Coordinating Mathematical and Pedagogical Perspectives in Practice-Based and Discipline-Grounded Approaches to Studying Mathematical Knowledge for Teaching (K-8)

by

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To my family—for steadfast and loving support.
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ABSTRACT

This dissertation develops a design for research that studies classroom practice as a way to identify mathematical knowledge for teaching. It develops the idea of using the discipline of mathematics to analyze practice and examines what is required to make that analysis productive of insight into the mathematical entailments of teaching. Such analysis represents a novel approach to studying teacher content knowledge and, hence, is central to the improvement of teaching and learning.

Ball and Bass (2003b) have developed a practice-based and discipline-grounded approach that examines the work of teaching and its mathematical demands, yet issues of method and design have remained underspecified (Ball, 1999). Addressing this gap, this dissertation analyzes documents from their project (on which the author worked) — not with the goal of making claims about these scholars’ work, but with the goal of specifying better the design of such research. To this end, data from 1500 project documents produced from 1997 to 2004 (e.g., draft proposals, annotated transcripts, internal memos, meeting notes, and presentations) were analyzed. The initial sample of documents was based on analytical (rather than chronological) distinctions, but as the analysis progressed additional documents were incorporated in order to elaborate, revise, and sharpen assumptions and claims. Analysis worked both “top-down” with a set of structural hypotheses and “bottom-up” with an examination of comments — what they were about, and what about that “what” they addressed.

This dissertation extracts three alternative designs from this program of work and specifies three key components of such designs, including: (i) a mathematical analysis of practice; (ii) a pedagogical analysis of practice; and (iii) ways of coordinating these two analyses and the distinct perspectives on which they are based. This third component is particularly important because the two perspectives, while not necessarily at odds, often seem unable to “talk” to one another and remain unconnected. This study clarifies how these two perspectives can be conceived so as to produce plausible and useful analyses of practice and explains how an understanding of the coordination of the two perspectives is central to conceptions of the perspectives themselves, to integration of the distinct analyses they produce, and to the conception of mathematical knowledge for teaching itself.
CHAPTER 1: STUDENT LEARNING, TEACHERS’ MATHEMATICAL KNOWLEDGE, AND THE STUDY OF PRACTICE

Concern about teachers’ mathematical knowledge at the elementary school level has in recent years shifted to concern about identifying a body of mathematical knowledge that actually matters for elementary school mathematics teaching. It turns out, however, that this is not so easy. Much of the recent national and international attention has focused on ways in which best to conceive, identify, and express that knowledge. Adler (in press) writes:

Working with a social epistemology, we understand that what comes to be MfT [Mathematics for Teaching] in any pedagogical practice is dialectically structured by pedagogic discourse (Bernstein, 2000). In other words, there is a structuring of mathematics by the institutions of schooling and curriculum, and by the activity of teaching within these. Mathematics for teaching can thus only be grasped through a language that positions it as structured by, and structuring of, the pedagogic discourse in which it ‘lives’. There is, thus, some similarity between our orientation to the structuring of mathematics by pedagogy, and the notion of institutionalisation as developed by (Chevallard, 1992). Our central concern is that of answering the question: what is constituted as mathematics for teaching (MfT) across varying sites of educational practice in South Africa and how is it so constituted? (pp. 3-4)

Part of my reason for starting with this dense and technical quote is to make the point that the specification of the mathematics a third grade teacher needs to know and research designed to conceptualize, identify, measure, and render learnable the mathematics teachers need to know can be dense and difficult terrain. In saying that, “what comes to be MfT in any pedagogical practice is dialectically structured by pedagogic discourse,” Adler is pointing to the ways in which the mathematical understandings that teachers use when teaching can be understood to be created as an initial mathematical understanding interacts with classroom activities of teaching and learning and with the institutions in which those sit. This echoes Shulman’s (1986) characterization of pedagogical content knowledge (the amalgam of knowing content and knowing students and teaching) and subsequent notions of the transformation of content knowledge in practice (Shulman, 1987; Wilson, Shulman, & Rickert, 1987; Magnusson, Krajcik, & Borko, H., 1999). Traditional notions of thinking about mathematical knowledge do not seem to serve well for
talking about the mathematics that teachers need. Understanding and studying this issue is important and challenging.

Using different language and a different framing of the problem, Ball and Bass (2003b) write:

The substantial efforts to trace the effects of teacher knowledge on student learning, and the problem of what constitutes important knowledge for teaching, led our research group to the idea of working bottom up, beginning with practice. We were struck with the fact that, although what teachers need to know may seem obvious, the nature of the knowledge required for teaching is underspecified. On one hand, the answer seems obvious. Who can imagine teachers being able to explain how to find equivalent fractions, answer student questions about primes or factors, or represent place value, without understanding the mathematical content? On the other hand, less obvious is what “understanding mathematical content” for teaching entails: How do teachers need to know such mathematics? What else do teachers need to know of and about mathematics? And how and where might teachers use such mathematical knowledge in practice?

Hence, instead of investigating what teachers need to know by looking at what they need to teach, or by examining the curricula they use, we decided to focus on their work. What do teachers do, and how does what they do demand mathematical reasoning, insight, understanding, and skill? We seek to unearth the ways in which mathematics is entailed by its regular day-to-day, moment-to-moment demands. These analyses help to support the development of a practice-based theory of mathematical knowledge for teaching. (p. 5)

Important similarities and differences exist between these two expressions of the problem of identifying mathematics for teaching and between the two approaches suggested for engaging the problem. Regarding similarities, both statements convey a sense that the fundamental problem for understanding mathematics for teaching is one of bridging between the subject of mathematics and the world of pedagogical practice; it is about understanding the nature of how mathematics and pedagogy meet and how they shape and are shaped by one another. Adler says that mathematics for teaching is “structured by, and structuring of, the pedagogic discourse in which it ‘lives’.” Ball and Bass say that teachers need to know mathematics in and for their work, but they also characterize that mathematics as emerging from that work. Where Adler refers to mathematics constituted by practice, Ball and Bass refer to mathematics entailed in teaching. What is behind the use of these subtle words? I do not answer these questions at the moment, but will return to them repeatedly throughout this study.
Another similarity between the two statements is that both simultaneously address the issue of how mathematics for teaching might be conceptualized and the issue of how to go about investigating mathematics for teaching. In other words, the notions of what it is and how to study it are deeply intertwined. In this study I take up the latter issue, but the former is never far off. In particular, I investigate the design of research that analyzes practice for the purpose of identifying mathematical knowledge for teaching along the lines described by Ball and Bass. For this, I analyze documents from the research conducted by Ball and Bass and their colleagues at the University of Michigan — not with the goal of making claims about this group or their work, but with the goal of understanding better what might be involved in the study of practice for this purpose (of understanding what is constituted in, or entailed in, such a study of practice).

Ball, Thames, and Phelps (2008) define *mathematical knowledge for teaching* as the mathematical knowledge, skills, and sensibilities needed to perform the recurrent tasks of teaching mathematics to students. This definition suggests the need to identify the recurrent tasks of teaching and then examine the mathematical demands of engaging in those tasks. Thus, the approach of the Michigan researchers involves *examining practice* in ways that are *attuned to the discipline* in order to unearth the knowledge demands of teaching, but as Ball (1999) argues, the research design and methods are underspecified.

Ball calls for the development of new methods to support this novel approach to determining mathematical knowledge for teaching, but she also expresses a sense that this is no small matter. First, this approach requires the cultivation of something new, something that might be called a “mathematical analysis”; this requires not only the development of method per se, but also the development of a theoretical perspective that is grounded in the discipline. Second, she points to the need for developing a mathematical analysis and for intertwining perspectives that draw on different orientations and training; differences in perspective and the need to tune them to each other are significant and constitute an important challenge for this program of research. A third challenge discussed in Grossman and McDonald (2008), Ball and Cohen (1999), and Ball, Sleep, Boerst, and Bass (2009) is the under-specification of what it might mean to conduct a “pedagogical analysis”; the lack of a robust, agreed-on decomposition of teaching constitutes
both an underdeveloped curriculum of professional education and an underdeveloped specification of tools for conducting a pedagogical analysis, in particular one attuned to mathematical issues.

This dissertation responds to this call and to the challenges that it raises by examining the question of design of research for analyzing practice. It asks:

\[ \text{What design of research would be suited for studying practice for the purpose of identifying mathematical knowledge for teaching?} \]

This is a question about the design of studies, not the specification of a set of research methods. I agree with Lesh, Lovitts, and Kelly (2000) that the construction of study design deserves a place of its own, independent of specific methods for implementing a design, especially in research domains engaged in the exploration of new designs and methods suited to the distinctive problems of the field.

To respond to the need for formulating a study design suited for a practice-based and discipline-grounded approach, I develop three distinct features of a practice-based and discipline-grounded approach to studying practice: (i) a mathematical analysis of practice (namely, one grounded in a disciplinary perspective); (ii) a pedagogical analysis of practice (grounded in a pedagogical perspective); and (iii) ways of coordinating these two sets of analyses and the distinct perspectives on which they are based. This third feature is particularly important because the two perspectives, while not necessarily at odds, are based on distinct experiences, views, and allegiances and often seem unable to “talk” to one another. In this study, I clarify how these two perspectives can be conceived so as to produce plausible and useful analyses of practice and explain how an understanding of the coordination of the two perspectives is central to these conceptions, to the integration of these distinct analyses, and to the very conception of mathematical knowledge for teaching — how the coordinating of perspectives represents an alternative view of Adler’s mathematics constituted by practice and of Ball and Bass’ mathematics entailed in teaching.

The remainder of this chapter consists of three sections. The first section situates the work in a broader context. This dissertation is partially motivated by questions in the research literature, but its more basic motivation is the improvement of student learning of mathematics. This practical goal motivates the work and guides its conduct. The first section argues for the practical importance of identifying the mathematical knowledge needed for teaching and makes explicit the underlying model for why and how such knowledge would matter in the world. This provides an important backdrop for the more focused study on practice-based and discipline-grounded approaches to studying practice to identify mathematical knowledge for teaching.
The second section describes the specific problem and research question taken up in this dissertation. It argues that the empirical study of teaching and learning, of practice itself, represents an important underdeveloped resource for solving the problem of teacher content knowledge, but that to make use of this resource we need to better understand what is involved in the study of practice, and in particular what is involved when the purpose is to determine what mathematical knowledge and skill would be useful to, and usable by, teachers. The last section of the chapter discusses the research questions, how the research has been carried out, a rationale for the approach taken, and an overview of the remaining chapters.

1.1 The Practical Problem of Mathematical Knowledge for Teaching

We live in a scientific and technological age. We face major social issues nationally, and globally, that are deeply mathematical in nature. However, far too few people in the country know enough mathematics to comprehend basic notions of science or to judge quantitative evidence central to public debate. On the 1996 NAEP, only one-in-five eighth graders were able to recognize what information two graphs of the same data, but with different vertical scales, made apparent or obscured, and less than one-in-four twelfth graders were able to decide which of two mixtures had the greatest concentration (6 oz. of cherry syrup and 53 oz. of water versus 5 oz. of syrup and 42 oz. of water) (Reese, Miller, Mazzeo, & Dossey, 1997). Indeed, in the United States, not being good at math serves as a kind of badge of honor, a prerequisite for membership in the mainstream. In international comparisons, U.S. performance is mediocre at best, and to an unconscionable extent, achievement gaps are persistently predictable by social group. Moreover, perennial attempts to reform mathematics education do little to change these basic features.

What, precisely, is the problem? And, how can it be addressed? Consider for a moment an example of the all-too-common breakdowns that occur in mathematics teaching and learning, as reported by Borko, Eisenhart, Brown, Underhill, Jones, and Agard (1992).

Ms. Daniels, a recent college graduate with extensive coursework in undergraduate mathematics, was reviewing division of fractions with her sixth graders during “morning math.” As an example, she wrote $\frac{3}{4}$ divided by $\frac{1}{2}$ on the board and walked through the steps of the standard procedure for inverting and multiplying. A student, Elise, then asked, “I was just wondering why, up there when you go and divide it and down there when you multiply it, why do you change over?” Ms. Daniels was caught off guard—she had only planned to review the rule. She recognized, however, that Elise was asking for a justification of the procedure, so she tried to generate a situation and a drawing that would help Elise make sense of the rule.
Well, as you learned before, when you divide a fraction into a fraction, the process is to flip the second one and multiply. And say we have a wall, okay, and we divide it into fourths. One fourth of it is already painted, okay. So we have three fourths of it left to paint. Right? You agree with me?"

Ms. Daniels then drew a rectangle on the board, drew four vertical lines, dividing it into four congruent parts, and shaded one part. “But we only have enough paint to paint half of these three fourths. So half of three fourths would be between about right there. Right, do you agree with that?” Ms. Daniels drew a line down the middle of the unshaded portion to divide it in half. Elise replied, “Yes.”

Regrettably, though, Ms. Daniels was representing \( \frac{3}{4} \) times \( \frac{1}{2} \), or \( \frac{3}{4} \) divided in half, not \( \frac{3}{4} \) divided by \( \frac{1}{2} \). She continued her explanation, but when she got to an answer of three eighths, she realized that she has made an error. “Okay, oh wait. I did something wrong here.” She then paused for about 2 minutes, studying the board, and then abandoned her example.

Well, I am just trying to show you so you can visualize what happens when you divide fractions, but it is kind of hard to see. We’ll just use our rule for right now and let me see if I can think of a different way of explaining it to you. Okay? But for right now, just invert the second number and then multiply.

While students continued working on problems independently, Ms. Daniels stood at the board, considering the problem. She then consulted the textbook and commented to another adult in the room that she had just done multiplication, but she did not tell students this and did not attempt to correct the explanation the following day.

Breakdowns in the teaching and learning of mathematics, such as the one in the example above, play out everyday in classrooms across the country. In this chapter, I argue the following: that such breakdowns are the result of teachers’ inadequate mathematical knowledge and skill, which is due to an underdeveloped understanding of the mathematics teachers need for their work, which is due in turn to underdeveloped approaches for studying the mathematical demands of teaching.

The importance of content knowledge for the improvement of teaching and learning has been argued compellingly by a number of scholars. As incoming president of the American Educational Research Association, Shulman (1986, 1987) argued that an understanding of teaching and learning of particular content requires thoughtful attention to that content and that content knowledge was the “missing paradigm” in research on teaching and teacher education. Specific to mathematics, Ball, Lubienski, and Mewborn (2001) argued that teachers’ mathematical knowledge had been a central focus of research in the intervening 15 years, was
prominent in practice and policy debates, and offered a programmatic agenda that combines theory, practice, and empirical inquiry.

Drawing on these scholars, I argue that the study of mathematical knowledge for teaching is an important practical problem in the field. I begin by arguing that the problem of improving student achievement is one of improving the instructional dynamics in classrooms, which depends on teachers, the teaching they do, and the resources they have at their disposal for designing, enacting, and overseeing instruction. I argue, further, that teacher effectiveness is significantly dependent on their mathematical knowledge and skill. In other words, I argue that teacher content knowledge is key to improving teaching, which is key to improving instruction, which is key to improving student learning.

Figure 1.1. Model for effects of teacher content knowledge on student learning.

This discussion provides a backdrop for the focus and goals of this dissertation—to understand better what is involved in the study of practice as a way to determine the mathematics that K-8 teachers need to know.

1.1.1 The Problem of U.S. Student Achievement in Mathematics.

Bemoaning student achievement in mathematics is commonplace in the United States. Is there, however, really a problem or is the phenomenon just a kind of Lake Wobegon effect—where average people, because of their tendency to see themselves as above average, are alarmed by information about being average? The short answer is no. Student achievement in mathematics is indeed a significant social and intellectual problem deserving serious attention. Three kinds of complementary evidence support this claim: poor performance, poor public image, and inexcusable inequities.

Unfortunately, claims about U.S. students’ poor performance in mathematics are not easily established. For instance, the National Assessment of Educational Progress (NAEP) was established in 1969 to provide national data on how students were performing and progressing, but interpreting the results has proven problematic. Although the NAEP content framework was developed with broad input and agreement, early versions reflected content deemed important during the 1960s and 1970s. What was seen as important, however, changed. This led to two different NAEP tests: a main test that reflects these changes and a trends test that continues the original content. Not surprisingly, they provide very different pictures and are the focus of much debate (Loveless, 2002). In addition, to help with the interpretation of results, achievement levels
of basic, proficient, and advanced were set. Performance on these has been very low: From 1990 to 2007, the number of US 8th graders performing below basic has ranged from 29% to 48%, with only 15% to 32% performing above basic (Lee, Grigg, & Dion, 2007). Such low performance, however, has led many to question the cut points for these achievement levels (Phillips, 2007; Loveless, 2007). Phillips links NAEP to TIMSS to argue that with these cut points even the highest performing countries internationally do not fare well. In short, the problems of what to measure and where to set the bar are fundamentally political in nature. This leads to a common practice of comparing individuals and groups to others or to their earlier performance rather than to standards anchored in the curriculum.

Still, though, student performance in mathematics seems to be a compelling problem. Students at every level are seen as ill prepared for subsequent mathematical study and for the workforce. As Paulos (1989) argues, Americans are innumerate. As Smith (2002) argues, a glass wall keeps people from moving from the familiar physical world they know to the distinctive mathematical world on the other side of the wall. Mathematics has a poor image in the United States. Many Americans readily admit that they do not know mathematics and were never good at mathematics, and they do so with a pride that one has trouble imagining anyone would ever have in saying that they cannot read. In addition, performance in mathematics is inexcusably predictable by race, class, and other social groupings. Poor mathematics performance keeps many students from being able to pursue a variety of careers that require mathematics, and it does so disproportionately for students from under-resourced, under-represented groups. Moreover, international comparisons, such as TIMSS and PISA, consistently place the United States close to the middle of the pack internationally, with several Asian and several European countries, including our main economic competitors, routinely out-performing the United States. The general consensus is that performance is a problem and that it needs to be addressed.

What, though, is the cause of this problem, and how can it be best addressed?

1.1.2 The Centrality of Instruction to Improving Achievement

Cohen, Raudenbush, and Ball (2002) argue that it is in the dynamics of instruction that resources are used, or not used, in ways that influence learning. They write that attempts to improve education in America have historically focused on money, curriculum materials, and facilities as though these familiar resources directly cause learning. From correlational studies, researchers refer to the “effects” of class size, of a particular curriculum, or of increased teacher salaries without an underlying model of how such resources are used to bring about learning. They argue that a basic feature of such a model is that it is what happens in classrooms among teachers and students around content that matters for improving learning. A curriculum does not
teach itself, state standards do not assure that students have the capacity, or even the will, to achieve those standards, and money does not help a teacher who cannot make sense of her students’ misunderstandings or does not know how to address them.

These define instruction to be, “the interactions among teachers, students, and content in various environments” (p. 88).

![Diagram of Instruction as interaction of teachers, students, and content, in environments.](image)

They describe instruction, not as an event, but as a stream of interactions among teacher, students, and content, flowing in the environment of other teachers and students, school leaders, parents, local districts, state agencies, textbooks, tests, and more. For these participants (teacher, students, and content) to be effective, they need to solve four central problems: (i) use the resources that are available; (ii) coordinate instruction so that all of its pieces are working in concert; (iii) mobilize incentives for everyone’s performance; and (iv) manage the environments, working in, with, and in spite of its elements.

Seeing instruction as key to improving teaching and learning suggests different sorts of resources. Cohen and his colleagues (2002) identify three types. Conventional resources are those that have typically been considered: money, textbooks, time, libraries, and so forth. Personal resources are the knowledge, skill, and will of teachers and students that affect their view of and their use of conventional resources. Environmental resources are those such as a principal, other teachers, parents, state policy, and more, which are available for use by teachers and students. They argue that although there is no one resource that causes all others, the personal resources of teachers and students deserve particular attention. Teachers and students would not do well without books, without schools to work in, without time or financial support.
However, conventional resources only matter for instruction if and when teachers and students notice them and have the knowledge, skill and will to use them productively. As they write:

When added conventional resources appear to affect learning accomplishments, it is because they were usable, because teachers and students knew how to use them effectively, and because environments enabled or did not impede use by these particular teachers and students. … This discussion also implies that when added resources lie outside the range of teachers’ and students’ practices, knowledge, norms, and incentives, they will have no discernable effect on learning. (pp. 102-103)

The improvement of student achievement, then, is centrally a problem of resource use in instruction. Cohen et al. go on to describe what these ideas imply for research.

Given our theoretical position, the overarching research question cannot be “Do resources matter”? No deliberate attempt to learn or teach is conceivable in the absence of conventional resources, and there is ample evidence that teaching is causally related to learning. The overarching question must be: “What resources matter, how, and under what circumstances?” (pp. 108-109)

Hence, the question of how to improve student achievement is not one of deciding which of the many resources matters most, but rather one of understanding how and when specific resources get used in instruction to the benefit of teaching and learning. Because the knowledge, skill, and will of teachers and students mediate the use of other resources, these become particularly important to study if we hope to make significant inroads into improving teaching and learning.

1.1.3 The Centrality of Teachers—and Teaching—in Improving Instruction

Putting instruction at the center of the education enterprise makes teachers and teaching central as well. To some degree, students’ success depends on themselves, on their parents, on their community, on their peers, on school officials, and on the many other people who influence instruction. The teacher, however, is the professional directly responsible for orchestrating the dynamics of instruction and for assuring that learning occurs. As such, teaching—“what teachers do, say, and think with learners, concerning content, in a particular organization of instruction, in environments, over time”—is particularly important, as is teachers’ knowledge, skill, and will for using resources in the service of instruction (Cohen et al., 2002, p. 90).

Moreover, evidence that teachers matter has been abundant and compelling. Correlational studies, in varied contexts with varied designs spanning the past 30 years, consistently indicate that, adjusting for student characteristics, roughly one tenth of the variance
in student achievement gains is associated with teachers, that cumulative effects are even larger, and that effective teaching substantially lessens differences in achievement predictable by student characteristics, in particular differences predictable by race/ethnicity and social class (Goldhaber, 1999; Rivkin, Hanushek, & Kain, 2005; Rockoff, 2004; Sanders & Rivers, 1996). In addition, using the Tennessee class-size experimental data with random assignment of teachers and students, Nye, Konstantopoulos, and Hedges (2004) found similar results.

Although the report by Coleman et al. (1966) is often interpreted to mean that schools, and the teachers in them, have little effect on student outcomes, the study actually reveals that teachers and teaching account for a substantial portion of variation in student test scores, second only to student characteristics such as socioeconomic background. Methods and understanding of this complex set of relationships have increased substantially since the report by Coleman et al. For example, in a recent hierarchical-linear-modeling analysis of three national probability samples of high school seniors spanning three decades, Konstantopoulos (2006) found that, adjusting for student characteristics, variation in student achievement within schools is five times as large as between schools and that between-teacher variation is a large part of the variation within schools. Konstantopoulos’ analysis differentiates teacher effects and school effects. He writes that “an important part of achievement differences within schools is due to teachers” and that “it appears that the teachers whom students are assigned to may be more important than the schools they attend” (p. 2577). This result does not provide the direct observation of practice that Raudenbush (2004) calls for, but it provides evidence that the effect is due more to teachers than to contextual factors or to other school factors.

Although it is now clear that substantial differences in achievement gains and in the achievement gap between social groups are associated with differences in teachers within (and independent of) schools, our ability to effectively conceptualize and to measure teacher quality remain limited. As Goldhaber and Anthony (2007) point out: “There is a seeming contradiction between the fact that teachers have a large impact on student achievement, but specific teacher attributes are not consistently found to directly impact student achievement” (p. 135). Using data for third-grade, fourth-grade, and fifth-grade teachers from the Los Angeles Unified School District, Gordon et al. (2006) make the point that differences due to teachers matter, but are largely unrelated to certification. Examining classroom-level impacts of teachers on average student performance, (controlling for baseline scores, student demographics, and program participation) they find no statistically significant differences between certified and uncertified teachers.
In contrast, they sorted teachers into quartiles based on the teachers’ impact on student performance across two years, and then looked at their impact on student performance in the third year. While certification status was not helpful in predicting teacher impact on student performance, teachers’ effectiveness during the first two years is an informative predictor of effectiveness in the third year.

Figure 1.3. Teacher impacts on math performance by initial certification.

In effect, we know that teachers and teaching matter, but we do not adequately understand what about them matters, and what it is about what they do that matters. Goldhaber (2002) writes:

Only about 3 percent of the contribution teachers made to student learning was associated with teacher experience, degree attained, and other readily observable characteristics. The
remaining 97 percent of their contribution was associated with qualities or behaviors that could not be isolated and identified. (p. 53)

This is tantalizing—it is discouraging in that compelling ideas about what should matter, and numerous attempts to show that these things do indeed matter, have yielded so little, but reassuring in that real gains are there to be made if we can identify what it in fact is about teachers and teaching that produce substantial teacher effects. Where, then, should we focus our efforts to determine what factors matter for improving teacher quality, teaching quality, and student learning? Currently, this is a site of heated debate.

1.1.4 The Importance of Teacher Content Knowledge

In this debate, researchers have tried to muster evidence to show effects for programs and policies, but effects tend to be inconsistent and weak. Instead of arguing from an overly thin knowledge base, the debate is in need of research that more fully articulates assumptions and hypothesized models and that tests these models. What specific knowledge, skill, and understanding matter for teaching, how do they matter, and under what conditions? Similarly, we need better understanding of the conditions under which incentives matter, or of any other factor proposed to be key to teaching, and of the mechanisms at play in shaping its impact on instruction and learning. Short of this, arguments about effects are unlikely to lead to agreement or to improvement.

A prime candidate for such investigation is teacher content knowledge. If student learning of content is a central goal, then teachers certainly need to know that content. Moreover, conceptual work on teaching consistently suggests that there is something additional, subtle, and substantial about the content knowledge teachers need. Dewey (1964/1904) argues that there is method in subject matter itself and that it is by “carrying back subject matter to its common psychical roots” (p. 23) that teachers are able to direct the mental movement of students. Schwab (1978) argues that this deeper knowledge of the subject requires an understanding both of important ideas and skills in the domain and of how new ideas are added and deficient ideas are dropped. Both Dewey and Schwab see content knowledge as key to teaching and see conceptions of content knowledge as underdeveloped for the purpose of teaching—that teaching demands a kind of understanding, appreciation, and depth of content that is a hallmark of the endeavor. As Elbaz (1983) writes, “the single factor which seems to have the greatest power to carry forward our understanding of the teachers’ role is the phenomenon of teachers’ knowledge” (p. 45).

Building on these scholars, Shulman (1987) introduces the notion of pedagogical content knowledge as a further proposal of the distinctive understanding of content needed by teachers:
“that special amalgam of content and pedagogy that is uniquely the providence of teachers” (p. 8). And, he (1986) describes three forms of knowledge: propositional, case knowledge, and strategic knowledge, where the latter extends understanding “beyond principle to the wisdom of practice” (p. 13). In other words, the teacher must be “capable not only of practicing and understanding his or her craft, but of communicating the reasons for professional decisions and actions to others” (p. 13). And, such capacity requires more than knowing subject matter in ways that non-teachers must know subject matter; it requires capacity for explaining why something is done and dealing with situations in which principles collide and no simple solution is possible. This suggests a kind of skill with subject matter that has, for instance, the depth and flexibility to evaluate a particular representation of an idea, to weigh alternatives, to reason about the ends and means of using the representation, and to act while reflecting on these choices. According to Borko et al. (1992), it is the lack of such knowledge, and the ability to use such knowledge in practice, that causes Ms. Daniels problems in her review of division of fractions.

There is no clear case for saying that teacher content knowledge is the key factor behind the largely unexplained teacher effect. Instead, my point is that the debate about improving teaching is being argued by input-output studies that do not help us understand the mechanism that connect inputs to outputs or the conditions that support or impede improvement. The field needs a better understanding of how inputs shape outputs before it is likely to make much progress on explaining currently unexplained effects. With this in mind, I am arguing that content knowledge—of what content knowledge matters and how it matters—is a prime candidate for investigation: (i) because commonsense suggests that teacher content knowledge should matter and anecdotal evidence suggests that it is a problem; (ii) because current measures fail to identify the effects one would expect to see; and (iii) because the conceptual literature suggests that the kind of content knowledge and skill needed by teachers is different from that needed by non-teachers and that it is poorly understood.

The next section reviews what we currently know about teacher content knowledge and its role in teaching and learning. It begins by reviewing what we know from studies of the effects on student learning, and then describes research that uses the study of practice to investigate what mathematical knowledge matters for teaching and how it matters.
1.2 Studying Practice to Determine Mathematical Knowledge for Teaching

Studies attempting to link student achievement to their teachers’ content knowledge consistently suggest that mathematical knowledge closely related to practice is more likely to have a positive effect on teaching and learning. This has led some researchers to go back to the classroom to study practice as a way to determine the mathematics needed for the work of teaching. Unfortunately, the study of practice is not straightforward or easy. It is this challenge that leads to the specific focus for this dissertation.

