, h(x)dx.$$

At equilibrium, $p = \gamma(k)$, so FOC $\frac{dR(p)}{dp} = 0$ reads

$$\{k - [k \wedge (\xi - k)]\}\, h(k)\frac{\gamma(k)}{\gamma'(k)} + kH(k) + \int_{k}^{\bar{k}} [k \wedge (\xi - x)]\, h(x)dx = 0.$$

That is

$$\frac{\gamma'(k)}{\gamma(k)} = -\frac{\{k - [k \wedge (\xi - k)]\}\, h(k)}{kH(k) + \int_k^{\bar{k}} [k \wedge (\xi - x)]\, h(x)dx} \tag{4.12}$$

Note that the above equation suggests $\gamma'(k) \leq 0$, which is consistent with our assumption. The general solution to the above ODE is

$$\gamma(k) = C_D \cdot \exp\{-\int_{\underline{k}}^{k} \frac{y - [y \wedge (\xi - y)]}{yH(y) + \int_y^{\bar{k}} [y \wedge (\xi - x)]\, dH(x)} dH(y)\} \tag{4.13}$$

where $C_D$ is a constant. Next we determine the boundary condition. The function can be “fixed” by either the left end $\gamma(\underline{k})$ or the right end $\gamma(\bar{k})$. Furthermore, we conjecture either $\gamma(\underline{k}) = B$ or $\gamma(\bar{k}) = c$. Expression( 4.13) suggests that if $\gamma(\bar{k}) = c$ then, $C_D = 0$ implies $\gamma(k) = 0$ for all $k$, which is clearly not optimal for a bidder with type $\underline{k} < \xi$. This possibility is ruled out then. Now consider the possibility $\gamma(\underline{k}) = B$. A special consideration needs to be paid to the case $\underline{k} < \frac{\xi}{2}$. For all $k \leq \frac{\xi}{2}$, ODE(4.12) reduces to $\frac{\gamma'(k)}{\gamma(k)} = 0$, suggesting $\gamma(k)$ is a constant. Therefore, we have that for $k \in \left[\underline{k} \wedge \frac{\xi}{2}, \underline{k} \vee \frac{\xi}{2}\right]$, we have a *pooling* strategy $\gamma(k) = B$.

**Proposition 6** *The symmetric equilibrium bidding strategy for DAK-2 is*

$$\gamma_D(k) = \begin{cases} B & fork \leq (\underline{k} \vee \frac{\xi}{2}) \\ B \exp\left[-\int_{\underline{k} \vee \frac{\xi}{2}}^{k} \theta(y) dH(y)\right] & fork > (\underline{k} \vee \frac{\xi}{2}) \end{cases}. \tag{4.14}$$
$$where\theta(y) = \frac{2y - \xi}{yH(y) + \xi\bar{H}(y) - \int_y^{\bar{k}} x dH(x)}.$$

**Proof.** First notice that the simplification of $\theta(y)$ is due to $y \wedge (\xi - y) = \xi - y$ and $y \wedge (\xi - x) = (\xi - x)$ for $x \geq \frac{\xi}{2}$ and $y \geq \frac{\xi}{2}$.

To show the above bidding strategy is indeed a PBE, we only need to verify that $R(p)$ is maximized at $p = \gamma(k)$ when a bidder's type is $k$. Similar to the case with uncertain cost, we define

$$
\begin{aligned}
R(l,k) &= \gamma(l)\left[kH(l)+\int_l^{\overline{k}}(\xi-x)h(x)dx\right] \\
&= B\exp\left[-\int_{\underline{k}\vee\frac{\xi}{2}}^{l}\theta(y)h(y)dy\right]\left[kH(l)+\int_l^{\overline{k}}(\xi-x)h(x)dx\right].
\end{aligned}
$$

We only need to show that $R(l,k)$ is maximized at $l=k$. It is sufficient to show $\frac{\partial R_D(l,k)}{\partial l}\geq 0$ for $l\leq k$ and $\frac{\partial R(l,k)}{\partial l}\leq 0$ for $l\geq k$. We compute the partial derivative

$$
\begin{aligned}
\frac{\partial R(l,k)}{\partial l} &= -B\exp\left[-\int_{\underline{k}\vee\frac{\xi}{2}}^{l}\theta(y)h(y)dy\right]\theta(l)h(l)\left[kH(l)+\int_l^{\overline{k}}(\xi-x)h(x)dx\right] \\
&\quad +B\exp\left[-\int_{\underline{k}\vee\frac{\xi}{2}}^{l}\theta(y)h(y)dy\right](k+l-\xi)h(l) \\
&= B\exp\left[-\int_{\underline{k}\vee\frac{\xi}{2}}^{l}\theta(y)h(y)dy\right]h(l)\eta(k,l)
\end{aligned}
$$

where

$$
\begin{aligned}
\eta(l,k) &\equiv 2k-\xi-\frac{\left[kH(l)+\int_l^{\bar{k}}(\xi-x)h(x)dx\right](2l-\xi)}{lH(l)+\int_l^{\bar{k}}(\xi-x)h(x)dx} \\
&= 2k-\xi-(2l-\xi)-\frac{(k-l)H(l)(2l-\xi)}{lH(l)+\int_l^{\bar{k}}(\xi-x)h(x)dx} \\
&= (k-l)\left[2-\frac{H(l)(2l-\xi)}{lH(l)+\int_l^{\bar{k}}(\xi-x)h(x)dx}\right] \\
&= (k-l)\frac{2\int_l^{\bar{k}}(\xi-x)h(x)dx+H(l)\xi}{lH(l)+\int_l^{\bar{k}}(\xi-x)h(x)dx}.
\end{aligned}
$$

Obviously, $\eta(l,k)>0$ for $l<k$ and $\eta(l,k)<0$ for $l>k$, and we have the desired property for $\frac{\partial R_D(l,k)}{\partial l}$. ■

Plugging (4.14) into expression $R(k,k)$, we have the expected revenue for supplier with capacity $k$,

$$
R_D(k)=\gamma_D(k)\left[kH(k)+\int_k^{\overline{k}}\left[k\wedge(\xi-x)\right]dH(x)\right]. \tag{4.15}
$$

It is easy to show that $R_D(k)$ is strictly increasing in $k \in \left[\underline{k} \vee \frac{\xi}{2}, \bar{k}\right]$

$$\begin{aligned}
\frac{dR_D(k)}{dk} &= \gamma'(k)\left[kH(k) + \int_k^{\bar{k}} [k \wedge (\xi - x)]\, dH(x)\right] \\
&\quad + \gamma(k)\left\{H(k) + kh(k) - h(k)\left[k \wedge (\xi - k)\right]\right\} \\
&= -\gamma(k)\left\{k - [k \wedge (\xi - k)]\right\} h(k) + \gamma(k)\left\{H(k) + h(k)k - h(k)\left[k \wedge (\xi - k)\right]\right\} \\
&\quad [\text{by} \frac{\gamma'(k)}{\gamma(k)} = \frac{\left\{k - [k \wedge (\xi - k)]\right\} h(k)}{kH(k) + \int_k^{\bar{k}} [k \wedge (\xi - x)]\, h(x)dx}] \\
&= \gamma(k)H(k) > 0
\end{aligned}$$

### 4.5.2 Uniform Auction (UAK-2)

*Derivation.* For demand $\xi \in \left[\underline{k}, \bar{k}\right]$, similar to model DAK-2, we will focus only on the case where equilibrium bidding strategy is a decreasing function of the capacity $k$. The heuristic derivation starts from calculating the expected payoff of supplier 1 whose capacity is $k$ and chooses price $p$, given that supplier 2 uses equilibrium bidding strategy $p_2 = \gamma(\cdot)$. As noted above, $\bar{k} < \xi$ has to be assumed.

$$R(p) = \int_{\underline{k}}^{\gamma^{-1}(p)} k\gamma(x)h(x)dx + p\int_{\gamma^{-1}(p)}^{\bar{k}} [k \wedge (\xi - x)]\, h(x)dx.$$

It follows

$$\begin{aligned}
\frac{dR(p)}{dp} &= k\gamma(\gamma^{-1}(p))h(\gamma^{-1}(p))\frac{d\gamma^{-1}(p)}{dp} - p[k \wedge (\xi - \gamma^{-1}(p))]h(\gamma^{-1}(p))\frac{d\gamma^{-1}(p)}{dp} \\
&\quad + \int_{\gamma^{-1}(p)}^{\bar{k}} [k \wedge (\xi - x)]h(x)dx \\
&= h(\gamma^{-1}(p))\frac{d\gamma^{-1}(p)}{dp}\left\{k\gamma(\gamma^{-1}(p)) - p[k \wedge (\xi - \gamma^{-1}(p))]\right\} + \int_{\gamma^{-1}(p)}^{\bar{k}} [k \wedge (\xi - x)]h(x)dx.
\end{aligned}$$

At equilibrium $k = \gamma^{-1}(p)$ and $p = \gamma(k)$, and we have an ODE following FOC $\frac{dR(p)}{dp} = 0$,

$$\begin{aligned}
\frac{h(k)\gamma(k)\left\{k - [k \wedge (\xi - k)]\right\}}{\gamma'(k)} &= -\int_k^{\bar{k}} [k \wedge (\xi - x)]\, h(x)dx \\
\frac{\gamma'(k)}{\gamma(k)} &= -\frac{h(k)\left\{k - [k \wedge (\xi - k)]\right\}}{\int_k^{\bar{k}} [k \wedge (\xi - x)]\, h(x)dx}
\end{aligned}$$

Notice that $\frac{\gamma'(k)}{\gamma(k)} = 0$ for $k < \frac{\xi}{2}$, suggesting all type $k \in \left[\underline{k} \wedge \frac{\xi}{2}, \underline{k} \vee \frac{\xi}{2}\right]$ will choose $\gamma(k) = B$. The above ODE only applies for $k > \left[\underline{k} \vee \frac{\xi}{2}, \bar{k}\right]$, implying $0 \leq \xi - k \leq k$, so the ODE is simplified as

$$\frac{\gamma'(k)}{\gamma(k)} = -\frac{h(k)(2k-\xi)}{\int_k^{\bar{k}}(\xi - x)h(x)dx}. \tag{4.16}$$

Summarizing the three cases, we have the following proposition.

**Proposition 7** *The symmetric equilibrium bidding strategy for UAK-2 is*

$$\begin{aligned} \gamma_U(k) &= \begin{cases} B & fork \leq (\underline{k} \vee \frac{\xi}{2}) \\ B\exp\left[-\int_{\underline{k}\vee\frac{\xi}{2}}^{k} \tau(y)dH(y)\right] & for(\underline{k} \vee \frac{\xi}{2}) < k < \bar{k} \end{cases} \\ where\tau(y) &= \frac{2y-\xi}{\int_y^{\bar{k}}(\xi - x)h(x)dx}. \end{aligned}$$

**Proof.** Similarly to the DA case, we examine the below function for any $k \in ((\underline{k} \vee \frac{\xi}{2}), \bar{k})$

$$\begin{aligned} R_U(l,k) &= k\left[BH(\underline{k} \vee \frac{\xi}{2}) + \int_{\underline{k}\vee\frac{\xi}{2}}^{l} \gamma(x)h(x)dx\right] + \gamma(l)\int_l^{\bar{k}}(\xi - x)h(x)dx \\ \frac{\partial R_U(l,k)}{\partial l} &= k\gamma(l)h(l) - \gamma(l)(\xi - l)h(l) + \gamma'(l)\int_l^{\bar{k}}(\xi - x)h(x)dx \\ &= \gamma(l)h(l)(k + l - \xi) + \gamma'(l)\int_l^{\bar{k}}(\xi - x)h(x)dx \\ &= \gamma(l)h(l)\left[k + l - \xi + \frac{\gamma'(l)}{\gamma(l)h(l)}\int_l^{\bar{k}}(\xi - x)h(x)dx\right] \end{aligned}$$

Notice that $\frac{\gamma'(l)}{\gamma(l)h(l)} = -\frac{2l-\xi}{\int_l^{\bar{k}\wedge\xi}(\xi-x)h(x)dx}$, so

$$\frac{\partial R_U(l,k)}{\partial l} = \gamma(l)h(l)(k-l).$$

Clearly $R_U(l,k)$ is quasi-concave in $l$ and maximizes at $l = k$. ■

The expected payoff for supplier with capacity $k$ is

$$R_U(k) = k\int_{\underline{k}}^{k} \gamma_U(x)dH(x) + \gamma_U(k)\int_k^{\bar{k}} [k \wedge (\xi - x)]\, dH(x) \tag{4.17}$$

We can also show that the expected revenue is increasing in supplier's capacity $k$.

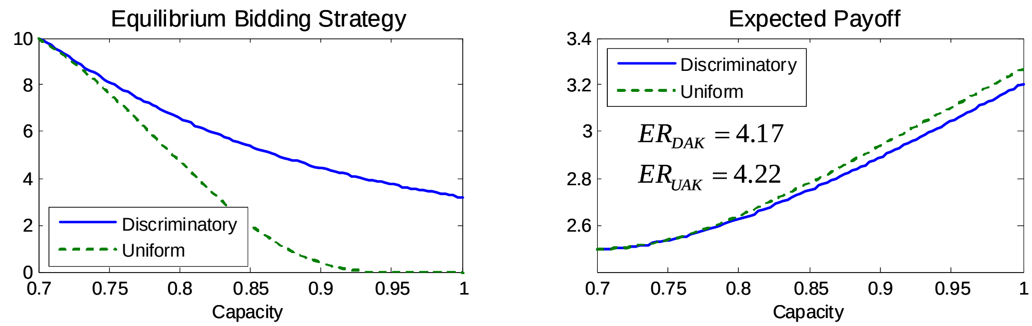
$$\begin{aligned}
& \frac{dR_U(k)}{dk} && (4.18)\\
= & \int_{\underline{k}}^{k} \gamma(x)dH(x) + k\gamma(k)h(k) + \gamma'(k)\int_{k}^{\bar{k}}[k \wedge (\xi - x)]dH(x) - \gamma(k)[k \wedge (\xi - k)]h(k) \\
= & \int_{\underline{k}}^{k} \gamma(x)dH(x) + \gamma(k)h(k)\{k - [k \wedge (\xi - k)]\} - \gamma(k)h(k)\{k - [k \wedge (\xi - k)]\} \\
& [\text{by}\frac{\gamma'(k)}{\gamma(k)} = -\frac{h(k)\{k - [k \wedge (\xi - k)]\}}{\int_k^{\bar{k}}[k \wedge (\xi - x)]h(x)dx}] \\
= & \int_{\underline{k}}^{k} \gamma(x)dH(x) > 0.
\end{aligned}$$

*Implication to Capacity Withholding.* The property of monotone revenue has an interesting implication about the issue of capacity withholding in uniform auctions. In the above analysis, we have restricted each supplier to fully commit its available capacity and the only decision is the unit price. If we relax this assumption and let supplier $i$ to choose a price-quantity pair $\{p_i, q_i\} \in [0, B] \times [0, k]$. We name such an auction as *unit price-quantity auction.* The following proposition follows,

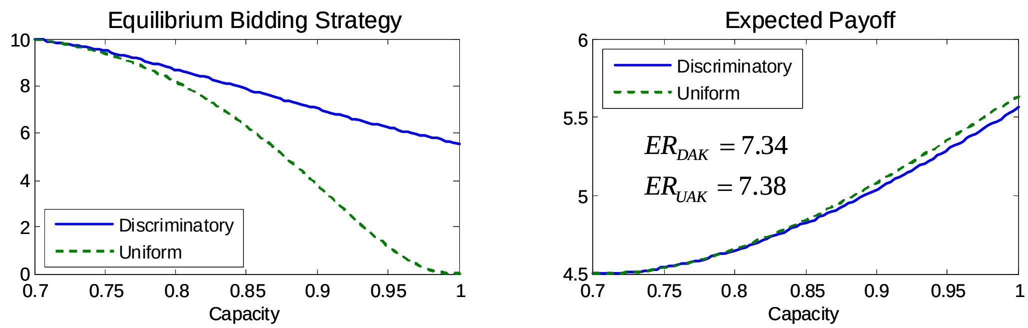
**Proposition 8** *In a two-bidder unit price-quantity auction with uniform payment rule, there is a symmetric Bayesian Equilibrium,* $\{p^*, q^* | k\} = \{\gamma_U(k), k\}$.

**Proof.** We only need to prove that if supplier 2 follows such a strategy, it is optimal for supplier 1 to use this strategy too. We show this by constructing contradiction. Suppose there is another strategy $\{\hat{p}_1, \hat{q}_1\} \neq \{\gamma_U(k_1), k_1\}$ such that $R_1(\{\gamma_U(k), k\}) < R(\hat{p}_1, \hat{q}_1)$. If $\hat{q}_1 = k$, then existence of $\hat{p}_1$ contradicts to the optimality of $\gamma(k)$ in game UAK-2. Suppose $\hat{q}_1 < k$, we must have $R_1(\hat{p}_1, \hat{q}_1) \leq R_1(\gamma(\hat{q}_1), \hat{q}_1)$ because of the optimality of $\gamma(\hat{q}_1)$ in game UAK-2. It implies that $R_1(\gamma(\hat{q}_1), \hat{q}_1) > R_1(\{\gamma_U(k), k\})$, which is a contradiction to 4.18. ■

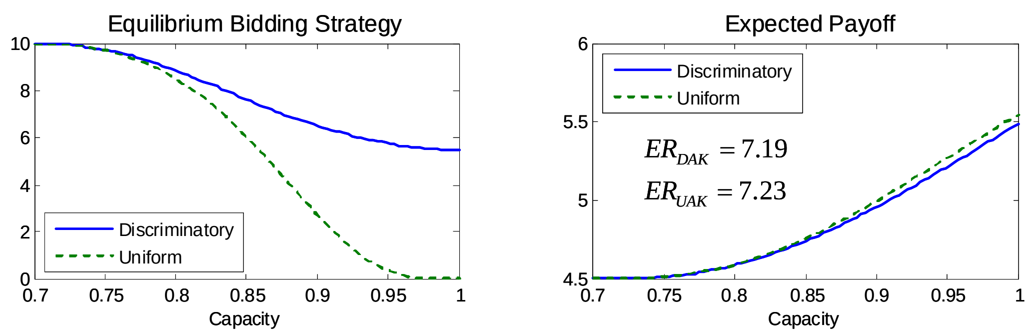
*Numerical Studies.* In Figure 4.2, four auctions are examined. For each auction, the equilibrium bidding strategies are plotted in the left and the expected payoffs $R(k)$ are displayed in the right. In order to illustrate the average revenue, we also compute $ER = \int R(k)dH(k)$ and mark the values in the right graph for each game.



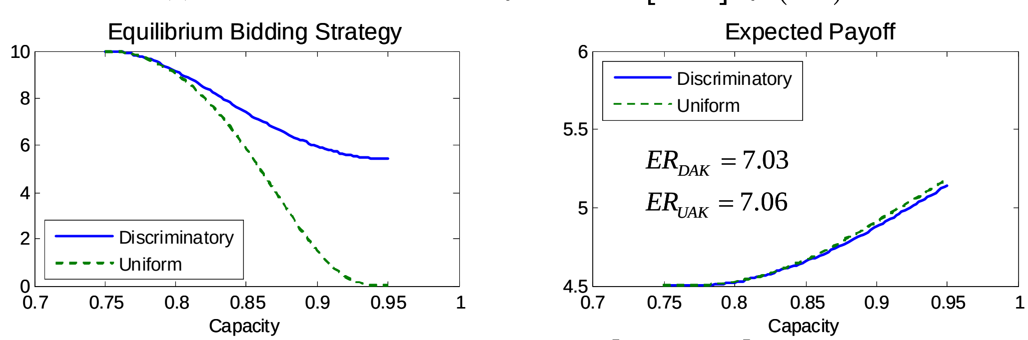

(a) Auction: N = 2, B = 10, $\xi = 1.1$, $k \in U[0.7,1]$




(b) Auction: N = 2, B = 10, $\xi = 1.3$, $k \in U[0.7,1]$




(c) Auction: N = 2, B = 10, $\xi = 1.3$, $k \in [0.7,1] * \beta(2,2)$




(d) Auction: N = 2, B = 10, $\xi = 1.3$, $k \in [0.75, 0.95] * \beta(2,2)$

**Figure 4.2.** Comparison of DAK-2 and UAK-2

Fixing the price cap $B = 10$ and the capacity range within $[0.7, 1]$, we change the level and shape of capacity distribution function. The following observations are reported.

- The expected revenues for the two auctions are not equivalent, and uniform auction yields higher expected payoff. For low-capacity types close to $\underline{k}$, the difference is marginal while for the high-capacity types the difference becomes noticeable.

- The upper bound of equilibrium bids is $B = 10$; the lower bound of discriminatory auction $\gamma_D(\bar{k})$ changes as game setting is modified, while the uniform auction always has $\gamma_U(\bar{k}) = 0$.

- Effect of higher demand: When demand level is increased ($\xi_b = 1.3 > \xi_a = 1.1$), equilibrium bids are increased for both discriminatory and uniform auctions. The expected payoffs for all types are lifted significantly, indicating the demand level is still the dominant force for auctions with capacity constraints.

- Effect of different distribution function: With other parameters identical, the uncertain capacity in auction in Figure 2(b) has a uniform distribution between $[0.7, 1]$, while the ones in Figure 2(c) has a scaled beta distribution, i.e., $k = 0.7 + 0.3x$ where $x\tilde{\ }\beta(2, 2)$. The impacts are not significant and expected revenue reduces slightly.

- Effect of reduced capacity range: In Figure 2(d), we only reduces the range of capacity distribution from Figure 2(c) to $[0.75, 0.95]$ , while keep the mean to be 0.85. This change has sizable impact, in terms of average revenue $ER$, since impacts is not proportional on all capacity types. For the auctioneer, the benefits from eliminating the high-capacity types dominates the loss due to removal of low-capacity suppliers.

### 4.5.3 N-Bidder Equilibrium Solution

*Monotone Equilibrium.* The analysis of N-bidder cases are considerably more complicated, involving intensive probability manipulations. Fortunately, we manage to derive the equilibrium bidding strategies under the assumption that the curves are decreasing in capacity $k$. We have not been able to show the derived solution is a Nash solution, but it can be numerically verified by computing function $R(l, k)$ by checking if $\gamma(k) = \arg\max_l R(l, k)$ holds for all $k$. Here we first present the solutions.

**Proposition 9** *If auction DAK-N has a symmetric Bayesian equilibrium with de-*

*creasing bidding strategy, the solution is*

$$\gamma_D(k) = B \exp\left[-(N-1)\int_{\underline{k}}^{k} \Delta(x) dH(x)\right] \tag{4.19}$$

*where*

$$\begin{aligned}\Delta(x) &= \frac{\sum_{i=0}^{N-2}\binom{N-2}{i}H(x)^{N-i-2}\left[U_i(x,\xi)-U_i(x,\xi-x)\right]}{\sum_{i=0}^{N-1}\binom{N-1}{i}H(x)^{N-i-1}U_i(x,x,\xi)}\\ U_i(x,d) &= \begin{cases} x\wedge(d)^+ & i=0\\ \int_x^{\bar{k}}\cdots\int_x^{\bar{k}}[x\wedge(d-\sum_{j=1}^{i}k_j)^+]dH(k_1)\dots dH(k_i) & i\geq 1\end{cases}.\end{aligned}$$

*The expected equilibrium payoff for supplier with capacity $k$ is,*

$$R_D(k) = \gamma_D(k)\sum_{i=0}^{N-1}\binom{N-1}{i}H(k)^{N-i-1}U_i(k,k,\xi).$$

**Proposition 10** *If auction UAK-N has a symmetric Bayesian equilibrium with decreasing bidding strategy, the equilibrium solution is*

$$\gamma_U(k) = B \exp\left[-(N-1)\int_{\underline{k}}^{k} \Theta(x) dH(x)\right] \tag{4.20}$$

*where*

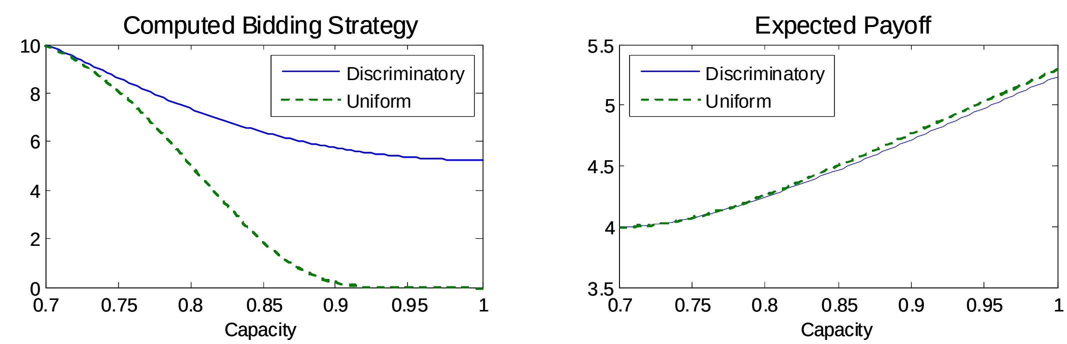
$$\Theta(x) = \frac{\sum_{i=0}^{N-2}\binom{N-2}{i}H(x)^{N-i-2}\left[W_i^{N-1}(x,\xi)-W_i^{N-2}(x,\xi-x)+xV_i^{N-2}(x,\xi-x)\right]}{\sum_{i=0}^{N-1}\binom{N-1}{i}H(x)^{N-i-1}W_i^{N-1}(x,\xi)}$$

$$\begin{aligned}&W_i^M(x,d)\\ =&\begin{cases}(x\wedge d)\delta_{\{0<d\}} & i=M=0\\ d\delta_{\{0<d\leq x\}} & i=0<M\\ \int_x^{\bar{k}}\dots\int_x^{\bar{k}}(d-\sum_{j=1}^{i}k_j)\delta_{\left\{d-x\leq\sum_{j=1}^{i}k_j<d\right\}}dH(k_1)\dots dH(k_i) & 0<i<M\\ \int_x^{\bar{k}}\dots\int_x^{\bar{k}}\left[x\wedge(d-\sum_{j=1}^{M}k_j)\right]\delta_{\left\{\sum_{j=1}^{M}k_j<d\right\}}dH(k_1)\dots dH(k_i) & 0<i=M\end{cases}\end{aligned}$$

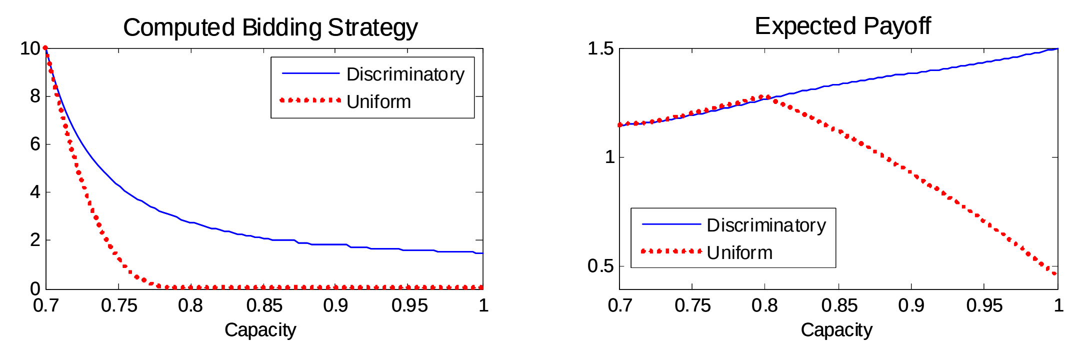
$$V_i^M(x,d) = \begin{cases}\delta_{\{0<d\}} & i=M=0\\ \delta_{\{0<d\leq x\}} & i=0<M\\ \int_x^{\bar{k}}\dots\int_x^{\bar{k}}\delta_{\left\{d-x\leq\sum_{j=0}^{i}k_j<d\right\}}dH(k_1)\dots dH(k_i) & 0<i<M\\ \int_x^{\bar{k}}\dots\int_x^{\bar{k}}\delta_{\left\{\sum_{j=0}^{M}k_j<d\right\}}dH(k_1)\dots dH(k_i) & 0<i=M\end{cases}.$$

*The expected equilibrium payoff for supplier with capacity* $k$*,*

$$\begin{aligned} R_U(k) &= \gamma_U(k) \sum_{i=0}^{N-1} \binom{N-1}{i} H(k)^{N-i-1} W_i^{N-1}(k,\xi) \\ &\quad + (N-1)k \int_{\underline{k}}^{k} \left[ \gamma_U(x) \sum_{i=0}^{N-2} \binom{N-2}{i} H(x)^{N-i-2} V_i^{N-2}(x, \xi - k) \right] dH(x). \end{aligned}$$




(a) Auction with Monotone UAK equilibrium: N = 3, B = 10, $\xi = 2.1$, $k \in U[0.7,1]$




(b) Auction without Monotone UAK equilibrium: N = 3, B = 10, $\xi = 1.8$, $k \in U[0.7,1]$

**Figure 4.3.** Auctions with/without Monotone UAK Equilibrium

*Numerical Studies* Numerical studies are conducted for $N$-bidder auctions and two examples are provided in Figure 4.3. We have the following observations:

- Regularity of DAK-N: For all derived solutions, we numerically check the equilibrium condition, $\gamma(k) \in \arg\max_l R(l,k)$. All solutions in DAK survive the test, indicating the robustness of our monotone equilibrium solution. Furthermore, all properties we observe in the DAK-2 model remains to hold qualitatively. For example, $\frac{dR(k)}{dk} > 0$.

- Nonexistence of Monotone Equilibrium for UAK-N: Although the solution (4.20) can be computed for all models, not all of the solutions can survive the optimality test.

It suggests that UAK-N with $N \geq 3$ may not have a monotone Bayesian equilibrium, and the search of equilibrium needs to expand to the strategy space by including non-monotone bidding strategy or even mixed strategies. The analysis becomes extremely complicated and is therefore out of the scope of this paper. Numerical studies also indicates that, when other parameters are fixed, there is a demand threshold such that all demand lower than the threshold fails to have monotone equilibrium, while all demand above it sustains the monotone solution (4.20).

- Revenue monotonicity sustains in monotone equilibrium. Our numerical test indicates that, the survival of monotone equilibrium in UAK-N always couples with the revenue monotonicity. In other words, the results in Proposition 8 remain to be true if there is a monotone equilibrium. In Figure 3, we display two games, where auction (a) has monotone equilibria for both DAK and UAK, while auction (b) has no monotone equilibrium for UAK. The derived "solution" yields a non-increasing expected revenue, and numerical study indicates $\gamma(k) = \arg\max_l \{R_U(l,k)\}$ does not hold for all $k$.

## 4.6 Concluding Remarks

In order to understand the impacts of supply uncertainty in share procurement auctions, we investigate the unit-price procurement auction models under uncertain supplier cost and capacity. One main contribution of this paper is to analytically characterize the equilibrium solutions, so that economic insights can be obtained. To keep analytical tractability, we present two models, exploring the uncertainties of costs and capacities separately. Among many interesting aspects of share auctions, we focus on (a) the interplay of information rent and market power, and (b) comparison of discriminatory and uniform auctions.