This section reviews what is known about the effects of teacher content knowledge on student achievement in mathematics and recent work that studies practice in order to determine the mathematics elementary teachers need to know. Based on these, I argue that, if we are to effectively leverage the study of practice for the purpose of identifying the mathematics that elementary teachers need to know, we need to better understand what is involved in the study of practice for such a purpose.

1.2.1 Effects of Elementary Teachers’ Mathematical Education on Student Achievement

While most assume that teaching mathematics demands knowledge of the subject, evidence for this claim has been elusive. As Begle (1979) concluded from his meta-analysis of studies conducted during the 1960s and 1970s:

There is no doubt that teachers play an important role in the learning of mathematics by their students. However, the specific ways in which teachers’ understanding, attitudes, and characteristics affect their students are not widely understood. In fact, there are widespread misconceptions, on the part not only of laypersons but also mathematics educators, about the ways in which teachers influence mathematics learning by their students. (Begle, 1979, p. 27)

Begle’s work was followed by a spate of studies during the 1980s and 1990s that examined effects of teacher characteristics on student performance. Unfortunately, the question is surprisingly complex, and existing research is inconclusive. To date, the most comprehensive studies have adopted production-function models from economics to study home, school and teacher effects on student achievement gains. For these studies, however, the subject matter, grade level, and context vary. In addition, proxies are often used to measure subject matter preparation—such as institutional ratings, highest degree, certification, sundry non-mathematical
tests, and others. In using such proxies, little attention has been given to distinguishing between effects of general knowledge (general aptitude, verbal competence, level of education) and effects of content knowledge (quantitative competence, mathematics coursework, knowledge of the elementary school curriculum). Although there is general consensus that teachers’ mathematical knowledge makes a difference, the field has yet to address adequately the question of what mathematics matters most for teaching and to distinguish differences in effects of specific mathematics content and preparation for teachers (Ball, Lubienski, & Mewborn, 2001).

In reviewing teacher characteristics and student achievement gains, Wayne and Youngs (2003) indicate the limited nature of what we know from this eclectic group of studies.

The studies confirm that students learn more from teachers with certain characteristics. In the case of teachers’ college ratings and test scores, positive relationships exist and should be investigated further to learn about the relative importance of specific college characteristics and tested skills and knowledge. In the case of degrees, coursework, and certification, findings have been inconclusive except in mathematics, where high school students clearly learn more from teachers with certification in mathematics, degrees related to mathematics, and coursework related to mathematics. (p. 107)

What this implies for elementary school, though, is unclear. In addition, even at the secondary level, effect sizes are typically small, interaction effects are substantial, and the effects of coursework and major are even less determinate in other subject areas. Several studies show no effects for teachers’ study of conventional college mathematics when they teach below ninth grade (Rowan, Correnti, & Miller, 2002; Harris & Sass, 2007; and Hill, Rowan, & Ball, 2005). Thus, as described above, although elementary school teacher effects on student achievement gains are considerable, researchers have not yet pinpointed what mathematical knowledge matters for elementary school teaching and learning.

In another review of the literature, Wilson, Floden, and Ferrini-Mundy (2002) summarize:

We found no reports meeting our selection criteria that directly assessed prospective teachers’ subject matter knowledge and evaluated the relationship between teacher subject matter preparation and student learning. To date, researchers conducting large-scale studies have relied on proxies for subject matter knowledge, such as majors or coursework. The research that does exist is limited, and in some cases, the results are contradictory. The conclusions of the few studies in this area are especially provocative because they undermine the certainty
often expressed about the strong link between college study of a subject matter and teacher quality. (p. 191)

What then are we to conclude, if anything, from the fragmentary and, at times, counter-intuitive nature of research findings? A plausible interpretation is that the mathematical knowledge that matters most is that which is closer to the content teachers teach and the work they do. For instance, conventional college mathematics courses may be a better proxy for the actual knowledge teachers need at the high school level than at the elementary school level. Knowing advanced mathematics may not be particularly helpful for teaching second graders basic ideas about number and operation. Many of us can probably identify with the experience of learning mathematics from someone who is, indeed, very knowledgeable about mathematics, but whose very success in mathematics gets in the way of his or her being able to communicate basic ideas effectively. Simply requiring more mathematics does not necessarily lead to better teaching. This fits with results reviewed by Wilson, Floden, and Ferrini-Mundy (2001) where degree level has no effect (or negative effect) and undergraduate major often has small positive effect, but specific coursework has more consistent and larger effects.

For instance, as Monk’s (1994) study suggests, at the secondary level, effects vary across topics and courses. For instance, he found that teachers’ introductory college mathematics course taking contributed more to performance gains of their students in classes for juniors (typically algebra) than in classes for sophomores (typically geometry). It makes sense that effects would be greater for junior algebra classes than for sophomore geometry classes because introductory college courses tend to provide additional exposure to and practice with algebra but do little with geometry. And, he found that the number of mathematics courses in a teacher’s background has a positive effect on students in advanced courses, but no effect on students in remedial courses. Again, this suggests that specific content can support effective teaching of some mathematics but not other mathematics. He also found a threshold effect—that after 4 or 5 courses, the addition of courses has a smaller effect on performance gains and that having a major has no apparent bearing. (Similarly, he found differential effects for teachers’ coursework in the physical sciences and the life sciences.) These results, and others like them, suggest that the specific nature of the relationship between the mathematics taught to teachers and the mathematics those teachers teach does, indeed, make a difference.

In short, the evidence suggests that teachers’ mathematical knowledge matters, but that we lack an understanding of what mathematical knowledge is important for elementary school mathematics teaching. We need studies that directly assess teachers’ mathematical knowledge and that examine effects of specific mathematical knowledge of teachers on the quality of
instruction and on improved student learning. In particular, we need studies that test specific hypotheses about what mathematical knowledge matters and how it matters.

Two hypotheses seem worth considering. It may be that teachers need to know the elementary school curriculum, several levels below and beyond the grades they are responsible for teaching, but perhaps a more rigorous and ambitious vision of that content (as evident in some international curricula, with multi-step problems, hard geometry problems, non-routine word problems, together with high expectations for computational fluency). Instead of advanced topics, perhaps teachers need deep understanding of the curriculum they will eventually teach. Ma (1999) provides a characterization of what she calls profound understanding of fundamental mathematics. In general, though, what is meant by such “deep understanding” is not completely clear.

Another possibility is that teachers need to know mathematics as it is needed in teaching practice. The assumption here is that teachers’ content knowledge needs to be relevant to their daily work, and that they need to hold that knowledge in ways that make it usable as they plan lessons, interact with students, assess learning, and talk to parents. Under this assumption, teachers need mathematics courses that prepare them for the mathematical demands of their professional work. Likewise, they need methods courses that provide instruction on lesson planning, classroom management, or the design of instructional formats, but with serious treatment of the content to be taught. For this hypothesis, the question of what makes knowledge “usable” or “clearly connected to teaching” needs clearer specification.

Although these are both viable hypotheses and possibly connected, some significant evidence supports the second one. First is that effects of methods courses, when they are studied, have been more consistently positive than effects of other variables (Begle, 1979; Ferguson & Womack, 1993; Guyton & Farokhi, 1987; and Monk, 1994). Indeed, at least at the secondary level, Monk found that teachers’ courses in undergraduate mathematics pedagogy contributed more to student performance gains than did undergraduate mathematics courses.

Second is evidence from the only large-scale evaluation of effects of directly measured subject matter knowledge on student performance gains done since the reviews of Wilson et al. and of Wayne & Youngs. This study, reported by Hill, Rowan, and Ball (2005), used a measure specifically focused on mathematical knowledge as it arises in the work of teaching. These researchers found significant effects on student performance gains and tested their mathematics-specific measure against other general measures, such as a test of reading knowledge. Although, as is to be expected, the effect size was small, it was comparable to that of socio-economic status, which is typically one of the largest predictors of student performance gains (p. 396).
These results suggest that the challenge we face is not whether teachers can do the mathematical work they give to their students, but whether they have the wealth of other mathematical knowledge and skill that they need for the work they specifically do as teachers responsible for children's learning of mathematics. Breakdowns in classroom teaching of mathematics are commonplace and can originate either in a teacher’s lack of understanding of the topic at hand or in a teacher’s understanding that is fragile or superficial, but can also originate in teachers’ lack of mathematical knowledge and skill needed for what they have to do in the classroom.

Teachers should be able to divide 24.56 by 0.004 correctly, or 1 3/4 by 1/2, and be able to solve multi-step word problems. These are important problems that reveal important gaps in, not just teachers' understanding, but in the understanding of people in this country more broadly. Proficiency with the mathematics of such problems is part of the mathematical knowledge needed for teaching. In addition, however, teachers need mathematical knowledge that goes beyond the elementary school curriculum (Ball, Hill, & Bass, 2005). For instance, Ball and Bass (2003b) have argued that elementary school teachers must be able to do tasks such as: “design mathematically accurate explanations that are comprehensible and useful for students; use mathematically appropriate and comprehensible definitions; represent ideas carefully, mapping between a physical or graphical model, the symbolic notation, and the operation or process; interpret and make mathematical and pedagogical judgments about students’ questions, solutions, problems, and insights (both predictable and unusual); be able to respond productively to students’ mathematical questions and curiosities; make judgments about the mathematical quality of instructional materials and modify as necessary; be able to pose good mathematical questions and problems that are productive for students’ learning; assess students’ mathematics learning and take next steps” (p. 11). Each of these tasks, though pedagogically situated, requires substantial mathematical knowledge. It is this knowledge that teachers need and that is more likely to improve teaching and learning.

In sum, studies attempting to link student achievement to their teachers’ content knowledge consistently suggest that mathematical knowledge more closely related to practice is more likely to have a positive effect on teaching and learning. It is not just any type of mathematical knowledge that matters for teaching, but rather knowledge of mathematics that is useful to—and usable in—the work. Thus, going back to observe teaching in ways that are attentive to its mathematical demands provides a sensible way to determine the mathematical knowledge and skill likely to be useful in teaching.
1.2.2 A Practice-Based Theory of Mathematical Knowledge for Teaching

In a chapter of the Handbook of Research on Teaching, Ball, Lubienski, and Mewborn (2001) characterize a shift in research on mathematics teaching from studies largely of teaching behaviors not directly related to students’ learning of content to studies emphasizing teacher content knowledge as an important feature of effective teaching. They review research on effects of teachers’ mathematical knowledge on student performance and research on teachers’ mathematical understanding and lack of understanding, but argue that this research does not provide an answer to the question of how teacher content knowledge can be leveraged to improve teaching and learning. They conclude that little evidence exists for the benefit of a particular course, having a major, or passing a licensure exam and that the examination of teachers and what they know tells us little about what they need to know to be more effective. To address these issues they propose redefining the problem from one about teachers’ mathematical knowledge to one about knowledge of mathematics in and for teaching. They offer Lampert’s research as an example (in particular, Lampert, 1986, 1989, 1990, and 1992).

Her writing, not about the teacher but about practice, reveals vividly the mathematical reasoning entailed in choosing and using particular representations, in managing a complex classroom discussion, and in designing a problem or figuring out how to formulate a good question. Her work shows some of the mathematics needed in teaching, as well as where and how those mathematics have to be put to use. (p. 450)

They review three additional studies (Swafford, Jones, & Thornton, 1997; Sowder, Philipp, Armstrong, & Schappelle, 1998; and Fennema & Franke, 1992) that follow teachers from a professional development experience back into the classroom to document changes in their classroom practice. Some of the changes are about attitudes, confidence, and expectations, but Ball, Lubienski and Mewborn propose that some changes, such as expecting more conceptual explanations, probing student thinking more often, and engaging students in more mathematical discussions may reflect changes in what teachers are able to do after having developed their mathematical understanding and skill. In summarizing, they identify two important kinds of results from these studies and give an example of each.

One has to do with what mathematical knowledge matters (classification of problem types within addition and subtraction) and the other has to do, mostly implicitly, with uses to which teachers must put such knowledge (managing productive discussions in class and helping students develop multiple solutions to problems). (p. 450)
They also discuss four studies that reveal the potential gap between a teacher knowing particular mathematics and being able to use that knowledge while teaching in the classroom (Thompson & Thompson (1994); Schram, Wilcox, Lanier, & Lappan, 1988; Eisenhart, Borko, Underhill, Brown, Jones, & Agard, 1993; and Heaton, 2000). They argue that these studies point the way for future work—that what is needed is research that, instead of analyzing curricula or interviewing teachers, analyzes core activities of mathematics teaching in order to identify the pedagogically functional mathematical knowledge central to effective teaching. Such an investigation of practice, of the dynamics of instruction as described by Cohen et al. (2002), would, according to Ball et al., solve two recurring problems with past work.

On one hand, teachers cannot always use what they know. On the other, what they know may not fully reflect the demands of their work in context. For both of these reasons, researchers working in this third approach would begin instead with an examination of practice itself. (p. 453)

Analysis of practice along the lines called for by Ball et al. (2001) has been the focus of research conducted here at the University of Michigan over the past decade. Ball and Bass (2003b) describe this line of work as a kind bottom-up, job analysis and its products as contributing to a practice-based theory of the mathematical knowledge for teaching.

What do teachers do, and how does what they do demand mathematical reasoning, insight, understanding, and skill? We began to try to unearth the ways in which mathematics is entailed by its regular day-to-day, moment-to-moment demands. These analyses help to support the development of a practice-based theory of mathematical knowledge for teaching. We see this approach as a kind of “job analysis”, similar to analyses done of other mathematically intensive occupations, from nursing to engineering and physics (Hoyle, Noss, & Pozzi, 1997), to carpentry and waiting table. In this case, we ask:

- *What* mathematical knowledge is entailed in the work of teaching mathematics?
- *Where* and *how* is mathematical knowledge useful and used within the work of teaching mathematics? How is mathematical knowledge intertwined with other knowledge and sensibilities? (p. 5)

Central to the research of this group is a large longitudinal database documenting an entire year of the mathematics teaching and learning in a third grade classroom during 1989-90.

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1 These data were collected under a 1989 National Science Foundation grant to Deborah Ball and Magdalene Lampert, then at Michigan State University.
The records collected across that year include videotapes and audiotapes of the classroom lessons, transcripts, observation notes, copies of students’ written class work, homework, and quizzes, student interviews, as well as teacher lesson plans, notes, and reflections. These records were designed to represent teaching and learning for the purpose of studying it (Ball & Lampert, 1999). Some records are the product of instruction itself, such as student work or the teacher’s lesson plans. Other records provide additional documentation, not part of the usual activities of teaching and learning, such as classroom videotape or student interviews. Records of this kind, gathered directly from practice, capture key features of instruction, with detail and particularity, and make them available for review, scrutiny, and public viewing. As such, they constitute an important resource for the study of practice.

The goal of the research is to analyze practice for the purpose of identifying mathematical knowledge useful to and usable in teaching. One caveat given by Ball (1999) is that the knowledge identified is generally applicable to many classrooms, it is not necessarily applicable to all.

Ours is also not a study of mathematics and pedagogy in any approach to mathematics teaching. The study focuses on a kind of teaching that aims to take seriously both mathematics as a discipline and children’s mathematical ideas, and that sees mathematics as a collective intellectual endeavor situated within community. What it means to take seriously either mathematics or children’s ideas, or to treat mathematical work as collective, is of course highly contestable. “Mathematics as a discipline” is hardly a monolith; and “taking children’s ideas seriously” can imply wildly different actions. … Still, it is important to appreciate that these imperatives — to treat both mathematics and students with respect, and to consider mathematical work as a collective endeavor — do not prescribe a single approach, or a tightly-specified strategy of teaching. (p. 29)

In addition, the analysis of the data is not about describing this teacher, her knowledge, or her teaching, but to use it as a starting point for considering what teaching is as an endeavor and what, as an endeavor, it demands in the way of mathematical knowledge and skill. Ball and Bass (2000b, p. 200) describe two analytic tasks for the work. One is to probe the particulars of the cases being examined and the other is to identify what is generalizable and what is specific to particular approaches to teaching, or to the specific cases being studied. She describes two techniques used to establish generalizability.

This is done both through logical analysis (focused on identifying elements of mathematical knowledge or insight that
would be important in any approach to teaching) and through comparisons derived from further empirical analyses of other classroom data. (p. 200)

Ball and Bass (2003b) go on to describe the approach as “a kind of methodology of interdisciplinary observation of teaching” where a team of researchers with backgrounds in teaching practice, in research mathematics, in cognitive and social psychology, and in education research conduct mathematical and pedagogical analyses of teaching focusing on mathematics as it emerges within the core task domains of teachers’ work (p. 6).

This work has produced numerous examples of mathematical demands faced by teachers and of mathematical knowledge and skill useful in teaching. Many of the examples are characterized by mathematically focused tasks of teaching, such as explaining why mathematical content makes sense or evaluating non-standard student work. Ball, Hill and Bass (2005) give an example using $35 \times 25$. They argue that being able to carry out this multiplication is essential, but insufficient, knowledge for teaching. They give several wrong answers and demonstrate a kind of “error analysis” that is common in teaching. They then discuss the mathematical nature of the work teachers do in choosing the numbers for an example, in sequencing examples, in developing representations of multiplication, and in using these to represent, meaningfully and effectively, the standard algorithm. In finishing the example, they point out that none of the work they have described involves knowing about students or about how to teach multiplication or remediate errors. Here are some other mathematical tasks of teaching taken from presentations and publications developed by members of the group.

- Producing and analyzing explanations
- Generating simpler and more complex versions of a problem
- Using and inventing notation
- Defining terms and attending closely to language
- Asking mathematical questions (Why does this work? Does this work in all cases? Do we have all the solutions? How are these two representations related?)
- Thinking of special cases and choosing examples (boundary cases, or examples that might push an initial idea)
- Interpreting and evaluating alternative solutions and alternative thinking
- Unpacking and decomposing mathematical ideas
- Analyzing mathematical treatments in textbooks
- Choosing or designing tasks to assess student understanding
- Communicating with parents about mathematical issues

Another type of result from this work has been the identification of mathematical practices central to teaching (such as giving and evaluating mathematical explanations, choosing and using representations, and using definitions and mathematical language) and a fuller
characterization of the nature of mathematical work (Ball & Bass, 2000b, 2003a; Ball, Lewis & Thames, 2008). For instance, Ball and Bass (2000b) identify three elements that connect the disciplinary construction of mathematical knowledge to the work of teaching: the reasoning of justification which provides a chain of supporting knowledge, linked logically, in ways that establish the validity of a claim and warrants conviction for oneself and others; the identification, use and ongoing development of a base of publicly shared knowledge to support reasoning; and the significance of mathematical language in reasoning, including definitions, notation, terms, and the syntax of their use. Together, these provide the mathematical underside of what is involved in facilitating mathematically reasoned discussions in classrooms and in supporting the development of mathematically reasoned knowledge in individual students and in the class.

Their analysis of practice contributed to an understanding of the composition and structure of mathematical knowledge for teaching. In synthesizing results from their analyses, Ball, Thames, and Phelps (2008) define mathematical knowledge for teaching (MKT) as mathematical knowledge needed to perform the recurrent tasks of teaching mathematics to students. The conception of knowledge is broad, including knowledge, skill, habits of mind, sensibilities, and so forth. And, by “tasks of teaching” or “work of teaching,” they mean everything teachers do to support the instruction of their students, in and outside of the classroom.

Ball et al. (2008) go on to report:

Perhaps most interesting to us has been evidence that teaching may require a specialized form of pure subject matter knowledge—“pure” because it is not mixed with knowledge of students or pedagogy and is thus distinct from the pedagogical content knowledge identified by Shulman and his colleagues; “specialized” because it is not needed or used in settings other than mathematics teaching. It is this formulation in terms of the work of teaching that makes this content knowledge special. (p. 24)

This research on mathematical knowledge for teaching has provided a foundation for two other supporting lines of work—the development of instruments for detecting, measuring, and categorizing mathematical knowledge for teaching and the development of tasks and materials for teaching such knowledge to teachers and other professionals who work with teachers in various settings. The development of large-scale measures of mathematical knowledge for teaching has included validation using cognitive interviews, studies of teaching practice, and links to student achievement. This work has provided evidence that there is mathematical knowledge specific to teaching and that it can be measured (Hill, Schilling & Ball, 2004). Furthermore, it has shown that a teacher’s mathematical knowledge for teaching is linked to the mathematical quality of
instruction (Learning Mathematics for Teaching, in press) and is a significant predictor of gains in student achievement (Hill, Rowan, & Ball, 2005).

In addition, these researchers have developed and evaluated approaches to helping teachers and other professionals learn mathematical knowledge for teaching. For example, they teach mathematics content and methods courses for teacher education students, facilitate MKT-focused professional development for practicing teachers, organize regional study groups with fellow mathematics teacher educators, and offer institutes for teacher educators, mathematicians, and other teacher developers. In doing this work, they have developed a wide range of tasks and materials that create opportunities for learning MKT. Designing MKT-focused materials has enabled us to better understand the nature of MKT, to learn more about how teachers develop this knowledge, and to consider the challenges of helping faculty develop the capacity to teach MKT to future teachers.

From this combination of work, they have developed a framework for categorizing mathematical knowledge for teaching that distinguishes four main domains: (1) common content knowledge; (2) specialized content knowledge; (3) knowledge of content and students; and (4) knowledge of content and teaching (Ball, Thames, & Phelps, 2008). Articulating domains of mathematical knowledge for teaching has made theoretical, empirical, and practical contributions to the problem of teachers’ mathematical knowledge. The domains extend and elaborate the idea of pedagogical content knowledge and enable specific aspects of teachers’ mathematical knowledge to be measured and linked to teaching practice and to student achievement. In addition, a better understanding and articulation of the mathematical knowledge needed for teaching helps to structure the design of teacher education materials that develop mathematical knowledge for teaching.

This program of work has been productive and has made significant contributions to the field. Despite its success, however, its approach to the study of practice for the purpose of identifying the mathematical demands for teaching remains largely underspecified. This shortcoming reflects both a larger problem in the field and a particular problem for the study of mathematical knowledge for teaching. And, as we shall see, an understanding of what might be done to study practice in order to identify mathematical knowledge for teaching can shed light on the options available for conceptualizing mathematical knowledge for teaching and some of their pros and cons.

1.2.3 The Problem of Studying Practice

Practice itself is complex, as is the study of practice (Lampert, 2001). The goal of studying mathematics in the context of practice presents an added challenge. In reviewing an
edited volume of papers that use the study of communication as a means to get at core features of classroom teaching and learning, Thames and Ball (2004) argue that mathematics often slips from view in analyses of practice.

The classroom interactions described in this book are replete with descriptions of the mathematics and mathematical practices. Still, the tools used to analyze the mathematics of teaching and learning seem still under development. As analyses proceed, the mathematics seems to slip out of view. In many ways, attention to a theory of learning is the culprit—in this book and in the field at large. A theory of learning, socio-cultural or otherwise, is less likely to hold onto the mathematics of teaching and learning. (p. 430)

What, then, does it mean, and what does it take, to study mathematics in practice or to conduct a mathematical analysis of practice? Synthesizing descriptions given in Ball (1999), in Ball and Bass (2000a, 2000b), and in Ball, Bass, and Hill (2004), I describe three central features of the aims and structure of the approach: (i) analysis of practice from a disciplinary perspective; (ii) analysis of teaching and learning from a pedagogical perspective; and (iii) multidisciplinary collaboration to meld these perspectives. I describe each of these in turn.

In general, the approach belongs within an interpretive tradition of research on teaching, where particular instances are probed in detail and generalization is established either logically or through comparison and testing with other instances (Ball, 1999, p. 33). Logical and comparative analyses are about establishing meaningful interpretations that are consistent and coherent. The initial inspiration for the work was to bring a mathematical eye to the observation of teaching and learning in order to gain new insight into mathematical issues—beyond topical descriptions of the curriculum, yet ones that have implications for the mathematics that teachers should know. Ball and Bass (2000b) offer a version of what this might mean by describing it in terms of Wineburg’s (1996) notion of an “applied epistemology”. They cite Wilson’s (2001) elaboration of the idea in the context of history teaching:

An applied epistemology would … require that the historian and the psychologist listen to classrooms in new ways. As they do so, their own understanding might continue to unfold, as the words of a child or discourse of a classroom send them back to history and psychology in search of resonant ideas. They might see in a child’s words central historical concepts like the role of human agency. (p. 195)

They describe their approach as being similar to Rose’s (1999) study of fourth graders learning history, where historical perspectives helped her see and hear how her students “came to
know the past and constructed meaning of historical events in ways that were much more rooted
in the nature of historical sense-making” than she had realized previously (p. 196).

They draw a parallel in their own work.

The children were encountering a central problem of
constructing mathematical knowledge in the discipline: the
challenge of proving that something is true when all cases cannot
be checked. That this can be done is a fundamental achievement
of mathematics, and these children were arriving at an
appreciation of the need to be able to do so. (p. 197)

Here, applied epistemology suggests that the disciplinary practice of deductive mathematical
justification offers a lens for viewing what children are doing and thinking about in trying to
establish general claims about numbers and that this lens helps make visible the deeply
mathematical nature of children’s talk and work in the classroom. As such, it is mathematical
knowledge, beyond the curriculum, that might be important for teachers to know and be able to
use when teaching.

Elsewhere, Ball (1999) points out that the goal of the research program is not to revise
the curriculum or make claims about what children ought to be taught. Instead, the point is that,
at least in a classroom committed to treating the discipline with integrity, to taking children’s
mathematical ideas seriously, and to seeing mathematical work as a collective endeavor, there is
much to be seen when viewed from a disciplinary perspective that is not typically noticed by
teachers, teacher education students, or education researchers and that these mathematical
interpretations of practice might contribute in important ways to what teachers are able to see and
do.

Further on, Ball and Bass (2000b) write:

Using a mathematical eye as a tool of the inquiry, we seek to
annotate and index mathematical issues that shape an account of
what is happening in the class. (p. 200)

What is involved in “using a mathematical eye” is, however, not specified. Indeed, Ball (1999, p.
20) writes that the genre of “mathematical commentary” developed for the work was purposely
not specified so as to remain open to generative possibilities that a disciplinary perspective might
bring to the observation of practice. If existing notions of the mathematical knowledge that
matters for teachers is off the mark, too much direction might keep the analysis from breaking the
new ground being sought.

The second and third features of the approach go hand in hand—to analyze teaching and
learning from the perspective of practice and to combine such analyses with mathematical
analyses. Both dimensions of the work are evident in the layered nature of the research question. The driving research question is about the mathematics teachers need to know, but this question is situated in a question about what it takes to teach mathematics, which is situated in a question about what teaching is. Ball and Bass (2000b) write:

Our work turns the usual approach to the study of teacher knowledge on its head. Instead of asking what do teachers need to know, and making lists of topics based on the curriculum for which teachers are responsible, we ask, “What is teaching, and what does it take to teach mathematics?” (p. 200)

Here, in the final formulation of the question, the reference to “mathematical knowledge” remains implicit. Central to the approach is the combining of the three related questions: one about mathematics for teaching, another about what teaching is, and a third about connecting them. These three are combined in the question of what, where, how, and to what ends mathematical resources might be useful in the teaching of mathematics.

In project publications on mathematical knowledge for teaching, little is said about the analysis of teaching. Ball and Bass (2000b) write:

In interaction with other members of the research group, our written and oral exchanges over the mathematical analyses enable us to meld mathematical issues to pedagogical patterns, issues, and events that we identify as intersecting with them. These emanate from analyses of teaching and learning from the perspective of practice. (p. 201)

The same two sentences occur in Ball (1999, p. 34) as well. For the study of teaching, footnotes reference Lampert (1998), Ball and Lampert (1999), and Ball (2000b). Similar comments are made in Ball, Hill, and Bass (2005), with references to Ball (1993) and Lampert, (2001). These writings describe the study of teaching as one of illuminating intricacies and, in the words of Ball and Lampert (1999), adding to the cacophony of pedagogical interpretation, rather than as one of establishing solutions or making normative claims. Synthesizing their own studies and those of others, Ball and Lampert (1999) observe that multiplicities along three dimensions—time, evidence, and interpretation—are useful in analyses of practice. It is important to be able to look at what came before or after an episode, to pause time, and to back up and reexamine events. In addition, having a variety of materials that represent practice provides different information and allows for richer exploration of a question. It also increases the likelihood of getting conflicting views, as one does in teaching, and of needing to confront and consider them as part of understanding teaching. Third, it is in the multiple interpretations around a common text that webs of complex insight contribute to views of teaching and learning.
This last point, about multiple interpretations, is the central methodological tenet of the study of practice argued for by Ball and Lampert. The combination of outside perspectives, both from disciplines such as mathematics, sociology, philosophy and so on, and from interested parties such as parents, policymakers, and others, as well as inside perspectives, from the teacher or the students who participated, can add to people’s capacity to see, understand, and even act.

Many of the questions, central as they are, are also fundamentally unknowable. … Yet investigation and careful analysis can finely map an issue in such a way that it can be seen and considered from more perspectives. Although this does not lead to answers, it does lead to improved understanding of the multiple considerations and interactions within any particular slice of teaching and learning. (p. 380)

They go on to point out that there is no algorithm for combining multiple perspectives into a more refined perspective and that, more often than not, these perspectives are not able to “talk” to one another. Given this, they argue for the development of border crossers and ambassadors to help with translation and for practices of knowing and norms of discourse that can facilitate communication about teaching within and across different communities. They conclude by identifying what an epistemology of practice would entail.

A discourse of practice needs discussable standards for validating different kinds of claims and for distinguishing better from less well-grounded assertions. Finally, and perhaps most complex, the study of practice requires methods for mediating across multiple claims. If “polyfocal conspectus” (Schwab, 1961) on teaching is to yield the richness of understanding it promises, then mechanisms are needed for reconciling or holding in interpretive complement different interpretations of teaching and learning. (p. 396)

Ball and Lampert argue for, and give examples of, multidisciplinary work, however, there is still much to understand about what it takes to analyze practice.

Ball (1999) gives a similar account of her collaboration with mathematician Hyman Bass in their joint effort to analyze practice in order to identify mathematical knowledge needed for teaching.

Dialogue and exchange are critical components of the work. Negotiating the different interpretations of and perspectives on what is going on will push and extend the plausible accounts of classroom episodes: the core analytic work of the project will be dialogic. Such discussions do not necessarily yield joint or single interpretations, but the interactions are a crucial medium of the work. The goal is to produce plausible analyses of
teaching and learning that interplay mathematical perspective with pedagogy, with an eye to expand the range of mathematical possibility that might be seen, heard, located, and, in turn, nurtured, in teaching and learning. (p. 31).

The characterization of engaging multiple perspectives in the analysis of practice and the nature of this engagement, both as a dialogical process and as one requiring people and roles of border crossing provides important insight into the process. However, questions remain about the systematic design of research that combines mathematical and pedagogical interpretations to produce plausible and useful analyses of practice. To address this need, Ball goes on to describe how the development of effective collaboration is made an explicit part of the study. She describes the extensive documentation of analytic memos, layered sets of annotations and comments, audiotapes and much more, and she describes three foci of this meta-analysis.

We focus on (1) the preservation and separation of individual perspectives; (2) the exploration and mediation of differences in ideas, perspectives, and interpretations; (3) the construction of collective and common knowledge. Ultimately, we would like to understand better what it takes to learn to do collaboration of this kind, and what the likely recurrent challenges as well as profits are. (pp. 32-33)

She concludes by underscoring that:

[T]his work also precipitates a need for new developments in method, in the cultivation of something we might be able to call “mathematical analyses” of teaching and learning. In complement to this, this research will need to continue to consider the development of methods for intertwining pedagogical and mathematical perspectives where the perspectives draw on different orientations and even different disciplinary training. (p. 36)

It is this set of questions about what is involved in studying practice — in order to investigate mathematical knowledge for teaching and in using and intertwining the perspectives of mathematics and teaching — that this dissertation seeks to illuminate.

From the description above, while important ideas about what is involved in studying practice for the purpose of identifying mathematical knowledge for teaching have been developed, there is more to understand about the design and conduct of such research. As Ball says, there is a need to understand better what is involved and it is this need that prompted the research group to create a paper trail of the work and to make the study of method an explicit part of the study. This dissertation is meant to contribute to that agenda.
CHAPTER 2: STUDYING THE STUDY OF PRACTICE

The aim of this dissertation is to explore the design of research that studies practice for the purpose of identifying mathematical knowledge for teaching — in particular, aspects of design that might be obscured on casual inspection. Are there aspects that are not readily apparent but that are playing a key role? What are the practices, conduct, substance, and conditions important for studying practice in order to identify mathematical knowledge for teaching? In this chapter, I clarify the research questions, describe the data used in addressing these questions, and outline the analyses reported in subsequent chapters.

2.1 Clarifying the Research Questions

The overarching research question for this study, then, is:

How can teaching practice itself be studied in order to identify common mathematical demands of teaching and, from that, useful hypotheses about the mathematical knowledge needed for practice?

And, this question provides the basis for three more focused questions addressed by this study.

1. What might it mean to analyze practice from a mathematical perspective in order to identify mathematical knowledge for teaching?

2. What might it mean to analyze practice from a pedagogical perspective in order to identify mathematical knowledge for teaching?

3. What might it mean to “intertwine pedagogical and mathematical perspectives” in an analysis of practice designed to identify mathematical knowledge for teaching?

To understand how practice might be studied in order to identify mathematical knowledge needed for practice, I propose to study the work of our research group at the University of Michigan. This dissertation, then, is not directly a study of mathematical knowledge for teaching, but a study of the study of mathematical knowledge for teaching. It is a study of the one group’s attempt to study the practice of teaching for the purpose of identifying mathematical knowledge useful to teaching, carried out with the aim of understanding what might
be involved in doing so, how it might be done, and what these imply about what it is. It analyzes documents produced by the group to understand better how the work was carried out, with the intention of informing what might be done in similar, future research of this kind and of informing the conceptual foundation of ideas about mathematical knowledge for teaching.

Figure 2.1. Levels of study informing an understanding of MKT.

The question is broadly about what is involved in such a study, about what it takes to study practice in a way that keeps an eye on mathematics and another eye on the work of teaching for which that mathematics might matter. The goal of this dissertation is to shed light on the undertaking of researchers at the University of Michigan in ways that would provide a more theorized description of how one might conduct a study of practice for the purpose of identifying mathematical knowledge and what such study of practice can contribute. And, this dissertation should support future efforts, by ourselves or others, to extend the study of mathematical knowledge for teaching to the secondary level, to explore content-knowledge demands in other countries and in other subject areas, and to produce a more complete picture of the mathematical knowledge needed for teaching at the elementary level. The goal is to illuminate the practical question of “what it takes” of “what is involved” without expecting definitive conclusions. In other words, the goal is not a matter of establishing a correct answer for a well-defined problem. Instead, this study explores, rather than settles its central question. Below, I discuss what I mean by this. In doing so, I mean to clarify the intent of this study and to defend my choice of research questions and my choice of studying the work of the research group of which I have been a member. I then address the challenges and pitfalls of conducting this study as insider research.
2.1.1 Research as Exploration

Suzanne Wilson’s dissertation (1988) is also an exploration in this sense. When Wilson wrote her dissertation, work on subject matter knowledge was in full swing. She was a member of the Knowledge Growth of Teaching project at Stanford and was armed with the categories of knowledge and the cycle of pedagogical reasoning that Shulman and the research group had identified. (See, for instance, Shulman (1997).) It seemed to Wilson, and to others, that there was something about the nature of subject matter knowledge for teaching that was distinctive and poorly understood. Wilson was interested in digging into its “nature.” She wanted to get a clearer sense of ways in which it was distinctive. She had a suspicion that a key feature of what made it distinctive was its use in teaching. For this reason, she asked two questions: What is the nature of historical understanding? What role does subject matter knowledge play in the pedagogical reasoning of U.S. history teachers?

Like Wilson, I feel there is something distinctive about the nature of the phenomenon — the practice-based study of mathematical knowledge for teaching — that is distinctive and poorly understood. Also like Wilson, I find myself in the midst of a productive line of work that has broken new ground and has already generated significant contributions to the field: a practice-based theory of mathematical knowledge for teaching, specific domains of that knowledge, and instruments for measuring them. Her research questions could be thought of as bringing together the two essential aspects that seemed to her to be interacting in practice. Broadly, she was asking about the nature of subject matter in teaching. In her more focused research questions, she first takes subject matter, but considers it as a form of reasoning, which might be a form more readily adapted to its use in teaching. Second, she considers pedagogical reasoning, but asks what role subject matter knowledge plays in it. My question about what is involved in the study of practice is also a broad question about the nature of an endeavor. And, the more focused research questions I gave above are meant to provide a similar sort of attention to interactive dynamics of subject matter knowledge and pedagogy, but in the context of mathematics and of research methods for the study of practice, rather than practice itself.

Based on a conception of good history teaching that involves teaching students about both the narratives of history and the nature of historical knowledge, Wilson explored qualitative aspects of historical understanding and relationships between those ways of knowing and instructional decisions teachers make as they plan instruction. To explore this, she interviewed ten teachers, noted significant passages, coded types of knowledge, wrote summaries, and then, from these results, developed “strategies that would let me grasp” the emerging differences (p. 104).
My approach will be similar. Instead of interviews, I will analyze documents. Many of these documents were produced with the explicit intent of documenting the process of the work and the interactions that constituted it. Others represent a kind of “record of practice” described by Ball and Cohen (1999). In this case, the “practice” is research instead of teaching, but the way in which these are documents are produced as part of doing the work is analogous to records produced in and for teaching that serve simultaneously to constitute the work and to document it. Likewise, the analyses I produce for this study are meant to provide strategies that would let me grasp more about the differences in ways that practice could be studied and differences in ways of intertwining mathematics and pedagogical perspectives in the analysis of practice.

Three results emerged from Wilson’s study. She produced a framework for considering historical understanding but grounded in the language of history instead of the generic language of psychology. This framework suggests that there are subject-specific ways in which to discuss forms of knowing. Second, the content knowledge of the teachers she interviewed is examined using this framework, shedding light on differences in their historical knowledge that would not be captured by traditional measures of recall, in particular in the degree to which their knowledge was differentiated, detailed, qualified, and integrated. Third, Wilson used the differences as a way to frame four patterns of pedagogical reasoning that surfaced in the subject matter knowledge of the teachers. While deep subject matter knowledge played an important role in how effectively participants could plan instruction, other forms of knowledge were also influential, including knowledge of teaching, learning, and learners. Participants with richer pedagogical knowledge were able to consider alternative strategies, had well-developed and detailed scripts for enacting pedagogy, and were more inclined to incorporate concerns for students — both their general characteristics and their prior knowledge and experiences — in their deliberations about teaching.

In these results, one can see that Wilson used the study less as a way to describe these teachers, or even teachers more generally, and more as a way to understand the nature of subject matter knowledge for teaching. My goal, too, will be about understanding the nature of studying practice as set out in the discussion above, not about producing a history of the group. In other words, the goal is to understand the analytic space of such work and to use this understanding to inform the future design of studies in this domain. The nature of claims will be about this analytic space and its implications for the design of research. For instance, given an examination of project documents, what can be said about choices that need to be made, about challenges to manage, and about aspects and issues that deserve focus? This is not the same as making claims about what was done or about the specific identity and conduct of this research group.
Part of getting a better grasp on what is involved has to do with understanding a basic divide between disciplinary knowledge (of mathematics) and professional knowledge (of teaching). As earlier discussions have suggested, a clue to addressing the problem of content knowledge for teaching lies in understanding that the knowledge teachers must know has got to be pertinent to what teachers do with mathematical knowledge in teaching — it has to be usable, actionable knowledge. Perhaps it would be a good thing, if whenever we want to claim that certain mathematics is important for teachers to know, we be required to make explicit where, when, and how teachers would use such knowledge in teaching, or at least where, when, and how it would inform or otherwise be relevant in some way. At least we should recognize that considering what teachers do with mathematical knowledge in teaching, and to what effect, might provide valuable guidance for figuring out what mathematical knowledge really matters. These issues suggest a fundamental challenge for efforts to identify mathematical knowledge for teaching — that disciplinary knowledge is one kind of thing and the practice of teaching is another.

Research on teacher content knowledge is built on disciplinary conceptions of knowledge, where knowledge is general, formal, and academic. These conceptions of knowledge shape teacher education and education policy. However, research on the knowledge that teachers seek out and use in practice suggests that such knowledge is personal, specific, and action oriented (Hubermann, 1983). It is this discrepancy between disciplinary knowledge of the academy and practical knowledge use in the profession that represents a fundamental challenge for the field and lies at the heart of understanding what is involved in studying practice to determine MKT. In order to illuminate “the territory to which Dewey called attention almost a century ago, bridging the divide between content and pedagogy,” Ball and Bass (2000a) point out that we need to understand better what makes mathematical knowledge usable for teaching? They write:

A second problem concerns how subject matter must be understood in order to be usable in teaching. We need to probe not just what teachers need to know, but to learn how that knowledge needs to be held and used in the course of teaching.

(p. 97)

Past studies of teachers’ knowledge-use suggest that disciplinary knowledge and research results are not readily picked up and deployed by teachers. One reason this process is not straightforward is because teachers do not just need to know mathematics: they need to be able to use it to do a particular kind of work. This helps to explain, in part, why just reviewing topics of the curriculum might not actually get at the things teachers need to know (and have to do). What
does it take, then, to take something that already has an independent structure and figure out what its structure is closer to the work?

2.1.2 Researching Research from the Inside

To address these issues, I begin with an exploratory analysis of what the University of Michigan research group, of which I was a member, was doing and learning. I then move to a more focused analysis of the theoretical perspectives we were using and then end with an analysis of different ways we coordinated these perspectives. The empirical basis for these analyses is a collection of roughly 1500 electronic documents. An overview of these documents and of the methods used to analyze them is given in the next chapter.

This study, then, is an examination of research from the inside, from a first-person perspective, analogous to studying teaching from the inside as described by Ball (2000b). In this case, it is a researcher of mathematical knowledge for teaching (myself) who is also the principal investigator of the study of the study. This first-person research blends the construction of research with its analysis. And, adapting Lampert’s 1986 comments about first-person teacher research (p. 306) to the context of this study, the research reported here is not oriented toward creating a single, prescriptive research design for studying practice for the purpose of identifying mathematical knowledge for teaching. Instead, it develops a working proposal and a set of critical issues for the development of methods and for the development of alternative designs — and is grounded in an investigation of an actual instance of research conducted in this vein.

In considering the special value of, and the potential pitfalls of, first-person perspective in research on teaching, Ball (2000b) identifies three issues: (i) research questions: connecting problem and perspective (ii) scholarly stance: composing distance and insight; and (iii) claims: navigating the general and particular. The first is about asking, for a given research question, whether and how the first-person perspective can play a role. The second grows out of the ways in which teaching is relational and about persons, and therefore calls for scholarship that uses the personal as a resource to gain insights not accessible to the outsider; with this, however, come a need for distance that can allow for the conversation between researcher and phenomenon that makes it possible to excavate, name, and analyze aspects of experience unseen by the outsider. The third asks what the claims of the research are about and whether they effectively move beyond a retelling of a personal story, yet do so without inflating the generality and applicability of the claim. Modifying these three issues to the context of this study, I use them to explain and to cross-examine my first-person analysis of our University of Michigan research project.

Connecting the Research Problem and My Insider Perspective. For the first issue, Ball formulates three crucial questions. Does the phenomenon in which the researcher is interested
exist elsewhere? Second, is what the researcher wants to know uniquely accessible from the inside or would an outsider be able to access this issue as well, or perhaps better? Last, is the question at hand one in which other scholars have an interest, or should have an interest, and if so, will probing the inside of a particular design offer perspectives crucial to a larger discourse? (2000b, p. 391)

The University of Michigan group has pioneered efforts to analyze practice for the purpose of identifying mathematical knowledge for teaching. Other scholars have engaged in related research, in part built on the research of the university of Michigan group, but with somewhat different agendas. For instance, Adler and Davis (2006) have analyzed teacher education programs in South Africa and Davis and Simmt (2006) have analyzed professional development activities in Canada. (The first is Z. Davis, and the second is B. Davis.) The agenda in both of these studies is to explore the nature of “unpacked” mathematics for teaching that seems to arise in the context of teacher learning of mathematics when that learning is grounded in thinking about the teaching of mathematics. Instead of analyzing classroom teaching, these researchers analyze teachers’ engagement with mathematical knowledge for teaching in the context of its being taught to them. These studies focus on understanding why mathematics for teaching is not readily or effectively taught in teacher education programs (Adler and Davis) and on the qualities that make it different from the mathematics children are expected to learn (Davis and Simmt). In more recent work, Alder (in press) writes about methods her group is developing for studying practice with the purpose of understanding mathematics for teaching, but little of this work has been published yet, and for this reason I do not review this work in this dissertation.

Another example of related research is a study by Rowland, Huckstep and Thwaites (2005) designed to examine how and where subject matter knowledge and pedagogical content knowledge contribute to teaching and to use this analysis to generate a framework that would help focus discussions among teacher educators, student teachers, and mentor teachers on the use of mathematical knowledge in teaching. Their research involves analysis of student teachers’ classroom teaching and a grounded theory approach to generating codes for:

… aspects of trainees’ actions in the classroom that seemed to be significant in the limited sense that it could be construed to be informed by a trainee’s mathematics content knowledge or their mathematical pedagogical knowledge. (p. 258)

These codes are then organized into four groups: a first of which they argue is indicative of propositional knowledge of, and beliefs about, mathematics and three other groups of codes that
“refer to ways and contexts in which knowledge is brought to bear on the preparation and conduct of teaching” (p. 261).

These three studies could be seen as existing cases that might be studied. However, there would be some practical challenges to doing so: all three are conducted by researchers outside of the United States (South Africa, Canada, and Great Britain); all three are temporal in nature (conducted in the past and with indefinite futures); and none of the three systematically documented the research activity for the purpose of studying its conduct. Moreover, my central question is about what is involved in examining practice for the purpose of identifying the mathematical knowledge for teaching that has seemed to be largely missing from programs responsible for the mathematical education of teachers and from assessments used to evaluate effects of teachers’ mathematical knowledge on children’s learning. As such, the agendas of these other researchers are not as well suited as is the agenda of the research group at the University of Michigan.

In addition to the issue of whether there exists a different research effort that I could have studied is the issue of the practicality, advantages and disadvantages of studying such an effort from the inside or from the outside. The broad goal of this study, as described earlier in this chapter, is to contribute to the conceptual and operational base of research on mathematical knowledge for teaching. It is a goal of probing the tensions in research designed to intertwine disciplinary knowledge of mathematics with the complex and messy practice of teaching. I am arguing that the research we were conducting offers an example to be scrutinized about the ways of intertwining these disparate elements. For instance, later I will argue that researching practice for the purpose of identifying mathematical knowledge for teaching is aided by the use of analytic tools that have, built into them, ways of coordinating mathematical and pedagogical perspectives. Goals such as these might be addressed by an outsider, with the development of an appropriate research design, but knowledge of the content and of the intent of the work are likely to be important resources for such study. The immediate implication of this is that being an insider, while having potential pitfalls, may indeed offer important advantages that should not too readily be dismissed.

As for whether the research question I am posing is of interest to other scholars, I have argued throughout this chapter that it is — that other scholars want insight into bridging the discipline of mathematics and the practice of the classroom in ways that provide an articulation of mathematical knowledge for teaching believed to be important for improving teaching and

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2 This is a prospect to which I will return at then end of this study, where I propose that the results of this study might well be used to frame an examination of these other efforts.
learning but elusive in past attempts to identify, measure and change it. My goal with this study is, by probing the inside of a particular effort, to provide conceptual and analytic tools for building such a bridge.

_Composing Distance and Insight._ A second challenge that Ball identifies for first-person research has to do with maintaining and using the self in the study while creating sufficient distance between the researcher and the context and problems of the study.

Because research methodology is so often fraught with imperatives to separate the self from the inquiry, we need to develop disciplined methods that deliberately use the self as a tool to construct insights, perspectives, and knowledge that expand our capacity to know (Krieger, 1991). At the same time we need to guard against the tendency toward the personal on the basis of some kind of basic appeal or, worse, naïve ideas about what constitutes knowledge. That it is asserted from the first-person perspective cannot make it automatically true. (p. 13)

She argues that first-person researchers must be able to view the object of study in the context of the experience, yet apart from their efforts and desires as one researching that experience and that this requires both an unusual concentration on, and use of, self, combined with an almost unnatural suspension of the personal.

I have dealt with this in several ways. To create more of a distinction, or boundary, between myself as a participant in the research and myself as a researcher of the research, I have focused the analysis on research conducted, and documents produced, in the past. Limiting the analysis to research conducted done from 1997 to 2004 created a buffer of three or four years between my current analysis and my participation in the research being studied. In addition, I wrote initial, best-attempt answers to the research questions at the outset as a way to put a boundary between the perceptions of the Mark who participated in the research and those of the Mark conducting this study. I then revisited these answers periodically to check that what I was writing about was distinct from the understanding I brought to the study.

The deliberate use of pronouns also served as a tool for creating distance and insight. In subsequent chapters, I use third-person pronouns for statements meant to represent the Mark1 and the research group in which he participated and first-person pronouns to represent the Mark2, several years later, who is conducting an analysis of documents produced by the project and who knows the experiences intimately but who has increased conceptual distance from them. In other words, I refer to the University of Michigan researchers as “they” or “them.” This may seem

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3 In a few instances, references are given to more recent publications, but only when those references represented research conducted earlier and offered clearer and more authoritative expressions of the research conducted.
awkward at times and may seem misleading, or even deceptive, hiding the fact that I am referring to a group of which I was a member and may even be referring to myself in particular. It is not meant to be deceptive or to claim a false distance. Instead, it is a device for creating distance — for aiding me in separating myself as a researcher of the research from myself as someone participating in the earlier research with investment in the positions and roles of that earlier time. For instance, in making a claim, the simple task of deciding between first and third person has helped me systematically question which hat I am wearing in making the claim. It has provided me with a disciplined way of asking whether my experience as a participant about what the researchers were doing is the basis for the claim or my current analysis is the basis for the claim. If it is the latter, what is the evidence for it and what parts of that evidence are to be attributed to the insider (what are these and what is the basis for them) and what to the outsider (what are these and what is the basis for them)? The pronouns have helped me create the conversation with myself that makes it possible to excavate, name, and analyze aspects of experience unseen by one who is only an insider or only an outside.

Navigating the General and Particular Nature of Claims. Beyond the usual concerns about the generality, or reach, of claims made in research, first-person research raises additional challenges because the fusion of researcher of research and research participant can make it harder still to determine the location of claims. The question for this current study is: Are the claims made here about what the University of Michigan research group did, are they about researching mathematical knowledge more generally, or are they about the analysis of practice for this purpose more generally?

The research question posed in the funding proposal to the Spencer Foundation in 1997 was: What is entailed in collaboration across the divide of mathematics and educational research? And, the stated aim was to make the collaborative inquiry itself also an object of inquiry and analysis (97SpPrp, p. 11). Likewise, the research questions posed above are not specific to the activities of the University of Michigan research group, but are about the broader challenge of researching practice for the purpose of identifying mathematical knowledge for teaching and the broader challenge of coordinating mathematical and pedagogical perspectives in such work. My claims are not intended to be about what to do, but about the nature of using practice for such a purpose and of coordinating perspectives in doing so. It is my intent to make claims about what is involved in, and to provide abstract models for, researching mathematical knowledge through the analysis of practice. Whether I am successful in doing so is matter to be judged based on the overall substance and conclusions of this study. Hence, I return to this issue in the conclusion of the study, arguing there that the claims made are not limited to the University of Michigan.
research group, that the products of the study have a broader applicability to other work being
done in the field, and that what this study illuminates, the questions it raises, and the things it
prompts us to think about go beyond the particulars of data and context in which it is placed.

2.2 Overview of the Data and the Selection of Samples

The research I analyze spans a seven-year period from 1997 through 2004. It was
carried out by a changing group of roughly 3 to 12 people at any one time, with a total
involvement of about 4 faculty, 8 graduate students and post-docs, 3 undergraduates, and 4
visiting scholars. (These numbers do not include occasional visitors.)

Figure 2.2. Participation from 1996 to 2005 for different members of the group.

Soon after the project started, the work of the research group was organized around
weekly three-hour project meetings with additional individual and small group work. For this
period, the roughly 1500 available documents are of several different types, falling into three
broad categories.

- Planning documents

\[4\] A few documents produced in 1996 or since 2004 are included, but the majority of the work analyzed was
conducted during this seven-year span.
(e.g., proposals, work plans, and planning memos)

- Primary working documents
  (e.g., analyses, analytic exercises, analytic memos, and meeting notes)

- Products and uses
  (e.g., reports, papers, presentations, and materials for courses)

These three categories represent different stages of the research process—beginning, middle, and end. As such, they offer three different views on the research being done, each with a different emphasis.

Planning documents provide a view of initial questions, assumptions, and plans for the work. They represent insiders’ views of what is to be done. Primary working documents represent the work being done. They provide a view of the actual methods enacted by the researchers and of initial results. They can also be analyzed for implicit and explicit versions of the questions and assumptions, but their central contribution is the window they offer onto the actions taken to analyze practice and onto the yield of those actions. Planning documents and products contribute perspective on the work, but it is the working documents themselves that are closest to the interactional conduct of the group.

The third category, products and uses, consists of documents whose purpose is to represent for others the lessons learned and to try out the ideas in the service of improving teaching and learning. For academic research, reports and papers are the primary venues for communicating about the work. However, throughout the project, the research group regularly used the ideas generated to inform their teaching of methods courses for prospective elementary school teachers. Also distinctive were numerous presentations and the development of instruments as testing grounds for ideas. The products again offer an insider’s view of the work, but from the vantage of extended analysis and experience.

In doing the research, initial documents were often distributed to other group members, who wrote reactions and comments, often as colored text or comments in the original document, such as in the figure below.
Figure 2.3. An example of layers of comments and reactions developed in the analysis of the mathematics of a classroom episode.

In this excerpt, a member of the group writes about what he thinks a student, Amare, is thinking. His original text is black (darkest). Another member of the group responds in blue (second darkest), asking for clarification and questioning his interpretation. The original commentator, in turn, replies in magenta (third darkest), and the same responder replies again in green (lightest).

There might then be several layers of such back and forth exchange in a single document, punctuated then by new reversions of the original document and possibly more layers of exchange. Thus, to capture the interactive nature of the research better, instead of selecting single documents, I have selected, when possible, small collections of documents that contain the most extensive interactions and that seemed to bear most directly on the focus of study. These collections include different drafts of the same document, files with reactions and comments from different people, or related files written by different people. Each small collection typically includes documents written at about the same time, though a few are not. In all cases, the collection is part of a single chunk of work, focused around a coherent research task. I refer to these small collections of documents as “episodes” to highlight their unity.
For example, in the following set of files, developed in preparation for a presentation, Def 1a and Def 1b are the same file, but one has responses from one member of the group and the other has responses from another. The same is true for Benny 1a and Benny 1b as well. The work on definitions, begun in Def1a and Def1b, includes a second draft, Def 2. The three strands are then combined into talking notes for the presentation in Final.

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Figure 2.4. An example of a collection of files produced in preparation for a presentation.
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The selection of several documents around a single activity provides greater insight into the activity of the group than would the selection of a single document. The clump of documents includes back and forth exchanges among researchers about what is being done and captures changes being made. Selecting a variety of research episodes from different points in time allows for sampling of different lines of work and provides insight into the ways in which the production of different types of documents was different and interacted with and contributed to the production of other documents and to comprising the research being done.

The focus here in selecting documents, though, is not chronological: instead, it is analytic. Different lines of work may represent different approaches taken at different times, or may represent, contemporaneous, yet as distinct efforts. Shifts are typically temporal—they have a narrative and represent learning and improvement—but this dissertation is concerned with the approach or approaches taken by the group and if multiple, the analytic differences among them. I am concerned with what it took to carry out the different approaches and with how their yield was different. Thus, sampling has often focused on analytically distinct approaches and not specifically on chronology. Of course, these two variables are closely related. This study, however, will not be adequate to separate these two variables. The focus will be on analytic differences in the approaches, and time will be relegated to the larger context used in interpreting the approaches.
Another feature that shaped the interactive character of the research was the people involved. The project was centrally about crossing professional boundaries and coordinating multiple perspectives. In part, this was accomplished through the engagement of a variety of people with different backgrounds. One key difference was their experience in mathematics and in elementary school classrooms. Other differences included their professional background, research experience, and grasp of the project. Project members included the PIs, who were senior faculty, but also junior faculty, graduate students, and several undergraduates.

To understand the nature, conduct, and yield of the work requires both an examination of the work overall, as well as an examination of distinct lines of work that characterize different initiatives taken along the way. The selected documents to be analyzed need to afford a view of the data that will illuminate characterizations both of the broad scope of work and more specific activities. Several factors, then, are important for sampling episodes and documents: type of document, date, visibility of interactions among researchers, relevance to the focus of study, and people.

Still, however, it is not clear how best to sample the data in ways that are disciplined but not overly constrained. The approach I took is twofold. I used the factors identified above to establish a systematic sample for analysis and I also moved purposefully across the data, browsing episodes and documents with an eye for those that seemed promising. In other words, I pre-selected a basic set of documents to be analyzed, but I also explored the data opportunistically. These two activities — analysis of the pre-selected sample and opportunistic analyses — were carried out in parallel. And, this two-fold approach was used for each of the analyses carried out in this study.

Listed chronologically, the basic set of sample episodes includes the following.