*Impacts of Information Rent and Market Power.* The main driver of equilibrium pricing in private-value auctions is a bidder's request for a rent to reveal its true type. In our model of uncertain cost, this request remains. Additionally, a supplier's marginal contribution to the industry's capability to meet demand, namely market power, is the primary driver in oligopoly pricing, such as in a Bertrand-Edgeworth scenario. In unit-price auctions with uncertain costs, these two drivers jointly control the market outcome, and simultaneously result in the generalized revenue(payment) equivalence. In auctions with private knowledge about capacities, the two factors are intervened and our paper is the first one to study the scenario. Revenue equivalence collapses and discriminatory auctions result in slightly lower expected payment for the auctioneer under monotone equilibria.

*Comparison of Discriminatory and Uniform Auctions.* Our paper brings new insights about the two competing market designs. We disclose the conditions under which revenue equivalence is likely to hold and when the opposite might be true. Considering both models (one with uncertain cost and one with uncertain capacity), we find discriminatory auction is a more stable and tractable design. The market outcome is easy to predict, since unique equilibrium with well behaved bidding strategy always exists. At the same time, the pricing range is small. Our numerical examples clearly indicates the overall price variability for discriminatory auction is smaller, a further confirmation of Chapter 2's conclusion. Uniform auction involves higher uncertainty, in that the equilibrium outcome is hard to find and may be multiple (as in Chapter 2). Within the equilibria we identify, bidders choose their prices in a wider range than in discriminatory auctions. What further upsets an auctioneer is that when suppliers have private capacity information, uniform auctions could result in higher expected payment for the auctioneer. Based on this paper, we would put discriminatory auction ahead of uniform auction when making recommendation to a procurement auction designer.

*Capacity Withholding.* For many reasons, uniform auction is still the dominant market design in public sector. Its obvious advantages include ease of implementation and the seeming fairness among bidders. Our paper contributes to the understanding of capacity withholding in uniform auctions. The theoretical research about this widely discussed issue (particularly after the 2000 California Energy Crisis) is limited and this is a first paper with explicit analysis in static game setting. This encouraging news for uniform auction advocates is that it might not be incentive compatible for bidders to withhold capacity, particularly when the demand is relatively high. The speculated capacity withholding would hurt auctioneer badly in this scenario, but the competitive nature of auctions induces suppliers to fully commit. We note that our finding about capacity withholding is preliminary since we have not solved the auctions when monotone equilibrium does not exist, due to prohibitive analytical difficulty. It opens room for future research studies.

# APPENDICES

# APPENDIX A

# Proofs and Analysis for Chapter 2

Note: When context is clear, we ignore the superscripts of auction formats, $d$ and $u$.

## A.1 Continuity and Monotonicity for $F^d$.

**Lemma 3** *Consider a symmetric DA with $(N-1)k < \xi < Nk$. For a symmetric mixed-strategy equilibrium, $F^d(p)$ is continuous and strictly increasing in $p \in [\underline{p}^d, \bar{p}^d] \cap (c, B]$.*

**Proof.** (a) *Continuity of $F$ for $p > c$.* Suppose $m(p) > 0$ for certain $p \in (c, B]$, implying $\hat{R}(p) = ER$. Note that

$$\begin{aligned}
\hat{r}^-(p) - \hat{r}(p) &= \mathbf{E}_{\sigma^*_{-i}}[r_i^-(p) - r_i(p)] \\
&\geq Pr(\sigma^*_n = p)_{n \neq i} \mathbf{E}_{\sigma^*_{-i}}[r_i^-(p) - r_i(p) | (\sigma^*_n = p)_{n \neq i}] \\
&= Pr(\sigma^*_n = p)_{n \neq i}[(1 \wedge \frac{\xi}{k}) - (1 \wedge \frac{\xi}{Nk})] \qquad \text{[by (2.1-a)]} \\
&= m(p)^{N-1}[1 - \frac{\xi}{Nk}] > 0. \qquad \text{[by independence of all } \sigma^*_n\text{'s]}
\end{aligned}$$

Since $p > c$, by (2.2), we have $\hat{R}^-(p) - \hat{R}(p) = (p-c)k[\hat{r}^-(p) - \hat{r}(p)] > 0$, implying a contradiction to the optimality of $ER = \hat{R}(p)$.

(b) *Monotonicity of $F$.* Suppose there exist $\alpha < \beta$ such that $F(p') < F(\alpha) = F(\beta) < F(p'')$ for all $p' < \alpha$ and all $p'' > \beta$. Note that any player gets at least a sales of $\xi - (N-1)k$, so the equilibrium payoff must be positive. From (2.3), we have $\hat{R}^+(\underline{p}) = ER > 0$, implying $\underline{p} > c$. Part (a) implies $\hat{R}^-(p) = \hat{R}(p) = \hat{R}^+(p)$ for all $p \in [\underline{p}, B]$. Now initial assumption implies two results contradicting to each other, (i) $\hat{R}(\alpha) = ER = \hat{R}(\beta)$ by (2.3) and (ii) $\hat{r}(\beta) = \sum_{n=0}^{N-1} \binom{N-1}{n} F(\alpha)^n \bar{F}(\alpha)^{N-n-1} \cdot [1 \wedge \frac{(\xi - nk)^+}{k}] = \hat{r}(\alpha)$, implying $\hat{R}(\alpha) < \hat{R}(\beta)$.

## A.2 Uniqueness of Mixed-strategy Equilibrium for a Symmetric DA.

**Proposition 11** *For a symmetric DA with $(N-1)k < \xi < Nk$, if $F_i(p)$ is continuous and strictly increasing in $p \in [\underline{p}_i, \bar{p}_i] \cap (c, B)$ for all $i$ with $\underline{p}_i < \bar{p}_i$, the symmetric equilibrium defined in (2.4) is the unique Nash equilibrium.*

**Proof.** (a) *$ER_i \geq (B-c)[\xi - (N-1)k]$ and $\underline{p}_i \geq \frac{(B-c)[\xi-(N-1)k]}{k} + c > 0$.* For any supplier $i$ and any price $p$, we have $\hat{r}_i(p) \in [\frac{\xi-(N-1)k}{k}, 1]$. Therefore, optimality of $ER_i$ implies $ER_i \geq \hat{R}_i(B) \geq (B-c)[\xi-(N-1)k] > 0$. By (2.3-c), we also have $ER_i = \hat{R}_i^+(\underline{p}_i) \leq (\underline{p}_i - c)k$, implying $\underline{p}_i \geq \frac{ER_i}{k} + c$.

(b) *$\underline{p}_i = \underline{p} \equiv \min_n\{\underline{p}_n\}$ and $ER_i = (\underline{p}-c)k$ for all $i$.* Suppose there is a supplier $i$ such that $\underline{p}_i > \underline{p}$. For supplier $j \neq i$, consider her expected payoff at a price $\tilde{p} < \underline{p}_i$. With probability 1, supplier $j$ achieves a sales fraction of $r_j(\tilde{p}) = 1 \wedge \frac{\xi - \sum_{n\neq i} k\delta_{(p_n<\tilde{p})}}{\sum_{n\neq i} k\delta_{(p_n=\tilde{p})}} = 1$ because $\xi - \sum_{n\neq i} k\delta_{(p_n<\tilde{p})} > \sum_{n\neq i} k\delta_{(p_n=\tilde{p})}$. [by $\xi - \sum_{n\neq i} k\delta_{(p_n<\tilde{p})} - \sum_{n\neq i} k\delta_{(p_n=\tilde{p})} = \xi - \sum_{n\neq i} k\delta_{(p_n\leq\tilde{p})} \geq \xi - (N-1)k > 0$] Therefore, $\hat{r}_j(p) = 1$ and $\hat{R}_j(p) = (p-c)k$ for all $p < \underline{p}_i$. As it is strictly increasing in $p$, the optimality of $\sigma_j^*$ requires $\Pr\{\sigma_j^* \in [0, \underline{p}_i)\} = 0$, implying $\underline{p}_j \geq \underline{p}_i$ for all $j \neq i$. It implies $\min_{j\neq i}\{\underline{p}_j\} \geq \underline{p}_i$, a contradiction to $\underline{p}_i > \underline{p} = \min_n\{\underline{p}_n\}$, so we must have $\underline{p}_i = \underline{p}$ for all $i$. Now continuity of $F_i$ at $\underline{p} > c$ implies $m_i(\underline{p}) = 0$ for all $i$ and therefore $\hat{r}_i^-(\underline{p}) = \hat{r}_i(\underline{p}) = \hat{r}_i^+(\underline{p}) = 1$. Applying (2.3-c), we obtain $ER_i = \hat{R}_i^+(\underline{p}) = (\underline{p}-c)k\hat{r}_i^+(\underline{p}) = (\underline{p}-c)k$ for all $i$.

(c) *There is at most one player with $m(\bar{P}) > 0$ for $\bar{P} \equiv \max_n\{\bar{p}_n\}$.* If $\bar{P} < B$, the desired result follows directly our initial assumption (continuity of $\{F_n\}$). Suppose $\bar{P} = B$ and there are $M \geq 2$ suppliers having $m(B) > 0$. From (2.3-a), we have $ER_i = \hat{R}_i(B)$ for all $i \in I_M \equiv \{n : m_n(B) > 0\}$. However, similarly to part (a) of Lemma A1's proof, we can establish

$$\hat{r}_i^-(p) - \hat{r}_i(p) \geq \prod_{n \in I_M \setminus \{i\}} m_n(B) \cdot [1 - \frac{\xi - (N-M)k}{Mk}] > 0,$$

implying $\hat{R}_i^-(B) > \hat{R}_i(B) = ER_i$, a contradiction to the optimality of $ER_i$.

(d) *$ER_i = (B-c)[\xi-(N-1)k]$ for $i = 1, 2, \ldots, N$ and $\underline{p} = c + \frac{(B-c)[\xi-(N-1)k]}{k}$.* Part (c) implies at price $\bar{P}$, either $m_i(\bar{P}) = 0$ for all $i$ or only one supplier has

$m(\bar{P}) > 0$. For both cases, we can find one supplier, say $h$, such that $m_n(\bar{P}) = 0$ for all $n \neq h$. Clearly, supplier $h$ has $\hat{r}_h^-(\bar{P}) = \hat{r}_h(\bar{P}) = \frac{\xi-(N-1)k}{k}$, and consequently, $ER_h = \hat{R}_h^-(\bar{P}) = (\bar{P}-c)[\xi-(N-1)k]$ by (2.3-b). Optimality of $ER_h$ requires $\bar{P} = B$ and part (b) yields $\underline{p}$.

(e) *Derivation of equilibrium solution.* Consider supplier $i$'s expected payoff at any $p \in [\underline{p}, \bar{p}] \cap [\underline{p}, B)$ where $\bar{p} \equiv \min_n \{\bar{p}_n\}$. As all $F_n$'s are continuous, we have $m_n(p) = 0$ for all $n \neq i$, and the possibilities of price-tie at $p$ can omitted. Now given any bid vector $\mathbf{p}_{-i}$, as long as $b_{(1)}^{(-i)} = \max\{\mathbf{p}_{-i}\} > p_i$, supplier $i$ sells $z_i = k$, while if $b_{(1)}^{(-i)} < p_i$, $z_i = \xi - (N-1)k$. Note that $G_{(1)}^{(-i)}(p) = \Pr\{\max\{\mathbf{p}_{-i}\} < p\} = \Pi_{n\neq i} F_n(p)$, and we have

$$\hat{R}_i(p) = (p-c)[(\xi - Nk + k)G_{(1)}^{(-i)}(p) + k\bar{G}_{(1)}^{(-i)}(p)] = ER_i = (\underline{p}-c)k.$$

It implies

$$G_{(1)}^{(-i)}(p) = \Pi_{n\neq i} F_n(p) = \frac{k}{Nk-\xi}\frac{p-\underline{p}}{p-c} \quad \text{for all } i.$$

Therefore, $\Pi_{i=1}^N G_{(1)}^{(-i)}(p) = \Pi_{i=1}^N F_i(p)^{N-1} = [\frac{k}{Nk-\xi}\frac{p-\underline{p}}{p-c}]^N$, implying $\Pi_{i=1}^N F_i(p) = [\frac{k}{Nk-\xi}\frac{p-\underline{p}}{p-c}]^{\frac{N}{N-1}}$. Now we have

$$F_i(p) = \frac{\Pi_{i=1}^N F_i(p)^{N-1}}{G_{(1)}^{(-i)}(p)} = [\frac{k}{Nk-\xi}\frac{p-\underline{p}}{p-c}]^{\frac{1}{N-1}} \quad \text{for } p \in [\underline{p}, \min\{\bar{p}_n\}]$$

Since $F_i(p)$ is strictly increasing and takes value 1 at $B$, we must have $\bar{p}_i = B$ for all $i$.

## A.3 Structural Properties of Mixed-Strategy Equilibria for Symmetric UA.

**Proposition 12** *Consider a symmetric UA with $(N-1)k < \xi < Nk$. For an irreducible equilibrium $\sigma_u^*$, if $F_i(p)$ is continuous and strictly increasing in $p \in [\underline{p}_i, \bar{p}_i] \cap (c, B)$ for all $i$ with $\bar{p}_i > c$, then*

*(a) $\bar{P} \equiv \max\{\bar{p}_i\} = B$ and at most one supplier has $m(B) > 0$; moreover, at least one supplier $h$ with $\bar{p}_h = B$ has $ER_h = (B-c)[\xi-(N-1)k]$;*

*(b)* $\underline{p}_i = c$ *and* $m_i(c) = 0$ *for all* $i$ *with* $\bar{p}_i > c$*;*

*(c) for* $p \in (c, B)$ *and* $i \in I_p \equiv \{n : p < \bar{p}_i\}$*,* $F_i(p) = C^i_{I_p}(p-c)^{\frac{\xi-(N-1)k}{(N_p-1)(Nk-\xi)}}$*, where* $N_p \geq 2$ *denotes the number of players in* $I_p$ *and* $C^i_{Ip} > 0$ *remains constant for given* $I_p$*.*

**Proof.** First, due to the same logic as part (c) of Appendix A.2, there is at most one player with $m(\bar{P}) > 0$. Similarly to part (d) of Appendix A.2, we can show there exists supplier $h$ such that $\bar{p}_h = \bar{P}$ and $ER_h = (\bar{P} - c)[\xi - (N-1)k]$. The optimality of $ER_h$ requires $\bar{P} = B$.

Denote $\underline{p}^* = \max\{c, \max_n\{\underline{p}_n\}\}$ and irreducibility implies $\underline{p}^* < B$. For any $p \in (\underline{p}^*, B)$, denote $J_p \equiv \{n : \underline{p}_n < p < \bar{p}_n\}$ and $M_p = \|J_p\|$. We first show $M_p \geq 2$. Otherwise, if $M_p = 1$ ($M_p = 0$ is impossible), then we must have $\underline{p}_h < p$ and $\bar{p}_j < p$ for all $j \neq h$ [by continuity and monotonicity of $F_i$ for $i \in J_p$]. Due to the monotonicity and continuity of $F_h$ in $[p, B]$, we establish a contradiction $ER_h = \hat{R}_h(p) < \hat{R}_h(B) = ER_h$ where the two equalities result from statements (4c) and (4b).

We next derive the functional form of $F_i(p)$ for $i \in J_p$ and $p \in (\underline{p}^*, B)$. By (2.3) again, we have

$$\hat{R}_i(p) = G^{(-i)}_{(1)}(p)(p-c)[\xi - (N-1)k] + k\int_p^B (v-c)dG^{(-i)}_{(1)}(v) = ER_i.$$

The first order condition is identical to (9), so it has the same solution

$$G^{(-i)}_{(1)}(p) = \prod\nolimits_{n \in J_p \setminus \{i\}} F_n(p) = D_i(p-c)^{\frac{\xi-(N-1)k}{Nk-\xi}} \text{ where } D_i > 0.$$

Now we have

$$F_i(p) = \frac{\sqrt[M_p-1]{\Pi_{i \in J_p} G^{(-i)}_{(1)}(p)}}{G^{(-i)}_{(1)}} = C^i_{J_p}(p-c)^{\frac{\xi-(N-1)k}{(M_p-1)(Nk-\xi)}} \text{ where } C^i_{J_p} > 0.$$

It implies, for all $i \in J_p$, $F_i(p) > 0$ for $p > \underline{p}^*$, i.e., $\underline{p}_n \leq \underline{p}^*$. Note that the functional form has $F_i(p) > 0$ for all $p > c$ and $F_i^+(c) = \lim_{p' \downarrow c} F_i(p) = 0$. As $F_i(p)$ is continuous and monotone in $p \in [\underline{p}_i, \bar{p}_i] \cap (c, B)$, we must have $\underline{p}_i = c$. It further implies $\underline{p}^* = c$ and $J_p = I_p$ everywhere for $p \in [c, B]$. Now part (c) follows.

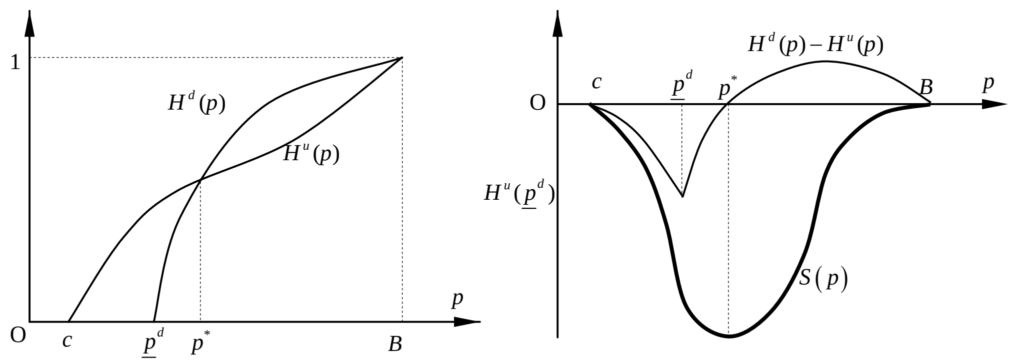



**Figure A.1.** Graphic Illustration of Stochastic Ordering

## A.4 Cumulative Distribution Functions of $P_S^d$, $P_B^d$, and $P^u$.

$$H^u = (F^u)^N, \tag{A.1}$$

$$\begin{aligned} H_S^d &= \sum_{n=2}^{N} \frac{k}{\xi} G_{(n)}^d + \frac{\xi - Nk + k}{\xi} G_{(1)}^d = \frac{\xi - Nk}{\xi} G_n + \frac{k}{\xi} \sum_{n=1}^{N} G_n \quad [\text{by } G_{(n)} = G_{N+1-n}] \\ &= \frac{\xi - Nk}{\xi} F^N + \frac{k}{\xi} \sum_{n=1}^{N} \sum_{m=n}^{N} \tbinom{N}{m} F^m \bar{F}^{N-m} = \frac{\xi - Nk}{\xi} F^N + \frac{k}{\xi} \sum_{m=1}^{N} \sum_{n=1}^{m} \tbinom{N}{m} F^m \bar{F}^{N-m} \\ &= \frac{\xi - Nk}{\xi} F^N + \frac{k}{\xi} \sum_{m=1}^{N} m \tbinom{N}{m} F^m \bar{F}^{N-m} = \frac{\xi - Nk}{\xi} F^N + \frac{kN}{\xi} F \sum_{m=1}^{N} \tbinom{N-1}{m-1} F^{m-1} \bar{F}^{N-m} \\ &= \frac{\xi - Nk}{\xi} (F^d)^N + \frac{kN}{\xi} F^d, \end{aligned}$$

$$H_B^d = B_1 * B_2 * \ldots * B_n$$

where $B_n(p) = \begin{cases} G_{(n)}^d(\frac{\xi}{k} p) & \text{for } n < N \\ G_{(n)}^d(\frac{\xi}{\xi - Nk + k} p) & \text{for } n = N \end{cases}$ and $(*)$ denotes convolution operator.

## A.5 Proof of Proposition 1.

**Proof.** As explained in the paper, for any realized bid vector, $P_B^d$ is a constant while $P_S^d$ is a random variable. Therefore, $P_S^d$ is a mean-preserving spread of $P_B^d$, so we only need to compare $P_S^d$ and $P^u$.

(a) *Equal Expected Price.* It results from $\sum_{i=1}^{N} ER_i^A + c\xi = \mathbf{E}[P^A]\xi$ for $A = d$ and $u$. Propositions 1 and 2 imply $ER_i^A = \hat{R}_i^A(B) = (B - c)[\xi - (N-1)k]$, so the desired result follows.

(b) *Stochastic Ordering.* We illustrate the proof in the above Figure A.1. Define function $S(p) \equiv \int_c^p [H_S^d(v) - H^u(v)]dv$ for $p \in [c, B]$. From part (a), we have $\int_c^B \bar{H}_S^d(v)dv + c = \mathbf{E}[P_S^d] = \mathbf{E}[P^u] = \int_c^B \bar{H}^u(v)dv + c$, implying $S(B) = \int_c^B [H_S^d(v) - H^u(v)]dv = \int_c^B [\bar{H}^u(v) - \bar{H}_S^d(v)]dv = 0$. To show $P^u$ is stochastically more variable than $P^u$, we only need to prove $S(p) \leq 0$ for all $p \in [c, B]$.

For $p \in [c, \underline{p}^d]$, as $H_S^d(p) = 0$ and $H^u(p) > 0$, $S(p) = -\int_c^p H^u(v)dv < 0$. For $p \in [\underline{p}^d, B]$, as $S(\underline{p}^d) < 0$ and $S(B) = 0$, it is sufficient to show that $S(p)$ is quasi-convex in $p \in [\underline{p}^d, B]$, that is, there exists $p^* \in (\underline{p}^d, B)$ such that $H_S^d(p) - H^u(p) < 0$ for $p < p^*$ and $H_S^d(p) - H^u(p) > 0$ for $p > p^*$. Given that $H_S^d(\underline{p}^d) - H^u(\underline{p}^d) < 0$ and $H_S^d(B) - H^u(B) = 0$, we only need to show $H_S^d(p) - H^u(p)$ is quasi-concave in $p \in [\underline{p}^d, B]$. A sufficient condition for quasi-concavity is that there exist $T_1(p)$ and $T_2(p)$ such that $\dot{H}_S^d(p) - \dot{H}^u(p) = T_1(p)T_2(p)$, $T_1(p) > 0$, and $T_2(p)$ is decreasing in $p$. Next we construct such $T_1$ and $T_2$. From (A.1),

$$\begin{aligned}
\dot{H}_S^d(p) &= \frac{\underline{p}^d - c}{\rho(N - N\rho)^{\frac{1}{N-1}}(N-1)(p-c)^2}[(\frac{p - \underline{p}^d}{p - c})^{\frac{1}{N-1}-1} - (\frac{p - \underline{p}^d}{p - c})^{\frac{1}{N-1}}] \qquad \text{(A.2)} \\
&= C_d(p - \underline{p}^d)^{\frac{2-N}{N-1}}(p-c)^{-2-\frac{1}{N-1}} \\
\dot{H}^u(p) &= (B-c)^{-\frac{N\rho - N+1}{(N-1)(1-\rho)}} \frac{N\rho - N + 1}{(N-1)(1-\rho)}(p-c)^{\frac{N\rho-N+1}{(N-1)(1-\rho)}-1} = C_u(p-c)^{\frac{\rho}{(N-1)(1-\rho)}-2}
\end{aligned}$$

where $C_d = \frac{(\underline{p}^d - c)^2}{\rho(N-N\rho)^{\frac{1}{N-1}}(N-1)} > 0$ and $C_u = (B-c)^{-\frac{N\rho-N+1}{(N-1)(1-\rho)}} \frac{N\rho-N+1}{(N-1)(1-\rho)} > 0$. Thus,

$$\begin{aligned}
\dot{H}_S^d(p) - \dot{H}^u(p) &= C_d(p - \underline{p}^d)^{\frac{2-N}{N-1}}(p-c)^{-2-\frac{1}{N-1}} - C_u(p-c)^{-2-\frac{\rho}{(N-1)(\rho-1)}} \\
&= (p-c)^{-2-\frac{\rho}{(N-1)(\rho-1)}}[C_d(p-\underline{p}^d)^{-\frac{N-2}{N-1}}(p-c)^{-\frac{1}{(\rho-1)(N-1)}} - C^u].
\end{aligned}$$

Let $T_1(p) = (p-c)^{-2-\frac{\rho}{(N-1)(\rho-1)}}$ and $T_2(p) = C_d(p-\underline{p}^d)^{-\frac{N-2}{N-1}}(p-c)^{-\frac{1}{(\rho-1)(N-1)}} - C^u$. As $\dot{T}_2(p) = -\frac{N-2}{N-1}C_d(p-\underline{p}^d)^{-\frac{N-2}{N-1}-1}(p-c)^{-\frac{1}{(\rho-1)(N-1)}} - \frac{1}{(\rho-1)(N-1)}C_d(p-c)^{-\frac{1}{(\rho-1)(N-1)}-1} < 0$ and $T_1(p) > 0$, we obtain the desired functions $T_1$ and $T_2$.

## A.6 Derivation of Expressions (2.13) and (2.14).

$$\hat{R}_i^u(p|\mathbf{b}^{-i}) = \mathbf{E}_\xi[R_i^u(p|\xi,\mathbf{b}^{-i})] = (p-c)\sum_{n=1}^{N}\delta_{(b_{n-1}<p<b_n)}\int \delta_n(\xi)[k\wedge(\xi-nk+k)]d\Phi(\xi)$$
$$+\sum_{n=1}^{N-1}\delta_{(p<b_n)}(b_n-c)\int k\delta_{n+1}(\xi)d\Phi(\xi)$$
$$=(p-c)\sum_{n=1}^{N}\delta_{(b_{n-1}<p<b_n)}Y_n + k\sum_{n=1}^{N-1}\delta_{(p<b_n)}(b_n-c)\Phi_{n+1}$$
$$\hat{R}_i(p) = \mathbf{E}_{\mathbf{b}^{-i}}[\hat{R}_i^u(p|\mathbf{b}^{-i})] = (p-c)\sum_{n=1}^{N}[G_{n-1}(p)-G_n(p)]Y_n$$
$$+k\sum_{n=1}^{N-1}\Phi_{n+1}\int_p^B(b_n-c)dG_n(b_n). \text{ [A rearrangement provides (2.13).]}$$
$$\frac{d\hat{R}_i(p)}{dp} = \sum_{n=1}^{N}[G_{n-1}(p)-G_n(p)]Y_n + (p-c)\sum_{n=1}^{N-1}\dot{G}_n(p)(Y_{n+1}-Y_n-k\Phi_{n+1})$$
$$=\sum_{n=1}^{N}[G_{n-1}(p)-G_n(p)]Y_n + (p-c)\sum_{n=1}^{N-1}\dot{G}_n(p)(Z_{n+1}-Z_n)$$
$$=\sum_{n=0}^{N-1}\binom{N-1}{n}F^n\bar{F}^{N-n-1}Y_{n+1}$$
$$+(p-c)\dot{F}\sum_{n=1}^{N-1}(N-n)\binom{N-1}{n-1}F^{n-1}\bar{F}^{N-n-1}(Z_{n+1}-Z_n),$$

where the last equality follows the established results of order statistics, $G_n = \sum_{m=n}^{N-1} F^m\bar{F}^{N-m-1}$ and $\dot{G}_n = (N-n)\binom{N-1}{n-1}F^{n-1}\bar{F}^{N-n-1}\dot{F}$. The F.O.C. $\frac{d\hat{R}_i(p)}{dp} = 0$ yields ODE (2.14).

## A.7 Proof of Proposition 2

**Proof.** Parts (a) and (b) follow exactly the same argument as in the deterministic case. We only consider the case for $\frac{\underline{\xi}}{N} < k < \frac{\bar{\xi}}{N-1}$. First, it is easy to verify that if $F^d$ satisfying (*2.12*) exists, it must be an equilibrium solution by noticing (a) $\hat{R}^d(p) < ER^d$ for all $p < \underline{p} = c + \frac{(B-c)Z_n}{Z_1}$ and (b) $\hat{R}^d(p) = ER$ for all $p \geq \underline{p}$. Similarly, it can be verified that a solution to (*2.14*) defines a symmetric mixed-strategy equilibrium for a UA. Therefore, the only thing we need to show that the solutions to (*2.12*) and (*2.14*) exist and are unique.

Unlike the deterministic cases, where unique solutions are identified in closed forms, here we have to study in more abstract forms. First consider symmetric DA. A necessary condition for equation (*2.12*) to hold is $\frac{d\hat{R}}{dp} = 0$ for $p \in [\underline{p}, B]$. It leads to the following ODE

$$\dot{F}(p) = \frac{\sum_{n=0}^{N-1}[\binom{N-1}{n}F^n(1-F)^{N-n-1}Z_n]}{(p-c)\sum_{n=1}^{N-1}[\binom{N-1}{n}nF^{n-1}(1-F)^{N-n-1}(Z_n - Z_{n+1})]} \quad \text{(A.3)}$$
$$F(B) = 1.$$

Notice that $\frac{\underline{\xi}}{N} < k < \frac{\bar{\xi}}{N-1}$ implies $Z_1 \geq Z_2 \geq \ldots \geq Z_n$ and $Z_{N-1} > Z_n$. Thus, the difference $Z_n - Z_{n+1} \geq 0$ for all $n \in \{1, \ldots, N-1\}$ and $Z_{N-1} - Z_n > 0$. It implies function $\Pi(p, F)$ defined by RHS of (A.3) is continuous and $\frac{\partial \Pi}{\partial F}$ is continuous in $F$ for any $\{F, p\} \in (0, 1] \times (c, B + \varepsilon)$ with small $\varepsilon > 0$. Therefore ODE (A.3) has one and only one solution $F(p)$ in some neighborhood of $\{F^*, p^*\} \in (0, 1] \times (c, B+\varepsilon)$ satisfying $F(p^*) = F^*$.[1] As boundary condition $\{1, B\} \in (0, 1] \times (c, B+\varepsilon)$, the solution to (A.3) exists uniquely.

For UA, clearly, ODE (*2.14*) can be rearranged into

$$\dot{F}(p) = \frac{\sum_{n=0}^{N-1}\binom{N-1}{n}(F^u)^n(\bar{F}^u)^{N-(n+1)}Y_{n+1}}{(p-c)\sum_{n=1}^{N-1}(N-n)\binom{N-1}{n-1}(F^u)^{n-1}(\bar{F}^u)^{N-n-1}(Z_n - Z_{n+1})}$$
$$F(B) = 1.$$

Similarly to the above case of DA, we can show the existence and uniqueness of the solution to the above ODE.