1. Early planning documents that lay out what is to be done (1996ss)
2. Analytic commentaries written in preparation for a conference sponsored by EDC (1997s)
3. Initial Spencer proposal and response to questions (1997sf)
4. A set of project meeting notes selected because they seem to address MKT (1997-2002)
5. An early assignment to write an episode for one of the themes of the book (1998s)
6. A collection of memos written to guide the writing of a book on the work of teaching at the beginning of the year (1998f)
7. Teaching and problem solving memos (1999f)
8. Proposal to NSF for continuation of the work (2001s)

5 These episodes were pre-selected after an initial stage of perusing documents in the data set and contemplating approaches to selecting, compressing and analyzing them for this study.
10. The framing of mathematical knowledge for teaching given to students at the Teaching Future Teachers workshop (2002w)
11. Text written for a chapter analyzing the tasks used at the beginning of the year (2002s)
13. Two research presentations given at California State University Northridge (2004)
14. Final report written to NSF (2004f)

Each of these selections offers an important window on the work. Below, I group these into the three categories described earlier and explain the way in which each represents a case of something and represents the larger body of work.

The planning documents were written primarily by the principal investigator, but they also include reactions from others.

- Early planning documents that lay out what is to be done (1996ss)
- Initial Spencer proposal and response to questions (1997sf)
- Proposal to NSF to continue the work (2001s)

The documents in the first bullet are some of the earliest records available. They provide an initial plan for the work and include extensive reaction from a research mathematician enlisted to help carry out the work. This initial work plan was a precursor to a funding proposal written the following year. The proposal provides an extensive articulation of the work to be done. Questions from reviewers, and responses to those questions touch on some of the subtleties and challenges of the work. Four years later, a second proposal was written to continue the work.

The work was conducted in different ways over time. Early on, the effort involved only a few people (3 to 4) and was mostly conducted via email, text documents, and informal conversations. Later, the group expanded to a dozen people or more and had weekly project meetings. In later stages, it was carried out in more fragmented ways, among a smaller group or 4 to 6 people. The working documents selected vary in nature, but they represent intermediate products of the work and they capture interactions among project members engaged in the work.

- A set of project meeting notes selected because they seem to address MKT (1997-2002)
- Analytic commentaries written in preparation for a conference sponsored by EDC (1997s)
- An early assignment to write an episode for one of the themes of the book (1998s)
- A collection of memos written to guide the writing of a book on the work of teaching at the beginning of the year (1998f)
- Teaching and problem solving memos (1999f)
- Beginning book re-focusing memos (2001f)
Project meeting notes are uneven, in content and completeness, but project meetings were a key site for conducting the work and they offer a potentially useful window on interactions among project members about the content and conduct of the work. Early work focused on writing mathematical commentaries about the mathematics arising in classrooms, and such commentaries, on a smaller, more informal scale, continued to be central to the work. The second set of documents is the most extensive and well-documented collection, with layers of comments and revisions. As the project shifted in focus, included more researchers, and invested in collective analyses of practice, the group used “homework assignments” to structure and to advance its efforts. The third bullet above is one such assignment. It is included because it includes direct contributions of a large number of people and because it involves writing about an episode from the classroom. As such, it provides an interesting contrast with other written descriptions of practice produced for the project, but does so across a wider group of people.

Another type of document produced by and for the project is referred to as project memos, mostly written by the principal investigator in an effort to capture progress of the group, their ongoing ideas and insights, and to orient and focus people’s efforts. The memos selected were ones that became important for the group over time. They occur at three different points in time and were referred to and used by the group years after their production. The first set listed above were written to synthesize past analyses and guide the writing of a book on the mathematical work of teaching at the beginning of the year. The second set, written a year later, address two ideas that became central to the group. The third were written two years later as the group geared up for a second push to complete the book. (The book remains unfinished, but may yet be revived.)

These documents include the most important and repeatedly used working documents of the project, they include samples from each of the major types of working documents (meeting notes, written analyses, analytic exercises, and conceptual memos), and they provide contributions from a wide range of people associated with the project. The first three involve analytic work with the data of the project and the last three discuss conceptual frameworks used in those analyses. Together, they shed light on central activities of the endeavor.

The third category is products of the project. These documents report on the work accomplished by the group, both results of analyses and characterizations of what was done.

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6 The memos written to guide the writing of the book might be thought of as planning documents because they provide a kind of proposal for work to be done, but their content and purpose is more fully about synthesizing work being done at the time and using it to generate conceptual frameworks that could be used to continue the work. They also might be thought of as products of the work, but I have included them here because they were produced for internal use and represent intermediate products in ways that are similar to other working documents in this category.
They report on analyses of practice, but they do so in the context of moving local ideas with local meanings to a broader, public audience, either communicating it as research or using it in practice. In doing so, they provide a retrospective on the work and contribute to an understanding of what different approaches to the problem yielded.

- The framing of mathematical knowledge for teaching given to students at the Teaching Future Teachers workshop (2002w)
- Text written for a chapter analyzing the tasks used at the beginning of the year (2002)
- Excerpts from a paper articulating a practice-based theory of mathematical knowledge for teaching (2003)
- Two research presentations given at California State University Northridge (2004)
- Final report to NSF (2004)

The project produced numerous drafts of papers and chapters, many of which have not, or have not yet, been adequately developed for publication. From among these, the chapter on task analysis was chosen for three reasons. First, it represents a theme — task analysis — that runs through much of the group’s work but is not represented by other selected documents. It is an important one, and it contrasts themes already represented, ones that focus more directly on classroom talk. Second, it was written by the mathematician who was the primary author of the mathematical commentaries included above. The mathematical commentaries were developed early in the work. Like the commentaries, this chapter includes detailed analysis of classroom interactions, but in contrast, it was written significantly later and uses a different approach to analysis. Third, it includes significant comments and responses.

The paper chosen is the one that most closely addresses the issues of interest to this study. Other papers address the issues, but in the context of other topics, such as equity, mathematical reasoning, and specific aspects of the work teachers need to do in order to teach effectively. This paper includes analyses of practice, but also more extensive discussions about the research endeavor and the use and coordination of mathematical and pedagogical perspectives.

In addition to papers, presentations were a central product of the project. This is described and defended in proposals and reports to funding agencies. The two presentations chosen here were selected because they address a strand of the work that is otherwise missing from selected documents. In particular, research focused on identifying mathematical tasks of teaching was done more informally and has less documentation. Presentations were a primary site for developing and communicating this line of work. One of the presentations was selected because it was an early presentation of examples of mathematical tasks of teaching and slides.
from it were used repeatedly in later presentations. Unfortunately, there are few documents that capture the exchanges around planning presentations and there is no recording of this particular presentation. This limits the analysis to the slides only. The second presentation selected was recorded, so it offers a more extensive representation of the product. It was selected because it was recorded and it presents the notion of mathematical tasks of teaching.

Less prominent products of the project include reports written to funding agencies. I selected the final report to the National Science Foundation because it occurs late in the work and is more likely to include ideas not apparent in other documents. Reports written to the Spencer Foundation were more extensive than those to the National Science Foundation, but ideas expressed in those reports show up in other documents.

As stated earlier, traditional academic publications were not the only outlet for the ideas developed by the group. In addition to presentations given to a variety of different audiences, teachers, mathematicians, policy makers, and researchers, the group used ideas from the work in several other activities. Three of these “uses” of the products of the group seem significant: (i) in the mathematical education of teachers; (ii) in study groups and workshops with mathematicians and mathematics teacher educators for use in mathematics content courses with teachers; and (iii) in the development of instruments for measuring mathematical knowledge for teaching. For this study, I selected part of a lesson designed to orient prospective teachers to an approach to their mathematical education that was based on the notion of mathematical knowledge for teaching. This lesson was taught to a lab class that was situated in a larger summer institute designed to engage professionals engaged in the mathematical education of teachers. Thus, complementing the lesson segment from the lab class, are discussions of this segment that were conducted with these professionals.

The selected episodes sample from across types of documents and types of research activities. They sample from across the time span, and consist of about 30-40 documents, or roughly about 200 pages of text. The selected episodes sample from a wide range of people; they also focus on key people (in particular the two faculty members responsible for the work). For instance, the analytic assignment selected includes the analysis and writing of nine project members (two faculty, one post-doc, four graduate students, and one undergraduate). Together these documents provide many different windows into the work, with consideration for the relevant factors given above: type of document, date, visibility of interactions among researchers, relevance to the focus of study, and people.
2.3 Overview of the Analyses

The object of this study is research itself. Research is an investigation directed to the discovery of insight and the production of new knowledge. Synthesizing my review of numerous textbooks on the conduct of research, I suggest that it is a process of formulating research questions, gathering data, analyzing the data, and generating and supporting claims. It also seems to be about harnessing curiosity in an effort to defy the paradox of learning something new (put by Fodor (1975) as: “[i]f the mechanism of concept learning is the projection and confirmation of hypotheses (and what else could it be), then there is a sense in which there is no such thing as learning a new concept” (p. 95)). So, research seeks to find the unexpected and to see anew.

To develop a design that studies practice for the purpose of identifying mathematical knowledge for teaching, I first consider macro level components of the research: What might the research questions be? What data might be analyzed, using what methods? What results are likely to be generated? And how are these questions, data, methods, and results related? Chapter 3 distinguishes three different approaches to studying practice for the purpose of identifying mathematical knowledge for teaching. I use documents produced by the research group at the University of Michigan to create images of what might be involved in each approach and to argue that each approach constitutes a coherent package of related questions, data, methods, and results. These three approaches emerged from an examination of these documents, but my argument is that these approaches represent distinct ways of going about the analysis of practice, each with its own advantages and disadvantages. Chapter 4 considers what is involved in taking either a mathematical or pedagogical perspective in the analysis of practice and in coordinating these perspectives. The conclusion synthesizes chapters 3 and 4 and considers implications.
CHAPTER 3: THREE BROAD APPROACHES TO ANALYZING PRACTICE

Research is a process of formulating questions, systematically collecting and analyzing relevant data, and interpreting the results, which then provide new knowledge. These features—research questions, data, methods, and findings—give a basic structure for analyzing a research endeavor. This analysis proceeded iteratively, as described by Erickson (1985), working back and forth between hypotheses about these features and available records. The analysis reveals three analytically distinct approaches to the research. Again, while these represent shifts occurring in the work over time, in positing them, I am not making historical claims as much as analytic claims. The fact that they occurred as three successive phases lends credence to the claim that they are distinct approaches, but I am not claiming that they need to be temporally distinct or that the order or progression is critical.

1. The first approach investigates the mathematics entailed in classroom teaching and learning. This approach seeks to identify and name the mathematical issues evident in classroom interactions.

2. The second approach investigates teaching and its mathematical demands. This approach examines the work of teaching and the mathematical resources needed to do this work.

3. The third approach identifies work of teaching that is distinctively mathematical work. This approach identifies mathematical tasks of teaching that are specifically mathematical in nature and constitute a form of mathematical problem solving entailed in teaching.

Each of these approaches conceives of the problem in a different way, leading to different formulations of the question to be studied and different methods for conducting the research. Each makes certain assumptions and yields particular kinds of results. The figure below gives an overview of the three approaches.
<table>
<thead>
<tr>
<th>Questions</th>
<th>Approach 1</th>
<th>Approach 2</th>
<th>Approach 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>#1: What kinds of mathematical knowledge are entailed in teaching?</td>
<td>The Mathematics Entailed in Classroom Teaching and Learning</td>
<td>The Work of Teaching That is Distinctively Mathematical Work</td>
<td></td>
</tr>
<tr>
<td>(97SpPrp, p. 5)</td>
<td>#1: What mathematical knowledge is entailed by the work of teaching mathematics?</td>
<td>#1: What is the mathematical work of teaching? What are mathematical tasks and sub-tasks of that work?</td>
<td></td>
</tr>
<tr>
<td>#2: What is entailed in collaboration across the divide of education</td>
<td>#2: Where and how is mathematical knowledge entailed, and how would it be useful and used within the work of</td>
<td>(04PrsCSUN)</td>
<td></td>
</tr>
<tr>
<td>research and the discipline?</td>
<td>teaching?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(97SpPrp, p. 11)</td>
<td>How is it intertwined with other knowledge and sensibilities in the course of that work?</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Data</td>
<td>Primary Data</td>
<td>Primary Data</td>
<td></td>
</tr>
<tr>
<td>1989-1990 third-grade records of practice (with focus on excerpts from</td>
<td>1989-1990 third-grade records of practice (with focus on the beginning of the year and on relationships</td>
<td>Records of practice from Ball’s 1989-1990 third-grade class and from ten 2nd-8th-grade teachers collected nine</td>
<td></td>
</tr>
<tr>
<td>individual lessons, mostly from the middle of the year)</td>
<td>between different periods in the year)</td>
<td>times from 2003-2004 (individual lessons) (Ball, Bass, &amp; Hill, 2004)</td>
<td></td>
</tr>
<tr>
<td>Secondary Data</td>
<td>Also, a limited number of records of practice from other classrooms (001028BkMem)</td>
<td>Secondary Data</td>
<td></td>
</tr>
<tr>
<td>Mathematical commentaries written for segments from the primary data</td>
<td></td>
<td>Materials designed for prospective teachers and their work with the materials; and items developed to measure</td>
<td></td>
</tr>
<tr>
<td>(97SpPrp, pp. 9-10)</td>
<td></td>
<td>MKT and feedback on the items (04PrpROLE)</td>
<td></td>
</tr>
<tr>
<td>Method</td>
<td>Interdisciplinary observation of teaching as developed in earlier work</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Interdisciplinary work to develop method grounded both in practice and</td>
<td>Two-stage analysis:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>in mathematical lenses and structures (97SpPrp, p. 11)</td>
<td>(i) analyze mathematics in primary data and write mathematical commentaries</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(ii) analyze patterns and themes in commentaries for mathematics that is pedagogically relevant</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(97SpPrp, p. 9-10)</td>
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</tbody>
</table>

Figure 3.1. Questions, data, and methods for three approaches to studying the mathematical knowledge that might matter for teaching.
Figure 3.2. Results for three approaches to studying the mathematical knowledge that might matter for teaching.

All three approaches are concerned with identifying mathematical knowledge that might matter for teaching, but they frame the question in slightly different ways, and this difference has implications for how the work is done and what is learned. I describe each of the three approaches and provide a detailed illustration of each.

3.1 Approach 1: The Mathematics of Classrooms

In the first approach, the pursuit seems to be one of looking at classroom interactions with a mathematical eye, in the hopes that what emerges might inform notions of the mathematics that would be important for teachers to know. Often, debates about the mathematics courses elementary school teachers should be required to take are argued logically or theoretically, with little attention to when, where, and how such knowledge might be pertinent (Thames, 2006, p. 2). By studying the mathematics that percolates up from analyses of classrooms, the first approach grounds such debates empirically. The approach consists of writing mathematical commentaries...
from examination of practice and then probing the commentaries for what they might reveal about mathematics that might be useful in teaching. In an early proposal, the researchers write:

[We] would analyze the mathematics of a collection of the lessons, as evident in the videotapes, the children's work, the teacher's notes, and the interviews. We would construct what we call "mathematical commentaries" on the class work. … The analytic commentaries then offer a field of material for analysis of the mathematical entailments of teaching. This is the next order of analysis in this research: to analyze the mathematical analyses generated from close examination of the primary source materials. They are the secondary data to be examined to develop better maps of the role and nature of mathematics entailed in teaching elementary mathematics. (97SpPrp, p. 10)

The approach seems to use mathematicians and their unique disciplinary perspective to explore the mathematics needed for teaching by having them examine classrooms and by then analyzing their characterizations of mathematics in order to identify new ways to conceptualize and catalog the mathematics that would be relevant for teaching. For Ball (1999), the choice of mathematicians is deliberate, and their contribution is intended to suggest new insights into mathematical knowledge for teaching. As Ball (1999) writes:

The principle here is that signs of the children’s and teacher’s mathematical activity may appear differently to someone who is engaged in the practice and may have a special eye for mathematical mind, insight, process, or flight of activity. (p. 31)

She goes on to explain that:

… any group of people could be chosen as collaborators for such joint work, and much of it could be illuminating to analyses of teaching and learning. But mathematicians who, like others, have untrained eyes for elementary school classrooms and elementary school pedagogy, bring eyes that are highly trained for mathematics. Thus, as practitioners of the field, they are uniquely situated to probe mathematics as its seems to be entailed in elementary school teaching. (p. 31)

Although the approach starts with open-ended writing of mathematical issues suggested by mathematicians’ observation of practice, these are then analyzed in a secondary analysis in order to understand the mathematical knowledge that might be relevant to the problems teachers confront in teaching. In the funding proposal mentioned above, the researchers write:

The goal is to produce plausible analyses of teaching and learning that interplay mathematical perspective with pedagogy, with an eye to expand the range of mathematical possibility that
might be seen, heard, located, and, in turn, nurtured, in teaching and learning. (97SpRsp, p. 8)

In contrast to the other approaches, this first approach uses mathematical perspective to suggest potentially new ideas about the mathematics that might matter for teaching, then uses a pedagogical perspective to judge the pertinence of those ideas. Looking across documents produced early in the life of the project, questions used to examine practice have a broad focus: What is happening, mathematically, in this classroom? Looking with a mathematical eye, what is there to notice? By looking at records of practice, the approach satisfies the basic notion of staying connected to the classroom, but it enlists mathematicians to look at classrooms and to put on the table anything and everything of a mathematical nature that comes to mind (Ball, 1999, p. 20). These observations can be examined for patterns and themes, and a pedagogical perspective can be brought to bear to pull out observations that seem relevant to teaching and to probe them in ways that bring relevant mathematical knowledge and skill to the surface.

This general observation about potential findings breaking out of pre-existing notions of the mathematical knowledge important for teachers holds for the particular findings of early project work. Looking at reports submitted to funding agencies after the first few years of the project, results fell into two broad categories. First, the researchers found that student work, thinking, and action can share features of research mathematics not typically thought of as being associated with school mathematics. Implicit in Bruner’s call for teaching the discipline with intellectual honesty (1960) lies a recognition that students bring significant intellectual resources to instruction and are capable of making sense, in their own ways, of sophisticated ideas when appropriate bridges are built between those ideas and the ways of thinking students already have available to them. The researchers found that, with support, students can engage in a kind of mathematical work (solving problems, generating ideas, defining terms, and reasoning and needing to articulate reasoning to others) that is readily familiar to mathematicians from their experiences doing research in mathematics. A fuller articulation of what is involved in carrying out mathematical work and of what students were mathematically doing, saying, and thinking stands out in this early phase of work, such as: defining, conjecturing, explaining, proving, choosing key examples, giving counterexamples, generalizing ideas, developing and using representations, and asking mathematically engaging questions. These mathematical practices and their prominence in the classroom became increasingly evident in early analyses, and the researchers seem to have become increasingly convinced that understanding these mathematical practices was an important part of the mathematical knowledge teachers need in order to orchestrate student learning in classrooms.
Another set of results from this phase of the work amounted to a realization that disciplinary mathematics is often highly “compressed” and needs to be “unpacked” in relation to K-8 classrooms. It appears that the researchers found that the ideas mathematicians expressed about what they observed, while relevant to teaching and learning in important ways, were at the same time often distant from teaching and learning in important ways. Likewise, it seems as if mathematicians were often unable to appreciate mathematical work going on in the classroom and seen as important by those researchers who had taught elementary school. This may have been because the mathematicians’ own knowledge of the content was too condensed. As Ball and Bass (2003b) write:

> When ideas are represented in compressed symbolic form, their structure becomes evident, and new ideas and actions are possible because of the simplification afforded by compression and abstraction. Mathematicians rely on this compression in their work. However, teachers work with mathematics as it is being learned, which requires a kind of decompression, or “unpacking”, of ideas. (p. 11)

For example, consider the following problem.

> If a car traveling at a constant speed goes 22 miles in 10 minutes, how far will it travel in 50 minutes?

With a problem such as this, a mathematician might jump to algebraic expressions of the relationship and overlook the transitional space between additive conceptions of the increases and multiplicative ones. In other words, a student who knows how to multiply numbers might still reason by doubling 22 and 10 to get 44 miles in 20 minutes, then doubling again to get 88 miles in 40 minutes, then adding 22 and 10 to get 110 miles in 50 minutes. In understanding this student solution, a mathematician might then be inclined to think that this student fully understands the proportional nature of the relationship and would readily see, if it were pointed out, that multiplying by 5 would be simpler and work as well. However, it is quite possible that the student would find the mention of multiplication confusing because by “doubling” the student might have meant adding 22 to 22 and 10 to 10 and the student’s conception of the problem might have been one of iteratively adding these values. A key instructional goal would be to help the student begin to see this proportional relationship as being about multiplication, but at this point in the students’ mathematical development the interpretation of multiplication as scaling might be new. In the context of thinking about the mathematical knowledge need for teaching, what is

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7 In more recent interviews of mathematicians, as part of a validation study for the measures of MKT, I found that many research mathematicians have difficulty making sense of student reasoning of this kind.
significant is not the issue of deciding on an instructional move, but instead, the issue of the mathematics of the problem. Where a mathematician might see the mathematics of this problem as being about solving an equation in one unknown or perhaps recognizing that the new time is five times the given one, so the unknown distance can be found by multiplying the given distance by five, an elementary school teacher might find it helpful to recognize that increase can be thought of in additive terms or multiplicative terms and that the former can support thinking about the later by leveraging basic relationships, such as doubling. Elementary teachers are likely to find it helpful to be familiar with additive strategies for solving the problem that are often invisible to those with advanced training.

Through repeated exchanges that integrated mathematical and pedagogical perspectives, the researchers seemed to find mathematical ground that lay between, on the one hand, the mathematics articulated by mathematicians and identifiable as part of the mathematical canon and, on the other hand, the mathematical issues seen as important for teaching. The researchers’ explorations identified new instances of such knowledge and proposed the idea that “unpacked” knowledge is a distinctive form of mathematical knowledge for teaching.

To illustrate this initial phase of the work more fully, I provide a more detailed example of work using this first approach: an analysis of definitions for even and odd numbers that were at play in the class in which Sean numbers were developed. This analysis was one of three mathematical commentaries produced for a keynote presentation at a conference sponsored by the Education Development Corporation in the spring of 1997. The conference was designed to facilitate exchange among research mathematicians, mathematics education researchers, and mathematics education practitioners. This analysis exemplifies the two kinds of results mentioned above — the common features between research mathematics and students’ mathematical work in the classroom and the compressed nature of disciplinary mathematics. It also exemplifies the payoffs and pitfalls of this approach.

3.1.1 Unearthing Definitions of Even and Odd Numbers

I begin by describing part of a classroom lesson and a mathematical commentary written about it. Then, in subsequent sections, I describe some of the interactions among research members about the commentary and contributions the commentary and the interactive analysis make to an understanding of the mathematical issues that bear on teaching.

It was mid-year in the third-grade class. The class had been discussing whether zero was even or odd when Sean announced that he had been thinking that the number six could be even

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8 This classroom episode is written about in a number of publications.
and odd. He explained that because six was made up of three twos and three is odd, and then six could be odd. Cassandra disagreed. She picked up a yardstick and, pointing to numbers on the number line above the chalkboard, argued that six cannot be odd because, starting with zero, numbers alternate even, odd, even, odd, even, odd, even. Sean then repeated his explanation: “Because there can be three of something to make six, and three of something is like odd.” He continued, “Three twos to make that and two threes make six.” But, Keith objected, arguing that: “just because two odd numbers add up to an even number doesn’t mean it has to be odd.”

The teacher then returned the class to its “working definition” of an even number, that a number was even “if you can split it in half without having to use halves.”

![Six is even because you can split it in half without having to use halves.](image)

![Five is not even because you have to split one in half. Five is odd.](image)

Figure 3.3. The class’ “working definitions” for even numbers and odd numbers.

After the class considered proposals for a definition of an odd number and the teacher tried to get Sean to use this to reexamine his claim, Sean persisted in claiming that six was both even and odd. Challenged by Tembe to “prove it to us,” Sean explained his reasoning once again. This time, though, Mei thought she understood what he was saying and went to the board to draw ten circles divided into five groups of two.

![Mei’s drawing of ten divided into five groups of two.](image)

She asked why he did not consider ten to be both even and odd. On further reflection, Sean decides that ten too is both even and odd. Understanding his idea better, the class subsequently went on to identify numbers with an odd number of groups of two — even numbers not divisible by 4 — and to label them Sean numbers.

Many observers are impressed with the mathematical character of this lesson. Students provide mathematical reasons for their claims, listen and respond thoughtfully to ideas, and use technical mathematical language such as “prove” and “definition”. The classroom talk resonates
well with mathematicians’ sense of substantive mathematical work. However, what is it about what is happening here that makes it so impressive mathematically?

A research mathematician on the project wrote a commentary for this classroom segment that highlighted the important place of definitions of even and odd. While the notion of “definition” was explicitly mentioned in the class, what stood out for the mathematician was the important role definitions played both in the mathematical work that was unfolding in the class and the role they played in understanding and interpreting what was happening. About midway through several months of developing his commentary, he produced a two-page draft on which two other researchers commented. The first page describes several ways in which the nature and role of mathematical definition and the inventing of definitions figured into the lesson. In this commentary, the mathematician argues that Sean “violates convention” in claiming that six is both even and odd, but that Sean is introducing a mathematically significant idea of even numbers with an odd number of groups of two, “… namely, that when one views an even number, $2k$, as a certain number, $k$, of groups of two, one can give attention to that number $k$, noting in particular when it is odd.” He claims that Sean finds this interesting, does not want to let go of it, and that because he does not have a better name for it, borrows one that only partially fits. He writes that:

… instead of focusing on whether one is entitled to call twice an odd number “odd”, one could, instead, introduce a name for such numbers, so eliminating the terminological conflict, and thus transforming the discussion into one of investigating properties of such numbers.

Because of earlier challenges from other project researchers about whether he was in a position to defend this claim, he also wrote, in this draft, that he is not saying that this should have happened, only that it was a possibility.

This part of the commentary concludes with comments about the “invention of definitions”.

Definitions tend to be regarded by school teachers and students as somehow only decreed by higher authority. In fact, an important part of mathematical practice consists of judging what kinds of definitions to introduce, at what moment, and with what purpose. I would argue that this process has a legitimate place in school mathematics instruction, and that the Sean episode well illustrates a moment when the invention of terminology could possibly be appropriate and fruitful. Mathematical language should, like mathematical ideas, become something that children have the opportunity to engage, manipulate, and invent.
The second page of the commentary offers three definitions for even numbers and discusses these definitions in relation to the lesson.

<table>
<thead>
<tr>
<th>Fair even:</th>
<th>n = k + k = 2k = two k’s;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>You can divide n fairly into two (equal) parts (without cutting one in half).</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Pair-even:</th>
<th>n = 2 + 2 + 2 + ... + 2 = k2 = k twos;</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>You can divide n into groups (of two) with none left over.</td>
</tr>
</tbody>
</table>

| Alternating-even-odd: Starting with zero, the numbers on the number line alternate, even-odd-even-odd-even-odd-even-...... |

Figure 3.5. The mathematician’s articulation of definitions for even numbers at play in the classroom.

The mathematician points out which students are using which definition and describes characteristics of the definition that either come into play or might have been relevant. For instance, he points out that the alternating definition makes the exclusive nature of even and odd apparent and that it is this feature that several students use to counter Sean’s claim about six being both.

The mathematician also raises mathematical issues that were prompted for him. He describes the “interesting mathematical question of establishing the equivalence of the definitions” and ways in which the question affords “a natural opportunity to brush up against some significant mathematical ideas,” in particular, the commutative property of multiplication and its role in distinguishing the fair-even definition from the pair-even definition.

```
0 0 0
0 0 0
```

Figure 3.6. The mathematician’s rectangular array to illustrate the Commutative Property of Multiplication for 2x3.

He writes that this geometric device can be generalized to show that $2k = k2$, and that the fair-even and pair-even definitions are equivalent. The commentary goes on to address the alternating-even-odd definition.

To connect Alternating-even-odd to Pair-even, one can observe that, from Alternating-even-odd, the even numbers are obtained by starting from zero, and then stepping two numbers at a time, i.e. by successively adding two.

The commentary ends by mentioning that there are also several definitions of odd numbers in the class and comments on Ofala’s.
Ofala (1:18:10) speaks of odd numbers as those that have a “one in the middle,” and in fact depicts the number five on the board by drawing strokes in the following order:

```
1 2 5 3 4
```

This introduces a nice notion of geometric symmetry into these concepts, which might have been interesting to spend some time with. Algebraically, Ofala’s idea would correspond to writing \( n = k + 1 + k \), as opposed to \( 2k + 1 \) or \( k^2 + 1 \), so that the very algebraic notation expresses the same symmetry that she has in mind.

This commentary provided what was asked for — a research mathematician’s discussion of the mathematical issues that arose from observing this episode of teaching and learning. But, what does this commentary suggest about mathematics entailed in teaching? I take up three contributions this commentary makes to an understanding of mathematics entailed in classrooms — the mathematics of hearing students, seeing mathematics that could be taught, and inventing definitions — and end by describing dynamics in combining the mathematician’s perspective with the teaching perspective taken by the other two researchers, Researcher 2 and Researcher 3.