## A.8 $H^d$ and $H^u$ for Symmetric Auctions with Random Demand

*Symmetric DA:* (Numerical Scheme) Similarly to Appendix A.4, given equilibrium distribution function $F$, we have $G^d_{(n)}(p) = \sum_{m=n}^{N}\binom{N}{m}F^m\bar{F}^{N-m}$. Now cdf of $P^d$, conditional on demand $\xi$ is

$$H^d(\cdot|\xi) = B_1(\cdot|\xi) * B_2(\cdot|\xi) * \ldots * B_{\bar{N}(\xi)}(\cdot|\xi)$$

[1] **Existence and Uniqueness Theorem for Regular ODE:** For ODE $\dot{\mathbf{x}} = f(t, \mathbf{x})$, if $f$ is continuous and $\frac{\partial f}{\partial x_i}$ are continuous in $\mathbf{x} \in D$, $t \in I$ for all $i$, where $D$ is a domain and $I$ is an open interval, then the ODE has a solution $\mathbf{x}(t)$, defined uniquely in some neighborhood of $(\mathbf{x}^* \in D, t^* \in I)$ which satisfies $\mathbf{x}^* = \mathbf{x}(t^*)$.

where $B_n(p) = G^d_{(n)}(\frac{\xi}{z_n}p)$, $z_n = \min\{k, \xi-(n-1)k\}$ and $\bar{N}(\xi) = \left\lceil \frac{\xi}{k} \right\rceil$. Thus, $H^d(p) = \mathbf{E}_\xi[H^d(p|\xi)]$ can be computed.

*Symmetric UA:* (Analytical Scheme) From $P^u = \sum_{n=1}^N b_n^{(N)} \delta_{(nk-k<\xi\le nk)} + b_{(N)}^{(N)}\delta_{(\xi>Nk)}$,

$$\begin{aligned} H^u(p|\mathbf{b}) &= \mathbf{E}_\xi[Pr\{P^u < p|\mathbf{b}\}] = \sum_{n=1}^N \delta_{(b_n<p)}\Phi_n + \xi_{(b_n<p)}\bar{\Phi}(Nk) \\ H^u(p) &= \mathbf{E}_\mathbf{b}[H^u(p|\xi,\mathbf{b})] = \sum_{n=1}^N \Phi_n \sum_{m=n}^N \tbinom{N}{m} F^m \bar{F}^{N-m} + \bar{\Phi}(Nk)F(p)^N \\ &= \sum_{m=1}^N \tbinom{N}{m} F^m \bar{F}^{N-m} \sum_{n=1}^m [\Phi(nk) - \Phi(nk-k)] + \bar{\Phi}(Nk)F(p)^N. \end{aligned}$$

## A.9 Equilibrium Analysis for Two-Bidder Auctions with Random Demand

The following notation is useful for both auctions. For supplier $i = 1, 2$, define $K_i \equiv \mathbf{E}[k_i \wedge \xi] = \int_{\underline{\xi}}^{k_i} \xi d\Phi(\xi) + k_i\bar{\Phi}(k_i) = \int_0^{k_i} \bar{\Phi}(\xi)d\xi$ as her expected sales for $p_i < p_j$. Define $X \equiv \mathbf{E}[(k_1+k_2)\wedge\xi] = \int_0^{k_1+k_2} \bar{\Phi}(\xi)d\xi$ as the expected total sales. It is easy to verify that, in case of $p_i > p_j$, supplier $i$'s expected sales is $X - K_j$.

1. *Discriminatory Auctions.* We partition the space of capacities in the below table and illustrate them the right figure. The unique equilibrium solution is as follows.

*(a) Pure strategy equilibrium.* (a-i) For $\mathbf{k} \in \Omega_1^d$, it is a pure-strategy equilibrium with $\left\{p_1^{d*} = p_2^{d*} = B\right\}$; (a-ii) For $\mathbf{k} \in \Omega_4^d$, it is a pure-strategy equilibrium with $\left\{p_1^{d*} = p_2^{d*} = c\right\}$.

*(b) Mixed-Strategy Solution.* For $\mathbf{k} \in \Omega_3^d \cup \Omega_4^d$, it is a mixed-strategy equilibrium $(\sigma_1^*, \sigma_2^*)$ with

$$\begin{aligned} F_h^d(p) &= \frac{(p-\underline{p}^d)K_l}{(p-c_l)[K_h+K_l-\xi]} \text{ for } p \in [\underline{p}^d, B) \text{ and } m_h^d(B) = 1 - \frac{(B-c_h)K_l}{(B-c_l)(K_h\wedge\xi)}; \\ F_l^d(p) &= \frac{(p-\underline{p}^d)K_h}{(p-c_h)[K_h+K_l-\xi]} \text{ for } p \in [\underline{p}^d, B], \end{aligned}$$

where $\underline{p}^d = \frac{(B-c_h)(\xi-K_l)}{K_h\wedge\xi} + c_h$ and $(h,l) = (1,2)$ for $\mathbf{k} \in \Omega_3^d$, while $(h,l) = (2,1)$ for

$\mathbf{k} \in \Omega_4^d$.

$$\Omega_1^d \equiv \{\mathbf{k} : k_1 + k_2 \leq \underline{\xi}\}$$
$$\Omega_2^d \equiv \{\mathbf{k} : k_1 \geq \bar{\xi}, \tfrac{K_2}{B-c_2} \geq \tfrac{X}{B-c_1}\}$$
$$\Omega_3^d \equiv \{\mathbf{k} : k_1 + k_2 > \underline{\xi}, \tfrac{K_2}{B-c_2} < \tfrac{X}{B-c_1}, \tfrac{K_1}{B-c_1} \geq \tfrac{K_2}{B-c_2}\}$$
$$\Omega_4^d \equiv \{\mathbf{k} : k_1 + k_2 > \underline{\xi}, k_1 < \bar{\xi}, \tfrac{K_1}{B-c_1} \leq \tfrac{K_2}{B-c_2}\}$$

2. *Uniform Auctions.* The partition of capacity combinations is as follows and illustrated by the right figure. Note that set $\Omega_2^u$ is denoted by the grey area which may overlap with set $\Omega_{5d}^u$ (and possibly $\Omega_3^u$). We formally present the equilibrium solution.

$$\Omega_1^u \equiv \{\mathbf{k} : k_1 + k_2 \leq \underline{\xi}\}$$
$$\Omega_2^u \equiv \left\{\mathbf{k} : k_1 \geq \bar{\xi}, \tfrac{K_2}{B-c_2} \geq \tfrac{X}{B-c_1}\right\}$$
$$\Omega_3^u \equiv \{\mathbf{k} : k_1 \geq \bar{\xi}, k_2 \leq \underline{\xi}\}$$
$$\Omega_4^u \equiv \{\mathbf{k} : k_1 \geq \bar{\xi}, k_2 < \bar{\xi}\}$$
$$\Omega_{5a}^u \equiv \{\mathbf{k} : k_1 + k_2 > \underline{\xi}, k_1 \leq \underline{\xi}, k_2 \leq \underline{\xi}\}$$
$$\Omega_{5b}^u \equiv \{\mathbf{k} : k_1 \leq \underline{\xi}, \underline{\xi} < k_2 < \overline{\xi}\}$$
$$\Omega_{5c}^u \equiv \{\mathbf{k} : \underline{\xi} < k_1 < \overline{\xi}, k_2 \leq \underline{\xi}, \}$$
$$\Omega_{5d}^u \equiv \{\mathbf{k} : \underline{\xi} < k_1, \underline{\xi} < k_2, (k_1 \wedge k_2) > \bar{\xi}\}$$

*(a) Pure-Strategy Solution.* (a-1) For $\mathbf{k} \in \Omega_1^u$, it is unique with $\{p_1^{u*} = p_2^{u*} = B\}$; (a-2) For $\mathbf{k} \in \Omega_2^u$, $\{p_1^{u*} = p_2^{u*} = c_2\}$ is an equilibrium; (a-3) For $\mathbf{k} \in \Omega_3^u \cup \Omega_4^u \cup \Omega_{5a}^u \cup \Omega_{5b}^u \cup \Omega_{5c}^u$, $\{p_h^{u*} = B, p_l^{u*} = \frac{(B-c_2)(X-k_l)}{K_h} + c_h\}$ is an asymmetric pure-strategy equilibrium, where $(h,l) = (1,2)$ for $\mathbf{k} \in \Omega_{5a}^u \cup \Omega_{5c}^u \cup \Omega_3^u$; and $(h,l) = (2,1)$ for $\mathbf{k} \in \Omega_{5a}^u \cup \Omega_{5b}^u \cup \Omega_4^u$; (a-4) For $\mathbf{k} \in \Omega_{5d}^u \backslash \Omega_2^u$, there is no pure-strategy equilibrium.

*(b) Mixed-Strategy Solution.* Irreducible mixed-strategy equilibrium exists for $\mathbf{k} \in \Omega_{5(a,b,c,d)}^u$ and satisfies the following distribution function,

$$F_i^*(p; m_B^i) = \begin{cases} (\frac{\beta_i}{\lambda_i} + 1 - m_B^i)(\frac{p-c_j}{B-c_j})^{\lambda_i} - \frac{\beta_i}{\lambda_i} & \text{if } \lambda_i \neq 0 \\ \beta_i \ln \frac{p-c_j}{B-c_j} + 1 - m_B^i & \text{if } \lambda_i = 0 \end{cases} \quad \text{for } p \in [\underline{P}, B), \qquad \text{(A.4)}$$

$$\text{where } \underline{P} = \max\{\underline{p}_1, \underline{p}_2\},\ \beta_i = \frac{\int_{\underline{\xi}}^{k_j} \xi d\Phi(\xi)}{K_1 + K_2 - X} \text{ and } \lambda_i = \frac{X - K_i - \int_{\underline{\xi}}^{k_j} \xi d\Phi(\xi)}{K_1 + K_2 - X}.$$

Moreover, price bound $\underline{P}$ satisfies $\underline{P} = \max\{L_1(0), L_2(0)\}$, where function $L_i(\cdot)$ and its inverse function $M_B^i(\cdot) = L_i^{(-1)}(\cdot)$ are defined from equation $F_i^*(L_i; m_B^i) = 0$ for

$i = 1, 2,$

$$\begin{cases} L_i(m_B^i) \equiv c_j + (B - c_j)[\frac{\beta_i}{\beta_i + \lambda_i(1 - m_B^i)}]^{\frac{1}{\lambda_i}}; & M_B^i(\underline{p}_i) \equiv 1 + \frac{\beta_i}{\lambda_i} - \frac{\beta_i}{\lambda_i}(\frac{B - c_j}{\underline{p}_i - c_j})^{\lambda_i} \quad \text{if } \lambda_i \neq 0 \\ L_i(m_B^i) \equiv c_j + (B - c_j)\exp(\frac{m_B^i - 1}{\beta_i}); & M_B^i(\underline{p}_i) \equiv 1 + \beta_i \ln(\frac{\underline{p}_i - c_j}{B - c_j}) \quad \text{if } \lambda_i = 0 \end{cases}.$$

The probability masses $m_B^1$ and $m_B^2$ determine the set of irreducible equilibria. (b-1) For $\mathbf{k} \in \Omega_{5a}^u$, $\underline{P} = c_2$. Any $\{m_B^1, m_B^2\} \in [0, 1)^2$ with $m_B^1 \cdot m_B^2 = 0$ defines an equilibrium *satisfying* $F_2(p) \geq F_2^*(p, m_B^2)$ *for* $p < c_2$; (b-2) For $\mathbf{k} \in \Omega_{5d}^u$, the equilibrium is *unique* with $m_B^l = 0$ for $l = \arg\max_i \{L_i(0)\}$ and $m_B^h = M_h(\underline{P})$ for $h \neq l$; (b-3) For $\mathbf{k} \in \Omega_{5b}^u$, $m_B^1 = 0$ and any $m_B^2 \in [0, 1)$ defines an irreducible equilibrium with $m_2(\underline{P}) = F_2(\underline{P}; m_B^2)$; (b-4) For $\mathbf{k} \in \Omega_{5c}^u$ if $c_2 \leq L_2(0)$, then $m_B^2 = 0$ and any $m_B^1 \in [0, 1)$ defines an irreducible equilibrium; if $c_2 > L_2(0)$, the equilibrium is *unique* $m_B^1 = 0$, and $m_B^2 = M_B^2(c_2)$.

# APPENDIX B

# Proofs and Analysis for Chapter 3

## B.1 Pure-Strategy Equilibrium Analysis

Based on Lemma 16, equation (3.1) can be simplified as,

$$\begin{aligned} r_i(p_i, \mathbf{p}_{-i}) &= \min\left\{1, \frac{[d - Q_i^L(p_i, \mathbf{p}_{-i})]^+}{x_i + Q_i^E(p_i, \mathbf{p}_{-i})}\right\} \\ \text{where } Q_i^L(p_i, \mathbf{p}_{-i}) &= \sum_{k \neq i} x_k \delta_{(p_k < p_i)} + \sum_{k \neq i} x_k \delta_{(p_k = p_i, c_k < c_i)} \\ Q_i^E(p_i, \mathbf{p}_{-i}) &= \sum_{k \neq i} x_k \delta_{(p_k = p_i, c_k = c_i)}. \end{aligned} \tag{B.1}$$

Note that, $p_k \in [c_k, b]$ implies,

$$Q_i^L(p_i, \mathbf{p}_{-i}) + Q_i^E(p_i, \mathbf{p}_{-i}) = \sum_{k \neq i} x_k \delta_{(p_k \leq p_i)} - \sum_{k \neq i} x_k \delta_{(p_k = p_i, c_k > c_i)}. \tag{B.2}$$

Notice that, when the total active capacities at price $p$ do not exceed the demand, all suppliers pricing below or at $p$ have sales ratio of 1. That is,

$$\textstyle\sum_k x_k \delta_{(p_k \leq p)} \leq d \text{ implies } r_k(p_k, \mathbf{p}_{-k}) = 1 \text{ for all } k \text{ with } p_k \leq p. \tag{B.3}$$

Similarly, we have

$$\textstyle\sum_k x_k \delta_{(p_k < p)} \leq d \text{ implies } r_k(p_k, \mathbf{p}_{-k}) = 1 \text{ for all } k \text{ with } p_k < p. \tag{B.4}$$

Consider a pure-strategy equilibrium outcome $\mathbf{p}^* = \{p_1^*, p_2^*, \ldots, p_N^*\}$ and recall $A = \{i : r_i(p_i^*, \mathbf{p}_{-i}^*) > 0\}$, the set of active suppliers (those with positive market share). Lemma 17 shows that all active players must choose the same bid price in equilibrium, and also describes features of the set of active suppliers.

**Lemma 4** *For any pure-strategy equilibrium $\mathbf{p}^*$, there exists a constant $P^*$ such that (a) $i \in A$ implies $p_i^* = P^*$; (b) $c_i < P^*$ implies $i \in A$, so $i \notin A$ implies $c_i \geq P^*$; (c) $c_i > P^*$ implies $i \notin A$; (d) $p_i \geq P^*$ for all $i$.*

**Proof.** Proof of Lemma 17 For part (a) it is equivalent to show that any $i, j \in A$ must have $p_i^* = p_j^*$. Suppose $p_i^* < p_j^*$ and $r_j(p_j^*, \mathbf{p}_{-j}^*) > 0$, implying $\sum_k x_k \delta_{(p_k^* < p_j^*)} < d$. This relation will not change if supplier $i$ raises her price to $p_i' = \frac{p_i^* + p_j^*}{2}$ since $\delta_{(p_i^* < p_j^*)} = \delta_{(p_i' < p_j^*)} = 1$, suggesting $r_i(p_i', \mathbf{p}_{-i}^*) = r_i(p_i^*, \mathbf{p}_{-i}^*) = 1$, implying $R_i(p_i') > R_i(p_i^*)$, a contradiction to the optimality of $p_i^*$.

(b) Suppose $c_i < P^*$ but $r_i(p_i^*, \mathbf{p}_{-i}^*) = 0$ so $R_i(p_i^*, \mathbf{p}_{-i}^*) = 0$. But, if supplier $i$ bids $p_i' = P^*$, she would have $R_i(P^*, \mathbf{p}_{-i}^*) = (P^* - c_i) x_i r_i(P^*, \mathbf{p}_{-i}^*) > 0$, contradicting the optimality of $p_i^*$.

(c) As $p_i \geq c_i$ always, we must have $p_i^* \geq c_i > P^*$, implying $i \notin A$ from the converse of part (a).

(d) From (a) and (b) if $c_i < P^*$ then $p_i = P^*$. If $c_i \geq P^*$ then $p_i \geq c_i \geq P^*$. ■

The next lemma shows that agent $i$'s sales ratio $r_i(\mathbf{p})$, or equivalently the sales $z_i(\mathbf{p})$, is (weakly) increasing in her opponents' bid price and (weakly) decreasing in her own bid price.

**Lemma 5** *$r_i(\mathbf{p})$ is nondecreasing in $p_j$ for $j \neq i$ and nonincreasing in $p_i$.*

**Proof.** Proof of Lemma 18 (i) We prove the monotonicity of $r_i(\mathbf{p})$ in $p_j$ with $j \neq i$ in three steps.

(i-a) $Q_i^L$ is weakly decreasing in $p_j$. From (C.1), $Q_i^L$ involves two groups of indices. When $p_j$ increases, index $j$ can disappear from the first group ($\delta_{(p_j < p_i)}$ changing from 1 to 0), or disappear from the second group ($\delta_{(p_j = p_i, c_j < c_i = p_i)}$ changing from 1 to 0), or move from the first group to the second. For any one of above cases, $Q_i^L$ does not increase.

(i-b) $Q_i^L + Q_i^E$ is weakly decreasing in $p_j$. From (C.2), $Q_i^L + Q_i^E$ is a difference of two groups of indices. When $p_j$ increases, index $j$ can disappear from the first group ($\delta_{(p_j \leq p_i)}$ changing from 1 to 0), or disappear from group 2 ($\delta_{(p_j = p_i, c_j > c_i)}$ changing from 1 to 0) or both. When it disappears from the second group, it also disappears from the first group.

(i-c) Consider any $p_j' < p_j''$. We want to show that

$$1 \wedge \frac{[d - Q_i^L(p_j', \mathbf{p}_{-j})]^+}{x_i + Q_i^E(p_j', \mathbf{p}_{-j})} \leq 1 \wedge \frac{[d - Q_i^L(p_j'', \mathbf{p}_{-j})]^+}{x_i + Q_i^E(p_j'', \mathbf{p}_{-j})}, \tag{B.5}$$

If $Q_i^L(p_j', \mathbf{p}_{-j}) \geq d$, then the left hand side of (C.6) is 0 and the desired inequality holds trivially. If $Q_i^L(p_j', \mathbf{p}_{-j}) < d$, from (i-a) and $p_j' < p_j''$, we have $Q_i^L(p_i, p_j'', \mathbf{p}_{-i-j}) \leq Q_i^L(p_j', \mathbf{p}_{-j}) < d$. Now it is sufficient to show that

$$\frac{x_i + Q_i^E(p_j', \mathbf{p}_{-j})}{d - Q_i^L(p_j', \mathbf{p}_{-j})} \geq \frac{x_i + Q_i^E(p_j'', \mathbf{p}_{-j})}{d - Q_i^L(p_j'', \mathbf{p}_{-j})},$$

i.e., $1 + \frac{x_i - d + Q_i^E(p_j', \mathbf{p}_{-j}) + Q_i^L(p_j', \mathbf{p}_{-j})}{d - Q_i^L(p_j', \mathbf{p}_{-j})} \geq 1 + \frac{x_i - d + Q_i^E(p_j'', \mathbf{p}_{-j}) + Q_i^L(p_j'', \mathbf{p}_{-j})}{d - Q_i^L(p_j'', \mathbf{p}_{-j})}$. Using (i-a) and (i-b), the above inequality follows.

(ii) Now we show $r_i(\mathbf{p})$ is nonincreasing in $p_i$. Part (i) implies that for $k \neq i$, $z_k(p_k, p_i, \mathbf{p}_{-i-k}) = x_k r_k(p_k, p_i, \mathbf{p}_{-i-k})$ is nondecreasing in $p_i$. Since $\sum_k z_k(p_k, p_i, \mathbf{p}_{-i-k}) = \min\{d, \sum_k x_k\}$ is nonincreasing in $p_i$, we must have that $z_i(p_i, \mathbf{p}_{-i}) = \min\{d, \sum_{k=1}^N x_k\} - \sum_{k \neq i} z_k(p_k, p_i, \mathbf{p}_{-i-k})$ is nonincreasing in $p_i$. ■

Denote $r_i^-(p_i, \mathbf{p}_{-i}) = \lim_{p_i' \uparrow p_i} r_i(p_i', \mathbf{p}_{-i})$ and $r_i^+(p_i, \mathbf{p}_{-i}) = \lim_{p_i' \downarrow p_i} r_i(p_i', \mathbf{p}_{-i})$. Similar notation applies to $R_i^-(p_i, \mathbf{p}_{-i})$ and $R_i^+(p_i, \mathbf{p}_{-i})$.

**Lemma 6** *For any price $p_i > c_i$,*
*(a) $r_i^-(p_i, \mathbf{p}_{-i}) = 1 \wedge \frac{(d - \sum_{k \neq i} x_k \delta_{(p_k < p_i)})^+}{x_i}$ and $r_i^+(p_i, \mathbf{p}_{-i}) = 1 \wedge \frac{(d - \sum_{k \neq i} x_k \delta_{(p_k \leq p_i)})^+}{x_i}$;*
*(b) $r_i^-(p_i, \mathbf{p}_{-i}) \geq r_i(p_i, \mathbf{p}_{-i}) \geq r_i^+(p, \mathbf{p}_{-i})$.*
*(c) $p_i^* > c_i$ at equilibrium implies $r_i^-(p_i^*, \mathbf{p}_{-i}^*) = r_i(p_i^*, \mathbf{p}_{-i}^*)$.*

**Proof.** Proof of Lemma 19 (a) First notice that for any $p_i$,

$$\lim_{p_i' \uparrow p_i} \delta_{(p_k < p_i')} = \delta_{(p_k < p_i)}, \quad \lim_{p_i' \downarrow p_i} \delta_{(p_k < p_i')} = \delta_{(p_k \leq p_i)}, \text{ and } \lim_{p_i' \to p_i} \delta_{(p_k = p_i')} = 0.$$

Substituting the above expressions into (C.1), we have $\lim_{p_i' \uparrow p_i} Q_i^L(p_i', \mathbf{p}_{-i}) = \sum_{k \neq i} x_k \delta_{(p_k < p_i)}$, $\lim_{p_i' \downarrow p_i} Q_i^L(p_i', \mathbf{p}_{-i}) = \sum_{k \neq i} x_k \delta_{(p_k \leq p_i)}$, and $\lim_{p_i' \to p_i} Q_i^E(p_i', \mathbf{p}_{-i}) = 0$, which implies (a).

(b) Part (b) is directly implied by monotonicity of $r_i(p_i, \mathbf{p}_{-i})$ established in Lemma 18.

(c) $R_i^-(\mathbf{p}^*) = (p_i^* - c_i) x_i r_i^-(\mathbf{p}^*) \leq R_i(\mathbf{p}^*)$ at equilibrium or else agent $i$ would deviate downward from $p_i^*$ if possible. When $p_i^* > c_i$ this implies $r_i^-(\mathbf{p}^*) \leq r_i(\mathbf{p}^*)$. But by part (b), $r_i^-(\mathbf{p}^*) \geq r_i(\mathbf{p}^*)$, so $r_i^-(\mathbf{p}^*) = r_i(\mathbf{p}^*)$ at equilibrium. ■

We can now formally present the necessary conditions for a pure strategy equilibrium. Denote the set of profitable suppliers at equilibrium $\mathbf{p}^*$ by $\Pi = \{i : R_i(\mathbf{p}^*) > 0\}$.

clearly, $\Pi = \{i : c_i < P^*\} \subset A$ (from Lemma 17b) and $i \in A\backslash\Pi$ implies $c_i = P^*$. Proposition 17 justifies the intuition established earlier regarding the bid prices and also shows that only a profitable player uses all of her capacity at equilibrium.

**Proposition 13** *A pure-strategy equilibrium* $\mathbf{p}^*$ *satisfies (i)* $p_i^* = P^*$ *for all* $i \in A$ *where* $P^*$ *is defined in (C.7); (ii)* $r_i(\mathbf{p}^*) = 1 \wedge \frac{d}{x_i}$ *for all* $i \in \Pi$*; and (iii)* $r_i(\mathbf{p}^*) = \frac{d-\sum_{k\in\Pi} x_k}{\sum_{k\in A\backslash\Pi} x_k} < 1 \wedge \frac{d-\sum_{k\in\Pi} x_k}{x_i}$ *for all* $i \in A\backslash\Pi$.

**Proof.** Proof of Proposition 17 (i.a) Suppose $P^* < c_2$. It is impossible when $c_1 = c_2$, since $P^* \geq c_k \geq c_1$ for all $k$. When $c_1 < c_2$, $P^* < c_2$ implies that $A = \{1\}$ and any price $p_1' \in (P^*, c_2)$ yields the same ratio $r_1$ and a strictly higher payoff, which is a contradiction. Suppose $P^* > c_2$. From Lemma 17(b), we have $\{1, 2\} \subset \Pi$, and therefore, $r_1(P^*, \mathbf{p}_{-1}^*) = \frac{d}{\sum_{k\in A} x_k} \leq \frac{d}{x_1+x_2} < \frac{d}{x_1} = r_1^-(P^*, \mathbf{p}_{-1}^*)$, which contradicts Lemma 19(c). Therefore, we must have $P^* = c_2$.

(i.b) Suppose $P^* < c_j$. Then $j \notin A$, by Lemma 17(c), and $r_j(\mathbf{p}) = 0$. Condition (b) implies $\sum_{k=1}^{j-1} x_k \leq d$ and $r_k = 1$, for $k = 1$ to $j - 1$. But, $r_k = 1$ is equal to 1, as long as $p_k < c_j$. Thus, these agents will want to defect to a price greater than $P^*$, contradicting the optimality of $p_k^* = P^*$, and we must have $P^* \geq c_j$. $P^* > c_j$ cannot occur, because then at least one agent $k$ $(1 \leq k \leq j)$ must have $r_k(\mathbf{p}) < 1$, which means $r_k^-(\mathbf{p}) = 1$ contradicting Lemma 19(c). So $P^* = c_j$.

(i.c) From (C.4), if $d \geq \sum_k x_k$, then $r_i(p) = 1$ for all $i$ and $p$ and the only possible profit maximizing price for any agent is $b$.

(ii) From Lemma 17(b) and (c), $\{i|c_i < P^*\} \subseteq A \subseteq \{i|c_i \leq P^*\}$. So $\Pi = A \cap \{i|c_i < P^*\} = \{i|c_i < P^*\} \subseteq A \subseteq \{i|c_i \leq P^*\}$. For case (a) in equation (C.7), we have either $\Pi = \varnothing$, when $c_1 = c_2$ or $\Pi = \{1\}$ and $r_1(\mathbf{p}^*) = \frac{d}{x_1} < 1$, when $c_1 < c_2$, so part (ii) holds in that case. For cases (b) and (c), we have from equation (C.7), that $\sum_{k\in\Pi} x_k = \sum_k x_k \delta_{(c_k<P^*)} \leq d$ so $r_i(\mathbf{p}^*) = 1$, for all $i \in \Pi$ and again part (ii) hold.

(iii) For any firm $i$ in $A\backslash\Pi$, $Q_i^E(\mathbf{p}^*) = \sum_{k\neq i, k\in A\backslash\Pi} x_k$ so $x_i + Q_i^E(\mathbf{p}^*) = \sum_{k\in A\backslash\Pi} x_k$ is a constant for all such firms. Also, for $i$ in $A\backslash\Pi$, $Q_i^L(\mathbf{p}^*) = \sum_{k\in\Pi} x_k$, is also constant. That is, the allocation $r_i = r$ is a constant for all firms in $A\backslash\Pi$. But that means

$$d = \sum_{k\in A} x_k r_k(\mathbf{p}^*) = \sum_{k\in\Pi} x_k + r \sum_{k\in A\backslash\Pi} x_k, \text{ so } r = \frac{d - \sum_{k\in\Pi} x_k}{\sum_{k\in A\backslash\Pi} x_k} \text{ for all firms in } A\backslash\Pi.$$

All firms in $A$ bid $P^*$, so either all firms are in $A$ (in which case $P^* = b$ and $r_i = \frac{d}{\sum_k x_k}$ for all $i$) or the next highest bid price (from a firm not in $A$) is strictly greater than $P^*$. Suppose $r = 1$ for all firms in $A \backslash \Pi$ and $d = \sum_{k \in A} x_k$. Then, $r_i$ would remain equal to one if firm $i$ raised her price bid. This cannot be in equilibrium, so we must have $r_i < 1$ for $i \in A \backslash \Pi$. If there is more than one firm in $A \backslash \Pi$ then $r_j = \frac{d - \sum_{k \in \Pi} x_k}{\sum_{k \in A \backslash \Pi} x_k} < 1 \wedge \frac{d - \sum_{k \in \Pi} x_k}{x_j}$ for all $j \in A \backslash \Pi$. Suppose now that just one firm $j$ is in $A \backslash \Pi$, then $r_j = \frac{d - \sum_{k \in \Pi} x_k}{x_j}$, but she would defect to a higher price since $r_j = \frac{d - \sum_{k \in \Pi} x_k}{x_j}$ for all $p \in [c_j, c_{j+1}]$. It contradicts the optimality of $p_j^* = P^*$. ■

Lemma 20 formally shows that, for any equilibrium in the bidding game, there exists a normalized, payoff equivalent equilibrium. It further implies that for any game, if pure-strategy equilibrium exists, there is a unique normalized equilibrium.

**Lemma 7** *(a) If a pure-strategy equilibrium* $\mathbf{p}^*$ *has* $p_i^* > c_i$ *for any* $i \notin \Pi$, *then* $(c_i, \mathbf{p}_{-i}^*)$ *is also an equilibrium and it is payoff-equivalent to* $\mathbf{p}^*$*; (b) If a pure-strategy equilibrium exists, then the game has a unique normalized equilibrium* $\{p_i^* = c_i \wedge P^*$ *for all* $i\}$ *with* $P^*$ *given by (C.7).*

**Proof.** Proof of Lemma 20 (a) $i \notin \Pi$ implies $R_i(\mathbf{p}^*) = 0$ and agent $i$ is not economically affected by changing $p_i^*$ to $c_i$. We need, however, to show that no other agent $j \neq i$ has an incentive to defect from $p_j^*$, after this adjustment. From Lemma 18, $r_j$ is nonincreasing in $p_i$, so $r_j(p_j^*, c_i, \mathbf{p}_{-i-j}^*) \geq r_j(p_j^*, p_i^*, \mathbf{p}_{-i-j}^*)$ and $R_j(p_j^*, c_i, \mathbf{p}_{-i-j}^*) \geq R_j(p_j^*, p_i^*, \mathbf{p}_{-i-j}^*)$ for all $j$. If $j \in \Pi$, implying $c_j < p_j^* = P^* \leq c_i < p_i$, agent $j$ has a strictly higher allocation priority over agent $i$ regardless of whether agent $i$ bids $p_i$ or $c_i$, so agent $j$'s allocation and optimal response are unaffected by this change. If $j \notin \Pi$ then the above and the non-negativity of $R_j$ imply $0 \leq R_j(p_j, c_j, \mathbf{p}_{-i-j}) = 0$ so again there is no effect.