### 3.1.2 Mathematical Demands in Hearing Children and Deciding What To Do

Perhaps first and foremost, the commentary would seem to add significantly to hearing what these students are saying and might help to inform decisions about what to do. In particular, having the three definitions provides a new way of seeing what is happening mathematically. The definitions might help a teacher recognize that when Cassandra uses the yardstick to point to numbers on the number line and chants “even, odd, even, odd, …” that she is implicitly using a definition of even numbers as every other number on the number line, that her implicit definition is mathematically sensible but may be at variance with the foundational meaning others are using. Likewise, recognizing that Sean is reasoning from a definition of even numbers as being those that you can split in two without cutting one in half might help to realize that his reference to six being three twos is not a reason for its being even, but a reason for its being odd. Students who think that a number is even if it can be grouped in pairs with none left over are likely to misunderstand Sean because they think that six being three twos is grounds for its being even, not odd. This, in fact, may be happening when Keith objects to Sean, “because two odd numbers add up to an even number doesn’t mean it has to be odd.” Here, he has misunderstood Sean’s reason for saying six is odd. Six is odd for Sean because it is three twos; it is even for Sean because it is
two threes, as stipulated in the class’ working definition of even numbers. Likewise, it is Mei’s understanding of Sean’s reasoning — in particular his use of the definition — that permits her to generalize his idea to ten and to lead the class into a deeper mathematical exploration of his idea.

As Ball (1997) argues, hearing student thinking, for many reasons, is not an easy matter. Having these definitions available aids a teacher in hearing students — in hearing across the divides of different ways of thinking and speaking and in hearing through the fog of new and emerging thoughts. Furthermore, in subsequent work, Ball and Bass (2000a) argue that a disciplinary lens reveals how students are constructing meaning of mathematical ideas in ways that a cognitive, sociocultural, or other perspectives cannot. A disciplinary perspective provides a focus on subject-specific aspects of a child’s thinking and of a class’ collective work. It is hard to imagine a teacher responding to students’ mathematical claims and reasoning without first understanding its subject matter significance. In this commentary, attention to definitions provides a focused example of mathematical knowledge that can be an important resource for hearing students and for positioning a teacher to lead students in productive mathematical exchange.

3.1.3 Seeing Mathematics That Could Be Taught.

A second way in which the commentary provides mathematical knowledge that might inform teaching is that it opens up features of what might be taught — of the situation-specific mathematics that arises in instruction — some of which is impossible to recognize before, or independent from, the actual interactions. For instance, the commentary points out the mathematical importance of showing equivalence among different definitions. Equivalence is important in different ways. Deciding to engage students in establishing equivalence among the definitions depends on many non-mathematical factors, but knowing that this is relevant mathematical work that could be done is important. A teacher needs to recognize when and where establishing equivalence will help to reconcile ideas being discussed. It is also helpful for a teacher to recognize what reasoning about equivalence among particular definitions is likely to afford mathematically. For instance, in discussing the equivalence of the fair-even definition and the pair-even definition, the commentary points out that this amounts to a case of the commutative property, that $2k = k2$, and suggests a rectangular array for illustrating it.

Awareness that the commutative property is lurking in the mathematical wings of reasoning about the equivalence of these definitions might allow a teacher to connect this work with other work in the class, reinforcing ideas from earlier lessons or setting the stage for lessons to come. The rectangular array provides a particularly powerful representation for this content in a way that fits with Shulman’s notion of pedagogical content knowledge.
Similarly, the commentary points out that: “To connect Alternating-even-odd to Pair-even, one can observe that, from the Alternating-even-odd, the even numbers are obtained by starting from zero, and then stepping two numbers at a time, i.e. by successively adding two.” This suggests the possibility of using skip counting and representations such as a 2 by $k$ array to connect the ideas of students who are thinking in terms of an alternating pattern with those who are thinking in terms of pairs. Whether to introduce skip counting, arrays, or the commutative property, however, and whether to do so at the start of instruction or to have students come to it on their own, are pedagogical issues that go beyond the scope of unearthing these mathematically relevant possibilities.

Another example of a kind of mathematical space that the commentary opens up is the tie to algebra, and simultaneously, to geometry. Ways of talking about “any number” can be developed in the context of talking about even numbers and the need to talk about a generic one in the category. This notion extends through school mathematics and beyond. Likewise, the simple expression, $2k$, is a powerful notion and notation. The point here is not that algebraic notation be used to define even numbers for third graders, but that a teacher recognize that the defining of even numbers calls for some way to talk about general and infinite cases of numbers and that this is a central function of algebraic notation. The commentary goes on to point out the terrain opened up by considering the different geometry suggested by the expressions: $2k + 1$, $k^2 + 1$, and $k + 1 + k$. While these three expressions are equivalent, they fit with different ways of thinking about the situation: one more than the double of some number; a set of pairs with one leftover; and a number with a middle “one”. Recognizing these features can help a teacher in her attempts to build the bridges between student thinking and standard mathematics crucial to supporting children’s learning.

### 3.1.4 Inventing Definitions as Part of School Mathematics

Another broad issue that the commentary suggests is that the invention of definitions might be an important mathematical activity to address in school, or at least to understand and appreciate in teaching school mathematics. While the mathematician’s comments are brief and his reasons under-developed, it is important to note that his call for giving the invention of definitions “a legitimate place in school mathematics instruction” grows out of the classroom interactions he observed and out of his sense as a research mathematician that Sean’s idea is not only interesting to Sean, but is mathematically interesting in its own right. If nothing else, the commentary suggests that when a student announces that he thinks six is even and odd, that it is important for a teacher to recognize this as both counter to standard mathematical ideas and as a valuable mathematical impulse to notice and name, as a pattern in the structure of numbers, and
that a teacher’s response to Sean should not invalidate or quash his mathematically legitimate interest.

The larger point, that knowledge-production practices of the discipline, such as inventing definitions, should have a place in school mathematics, has roots in Dewey, Schwab, Bruner, and others. Ball & Bass (2000b) argue that for students to learn mathematics they must contend, to some degree, with these fundamental knowledge-production practices.

We do not argue that students “make believe” they are mathematicians. Instead, we argue that “making believe,” or producing conviction, is central to constructing mathematics and that it has special subject-specific characteristics with which even young children contend. (p. 218)

As such, practices such as inventing definitions constitute an important part of the curriculum of school mathematics—whether recognized or not.

Independent of this issue, though, one can argue that teachers still need to be able to “judge what kinds of definitions to introduce, at what moment, and with what purpose.” The mathematician’s analysis of the definitions and reasoning at play in this episode draws attention to the way in which Sean’s ideas create the need for a definition of odd numbers, a definition “not having earlier been thought to be necessary,” or rather, a definition that may have been presumed to be implied by the definition for even numbers — as those numbers that are not even. Sean’s statement that six is odd made it apparent that the assumption of even numbers and odd numbers being mutually exclusive was not shared by everyone. Responding effectively as a teacher requires an understanding of the nature and role of definitions in the development of mathematical ideas.

3.1.5 Complementing Mathematical Expertise With Teaching Experience

In addition to informing an understanding of the mathematics entailed in classroom interactions, the commentary also suggests some of the challenges and problems involved in this process. To revise the initial commentary to one that would contribute productively to the research questions required significant input from others who could bring a sensibility and sensitivity for classroom teaching to the picture.

One issue evident in written comments was that language that might be seen as a statement of fact by a mathematician might be read, by a teacher, as overstated or even offensive. The mathematician’s midway commentary was produced after extensive exchange and reaction, yet it still contains language that might be seen as potentially problematic. The first full paragraph refers to “violating convention”, “being indifferent”, and “mathematically
troublesome”. While some teachers might see Sean’s claim as simply wrong and accurately described as a “violation of convention”, others might see it as a fledgling idea, expressed by a trusting child who is trying to learn an alien language and way of thinking. People with the latter perspective might take exception to labeling Sean’s idea as a violation and instead refer to it as thinking that is under development and not yet aligned with standard mathematical ideas. My point here is not about what the mathematician meant, but about his choice of words, influenced by his life as a research mathematician, and about the ways in which his words might be heard by a teacher or professional educator. The etymology of the word violate refers to injure, dishonor, and outrage; its meaning is often associated with moral transgression and rape; it is an emotionally loaded word for many, especially women and people who work with and care for children. I do not mean to be melodramatic in saying this, but to name a problem that the researchers seemed to face in this work, where the word choice of mathematicians often appears to have created an impediment to get at, name, and express mathematical knowledge that matters for the profession in ways that can be heard and used by professionals.

Similarly, consider the mathematician’s statement that Sean “is allowing six to be both odd-and-even, thus being indifferent to their normally understood mutually exclusive character.” Again, the word “indifferent” as used here means “regarding it as not mattering,” but this statement, itself, can be interpreted in different ways. It seems as if the mathematician is saying that Sean disregards, or pays no attention to, the mutually exclusive nature of even and odd numbers, but the word also connotes apathy and disinterest that may be at odds with a teacher’s sensibility and sensitivity to how Sean might be feeling at this point in the class. I do not mean to leave the reader with the impression that the linguistic usages of the mathematician implied any of the scornful or unsympathetic attitudes toward the students that others might read into this language. It is really a cross-cultural linguistic, not necessarily affective, dissonance.

In part, words are charged for some groups in ways that they may not be charged for others. (Might some react to the phrase “normally understood” in this sentence as well? Normal for whom? Does this mean that Sean is abnormal?) Additionally though, mathematicians often express their ideas in ways that project, sometimes unintentionally, interpretations of students’ feelings and emotional states that are often seen as problematic to teachers. Teachers learn to read children, and learn, over time, that many of their initial reads of children are incorrect. What is the evidence that Sean was dismissive or lacked an opinion about the mutually exclusive character of even and odd numbers? Many teachers also learn to label the behavior, not the child, as a way to avoid offense and support the change of behavior. As empathetic beings we may all have legitimate opinions about what students we observe are thinking and feeling, but this is not
the special expertise mathematicians bring, and indeed, their impressions are often at odds with the experience, judgment, and sensibilities of teachers.

Similarly, mathematicians often make inappropriate claims about what should happen instructionally. This problem seems to have both an overt aspect and a subtle one. In his work on the project, this mathematician generally recognized and avoided the overt version, of pronouncing what the teacher did wrong and should have done instead, but the subtle version is, in many ways, unavoidable. In asking a mathematician to write mathematical commentaries of classroom interactions, the project was asking him to comment on mathematical issues and to situate his comments in the context of the particulars of student and teacher interactions, yet without overstepping his expertise into claims about teaching and learning. In effect, though, it was asking him to use the classroom to generate mathematical comments that might be pertinent to teaching and learning. Random associations would not be helpful. But how could a mathematician decide what is pertinent without establishing interpretations and opinions about the unfolding instruction?

An even subtler version of this problem arises just from the fact that the commentary is situated in practice, so the mathematician is being asked to situate his comments in a domain in which he has little experience. For example, in his opening paragraph, the mathematician says that Sean’s claim is at variance with common understanding and is indifferent to, and challenged, what is normally understood. In response, another researcher writes:

It might be important to note that Sean’s ideas are being compared with standard mathematical ideas, not necessarily claiming anything about the stability or consensus of the class’ ideas, right? Since the definitions were under development in the class, we don’t exactly know what his claim challenged.

Here, the researcher, who had significant experience teaching elementary school and doing educational research, draws a distinction between claims about “standard mathematical ideas” and claims about the “stability or consensus of the class’ claims.” Expertise necessary for evaluating these claims is different. The former is the professional purview of mathematicians, while the latter requires the coordination of expertise from both domains.

The point is not that mathematicians should never make claims about students or teachers, but that two potential problems may arise. First, such claims may move afield from the research agenda of identifying mathematical knowledge needed for teaching. Second, such claims often require significant review by, and negotiation with, those who have expertise with teaching and learning. The researcher’s comment proposes a clarification for the claim the
mathematician is making and suggests that if he wants to make a claim about students’ ideas, he needs to provide more extensive evidence and be prepared for rigorous interrogation.

In summarizing, then, the first approach, as described above, can provide significant insights into mathematical knowledge that might matter for teaching, but it can also present challenges. For instance, on several occasions the researchers seemed to find themselves at loggerheads about an interpretation of what was happening mathematically in the classroom and whether certain mathematical issues were relevant to teaching. In particular, it seems as if the twin mathematical and pedagogical perspectives were at times at odds, but in the end, need to be reconciled and made to work in concert. The coordination appears, however, often hard to achieve.

One problem is that mathematicians, not being elementary school teachers, lack expertise that allows them to situate their interpretations of classroom events in the pedagogical context in which those events occur. For instance, they are likely to miss behavioral issues with which a teacher may be contending or to be unfamiliar with common sequencing of particular topics for the purpose of instruction. Their lack of knowledge about elementary school students and about the teaching of elementary school mathematics leads them to misread classroom events and to argue for impractical approaches to instruction. Finally, their lack of experience with elementary school teaching can cause them to see a mathematical issue as pedagogically important, even in the face of contrary opinions of others with more experience, and to perseverate the point.

Another problem seems to arise from the condensed nature of mathematicians’ mathematical knowledge. For instance, as mathematicians routinely carry out mathematical procedures or use mathematical ideas, the procedures and ideas may become so automatic that their origins and detail become obscure. This too can make it difficult for them to make sense of what is happening mathematically in a classroom or to hear the perspective of others about what is happening in classrooms.

In addition, there may be a tendency on the part of mathematicians, and perhaps on the part of any group with limited experience teaching elementary children, to overstep into commenting on what teachers should say and do. As Lortie (1976) points out, everyone sees teaching for many years as a student and thus feels qualified to judge teaching. This problem is confounded by the nature of the task. In asking mathematicians to write commentaries, education researchers are asking them to write about any mathematical aspect suggested to them from observing classroom teaching and learning, but then the researchers are also prodding and selecting mathematics based on its relevance to the purposes and goals of instruction. This can put mathematicians in an awkward position of trying to second-guess what education researchers
want. In this particular research group, this problem seemed to lead either to an attempt to try to teach mathematicians about teaching or to ride roughshod over the commentaries mathematicians produced. Both approaches were problematic. Learning to teach elementary school is not readily accomplished with a quick tutorial and much of the sensibility and judgment that goes into it is very foreign to research mathematicians. Likewise, because mathematicians are commenting on what they see as mathematically significant, they are unlikely to let go readily of interpretations and convictions they develop.

This issue has several implications for the study of mathematics entailed in classroom teaching and learning. First, condensed mathematics goes hand in hand with not seeing how big an idea might be for learners or what it implies for teaching. Second, it suggests an important body of mathematical knowledge teachers need but that mathematicians are not well situated to identify. While mathematicians’ interpretations of classroom events that are at odds with teachers’ interpretations of those events can suggest places where mathematical content needs to be “unpacked” and more fully articulated, these additional mathematical issues are not readily visible to mathematicians and their invisibility can limit what they address in their commentaries and can impede productive communication and exchange.

3.2 Approach 2: Mathematical Demands of the Work of Teaching

The second approach might be thought of as reversing the first. Whereas the first explores the mathematics suggested by looking at classroom interactions with a mathematical eye and then analyzes the products to determine the mathematical knowledge and skill that might be relevant to teaching, I will show in this section that the second seems to explore teaching first, with a pedagogical eye, and then to consider the mathematical resources that might help to carry out that work. It begins by analyzing classroom instruction to map the work that the teacher is doing — or needs to do, as necessitated by the goals and the circumstances. In particular, it seeks to identify work of teaching that seems as if it might place significant mathematical demands on a teacher. Once a piece of the work of teaching is clearly articulated, then the approach analyzes the mathematical demands of that work. Reversing the process would seem to foreground teaching and to provide greater focus for mathematical analysis.

Looking over documents for the project, two potential motives seem likely for the shift from the first approach to the second. One reason for the shift is suggested by the challenges that arose in early work where no equally available and agreed-on criteria existed to decide which mathematical issues were relevant and which were not. A second reason is suggested by the first phase, but is more about building on results from the first phase rather than adjusting to its
problems. It seems as if the research group, by watching videotaped lessons, became convinced that impressive mathematical contributions being made by students midway through the year were the result of a foundation laid by the teacher at the beginning of the year and that this work on the part of the teacher required significant mathematical knowledge and skill (020327BkPpt). However, as the group went back to the beginning of the year to watch lessons, they did not find it easy to say exactly what it was that the teacher was doing or what it was that the teacher knew and used in her work. These issues appear to have precipitated the second approach, where the researchers first analyzed the work of teaching and then considered the mathematical demands of that work.

In contrast to the other approaches, this second approach uses a pedagogical perspective to suggest key aspects of the work and then uses mathematical perspective to identify the mathematical knowledge and skill needed for that work. This approach would seem likely to be generative in different ways from the first. Whereas the first relies on perspectives from the discipline to suggest new ideas about mathematical issues that might bear on teaching (and thus on mathematical knowledge for teaching), the second relies on perspectives from research on teaching to suggest new ideas about work teachers have to do, which then might suggest new ideas about the mathematical demands of teaching (and thus about mathematical knowledge for teaching).

During this second phase of the research, the group seems to have focused much of its time and energy on writing a book about the work of teaching at the beginning of the year and the mathematical nature of such work. This phase of the research group’s activity is characterized by extensive project meetings — often weekly and lasting for several hours — with roughly 6 to 10 people attending. For the book, the researchers collectively watched, discussed, and analyzed the first month of instruction in particular, scrutinizing the teaching and considering what is involved in getting started at the beginning of the year, including what it takes for a teacher to get students to a place where they can engage productively in learning mathematics and the mathematical demands on the teacher in doing so.

Results from this line of work were broad in scope. One set of results is about understanding more fully what students must do and learn in order to use classroom instruction as a place for them to learn mathematics (001030PrcMem). Yet another is about the very invisibility of the teaching the group sought to study (990421CnfPpt; 00405CnfPrp). The majority of the results, however, can be grouped around the work of teaching and the mathematical demands of this work. These results fall into two broad categories: (i) core tasks of
teaching and the mathematics entailed in them; and (ii) different ways of thinking about teaching
and subsequently of the relationship between mathematical knowledge and teaching.

Core tasks of teaching. First, using this approach to the work, as specified in funding
proposals, the researchers identified core tasks of teaching and then analyzed their mathematical
demands. In particular, they identified tasks of teaching key to getting started at the beginning of
the year in ways that are respectful of students, respectful of the discipline, and that set the stage
for collective mathematical work (010902BkMem). Results from this second approach seem to
contribute to an understanding of mathematical knowledge for teaching in three distinctive ways.
The researchers: (a) identified core tasks of teaching that have significant mathematical demands
and identified the nature and substance of mathematical knowledge and skill entailed in such
work; (b) explored ways in which many tasks of teaching not typically thought of as
mathematical in nature could be deeply mathematical; and (c) considered comprehensive pictures
of teaching and what such pictures of the work suggest for mathematical knowledge for teaching.
I discuss each of these briefly.

The researchers identified many different core tasks of teaching that have significant
mathematical entailments and they analyzed the mathematical demands of those tasks. For
instance, they analyzed what it takes to figure out what students know mathematically and what
students bring to instruction and they analyzed the mathematical knowledge and skill a teacher
needs to do so, such as formulating questions to probe student thinking, listening to and
interpreting alternative formulations of an idea, and generating mathematical problems to test
hypotheses about students’ ideas (981129TskMem; 001001PrcMem; 010705HwMem;
010907BkMem; 030423EqTxt). They also analyzed the work of selecting, of analyzing, and of
setting up mathematical tasks, and they analyzed the mathematical knowledge and skill needed
for that work, such as knowing, or being able to produce, different ways of doing a problem or
appraising the mathematical prerequisites of a problem (980702BkMem; 981129TskMem;
990617PsMem).

In addition to this perhaps more straightforward line of inquiry, this second approach to
the research also opened up unexpected terrain. As the researchers analyzed the work of
teaching, they seemed to become interested in the mathematical demands of tasks of teaching
typically thought of in general and generic terms, tasks such as managing classroom behavior,
establishing classroom culture, attending simultaneously to students as individuals and as
members of a class, and preparing students for participation in a democratic society. Each of
these aspects of the work of teaching was found to have significant mathematical entailments.
For example, well-placed questions can be used to hold students’ attention or to draw them back
from distractions. Likewise, the admonitions a teacher uses or the “ground rules” she sets at the beginning of the year, about listening attentively and respecting others, can gain power when framed in mathematical terms. Telling students, for instance, to listen so that they can decide whether they agree or disagree provides students with motivation and with something specific to do. And, using mathematical reasoning in place of social authority as the basis for establishing agreement can make exchanges less personally contentious.

Along with this work, documents show that the researchers invested significant time in analyzing teaching as a whole and using these analyses to consider other unexplored, potentially important mathematical demands that might be entailed in teaching. For instance, a memo on tasks of teaching identifies several broad domains:

1. Setting and using instructional tasks
   a. Selecting tasks
   b. Remodeling tasks
   c. Setting tasks
   d. Enacting tasks
   e. “Making something of the task” toward an instructional goal
   f. Making a transition from one task to the next
2. Establishing ways of working, talking, and behaving in the classroom in ways that support learning
   a. Creating and maintaining basic standards for behavior in the classroom
   b. Establishing and maintaining routines
   c. Establishing and managing how knowledge is to be viewed and treated
   d. Establishing and managing the exchange of knowledge
3. Figuring out whether students are “getting it”
   a. Assessing what students can do prior to instruction
   b. Anticipating what students may say or do in response to an assignment, question, task
   c. Listening to students’ responses during a lesson, interpreting what they are saying
   d. Correcting student papers
   e. Listening to what kids say or reading their work and deciphering it, comparing it with goals;
   f. Reading, interpreting, and evaluating students’ answers to a quiz
   g. Sizing up what the group (not the individuals within the class) knows and can do
   h. Mapping kids’ progress against objectives, goals, curricula
4. Other
   (From 981129TskMem, with detailed text for each item omitted)

While this map is incomplete, it suggests ways in which this second approach might be used to develop a comprehensive map of the work of teaching and then of the mathematical knowledge needed for that work. For instance, from the list above, one could ask about the mathematical knowledge and skill helpful in making a transition from one task to another or in managing the exchange of knowledge in the classroom. What might be useful? What is distinctive about the
mathematical knowledge needed to accomplish this task and where in their preparation might teachers develop the mathematical knowledge and skill needed to do so? As a complete and robust map of teaching is developed, it could be used to consider the complete body of mathematical knowledge and skill comprehensively needed for teaching. Results of this kind are a highlight of this second approach to the analysis of practice.

**Conceptualizations of teaching and of mathematical knowledge.** In conjunction with analyses of core tasks of teaching and their mathematical entailments, numerous documents suggest that the researchers were simultaneously exploring different ways to conceptualize teaching, what is involved in teaching, and the relationship between teaching and mathematical knowledge. The researchers’ initial conception of teaching seems to be influenced by Lampert’s notion of teaching as managing dilemmas (971002LngMem; 980321PrjMem; 990608ResMem; 990617PrbMem; 010502MtgNot; 010620MtgNot). From this starting point, the researchers developed different ways to conceptualize what it is that teachers do and to relate mathematical knowledge to that work. Early on, they speak of identifying resources for managing the dilemmas of teaching, in particular mathematical resources of the teacher (990608ResMem). This image and language continue throughout the work, but several other notions are used, each with its own implications for thinking about the nature and role of mathematical knowledge for teaching.

Throughout the documents the researchers refer to work of teaching, comprised of tasks of teaching, but they also explored resources for teaching (990608ResMem), problems of teaching (990603PrbMem), teaching as problem solving (990704PrbSlvMem), mathematical teaching practices and pedagogical moves (981112MvsMem).

For instance, one way of thinking about the relationship between teaching and mathematical knowledge is the idea of identifying resources, in particular mathematical resources, for managing the uncertain and complex work of teaching. In a memo to the group, one researcher writes about reasons for considering resources and how such a framing might serve the ultimate goal of improving teaching and learning.

The turn in our conversation in the project meeting was to ask what resources teachers have to cope with challenges? Beyond drawing attention to the complexity and difficulty of teaching, can we organize our analyses around resources for managing the challenges without oversimplifying the nature of the work? … In focusing on the work of teaching, it would seem that we would want to be as interested in the response to challenges as in the challenges themselves. It is as if resources offer another way to approach and talk about central work of teaching, as if the actual work lie[s] at the interface of challenges and resources for
managing. Resources also provide us with a starting point and an optimism to counteract the frustration and pessimism that can accompany challenges (though likewise, resources have the potential for complacency and challenges have the potential for stimulation). Pointing out potential resources in the practice of teaching can attend to the complexity and difficulty of teaching while also opening horizons of possible action. (990608ResMem)

For the researchers, one implication of this conception seems to have been to use challenges and resources to structure the analysis of practice as indicated by the matrix below.

<table>
<thead>
<tr>
<th>Students as Resource</th>
<th>Mathematics as Resource</th>
<th>Teacher as Resource</th>
</tr>
</thead>
<tbody>
<tr>
<td>Taking a collection of individuals and making them into a group, an intellectual collective in which mathematical work can be done.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Creating a set of expectations and (positive) anticipations about what doing mathematics is going to entail this year.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Finding our where students are mathematically (with a broad conception of what there is to find out.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Figure 3.7. Matrix for structuring the analysis of practice (990617ResMem)

The two dimensions of the matrix suggest a two-pronged analysis.

Together, they offer a cross-hatched, twofold approach to analysis. We could analyze the data for problems and we could analyze it for resources, with each analysis shedding light on the other—problems call for resources and resources are responses to some felt problem. (990617ResMem)

This notion of dilemmas, or challenges, and resources for responding to them apparently grew out of past work, but other notions also emerged and were used. One development that seemed unexpected by the researchers seems to have grown from the researchers’ experiences with the first approach where the mathematical nature of students’ activity in the classroom had been so significant. If mathematical work is central to an understanding of mathematics instruction, and if teachers need to teach children how to engage in mathematical practices of explanation, representation, and the use of language, then teachers need to understand these mathematical practices themselves. More than this, though, as the researchers analyzed the work of teaching, this observation appears to have grown into the observation that certain mathematical
practices are central in much of that work (001002MtltMem; 040223CsunPrs). For instance, Ball, Lewis, and Thames (2008) report on how their analysis of a short video segment led to identifying mathematical practices that are, simultaneously, teaching practices.

In this segment our attention was drawn to talk: to how the teacher’s and students' work can be understood through a close examination of their talk — an examination that keeps its eye on both the discipline of mathematics and the practice of teaching. Although students do much of the talking during this segment, we sought in our analysis to uncover the teaching that structures students' mathematical talk. Our study of that talk led us to identify three aspects of the mathematical work in which students are engaged through that talk, and we used those three aspects to probe the work of the teacher. These three aspects are naming and using names, making and interpreting claims, and evaluating mathematical assertions. We will show that they are, at once, both mathematical practices and teaching practices. (p. 17)

In this development, the categories of mathematical knowledge and skill, on the one hand, and work of teaching, on the other, are the same.

Another notion developed by the group was to conceive of teaching as problem solving, with an analogy to problem solving in mathematics.

What are the enduring problems of teaching? This is a guiding question for our work. Teaching is not only problems, it is also moments of inspiration, insight, and deep emotional connection with others; it is artful, lyrical, prosaic; the act of teaching often contains clean answers to simple questions and organizational schemes. But though teaching is not only problems, it is fraught with problems, and even some of these alternative metaphors I’ve listed to describe the work of teaching are often outcomes or predecessors or springboards to the enduring problems that teaching poses. We don’t have to say that teaching is problems for this to be helpful. We want to be able to argue that “problem solving” offers a useful perspective for some essential features of the work. (990704PsMem)

In this memo, the researchers write about three advantages that the metaphor of teaching as problem solving might offer. It creates a nice symmetry with mathematics. It suggests a practice-centered approach for the education of teachers, focused on core problems of the work. As such, it contrasts with knowledge-centered approaches and might better address the issue of applying knowledge appropriately to practice. Third, it fits with the dilemma-filled characterization of teaching evident in the research literature. However, the metaphor was also seen as problematic.
So what, then, do we mean by the problems in teaching? What species of problems are these? How are they like and unlike the problems in mathematics, or in other arenas?

I think when we say that teaching is characterized by the existence of enduring problems, we don’t mean the kind of problems that are math exercises-- the practice of certain algorithms, or the school math problem that asks us to recount a proof in a standard format. These kinds of problems have ready solutions-- that is, even if the student doesn’t immediately see the solution, the expert does. There is a single, satisfying answer at the end. I’m on shaky ground here, but it seems to me that even in mathematics far beyond that which I understand, there is a push towards closure, towards solution, towards consensus around a single view. If I’m right, perhaps this is where mathematics and teaching diverge ultimately: the presumption that even the most complex mathematical questions have answers, however distant they may be from even the most expert mathematician. The problems in teaching, however, are a different sort than this. The problems we speak of defy solution. There is no urge to closure, but rather a drawing towards complexity, towards a further multiplicity of what we can see. We seek not to solve or simplify our problems in teaching, but to ride them. We follow them, with keen awareness and interest (in the best case), and see them as resources, as generative. We are bedeviled by them but bewitched by them too. This is one key place for us to think and to be clear about what we want to imply about problems and problem solving in mathematics and education. (990704PsMem)

Following a few months of lively debate about the metaphor, the metaphor of problem solving seems to have faded into the background, though the phase continues to get used in suggestive ways in presentations and publications.

Yet another development evident in project documents was the notion of pedagogical moves. This conception seems to have been helpful in making sense of teaching and in connecting events in the classroom to goals that a teacher might have.

Making moves in teaching is also a function of having goals, having a repertoire of moves, and being able to use them in the service of those goals (981028MovMem)

In approaching the study of teaching in this wide variety of ways, mathematical knowledge for teaching might be seen, correspondingly, as a resource for the work, or as contributing skills that constitute moves, or as framing goals and providing orientation for the work. Each of the different ways of thinking about teaching has implications for thinking about
the relationship between teaching and mathematical knowledge as well as for how to conduct research on mathematical knowledge for teaching.

To illustrate this second approach in more detail, I next describe a presentation given at the 2002 Annual Meeting of the American Educational Research Association. The talk was designed to report on the group’s progress on the book being developed and was entitled: *Beginning the year: What is the work of the teacher to help students learn to do mathematics.*

Files for this episode include drafts of the presentation proposal, proposal reviews, draft work on four chapters of the book, a memo on the book, a memo to the discussants, an outline of the talk, and drafts of the presentation. In particular, I use the slides and notes from the presentation to illustrate the kind of digging into mathematical aspects of teaching that typifies much of the work done using this second approach. This episode exemplifies the two kinds of results mentioned above — in particular, the mathematical character of general features of pedagogy and a conception of teaching as the cultivation of mathematical practices. I then discuss some issues and pitfalls of the approach.

### 3.2.1 Cultivating Practices of Mathematics, of Learning, and of Collective Work

I begin by describing the overall design and content of the presentation and the events of the two video clips used in the talk. Then, I describe the three analysis sections of the talk and the contributions these analyses make to an understanding of the mathematical demands of teaching.

In a memo to the two discussants for the session, the researchers describe the design of the talk as: (i) showing two short video clips from a third-grade classroom, one from midway through the year and the other from the beginning of the year; (ii) using the contrast between the videos to frame the problem of what might have to happen at the beginning of the year to help develop what the kids and the teacher are doing at midyear; (iii) providing three analytic examinations of what is going on related to beginning to do such mathematics learning and teaching in school; and then (iv) synthesizing the three analyses into a picture of what would be involved in making possible teaching and learning of the sort they are interested in (020327DscMem).

The initial framing of the problem is given in terms of three commitments or goals.

Suppose that you wanted to create a classroom in which mathematics was treated with integrity, where students’ mathematical thinking was taken seriously, and where students and their teacher worked collectively on mathematics — none of which is exactly business as usual in school. What, then, would the work be for the teacher at the beginning of the school year, to make this sort of mathematics learning and teaching possible? (020327DscMem)
To address this problem, from the viewing of the two video clips, the researchers argue that the cultivation of practices of mathematics, of learning mathematics and of working collectively are key.

In the video clip from the first day of mathematics class, the class discusses the list of six solutions they have generated for the following problem.

I have pennies, nickels, and dimes in my pocket. Suppose I pull out two coins. How much money might I have?

Figure 3.8. Mathematics problem given on the first day of class.

They have generated 15 cents, 20 cents, 6 cents, 11 cents, 2 cents, and 10 cents. The teacher presses them for another solution, one not already on the list. Asking how many different possible answers they came up with, she then asks how they know they have them all. “How do we know there isn’t a seventh one? Or an eighth one that we didn’t find yet?” Ofala answers that there are not any quarters. The teacher encourages students to build on Ofala’s initial attempt.

Ofala thinks--maybe somebody can pick up on what you’re saying--Ofala thinks that because we don’t have any quarters, this is all we can make. (pause) Mei, what do you think about that? (890911BkTrn)

Several students respond that they think they have them all because no one found more, or because they have “tried lots of them,” and “if you keep picking them up you’re just going to get the same answers.” The teacher asks if anyone used a different strategy, and Mei describes “writing them down and then thinking about it some more.” With the students unable to provide a mathematical basis for establishing that they have them all, the teacher ends class by telling them to “try very hard to think, before math class tomorrow, if you can find another answer.”

In the second video clip, of the same class four months later, the class has been discussing a conjecture made by Betsy about the sum of two odd numbers.

Betsy's Conjecture
An odd number plus an odd number equals an even number.

Figure 3.9. Betsy’s conjecture about the sum of two odd numbers.

The children have been providing examples of the conjecture, when the teacher asks whether anyone has found an example with an odd number and an odd number not equal to an even number. She calls on Jeannie, who says that she and Sheena did not find one that did not work, but that they were “trying to prove that you can’t prove that Betsy’s conjecture always works.”
She reasons that, “numbers go on and on forever and that means that odd numbers and even
numbers go on forever, so you couldn’t prove that all of them work.” The teacher then asks what
other people think. First, Ofala says she thinks it “can always work” because she tried many and
they all worked. Then, Mei objects, not to their conclusion, but to their argument. She says that,
if the class accepts the principle of Jeannie’s argument, then that same principle would defeat
several previous conjectures, which the class had already convinced itself were true.

The three central sections of the presentation analyze the cultivation of practices: of
mathematics, of learning, and of collective work. I provide more detail for each of these sections,
and then consider what these analyses imply for the study of mathematical knowledge for
teaching.

*What mathematics to work on?* The first section analyzes the sparse nature of
mathematics problems and topics addressed in first month of school. For instance, the topics for
the first three days are few in number and nominal in scope.

<table>
<thead>
<tr>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
</tr>
</thead>
</table>
| 11           | Identifying monetary value of coins (pennies, nickels, dimes).
|              | Addition of two whole numbers $\leq 10$. | Permutations of two and three objects (numerals, children in a queue). |
| 12           | Permutations of four objects; finding all of them. |
|              | Place value meaning of four-digit numbers. | |

Figure 3.10. Mathematical topics addressed during the first three days of class.
(excerpt from 020329TopTbl)

The researchers argue that such a view of the mathematics addressed in the class is impoverished.
They write that in trying to understand what students might work on at the beginning of the year
to support improved learning of mathematics, they needed a finer lens, or another perspective.
The lens they develop is one of *mathematical practices*. They contrast the abundance of
mathematical practices being cultivated with the sparse nature of the topics, as suggested by the
excerpt below.

<table>
<thead>
<tr>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
</tr>
</thead>
<tbody>
<tr>
<td>11</td>
<td>Keeping written records of mathematical work</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Inquiring about dates, birthdays, and paradoxes in calculating ages.</td>
<td>Keeping written records of mathematical work</td>
</tr>
<tr>
<td>12</td>
<td>Keeping written records of mathematical work</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Interpreting and working on unfamiliar kinds of problems. (permutations)</td>
<td>Keeping written records of mathematical work</td>
</tr>
<tr>
<td>13</td>
<td>Keeping written records of mathematical work</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Solving problems with multiple solutions, and reporting and explaining</td>
<td></td>
</tr>
</tbody>
</table>
• Exploring the “solution space” of a problem with multiple solutions.
• Reporting and explaining solutions.
• Listening critically to other students’ productions.
• Trying to find all solutions to a problem.
• Compiling and organizing lists.
• Trying to prove that you have them all: empirical methods (“If you keep picking them up you're just going to get the same answers.” “you first think of what you can make first, you can make out of nickels, dimes and pennies and then you take, and then you write them down and then, then you think about it some more, then you’ll... then you’ll get them all.”
• Initiating work on a scaled up version (3 coins) of the problem, and speculating about the number of solutions.
• Reflecting on proof.

- Sizing up a problem; speculating about the answer.
- Exploring the “solution space” of a problem with multiple solutions.
- Reporting and explaining solutions.
- Listening critically to other students’ productions.
- Trying to find all solutions to a problem.
- Compiling and organizing lists.
- Thinking about how to tell when you have all the answers. (“I tried it in my brain and I couldn't find anymore.”).
- Reflecting on proof.
- Making a structured list, and using this to prove completeness. (Bernadette)
- Defending an argument before skeptical questioning.
- Collective interpretation and elaboration of an explanation.
- Creation of compressed and unambiguous notation.
- Working with different models of the same mathematical problem.
- Working with a scaled up version of the problem; speculating about the solution.

• Making a structured list.
• Interacting critically and constructively with a student’s demonstration of a complex solution and justification.
• Proving completeness of the list. (Bernadette & class.)
• Looking for patterns (collective).
• Explaining ideas so that classmates can understand.
• Negotiating mathematical sameness and difference (collective).
• Reflecting on the problem’s substance, and on one’s understanding: “How would you pose this problem to someone you know (parent, baby-sitter)? How well do you think they could do it? Would you be able to help them if they got stuck?”

Figure 3.11. Mathematical practices cultivated during the first three days of class. (excerpt from 020329PrcTbl)

This first analysis, then, examines the practice of mathematical justification in particular and shows that justification is established from the outset as an imperative of the mathematical culture of the class.

What learning practices to cultivate? A second analysis examines “quite ordinary” practices of learning that students bring to the classroom and how these might be cultivated. In particular, it identifies remembering, explaining, and listening as three such practices and then focuses its analysis on listening, then focuses on the cultivation of listening. It begins by giving a list of ways the word listening might commonly be used in talk both inside and outside classrooms.

- Listen up
• Listen harder
• Pay attention
• Listen to me
• Did you hear what I said?
• Did you hear what Susie (or Billy) said?
• You weren’t listening
• Listen to me
• I can’t hear you
• I didn’t hear you
• Could you repeat that?
• This is the last time I going to tell you
• Did you hear what she said?
• He never listens
• She is a good listener

It asks, what might be meant by “hard” in “listen harder” and “good” in, “She’s a good listener.” It then juxtaposes these uses with a dictionary definition of listening. It uses the distinction between “hearing,” on the one hand, with a more Deweyian, “minds-on” listening, on the other, and then ties this to the mathematics being taught and learned. In the notes to the slides, the researchers write:

> As a case in point, consider how we might categorize, for instance, “She’s a good listener.” We might mean that she sits with her head up and her eyes on the speaker doing an admirable job of paying attention. Or we might mean that she always does or heeds what she is told. But, by the phase good listening we might be referring to her quality of listening, something that does not seem to be usefully captured either by simple hearing or heeding. To think about what might constitute quality, I need to take a step back into the mathematics classroom.

The next slide of the presentation gives an excerpt from the transcript of the first day.
Figure 3.12. Presentation slide 33 with transcript. (020327BkPpt)

While the statement, “Listen to Lisa,” might, in isolation, be interpreted to mean pay attention or take heed, the researchers propose two other interpretations. The first is about the presence of an audience and the need to speak with this audience in mind.

This request to listen is juxtaposed with "Talk so that other people can hear you, though." Thus listening seems to be marked as something a speaker should be aware of/should expect. Lisa is to talk so other people can hear her - she has listeners. This statement by Ms Ball might indicate that the listening of other peers is something to be taken in account by those who speak, it is something towards which a speaker should address her speech. (020327BkPpt)

The second interpretation is about listening as mathematical work.

And this request to occurs in the context of a mathematics question: "How do we know we have them all?" Lisa is answering a question that has been posed for the class – that Ofala and Mei have already attempted and Sheena will attempt. Thus, the request for you guys to listen seems to mark listening as a way of participating in the doing of mathematics. You - a student - may very well be asked to speak your thoughts of the mathematics in play. Listening appears here as a way and a tool for doing mathematics with others. (020327BkPpt)

The point that listening is a form of mathematical work is emphasized by comparing the references to listening from the video on the first day with those from midyear.
On the first day, the teacher asks the students how they know they have all of the solutions. When one student says it is because there are not any quarters, the teacher suggests that others might pick up on what she is saying.

3. Ball: Six answers. How do we know that we have them all though? How do we know there isn’t a seventh one? Or an eighth one that we didn’t find yet? Ofala?
4. Ofala: There aren’t any quarters.
5. Ball: Why would that matter?
6. Ofala: Because you can only use um, pennies, um . . .
7. Ball: Ofala thinks, maybe somebody can pick up on what you’re saying – Ofala thinks that because we don’t have any quarters, this is all we can make. Mei, what do you think about that?

Figure 3.13. Presentation slide 34 with transcript. (020327BkPpt)

Instead, though, students’ subsequent comments are about their own ideas about why they think they have all of the solutions.

How do we know there isn’t a seventh one?

4. Ofala: There aren’t any quarters.
8. Mei: I think we have them all - ‘cause we had lots of them already and all the people at my table had six.
12. Lisa: We can only pick up two coins and if we pick up 7 then we would be picking up 3 or 4.
18. Sheena: If you keep picking them up you’re just going to get the same answers.

Figure 3.14. Presentation slide 35 with transcript. (020327BkPpt)

The researchers write that, on the first day, when the teacher asks a question:

… students, in what is often interpreted as a school behavior of hearing, raise their hands. It is, as if, listening - a listening perhaps focused on the teacher - is here a way through which students compete in their answers. (020327BkPpt)

They then contrast this with exchanges from midyear, when students refer to one another’s ideas and use one another’s words.
Here, Jeannie refers to Betsy’s conjecture, and then the teacher and other students use Jeannie’s phrase of “always works” to formulate their own comments and questions. Returning again to the beginning of the year, the researchers argue that this pattern in the children’s talk midway through the year is foreshadowed in the teacher’s talk at the beginning of the year.

Figure 3.15. Presentation slide 36 with transcript. (020327BkPpt)

The researchers observe that each of the teacher’s turns uses words and ideas from the children’s talk. What is involved in teaching children to adopt these ways of building on the talk of others may not be clear, but according to the researchers, the attention to and the modeling of listening as a central feature of doing mathematical work seems to play an important role in supporting children’s engagement in learning and doing mathematics in school.

Figure 3.16. Presentation slide 37 with transcript. (020327BkPpt)
**How to work collectively on mathematics?** A third analysis argues that attention to collective mathematical work provides a way of addressing the issues of classroom culture, classroom management, and cooperative learning while still keeping mathematics at the fore. In a slide and accompanying notes in that presentation, the researchers write:

> How we got to this idea of collective mathematical work.

As we watched video from across the year we were often struck by the way students interacted, by how polite and respectful they were, by the way they initiated ideas and talked directly with one another, and by how well they worked together as a group.

Attention to group work in classrooms is not new, even in mathematics classrooms. Here are three lines of thinking in the literature. Each offers a reason.

The literature on classroom culture argues that classroom norms and classroom environment shape mathematics instruction, providing constraints and affordances.

The literature on classroom management argues that group structures are an effective means of managing crowded places.

The literature on the situated nature of learning argues that interactions among people play a significant role in learning.

We agree with all 3 of these reasons, but even together they didn’t seem adequate.

As we watched we saw mathematics playing a significant role in shaping the group work. Our notion of collective mathematical work addresses the concerns raised by all three of these lines of thinking but also puts mathematics at the fore. (020327BkPpt)
This section of the talk then returns to the video to give people the sense that the work is somehow collective, but that what is going on is a blur of collective activity and that discerning it analytically is not so simple. It makes the point that some categories, or ways of parsing the activity, would help, not just to know how to do collective work or how to create it, but also to understand it better. It then examines how collective mathematical work might be cultivated by specific strategies designed to: (i) establish common understanding; (ii) produce and use one another’s ideas; and (iii) arrive at agreed-upon solutions and methods.

This presentation reflects many features of the work done by the group in this second phase. Not only did the group consider the collective nature of mathematical work in the classroom, they conducted the research collectively. Six people actively participated in giving the presentation. The three analyses grew from extensive deliberation in project meetings and are threaded through project meeting notes for over two years prior to the presentation. As described in a number of published articles, the different perspectives represented on the research team helped to shape the direction and substance of the analyses. But, what does this presentation suggest about mathematics entailed in teaching?

Together, the three examinations demonstrate the breadth of insights into mathematical knowledge for teaching that this second approach can generate. They also suggest the ways in which this research group used the approach to consider more fundamental issues of nature of teaching, the nature of its mathematical demands, and different ways in which teaching and mathematical knowledge for teaching might be conceptualized. Below I discuss only two of these: (i) the exploration of ways in which many tasks of teaching not typically thought of as mathematical in nature can be deeply mathematical; and (ii) the picture of teaching that these investigations provide and what such pictures of the work of teaching suggest about mathematical knowledge for teaching. Again, I end with a discussion of the payoffs and pitfalls of this approach.

3.2.2 The Mathematical Character of General Features of Pedagogy

It appears as if, early on, the researchers were struck by the mathematical sophistication of what they saw the children doing. This led them to investigate what was happening at the beginning of the year that precipitated later work. What they found was that the very generic features of classroom instruction were, from the start, deeply informed by mathematical sensibility.

The second analysis above seems to argue that ordinary activities, typically thought of as generic aspects of learning in school — remembering, listening, and explaining — can be thought of as fundamental mathematical activities and that, in doing so, they are likely to contribute to
productive engagement in doing and learning mathematics. Listening, as they characterized it above, is a mathematical activity. It is about comprehending the mathematical ideas being advanced by others and about deciding whether you agree or disagree. They suggest that this characterization of listening, in contrast to listening as a form of politeness, provides students with something active to do when others are talking, and something likely to contribute to doing and learning mathematics.

Likewise, implicit in the argument about collective mathematical work is the idea that skill in mobilizing and managing collective mathematical work and the mathematical knowledge needed to do so are key to establishing an effective classroom culture, to managing classroom behavior, and to supporting cooperative learning — teaching tasks typically thought of in generic terms and as independent of the content being taught. Signs of this argument also appear in a draft chapter of the book (020328BkChp6).

Furthermore, to the extent that these ideas are important for teaching, they seem to imply that teachers would need a deep understanding of the character and functioning of mathematical work. It suggests that teachers not only need to be knowledgeable about commonly recognized mathematical practices, such as those of using and connecting representations and giving mathematical explanations and justifications for claims, but that they need to know what it means to listen mathematically and to work together on mathematics. My point here is not that these researchers are right or wrong about these insights. Nor is it that these are necessary conclusions of the second approach. Instead, my point is that these are the kinds of results that the second approach makes possible. By beginning with the question of identifying the work of teaching and then asking what might be mathematical about that work, the second approach opens up the possibility of seeing mathematics in aspects of teaching not typically thought of in mathematical terms.

### 3.2.3 Teaching as the Cultivating of Mathematical Practices

All three of the investigations presented in the talk described above put mathematical practice at the center of classrooms and at the center of teaching. Indeed, the “working hypothesis” given in the talk was that teaching that treats mathematics with integrity, treats students’ mathematical thinking with respect and works on mathematics as a collective (as well as individual) endeavor would require that teachers cultivate practices of mathematics, of learning mathematics, and of working collectively. The first investigation identifies mathematical practices as key goals for the beginning of the year. Given my comments above, though, the latter two investigations view practices of learning and of collective work as kinds of mathematical practices. Thus, the vision of classrooms here would seem to be one of engaging in
mathematical practice, or at least one where engagement in mathematical practices is central. This then seems to imply that teaching is fundamentally about understanding mathematical practice and about ways of cultivating such practice in others.

In the first investigation given in the talk above, the point seems to be that the mathematics being taught and learned in the classroom might be better characterized as being about mathematical practices than about topics or problems. If the mathematics central to what is happening in the classroom is as much about the mathematical practices being cultivated as it is about the topics being covered, this suggests that teachers might need to know as much about the nature and role of mathematical practices as they would need to know about the mathematical topics of the curriculum. This might be seen as having significant implications for the mathematical education of teachers. Of course, this might only be an artifact of the classroom being studied by the group or might only be pertinent for classrooms where the cultivation of mathematical practices is itself a major goal. However, it seems that the researchers are arguing that these practices are perhaps key to any classroom committed to integrity of mathematics, respect for students, and collective work on mathematics, even if the instructional goals are restricted to the learning of basic skills. For instance, in several publications Ball and Bass write that if mathematical understanding is a goal then mathematical reasoning and justification are basic skills.

Our point is that mathematical reasoning is as fundamental to knowing and using mathematics as comprehension of text is to reading. Readers who can only decode words can hardly be said to know how to read. … Making mathematics reasonable means making it reasoned and, therefore, known in useful and usable ways. (Ball and Bass, 2003a, p. 29)

Thus, the focus on cultivating mathematical practices might be seen as revealing that mathematical practices undergird basic mathematical proficiency and the teaching of basic mathematical proficiency to a greater extent than might, at first, be expected.

Again, my point here is not that the researchers are right about the importance of mathematical practices. Instead, my comments are about the nature of the results of the group in the second phase of their work. Mathematical practices are clearly central in the work. While they might be seen as an important end of instruction, in and of themselves, the implication from the research being done here seems to be that mathematical practices may be an important means as well as end and that this observation has important implications for mathematical knowledge for teaching. In other words, the first investigation might be seen as suggesting that teaching and learning, even very traditional conceptions of the curriculum, may be more about mathematical
practices than typically recognized. This seems reinforced by the observations of the second and third investigations, where ordinary learning practices and basic features of instructional organization in classrooms are deemed to be deeply mathematical in nature. Together, these results suggest that mathematical practices are profoundly central to understanding instructional interactions in classrooms and to understanding teaching. The researchers seem to be putting forward a view of teaching as being fundamentally about engaging in and managing mathematical practices and mathematical work in the classroom. Thus, the diverse activities of establishing a classroom culture, of getting students to listen, and of deciding what mathematics to work on are all centrally about engaging in and managing mathematical practices. The culminating point here, then, is that mathematical practices are central to the work of teaching and to notions of mathematical knowledge for teaching.

### 3.2.4 Finding the Mathematics

Work done using this second approach has added to an understanding of mathematical knowledge for teaching. It seems to have been used to establish a more complete picture of the mathematical knowledge, skill and sensibility involved in analyzing tasks and in considering the mathematics of a curriculum and of instruction. It has also revealed mathematical demands inside work of teaching typically seen as general in character and has raised questions about how best to conceive of teaching and has generated proposals for how it might be conceived.

Still, though, a concern with the work reported here is that it seems to run a risk of attending to issues of teaching with no clear and direct implication for mathematical knowledge for teaching — and for the mathematical education of teachers. For instance, in noticing that listening can have a deeply mathematical character to it and that when such listening is taught to children it can significantly enhance their ability to learn mathematics and to contribute to the learning of others, we are still left to wonder what mathematics teachers need to know. What does the observation imply for changes in the content of a mathematics course for teachers? Perhaps if teachers knew the topic well, then teaching children to listen mathematically is only a pedagogical issue not requiring anything further with regard to mathematical knowledge or skill on the teacher’s part. These are important issues, and this second approach, at least as it played out in this case, seems to raise as many questions about mathematical knowledge for teaching as it answers.

It is as if the second approach, which might be seen as reversing the first approach, runs into the reverse problem. Instead of being susceptible to mathematical considerations that are not usefully relevant to teaching, the second approach is susceptible to pedagogical pursuits that leave
obscure the exact implications for the mathematical knowledge and skill that teachers would need. It appears as if it may have been this problem that led to the third approach.

3.3 Approach 3: Mathematical Work and Mathematical Tasks of Teaching

The third approach focuses more directly or immediately on mathematics. Instead of asking more generally about the work of teaching and then following this with the question of what such work implies for mathematical knowledge for teaching, the third approach asks about the mathematical work of teaching. In other words, in contrast to the second approach, which explores the extent to which mathematical knowledge can inform the work of teaching, even work of teaching typically thought of as generic pedagogical work, the third approach turns the spotlight onto more clearly identifiable mathematical activities of mathematics teaching. The approach consists of analyzing, specifically, the mathematical work of teaching (comprised of mathematical tasks of teaching). Observation of practice, instead of watching for mathematical issues broadly and instead of watching for the general work of teaching, turns an analytic eye on the distinctively mathematical work of teaching.

As Ball and Bass (2003b) write, the mathematical work of teaching:

… involves teachers in a kind of mathematical reasoning, unencumbered by considerations of students, but applied in a pedagogical context. Our analyses have helped us to see that teaching is a form of mathematical work. Teaching involves a steady stream of mathematical problems that teachers must solve. (p. 6)

In their discussion of the mathematical work of teaching, however, they present this idea as an insight that grew out of earlier work and not, in and of itself, a method that the group consciously employs. I argue here, though, that it is indeed different as an approach to the study of practice, and in arguing this I mean to point to the distinctive character it seems to have as a form of research with a coherent set of questions, data, methods and results.

Analytically, it is different to ask about the distinctive mathematical work or mathematical tasks of teaching than it is to ask about the mathematical knowledge demands of the pedagogical work of teaching. The former foregrounds the mathematical nature of the work; the latter backgrounds it. And, the two approaches imply different methods. The latter would seem to imply first identifying the work of teaching and then, or simultaneously, determining implications of that work for mathematical knowledge, while the former suggests a more immediate focus on identifying the mathematical work to be done.
From an examination of the documents, however, I propose that the data and methods for this third approach also consist of a two-pronged process, albeit a different one from the other two approaches. The first prong uses primary records of practice described in the earlier approaches and applies an analytic focus on the distinctive mathematical work and mathematical tasks of teaching. The second turns to sites of secondary use of mathematical knowledge for teaching. By “secondary use,” I mean to distinguish between the use of mathematics in the context of teaching (implied in the phrase mathematical knowledge for teaching) and a secondary use, or exercising, of both the notion and the substance of mathematical knowledge in other, yet relevant, sites, such as in discussions about teaching that occur in teacher education or in contexts where teaching is being evaluated. As discussed in Chapter 1, the goal is the improvement of student learning; the sites I refer to are the social, political, economic and historical settings of human activity where such potential improvement would need to take place. These include, in particular, the teaching of mathematical knowledge for teaching (to prospective or practicing teachers) and the evaluation of teachers’ mathematical knowledge for teaching. Learning, application, and evaluation constitute a complete cycle of activity, where the primary site of use for mathematical knowledge for teaching is teaching practice and secondary sites include the education of teachers and the evaluation of teachers.

Perhaps there are other relevant sites, but these seem to stand out. For instance, one might imagine that policy debates would be another site where formulations of mathematical knowledge for teaching get used and could be studied, but most policy debates that consider mathematical knowledge for teaching are about the education of teachers, the practice of teaching, or the evaluation of teachers’ mathematical knowledge.

In the next three sections, I first describe threads of this third approach, focused on mathematical problem solving in teaching, mathematical work of teaching and mathematical tasks of teaching, as they are visible in disperse ways across the data, then describe ways in which the two secondary sites of teacher education and measures development extend the study of practice as it is used to understand and identify mathematical knowledge for teaching. I then summarize some of the advantages and drawbacks of this approach.
3.3.1 Focusing on Mathematical Problems, Mathematical Work and Mathematical Tasks

The phrases “mathematical problem solving,” “mathematical work of teaching” and “mathematical tasks of teaching” occur in only a handful of the project documents and these terms were not extensively used to describe methods for proposed research. Instead, it appears as if these phrases were first developed and used descriptively — to draw attention to the fundamentally mathematical character of teaching and to the fundamentally mathematical character of the ideas and results being generated by the researchers.

The first occurrence in presentations appears to be in an invited address to the Michigan Mathematics Forum at Michigan State University in April 2001 entitled Understanding Teaching as a Form of Mathematical Problem Solving (010317MsuPpt). This talk asked the question: “What mathematical problems do teachers need to solve.” The language here makes use of earlier explorations described under the second approach where the notion of teaching as problem solving gets retooled to be teaching as mathematical problem solving. The talk includes the kinds of examples of tasks of teaching and lists of such tasks that continue to be prominent in the research reported by the group, even up to the present, but it referred to these as “mathematical problems teachers have to solve.” This wording is used in several talks over the next two years, until a presentation entitled Learning Mathematics for Teaching, given in April 2003 at the annual meeting of the National Council of Supervisors of Mathematics in San Antonio, Texas (030416NcsmPpt). This talk introduces the idea of “seeing teaching as mathematical work.” The specific phrase, “mathematical tasks of teaching,” appears in a presentation, Seeing Teaching as Mathematical Work, given a year later, in June 2004, at a Preparing Mathematicians to Educate Teachers workshop in San Diego, California (040429PmetPpt).

Looking across the documents, it seems worth noting that the phrase “mathematical work of teaching” seems to get used to describe a more general character of the work of teaching with a more expansive sense of the activity. The phrases “mathematical tasks of teaching” and “mathematical problems of teaching” seem to get used more narrowly, to indicate specific examples and to focus attention on the fact that the thing being talked about is indeed mathematics, in other words that it is fundamentally mathematical in nature and not simply an aspect of teaching.