(b) From Proposition 17 and part (a), for any game with a pure-strategy equilibrium, $p_i^* = c_i \vee P^*$ defines a normalized pure-strategy equilibrium. Uniqueness follows because $P^*$ is uniquely defined in (C.7) and stays the same when we interchange the indices among players with the same cost. ■

Evaluation of the profits from defection allows us to state necessary and sufficient conditions for a pure-strategy equilibrium to exist. We only need to consider incentives for agents to raise price to higher competitors' bid levels, where the upper limit $b$ is also a potential target level. To simplify notation, let $c_{N+1} = b$.

**Proposition 14** *A unique normalized pure-strategy equilibrium exists with $p_i^* = P^* \vee c_i$ and $P^*$ given by (C.7) if and only if*

$$(P^* - c_i)(x_i \wedge d) \geq (c_j - c_i)(d - \sum_{k \neq i} x_k \delta_{(c_k < c_j)})^+ \tag{B.6}$$

*for all $i$ and $j$ such that $c_i \leq P^*$ and $j > i$, where $c_{N+1} = b$.*

**Proof.** Proof of Proposition 18 If a normalized pure-strategy equilibrium exists, by Lemma 20 (b), $p_i^* = P^* \vee c_i$ must be the equilibrium bid price. We seek conditions under which agents wish to defect from $p_i^* = P^* \vee c_i$. Consider three cases, $c_i < P^*$, $c_i > P^*$, and $c_i = P^*$.

(a) $c_i < P^*$: In this case $i \in \Pi \subseteq A$, $p_i^* = P^*$, and $r_i = 1 \wedge \frac{d}{x_i}$ from Lemma 17 and Proposition 17, so $R_i(P^*, \mathbf{p}_{-i}^*) = (P^* - c_i)(x_i \wedge d)$. $r_i$ is unaffected if supplier $i$ lowers her price, so there is no incentive to do that. If supplier $i$ raises her price she will raise it up to a level equal to some $c_j > P^* \geq c_i$, because there is no benefit to stopping between cost levels. If agent $i$ bids $c_j > P^*$ she will be allocated the residual demand $(d - \sum_{k \neq i} x_k \delta_{(c_k < c_j)})^+$. (Any agent $k$ with $c_k = c_j$ is dominated by agent $i$ by (C.1).) Thus, agent $i$ with $c_i < P^*$ will not defect to $c_j > P^*$ if and only if (C.8) holds.

(b) $c_i > P^*$: In this case $i \notin A$, $p_i^* = c_i$, and $R_i(\mathbf{p}^*) = 0$, by Lemma 17 and normalization. Agent $i$ cannot reduce her price below $c_i$ profitably, and since $r_i$ is non-increasing it will remain zero for $p_i$ above $c_i$. So, player $i$ has no incentive to deviate.

(c) $c_i = P^*$: In this case, $p_i^* = P^* = c_i$ and agent $i$ is making no profit, $R_i(p^*) = 0$. Lowering price cannot be profitable. If $r_i = 0$ then raising price cannot be profitable, either because $r_i$ is non-increasing. If $r_i > 0$ then by raising price to $c_j > c_i$ supplier $i$ will capture the residual demand $(d - \sum_{k \neq i} x_k \delta(c_k < c_j))^+$, and will be strictly better off (relative to zero profits) if this residual demand is positive. That is, agent $i$ will defect if and only if $d - \sum_{k \neq i} x_k \delta_{(c_k < c_j)} > 0$, which together with $R_i(p^*) = (P^* - c_i)(x_i \wedge d) = 0$ is equivalent to $(P^* - c_i)(x_i \wedge d) < (c_j - c_i)(d - \sum_{k \neq i} x_k \delta_{(c_k < c_j)})$.

Putting these three parts together, we see that agents with cost greater than $P^*$ never have an incentive to defect, and all other agents will defect if and only if $(P^* - c_i)(x_i \wedge d) < (c_j - c_i)(d - \sum_{k \neq i} x_k \delta(c_k < c_j))$. ■

## B.2 Mixed-Strategy Equilibrium

Proposition 18 suggests that there is no pure-strategy equilibrium if there exist $i$ and $j$ such that

$$(P^* - c_i)(x_i \wedge d) < (c_j - c_i)(d - \sum_{k \neq i} x_k \delta_{(c_k < c_j)})^+ \text{ for } c_i \leq P^* \text{ and } c_j > P^*. \quad \text{(B.7)}$$

*Throughout this section, we assume (C.9) holds for certain $i$ and $j$ and seek mixed-strategy Nash equilibria.* We first show, in Lemma 22, that the upper bound of active bids, $\overline{P}$, separates the profitable players and non-profitable ones according to whether an agent's cost is lower than $\overline{P}$ or not. Lemma 22 generalizes Lemma 17 in mixed-strategy space, which helps to justify our normalization of the mixed-strategy equilibria in Lemma 23.

Denote $\bar{r}_i^-(p, \sigma_{-i}) = \lim_{p' \uparrow p} \bar{r}_i(p', \sigma_{-i})$ and $\bar{r}_i^+(p, \sigma_{-i}) = \lim_{p' \downarrow p} \bar{r}_i(p', \sigma_{-i})$. Correspondingly, we have $\bar{R}_i^-(p, \sigma_{-i}) = (p - c_i) x_i \bar{r}_i^-(p, \sigma_{-i})$ and $\bar{R}_i^+(p, \sigma_{-i}) = (p - c_i) x_i \bar{r}_i^+(p, \sigma_{-i})$. As $r_i(p, \mathbf{p}_{-i}) \in [0, 1]$ for all $\mathbf{p}$, from *bounded convergence theorem*[1] and Lemma 19(a), we must have

$$\begin{aligned} \bar{r}_i^-(p, \sigma_{-i}) &= \mathsf{E}[r_i^-(p, \sigma_{-i})] = \mathsf{E}[1 \wedge \frac{(d - \sum_{k \neq i} x_k \delta_{(\sigma_k < p)})^+}{x_i}] \\ \bar{r}_i^+(p, \sigma_{-i}) &= \mathsf{E}[r_i^+(p, \sigma_{-i})] = \mathsf{E}[1 \wedge \frac{(d - \sum_{k \neq i} x_k \delta_{(\sigma_k \leq p)})^+}{x_i}]. \end{aligned} \quad \text{(B.8)}$$

Applying Lemma 19(b), we have

$$\bar{r}_i^-(p, \sigma_{-i}) \geq \bar{r}_i(p, \sigma_{-i}) \geq \bar{r}_i^+(p, \sigma_{-i}) \quad \text{(B.9)}$$

The next lemma is instrumental and it shows that, if a certain price is chosen with a positive probability, it must yield the expected equilibrium payoff $ER_i$. Similarly, if some prices are chosen in any small neighborhood of $p$, i.e., $F_i(\cdot)$ is strictly increasing

[1] Refer to standard textbooks of probability and measure theory such as Billingsley (1995).

in the left or right neighborhood of price $p$, then supplier $i$'s equilibrium payoff can be achieved at the left limit of her function at $p$. Lemma 21 states a slightly more general result.

**Lemma 8** *A mixed-strategy equilibrium satisfies the following, for any* $p \in [\underline{p}_i, \bar{p}_i]$,

*(a) if* $m_i(p) > 0$, *then* $\bar{R}_i^-(p) = \bar{R}_i(p) = ER_i$;

*(b) if* $F_i(p') < F_i^-(p)$ *for all* $p' < p$, *then* $\bar{R}_i^-(p) = ER_i$;

*(c) if* $F_i(p) < F_i(p')$ *for all* $p' > p$, *then* $\bar{R}_i^-(p) = \bar{R}_i(p) = \bar{R}_i^+(p) = ER_i$.

**Proof.** Proof of Lemma 21 Notice that for any $p > c_i$, from inequality (C.12) and optimality of $ER_i$, we have

$$\bar{R}_i^+(p) \leq \bar{R}_i(p) \leq \bar{R}_i^-(p) \leq ER_i. \tag{B.10}$$

For part (a), it is sufficient that $ER_i = \bar{R}_i(p)$, which clearly holds. For part (b), suppose there exists $p$ such that $F_i(p') < F_i^-(p)$ for all $p' < p$, but $ER_i > \bar{R}_i^-(p)$. It implies $\bar{r}_i^-(p) = \frac{\bar{R}_i^-(p)}{(p-c_i)x_i} < \frac{ER_i}{(p-c_i)x_i}$, that is, there exists $\delta_L > 0$ such that $\bar{r}_i(p') < \frac{ER_i}{(p'-c_i)x_i}$ for all $p' \in \mathbf{s}_L \equiv (p - \delta_L, p)$. We then have $\bar{R}_i(p') = (p' - c_i)x_i\bar{r}_i(p') < ER_i$ for all $p' \in \mathbf{s}_L$, implying $\Pr\{\sigma_i^* \in \mathbf{s}_L\} = 0$, a contradiction to $F_i(p') < F_i^-(p)$ for all $p' < p$. For part (c), by inequality (C.13), we only need to show $ER_i = \bar{R}_i^+(p)$. Since $F_i(p')$, as a cdf, is right continuous, we have $F_i^+(p) = F_i(p)$. Suppose $F_i^+(p) = F_i(p) < F_i(p')$ for all $p' > p$ but $ER_i > \bar{R}_i^+(p)$. Similar to part (b), there exists $\delta_R > 0$ such that $\bar{R}_i(p') < ER$ for all $p' \in \mathbf{s}_R = (p, p + \delta_R)$, implying $\Pr\{\sigma_i^* \in \mathbf{s}_R\} = 0$. This is a contradiction to $F_i^+(p) = F_i(p) < F_i(p')$, for all $p' > p$. ■

Notice that, for price $\underline{p}_i = \inf\{p|F_i(p) > 0\}$, we have either $m_i(\underline{p}_i) > 0$ or $\{m_i(\underline{p}_i) = 0$ and $F_i(p) > 0$ for all $p > \underline{p}_i\}$, so parts (a) or (c) of Lemma 21 apply. Similarly, for $\bar{p}_i$, part (a) or (b) of Lemma 21 can be applied. Therefore, we have a useful relationship,

$$\bar{R}_i^-(\underline{p}_i) = \bar{R}_i(\underline{p}_i) = ER_i = \bar{R}_i^-(\bar{p}_i). \tag{B.11}$$

Define the set of profitable suppliers as $\Pi := \{k : ER_k > 0\}$. Next, we show that price $\overline{P} = \max_{i \in \Pi}\{\bar{p}_i\}$ separates profitable and non-profitable suppliers according to whether their costs are less than $\overline{P}$ or not. Note that our proof relies crucially on the independence of suppliers' mixed-strategies.(That is, randomizing suppliers do not condition their behavior on any jointly observable events.) It allows us to use product of probabilities for the joint outcome of the bid vector. This assumption is used through the whole section and will also be used in the proofs of Lemma 25 and Proposition 16.

**Lemma 9** *$ER_i > 0$ if and only if $c_i < \overline{P}$.*

**Proof.** Proof of Lemma 22 Necessity is obvious from how $\overline{P}$ is defined. For sufficiency, suppose there exists $i$ satisfying $c_i < \overline{P}$ but $ER_i = 0$. We must have $\bar{r}_i(p) = 0$ for any $p \in (c_i, \overline{P})$. It implies $\Pr\{d \leq \sum_{k \neq i} x_k \delta_{(\sigma_k^* < p)}\} = 1$, or equivalently, $\Pr\{d > \sum_{k \neq i} x_k \delta_{(\sigma_k^* < p)}\} = 0$. On the other hand, $\overline{P} > p$ indicates that there must be certain player $h \in \Pi$ pricing over $(p, \overline{P}]$ with a positive probability, i.e., $\Pr\{\sigma_h^* > p\} > 0$. As supplier $h$ achieves $ER_h > 0$ by choosing certain price higher than $p$, her expected sales at price $p$ must be positive, implying, $\Pr\{d > \sum_{k \neq h} x_k \delta_{(\sigma_k^* < p)}\} > 0$. This leads to a contradiction,

$$\begin{aligned} 0 &= \Pr\{d > \sum_{k \neq i} x_k \delta_{(\sigma_k^* < p)}\} \geq \Pr\{d > \sum_{k} x_k \delta_{(\sigma_k^* < p)}\} \\ &\geq \Pr\{\sigma_h^* > p\} \Pr\{d > \sum_{k} x_k \delta_{(\sigma_k^* < p)} | \sigma_h > p\} \\ &= \Pr\{\sigma_h^* > p\} \Pr\{d > \sum_{k \neq h} x_k \delta_{(\sigma_k^* < p)}\} > 0. \end{aligned}$$

■

Similarly to the pure-strategy equilibrium analysis, in the rest of this section, we restrict ourselves to normalized equilibria by fixing non-profitable players' bidding strategy. The following lemma justifies the generality of our assumption.

**Lemma 10** *If $\sigma^* = \{\sigma_i^*, \sigma_{-i}^*\}$ is an equilibrium with $ER_i = 0$ and $\bar{p}_i > c_i$, then $\{p_i = c_i, \sigma_{-i}^*\}$ is a payoff-equivalent equilibrium to $\sigma^*$.*

**Proof.** Proof of Lemma 23 As $ER_i = \bar{R}_i(\sigma^*) = 0 = \bar{R}_i(c_i)$, agent $i$ herself is not economically affected by changing from $\sigma_i^*$ to $c_i$. For any other player $j \neq i$, since $\sigma_i^* \in \mathcal{P}([c_i, b])$ is (weakly) larger than $p_i = c_i$, the monotonicity of $r_j(\cdot)$ in $p_i$ (Lemma 18) implies $\Pr\left\{r_j(p_j, c_i, \sigma_{-j-i}^*) \leq r_j(p_j, \sigma_i^*, \sigma_{-j-i}^*)\right\} = 1$ for any $p_j$, and therefore,

$$\bar{R}_j(p_j, c_i, \sigma_{-j-i}^*) \leq \bar{R}_j(p_j, \sigma_i^*, \sigma_{-j-i}^*) \leq ER_j \quad \text{for all } p_j \geq c_j. \tag{B.12}$$

That is, given $c_i$ and $\sigma^*_{-j-i}$, $ER_j$ is an upper bound of supplier $j$'s payoff. Consequently, it is sufficient to show that strategy $\sigma^*_j$ maximizes $j$'s expected payoff and $\bar{R}_j(\sigma^*_j, c_i, \sigma_{-j-i}) = \bar{R}_j(\sigma^*_j, \sigma_i, \sigma_{-j-i}) = ER_j$. For a nonprofitable supplier $j \neq i$, this is trivially true since (C.15) implies $j$'s expected payoff is 0 everywhere. For a profitable supplier $j$ with $ER_j > 0$, Lemma 22 implies $c_j < \bar{p}_j \leq \overline{P} \leq c_i \leq \underline{p}_i$. By allocation rule (C.1), $\sigma^*_j$ yields a strictly higher allocation priority than both $\sigma^*_i$ and $p_i = c_i$. Therefore, her expected payoff by choosing $\sigma^*_j$ will not be affected by player $i$'s strategy adjustment.

■

Next we characterize the analytical properties of the equilibrium payoff functions. First we point out a useful inequality

$$\bar{r}_i^-(p) \geq \frac{(d - \sum_{k\neq i} x_k \delta_{(c_k<p)})^+}{x_i} \quad \text{for all } p \geq \max\{P^*, c_i\}. \tag{B.13}$$

Notice that $r_i^-(p) = 1 \wedge \frac{(d-\sum_{k\neq i} x_k \delta_{(\sigma^*_k<p)})^+}{x_i}$ is decreasing in $\sigma^*_k$ for $k \neq i$. As $\sigma^*_k \geq c_k$ and $d < \sum_k x_k \delta_{(c_k<p)} \leq \sum_{k\neq i} x_k \delta_{(c_k<p)} + x_k$, the term in the right-hand side of (C.16) is the lowest value for $r_i^-(p)$, and (C.16) follows directly. Our analysis starts with the lower price bound $\underline{P}$.

**Lemma 11** *(a) If $i \in \Pi$ and $\underline{p}_i = \underline{P}$, then $ER_i = (\underline{P} - c_i)(x_i \wedge d)$; (b) If $\sum_{k\neq i} x_k \delta_{(c_k<\underline{P})} \leq d$ (implied by $\sum_{k\in\Pi\setminus\{i\}} x_k \leq d$), then $\underline{p}_i = \underline{P}$; (c) $\overline{P} > \underline{P} > P^*$.*

**Proof.** Proof of Lemma 24 (a) It directly follows from (C.14) and $\bar{r}_i^-(\underline{P}) = 1 \wedge \frac{d}{x_i}$.

(b) Suppose there exists $i \in \Pi$ such that $\sum_{k\neq i} x_k \delta_{(c_k<\underline{P})}$ but $\underline{p}_i > \underline{P}$. Note that (C.14) implies $\underline{p}_k > c_k$ for all $k \in \Pi$, and therefore, any supplier $k$ with $c_k \geq \underline{P}$ has $\underline{p}_k > \underline{P}$. Denote $\tilde{p} := \underline{p}_i \wedge \min\{p_k : c_k \geq \underline{P}\}$ and we have $\tilde{p} > \underline{P}$. Now consider all suppliers with $c_j < \underline{P}$ and $j \neq i$, since their total capacity is less than the demand and all other suppliers price above $\tilde{p}$, they obtain $\bar{r}_j(p) = 1$ for all $p < \tilde{p}$, implying $\underline{p}_j \geq \tilde{p}$. It follows $\underline{P} = \min\{\underline{p}_k\} \geq \tilde{p}$, a contradiction to $\tilde{p} > \underline{P}$.

(c) $\overline{P} > \underline{P}$ follows from nonexistence of pure-strategy equilibrium, so we only need to show $\underline{P} > P^*$. Suppose $\underline{P} \leq P^*$. Now consider supplier $i$ and cost $c_j$ that satisfy (C.9), which implies $\sum_{k\neq i} x_k \delta_{(c_k<\underline{P})} \leq \sum_{k\neq i} x_k \delta_{(c_k<c_j)} < d$. Part (b) implies $\underline{p}_i = \underline{P}$ and therefore $ER_i = (\underline{P} - c_i)(x_i \wedge d)$ by part (a). But inequalities (C.16) and (C.9) suggest

$\bar{R}_i^-(c_j) \geq (c_j - c_i)(d - \sum_{k\neq i} x_k \delta_{(c_k<p)})^+ > (P^* - c_i)(x_i \wedge d) > (\underline{P} - c_i)(x_i \wedge d) = ER_i$, a contradiction to the optimality of $ER_i$. ■

**Lemma 12** *For a mixed-strategy equilibrium, at most one profitable supplier has* $m_i(\overline{P}) > 0$.

**Proof.** Proof of Lemma 25 Suppose there are at least two suppliers with probability mass at $\overline{P}$. For any $i \in \Omega_{\overline{P}} := \{k : m_k(\overline{P}) > 0, ER_k > 0\}$, from (C.14), we have $\bar{R}_i^-(\overline{P}) = ER_i > 0$ and $\bar{r}_i^-(\overline{P}) > 0$. Now consider all players in $\Pi \backslash \Omega_{\overline{P}}$, who with probability 1 price lower than $\overline{P}$. For supplier $i \in \Omega_{\overline{P}}$, since $\bar{r}_i^-(\overline{P}) > 0$, we must have $d > \sum_{k\in\Pi\backslash\Omega_{\overline{P}}} x_k$. Together with $\sum_{k\in\Pi} x_k > d$, and $x_i < \sum_{k\in\Omega_{\overline{P}}} x_k$, it implies

$$1 \wedge \frac{d - \sum_{k\in\Pi\backslash\Omega_{\overline{P}}} x_k}{x_i} > \frac{d - \sum_{k\in\Pi\backslash\Omega_{\overline{P}}} x_k}{\sum_{k\in\Omega_{\overline{P}}} x_k}.$$

Due to independence of suppliers' strategies, with strictly positive probability $\prod_{k\in\Omega_{\overline{P}}\backslash\{i\}} m_k(\overline{P})$, all players in $k \in \Omega_{\overline{P}} \backslash \{i\}$ choose $\overline{P}$ and we have

$$\bar{r}_i^-(\overline{P}) - \bar{r}_i(\overline{P}) \geq \prod_{k\in\Omega_{\overline{P}}\backslash\{i\}} m_k(\overline{P}) \cdot \left[ (1 \wedge \frac{d - \sum_{k\in\Pi\backslash\Omega_{\overline{P}}} x_k}{x_i}) - \frac{d - \sum_{k\in\Pi\backslash\Omega_{\overline{P}}} x_k}{\sum_{k\in\Omega_{\overline{P}}} x_k} \right],$$

which is strictly positive. It implies $\bar{R}_i^-(\overline{P}) > \bar{R}_i(\overline{P})$, contradicting the initial assumption $m_i(\overline{P}) > 0$, by Lemma 21(a).

■

Define supplier $i_A$ as an *anchoring supplier* if she satisfies

$$\text{(a) } c_{i_A} < \overline{P}, \text{ (b) } \bar{p}_{i_A} = \overline{P}, \text{ and (c) } m_{i_A}(\overline{P}) \geq m_j(\overline{P}) \quad \text{for all } j \in \Pi, \tag{B.14}$$

Lemma 27 identifies four important properties that an anchoring supplier must satisfy.

**Lemma 13** *Supplier* $i_A$ *satisfies (a)* $ER_{i_A} = S_{i_A}(\overline{P})$*; (b)* $c_{i_A} \leq P^*$*; (c)* $\underline{p}_{i_A} = \underline{P} = \frac{ER_{i_A}}{d\wedge x_{i_A}} + c_{i_A}$*; (d)* $\bar{p}_{i_A} = \overline{P} = \arg\max\{S_{i_A}(c_j) : c_j > P^*\}$.

**Proof.** Proof of Lemma 27 (a) $\bar{p}_{i_A} = \overline{P}$ implies $ER_{i_A} = \bar{R}_{i_A}^-(\overline{P}) = (\overline{P} - c_{i_A}) x_{i_A} \bar{r}_{i_A}^-(\overline{P})$ by (C.14). As $\sigma_k^* <_{a.s.} \overline{P}$ for all $k \in \Pi \backslash \{i_A\}$ by Lemma 25 and $c_{i_A} < \overline{P} \leq c_l$ for all $l \notin \Pi$, applying (C.1) and PR, we have $\bar{r}_{i_A}(\overline{P}) = \frac{[d - \sum_{k\neq i_A} x_k \delta_{(c_k<\overline{P})}]^+}{x_{i_A}}$. Part (a) follows directly.

(b) Suppose $c_{i_A} > P^*$. From (C.7), we have $d \leq \sum_k x_k \delta_{(c_k \leq P^*)}$, implying $S_{i_A}(p) = 0$

for all $p \geq c_{i_A} > P^*$. It contradicts $ER_{i_A} = S_{i_A}(\overline{P}) > 0$ for $\overline{P} > c_{i_A}$.

(c) $ER_{i_A} = S_{i_A}(\overline{P}) = (\overline{P} - c_{i_A})[d - \sum_{k \neq i_A} x_k \delta_{(c_k < \overline{P})}]^+ > 0$ requires $\sum_{k \neq i_A} x_k \delta_{(c_k < \overline{P})} < d$. By definition of $i_A$, we have $\sum_{k \neq i_A} x_k \delta_{(c_k < \overline{P})} = \sum_{k \in \Pi \setminus \{i_A\}} x_k$, and Lemma 24(b) then proves part (c).

(d) We first show that $\overline{P} = \arg\max\{S_{i_A}(p) : p \in (P^*, b]\}$. Suppose $S_{i_A}(\tilde{p}) > S_{i_A}(\overline{P}) = ER_{i_A}$ for $\tilde{p} \neq \overline{P}$ and $\tilde{p} > P^*$. By (C.16), we have

$$\bar{R}_{i_A}^-(\tilde{p}) = (\tilde{p} - c_{i_A}) x_{i_A} \bar{r}_i^-(\tilde{p}) \geq S_{i_A}(\tilde{p}) > S_{i_A}(\overline{P}) = ER_{i_A},$$

a contradiction to the optimality of $ER_{i_A}$. Thus, $\overline{P}$ maximizes $S_{i_A}(\cdot)$. Notice that, due to piece-wise linearity of $S_{i_A}(p)$, its maximizer over domain $(P^*, b]$ must equal to certain cost $c_j > P^*$ where discontinuity takes place. ■

Notice that, it is possible that there are multiple maximizers to function $S_{i_A}(p)$. Lemma 29 states that $\overline{P}$ must be the smallest one for the anchoring supplier. We start with an auxiliary lemma, used in several other places.

**Lemma 14** *If profitable supplier $i \in \Pi$ has $\bar{R}_i^-(p) = S_i(p)$ for price $p > P^*$, then any profitable supplier $k \neq i$ with $c_k < p$ satisfies $\overline{p}_k \leq p$.*

**Proof.** Proof of Lemma 28 $\bar{R}_i^-(p) = (p - c_i) x_i \bar{r}_i^-(p) = S_i(p)$ and (C.18) imply $\bar{r}_i^-(p) = \frac{d - \sum_{k \neq i} x_k \delta_{(c_k < p)}}{x_i}$. Also, from $p > P^*$ and (C.7), we have $x_i + \sum_{k \neq i} x_k \delta_{(c_k < p)} \geq \sum_k x_k \delta_{(c_k \leq P^*)} > d$ and, therefore, $\bar{r}_i^-(p) = \frac{S_i(p)}{(p - c_i) x_i} = \frac{d - \sum_{k \neq i} x_k \delta_{(c_k < p)}}{x_i} < 1$. Notice that any supplier $k$ with $c_k \geq p$ has $\sigma_k^* \geq p$, i.e., $\delta_{(c_k \geq p)} \delta_{(\sigma_k^* < p)} = 0$, so

$$\begin{aligned}
\bar{r}_i^-(p) &= \mathsf{E}\left[1 \wedge \frac{[d - \sum_{k \neq i} x_k \delta_{(\sigma_k^* < p)} (\delta_{(c_k < p)} + \delta_{(c_k \geq p)})]^+}{x_i}\right] \\
&= \mathsf{E}\left[1 \wedge \frac{(d - \sum_{k \neq i} x_k \delta_{(c_k < p)} \delta_{(\sigma_k^* < p)})^+}{x_i}\right].
\end{aligned}$$

The above expression achieves its minimum $\frac{d - \sum_{k \neq i} x_k \delta_{(c_k < p)}}{x_i} < 1$ only if $\delta_{(\sigma_k^* < p)} =_{a.s.} 1$ for all $k$ with $c_k < p$. That is, $\overline{p}_k \leq p$ for all $k \neq i$ with $c_k < p$. ■

**Lemma 15** *Supplier $i_A$ satisfies $\overline{P} = \overline{P}_{i_A}^{\min} := \min\{\arg\max\{S_{i_A}(c_j) : c_j > P^*\}\}$.*

**Proof.** Proof of Lemma 29 From Lemma 27(d), $\overline{P}$ maximizes $S_{i_A}(p)$. Thus, both $\overline{P}_{i_A}^{\min}$ and $\overline{P}$ maximize $S_{i_A}(p)$. Assume $\overline{P}_{i_A}^{\min} < \overline{P}$. Lemma 27(a) implies $ER_{i_A} = S_{i_A}(\overline{P}) = S_{i_A}(\overline{P}_{i_A}^{\min})$ and inequality (C.16) implies $ER_{i_A} \geq \bar{R}_{i_A}(\overline{P}_{i_A}^{\min}) \geq S_{i_A}(\overline{P}_{i_A}^{\min})$, so we must have $\bar{R}^{-}_{i_A}(\overline{P}^{\min}_{i_A}) = S_{i_A}(\overline{P}^{\min}_{i_A})$. From Lemma 28, $\bar{p}_k \leq \overline{P}^{\min}_{i_A}$ *for all* $k \neq i_A$ with $c_k < \overline{P}^{\min}_{i_A}$. Now consider interval $[\overline{P}^{\min}_{i_A}, \overline{P})$. Since $\overline{P}^{\min}_{i_A}$ belongs to $\{c_j : c_j > P^*\}$ and $\overline{P}^{\min}_{i_A} < \overline{P}$, there is at least one player with cost $c_k \in [\overline{P}^{\min}_{i_A}, \overline{P})$. Notice that, for any supplier $k$ with $c_k \in [\overline{P}^{\min}_{i_A}, \overline{P})$, Lemma 22 and (C.14) imply $ER_k = \bar{R}^{-}_k(\underline{p}_k) = (\underline{p}_k - c_k)\bar{r}^{-}_k(\underline{p}_k) > 0$, and therefore, $\underline{p}_k > c_k \geq \overline{P}^{\min}_{i_A}$. Let $\underline{p}^* := \min\{\underline{p}_k : c_k \in [\overline{P}^{\min}_{i_A}, \overline{P})\} > \overline{P}^{\min}_{i_A}$. Now we have all players with $c_k < \overline{P}^{\min}_{i_A}$ except $i_A$ price lower than $\overline{P}^{\min}_{i_A}$ and all players with $c_k \geq \overline{P}^{\min}_{i_A}$ price not lower than $\underline{p}^* > \overline{P}^{\min}_{i_A}$. It implies a contradiction $ER_{i_A} \geq \bar{R}_{i_A}(\underline{p}^*) = (\underline{p}^* - c_{i_A})(d - \sum_k x_k \delta_{(c_k < \overline{P}^{\min}_{i_A})}) > (\overline{P}^{\min}_{i_A} - c_{i_A})(d - \sum_k x_k \delta_{(c_k < \overline{P}^{\min}_{i_A})}) = S_{i_A}(\overline{P}^{\min}_{i_A}) = ER_{i_A}$. ■

Based on Lemmas 27 and 29, if supplier $i_A$ is identified, we can uniquely determine the price bounds $\overline{P} = \overline{P}^{\min}_{i_A}$ and $\underline{P} = \frac{S_{i_A}(\overline{P}^{\min}_{i_A})}{d \wedge x_{i_A}} + c_{i_A}$. Therefore, identifying the anchoring supplier $i_A$ is the critical step towards derivation of $\underline{P}$ and $\overline{P}$. Since multiple players may satisfy the conditions of Lemmas 27 and 29, these two tasks have to be completed jointly, as shown in Proposition 19. By analyzing their payoff structures, we rule out unqualified candidates and guarantee uniqueness of $\underline{P}$ and $\overline{P}$. Define $\Omega := \{i \in \mathcal{N} : c_i \leq P^*$ and $\sum_{k \neq i} x_k \delta_{(c_k \leq P^*)} < d\}$. Notice for all $i \notin \Omega$, we have either $c_i > P^*$ or $\sum_{k \neq i} x_k \delta_{(c_k \leq P^*)} \geq d$, implying that, for all $c_j > P^*$, $S_i(c_j) = (c_j - c_i)\left[d - \sum_{k \neq i} x_k \delta_{(c_k < c_j)}\right]^+ = 0$. Therefore, we must have $i_A \in \Omega$. For each $i \in \Omega$, define trial values

$$\begin{aligned}
\bar{R}^T_i \quad &: \quad = \max\{S_i(c_j) : c_j > P^*\}, \; \underline{P}^T_i := \frac{\bar{R}^T_i}{d \wedge x_i} + c_i, \\
\overline{P}^T_i \quad &: \quad = \min\{\arg\max\{S_i(c_j) : c_j > P^*\}\} = \min\left\{c_j > P^* : S_i(c_j) = \bar{R}^T_i\right\}.
\end{aligned}$$

From $\sum_k x_k \delta_{(c_k < \overline{P}^T_i)} \geq \sum_k x_k \delta_{(c_k \leq P^*)} > d$, we have $x_i > d - \sum_{k \neq i} x_k \delta_{(c_k < \overline{P}^T_i)}$. Also from the above definitions, we have $\bar{R}^T_i = (\underline{P}^T_i - c_i)(x_i \wedge d) = (\overline{P}^T_i - c_i)[d - \sum_{k \neq i} x_k \delta_{(c_k < \overline{P}^T_i)}] > 0$. It implies two useful inequalities

$$\text{(a) } 0 < d - \sum_{k \neq i} x_k \delta_{(c_k < \overline{P}^T_i)} < (x_i \wedge d) \text{ and (b) } \overline{P}^T_i > \underline{P}^T_i \quad \text{for all } i \in \Omega. \qquad \text{(B.15)}$$

With the above notation, we claim that $\underline{P}$ and $\overline{P}$ can be determined as follows.