My point here is not to establish the history of the terms, but to use the history to understand what the phrases contribute to approaching a study of practice that aims to understand mathematical knowledge for teaching. The contributions seem to be of three kinds. These phrases put mathematics at the fore, focus attention on the use of mathematics in teaching, and
allow for discussion of ideas about mathematical knowledge for teaching at different levels of specificity or grain size.

To see these issues in greater detail, consider the Learning Mathematics for Teaching presentation given at the annual meeting of the National Council of Supervisors of Mathematics in April, 2003. This talk was given as part of a symposium organized around the question: How can practice be used as a site for professional study and learning?

The talk is designed to address the problem of learning mathematics in ways that enable its use in teaching. It begins by posing the problem of teachers’ mathematical knowledge, then describes the mathematical work of teaching by giving examples of several mathematical problems or tasks of teaching, and ends by describing the use of records of practice to study and learn to do the mathematical work of teaching. The last section uses the viewing of a video of classroom instruction to frame several mathematical problems of teaching that could be used with prospective and practicing teachers.

The introduction is similar to that given in about half of the nearly 30 presentations related to mathematical knowledge for teaching given by members of the research group from 1999 to 2004. It refers to a shift in research on mathematical knowledge for teaching from knowing to knowing and doing. This would seem to suggest that earlier work had separated the question of teaching from the question of the knowledge needed for teaching, but that the current view shifts the focus on knowledge to one that combines knowledge with its use in teaching. In other words, the shift is from mathematical knowledge to mathematical work as an analytic focus. The introduction then uses a mathematical problem (or task) of teaching to illustrate what is meant by mathematical work of teaching. Earlier talks often seemed to engage people in observing video of classroom interactions, which provided a vicarious, somewhat messy teaching experience, and then in seeing, from the inside, just how mathematical it could be. The turn to specific mathematical problems allows for a quicker honing in on the central point of mathematics needed for the work of teaching.

An entire second section of the talk, entitled Seeing Teaching as Mathematical Work, is then given to examples of mathematical problems of teaching. These mathematical problems are not typical categories of mathematical knowledge, yet they seem to fit in a recognizable notion of “mathematics problems” that need to be solved. They also do not seem to represent teaching in ways that are familiar.

- Analyzing errors
- Giving and evaluating explanations
- Appraising unexpected claims, solutions, and methods
• Choosing and using representations
• Investigating correspondences among representations and solutions
• **Choosing and using definitions**
• Interpreting and responding to students’ ideas

They seem, however, suggestive of mathematics that might be important for teaching. They identify something of a mathematical character and they seem plausibly relevant to teaching. Although they teeter between accomplishing both goals (of identifying mathematics and of informing teaching) and accomplishing neither, they seem to fit well with the agenda set by Shulman (1986, 1987) calling for professional knowledge at the intersection of the discipline and of teaching. In this sense, also, they seem compelling in that they point at something important yet not currently addressed in professional education or in research on teaching and teacher knowledge.

The third section of the presentation begins by again showing this slide listing mathematical problems that arise teaching. The logic of this section seems to be that if you want teachers to learn to solve such problems then one strategy would be to engage them directly in such problems. In other words, that mathematical tasks of teaching provide prompts for a mathematics curriculum of professional education. The next slide would seem to support this. It is entitled *Learning in and from Practice*, and has notes about the way in which this talk addresses the question of how practice can be used as a site for professional study and learning. It comments that this can be thought about as the “curricularizing” of practice for teachers’ professional study, where “study” is borrowed from Lampert’s book, *Teaching Problems and the Problems of Teaching* (2001).

Maggie uses the word, “study,” to refer to the activities of learning, to work and practices involved in learning. We use this word for its active and deliberate sense, to name a collection of practices that are likely central to profitable investigation and learning. (030416NcsmpPt)

In other words, mathematical problems, or tasks, of teaching provide a way to “curricularize” practice, where the study of such problems engages teachers in learning to do the mathematical work of teaching.

Following the viewing of the video are two slides that use mathematical problems of teaching as instructional tasks for teachers.
These tasks are set in the context of practice and, simultaneously, are deeply mathematical in nature. Take, for instance, Task #3. Without even hearing what Riba said and getting a glimpse of what her confusion is about three fourths of a dozen, a teacher’s mathematical awareness of, or ability to analyze, the relationship between one half of six and three fourths of twelve seems important for the teacher’s work. At the same time, while the mathematics of determining three fourths of twelve might well be regarded as trivial from the perspective of reasonably well-educated adults, articulating the relationship between these two problems is not obviously so. Noticing that half of six can serve as a sub-problem in finding three fourths of twelve is important, and generating this and other similar possibilities is critical to success in teaching.
The next slide is entitled, *Solving Problems in Learning Mathematics for Teaching*, and it has a blank column prompting people to consider answers to the problems and challenges laid out in the proposed tasks. This slide is then repeated, but this time filled in, not with answers, but with characterizations of the mathematical work involved in solving the problems or tasks presented.

**Figure 3.19.** Slide 31 characterizing the mathematical work involved in solving the proposed tasks for learning mathematics for teaching. (030416NesmPpt)

This third section of the presentation uses the observation of a video of teaching and learning to dive into the mathematical work of teaching and then identifies specific mathematics problems, or tasks, arising in this situation and then steps back from these to identify more general, recurrent mathematical tasks of teaching applicable to many situations in teaching. We see here how the constructs both of mathematical work and of mathematical problems/tasks of teaching contribute to an understanding of mathematical knowledge for teaching and provide leverage on the practical problem of professional education. We also get a glimpse here of how these might be generated from an analysis of practice, where specific problems are identified in the situation and then generalized to more widely applicable tasks of teaching. One striking feature of third section of the presentation is the shifting focus from a sense of mathematical work as experienced from viewing the video, to formulating mathematics problems arising from inside of the work, to identifying more general categories of mathematical tasks central to the mathematical work of teaching. These three elements seem to be simultaneously about the same
fundamental thing, at the heart of what is meant by mathematical knowledge for teaching, yet different in what they draw into relief.

These three elements seem to differ in two important ways. First of all, in their grain size and in how detailed, focused, and particular they are. The notion of “mathematical work” is broader in scope, with problems and tasks as subordinate units. The notion of specific and highly contextualized problems contrasts with the more general categories of tasks offered in the right column of the last slide described above. Second, the three foci differ in their exclusive attention to mathematics, or perhaps rather in the balance they strike between attention to mathematical knowledge on the one hand and teaching on the other. Consideration of mathematical issues and mathematical work in observing the video seems leaves the mathematics deeply intertwined with pedagogy. The move to identify specific problems and general categories makes it possible to isolate, or at least to draw further to the fore, mathematical issues.

In this presentation we begin to see elements of a distinctive third approach where a focus on mathematical aspects of teaching provides a particular analytic focus and we begin to see what the yield of this approach can be — a conception of mathematical knowledge for teaching that puts mathematics at the fore, focuses attention on the use of mathematics in teaching, and allows for discussion of ideas at different levels of specificity. This presentation also anticipates aspects of ways in which the two secondary sites of teacher education and measures development extend the study of practice as it is used to understand and identify mathematical knowledge for teaching. Many of the examples of mathematical knowledge for teaching are drawn from the development of instruments for measuring mathematical knowledge for teaching and the instructional tasks were generated for efforts to improve the teaching of methods and content courses for elementary teachers. Next I examine the ways in which these secondary sites can be used to exercise and study mathematical knowledge for teaching.

3.3.2 Teacher Education as a Site for Exercising and Studying MKT

The first of these secondary sites consists of the settings and activities where the mathematical education of teachers happens. In these settings, materials and activities are designed to support the teaching and learning of mathematical knowledge for teaching. This design process can be carried out as a kind of design experiment (Kelly & Lesh, 2000) where students’ engagement with the materials provides data for analysis and for further refinement of the ideas. The researchers write about this idea in a (rejected) proposal in 2004 in which they describe teacher education as a “laboratory” for their work.

Drawing on our analyses from Strand 1 [analyses of classroom practice], we use our work with teacher education students to test
ideas about the mathematical work of teaching and the kinds of knowledge and reasoning entailed. We deliberately use these courses to create a design and research loop, drawing on the analyses of teachers’ practice from the first strand. (040601 RolePrp)

This research loop, however, is more than just a testing to see whether the ideas work. The researchers seem to be suggesting that it is a site for studying mathematical knowledge for teaching.

Doing this has two implications for our research: First, we articulate more clearly what there is to know and how it might be used when we try to teach it to someone else. The design of tasks for prospective teachers requires us to elaborate our ideas, and to add detail and structure. Second, as prospective teachers engage with the activities we design, we will learn things about the tasks of teaching, and about beginning teachers, as well as about the mathematics we are trying to help them learn. (040601 RolePrp)

Two additional points are worth making here. First, the word tasks used above refers to instructional tasks, but central to these instructional tasks are the distinctively mathematical tasks of teaching. In other words, most of the instructional tasks used in these courses focus on a particular mathematical task of teaching. Second, because the goal is about identifying mathematical knowledge that matters for teaching, the way in which the practical world of teacher education “talks back” to the initial formulation of a mathematical task of teaching is a key resource for identifying mathematical knowledge that bears, not only on teaching, but on teacher learning and thus on improving teaching and learning. In other words, embedded in the phrase mathematical knowledge for teaching is the additional emphasis, described in the first chapter, on mathematical knowledge useful and useable for improving teaching and, ultimately, student learning.

An implication of this is that the world of practice to be studied extends beyond the immediate realm of teaching to also include the realms of teacher education, professional development, teacher evaluation and instructional policy that both entail and interact with teaching practice. My point here is that, while the description above emphasizes the ways in which trying out ideas about MKT in courses for teachers can help refine the ideas, it also suggests that the exercising of the ideas in teacher education can be a site that, to some extent, defines and shapes the ideas and the very concept of MKT itself. As such, it too can be studied in order to determine mathematical knowledge for teaching.
The approach described here is different from studying the content being taught in teacher education or in professional development as described by Adler and Davis (2006) where they evaluate the mathematics currently being taught in programs in South Africa in relation to the notion of mathematical knowledge for teaching. In that research, they focus on mathematical knowledge for teaching as “unpacked” knowledge — which they define as “an understanding of the syllogistic chains (explicit coherent reasoning) relevant to the knowledge” (p. 284). They categorize instructional tasks based on the degree to which the tasks unpack mathematics and the degree to which they unpack teaching. From their analysis, they hypothesize that the absence of tasks that require unpacking is due to mathematical knowledge for teaching being not well understood and its being hard to do in the context of formalized teacher education (p. 291). This hypothesis leads them to recommend study of successful cases of teaching mathematical knowledge for teaching in order to articulate more fully effective ways of teaching mathematical knowledge for teaching. They do not, however, use teacher education as a site for empirical investigation that would help to identify and refine an understanding of mathematical knowledge for teaching. The design of research suggested by my examination of the work done by the University of Michigan researchers proposes analyzing the engagement of instructors and prospective teachers to better understand the mathematical demands of teaching and its improvement. Teacher education is, thus, used as a site of practice for the study of mathematical knowledge for teaching.

The University of Michigan researchers go on to describe their data as including prospective teachers’ engagement with specifications of mathematical knowledge for teaching and with tasks designed to teach it. The tasks they discuss are fundamentally mathematical tasks of teaching and the engagement of prospective teachers with a task — their use of it and their talk about it — seems to be important data in these researchers’ efforts to study mathematical knowledge for teaching.

In studying closely their engagement with the tasks, we will be able to learn more about the mathematical reasoning involved in the task (of teaching) and that will, in turn, help us unearth more about the mathematics that is required for teaching. Prospective teachers work and thinking and talk will make more visible what there is to know and do mathematically, in ways that our own reasoning often hides or masks. For example, we might see that their approach to figuring out a good numerical example (in the task we might set) reveals something that is involved in doing this that we did not at first appreciate. It is in this way then, that tasks serve as a critical agent by which we may learn more about the nuances of mathematical and pedagogical thinking that arise in the work of prospective teachers. (040601 RolePrp)
These comments also suggest that important aspects of mathematical knowledge for teaching can be made apparent by their absence — an absence that becomes evident in the context of teacher learning. It suggests that what is important to identify (mathematical knowledge for teaching) may be defined not only by what can be made visible through analysis of classroom teaching, but also by the practical realities of what unskilled teachers struggle to learn and to do in the context of beginning to engage in such tasks and work.

The researchers describe ways in which both the development of materials and the field-testing of those materials are research sites for studying practice, within the context of teacher education, in order to develop the notion and substance of mathematical knowledge for teaching.

The use of development work as a site for research is unconventional, yet well suited for work that aims to develop theory in accord with the practical realities of day-to-day teaching. First of all, development tasks press for a full and particularly useful articulation of the mathematical ideas we are developing in Strand 1 and of their specific use in teaching. Likewise, the development of content guidelines for mathematics courses for teachers affords a specification of the terrain guided by the practical realities of what prospective teachers are likely to already know and understand and what is most likely to serve them well. In addition to developing tasks and specifications, we will look for patterns and develop codes to identify, for example, the mathematics called for, the work of teaching in which it is situated, and how explicit or buried these are.

Second of all, the piloting of these tasks and content guidelines provides important feedback on how people read or interpret what we are saying. We will observe instructors’ interpretation and use of the tasks and specifications and prospective teachers’ engagement in these. We will analyze what instructors and teachers do with these in the fullest sense of do. For instance, beyond analyzing “student work,” we will, as a research team, systematically observe, write field notes, and collect artifacts around people’s engagement with tasks and specifications. We will look for patterns across the data, seeking to learn about beginners’ use of and responses to the tasks. We will generate analytic memos to guide the development of codes or the identification of patterns from the examples of student work on the problems, coding for mathematical knowledge in use, as well as places where it is used or needed. These codes will draw on codes we have developed in validation studies for measures of teachers’ knowledge where we developed codes for analyzing mathematics in use in video of classroom teaching. Our aim is not to develop a curriculum and assess the effects of it on student learning, but to refine our ideas by engaging them in the world of practice. (040601 RolePrp)
These comments reinforce earlier points, but also add to them. They suggest that how people read, interpret, and use the particular formulations and tasks identified matters to deciding what is and is not mathematical knowledge for teaching. It would seem that a practice-based theory of mathematical knowledge for teaching, as evident in the work of this research group, is one that is not only grounded in analysis of classroom teaching practice as described in Ball and Bass (2003b), but is grounded in the analysis of a larger set of practices across sites that bear on its development, use, and evaluation. In other words, it is not just about usable knowledge for teachers, but usable mathematical knowledge for a larger system responsible for the improvement of mathematics teaching and learning.

### 3.3.3 Teacher Evaluation as a Site for Exercising and Studying MKT

The second site consists of settings where teacher content knowledge gets evaluated. In these settings, instruments are designed to measure mathematical knowledge for teaching, and this activity, though an additional step removed from the activity of teaching, serves as a site of practice that bears on the development and use of mathematical knowledge for teaching analogous to the description above for teacher education. The researchers developed instruments to measure mathematical knowledge with a particular focus on the use of that knowledge in teaching. In a grant proposal submitted in 2002, they wrote:

> These measures are unusual, for they are not measures of “straight” mathematics knowledge, such as the ability to compute or answer word problems, but instead measures of mathematical content as it is used in teaching. (020616 MSPPrp)

Arguments for these instruments, in this proposal and others, distinguish mathematical knowledge measured by other available instruments from the knowledge measured by these. These arguments emphasize the uniqueness, to teaching, of the knowledge being measured and describe how the researchers found it useful to illustrate the distinction between “straight” subject matter and subject matter as it is used in teaching by “anticipating how someone who has not taught children but who is otherwise an expert in ‘adult’ mathematics might experience these items.” The researchers describe the ways in which mathematics-as-used-in-teaching items resonate with teachers. They also describe ways in which mathematicians, and other non-teachers, can struggle with items that the mathematicians admit are fundamentally about mathematics, but unfamiliar and challenging.

By contrast, however, the mathematics experts might be surprised, slowed, or even halted by the mathematics-as-used-in-teaching-items; they would not have had access to or experience with opportunities to see, learn about, unpack and understand
mathematics as it is used at the elementary level. (020616
MSPPrp)

These comments seem to imply a focus in the measurement work on distinctive mathematical knowledge tied to use in a way that is about the distinctive mathematical work and mathematical tasks of teaching.

In a grant proposal submitted the following year, the researchers described five areas in which they wanted to make progress:

1. Developing measures of knowledge for teaching in major K-8 mathematics content strands. By developing new measures and augmenting established scales, we hope to enable studies of programs aimed at improving teachers’ knowledge and, by extension, student learning.

2. Building theory about knowledge for teaching mathematics as a product of this measures development work.

3. Validating our measures, to ensure they adequately and accurately represent individuals’ underlying knowledge for teaching mathematics.

4. Supporting high-quality evaluations and uses of these measures, through assistance with evaluation design, scale construction, and analysis.

5. Building a self-sustaining system of measures use. (030806 RETAPrp)

I argue that each of the first three of these provides a way to exercise the ideas of mathematical knowledge for teaching as it is situated within a larger system, as referred to in the last of the five.

**Item writing.** In describing item writing, the researchers characterize two phases (030806 RETAPrp). The first phase consists of drawing on eclectic research and teaching experiences of the team. These “seeds” are then developed into items in intensive one- to two-week, collective, item writing sessions where project members write and collectively critique items. Each item is structured around a mathematical task of teaching. The process of expressing a mathematical task of teaching as an item and collectively deliberating about its mathematical and pedagogical integrity extends the initial analysis of a mathematical task of teaching and the key part of the process of extracting from the particulars of an instance of teaching to the identification of a generic mathematical task of teaching. In other words, it serves, as did the formulation and use of instructional tasks in teacher education as described above, to advance an understanding of mathematical knowledge for teaching in three ways: (i) through refinement of initial notions of mathematical knowledge for teaching gleaned from analysis of teaching; (ii) through engagement in and continued learning about the mathematical reasoning and mathematical demands of
teaching; and (iii) through analysis of ways in which the ideas play out in a larger system responsible for the mathematical education of children.

To illustrate these points, consider the following two items, presented by Rathmell, Townsend, Gabriele, and Leutzinger (2008). These are items that one could imagine the University of Michigan researchers using, but first needing to modify because the connection to teaching is missing, implicit, or distorted. Unlike items produced by the Learning Mathematics for Teaching project, the item below on the comparison meaning of subtraction is missing a brief scenario that situates it in a mathematical task of teaching.

![Addition/Subtraction Concepts](image)

Figure 3.20. An item not explicitly situated in the work of teaching (taken from Rathmell et al. (2008)).

The implicit task of teaching would seem to be something about needing to identify a problem involving comparison to use with students. When might teachers do this and to what end? For example, the item could begin by describing a situation in which a teacher is searching a textbook to find a problem that uses a comparison meaning of subtraction, and she is trying to spot one. But this is not the only mathematical task of teaching that might be imagined for this item. One could imagine a teacher wanting to order these for use with her students, perhaps from easiest to hardest, or wanting to pick one that is most like or most unlike one she has just used with her class, or walking by a student doing individual or small group work and needing to quickly make sense of his or her thinking and decide whether or how to intervene (which might very well include recognizing that the problem the student is thinking about is one of comparison). Each of these might involve deciding whether problems involve comparison. Yet, each of these is slightly different, with slightly different demands. Developing a mathematical storyline for a particular sequence of problems is somewhat different from spotting or appraising difficulty, and these are somewhat different from identifying similarities and differences among problems. A different
kind of contrast among these tasks is that the first might be done more leisurely, while the last might demand quick thinking on one’s feet.

Making a decision about which of these possible tasks to pursue has implications for the specific nature of the context that needs to be provided in the set up for the item. Simultaneously, deciding on aspects of the context shapes the nature of the actual task. Not only does the situation need to be described, the item writer must decide what information is necessary to adequately respond to the item. For instance, if the question is to be which of these four problems involves comparing then the item writer has to figure out where in the process of any of these teaching tasks this question arises. Using perhaps the most direct task, a context might be as follows.

Ms. Lowry has been teaching her students to solve story problems involving comparison that can be solved using subtraction. The class has just discussed several, but she wants to give them another. She glances in the textbook and sees the following four problems. Which of these involves comparing?

The question asked here is the same as in the initial item, but the context provides a focus for the question and shapes what one does to answer it and what it measures.

Using perhaps a less immediate task, but one that still involves asking which of several problems is about comparing, a context might be developed around looking for a mathematically similar task.

In a lesson called Finding the Difference, Mr. Chang’s textbook gave the following problem.

**There were 23 chairs in the room. Twelve of the chairs were occupied. How many chairs were unoccupied?**

This was the only example the textbook gave, but Mr. Chang wanted to use a second example. Looking in other resources, he found several subtraction problems. Of the problems below, which is mathematically most similar to the problem given in the textbook?

This problem does not explicitly refer to “comparing.” It refers to “difference” and to identifying “mathematical similarities” among problems. These are subtle shifts, but the point here is that this process of developing an idea into an item involves careful consideration and specification of mathematical tasks of teaching and exercises one’s sense of what is involved in addressing them and of mathematical knowledge for teaching.
Another example of the way in which the development of an item can provide a site for clarifying and advancing an understanding of mathematical knowledge for teaching is suggested by a second item, where the formulation of the mathematical task of teaching seems a bit off the mark.

![Using Addition Fact Strategies](image)

Figure 3.21. An item with an inauthentic task (taken from Rathmell et al. (2008)).

The task here does not seem quite right because as a teacher you would not typically confront a question quite like this. Students might produce drawings like these, and you might need to figure out what students were thinking, but you would not get the thinking and then need to figure out which drawing fit it. With this in mind, the item might instead give a drawing and ask what the student who produced the drawing might be thinking. This task is common in teaching. Or, it might describe compensation as a strategy for solving a subtraction problem and ask the teacher to generate a drawing for the strategy. This too is common in teaching. It is true that one might conjure a scenario in which several solutions with drawings are given on the board and a student says that eight plus six is the same as seven plus seven, but if it wasn’t clear which drawing the student was referring to the teacher would probably just ask.

Throughout these comments about item writing are judgments and questions that need empirical support, but my goal here is to demonstrate ways in which item writing can serve as a site for the refinement of existing notions and the development of new notions about mathematical tasks of teaching and the mathematical knowledge demanded by them. Item development provides a venue for exercising and studying mathematical tasks in action.

This activity is extended with interviews of people responding to items as well. For instance, in analyzing these interviews the researchers identified a set of items that all seemed to
involve a general mathematical task of evaluating the difficulty children might have with a particular mathematics problem or concept, and they generated a general specification of the work involved in evaluating difficulty.

*Evaluating difficulty:*

1. Tracking on the instructional purpose of the problem and/or when in the students’ learning trajectory the problem is being used, and what that implies for the mathematical territory of the problem.

2. Working through the given problems, playing with different ways that students might solve them, and determining what is different mathematically about the problems and how these differences might impact students’ thinking or their approaches to solving the problem.

3. Identifying how the differences identified in the problems are easier or more difficult for students this point in their learning. (070926TskMem)

This general specification was built from analyses of the demands of specific items, such as the specification below for an item asking which of four problems on proportional reasoning would be most challenging for a particular group of students.

In particular:

1. Recognizing that, because the problems are being given at the beginning of unit to introduce students to proportional reasoning, students would not have formal methods (like cross multiplication) available and would therefore be using more intuitive reasoning strategies.

2. Considering how students might reason proportionally without using formal strategies, for example by adding proportional amounts, and then solving each of the problems, observing how the numbers in the problem impact the ease of these informal strategies.

3. Recognizing that (a) and (c) can be reasoned more readily in an additive manner, whereas (b) presses more for a scale factor of 5/4 or a rate of 7/4 buttons per paper clip, numbers that are not as easy to reason with intuitively and make the multiplicative relationship less apparent to students. (070926TskMem)

The researchers also developed insight into potential distinctions among tasks. For instance, they had identified a mathematical task of *choosing examples*, but from their task analyses of people’s work on items they developed two distinct tasks within this category.
Choosing examples – selecting a problem for an exercise:

1. Tracking on the instructional purpose for the exercise (e.g., introduce a procedure, assess student understanding, provoke error, highlight a special case, encourage multiple approaches, etc.).

2. Considering the features of or what happens with particular numbers or examples by working through the given problems, playing with different ways that students might solve them, and determining what is different mathematically about the examples and how these differences might impact students’ thinking, their approaches to solving the problems, or the mathematical issues that might arise.

3. Identifying what feature of the example addresses that instructional purpose and whether aspects of the examples obscure or get in the way of the instructional purpose.

Choosing examples – illustrating a concept

1. Considering what is not captured or what could be incorrectly overgeneralized from the examples:

2. Looking for other relationships/similarities among the examples that might hide (or be focused on instead of) the concept/relationship being exemplified

3. Looking for aspects of the concept that are not represented by the examples

4. Considering special cases that might not be captured by the set of examples (070926TskMem)

As these two specifications indicate, there is a notable difference between selecting an example problem and developing or appraising a set of examples to illustrate a concept. The former focuses more on a kind of task analysis, where consideration of different approaches and the details of working through these approaches are central. The latter focuses more on a kind of mathematical defining, where consideration of both mathematical precision and psychological suggestion are central.

In these examples, we see ways in which the development of items can serve as a site for extending the study of mathematical knowledge for teaching.

Building theory. A second way in which research on measures of mathematical knowledge for teaching can inform an understanding of mathematical knowledge for teaching is through an identifying and testing of measurable sub-domains. The researchers argue this point in several funding proposals.
To date, our efforts to develop measures of content knowledge for teaching mathematics have yielded advances toward understanding the component elements of this construct. This empirical work, for instance, has found that individuals’ basic, or “ordinary” mathematics knowledge plays a strong role in determining the probability of a correct answer to most items. At the same time, however, we are able to discern separate and distinct abilities which correspond to a) *content knowledge as it is used in teaching* – making and using representations, analyzing and appraising non-standard methods, evaluating conjectures, and constructing and judging the adequacy of mathematical explanations (Hill, Schilling, & Ball, 2002); and b) *interpreting and appraising students’ mathematical thinking and work*. (030806 RETAPrp)

Here, the relationship to practice is more distant, but components being identified are defined in terms of mathematical tasks of teaching and the efforts to isolate measurable components contributes to overall attempts to define, organize and understand mathematical knowledge for teaching. As the researchers write:

> Our work on measures development can continue to provide such theoretical insights, including identifying additional elements of content knowledge for teaching mathematics, and allowing more opportunities to investigate the structure and organization of this knowledge. (030806 RETAPrp)

Research on measures has contributed to the mapping domains and sub-domains of mathematical knowledge for teaching as described in Ball, Thames and Phelps (2008).

*Validating measures.* A third way in which research on measurement has advanced an understanding of mathematical knowledge for teaching has been through extensive work on validation of the instruments. Interviewing teachers, mathematicians, and “laypeople” as they work through the items has provided insight into the mathematical demands of particular items and particular tasks of teaching, and contrasts among these groups has illuminated the distinctive nature of the mathematical knowledge and skill necessary for addressing the tasks. The items engage people in a kind of pseudo-teaching activity where, analogous to the “job-analysis” of actual classroom teaching, one can study the character and the demands of the work.

In addition, the researchers developed a set of codes for analyzing the mathematical quality of instruction as a way to validate an underlying assumption that a teacher’s mathematical knowledge for teaching shapes the mathematical character of the teacher’s teaching, which shapes the mathematical character of the interactions that constitute instruction, which influences student learning. The researchers used mathematical tasks of teaching as the initial basis for the video codes, and the development and use of the video codes served as a site for exercising initial
conceptions of mathematical tasks of teaching in the practice-focused context of evaluating the mathematical qualities of instruction. For instance, here is a description of a code related to the selection of examples. In this code, the focus is on the selection of contexts, numbers and cases.

*Selection of numbers, cases & contexts for mathematical ideas:*
Use of real-world or pretend contexts as the settings for developing ideas and procedures, selection of numbers for problems and examples, and selection of figures, shapes and cases (e.g., in geometry). Code for the appropriateness of the teacher’s selections in relation to working on specific mathematical content with these particular students. This includes attention to staging and sequencing, care that the numeric or contextual detail matches and does not obscure or confound the development of the mathematical content.

(060501GlsMem)

The need to develop a reliable video code effective for identifying mathematical knowledge for choosing examples seems to have led to a shift away from choosing problems, which are often given by the curriculum materials, to choosing some of details that arise when trying to enact problems and explanations in real-time instruction. It is as if the need to develop a reliable measure creates a demand on the analytic focus of a specification of a mathematical task of teaching that further exercises that mathematical task with the demands of practice that are empirically evident across multiple instances of practice.

3.3.4 *Deciding What Matters and Building Coherence*

The third approach, then, identifies mathematical knowledge in relationship to its use, or relevance, in teaching and it expands the notion of “practice” to include wider arenas and wider communities in which mathematical knowledge for teaching is put to use, exercised, and taken through its paces. It has yielded compelling examples of a body of currently unrecognized and inadequately tapped knowledge that seems promising for improving mathematics teaching and resultant student learning.