**Proposition 15** $\underline{P} = \max_{i \in \Omega}\left\{\underline{P}^T_i\right\}$ *and* $\overline{P} = \min_{i \in \Omega}\left\{\overline{P}^T_i : \underline{P}^T_i = \underline{P}\right\}$.

**Proof.** Proof of Proposition 19 The proof proceeds in three steps. The first part is an auxiliary result.

(a) $ER_i \geq \bar{R}_i^T$ *and* $\underline{p}_i \geq \underline{P}_i^T$ *for all* $i \in \Omega$. By (C.16), for any price equal to a cost level $p \in \Psi := \{c_j : P^* < c_j \leq b\}$, we have $ER_i \geq \bar{R}_i^-(p) = (p - c_i)x_i\bar{r}_i^-(p) \geq (p - c_i)[d - \sum_{k\neq i} x_k\delta_{(c_k<p)}]$, which implies $ER_i \geq \bar{R}_i^T = \max_{p\in\Psi}\{(p - c_i)[d - \sum_{k\neq i} x_k\delta_{(c_k<p)}]^+\}$. To show the second inequality, suppose $\underline{p}_i < \underline{P}_i^T$ for certain $i$. By (C.14), we have $ER_i = \bar{R}_i^-(\underline{p}_i)$, implying a contradiction, $ER_i = \bar{R}_i^-(\underline{p}_i) = (\underline{p}_i - c_i)x_i\bar{r}_i^-(\underline{p}_i) \leq (\underline{p}_i - c_i)x_i(1 \wedge \frac{d}{x_i}) < (\underline{P}_i^T - c_i)(d \wedge x_i) = \bar{R}_i^T \leq ER_i$.

(b) $\underline{P} = \max_{i\in\Omega}\left\{\underline{P}_i^T\right\}$. We know $\underline{p}_{i_A} = \underline{P}_{i_A}^T = \underline{P} \in \left\{\underline{P}_k^T : k \in \Omega\right\}$. Suppose that there exists supplier $h \in \Omega$ such that $\underline{P}_h^T = \max_{k\in\Omega}\left\{\underline{P}_k^T\right\} > \underline{P}$. Part (a) implies $\underline{p}_h \geq \underline{P}_h^T > \underline{P} = \underline{p}_{i_A}$. Consider price interval $[\underline{p}_{i_A}, \underline{P}_h^T)$. Inequality (C.19-a) implies $\sum_{k\neq h} x_k\delta_{(c_k<p)} < d$ for all $p \leq \underline{P}_h^T$. That is, given supplier $h$'s absence in this price interval, the total capacity available is smaller than demand. Therefore, supplier $i_A$ obtains sales ratio 1 almost surely for any $p \in [\underline{p}_{i_A}, \underline{P}_h^T)$, implying a contradiction, $ER_{i_A} = \bar{R}_{i_A}(\underline{P}) = (\underline{P} - c_{i_A})x_{i_A} < (\underline{P}_h^T - c_{i_A})x_{i_A} = \bar{R}_{i_A}^-(\underline{P}_h^T) \leq ER_{i_A}$.

(c) $\overline{P} = \overline{P}_{i_A}^T = \min_{i\in\Omega}\left\{\overline{P}_i^T : \underline{P}_i^T = \underline{P}\right\}$. Given $\underline{P}$ derived in part (b), if only one player satisfies $\underline{P}_i^T = \underline{P}$, she must be the anchoring supplier and Lemma 29 guarantees part (c). Consider the case when there are more than one players in set $\Omega$ satisfying $\underline{P}_i^T = \underline{P}$. Suppose $j \in \Omega$ exists such that $\underline{P}_j^T = \underline{P}$ and $\overline{P}_j^T < \overline{P} = \overline{P}_{i_A}^T$. Inequality (C.16) implies $\bar{R}_j^-(\overline{P}_j^T) \geq S_j(\overline{P}_j^T)$. If $\bar{R}_j^-(\overline{P}_j^T) = S_j(\overline{P}_j^T)$, as $c_{i_A} < \underline{P} < \overline{P}_j^T$, Lemma 28 brings a contradiction $\overline{p}_{i_A} \leq \overline{P}_j^T < \overline{P} = \overline{p}_{i_A}$. Thus, we must have $\bar{R}_j^-(\overline{P}_j^T) > S_j(\overline{P}_j^T)$. Now from $ER_j \geq \bar{R}_j^-(\overline{P}_j^T) > S_j(\overline{P}_j^T) = \bar{R}_j^T$, we have $\underline{p}_j > \underline{P} = \underline{P}_j^T$. [Otherwise, if $\underline{p}_j = \underline{P} = \underline{P}_j^T$, Lemma 24(a) brings a contradiction $ER_j = \bar{R}_j(\underline{P}) = (\underline{P}_j^T - c_j)(x_j \wedge d) = \bar{R}_j^T < ER_j$.] Also notice that (C.19-b) implies $\overline{P}_j^T > \underline{P}_j^T = \underline{P}$. Therefore, $p^* := \min\{\underline{p}_j, \overline{P}_j^T\} > \underline{P} = \underline{P}_j^T$ and for all $p < p^*$, $\sum_{k\neq j} x_k\delta_{(c_k<p)} \leq \sum_{k\neq j} x_k\delta_{(c_k<p^*)} \leq \sum_{k\neq j} x_k\delta_{(c_k<\overline{P}_j^T)} < d$, where the last inequality follows from (C.19-a). Similarly to part (b), we can show $\bar{r}_{i_A}(p) = 1$ for all $p < p^* \leq \underline{p}_j$, which further leads to a contradiction $ER_{i_A} = \bar{R}_{i_A}(\underline{P}) = (\underline{P} - c_{i_A})x_{i_A} < (p^* - c_{i_A})x_{i_A} = \bar{R}_{i_A}^-(p^*) \leq ER_{i_A}$. ■

Next we establish that the equilibrium distribution function is continuous for any *critical* supplier at price $\overline{P}$.

**Proposition 16** *For any critical supplier $i$ at price $\overline{P}$, $m_i(p) = 0$ for all $p \in (c_i \vee \underline{P}, \overline{P})$.*

**Proof.** Proof of Proposition 16 Suppose supplier $i$ with $\sum_{k\in\Pi\backslash\{i\}} x_k < d$ has $m_i(\tilde{p}) > 0$ for certain $\tilde{p} \in (\underline{P}, \overline{P})$, implying $ER_i = \bar{R}_i(\tilde{p})$. For any supplier $j \in \Pi\backslash\{i\}$, we show that supplier $j$ will choose $\Pr\{\sigma_j^* \in [\tilde{p}, \tilde{p} + \Delta_j]\} = 0$ for certain $\Delta_j > 0$. If $c_j \geq \tilde{p}$, we have $\underline{p}_j > c_j \geq \tilde{p}$ and $\Delta_j := \underline{p}_j - c_j > 0$. For $c_j < \tilde{p}$, if we show (#) $\bar{r}_j^-(\tilde{p}, \sigma_{-j}^*) > \bar{r}_j(\tilde{p}, \sigma_{-j}^*)$, it then follows $ER_j \geq \bar{R}_j^-(\tilde{p}) > \bar{R}_j(\tilde{p}) \geq \bar{R}_j^+(\tilde{p})$, implying $\Delta_j > 0$. Now let $\Delta = \min_{j\in\Pi\backslash\{i\}}\{\Delta_j\}$ and consider supplier $i$. Notice that $\Pr\{\sigma_j^* \in [\tilde{p}, \tilde{p} + \Delta]\} = 0$ implies that any price $p \in [\tilde{p}, \tilde{p}+\Delta]$ must yields the same expected sales $\bar{r}_i$ for supplier $i$, which leads to a contradiction $ER_i \geq \bar{R}_i^-(\tilde{p} + \Delta) > \bar{R}_i(\tilde{p}) = ER_i$.

The remaining task is to show that $m_i(\tilde{p}) > 0$ for $\tilde{p} > c_j$ implies (#). It is sufficient to show $\Pr\{r_j^-(\tilde{p}, \sigma_{-j}^*) > r_j(\tilde{p}, \sigma_{-j}^*)\} > 0$. First notice $\underline{p}_k > c_k$ for all $k \in \Pi$, so expression of $r_j$ can be simplified as $r_j(p, \sigma_{-j}^*) = 1 \wedge \frac{d-\sum_{k\neq j} x_k \delta_{(\sigma_k^* < p)}}{x_j + \sum_{k\neq j} x_k \delta_{(\sigma_k^* = p)}}$. Due to independence of each player's bid, we have

$$
\begin{aligned}
&\Pr\{r_j^-(\tilde{p}, \sigma_i^*, \sigma_{-i-j}^*) \; > \; r_j(\tilde{p}, \sigma_i^*, \sigma_{-i-j}^*)\} \\
&\geq m_i(\tilde{p}) \Pr\{r_j^-(\tilde{p}, \sigma_i^*, \sigma_{-i-j}^*) > r_j(\tilde{p}, \sigma_i^*, \sigma_{-i-j}^*) \big| \, \sigma_i^* = \tilde{p}\} \\
&= m_i(\tilde{p}) \Pr\{1 \wedge \frac{(d-\sum_{k\in\Pi\backslash\{i,j\}} x_k \delta_{(\sigma_k^* < \tilde{p})})^+}{x_j} > 1 \wedge \frac{(d-\sum_{k\in\Pi\backslash\{i,j\}} x_k \delta_{(\sigma_k^* < \tilde{p})})^+}{x_j + x_i + \sum_{k\in\Pi\backslash\{i,j\}} x_k \delta_{(\sigma_k^* = \tilde{p})}}\},
\end{aligned}
$$

which is positive if we can show $(a) \Pr\{d > \sum_{k\in\Pi\backslash\{i,j\}} x_k \delta_{(\sigma_k^* < \tilde{p})}\} > 0$ and $(b) \Pr\{d - \sum_{k\in\Pi\backslash\{i,j\}} x_k \delta_{(\sigma_k^* < \tilde{p})} < x_i + x_j\} > 0$. Inequality (a) directly follows $d > \sum_{k\in\Pi\backslash\{i\}} x_k > \sum_{k\in\Pi\backslash\{i,j\}} x_k \delta_{(\sigma_k^* < \tilde{p})}$. To show inequality (b), suppose $\Pr\{d - \sum_{k\in\Pi\backslash\{i,j\}} x_k \delta_{(\sigma_k^* < \tilde{p})} < x_j + x_i\} = 0$, implying $d \geq \sum_{k\in\Pi} x_k \delta_{(\sigma_k^* < \tilde{p})}$ with probability 1. It follows that for any player $k$ with $\underline{p}_k < p$, we must have $r_k^-(p) =_{a.s.} 1$. Lemma 18 then implies $r_k(p') =_{a.s.} 1$ for all $p' < p$ which is a contradiction to $\underline{p}_k < p$. ■

## B.3 Algorithm for Computing Mixed-Strategy Equilibrium

### B.3.1 Algebraic Expression of a Mixed-Strategy Equilibrium

We first derive the equilibrium conditions (B.20) for active supplier $i$. Proposition 16 indicates that no critical supplier $i$ (i.e., $d > \sum_{k\in\Pi} x_k - x_i$) would choose any price $p$

between $\underline{P}$ and $\overline{P}$ with a positive probability. We assume this property for all players.

(**A1**) *$F_i(p)$ is continuous in $p \in [\underline{P}, \overline{P})$ for all profitable players.*

Under (A1), price tie will almost surely not take place, and given bid vector $\mathbf{p} = \{p, \mathbf{p}_{-i}\}$ with $p > c_i$, and active supplier $i$'s realized sales ratio is written as

$$r_i(p, \mathbf{p}_{-i}) = 1 \wedge \frac{[d - \sum_{k \neq i} x_k \delta_{(p_k < p)}]^+}{x_i + \sum_{k \neq i} x_k \delta_{(p_k = p)}}. \tag{B.16}$$

(A1) implies $Er_i^-(p, \sigma_{-i}^*) = Er_i(p, \sigma_{-i}^*) = Er_i^+(p, \sigma_{-i}^*)$. From (B.16), we have that, with probability 1

$$z_i(d; \langle p, x_i \rangle ; \left\langle \mathbf{x}_{-i}, \mathbf{p}_{-i} \right\rangle) = x_i r_i(p, \mathbf{p}_{-i}) = x_i \wedge [d - \sum_{k \neq i} x_k \delta_{(p_k < p)}]^+ \tag{B.17}$$

where $z_i(\cdot; \cdot; \cdot)$ denotes supplier $i$'s sales as a function of demand $d$, his bid $\langle p, x_i \rangle$, and bids from her competitors $\left\langle \mathbf{x}_{-i}, \mathbf{p}_{-i} \right\rangle$.

In our mixed-strategy analysis, we have identified the unique price bounds $\left\{\underline{P}, \overline{P}\right\}$ and the set of active agents $\Pi = \left\{k : c_k < \overline{P}\right\}$. For any $p \in [\underline{P}, \overline{P})$, we divide active suppliers into those with bids lower than $p$, higher than $p$, and active at $p$,

$$\begin{aligned} L(p) &= \{k : F_k(p) = 1\}, \\ H(p) &= \{k : F_k(p) = 0\}, \\ A(p) &= \{k : F_k(p) \in (0, 1)\} \text{ and } A_{-i}(p) = A(p) \setminus \{i\}. \end{aligned}$$

From (B.17), for price $p$, supplier $i \in A(p)$ has an expected demand allocation

$$\begin{aligned} \bar{z}_i(\mathbf{p}) &= \mathsf{E}_{\sigma_{-i}^*}[z_i(d_{L(p)}; \langle p, x_i \rangle; \langle \mathbf{x}_{A_{-i}(p)}, \mathbf{p}_{A_{-i}(p)} \rangle)], \\ &\quad \text{where } \ d_{L(p)} = \left[d - \textstyle\sum_{k \in L(p)} x_k\right]^+ . \end{aligned} \tag{B.18}$$

As $m_i(p) = 0$ for all $i$ and all $p \in (\underline{P}, \overline{P})$, $\bar{z}_i(\mathbf{x}, \mathbf{p})$ is a polynomial of $\{F_k(p), [1 - F_k(p)]\}_{k \in \Pi \setminus \{i\}}$. As $F_k(p) = 1$ for $k \in L(p)$ and $F_k(p) = 0$ for $k \in H(p)$, it is a polynomial with homogenous order of $\|A_{-i}(p)\|$, where $\|\cdot\|$ denotes the number of elements in a set. Given $\{F_k = F_k(p)\}_{k \in \Pi}$, (B.18) can be further simplified as

$$\begin{aligned} &\bar{z}_i(\mathbf{p}) \\ &= \sum_{A_{-i}^m(p) \subseteq A_{-i}(p)} \left[ \prod_{k \in A_{-i}^m(p)} F_k \cdot \prod_{k \in A_{-i}(p) \setminus A_{-i}^m(p)} (1 - F_k) \cdot (x_i \wedge [d_{L(p)} - \sum_{k \in A_{-i}^m(p)} x_k]^+) \right] \end{aligned} \tag{B.19}$$

In quasi-symmetric oligopoly games, we can show that any active agent $i$'s bidding strategy has strictly increasing cdf function. This observation is used as our second assumption.

(**A2**) *$F_i(p)$ is strictly increasing in $p \in (\underline{p}_i, \overline{p}_i)$* for $i \in \Pi$.

(A1) and (A2) imply $\bar{R}_i(p) = ER_i$ for any $p \in (\underline{p}_i, \overline{p}_i)$. Thus a mixed-strategy solution $\{F_i(p)\}_{i \in A(p)}$ satisfies the following, for $p \in [\underline{P}, \overline{P}]$

$$\begin{cases} (p - c_i)\mathsf{E}_{\sigma^*_{-i}}[z_i(d_{L(p)}; \langle p, x_i \rangle ; \langle \mathbf{x}_{A_{-i}(p)}, \mathbf{p}_{A_{-i}(p)} \rangle)] = ER_i & \text{for } p \in [\underline{p}_i, \overline{p}_i] \cap [\underline{P}, \overline{P}) \\ (p - c_i)\mathsf{E}_{\sigma^*_{-i}}[z_i(d_{L(p)}; \langle p, x_i \rangle ; \langle \mathbf{x}_{A_{-i}(p)}, \mathbf{p}_{A_{-i}(p)} \rangle)] \le ER_i & \text{otherwise} \end{cases} . \tag{B.20}$$

Note that $\overline{P}$ is excluded from supplier $i$'s maximization domain, since, if $m_{i_A}(\overline{P}) > 0$, it can be shown that $\bar{R}_i(\overline{P}) < \bar{R}_i^-(\overline{P}) = ER_i$ for all players in $\Pi \backslash \{i_A\}$ with $\overline{p}_i = \overline{P}$.

### B.3.2 Progressive Algorithm for Computing $\{ER_i, \underline{p}_i, F_i, \overline{p}_i\}_{i \in \Pi}$.

In Proposition 19, we established the unique price bounds $\left\{\underline{P}, \overline{P}\right\}$ and existence of the anchoring supplier(s) $i_A$ with price range covering $\left[\underline{P}, \overline{P}\right]$. The next steps are (1) deriving $ER_i$ for all $i \in \Pi \backslash \{i_A\}$; (2) segmenting active players into three groups $\{L(p)$, $A(p)$, and $H(p)\}$ for any $p \in (\underline{P}, \overline{P})$; and (3) finding the solution $\{F_k\}_{k \in A(p)}$ to a system of equations, for $i \in A(p)$

$$\frac{ER_i}{p - c_i} \tag{B.21}$$
$$= \sum_{A^m_{-i}(p) \subseteq A_{-i}(p)} \left[ \prod_{k \in A^m_{-i}(p)} F_k \cdot \prod_{k \in A_{-i}(p) \backslash A^m_{-i}(p)} (1 - F_k) \cdot (x_i \wedge [d_{L(p)} - \sum_{k \in A^m_{-i}(p)} x_k]^+) \right]$$

The three tasks are interconnected. We proceed with $p$ *progressively* moving from $\underline{P}$ to $\overline{P}$, and complete the above tasks iteratively. The algorithm implicitly assumes the following two assumptions, additional to (A1) and (A2),

(**A3**) *$\bar{R}_i(p)$ is strictly increasing in $p \in [c_i, \underline{p}_i)$ for $i \in \Pi$.*

(**A4**) *There is unique mixed-strategy equilibrium.*

For $X = L$, $A$, and $H$, denote $X^-(p) = \cap_{\varepsilon > 0} X(p - \varepsilon)$ and $X^+(p) = \cap_{\varepsilon > 0} X(p + \varepsilon)$. It can be shown[2] that, for any price $p$, $L(p) = L^+(p)$ and $H(p) = H^-(p)$. The

[2]For $i \neq i_A$, $F_i(p)$ is continuous and strictly increasing from 0 to 1 when $p$ moves from $\underline{p}_i$ to

procedure identifies a sequence of critical prices $\underline{P}_0, \underline{P}_1, \ldots$. At each price $\underline{P}_m$, at least one supplier joins the set of active suppliers. Formally, we denote the players who joined at $\underline{P}_m$ by $I_m = \{i : \underline{p}_i = \underline{P}_m\}$. Note that

$$L^+(\underline{P}_m) = L(\underline{P}_m),\ A^+(\underline{P}_m) = A(\underline{P}_m) \cup I_m, \text{ and } H^+(\underline{P}_m) = H(\underline{P}_m) \backslash I_m \qquad \text{(B.22)}$$

The algorithm is as follows.

*1. Initialization step, $m = 0$.* Applying Proposition 3, we calculate $\left\{\underline{P}, \overline{P}\right\}$ and identify the set of anchoring supplier(s) $I_A := \left\{i : \underline{P}_i^T = \underline{P} \text{ and } \overline{P}_i^T = \overline{P}\right\}$ and $ER_i := R_i^T$ for all $i \in I_A$. Set $\underline{P}_0 := \underline{P}$, $L_0 := \varnothing$, $A_0 := I_A$, and $H_0 := \Pi \backslash I_A$ where $\Pi = \left\{i : c_i < \overline{P}\right\}$. Clearly, $J^0 := I_A \subseteq I_0$.

*2. Identification of $I_m$.* For given $\underline{P}_m$, if we can identify $I_m$, then there exists $\varepsilon > 0$ such that, for all $p \in (\underline{P}_m, \underline{P}_m + \varepsilon)$, $A(p) = A^+(\underline{P}_m) = A(\underline{P}_m) \cup I_m$. That is, $\{F_i(p)\}_{i \in A^+(\underline{P}_m)}$ solves polynomial equation system (B.21) for $p \in (\underline{P}_m, \underline{P}_m + \varepsilon)$ where

$$ER_i = \bar{R}_i(\underline{P}_m) = (\underline{P}_m - c_i) \cdot \bar{z}_i(\underline{P}_m, \mathbf{p}_{A_{-i}(\underline{P}_m)}) \qquad \text{(B.23)}$$

and $\bar{z}_i(\cdot)$ is given by (B.19). Knowing $J^m \subseteq I_m \subseteq H(\underline{P}_m) = H^-(\underline{P}_m)$, we need to determine $N_m := I_m \backslash J_m$. Since $N_m$ must be a subset of $\{i \in H(\underline{P}_m) \backslash J_m : c_i < \underline{P}_m\}$ (including $\varnothing$), we enumerate all subsets $N_m^k \subseteq \{i \in H(\underline{P}_m) \backslash J_m : c_i < \underline{P}_m\}$. For each $N_m^k$, let $I_m^k := J_m \cup N_m^k$ be a trial set of $I_m$. We run the following two tests (2a) and (2b). Test (2a) corresponds to Assumptions (A1) and (A2), while test (2b) corresponds to Assumption (A3).

(2a) *Monotonicity and Regularity of $\{F_i\}_{i \in A^+(\underline{P}_m)}$.* Denote $A_m^{k,+} := A(\underline{P}_m) \cup I_m^k$ as the trial set of $A^+(\underline{P}_m)$. Given $A(p) = A_m^{k,+}$, solve the equation system (B.21) and (B.23) for $p = \underline{P}_m + \Delta p$ with very small $\Delta p$.[3] If the solution satisfies monotonicity

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$\overline{p}_i$. $F_{i_A}(p)$ is also continuous and strictly increasing for $p \in [\underline{P}, \overline{P})$, except there is a jump only at $\overline{p}_{i_A} = \overline{P}$. Therefore, we have the following results for any $i \in \Pi$ and any $p < \overline{P}$: if $p < \underline{p}_i$, then $i$ only exists in $H^-(p)$, $H(p)$, and $H^+(p)$; if $p = \underline{p}_i$, then $i$ only exists in $H^-(p)$, $H(p)$, and $A^+(p)$; if $p \in (\underline{p}_i, \overline{p}_i)$, then $i$ is in $A^-(p)$, $A(p)$, and $A^+(p)$; if $p = \overline{p}_i$, then $i$ only exists in $A^-(p)$, $L(p)$, and $L^+(p)$; if $p > \overline{p}_i$, then $i$ can only be in $L^-(p)$, $L(p)$, and $L^+(p)$. Based on above observations, we must have $L(p) = L^+(p)$ and $H(p) = H^-(p)$ for $p < \overline{P}$.

[3]For $p = \underline{P}_m + \Delta p$ with $\Delta p$ small enough, we have $A(p) = A^+(\underline{P}_m)$, $L(p) = L^+(\underline{P}_m)$, and $H(p) = H^+(\underline{P}_m)$.

(i.e., $F_i(p) > F_i(\underline{P}_m)$) and regularity (i.e., $F_i(p) \in (0,1]$) for all $i \in A(p)$, we say trial set $I_m^k$ survives test (2a) and go to step (2.b). Otherwise, trial set $I_m^k$ is rejected.

(2b) *Monotonicity of* $\left\{\bar{R}_j(p)\right\}_{j \in H^+(\underline{P}_m)}$. Denote $H_m^{k,+} := H(\underline{P}_m) \backslash I_m^k$ as the trial set of $H^+(\underline{P}_m)$. For given (increasing and within (0, 1) interval) cdf $\{F_i(p)\}_{i \in A_m^{k,+}}$ calculated in (2a) for $p = \underline{P}_m + \Delta p$, check the following value function for all $j \in H_m^{k,+}$,

$$\bar{R}_j(p) \quad = \quad (p - c_j) \cdot \bar{z}_j(p, \mathbf{p}_{A_{-i}(p)}) \qquad \text{(B.24)}$$

where $L(p) = L(\underline{P}_m)$ and $A_{-j}(p) = A_m^{k,+} \backslash \{j\} = A_m^{k,+}$. If $\bar{R}_j(p) > \bar{R}_j(\underline{P}_m)$, for all $j \in N_m^{k,+}$, then $I_m^k$ survives test (2b). Otherwise, $I_m^k$ is rejected.

From Assumption (A4), there must exist $N_m^k$ and $I_m^k = J_m \cup N_m^k$ surviving both tests. In fact, supposing the enumeration yields the unique $I_m$, we have used a stronger assumption than (A4).

(**A4'**) *There is unique $I_m$ satisfying tests (2a) and (2b) for every $\underline{P}_m$.*

Based on Assumptions (A1) — (A4'), the enumeration of $\{N_k^m\}$ should yield the unique $I_m$. Let $L_m = L(\underline{P}_m)$, $A_m = A(\underline{P}_m) \cup I_m$, and $H_m = H(\underline{P}_m) \backslash I_m$ and go to the next step.

*3. Computation of $\{F_i(p)\}_{i \in A(p)}$ in $\underline{P}_m$'s right neighborhood.* For given $L(p) = L_m$, $A(p) = A_m$ and $H(p) = H_m$, moving upwards from $\underline{P}_m$, pointwisely solve equation system (B.21) and (B.23). By Assumption (A2), the solution $\{F_i(p)\}_{i \in A^+(\underline{P}_m)}$ are strictly increasing in $p$. We continue until one of suppliers, $i^* \in A^+(\underline{P}_m)$ achieves $F_{i^*}(\bar{p}_{temp}) = 1$ at $\bar{p}_{temp}$. For $p \in [\underline{P}_m, \bar{p}_{temp}]$, check payoff function (B.24) for all supplier $i \in H(p) = H^+(\underline{P}_m)$. Let $\underline{p}_i' = \arg\max_{p \in [\underline{P}_m, \bar{p}_{temp}]} \bar{R}_i(p, \mathbf{F}_{-i})$.

(3a) If $\min_{i \in H^+(\underline{P}_m)}\{\underline{p}_i'\} = \bar{p}_{temp}$, remove supplier(s) $i^*$ satisfying $F_{i^*}(\bar{p}_{temp}) = 1$ from set $A_m$, and include it (them) in $L_m$. Repeat while (3a) holds. (3b) If $\min_{i \in H^+(\underline{P}_m)}\{\underline{p}_i'\} < \bar{p}_{temp}$, we have

$$\begin{aligned} \underline{P}_{m+1} \quad &: \quad = \min_{i \in H(\underline{P}_m)}\{\underline{p}_i'\} \\ \text{and } J_{m+1} \quad &: \quad = \{i \in H_m : \arg\max\{\bar{R}_i(p, \mathbf{F}_{-i})\} = \underline{P}_{m+1}\} \\ &= \quad \{i \in H_m : \underline{p}_i = \underline{p}_i'\}, \end{aligned}$$

Go to step 2.

# APPENDIX C

# Analysis of B-E Games with Elastic Demand

**Notation**

| | |
|---|---|
| $d(p)$ | Buyer's demand function |
| $x_i$ | Supplier $i$'s capacity |
| $b$ | Price cap for bid price |
| $c_i$ | Supplier $i$'s unit production cost |
| $(p_i, q_i)$ | Supplier $i$'s bid of price $p_i$ and quantity $q_i$ |
| $z_i$ | Supplier $i$'s realized sales |
| $r_i = \frac{z_i}{q_i}$ | Fraction of supplier $i$'s bid quantity that is accepted by the buyer |
| $R_i = (p_i - c_i)z_i$ | Supplier $i$'s realized profit |

$\mathbf{p} = (p_1, p_2, \ldots, p_N)$, $\mathbf{p}_{-i} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_N)$, $\mathbf{q} = (q_1, q_2, \ldots, q_N)$

**Assumption 1** *The demand function $d(p)$ is decreasing and concave in $p$.*

Assumption 1 implies that *the reverse demand function $p(d)$ exists and it is also a decreasing and concave function.*

The surplus maximizing rule with is

$$\begin{aligned} r_i(\mathbf{p},\mathbf{q}) &= 1 \wedge \frac{[d(p_i) - Q_i^L(\mathbf{p},\mathbf{q})]^+}{q_i + Q_i^E(\mathbf{p},\mathbf{q})} \qquad \text{(C.1)} \\ \text{where } Q_i^L(\mathbf{p},\mathbf{q}) &= \sum_{k\neq i} q_k \delta_{(p_k<p_i)} + \delta_{(p_i=c_i)} \sum_{k\neq i} q_k \delta_{(p_k=p_i, c_k<p_i)} \\ Q_i^E(\mathbf{p},\mathbf{q}) &= \delta_{(p_i>c_i)} \sum_{k\neq i} q_k \delta_{(p_k=p_i, c_k<p_i)} + \delta_{(p_i=c_i)} \sum_{k\neq i} q_k \delta_{(p_k=p_i, c_k=p_i)}. \end{aligned}$$

It can be verified that $Q_i^L(\mathbf{p},\mathbf{q})$ and $Q_i^E(\mathbf{p},\mathbf{q})$ satisfy

$$Q_i^L(\mathbf{p},\mathbf{q}) + Q_i^E(\mathbf{p},\mathbf{q}) = \sum_{k\neq i} q_k \delta_{(p_k\leq p_i)} - \delta_{(p_i>c_i)} \sum_{k\neq i} q_k \delta_{(p_k=p_i, c_k=p_i)} \qquad \text{(C.2)}$$

Notice that total realized demand is a function of the bid vector,

$$d(\mathbf{p},\mathbf{q}) = \max_{i\in\mathcal{N}} \left\{ d\left(p_i\right) : \sum q_k \delta_{(p_k<p_i)} < d\left(p_i\right) \right\}. \tag{C.3}$$

**Lemma 16** *For all suppliers, it is optimal to bid all their capacities into the game, that is $q_i = x_i$ for all $i$.*

**Proof.** As $R_i(\mathbf{p},\mathbf{q}) = (p_i - c_i) z_i(\mathbf{p},\mathbf{q})$ and $p_i \geq c_i$, it is sufficient that $z_i(\mathbf{p},\mathbf{q})$ is nondecreasing in $q_i$. By (C.1), we have

$$z_i(\mathbf{p},\mathbf{q}) = q_i r_i(\mathbf{p},\mathbf{q}) = q_i \wedge \frac{q_i[d\left(p_i\right) - Q_i^L(\mathbf{p},\mathbf{q})]^+}{q_i + Q_i^E(\mathbf{p},\mathbf{q})} = q_i \wedge \frac{[d\left(p_i\right) - Q_i^L(\mathbf{p},\mathbf{q})]^+}{1 + \frac{Q_i^E(\mathbf{p},\mathbf{q})}{q_i}}.$$

Notice that for fixed $\mathbf{p}$ and $\mathbf{q}_{-i}$, $[d\left(p_i\right) - Q_i^L(\mathbf{p},\mathbf{q})]^+$ and $Q_i^E(\mathbf{p},\mathbf{q})$ are non-negative constants by (C.1). Thus, $z_i$ is nondecreasing in $q_i$. ■

Lemma 1 implies that each supplier's bid quantity decision can be simplified to $q_i = x_i$, and the bidding game is reduced to an oligopoly pricing game. Henceforth, without loss of generality we assume that $q_i = x_i$ for all players $i$. Accordingly, in what follows we will use the condensed notation $r_i(\mathbf{p}) = r_i(\mathbf{p},\mathbf{x})$, $Q_i^L(\mathbf{p}) = Q_i^L(\mathbf{p},\mathbf{x})$, and $Q_i^E(\mathbf{p}) = Q_i^E(\mathbf{p},\mathbf{x})$. So bidder $i$'s problem in the game is,

$$\max_{p_i\in[c_i,b]} \; R_i(p_i,\mathbf{p}_{-i}) = (p_i - c_i) x_i r_i(p_i,\mathbf{p}_{-i}).$$

Notice that if there is no excessive capacity at price level $p$, all suppliers pricing below or at $p$ obtain a sales ratio 1. That is,

$$\sum\nolimits_k x_k \delta_{(p_k \leq p_i)} \leq d(p) \text{ implies } r_k(p_k,\mathbf{p}_{-k}) = 1 \text{ for all } k \text{ with } p_k \leq p, \tag{C.4}$$

$$\sum\nolimits_k x_k \delta_{(p_k < p_i)} \leq d(p) \text{ implies } r_k(p_k,\mathbf{p}_{-k}) = 1 \text{ for all } k \text{ with } p_k < p. \tag{C.5}$$

**Pure-Strategy Equilibrium Analysis.** For a pure-strategy equilibrium outcome $\mathbf{p}^* = \{p_1^*, p_2^*, \ldots, p_N^*\}$, define $A = \{i : r_i(p_i^*, \mathbf{p}_{-i}^*) > 0\}$, the set of active suppliers (those with positive market share).

**Lemma 17** *For any pure-strategy equilibrium $\mathbf{p}^*$, there exists a constant $P^*$ such that (a) $i \in A$ implies $p_i^* = P^*$; (b) $c_i < P^*$ implies $i \in A$; (c) $c_i > P^*$ implies $i \notin A$; (d) $p_i \geq P^*$ for all $i$.*

**Proof.** For part (a) it is equivalent to show that any $i, j \in A$ must have $p_i^* = p_j^*$. Suppose $p_i^* < p_j^*$ and $r_j(p_j^*, \mathbf{p}_{-j}^*) > 0$, implying $\sum_k x_k \delta_{(p_k^* < p_j^*)} < d(p_j^*)$. This relation will not change if supplier $i$ raises her price to $p_i' = \frac{p_i^* + p_j^*}{2}$ since $\delta_{(p_i^* < p_j^*)} = \delta_{(p_i' < p_j^*)} = 1$, suggesting $r_i\left(p_i', \mathbf{p}_{-i}^*\right) = r_i\left(p_i^*, \mathbf{p}_{-i}^*\right) = 1$ by (C.5). It follows $R_i\left(p_i'\right) = \left(p_i' - c_i\right) x_i > \left(p_i^* - c_i\right) x_i = R_i\left(p_i^*\right)$, a contradiction to the optimality of $p_i^*$.

(b) Suppose $c_i < P^*$ but $r_i(p_i^*, \mathbf{p}_{-i}^*) = 0$ so $R_i(p_i^*, \mathbf{p}_{-i}^*) = 0$. But, if supplier $i$ bids $p_i' = P^*$, she would have $R_i(P^*, \mathbf{p}_{-i}^*) = (P^* - c_i) x_i r_i(P^*, \mathbf{p}_{-i}^*) > 0$, contradicting the optimality of $p_i^*$.

(c) As $p_i \geq c_i$ always, we must have $p_i^* \geq c_i > P^*$, implying $i \notin A$ from the converse of part (a).

(d) From (a) and (b) if $c_i < P^*$ then $p_i = P^*$. If $c_i \geq P^*$ then $p_i \geq c_i \geq P^*$. ■

**Lemma 18** $r_i(\mathbf{p})$ *is nondecreasing in* $p_j$ *for* $j \neq i$ *and nonincreasing in* $p_i$*.*

**Proof.** (i) We prove the monotonicity of $r_i(\mathbf{p})$ in $p_j$ with $j \neq i$ in three steps. Notice that when $p_j$ changes, $d(p_i)$ is not changed.

(i-a) $Q_i^L$ is weakly decreasing in $p_j$. From (C.1), $Q_i^L$ involves two groups of indices. When $p_j$ increases, index $j$ can disappear from the first group ($\delta_{(p_j < p_i)}$ changing from 1 to 0), or disappear from the second group ($\delta_{(p_j = p_i, c_j < c_i = p_i)}$ changing from 1 to 0), or move from the first group to the second. For any one of above cases, $Q_i^L$ does not increase.

(i-b) $Q_i^L + Q_i^E$ is weakly decreasing in $p_j$. According to equation (C.2), $Q_i^L + Q_i^E$ is a difference of two groups of indexes. When $p_j$ increases, index $j$ can disappear from the first group ($\delta_{(p_j \leq p_i)}$ changing from 1 to 0), or disappear from group 2 ($\delta_{(p_j = p_i, c_j < p_i, c_i < p_j)}$ changing from 1 to 0) or both. When it disappears from the second group, it also disappears from the first group.

(i-c) Consider any $p_j' < p_j''$ and we want to show

$$1 \wedge \frac{[d(p_i) - Q_i^L(p_j', \mathbf{p}_{-j})]^+}{x_i + Q_i^E(p_j', \mathbf{p}_{-j})} \leq 1 \wedge \frac{[d(p_i) - Q_i^L(p_j'', \mathbf{p}_{-j})]^+}{x_i + Q_i^E(p_j'', \mathbf{p}_{-j})}, \tag{C.6}$$

If $Q_i^L(p_j', \mathbf{p}_{-j}) \geq d(p_i)$, then the left hand side of (C.6) is 0 and the desired inequality holds trivially. If $Q_i^L(p_j', \mathbf{p}_{-j}) < d(p_i)$, from (i-a) and $p_j' < p_j''$, we have

$Q_i^L(p_i, p_j'', \mathbf{p}_{-i-j}) \le Q_i^L(p_j', \mathbf{p}_{-j}) < d(p_i)$. Now it is sufficient to show that

$$\frac{x_i + Q_i^E(p_j', \mathbf{p}_{-j})}{d(p_i) - Q_i^L(p_j', \mathbf{p}_{-j})} \ge \frac{x_i + Q_i^E(p_j'', \mathbf{p}_{-j})}{d(p_i) - Q_i^L(p_j'', \mathbf{p}_{-j})} \text{ implying}$$

$$1+\frac{x_i - d(p_i) + Q_i^E(p_j', \mathbf{p}_{-j}) + Q_i^L(p_j', \mathbf{p}_{-j})}{d(p_i) - Q_i^L(p_j', \mathbf{p}_{-j})} \ge 1+\frac{x_i - d(p_i) + Q_i^E(p_j'', \mathbf{p}_{-j}) + Q_i^L(p_j'', \mathbf{p}_{-j})}{d(p_i) - Q_i^L(p_j'', \mathbf{p}_{-j})}.$$

Using (i-a) and (i-b), the above inequality holds.

(ii) Now we show $r_i(\mathbf{p})$ is nonincreasing in $p_i$. From (C.3) and monotonicity of $d(\cdot)$, we have that $d(p)$ is nonincreasing in $p_i$. Part (i) implies that for $k \neq i$, $z_k(p_k, p_i, \mathbf{p}_{-i-k}) = x_k r_k(p_k, p_i, \mathbf{p}_{-i-k})$ is nondecreasing in $p_i$. Since $\sum_k z_k(p_k, p_i, \mathbf{p}_{-i-k})$ $= \min\{d(\mathbf{p}), \sum_k x_k\}$ is nonincreasing in $p_i$, we must have that $z_i(p_i, \mathbf{p}_{-i})$ $= \min\{d(\mathbf{p}), \sum_{k=1}^N x_k\} - \sum_{k\neq i} z_k(p_k, p_i, \mathbf{p}_{-i-k})$ is nonincreasing in $p_i$. ■

Denote $r_i^-(p_i, \mathbf{p}_{-i}) = \lim_{p_i'\uparrow p_i} r_i(p_i', \mathbf{p}_{-i})$ and $r_i^+(p_i, \mathbf{p}_{-i}) = \lim_{p_i'\downarrow p_i} r_i(p_i', \mathbf{p}_{-i})$. Similar notation applies to $R_i^-(p_i, \mathbf{p}_{-i})$ and $R_i^+(p_i, \mathbf{p}_{-i})$.

**Lemma 19** *For any price $p_i > c_i$,*

*(a) $r_i^-(p_i, \mathbf{p}_{-i}) = 1 \wedge \frac{(d(p_i) - \sum_{k\neq i} x_k \delta_{(p_k < p_i)})^+}{x_i}$ and $r_i^+(p_i, \mathbf{p}_{-i}) = 1 \wedge \frac{(d(p_i) - \sum_{k\neq i} x_k \delta_{(p_k \le p_i)})^+}{x_i}$;*

*(b) $r_i^-(p_i, \mathbf{p}_{-i}) \ge r_i(p_i, \mathbf{p}_{-i}) \ge r_i^+(p, \mathbf{p}_{-i})$.*

*(c) $p_i^* > c_i$ at equilibrium implies $r_i^-(p_i^*, \mathbf{p}_{-i}^*) = r_i(p_i^*, \mathbf{p}_{-i}^*)$.*

**Proof.** (a) First notice that for any $p_i \lim_{p_i'\uparrow p_i} \delta_{(p_k < p_i')} = \delta_{(p_k < p_i)}$, $\lim_{p_i'\downarrow p_i} \delta_{(p_k < p_i')} = \delta_{(p_k \le p_i)}$, and $\lim_{p_i'\to p_i} \delta_{(p_k = p_i')} = 0$. Plugging the above expressions into (C.1), we have $\lim_{p_i'\uparrow p_i} Q_i^L(p_i', \mathbf{p}_{-i}) = \sum_{k\neq i} x_k \delta_{(p_k < p_i)}$, $\lim_{p_i'\downarrow p_i} Q_i^L(p_i', \mathbf{p}_{-i}) = \sum_{k\neq i} x_k \delta_{(p_k \le p_i)}$, $\lim_{p_i'\to p_i} Q_i^E(p_i', \mathbf{p}_{-i}) = 0$. As $\lim_{p_i'\uparrow p_i} d(p_i') = d(p_i)$, result (a) follows.

(b) Part (b) is directly implied by monotonicity of $r_i(p_i, \mathbf{p}_{-i})$ established in Lemma 18.

(c) $R_i^-(\mathbf{p}^*) = (p_i^* - c_i) x_i r_i^-(\mathbf{p}^*) \le R_i(\mathbf{p}^*)$ at equilibrium or else agent $i$ would deviate downward from $p_i^*$ if possible. When $p_i^* > c_i$ this implies $r_i^-(\mathbf{p}^*) \le r_i(\mathbf{p}^*)$. But by part (b) $r_i^-(\mathbf{p}^*) \ge r_i(\mathbf{p}^*)$, so $r_i^-(\mathbf{p}^*) = r_i(\mathbf{p}^*)$ at equilibrium. ■

Denote the set of profitable suppliers at equilibrium $\mathbf{p}^*$ by $\Pi = \{i : R_i(\mathbf{p}^*) > 0\}$. $\Pi = \{i : c_i < P^*\} \subseteq A$ (from Lemma 17b) and $i \in A \backslash \Pi$ implies $c_i = P^*$.

**Proposition 17** *A pure-strategy equilibrium* $\mathbf{p}^*$ *satisfies (i)* $p_i^* = P^*$ *for all* $i \in A$ *where*

$$P^* := \begin{cases} \underset{p\in[c_1,c_2]}{\arg\max}\{(p-c_1)\,[d\,(p)\wedge x_1]\} & \text{if } d\,(c_2) < x_1 & (a) \\ c_j & \text{if } \sum_{k=1}^{j-1} x_k < d\,(c_j) < \sum_{k=1}^{j} x_k \text{ and } j \geq 2 & (b) \\ p(\sum_{k=1}^{j} x_k) & \text{if } d\,(c_{j+1}) \leq \sum_{k=1}^{j} x_k \leq d\,(c_j) & (c) \\ b & \text{if } d\,(b) \geq \sum_k x_k & (d) \end{cases} \tag{C.7}$$

*(ii)* $r_i(\mathbf{p}^*) = 1 \wedge \frac{d}{x_i}$ *for all* $i \in \Pi$.

**Proof.** (i-a) Suppose $P^* > c_2$, implying $d(P^*) < x_1$. From Lemma 17(b), we have $\{1,2\} \subset \Pi$, and therefore, $r_1(P^*, \mathbf{p}_{-1}^*) = \frac{d(P^*)}{\sum_{k\in A} x_k} \leq \frac{d(P^*)}{x_1+x_2} < \frac{d(P^*)}{x_1} = r_1^-(P^*, \mathbf{p}_{-1}^*)$, which contradicts Lemma 19(c). Thus, we have $P^* \leq c_2$. If $c_1 = c_2$, from $P^* = p_1^* \geq c_1$ we have $P^* = c_2$. If $c_1 < c_2$, for all $p_1 \leq c_2$, (C.1) reads $r_1\,(p_1) = 1 \wedge \frac{d(p_1)}{x_1}$, implying $R_1(p_1) = (p_1 - c_1)[x_1 \wedge d(p_1)]$, a strictly concave function by Assumption 1. Optimality of $p_1^* = P^*$ requests $P^*$ to be unique maximizer of supplier 1's profit over the interval $[c_1, c_2]$.

(i-b,c) Denote $P_b = c_j$ and $P_c = p(\sum_{k=1}^{j} x_k)$. Suppose $P^* < P_b$ (or $P_c$), equation (C.7-b,c) implies $\sum_{k\in\Pi} x_k \leq d(P_b)$ (or $d(P_c)$), so $r_k = 1$ for all $k \in \Pi$. But, $r_k = 1$ will remain equal to 1 as long as $p_k \in [P^*, P_b$ (or $P_c$)]. Thus, these agents will want to defect to a greater price, contradicting the optimality of $p_k^* = P^*$. Suppose $P^* > P_b$ (or $P_c$), then $\sum_{k\in\Pi} x_k \geq \sum_{k=1}^{j} x_k \delta_{(c_k \leq P_b \text{ or } P_c)} \geq d\,(P_b)$ (or $d(P_c)$ )$> d(P^*)$, implying supplier $i \in \Pi$ has $r_i(\mathbf{p}^*) = \frac{d(P^*)}{\sum_{k\in\Pi} x_k} < [1 \wedge \frac{d(P^*)}{x_i}] = r_k^-(\mathbf{p}^*)$, contradicting Lemma 19(c). So $P^* = P_b$ (or $P_c$).

(i.d) From (C.4) if $d(b) \geq \sum_k x_k$ then $r_i(p) = 1$ for all $i$ and $p \leq b$ and the only possible profit maximizing price for any agent is $b$.

(ii) From Lemma 17(b) and (c), $\Pi = \{i | c_i < P^*\} \subseteq A \subseteq \{i | c_i \leq P^*\}$. For cases (a) in equation (C.7), we have either $\Pi = \varnothing$ when $c_1 = c_2$ or $\Pi = \{1\}$ and $r_1\,(\mathbf{p}^*) = \frac{d(P^*)}{x_1} < 1$ when $c_1 < c_2$, so part (ii) holds in that case. For cases (b), (c), and (d) we have from equation (C.7) that $\sum_{k\in\Pi} x_k = \sum_k x_k \delta_{(c_k < P^*)} \leq d$ so $r_i(\mathbf{p}^*) = 1$ for all $i \in \Pi$ and again part (ii) hold. ■

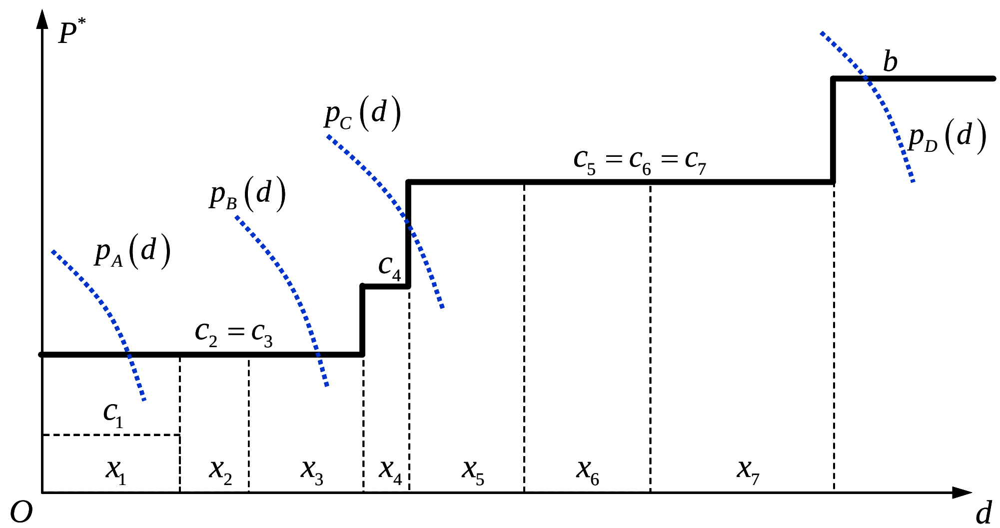


**Figure C.1.** Derivation of Pure-Strategy Equilibrium Prices under Different Demand Functions

Figure C.1 is a graphic illustration of Proposition 17.

**Lemma 20** *(a) If a pure-strategy equilibrium* $\mathbf{p}^*$ *has* $p_i^* > c_i$ *for any* $i \notin \Pi$*, then* $\left(c_i, \mathbf{p}_{-i}^*\right)$ *is also an equilibrium and it is payoff-equivalent to* $\mathbf{p}^*$*; (b) If a pure-strategy equilibrium exists, then the game has a unique normalized equilibrium* $\{p_i^* = c_i \wedge P^*$ *for all* $i\}$ *with* $P^*$ *given by (C.7).*

**Proof.** (a) $i \notin \Pi$ implies $R_i(\mathbf{p}^*) = 0$ and agent $i$ is not economically affected by changing $p_i^*$ to $c_i$. We need however to show that no other agent $j \neq i$ will have an incentive to defect from $p_j^*$ after this adjustment. From Lemma 18 $r_j$ is nonincreasing in $p_i$, so $r_j(p_j^*, c_i, \mathbf{p}_{-i-j}^*) \geq r_j(p_j^*, p_i^*, \mathbf{p}_{-i-j}^*)$ and $R_j(p_j^*, c_i, \mathbf{p}_{-i-j}^*) \geq R_j(p_j^*, p_i^*, \mathbf{p}_{-i-j}^*)$ for all $j$. If $j \in \Pi$, implying $c_j < p_j^* = P^* \leq c_i < p_i$, agent $j$ will have a strictly higher allocation priority over agent $i$ regardless of whether agent $i$ bids $p_i$ or $c_i$, so agent $j$'s allocation and optimal response are unaffected by this change. If $j \notin \Pi$ then the above and the non-negativity of $R_j$ imply $0 \leq R_j(p_j, c_j, \mathbf{p}_{-i-j}) = 0$ so again there is no effect.

(b) From Proposition 17 and part (a), for any game with a pure-strategy equilibrium, $p_i^* = c_i \vee P^*$ defines a normalized pure-strategy equilibrium. Uniqueness follows because $P^*$ is uniquely defined in (C.7) and stays the same when we interchange the indices among players with the same cost. ■

**Proposition 18** *A unique normalized pure-strategy equilibrium exists with* $p_i^* = P^* \vee c_i$ *and* $P^*$ *given by (C.7) if and only if*

$$(P^* - c_i)[x_i \wedge d(P^*)] \geq (p - c_i)[d\,(p) - \textstyle\sum_{k \neq i} x_k \delta_{(c_k < p)}]^+ \tag{C.8}$$

*for all* $i$ *with* $c_i \leq P^*$ *and* $p > P^*$.

**Proof.** If a normalized pure-strategy equilibrium exists, by Lemma 20 (b), $p_i^* = P^* \vee c_i$ must be the equilibrium bid price. We seek conditions under which agents wish to defect from $p_i^* = P^* \vee c_i$. Consider three cases, $c_i < P^*$, $c_i > P^*$, and $c_i = P^*$.

(a) $c_i < P^*$: In this case $i \in \Pi \subseteq A$, $p_i^* = P^*$, and $r_i = 1 \wedge \frac{d(P^*)}{x_i}$ from Lemma 17 and Proposition 17, so $R_i(P^*, \mathbf{p}_{-i}^*) = (P^* - c_i)[x_i \wedge d(P^*)]$. $r_i$ is unaffected if supplier $i$ lowers her price, so there is no incentive to do that. Therefore, agent $i$ has no incentive to defect from $p_i^* = P^*$ if and only if no higher price can lift her profit, i.e., (C.8) holds.

(b) $c_i > P^*$: In this case $i \notin A$, $p_i^* = c_i$ and $R_i(\mathbf{p}^*) = 0$ by Lemma 17 and normalization. Agent $i$ cannot reduce her price below $c_i$ profitably, and since $r_i$ is non-increasing it will remain zero and she cannot do better by increasing her price. So, she has no incentive to deviate.

(c) $c_i = P^*$: In this case, $p_i^* = P^* = c_i$ and agent $i$ is making no profit, $R_i(p^*) = 0$. Lowering price is not feasible, so agent $i$ has no incentive to defect from $P^*$ if and only if all higher price also yields zero profit, i.e., (C.8) holds. ■

**Mixed-Strategy Equilibrium Analysis.** Proposition 2 suggests that there is no pure-strategy equilibrium if there exist $i$ and $p$ such that

$$(P^* - c_i)[x_i \wedge d(P^*)] < (p - c_i)[d\,(p) - \textstyle\sum_{k \neq i} x_k \delta_{(c_k < p)}]^+ \tag{C.9}$$

*Throughout this section, we assume (C.9) holds for certain* $i$ with $c_i \leq P^*$ *and* $p > P^*$ *and seek mixed-strategy Nash equilibria.*

The following notation is used for describing mixed pricing strategies.

| | |
|---|---|
| $\mathcal{P}(B)$ | Probability space defined over set $B$ |
| $\sigma_i \in \mathcal{P}([c_i, b])$ | Supplier $i$'s mixed strategy (denoted as a random variable) |
| $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_N)$ | Mixed strategy outcome |
| $\sigma_{-i} = (\sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_N)$ | Mixed strategies of supplier $i$'s competitors |
| $F_i(p\|\sigma_i) = \Pr\{\sigma_i \le p\}$ | Cumulative distribution function of price for mixed strategy $\sigma_i$ |
| $m_i(p\|\sigma_i) = \Pr\{\sigma_i = p\}$ | Probability of price $p_i$ for mixed strategy $\sigma_i$ |
| $\overline{p}_i(\sigma_i) = \min\{p : F_i(p\|\sigma_i) = 1\}$ | Supremum of prices for mixed strategy $\sigma_i$ |
| $\underline{p}_i(\sigma_i) = \max\{p : F_i(p\|\sigma_i) = 0\}$ | Infimum of prices for mixed strategy $\sigma_i$ |
| $\bar{r}_i(p, \sigma_{-i}) = \mathsf{E}[r_i(p, \sigma_{-i})]$ | Expected fraction of supplier $i$'s sales at price $p$ |
| $\bar{R}_i(p, \sigma_{-i}) = (p_i - c_i)x_i\bar{r}_i(p, \sigma_{-i})$ | Supplier $i$'s expected return at price $p$ |
| $ER_i(\sigma^*)$ | Expected profit for player $i$ at mixed-strategy equilibrium $\sigma^*$ |

With the above notation, bidder $i$'s problem is as follows,

$$\max_{\sigma_i \in \mathcal{P}([c_i,b])} \bar{R}(\sigma_i, \sigma_{-i}) = x_i \mathsf{E}[(p_i - c_i) r_i(\sigma_i, \sigma_{-i})]. \tag{C.10}$$

For simplicity, when $\sigma_i = \sigma_i^*$, we suppress the notation by omitting $\sigma_i^*$ and $\sigma_{-i}^*$. For example, $F_i(p_i) = F_i(p_i|\sigma_i = \sigma_i^*)$ and $\overline{p}_i = \overline{p}_i(\sigma_i^*)$. Similarly, we use shorthand notation $\bar{R}_i(p_i) = \bar{R}_i(p_i, \sigma_{-i}^*)$, $\bar{r}_i(p_i) = \bar{r}_i(p_i, \sigma_{-i}^*)$, and $ER_i = ER_i(\sigma_i^*, \sigma_{-i}^*)$.

Denote $\underline{P} = \min\{\underline{p}_i : ER_i > 0\}$ and $\overline{P} = \max\{\bar{p}_i : ER_i > 0\}$ for the pricing bounds. Denote $\bar{r}_i^-(p, \sigma_{-i}) = \lim_{p' \uparrow p} \bar{r}_i(p', \sigma_{-i})$ and $\bar{r}_i^+(p, \sigma_{-i}) = \lim_{p' \downarrow p} \bar{r}_i(p', \sigma_{-i})$, and correspondingly, we have $\bar{R}_i^-(p, \sigma_{-i}) = (p - c_i)x_i\bar{r}_i^-(p, \sigma_{-i})$ and $\bar{R}_i^+(p, \sigma_{-i}) = (p - c_i)x_i\bar{r}_i^+(p, \sigma_{-i})$. As $r_i(p, \mathbf{p}_{-i}) \in [0, 1]$ for all $\mathbf{p}$, from *bounded convergence theorem*[1] and Lemma 19(a), we must have

$$\begin{aligned} \bar{r}_i^-(p, \sigma_{-i}) &= \mathsf{E}[r_i^-(p, \sigma_{-i})] = \mathsf{E}[1 \wedge \frac{(d(p) - \sum_{k \ne i} x_k \delta_{(\sigma_k < p)})^+}{x_i}] \\ \bar{r}_i^+(p, \sigma_{-i}) &= \mathsf{E}[r_i^+(p, \sigma_{-i})] = \mathsf{E}[1 \wedge \frac{(d(p) - \sum_{k \ne i} x_k \delta_{(\sigma_k \le p)})^+}{x_i}]. \end{aligned} \tag{C.11}$$

[1]Refer to standard textbooks of probability and measure theory such as Billingsley (1995).

Applying Lemma 19(b), we have

$$\bar{r}_i^-(p,\sigma_{-i}) \geq \bar{r}_i(p,\sigma_{-i}) \geq \bar{r}_i^+(p,\sigma_{-i}) \tag{C.12}$$

**Lemma 21** *A mixed-strategy equilibrium satisfies the following, for any* $p \in [\underline{p}_i, \bar{p}_i]$*,*

*(a) if* $m_i(p) > 0$*, then* $\bar{R}_i^-(p) = \bar{R}_i(p) = ER_i$*;*

*(b) if* $F_i(p') < F_i^-(p)$ *for all* $p' < p$*, then* $\bar{R}_i^-(p) = ER_i$*;*

*(c) if* $F_i(p) < F_i(p')$ *for all* $p' > p$*, then* $\bar{R}_i^-(p) = \bar{R}_i(p) = \bar{R}_i^+(p) = ER_i$*.*

**Proof.** Notice that for any $p > c_i$, from inequality (C.12) and optimality of $ER_i$, we have

$$\bar{R}_i^+(p) \leq \bar{R}_i(p) \leq \bar{R}_i^-(p) \leq ER_i. \tag{C.13}$$

For part (a), it is sufficient to show $ER_i = \bar{R}_i(p)$, which clearly holds. For part (b), suppose there exists $p$ such that $F_i(p') < F_i^-(p)$ for all $p' < p$, but $ER_i > \bar{R}_i^-(p)$. It implies $\bar{r}_i^-(p) = \frac{\bar{R}_i^-(p)}{(p-c_i)x_i} < \frac{ER_i}{(p-c_i)x_i}$, that is, there exists $\delta_L > 0$ such that $\bar{r}_i(p') < \frac{ER_i}{(p'-c_i)x_i}$ for all $p' \in \mathbf{s}_L \equiv (p-\delta_L, p)$. We then have $\bar{R}_i(p') = (p'-c_i)x_i\bar{r}_i(p') < ER_i$ for all $p' \in \mathbf{s}_L$, implying $\Pr\{\sigma_i^* \in \mathbf{s}_L\} = 0$, a contradiction to $F_i(p') < F_i^-(p)$ for all $p' < p$. For part (c), by inequality (C.13), we only need to show $ER_i = \bar{R}_i^+(p)$. Since $F_i(p')$, as a cdf, is right continuous, we have $F_i^+(p) = F_i(p)$. Suppose $F_i^+(p) = F_i(p) < F_i(p')$ for all $p' > p$ but $ER_i > \bar{R}_i^+(p)$. Similar to part (b), there exists $\delta_R > 0$ such that $\bar{R}_i(p') < ER$ for all $p' \in \mathbf{s}_R = (p, p+\delta_R)$, implying $\Pr\{\sigma_i^* \in \mathbf{s}_R\} = 0$. This is a contradiction to $F_i^+(p) = F_i(p) < F_i(p')$, for all $p' > p$. ■

Notice that, for price $\underline{p}_i = \inf\{p|F_i(p) > 0\}$, we have either $m_i(\underline{p}_i) > 0$ or $\{m_i(\underline{p}_i) = 0$ and $F_i(p) > 0$ for all $p > \underline{p}_i\}$, so parts (a) or (c) of Lemma 21 apply. Similarly, for $\bar{p}_i$, part (a) or (b) of Lemma 21 can be applied. Therefore, we have a useful relationship,

$$\bar{R}_i^-(\underline{p}_i) = \bar{R}_i(\underline{p}_i) = ER_i = \bar{R}_i^-(\bar{p}_i). \tag{C.14}$$

Define the set of profitable suppliers as $\Pi := \{k : ER_k > 0\}$.

**Lemma 22** $ER_i > 0$ *if and only if* $c_i < \overline{P}$*.*

**Proof.** Necessity is obvious from how $\overline{P}$ is defined. For sufficiency, suppose there exists $i$ satisfying $c_i < \overline{P}$ but $ER_i = 0$. We must have $\bar{r}_i(p) = 0$ for any $p \in \left(c_i, \overline{P}\right)$. It implies $\Pr\{d(p) \leq \sum_{k \neq i} x_k \delta_{(\sigma_k^* < p)}\} = 1$, or equivalently, $\Pr\{d(p) > \sum_{k \neq i} x_k \delta_{(\sigma_k^* < p)}\} = 0$. On the other hand, $\overline{P} > p$ indicates that there must be certain player $h \in \Pi$ pricing over $(p, \overline{P}]$ with a positive probability, i.e., $\Pr\{\sigma_h^* > p\} > 0$. As supplier $h$ achieves $ER_h > 0$ by choosing certain price higher than $p$, her expected sales at price $p$ must be positive, implying, $\Pr\{d > \sum_{k \neq h} x_k \delta_{(\sigma_k^* < p)}\} > 0$. Now we get a contradiction,

$$\begin{aligned} 0 &= \Pr\{d(p) > \sum_{k \neq i} x_k \delta_{(\sigma_k^* < p)}\} \geq \Pr\{d(p) > \sum_{k} x_k \delta_{(\sigma_k^* < p)}\} \\ &\geq \Pr\{\sigma_h^* > p\} \Pr\{d(p) > \sum_{k} x_k \delta_{(\sigma_k^* < p)} | \sigma_h > p\} \\ &= \Pr\{\sigma_h^* > p\} \Pr\{d(p) > \sum_{k \neq h} x_k \delta_{(\sigma_k^* < p)}\} > 0. \end{aligned}$$

■

Similarly to the pure-strategy equilibrium analysis, in the rest of this section, we restrict ourselves to normalized equilibria by fixing non-profitable players' bidding strategy. The following lemma justifies the generality of our assumption.

**Lemma 23** *If $\sigma^* = \{\sigma_i^*, \sigma_{-i}^*\}$ is an equilibrium with $ER_i = 0$ and $\bar{p}_i > c_i$, then $\{p_i = c_i, \sigma_{-i}^*\}$ is a payoff-equivalent equilibrium to $\sigma^*$.*

**Proof.** As $ER_i = \bar{R}_i(\sigma^*) = 0 = \bar{R}_i(c_i)$, agent $i$ herself is not economically affected by changing from $\sigma_i^*$ to $c_i$. For any other player $j \neq i$, since $\sigma_i^* \in \mathcal{P}([c_i, b])$ is (weakly) larger than $p_i = c_i$, the monotonicity of $r_j(\cdot)$ in $p_i$ (Lemma 18) implies $\Pr\left\{r_j\left(p_j, c_i, \sigma_{-j-i}^*\right) \leq r_j\left(p_j, \sigma_i^*, \sigma_{-j-i}^*\right)\right\} = 1$ for any $p_j$, and therefore,

$$\bar{R}_j\left(p_j, c_i, \sigma_{-j-i}^*\right) \leq \bar{R}_j\left(p_j, \sigma_i^*, \sigma_{-j-i}^*\right) \leq ER_j \quad \text{for all } p_j \geq c_j. \tag{C.15}$$

That is, given $c_i$ and $\sigma_{-j-i}^*$, $ER_j$ is an upper bound of supplier $j$'s payoff. Consequently, it is sufficient to show that strategy $\sigma_j^*$ maximizes $j$'s expected payoff and $\bar{R}_j(\sigma_j^*, c_i, \sigma_{-j-i}) = \bar{R}_j(\sigma_j^*, \sigma_i, \sigma_{-j-i}) = ER_j$. For a nonprofitable supplier $j \neq i$, this is trivially true since (C.15) implies $j$'s expected payoff is 0 everywhere. For a profitable supplier $j$ with $ER_j > 0$, Lemma 22 implies $c_j < \bar{p}_j \leq \overline{P} \leq c_i \leq \underline{p}_i$. By allocation rule (C.1), $\sigma_j^*$ yields a strictly higher allocation priority than both $\sigma_i^*$ and $p_i = c_i$. Therefore, her expected payoff by choosing $\sigma_j^*$ will not be affected by player $i$'s strategy adjustment. ■

The following inequality is very useful,

$$\bar{r}_i^-(p) \geq \frac{(d(p) - \sum_{k \neq i} x_k \delta_{(c_k < p)})^+}{x_i} \quad \text{for all } p \geq \max\{P^*, c_i\}. \tag{C.16}$$

Notice that $r_i^-(p) = 1 \wedge \frac{(d(p) - \sum_{k \neq i} x_k \delta_{(\sigma_k^* < p)})^+}{x_i}$ is decreasing in $\sigma_k^*$ for $k \neq i$. As $\sigma_k^* \geq c_k$ and $d(p) < d(P^*) \leq \sum_k x_k \delta_{(c_k \leq P^*)} \leq \sum_k x_k \delta_{(c_k < p)} \leq \sum_{k \neq i} x_k \delta_{(c_k < p)} + x_k$, the term in the right-hand side of (C.16) is the lowest value for $r_i^-(p)$, and (C.16) follows directly.

**Lemma 24** *(a) If $i \in \Pi$ and $\underline{p}_i = \underline{P}$, then $ER_i = (\underline{P} - c_i)(x_i \wedge d(\underline{P}))$; (b) If $\sum_{k \neq i} x_k \delta_{(c_k < \underline{P})} \leq d(\underline{P})$ (implied by $\sum_{k \in \Pi \setminus \{i\}} x_k \leq d(\underline{P})$ ), then $\underline{p}_i = \underline{P}$; (c) $\overline{P} > \underline{P} > P^*$.*

**Proof.** (a) It directly follows from (C.14) and $\bar{r}_i^-(\underline{P}) = 1 \wedge \frac{d(\underline{P})}{x_i}$.

(b) Suppose there exists $i \in \Pi$ such that $\sum_{k \neq i} x_k \delta_{(c_k < \underline{P})}$ but $\underline{p}_i > \underline{P}$. Note that (C.14) implies $\underline{p}_k > c_k$ for all $k \in \Pi$, and therefore, any supplier $k$ with $c_k \geq \underline{P}$ has $\underline{p}_k > \underline{P}$. Denote $\tilde{p} := \underline{p}_i \wedge \min\{p_k : c_k \geq \underline{P}\}$ and we have $\tilde{p} > \underline{P}$. Now consider all suppliers with $c_j < \underline{P}$ and $j \neq i$, since their total capacity is less than the demand and all other suppliers price above $\tilde{p}$, they obtain $\bar{r}_j(p) = 1$ for all $p < \tilde{p}$, implying $\underline{p}_j \geq \tilde{p}$. It follows $\underline{P} = \min\{\underline{p}_k\} \geq \tilde{p}$, a contradiction to $\tilde{p} > \underline{P}$.

(c) $\overline{P} > \underline{P}$ follows from nonexistence of pure-strategy equilibrium, so we only need to show $\underline{P} > P^*$. Suppose $\underline{P} < P^*$. From $\delta_{(\sigma_k^* < p)} \leq \delta_{(c_k < p)}$ and (C.7), we have $\sum_k x_k \delta_{(\sigma_k^* < P^*)} \leq \sum_k x_k \delta_{(c_k < P^*)} \leq d(P^*)$. From (C.5), we must have $r_k\left(p, \sigma_{-k}^*\right) = 1$ for all $k$ with $c_k < P^*$ and $p \in [\underline{P}, P^*)$, and therefore $\bar{r}_k\left(p, \sigma_{-k}^*\right) = 1$ and $\bar{R}_k\left(p, \sigma_{-k}^*\right) = (p - c_k)x_k$. It implies $\underline{p}_k \geq P^*$ for all $k$ with $c_k < P^*$, a contradiction to $\underline{P} < P^*$, so we must have $\underline{P} \geq P^*$. Suppose $\underline{P} = P^*$. Consider supplier $i$ and price $p > P^*$ that satisfy (C.9), which implies $\sum_{k \neq i} x_k \delta_{(c_k < P^*)} \leq \sum_{k \neq i} x_k \delta_{(c_k < p)} < d(p)$. Part (b) implies $\underline{p}_i = \underline{P} = P^*$ and therefore $ER_i = (P^* - c_i)(x_i \wedge d(P^*))$ by part (a). But inequalities (C.16) and (C.9) suggest $\bar{R}_i^-(p) \geq (p - c_i)(d(p) - \sum_{k \neq i} x_k \delta_{(c_k < \underline{P})})^+ > (P^* - c_i)(x_i \wedge d(P^*)) = ER_i$, a contradiction to the optimality of $ER_i$. So we must have $\underline{P} > P^*$. ■

Lemma 24(c) implies that we do not need to differentiate between profitable players (with $ER_k > 0$) and active players (with $\mathsf{E}[r_k] > 0$) in the rest of this section.

**Lemma 25** *For a mixed-strategy equilibrium, at most one profitable supplier has* $m_i(\overline{P}) > 0$.

**Proof.** Suppose there are at least two suppliers with probability mass at $\overline{P}$. For any $i \in \Omega_{\overline{P}} := \{k : m_k(\overline{P}) > 0, ER_k > 0\}$, from (C.14), we have $\bar{R}_i^-(\overline{P}) = ER_i > 0$ and $\bar{r}_i^-(\overline{P}) > 0$. Now consider all players in $\Pi \backslash \Omega_{\overline{P}}$, who with probability 1 price lower than $\overline{P}$. For supplier $i \in \Omega_{\overline{P}}$, since $\bar{r}_i^-(\overline{P}) > 0$, we must have $d(\overline{P}) > \sum_{k \in \Pi \backslash \Omega_{\overline{P}}} x_k$. Together with $\sum_{k\in\Pi} x_k \geq d(P^*) > d(\overline{P})$, and $x_i < \sum_{k \in \Omega_{\overline{P}}} x_k$, it implies

$$1 \wedge \frac{d(\overline{P}) - \sum_{k \in \Pi \backslash \Omega_{\overline{P}}} x_k}{x_i} > \frac{d(\overline{P}) - \sum_{k \in \Pi \backslash \Omega_{\overline{P}}} x_k}{\sum_{k \in \Omega_{\overline{P}}} x_k}.$$

Due to independence of suppliers' strategies, with probability $\prod_{k \in \Omega_{\overline{P}} \backslash \{i\}} m_k \left(\overline{P}\right) > 0$, all players in $k \in \Omega_{\overline{P}} \backslash \{i\}$ choose $\overline{P}$ and we have $\bar{r}_i^-(\overline{P}) - \bar{r}_i(\overline{P}) \geq \prod_{k \in \Omega_{\overline{P}} \backslash \{i\}} m_k \left(\overline{P}\right) \cdot \left[\left(1 \wedge \frac{d(\overline{P}) - \sum_{k \in \Pi \backslash \Omega_{\overline{P}}} x_k}{x_i}\right) - \frac{d(\overline{P}) - \sum_{k \in \Pi \backslash \Omega_{\overline{P}}} x_k}{\sum_{k \in \Omega_{\overline{P}}} x_k}\right] > 0$. It implies $\bar{R}_i^-(\overline{P}) > \bar{R}_i(\overline{P})$, contradicting the initial assumption $m_i(\overline{P}) > 0$, by Lemma 21(a). ■

From Lemma 25, at most one supplier, say $i$, chooses the upper price bound $\overline{P}$ with positive probability. As shown below, supplier $i$ also uses $\underline{P}$ as the lower bound for her bidding, $\underline{p}_i = \underline{P}$. It is possible that there is no supplier allocating positive probability on $\overline{P}$. If so, it can be shown that any supplier $i$ with $\bar{p}_i = \overline{P}$ satisfies $\underline{p}_i = \underline{P}$. We define supplier $i_A$ as an *anchoring supplier* if she satisfies

$$\text{(a) } c_{i_A} < \overline{P}, \text{ (b) } \bar{p}_{i_A} = \overline{P}, \text{ and (c) } m_{i_A}(\overline{P}) \geq m_j(\overline{P}) \quad \text{for all } j \in \Pi, \tag{C.17}$$

where "anchoring" stands for her function in our determining the price bounds $\underline{P}$ and $\overline{P}$.

Define the *residual supply payoff* functions $S_i(p, c)$ and $S_i(p)$ as follows. For $p \geq c_i$ and $c > c_i$,

$$S_i(p,c) \;:\; = (p - c_i) \min\{x_i, [d(p) - \sum_{k \neq i} x_k \delta_{(c_k < c)}]^+\} \tag{C.18}$$

$$S_i(p) \;:\; = S_i(p,p) = (p - c_i) \min\{x_i, [d(p) - \sum_{k \neq i} x_k \delta_{(c_k < p)}]^+\}$$

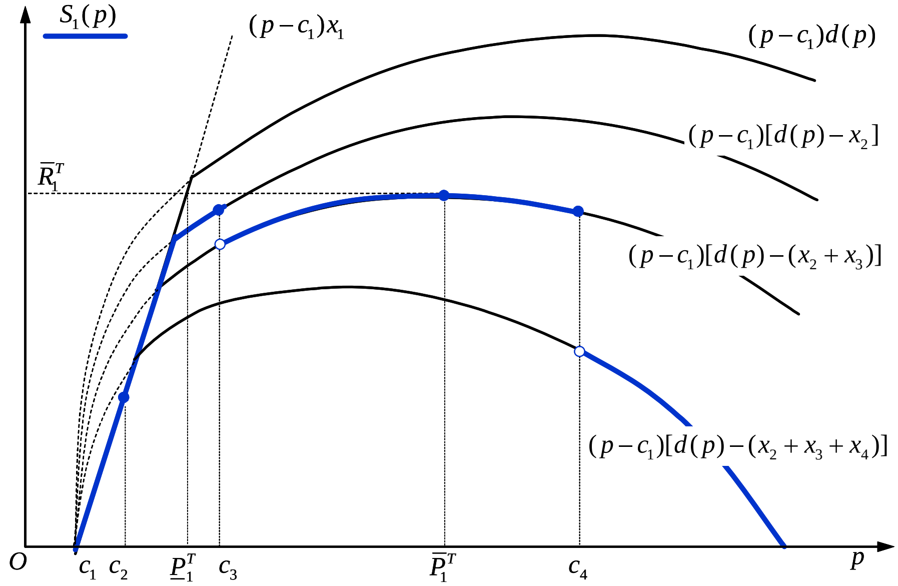


**Figure C.2.** Illustration of Functions $S_i(p)$, $S_i(p, c_k)$

Based on Assumption 1, we have that, when taking positive values, $S_i(p, c)$ is concave in $p$. Function $S_i(p)$ coincides to $S_i(p, c_k)$ where $c_k$ is the highest cost level not exceeding $p$, i.e.,

$$S_i(p) = S_i(p, \bar{c}(p)) \text{ where } \bar{c}(p) := \max\{c_k : c_k \leq p\}.$$

Thus, $S_i(p)$ is piece-wise concave and left continuous, and discontinuities happen at $p = c_j$ for certain $j > i$. Figure C.2 is a graphic illustration. The next lemma points out several properties of $S_i(p, c)$ and $S_i(p)$, which are useful for determining $\underline{P}$ and $\overline{P}$.

**Lemma 26** *(i) For $c_k < c_i < c_j$ and for $p$ such that $S_k(p, c_i)$ and $S_k(p, c_j)$ take positive values, (i-a) if $S_k(p, c_j)$ is increasing at $p$ then $S_k(p, c_i)$ is increasing at $p$; (i-b) if $S_k(p, c_i)$ is decreasing at $p$ then $S_k(p, c_j)$ is increasing at $p$.*

*(ii) If $S_k(p)$ takes positive values and has at least two maximizers, then such a maximizer $\hat{p}_k < \max\{\arg\max\{S_k(p)\}\}$ satisfies $\hat{p}_k \in \{c_i : c_i > c_k\}$.*

**Proof.** (i) Arranging the terms, we have

$$\begin{aligned} S_k(p, c_i) &= \min\left\{(p - c_k)x_k, (p - c_k)[d(p) - X_i]^+\right\} \text{ with } X_i = \sum\nolimits_k x_k \delta_{c_k < c_i}, \\ S_k(p, c_j) &= \min\{(p - c_k)x_k, (p - c_k)[d(p) - X_i - \sum\nolimits_k x_k \delta_{c_i \leq c_k < c_j}]^+\}. \end{aligned}$$

Based on Assumption 1, both $S_k(p,c_i)$ and $S_k(p,c_j)$ are unimodel. Comparing the above expressions, we have $\arg\max\{S_k(p,c_i)\} \geq \arg\max\{S_k(p,c_j)\}$, and parts (i-a) and (i-b) follows trivially.

(ii) From $\hat{p}_k < \hat{P}_k := \max\{\arg\max\{S_k(p)\}\}$, we must have $\bar{c}(\hat{p}_k) \leq \bar{c}(\hat{P}_k)$. Since $S_k(p,\bar{c}(\hat{P}_k))$ is concave and $\hat{P}_k = \arg\max_{p\leq\hat{P}_k}\{S_k(p,\bar{c}(\hat{P}_k))\}$, we must have $S_k(p,\bar{c}(\hat{P}_k))$ is strictly increasing in $p < \hat{P}_k$. As $\bar{c}(\hat{p}_k) \leq \bar{c}(\hat{P}_k)$, part(i-a) implies $S_k(p,\bar{c}(\hat{p}_k))$ is also strictly increasing in $p < \hat{P}_k$. Notice that if $\hat{p}_k \notin \{c_i : c_i > c_k\}$, there exists $\Delta > 0$ such that $S_k(p) = S_k(p,\bar{c}(\hat{p}_k))$ in $p \in [\hat{p}_k - \Delta, \hat{p}_k + \Delta]$. Monotonicity of $S_k(p,\bar{c}(\hat{p}_k))$ brings a contradiction $S_k(\hat{p}_k + \Delta) > S_k(\hat{p}_k) = \max\{S_k(p)\}$. ■

Lemma 27 identifies four important properties that an anchoring supplier must satisfy.

**Lemma 27** *Supplier $i_A$ satisfies (a) $ER_{i_A} = S_{i_A}\left(\overline{P}\right)$; (b) $c_{i_A} \leq P^*$; (c) $\underline{p}_{i_A} = \underline{P} = \frac{ER_{i_A}}{d\wedge x_{i_A}} + c_{i_A}$; (d) $\bar{p}_{i_A} = \overline{P} = \arg\max\{S_{i_A}(p) : p > P^*\}$.*

**Proof.** (a) $\bar{p}_{i_A} = \overline{P}$ implies $ER_{i_A} = \bar{R}^-_{i_A}(\overline{P}) = (\overline{P} - c_{i_A})x_{i_A}\bar{r}^-_{i_A}(\overline{P})$ by (C.14). As $\sigma^*_k <_{a.s.} \overline{P}$ for all $k \in \Pi\backslash\{i_A\}$ by Lemma 25 and $c_{i_A} < \overline{P} \leq c_l$ for all $l \notin \Pi$, applying (C.1) and PR, we have $\bar{r}_{i_A}(\overline{P}) = \frac{[d(\overline{P}) - \sum_{k\neq i_A} x_k\delta_{(c_k<\overline{P})}]^+}{x_{i_A}}$. Part (a) follows directly.

(b) Suppose $c_{i_A} > P^*$. From (C.7), we have $d(P^*) \leq \sum_k x_k\delta_{(c_k\leq P^*)}$, implying $S_{i_A}(p) = 0$ for all $p \geq c_{i_A} > P^*$. It contradicts $ER_{i_A} = S_{i_A}\left(\overline{P}\right) > 0$ for $\overline{P} > c_{i_A}$.

(c) $ER_{i_A} = S_{i_A}\left(\overline{P}\right) = (\overline{P} - c_{i_A})[d\left(\overline{P}\right) - \sum_{k\neq i_A} x_k\delta_{(c_k<\overline{P})}]^+ > 0$ requires $\sum_{k\neq i_A} x_k\delta_{(c_k<\overline{P})} < d\left(\overline{P}\right)$. By definition of $i_A$, we have $\sum_{k\neq i_A} x_k\delta_{(c_k<\overline{P})} = \sum_{k\in\Pi\backslash\{i_A\}} x_k$, and Lemma 24(b) then brings part(c).

(d) Suppose $S_{i_A}(\tilde{p}) > S_{i_A}\left(\overline{P}\right) = ER_{i_A}$ for $\tilde{p} \neq \overline{P}$ and $\tilde{p} > P^*$. By (C.16), we have $\bar{R}^-_{i_A}(\tilde{p}) = (\tilde{p} - c_{i_A})x_{i_A}\bar{r}^-_i(\tilde{p}) \geq S_{i_A}(\tilde{p}) > S_{i_A}\left(\overline{P}\right) = ER_{i_A}$, a contradiction to the optimality of $ER_{i_A}$. ■

Notice that, it is possible that there are multiple maximizers to function $S_{i_A}(p)$. Lemma 29 states that $\overline{P}$ must be the smallest one for the anchoring supplier. We start with an auxiliary lemma, used in several other places.

**Lemma 28** *If profitable supplier $i \in \Pi$ has $\bar{R}^-_i(p) = S_i(p)$ for price $p > P^*$, then all profitable supplier $k \neq i$ with $c_k < p$ satisfies $\overline{p}_k \leq p$.*

**Proof.** $\bar{R}_i^-(p) = (p - c_i)\, x_i \bar{r}_i^-(p) = S_i(p)$ and (C.18) implies $\bar{r}_i^-(p) = \frac{d(p) - \sum_{k \neq i} x_k \delta_{(c_k < p)}}{x_i}$. Also from $p > P^*$, (C.7), and Assumption 1, we have $x_i + \sum_{k \neq i} x_k \delta_{(c_k < p)} \geq \sum_k x_k \delta_{(c_k \leq P^*)} \geq d(P^*) > d(p)$, and therefore $\bar{r}_i^-(p) = \frac{S_i(p)}{(p - c_i) x_i} = \frac{d(p) - \sum_{k \neq i} x_k \delta_{(c_k < p)}}{x_i} < 1$. Notice that all supplier $k$ with $c_k \geq p$ has $\sigma_k^* \geq p$, i.e., $\delta_{(c_k \geq p)} \delta_{(\sigma_k^* < p)} = 0$, so we have

$$\begin{aligned} \bar{r}_i^-(p) &= \mathsf{E}\left[1 \wedge \frac{[d(p) - \sum_{k \neq i} x_k \delta_{(\sigma_k^* < p)} (\delta_{(c_k < p)} + \delta_{(c_k \geq p)})]^+}{x_i}\right] \\ &= \mathsf{E}\left[1 \wedge \frac{(d(p) - \sum_{k \neq i} x_k \delta_{(c_k < p)} \delta_{(\sigma_k^* < p)})^+}{x_i}\right]. \end{aligned}$$

The above expression achieves its minimum at $p$, $\frac{d(p) - \sum_{k \neq i} x_k \delta_{(c_k < p)}}{x_i} < 1$ only if $\delta_{(\sigma_k^* < p)} =_{a.s.} 1$ for all $k$ with $c_k < p$. That is, $\overline{p}_k \leq p$ for all $k \neq i$ with $c_k < p$. ■

**Lemma 29** *Supplier $i_A$ satisfies $\overline{P} = \overline{P}_{i_A}^{\min} := \min\{\arg\max\{S_{i_A}(p) : p > P^*\}\}$.*

**Proof.** From Lemma 27(d), $\overline{P}$ maximizes $S_{i_A}(p)$. Thus, both $\overline{P}_{i_A}^{\min}$ and $\overline{P}$ maximize $S_{i_A}(p)$. Suppose $\overline{P}_{i_A}^{\min} < \overline{P}$. Lemma 27(a) implies $ER_{i_A} = S_{i_A}\left(\overline{P}\right) = S_{i_A}(\overline{P}_{i_A}^{\min})$ and inequality (C.16) implies $ER_{i_A} \geq \bar{R}_{i_A}(\overline{P}_{i_A}^{\min}) \geq S_{i_A}(\overline{P}_{i_A}^{\min})$, so we must have $\bar{R}_{i_A}^-(\overline{P}_{i_A}^{\min}) = S_{i_A}(\overline{P}_{i_A}^{\min})$. From Lemma 28, $\bar{p}_k \leq \overline{P}_{i_A}^{\min}$ *for all* $k \neq i_A$ with $c_k < \overline{P}_{i_A}^{\min}$.

Now consider interval $[\overline{P}_{i_A}^{\min}, \overline{P})$. Since both $\overline{P}_{i_A}^{\min}$ and $\overline{P}$ are maximizers of $S_{i_A}(p)$ and $\max\{S_{i_A}(p)\} = ER_{i_A} > 0$, Lemma A1(ii) implies that $\overline{P}_{i_A}^{\min} = c_j$ for certain $j$, so the set of suppliers with cost $c_k \in [\overline{P}_{i_A}^{\min}, \overline{P})$ is not empty. For any supplier $k$ in the set, Lemma 22 and (C.14) imply $ER_k = \bar{R}_k^-(\underline{p}_k) = (\underline{p}_k - c_k)\bar{r}_k^-(\underline{p}_k) > 0$, and therefore, $\underline{p}_k > c_k \geq \overline{P}_{i_A}^{\min}$. Let $\underline{p}^* := \min\{\underline{p}_k : c_k \in [\overline{P}_{i_A}^{\min}, \overline{P})\} > \overline{P}_{i_A}^{\min}$. Now we have all players with $c_k < \overline{P}_{i_A}^{\min}$ except $i_A$ price lower than $\overline{P}_{i_A}^{\min}$ and all players with $c_k \geq \overline{P}_{i_A}^{\min}$ price no cheaper than $\underline{p}^* > \overline{P}_{i_A}^{\min}$. Consequently, for $p \in [\overline{P}_{i_A}^{\min}, \underline{p}^*)$, $\bar{R}_{i_A}(p) = (p - c_{i_A})[d(p) - \sum_{k \neq i_A} x_k \delta_{(c_k < c_j)}] = S_{i_A}(p, \overline{P}_{i_A}^{\min} = c_j)$.

As $S_{i_A}(p)$ coincides to the concave function $S_{i_A}(p, \bar{c}(\overline{P}))$ in $p \in (\bar{c}(\overline{P}), \overline{P}]$ and achieves maximum at $\overline{P}$, $S_{i_A}\left(p, \bar{c}(\overline{P})\right)$ must be strictly increasing in $p$ for $p \leq \bar{P}$. Since $\bar{c}(\overline{P}) \geq c_j = \overline{P}_{i_A}^{\min}$, Lemma A1(i-a) implies that $\bar{R}_{i_A}(p) = S_{i_A}(p, c_j)$ is striclty increasing in $p \in [\overline{P}_{i_A}^{\min}, \underline{p}^*)$. It follows $\bar{R}_{i_A}^-(\underline{p}^*) > \bar{R}_{i_A}(\overline{P}_{i_A}^{\min}) = ER_{i_A}$, a contradiction to the optimality of $ER_{i_A}$. ■

Define $\Omega := \{i \in \mathcal{N} : c_i \leq P^* \text{ and } \sum_{k\neq i} x_k \delta_{(c_k \leq P^*)} < d(P^*)\}$. Notice for all $i \notin \Omega$, we have either $c_i > P^*$ or $\sum_{k\neq i} x_k \delta_{(c_k \leq P^*)} \geq d(P^*)$, implying that, for all $p > P^*$, $S_i(p) = (p - c_i)\left[d(p) - \sum_{k\neq i} x_k \delta_{(c_k<c_j)}\right]^+ = 0$. Therefore, we must have $i_A \in \Omega$. For each $i \in \Omega$, define trial values

$$\begin{aligned}
\bar{R}_i^T &: = \max\{S_i(p) : p > P^*\}, \\
\overline{P}_i^T &: = \min\{\arg\max\{S_i(p) : p > P^*\}\} = \min\left\{p > P^* : S_i(p) = \bar{R}_i^T\right\}, \\
\underline{P}_i^T &: = \min\{p \in [P^*, \overline{P}_i^T] : (p - c_i)[d(p) \wedge x_i] = \bar{R}_i^T\},
\end{aligned}$$

Figure C.2 illustrates how $\bar{R}_i^T$, $\underline{P}_i^T$, and $\overline{P}_i^T$ are determined. Since $S_i(p)$ coincides to $S_i(p, \bar{c}(\overline{P}_i^T))$ at $\overline{P}_i^T$'s left neighborhood and achieves maximum at $\overline{P}_i^T$, the concave function $S_i(p, \bar{c}(\overline{P}_i^T))$ must be strictly increasing in $p \leq \overline{P}_i^T$. As $c_i \leq \bar{c}(\overline{P}_i^T)$, Lemma A1(i-a) suggests that

$$(\#)\ S_i(p, c_i) \text{ is strictly increasing for } p \in [c_i, \overline{P}_i^T].$$

From (#) and $S_i(\overline{P}_i^T, c_i) \geq S_i(\overline{P}_i^T) \geq S_i(P^*, c_i)$, intermediate value thoerem guarantees that equation $S_i(p, c_i) = S_i(\overline{P}_i^T)$ [or $(p - c_i)[d(p) \wedge x_i] = \bar{R}_i^T$] has a unique solution $p \in [P^*, \overline{P}_i^T]$. In other words, $\underline{P}_i^T \in [P^*, \overline{P}_i^T]$ always exists. Now from $\bar{R}_i^T = (\underline{P}_i^T - c_i)[x_i \wedge d(\underline{P}_i^T)] = (\overline{P}_i^T - c_i)[d(\overline{P}_i^T) - \sum_{k\neq i} x_k \delta_{(c_k<\overline{P}_i^T)}] > 0$, we have two useful inequalities

$$\text{(a) } 0 < d(\overline{P}_i^T) - \sum_{k\neq i} x_k \delta_{(c_k<\overline{P}_i^T)} \leq [x_i \wedge d(\underline{P}_i^T)] \text{ and (b) } \overline{P}_i^T \geq \underline{P}_i^T \quad \text{for all } i \in \Omega. \quad \text{(C.19)}$$

With the above notation, we claim that $\underline{P}$ and $\overline{P}$ can be determined as follows.

**Proposition 19** $\underline{P} = \max_{i\in\Omega}\left\{\underline{P}_i^T\right\}$ *and* $\overline{P} = \min_{i\in\Omega}\{\overline{P}_i^T : \underline{P}_i^T = \underline{P}\}$.

**Proof.** The proof proceeds in three steps. The first part is an auxiliary result.

(a) *$ER_i \geq \bar{R}_i^T$ and $\underline{p}_i \geq \underline{P}_i^T$ for all $i \in \Omega$.* By (C.16), for any price equal to a cost level $p \in (P^*, b]$, we have $ER_i \geq \bar{R}_i^-(p) = (p - c_i)x_i\bar{r}_i^-(p) \geq (p - c_i)[d(p) - \sum_{k\neq i} x_k \delta_{(c_k<p)}]$, which implies $ER_i \geq \bar{R}_i^T = \max_{p\in(P^*,b]}\{(p - c_i)[d(p) - \sum_{k\neq i} x_k \delta_{(c_k<p)}]^+\}$. To show the second inequality, suppose $\underline{p}_i < \underline{P}_i^T$ for certain $i$. By (C.14), we have $ER_i = \bar{R}_i^-(\underline{p}_i) = (\underline{p}_i - c_i)x_i\bar{r}_i^-(\underline{p}_i) \leq (\underline{p}_i - c_i)[x_i \wedge d(\underline{p}_i)] = S_i(\underline{p}_i, c_i)$.

Observation (#) then implies a contradiction, $ER_i = S_i(\underline{p}_i, c_i) < S_i(\underline{P}_i^T, c_i) < (\underline{P}_i^T - c_i)[d(\underline{P}_i^T) \wedge x_i] = \bar{R}_i^T \leq ER_i$.

(b) $\underline{P} = \max_{i\in\Omega}\left\{\underline{P}_i^T\right\}$. We know $\underline{p}_{i_A} = \underline{P}_{i_A}^T = \underline{P} \in \left\{\underline{P}_k^T : k \in \Omega\right\}$. Suppose that there exists supplier $h \in \Omega$ such that $\underline{P}_h^T = \max_{k\in\Omega}\left\{\underline{P}_k^T\right\} > \underline{P}$. Part (a) implies $\underline{p}_h \geq \underline{P}_h^T > \underline{P} = \underline{p}_{i_A}$. Consider price interval $[\underline{p}_{i_A}, \underline{P}_h^T)$. Inequality (C.19-a) and Assumption 1 implies $\sum_{k\neq h} x_k \delta_{(c_k<p)} < d(\underline{P}_h^T) \leq d(p)$ for all $p \leq \underline{P}_h^T$. That is, given supplier $h$'s absence in this price interval, the total capacity available is smaller than demand. Therefore, supplier $i_A$ obtains sales ratio 1 almost surely for any $p \in [\underline{p}_{i_A}, \underline{P}_h^T)$, implying a contradiction, $ER_{i_A} = \bar{R}_{i_A}(\underline{P}) = (\underline{P} - c_{i_A})\, x_{i_A} < \left(\underline{P}_h^T - c_{i_A}\right) x_{i_A} = \bar{R}_{i_A}^-(\underline{P}_h^T) \leq ER_{i_A}$.

(c) $\overline{P} = \overline{P}_{i_A}^T = \min_{i\in\Omega}\{\overline{P}_i^T : \underline{P}_i^T = \underline{P}\}$. Given $\underline{P}$ derived in part (b), if only one player satisfies $\underline{P}_i^T = \underline{P}$, she must be the anchoring supplier and Lemma 29 guarantees part (c). Consider the case when there are more than one players in set $\Omega$ satisfying $\underline{P}_i^T = \underline{P}$. Suppose $j \in \Omega$ exists such that $\underline{P}_j^T = \underline{P}$ and $\overline{P}_j^T < \overline{P} = \overline{P}_{i_A}^T$. Inequality (C.16) implies $\bar{R}_j^-(\overline{P}_j^T) \geq S_j(\overline{P}_j^T)$. If $\bar{R}_j^-(\overline{P}_j^T) = S_j(\overline{P}_j^T)$, as $c_{i_A} < \underline{P} < \overline{P}_j^T$, Lemma 28 brings a contradiction $\overline{p}_{i_A} \leq \overline{P}_j^T < \overline{P} = \overline{p}_{i_A}$. Thus, we must have $\bar{R}_j^-(\overline{P}_j^T) > S_j(\overline{P}_j^T)$. Now consider $\underline{p}_j$, if $\underline{p}_j = \underline{P} = \underline{P}_j^T$, thenLemma 24(a) brings a contradiction $ER_j = \bar{R}_j\,(\underline{P}) = \left(\underline{P}_j^T - c_j\right)[x_j \wedge d(\underline{P}_j^T)] = \bar{R}_j^T < ER_j$. So we must have $\underline{p}_j > \underline{P} = \underline{P}_j^T$. Also notice (C.19-b) implies $\overline{P}_j^T > \underline{P}_j^T = \underline{P}$. Therefore, $p^* := \min\{\underline{p}_j, \overline{P}_j^T\} > \underline{P} = \underline{P}_j^T$ and for all $p < p^*$. By (C.19-a), we have$\sum_{k\neq j} x_k \delta_{(c_k<p)} \leq \sum_{k\neq j} x_k \delta_{(c_k<p^*)} \leq \sum_{k\neq j} x_k \delta_{(c_k<\overline{P}_j^T)} < d(\overline{P}_j^T) < d(p)$. Similarly to part (b), we can show $\bar{r}_{i_A}(p) = 1$ for all $p < p^* \leq \underline{p}_j$, which further leads to a contradiction $ER_{i_A} = \bar{R}_{i_A}(\underline{P}) = (\underline{P} - c_{i_A})\, x_{i_A} < (p^* - c_{i_A}) x_{i_A} = \bar{R}_{i_A}^-(p^*) \leq ER_{i_A}$. ■

# APPENDIX D

# Proofs and Analysis for Chapter 4

## D.1 Impossibility of increasing equilibrium for DAK-2

This can be illustrated by contradiction. Consider model DAK-2 with $k \in \left[\underline{k}, \bar{k}\right]$ and assume there is a symmetric PBE with strictly increasing bidding strategy $p = \gamma(k)$. Suppose supplier 2 follows the bidding strategy $p = \gamma(k)$,and we consider supplier 1's payoff function $R(p)$, particularly for type $k = \underline{k}$. As $\gamma(k)$ is strictly increasing in $k$, it suggests that $\gamma(\underline{k})$ is the lowest bid possibly submitted by supplier 2. Since $\gamma(k)$ defines a symmetric PBE, we must have

$$R(\gamma(\underline{k})) \geq R(p) \text{ for all } p \in [0, B]. \tag{D.1}$$

If $\underline{k} < \frac{\xi}{2}$ (i.e, $\underline{k} < \xi - \underline{k}$) we instantly note a contradiction,

$$R(\gamma(\underline{k})) = \gamma(\underline{k})\, \underline{k} < \gamma(\xi - \underline{k})\, \underline{k} = R(\gamma(\xi - \underline{k})).$$

The second equality results from $\underline{k} < \xi - \underline{k}$ and strictly monoticity of $\gamma(\cdot)$.

If $\underline{k} \geq \frac{\xi}{2}$, we first derive the $\gamma(k)$ for $k$ in the right neighborhood of $[\underline{k}, k + \Delta)$ with $k + \Delta < \xi$.

$$\begin{aligned}
\max_p R(p) &= p\left[k\bar{H}\left(\gamma^{-1}\right) + \int_{\underline{k}}^{\gamma^{-1}} [k \wedge (\xi - k_2)]\, dH(k_2)\right] \text{ where } H \text{ is cdf of } k_2 \\
&= p\left[k\bar{H}\left(\gamma^{-1}\right) + \int_{\underline{k}}^{\gamma^{-1}} (\xi - k_2)\, dH(k_2)\right] \quad [\text{by } \xi - k_2 \leq \frac{\xi}{2} \leq \underline{k}]
\end{aligned}$$

$$\begin{aligned}
\frac{dR(p)}{dp} &= \left[k\bar{H}\left(\gamma^{-1}\right) + \int_{\underline{k}}^{\gamma^{-1}} (\xi - k_2)\, dH(k_2)\right] \\
&\quad + p\left[-kh\left(\gamma^{-1}\right) + \left(\xi - \gamma^{-1}\right) h\left(\gamma^{-1}\right)\right] \frac{d\gamma^{-1}(p)}{dp}
\end{aligned}$$

It implies the ODE defining the Nash solution with $p = \gamma(k)$:

$$\frac{\gamma'(k)}{\gamma(k)} = \frac{(2k-\xi)\,h(k)}{k\bar{H}(\gamma^{-1}) + \xi H(k) - \int_{\underline{k}}^{k} k_2 h(k_2)\,dk_2} \tag{D.2}$$

implying that

$$\gamma(k) = C_0 \exp\left\{\int_{\underline{k}}^{k} \frac{(2k-\xi)\,dH(k)}{k\bar{H}(k) + \xi H(k) - \int_{\underline{k}}^{k} k_2 h(k_2)\,dk_2}\right\} \text{ with } C_0 > 0.$$

Now we check if $p = \gamma(k)$ is the optimal solution to $R(p)$. Consider $p = \gamma(l)$ with $l > k$, we denote $R(p)$ as

$$\begin{aligned} R(l,k) &= \gamma(l)\left[k\bar{H}(l) + \int_{\underline{k}}^{l} (\xi - k_2)\,dH(k_2)\right] \\ \frac{\partial R(l,k)}{\partial l} &= \dot{\gamma}(l)\left[k\bar{H}(l) + \xi H(l) - \int_{\underline{k}}^{l} k_2 h(k_2)\,dk_2\right] + \gamma(l)(-k+\xi-l)\,h(l). \end{aligned}$$

Applying the property (D.2), we have

$$\begin{aligned} \frac{\partial R(l,k)}{\partial l} &= \gamma(l)\,h(l)\left\{\frac{(2l-\xi)\left[k\bar{H}(l) + \xi H(l) - \int_{\underline{k}}^{l} k_2 h(k_2)\,dk_2\right]}{l\bar{H}(l) + \xi H(l) - \int_{\underline{k}}^{l} k_2 h(k_2)\,dk_2} + \xi - k - l\right\} \\ &= \gamma(l)\,h(l)\left\{\frac{(2l-\xi)(k-l)\,\bar{H}(l)}{l\bar{H}(l) + \xi H(l) - \int_{\underline{k}}^{l} k_2 h(k_2)\,dk_2} + (2l-\xi) + \xi - k - l\right\} \\ &= \frac{\gamma(l)\,h(l)}{l\bar{H}(l) + \xi H(l) - \int_{\underline{k}}^{l} k_2 h(k_2)\,dk_2}(k-l) \\ &\quad \left[\int_{\underline{k}}^{l} k_2 h(k_2)\,dk_2 - (2\xi - l)\,\bar{H}(l)\right]. \end{aligned}$$

Notice for $k = \underline{k}$ and $l = \underline{k} + \Delta$, we have $k - l < 0$ and $\int_{\underline{k}}^{l} k_2 h(k_2)\,dk_2 - (2\xi - l)\,\bar{H}(l) < l - 2\xi\bar{H}(l) < 0$ for any $l$ close to $\underline{k}$ and $H(l) < \frac{1}{2}$. It suggests that $\frac{\partial R(l,k)}{\partial l} > 0$ for $l \in (\underline{k}, \underline{k} + \Delta)$, a direct contradiction to (D.1).

## D.2 Derivation of Symmetric Equilibrium for DAK-N

In this section, we heuristically derive the symmetric PBE for Model DAK-N. The derivation uses several assumptions and conclusions that are proven in the duopoly model (DAK-2). First, generalizing the observation from two-bidder case, we assume $\gamma'(k) \leq 0$.

Suppose all suppliers but $i$ follow the equilibrium bidding strategy $p_j = \gamma(k_j)$ for $j \neq i$ and consider supplier $i$'s payoff when choosing price $p$, given that his capacity is $k$,

$$R(p, \mathbf{p}_{-i}) = (p-c)\left\{\delta_{\{p<b_1\}}(k \wedge \xi) + \sum_{i=1}^{N-1} \delta_{\{b_i<p\leq b_{i+1}\}}[k \wedge (\xi - \sum_{j=1}^{i} \gamma^{-1}(b_i))^+]\right\}.$$

Note that, the above expression has assumed that no two bidders choose the same prices. Since we have $H(k)$ is continuous in $k$, we assume have $\rho(p, \mathbf{p}_{-i}) = 1$ almost surely for any $p$. Taking expectation of $R(p, \mathbf{p}_{-i})$ over $\mathbf{k}_{-i}$, and using the fact that $\gamma(\cdot)$ is a strictly decreasing function, we have

$$\begin{aligned} R(p) &= p(k \wedge \xi) H\left(\gamma^{-1}\right)^{N-1} + p \sum_{i=1}^{N-1} \left\{ \binom{N-1}{i} H\left(\gamma^{-1}\right)^{N-i-1} \cdot \right. \\ & \left. \int_{\gamma^{-1}}^{\bar{k}} \cdots \int_{\gamma^{-1}}^{\bar{k}} [k \wedge (\xi - \sum_{j=1}^{i} k_j)^+] dH(k_1) \ldots dH(k_i) \right\} \\ &= p \sum_{i=0}^{N-1} \binom{N-1}{i} H\left(\gamma^{-1}\right)^{N-i-1} U_i\left(\gamma^{-1}, k, \xi\right). \end{aligned}$$

where

$$U_i(x, k, d) = \begin{cases} k \wedge (d)^+ & i = 0 \\ \int_x^{\bar{k}} \cdots \int_x^{\bar{k}} [k \wedge (d - \sum_{j=1}^{i} k_j)^+] dH(k_1) \ldots dH(k_i) & i \geq 1 \end{cases}.$$

The first order condition $\frac{dR(p)}{dp} = 0$ is

$$\begin{aligned} 0 &= \sum_{i=0}^{N-1} \binom{N-1}{i} H\left(\gamma^{-1}\right)^{N-i-1} U_i\left(\gamma^{-1}, k, \xi\right) \\ &+ p \sum_{i=0}^{N-2} \binom{N-1}{i} (N-i-1) H\left(\gamma^{-1}\right)^{N-i-2} h\left(\gamma^{-1}\right) \frac{d\gamma^{-1}(p)}{dp} U_i\left(\gamma^{-1}, k, \xi\right) \\ &+ p \sum_{i=1}^{N-1} \binom{N-1}{i} H\left(\gamma^{-1}\right)^{N-i-1} \frac{dU_i\left(\gamma^{-1}, k, \xi\right)}{dp}. \end{aligned} \tag{D.3}$$

To further simplify the expression, $\frac{dU_i\left(\gamma^{-1}, k, \xi\right)}{dp}$ needs to be explicitly derived. We first note the following property,

$$\begin{aligned} & \frac{d}{dx} \int_{A_1(x)}^{\bar{A}_1} \cdots \int_{A_N(x)}^{\bar{A}_N} F(a_1, \ldots, a_N) da_1 \ldots da_N \\ = & -\sum_{j=1}^{N} \frac{dA_j(x)}{dx} \int_{A_1(x)}^{\bar{A}_1} \cdots \int_{A_N(x)}^{\bar{A}_N} F(a_1, \ldots, A_j(x), \ldots, a_N) da_1 \ldots da_{j-1} da_{j+1} \ldots da_N, \end{aligned}$$

where $A_j(x) < \bar{A}_j$ and $\bar{A}_j$ is constant for all $j = 1, 2, \ldots, N$. Applying this property, we have

$$\begin{aligned}
\frac{dU_i(\gamma^{-1}(p), k, \xi)}{dp} &= -\sum_{j=1}^{i} \frac{d\gamma^{-1}(p)}{dp} \cdot \\
&\int_{\gamma^{-1}(p)}^{\bar{k}} \cdots \int_{\gamma^{-1}(p)}^{\bar{k}} [k \wedge (\xi - \gamma^{-1}(p) - \sum_{m=1, m\neq j}^{i} k_m)^+] h(\gamma^{-1}(p)) \prod_{m=1, m\neq j}^{i} dH(k_m) \\
&= -i \frac{d\gamma^{-1}(p)}{dp} h(\gamma^{-1}(p)) \int_{\gamma^{-1}(p)}^{\bar{k}} \cdots \int_{\gamma^{-1}(p)}^{\bar{k}} [k \wedge (\xi - \gamma^{-1}(p) - \sum_{m=1}^{i-1} k_m)^+] \prod_{m=1}^{i-1} dH(k_m) \\
&= -i \frac{d\gamma^{-1}(p)}{dp} h(\gamma^{-1}(p)) U_{i-1}\left(\gamma^{-1}(p), k, \xi - \gamma^{-1}(p)\right).
\end{aligned}$$

FOC(D.3) now reads

$$\begin{aligned}
0 &= \sum_{i=0}^{N-1} \binom{N-1}{i} H\left(\gamma^{-1}\right)^{N-i-1} U_i\left(\gamma^{-1}, k, \xi\right) \\
&+ p \sum_{i=0}^{N-2} \binom{N-1}{i} (N-i-1) H\left(\gamma^{-1}\right)^{N-i-2} h\left(\gamma^{-1}\right) U_i\left(\gamma^{-1}, k, \xi\right) \frac{d\gamma^{-1}(p)}{dp} \\
&- p \sum_{i=1}^{N-1} \binom{N-1}{i} H\left(\gamma^{-1}\right)^{N-i-1} i \frac{d\gamma^{-1}(p)}{dp} h\left(\gamma^{-1}\right) U_{i-1}\left(\gamma^{-1}, k, \xi - \gamma^{-1}\right)
\end{aligned}$$

At equilibrium, $p = \gamma(k)$, $k = \gamma^{-1}(p)$, and $\frac{d\gamma^{-1}(p)}{dp} = \frac{1}{\gamma'(k)}$, so we have

$$\begin{aligned}
0 &= \sum_{i=0}^{N-1} \binom{N-1}{i} H(k)^{N-i-1} U_i(k, k, \xi) \\
&\quad + \frac{\gamma(k)}{\gamma'(k)} h(k) \sum_{i=0}^{N-2} \binom{N-1}{i} (N-i-1) H(k)^{N-i-2} U_i(k, k, \xi) \\
&\quad - \frac{\gamma(k)}{\gamma'(k)} h(k) \sum_{i=1}^{N-1} \binom{N-1}{i} i H(k)^{N-i-1} U_{i-1}(k, k, \xi - k) \\
&= \sum_{i=0}^{N-1} \binom{N-1}{i} H(k)^{N-i-1} U_i(k, k, \xi) \\
&\quad + \frac{(N-1)\gamma(k) h(k)}{\gamma'(k)} \sum_{i=0}^{N-2} \frac{(N-2)!}{i!(N-i-2)!} H(k)^{N-i-2} U_i(k, k, \xi) \\
&\quad - \frac{(N-1)\gamma(k)h(k)}{\gamma'(k)} h(k) \sum_{l=0}^{N-2} \frac{(N-2)!}{l!(N-l-2)!} H(k)^{N-l-2} U_l(k, k, \xi - k) \\
&\quad [\text{by } l = i-1].
\end{aligned}$$

Therefore, we have an ODE of $k$,

$$\begin{aligned} & \sum_{i=0}^{N-1} \binom{N-1}{i} H(k)^{N-i-1} U_i(k,k,\xi) \\ = & -\frac{(N-1)\gamma(k)h(k)}{\gamma'(k)} \sum_{i=0}^{N-2} \binom{N-2}{i} H(k)^{N-i-2} \left[U_i(k,k,\xi) - U_i(k,k,\xi-k)\right], \end{aligned}$$

or equivalently,

$$\frac{\gamma'(k)}{\gamma(k)} = -(N-1)h(k) \frac{\sum_{i=0}^{N-2} \binom{N-2}{i} H(k)^{N-i-2} \left[U_i(k,k,\xi) - U_i(k,k,\xi-k)\right]}{\sum_{i=0}^{N-1} \binom{N-1}{i} H(k)^{N-i-1} U_i(k,k,\xi)}. \tag{D.4}$$

The above differential equation has a general solution in format as

$$\gamma(k) = K_D \exp\left[-(N-1)\int_{\underline{k}}^{k} \Delta(x)\, dH(x)\right],$$

where $\Delta(x)$ is defined in (4.19) and $K_D > 0$. Note that, ODE (D.4) implies that $\gamma(k)$ is a decreasing function in $k$, which is consistent with our initial assumption. By the boundary condition $\gamma(\underline{k}) = B$, we have $K_D = B$, and therefore, we derive the equilibrium bidding strategy.

## D.3 Derivation of Symmetric Equilibrium for UAK-N

Similar to UAK-2, we assume $\gamma(k)$ is a decreasing function in $k$. Suppose all suppliers but $i$ follow the equilibrium bidding strategy $p_j = \gamma(k_j)$ for $j \neq i$ and consider supplier $i$'s payoff when choosing price $p$, given that his capacity is $k$,

$$R(p, \mathbf{p}_{-i}) = p\delta_{\{b=p\}}[k \wedge (\xi - \sum_{j\neq i} \delta_{\{p_i<p\}} k_j)] + b\delta_{\{p<b\}} k,$$

where $b$ denotes the market clearing price. Taking expectation of $R(p, \mathbf{p}_{-i})$, we have

$$\begin{aligned} R(p) &= R_p(p) + R_k(p) \\ \text{where } R_p(p) &= pE_{\mathbf{k}^{-i}}[[k \wedge (\xi - \sum_{j\neq i} k_j \delta_{\{\gamma(k_i)<p\}})] \cdot \delta_{\{b=p\}}] \\ \text{and } R_k(p) &= kE_{\mathbf{k}^{-i}}\left[b \cdot \delta_{\{p<b\}}\right]. \end{aligned}$$

We first derive the expressions of $R_p(p)$ and $\frac{dR_p(p)}{dp}$,

$$\begin{aligned} R_p(p) &= p\int \dots \int \left[k \wedge \left(\xi - \sum k_j \delta_{\{\gamma(k_j)<p\}}\right)\right] \delta_{\{b=p\}} dH(k_1) \dots dH(k_{N-1}) \\ &= p \sum_{i=0}^{N-1} \binom{N-1}{i} H\left(\gamma^{-1}\right)^{N-i-1} W_i^{N-1}\left(\gamma^{-1}, k, \xi\right), \end{aligned}$$

where we define

$$\begin{aligned} & W_i^M(x,k,d) \\ = & \begin{cases} (k \wedge d)\, \delta_{\{0<d\}} & i = M = 0 \\ d\delta_{\{0<d\le k\}} & i = 0 < M \\ \int_x^{\bar{k}} \dots \int_x^{\bar{k}} \left(d - \sum_{j=1}^{i} k_j\right) \delta_{\left\{\sum_{j=1}^{i} k_j < d \le \sum_{j=1}^{i} k_j + k\right\}} dH(k_1) \dots dH(k_i) & 0 < i < M \\ \int_x^{\bar{k}} \dots \int_x^{\bar{k}} \left[k \wedge \left(d - \sum_{j=1}^{M} k_j\right)\right] \delta_{\left\{\sum_{j=1}^{M} k_j < d\right\}} dH(k_1) \dots dH(k_i) & i = M > 0 \end{cases}. \end{aligned}$$

Next we derive a useful property of $W_i^M$. For $i = 0$, we have $\frac{\partial W_i^M(x,k,d)}{\partial x} = 0$. For $0 < i < M$,

$$\begin{aligned} & \frac{\partial W_i^M(x,k,d)}{\partial x} \\ = & -ih(x) \int_x^{\bar{k}} \dots \int_x^{\bar{k}} \left(d - x - \sum_{j=1}^{i-1} k_j\right) \delta_{\left\{d-x-k \le \sum_{j=1}^{i-1} k_j < d-x\right\}} dH(k_1) \dots dH(k_{i-1}) \\ = & -ih(x)\, W_{i-1}^{M-1}(x,k,d-x). \end{aligned}$$

It is easy to verify $\frac{\partial W_i^M(x,k,d)}{\partial x} = -ih(x)\, W_{i-1}^{M-1}(x,k,d-x)$ also holds for $i = M > 0$.

Applying the above property, we now have

$$\begin{aligned} \frac{dR_p(p)}{dp} = & \sum_{i=0}^{N-1} \binom{N-1}{i} H\left(\gamma^{-1}\right)^{N-i-1} W_i^{N-1}\left(\gamma^{-1}, k, \xi\right) \\ & + p \sum_{i=0}^{N-2} \binom{N-1}{i} (N-i-1)\, H\left(\gamma^{-1}\right)^{N-i-2} h\left(\gamma^{-1}\right) W_i^{N-1}\left(\gamma^{-1}, k, \xi\right) \frac{d\gamma^{-1}(p)}{dp} \\ & - p \sum_{i=1}^{N-1} \binom{N-1}{i} H\left(\gamma^{-1}\right)^{N-i-1} ih\left(\gamma^{-1}\right) W_{i-1}^{N-2}\left(\gamma^{-1}, k, \xi - \gamma^{-1}\right) \frac{d\gamma^{-1}(p)}{dp} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=0}^{N-1}\binom{N-1}{i}H\left(\gamma^{-1}\right)^{N-i-1}W_i^{N-1}\left(\gamma^{-1},k,\xi\right)\\
&\quad +ph(\gamma^{-1})\frac{d\gamma^{-1}(p)}{dp}(N-1)\sum_{i=0}^{N-2}\binom{N-2}{i}H(\gamma^{-1})^{N-i-2}W_i^{N-1}(\gamma^{-1},k,\xi)\\
&\quad -ph(\gamma^{-1})\frac{d\gamma^{-1}(p)}{dp}(N-1)\sum_{i=1}^{N-1}\binom{N-2}{i-1}H(\gamma^{-1})^{N-i-1}W_{i-1}^{N-2}(\gamma^{-1},k,\xi-\gamma^{-1})\\
&= \sum_{i=0}^{N-1}\binom{N-1}{i}H(\gamma^{-1})^{N-i-1}W_i^{N-1}(\gamma^{-1},k,\xi)+ph(\gamma^{-1})\frac{d\gamma^{-1}(p)}{dp}(N-1)\cdot\\
&\quad \sum_{i=0}^{N-2}\binom{N-2}{i}H(\gamma^{-1})^{N-i-2}\left[W_i^{N-1}(\gamma^{-1},k,\xi)-W_i^{N-2}(\gamma^{-1},k,\xi-\gamma^{-1})\right].
\end{aligned}
$$

Next we derive $R_k(p)$ and $\frac{dR_k(p)}{dp}$.

$$
\begin{aligned}
R_k(p) &= kE_{k_1,\dots,k_N}\left[b\cdot\delta_{\{p<b\}}\right]\\
&= (N-1)k\int E_{k_1,\dots,k_{N-2}}\left[\gamma(x)\cdot\delta_{\{p<\gamma(x),b=\gamma(x)\}}\right]dH(x)\\
&= (N-1)k\int_{\underline{k}}^{\gamma^{-1}}\left[\gamma(x)\sum_{i=0}^{N-2}\binom{N-2}{i}H(x)^{N-i-2}V_i^{N-2}(x,\xi-k)\right]dH(x),
\end{aligned}
$$

where

$$
V_i^M(x,d)\equiv\begin{cases}
\delta_{\{0<d\}} & i=M=0\\
\delta_{\{0<d\le x\}} & i=0<M\\
\int_x^{\bar{k}}\dots\int_x^{\bar{k}}\delta_{\left\{\sum_{j=0}^{i}k_j<d\le\sum_{j=0}^{i}k_j+x\right\}}dH(k_1)\dots dH(k_i) & 0<i<M\\
\int_x^{\bar{k}}\dots\int_x^{\bar{k}}\delta_{\left\{\sum_{j=0}^{M}k_j<d\right\}}dH(k_1)\dots dH(k_i) & 0<i=M
\end{cases}.
$$

It follows $\frac{dR_k(p)}{dp}=(N-1)h(\gamma^{-1}(p))pk\sum_{i=0}^{N-2}\binom{N-2}{i}H(\gamma^{-1}(p))^{N-i-2}V_i^{N-2}(\gamma^{-1}(p),\xi-k)\frac{d\gamma^{-1}(p)}{dp}$. For the supplier, optimal $p$ satisfies the FOC $\frac{dR(p)}{dp}=\frac{dR_p(p)}{dp}+\frac{dR_k(p)}{dp}=0$, i.e.,

$$
\begin{aligned}
0 &= \sum_{i=0}^{N-1}\binom{N-1}{i}H\left(\gamma^{-1}\right)^{N-i-1}W_i^{N-1}\left(\gamma^{-1},k,\xi\right)+(N-1)ph\left(\gamma^{-1}\right)\frac{d\gamma^{-1}(p)}{dp}\cdot\\
&\quad \sum_{i=0}^{N-2}\binom{N-2}{i}H\left(\gamma^{-1}\right)^{N-i-2}\left[W_i^{N-1}\left(\gamma^{-1},k,\xi\right)-W_i^{N-2}\left(\gamma^{-1},k,\xi-\gamma^{-1}\right)\right]\\
&\quad +(N-1)ph\left(\gamma^{-1}\right)\frac{d\gamma^{-1}(p)}{dp}k\sum_{i=0}^{N-2}\binom{N-2}{i}H\left(\gamma^{-1}\right)^{N-i-2}V_i^{N-2}\left(\gamma^{-1},\xi-k\right).
\end{aligned}
$$

At equilibrium, we must have $\gamma^{-1}(p)=k$, $p=\gamma(k)$, and $\frac{d\gamma^{-1}(p)}{dp}=\frac{1}{\gamma'(k)}$. Now FOC implies an ODE,

$$\begin{aligned} 0 &= \sum_{i=0}^{N-1} \binom{N-1}{i} H(k)^{N-i-1} W_i^{N-1}(k,k,\xi) + (N-1)\frac{\gamma(k)\,h(k)}{\gamma'(k)} \cdot \\ &\sum_{i=0}^{N-2} \binom{N-2}{i} H(k)^{N-i-2} \left[W_i^{N-1}(k,k,\xi) - W_i^{N-2}(k,k,\xi-k) + kV_i^{N-2}(k,\xi-k)\right] \end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned} &\frac{\gamma'(k)}{\gamma(k)}\frac{(-1)}{(N-1)\,h(k)} \\ = &\frac{\sum_{i=0}^{N-2}\binom{N-2}{i} H(k)^{N-i-2}\left[W_i^{N-1}(k,k,\xi) - W_i^{N-2}(k,k,\xi-k) + kV_i^{N-2}(k,\xi-k)\right]}{\sum_{i=0}^{N-1}\binom{N-1}{i} H(k)^{N-i-1} W_i^{N-1}(k,k,\xi)}. \end{aligned}$$

The above differential equation has a general solution in format as

$$\gamma(k) = K_U \exp\left[-(N-1)\int_{\underline{k}}^{k} \Theta(x)\, dH(x)\right]$$

where

$$\begin{aligned} &\Theta(x) \\ = &\frac{\sum_{i=0}^{N-2}\binom{N-2}{i} H(x)^{N-i-2}\left[W_i^{N-1}(x,x,\xi) - W_i^{N-2}(x,x,\xi-x) + xV_i^{N-2}(x,\xi-x)\right]}{\sum_{i=0}^{N-1}\binom{N-1}{i} H(x)^{N-i-1} W_i^{N-1}(x,x,\xi)}. \end{aligned}$$

By the boundary condition $\gamma(\underline{k}) = B$, we have $K_U = B$, and expression (4.20) follows. To verify the optimality of $p = \gamma(k)$, we also derive

$$\begin{aligned} R(l,k) &= \gamma(l)\sum_{i=0}^{N-1}\binom{N-1}{i} H(l)^{N-i-1} W_i^{N-1}(l,k,\xi) \\ &\quad + (N-1)\,k\int_{\underline{k}}^{l}\left[\gamma(x)\sum_{i=0}^{N-2}\binom{N-2}{i} H(x)^{N-i-2} V_i^{N-2}(x,\xi-k)\right] dH(x). \end{aligned}$$

If $\gamma(k) = \arg\max_l \{R(l,k)\}$ for all $k$, $\gamma(k)$ defines a PBE.

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