When looking comprehensively across the contributions of the third approach, several challenges seem to emerge. The contribution that stands out is the detailed set of compelling examples. However, these seem more suggestive than conclusive, at least at this point in their development. Missing is an adequately complete cataloging and organizing of such problems or tasks into a coherent body of semi-codified knowledge that would support the specification of a curriculum for the mathematical education of teachers and criteria for the evaluation of teachers’ professional knowledge. This may simply be a reflection of the need for more time to complete
the work, but it may be a more fundamental challenge that requires the development of better approaches to studying practice.

Another challenge for the approach has to do with knowing how to determine what counts as a mathematical task and what does not. What is the right grain size? In addition, what is the threshold of generality or frequency to qualify as a mathematical task of teaching? In part, this problem returns to the need for establishing a clearer picture of what is meant by work of teaching more generally, but it is also a problem specific to identifying mathematical tasks of teaching where the choice of what to distinguish and emphasize is driven by a need to improve the mathematical education for teachers in ways that will improve teaching and learning. In the end, it is probably an empirical problem that will require concerted work on the part of both practitioners and researchers.

3.4 Approaching the Study of Practice

My aim in this study is to understand more fully what might be involved in the study of practice for the purpose of identifying mathematical knowledge for teaching. To achieve this, I am examining the work of a research group engaged in such work, simultaneously attending to: (i) what the researchers have done in their work; (ii) what seems to be involved in such research; and (iii) what these suggest for how the study of practice might be done. In the previous sections, I have argued that an examination of the work done and produced by the research group yields three analytically distinct approaches, each forming a unique constellation of questions, data, methods, and results.

Differences in the research questions shape ensuing work and results. In the context of dissatisfaction with existing characterizations of the mathematical content teachers need to know, each approach offers a particular way of opening up new territory, new considerations, and new possibilities. A focus on the mathematics that arises in classrooms opens up a space for disciplinary perspectives to propose new considerations and novel language and terms for the examination of classroom practice. It also has the potential to increase attention to mathematics in the observation and examination of teaching and learning. A focus on the work of teaching, with systematic consideration of the mathematical demands of that work, can surface novel mathematical dimensions of mathematics teaching and has the potential of infusing mathematics into widely held conceptions and established cultural practices of mathematics teaching. In contrast, a focus on the mathematical work of teaching provides a way of wedding attention to mathematics and attention to teaching in research on teaching and in efforts to improve the mathematical education of teachers.
This analysis was not meant to establish that these are the only three approaches, or the only ways of carrying out the three approaches, or even that the University of Michigan research group used these three, and only these three, approaches. Instead, it was designed to create a broad vision for how such study might be done and what might be involved. This analysis suggests some of the different questions that might be asked and what these questions afford, and it gives an initial sense of the challenges, pitfalls, and limitations of each approach. This analysis also provides an important overview of the terrain in ways that set the stage for more focused investigation of the design of the underlying analyses used in these approaches: the development of a mathematical analysis for this work; the development of a pedagogical analysis; and the intertwining of these. This chapter has provided an initial answer to the question of what is involved in the study of practice for the purpose of identifying mathematical knowledge for teaching. This initial answer will serve as a reference point for pressing further into this study of the study of practice.
CHAPTER 4: COORDINATING MATHEMATICAL AND PEDAGOGICAL PERSPECTIVES IN THE ANALYSIS OF PRACTICE

In the last chapter, I examined broad components of studying practice for the purpose of identifying mathematical knowledge for teaching and identified three distinct approaches. In this chapter, I look more closely inside activities that constitute such study, in particular at the use of mathematical and pedagogical perspectives as theoretical lenses for the analysis of practice and the coordination of these perspectives in an analysis. What might it mean to use the discipline of mathematics as an analytic tool for studying practice? What might it involve and afford?

Second: What might it mean to use a pedagogical perspective as an analytic tool for studying practice, in particular when the purpose is specific to identifying mathematical knowledge for teaching? Other researchers have used pedagogical perspectives in the analysis of practice, but might this be distinctive in work specifically focused on identifying mathematical knowledge for teaching? What can such a consideration suggest back to a more general conversation about using a pedagogical perspective to analyze practice? Finally, how do these perspectives stand in relation to each other in such research? What might it mean to intertwine or coordinate these theoretical lenses?

This chapter has three major sections, one on using a mathematical perspective to analyze practice, one on using a pedagogical perspective to analyze practice, and a third on coordinating these perspectives.

4.1 Specifying a Mathematical Perspective

In Chapter 1, I argued that the notion of “using a mathematical eye as a tool of the inquiry,” as described by Ball and Bass (2000b), while provocative and appealing, calls for greater specification. What is a “mathematical eye” and how does one use it as a tool for analysis? Given the goal of developing new insights and formulations of mathematical knowledge for teaching, it perhaps makes some sense to consider bringing a disciplinary perspective to the examination of teaching and learning. Indeed, it might make some sense to develop a mathematical perspective for the analysis of mathematics teaching and learning in general, specifically because it is mathematics that is being taught and learned. Or, broadened in a different direction, it perhaps makes some sense to bring a disciplinary perspective to the
analysis of teacher knowledge demands for the teaching of that discipline, whatever the
discipline. All of these proposals seem quite plausible, but they beg a central question: what,
after all, is a disciplinary perspective? Or might be meant?

This section, then, attempts to answer the following questions:

• What is a disciplinary perspective that might be used to study practice — specifically for
  the purpose of identifying mathematical knowledge for teaching and perhaps for other
  purposes?
• What is involved in using such a perspective?
• What light might such a perspective shed on the study of practice, in particular when the
  purpose is to identify mathematical knowledge for teaching?

The focus will be on the first of these, but the other two questions are part of the discussion. To
some extent, I use the second, engagement of such a perspective, as a way of specifying a
perspective, and the third, the consideration of the light it sheds, to contribute to my argument for
the specific perspective I propose. Before I begin, though, I first clarify what I mean by
“disciplinary perspective.” I also argue that such a thing does not already exist, or rather, that
other notions of a “disciplinary perspective” that might be seen to exist in the literature do not
adequately address the needs I lay out here. Last, I attempt to say something more about why
such a perspective might be important, yet hard to develop, and what it might contribute. In these
preliminary remarks, I hope to clarify the underlying motivation and goal for this analysis.

In seeking to specify a disciplinary perspective, I do not mean to suggest that the one I
propose is the only, or even best, perspective. My goal is to formulate a possible perspective —
one that seems reasonably identified as a disciplinary perspective and yet is suited for the analysis
of teaching and learning. In this, by “disciplinary perspective” I have in mind more than an
epistemological outlook. Instead, I have in mind a perspective on the doing of mathematics.

As I argued in Chapter 1, the improvement of teaching and learning requires a focus on
the dynamics of instruction, in other words, on the interactions among teacher and students
around content. Thus, for the purpose of analyzing practice, I propose that a disciplinary analysis
that attends to the dynamics of doing mathematics is particularly well suited for the study or
practice. Put another way, teaching and learning mathematics are activities that include, at their
core, some version of doing of mathematics: on the one hand, there is a great deal about school
mathematics that might be seen as quite distinct from doing mathematics as professional
mathematicians do, and on the other hand, it is hard to say that when children are doing
mathematics problems in school, that they are not, in some sense, doing mathematics, and
although the mathematics done in school can be limited in scope, it certainly includes some degree of reasoning, of solving problems, of communicating mathematical ideas, and so forth.

This is related to an argument given by Ball and Bass (2000b) that disciplinary practices of building knowledge can provide insight into the mathematical knowledge building activities of learning mathematics. I propose a broadening of that argument to include a broad range of the activities of research mathematics and mathematics teaching and learning that may be, at least to some degree, analogous, beyond the more focused attention to mathematical reasoning that they give. For instance, disciplinary practices of verifying knowledge, communicating knowledge, searching for patterns, and posing problems might provide insight into classroom activities of checking solutions and claims, giving clear expositions, developing mathematical curiosity, and so forth. Thus, I seek to articulate what might constitute a mathematical perspective on the doing of mathematics that can be used to analyze teaching and learning.

Unfortunately, although a number of mathematics education researchers bring significant mathematical expertise to their analyses of teaching and learning and take up significant mathematical issues arising in classrooms, their research is typically conceived and organized from other theoretical perspectives, such as theories of learning and social interaction or theories of instruction. Systematic use of a mathematical (or disciplinary) perspective as a primary theoretical lens in education research remains uncommon. In addition, it is not clear where to turn for a formulation of a disciplinary perspective that would be well suited for studying teaching and learning. Two potential sources seem worth considering: the writings of research mathematicians and those of philosophers of mathematics. These are not completely distinct categories, but they provide a characterization of different bodies of work. I discuss each of them briefly.

Numerous mathematicians have written about the experience of being a mathematician and about the doing of mathematics from that experience. Certainly these are a potential source for ideas about formulating a mathematical perspective. However, two concerns emerge. One is that these rarely constitute disciplined study of mathematical practice. Mathematicians are not trained to do such work. Indeed, social science methodologies required for such work are typically unfamiliar and even foreign to research mathematicians. Furthermore, as numerous social-cultural anthropologists have argued there are good reasons to distrust the folk theories a group holds about itself and the things it does (Geertz, 1973). This is not to say that such theories are necessarily wrongheaded. They have potential, but they are not necessarily to be trusted. Also, this is not to say that they should not be consulted. As Kitcher (1984) argues, a proposed
interpretation of mathematical practice ought to be compelling to those who practice the profession.

A second potential source is the writings of philosophers. In contrast to the philosophy of science or the philosophy of history, the philosophy of mathematics has historically focused on epistemological issues (Mancuso, 2008). Over the past few decades, however, a growing body of work has been developed that examines the doing of mathematics (Lakatos, 1976; Kitcher, 1984; Mancosu, 2005; Mancosu, 2008). Much of this literature focuses on particular practices, such as the use of visual representations or proving, and few provided comprehensive frameworks for examining the doing of mathematics. This work could be mined for ideas about the constitution of mathematical practice and of a mathematical perspective and could be adapted to use in analyzing teaching and learning. However, my project here is different. I turn, instead, to an examination of project documents to do a kind of reverse engineering to create a proposal for a perspective, grounded in the discipline, for analyzing practice. Working inductively from past analyses, I am trying to bring to the surface an implicit focus of the researchers and to draw out features of that focus that seem to yield observations productive in the study of mathematical knowledge for teaching. One motivation for this approach is the goal of generating a perspective that, while still grounded in the discipline, is also suited for the analysis of teaching and learning. Much of the literature in the philosophy of mathematics is being developed for a different set of purposes and, perhaps because the philosophical study of mathematical practice is still in its relative infancy, has a certain degree of remoteness from teaching and learning.

This comment about a sense of remoteness, from teaching and learning, of the available literature raises an additional issue that motivates both the set of questions I posed above and the approach I take to address them. In Chapter 1, I argued that a challenge in creating a practice-based theory of mathematical knowledge has to do with a basic divide between disciplinary knowledge (of mathematics) and professional knowledge (of teaching). I described part of this divide as arising out of the need for usable knowledge — as arising out of the gap between the ways disciplinary knowledge is represented as part of a canon to be taught and the needs teachers have for personal, practical, craft knowledge readily deployed in teaching and learning. I argue here that, in addition, the divide grows out of other important differences between the world of mathematics and the world of teaching and that these differences matter for communicating and cooperating across this divide and for generating and specifying a disciplinary perspective that is specifically designed to bridge that divide.

One sign of such a divide lies in the susceptibility that mathematics education seems to have for controversies often dubbed the “math wars.” Ideological controversy is certainly not
new in education. However, in this case, it is largely a divide between research mathematicians and mathematics educators. In saying this, of course, I do not mean to suggest that all research mathematicians are aligned in a single position, but to suggest that perhaps there are fundamental differences in perspectives that help to explain why a group of research mathematicians seem to be the backbone of one group in this highly charged clash and a group of mathematics educators seem to be the backbone of the other. This clash between a mathematical and a pedagogical perspective is also suggested by the anecdotal stories of research mathematicians who experience certain kinds of angst about teaching because student thinking can be messy in ways that they perceive as an affront to a set of aesthetic sensibilities they associate with being a mathematician.

In making these claims, I do not mean to oversimplify or to overgeneralize, but I mean to point at a difference that creates a basic challenge for developing a disciplinary perspective that can effectively speak to issues of teaching and learning. Where teaching operates with insufficient knowledge, aims for multiple, often-conflicting outcomes, and is based on uncertainty and contingency, doing mathematics is, relative to teaching, often about exercising a prerogative to define terms, to establish constraints that manage complexity, and to remain somewhat detached from the social world around it. Accompanying these differences in purpose are different forms of knowledge and ways of knowing. Teaching is situational. Teacher knowledge can be triggered by the contexts in which it is used and by the colleagues with whom one works. It is also more tacit and analogical than is typical of mathematical knowledge and mathematical ways of knowing. Mathematical knowing, in contrast, tends to be logical and hierarchical. Its knowledge is, relatively speaking, definitive, unchanging, cumulative, and independent of time and context. Finally, differences in purpose and ways of knowing are related to differences in activities in each domain. Central to teaching are activities of enacting the mathematical knowledge one has in the context of teaching mathematics to children and of unpacking knowledge into forms that make it more readily examined, taught, and learned. Teachers need to motivate mathematical processes and concepts. Doing so involves making these ideas concrete, decompressing them, and reconnecting them to their origins. Mathematics, in contrast, is predominantly a process of abstracting, idealizing, and compressing processes and concepts into powerfully compact forms.

This is an argument about there being important differences between these two worlds in ways that matter for the analysis of teaching and learning. I do not mean to be making claims about the cause of these differences. However, one potential reason for them that is worth considering in this context is suggested by Waller’s (1932) argument that “teaching makes the teacher.” After arguing that teachers tend to have certain character traits, he argues that these
result from the kind of work that teaching is — that the work teachers do over time tends to turn them into certain kinds of people. The analogy is that doing mathematics over time might turn mathematicians into certain kinds of people, into people shaped by their work. Thus, differences in the work of these two groups of people lead to differences that matter for meaningful communication.

Thus, the differences in character, purpose, knowing, and practice, summarized in the table below, raise certain challenges for the proposal of using a disciplinary perspective to analyze practice. These differences are not prohibitive, but they have important implications. Figuring out how to bridge these differences is key.

<table>
<thead>
<tr>
<th>Character of work</th>
<th>World of Teaching</th>
<th>World of Mathematics</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flexible, fast-paced, uncertain, and relational</td>
<td>Aimed at precision &amp; rigor, often single-minded, solitary &amp; impersonal</td>
<td></td>
</tr>
<tr>
<td>Purpose</td>
<td>Focused on practical outcomes &amp; effecting change in others</td>
<td>Focused on establishing claims &amp; building theory</td>
</tr>
<tr>
<td>The nature of knowledge and ways of knowing</td>
<td>Situated, analogical, networked, and tacit</td>
<td>Independent of broader social context, logical, &amp; explicit</td>
</tr>
<tr>
<td>Activities in relation to knowledge</td>
<td>Unpacking and enacting</td>
<td>Compressing and abstracting</td>
</tr>
</tbody>
</table>

Figure 4.1. Differences between the world of teaching and the world of mathematics.

My goal, then, in this section is to specify a mathematical perspective that can be used to analyze practice in ways that effectively bridge the divide and can be used to speak to the world of teaching and learning. What might constitute a disciplinary perspective for analyzing teaching and learning — in general and for the specific purpose of identifying mathematical knowledge for teaching? What might it look like and what might be involved in using it? Embedded in these questions, however, is another: given the differences between the worlds of teaching and mathematics described above, what might it mean to take mathematics, with its independent agenda and structure, and figure out a version that is closer to the work of teaching?

To address these questions, I generated some initial hypotheses and then worked iteratively back and forth between project documents and revised hypotheses. This led me to
posit four strategic domains constituting a mathematical perspective potentially useful to the
analysis of practice — a kind of initial proposal for a theoretical perspective for analyzing
practice, one based in the discipline of mathematics seen as an intellectual heritage sustained by
an evolving system of professional norms and social practices. After providing an overview of
this process, I offer a detailed analysis of each of the four domains.

4.1.1 An Initial Hypothesis About the Composition of a Mathematical Perspective

According to the OED, the word *perspective* stems from a thirteenth century Latin
translation of Aristotle meaning to see through, look closely into, or discern. Its use as a noun
flourished with the development of perspective drawing and the science of optics in the fifteenth
and sixteenth centuries. A perspective is about a view of things from a certain angle or vantage
point. It involves standing in a particular position, viewing the world from there, and
systematically describing the view.

The angle, or vantage point, I have in mind is from the discipline — from the experience
of doing mathematics. Using mathematics as a theoretical lens, then, requires paying attention to
the ideas and sensibilities, the core questions and preoccupations, of the discipline, and using
these to examine the activity of elementary school teaching and learning. However, because the
analysis is of teaching and learning and because the purpose is to open up and clarify the
mathematics that would be useful to teachers, it may be worthwhile to frame the disciplinary
perspective with elements that intersect or seem particularly compatible with concerns of teaching
and learning. This suggests some potential foci.

First, such a mathematical perspective might include focused attention to issues of
mathematical language. This makes sense because teaching and learning are linguistically
intense. Here, I use the term language broadly as anything about the communicative act —
including talk, symbolization, and representation. Second, it should consider the nature and
substance of mathematical reasoning. This is important because teachers’ primary purpose is to
effect learning, and mathematical reasoning, as the disciplinary foundation for the generation of
new knowledge, provides a key support for learning. Third, mathematical perspective for the
analysis of teaching and learning should consider both issues of mathematical knowledge and of
mathematical activity. Teachers and students engage routinely in mathematical activity. What is
the nature of teaching and learning as mathematical activity? A mathematical perspective needs
to consider aspects of mathematical knowledge at play in the classroom, but also the nature and
role of the mathematical activity taking place. Last, the perspective should be eclectic in the
sense that it should explore a variety of epistemological stances and the sensibilities and
approaches of different branches of the discipline—algebra and geometry, logic and analysis, and pure and applied.

To further develop and test these ideas I analyzed project documents with an eye for what being done and said seemed to constitute the use of a mathematical perspective and might contribute either broadly to an understanding of teaching and learning or more narrowly to an understanding of mathematical knowledge for teaching. I also considered literature that might inform the analysis, but directly relevant literature seems limited and is only used sparingly. One principle evident in the comments above and used to guide the analysis is that the particular version of a disciplinary perspective that would be useful to an analysis of teaching and learning would be one that emphasizes elements that are important both to the discipline and to teaching and learning. In other words, as argued above, a focus on explanation is likely to be informative because it represents a key element both in the discipline and in teaching and learning. This principle has shaped the selection and framing of the four themes identified below.

To begin, I analyzed the content of comments. I read through a representative sample of project documents, considering each sentence as a means of commenting on something, of drawing something to the fore in the research. For each sentence, I asked: What is this a comment about? I wrote short descriptions for each and then reviewed these and created categories. Five broad foci emerged as salient.

• Mathematics
• Students
• Teacher
• Instruction
• Research

These foci make sense given the purpose of the research. I then looked more closely at documents and parts of documents engaged in the analysis of practice. Regarding comments about mathematics, I asked the question: What about mathematics is the comment about? From descriptions, I created 39 low-level categories arranged in 8 broader categories, which I iteratively reapplied to the data and revised. I then grouped these into the following four themes.

• Problems and answers
• Solutions and explanations
• Expressions and language
• Character of the work

If taking of a mathematical perspective is about looking at practice with a mathematical eye, then I am claiming here that part of what it might mean to look in this way, from this angle,
is about systematic examination of what is happening in regard to these four themes. It is about watching for the mathematical problems posed or engaged in and about the answers given for those problems. It is about watching for the genesis of solutions and rationale provided for them. It is about looking for ways in which ideas are expressed, with sensitivity to the nature of language use in relation to the work being done. Last, it is about examining characteristics about the mathematical work being done — who is doing it, what it is, when it is being done, and other key features. These four themes provide features on which to focus in conducting a mathematical analysis of practice. However, the four labels may be less important than the particular versions of them offered below, versions that are central to the discipline of mathematics yet selected and framed in ways that make them both fundamentally disciplinary in nature and yet relevant to teaching. The particular formulations developed in this chapter provide versions of each of these that link between the world of mathematics, on the one hand, and the world of teaching and learning on the other.

<table>
<thead>
<tr>
<th>Problems and Answers</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis of the mathematical structure of problems and answers and of the role of problems and answers in the conduct of mathematical work in the classroom can inform the selection, creation, and use of problems and tasks in classroom teaching.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Solutions and Explanations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis of twists and turns in solving and explaining in the classroom, particularly in terms of disciplinary structures and conventions, can inform mathematical motivation and sense making in classroom teaching and learning.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Expressions and Language</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis of expressions that fumble with mathematical ambiguity, including the ideas that might emerge from them, and the work involved in clarifying them and making them precise can inform the interpretation of classroom events and the development of student thinking and of collective mathematical work in the classroom.</td>
</tr>
</tbody>
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<table>
<thead>
<tr>
<th>Character(s) of the Work</th>
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</thead>
<tbody>
<tr>
<td>Analysis of global features of the mathematical work that goes on in classrooms, both overall characteristics of the work and the mathematical characters who do the work can inform the sequencing and structuring of work on instructional tasks, in particular the collective work as engaged by students learning to carry out specific tasks and roles in doing mathematics.</td>
</tr>
</tbody>
</table>

Figure 4.2. Overview of four themes (foci) of a mathematical analysis.

Each of the next four subsections provides a rationale for a theme, specifies what is involved, illustrates the theme, and gives examples of results.
4.1.2 Problems and Answers: Windows on Mathematical Structure and Work

Mathematics is in many ways about formulating problems and generating answers. In 1900, David Hilbert posed two-dozen open problems that provided direction and focus for over a century, with answers — and their existence or non-existence — remaining topics of lively debate even today. Can any even number greater than two be expressed as the sum of two primes? If \( n \) is an integer greater than two, does \( a^n + b^n = c^n \) have any positive integer solutions? Hilbert’s list, and his characterization of what constitutes a “good” mathematical problem, epitomizes the view that solving problems is the heart of doing mathematics professionally.

Problems and answers are also the focus of school mathematics, perhaps even its defining features. Teachers give students problems to solve, perhaps after demonstrating the solution of similar problems, and students demonstrate their proficiency by answering problems. However, the nature of problems and answers in school is rather different from those that are central to research mathematics. Much has been written about this difference. Lampert (1990) indicates some of the tension between disciplinary notions of problems and solutions and school-based notions of questions and answers. Her provocative title, *When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching*, is meant to juxtapose, on the one hand, a disciplinary “zig-zag between revising conclusions and revising assumptions” carried out with “courage and modesty,” and, on the other hand, school mathematics which is often about “being able to get to the right answer, quickly” (pp. 30-32). She argues that problems in school mathematics are often not problems to be solved as much as computations to be performed, and the division of labor, with teachers posing problems and students providing answers, breaks with the reciprocal nature of problem formulation and answer generation that typify work in the discipline. In some ways, these differences make sense; Teachers and students are not research mathematicians and their endeavor is different.

But how, then, might a disciplinary lens on problems and answers in the classroom be helpful in the analysis of practice? Can consideration of problems and answers provide a kind of point of intersection between the discipline and the teaching and learning of mathematics in schools? From analysis of project documents, I propose that analyses of the mathematical structure of problems and answers and of the role of problems and answers in the conduct of mathematical work in the classroom offer ways of formulating a version of the discipline that fit with the activities of teaching and learning. As such, a disciplinary perspective on problems and answers can provide an important bridge between the discipline and school mathematics.

Analyzing problems. I found that in project documents many of the comments about mathematics that seemed productive to the research endeavor were about mathematical problems,
or tasks, either engaged in the classroom or prompted by what was said or done in the classroom. Of the 39 low-level categories, five involved mathematical tasks or problems.

- Noticing tasks — what they are, when they start and stop, and similarities and differences among tasks
- Identifying sub-tasks and how they are related to the larger task
- Noticing when the criteria or constraints of a problem are changed or refined
- Expressing tasks in terms of their basic mathematical structure or mathematically salient features
- Evaluating the mathematical difficulty of a task (not for children, but what it takes mathematically)

Although many analyses of mathematics instruction are likely to consider the mathematics problem given to students, a disciplinary perspective seems to suggest an alert consideration of a wider and distinctly framed set of problems, or tasks, that may surface in the flow of instruction. I use the word task here because it seems to set a lower threshold on what is included. In a sense, many potential problems never achieve the status of a full-fledged problem to be worked on as the explicitly identified goal. Instead, they are suggested by the activity and talk in classroom interactions as part of what is, or might be, done. This term task is not common in the discipline of mathematics, but it seems to capture an important aspect of a distinctively disciplinary attention mathematicians typically give to research being done. For instance, in mathematics small shifts in wording can significantly change a problem; decomposition of a problem is often central to making progress on it; and close attention to the relationship among problems can help to organize the work and create new insights. It would seem that the activity of mathematical research is, in part, about being vigilant to what the tasks are, how they shift, and how they are related, even if they do not typically use the language of “tasks” per se. This is a distinctive disciplinary attention to mathematics problems that goes well beyond the usual attention given to school mathematics problems in other kinds of analyses.

Such a consideration begins simply by noticing tasks, as they show up in instruction, whether they are pursued or not. For instance, noticing asides made by the teacher or by a student

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9 In identifying what about mathematics a comment is about, I often found myself asking what the researcher seemed to have been looking for. The verb look for helped me to consider intention and where the researcher was coming from in ways that seemed in line with getting at the underlying perspective being used to make the comment. I used this language to frame the mid-level categories. For low-level categories, verbs such as notice, express, identify, and evaluate seemed to fit better with the actual use of the perspective in making a comment as a way to do or accomplish something.

10 Of course, the term task is common in the mathematics education literature where it is seen as a significant factor in teaching and learning (Doyle, 1983, 1988; Hiebert & Wearne, 1993; Watson & Mason, 2007). I choose it here because it also seems to best describe the activity-oriented character of a disciplinary analysis of practice.
about how many days it will take if four students leave the class for testing each day, or about when a student’s birthday is given the student’s age and year of birth. It also includes noticing when a particular problem is put in play, when it is laid down, and what its status is — whether it has been fully addressed or has parts that remain unresolved. It includes noticing sub-tasks, sequences of tasks, and transitions among tasks. For instance, in analyses across the first few days of the 1989 school year, a disciplinary framing seems to be used in attending to layers of tasks; for a problem that asks what amounts of money you could have if you have pennies, nickels, and dimes in your pocket and you pull out three coins, the analysis tracks on questions about what one amount might be, what other amounts might be, what all of the amounts might be, and how you know you have them all. These have a natural progression to them, but they might also arise as tangents, or asides. Another set of questions or tasks has to do with checking constraints. For a proposed solution (a group of coins), there are questions about whether it is three coins, what they add up to, and whether it is a new amount. A question arising in instruction, for instance one asked by the teacher, might be seen as indicating a different task to be done. In some cases, this “new” task might be a small part of a larger, coherent, overall task and, in other cases it might be a different task altogether — one that might be left unaddressed or that might be pursued in addition, or instead of, one already in place.

Another aspect of noticing tasks has to do with attending to changes or refinements in conditions that might clarify or fundamentally alter tasks. This might happen with a small change in wording that inadvertently changes the problem. It might happen as part of clarifying the problem. It is about sensitivity to subtleties in wordings and problem formulations. Such changes can be the source of potential difficulties or the source of additional related problems. For example, in the three-coin problem, one might ask how many different amounts are possible or one might ask how many different combinations of coins are possible. (In this version, the objects do not need to be coins; for example balls of three different colors would also serve.) Are these two questions equivalent? Under what conditions are they equivalent? Is one easier? Does answering one of the questions provide insight into answering the other? These pursuits represent a kind of mathematical work typical of research in mathematics.

In concert with noticing tasks is an examination of the mathematical structure of tasks. This includes identifying more general forms of the problem, abstracting out core versions of the problem, and selecting key variables that can then be varied (parameters) to produce families of related problems. Some of this might be suggested by the various problems and wordings that arise in instruction, but it can also be its own pursuit. It also includes an examination of mathematically salient features of tasks. For instance, in synthesizing analysis of the tasks from
the first month of the 1989 school year, one of the researchers identified three genres, each abstracted from the particulars of the classroom and each parameterized (paraphrased from 020322TskMem)

- **Coin problems**
  I have a bunch of coins in my pocket (pennies, nickels and dimes). I reach in and take out \( c \) coins. How much money could I have? What are the possibilities? How do you know when you have them all? (Could use more, or fewer, denominations of coins, for example by allowing quarters. An alternative formulation would be to ask what different combinations of coins are possible.)

- **Permutation problems**
  With some given collection of \( N \) objects: find permutations of them; find and list all permutations of them; prove that you have all of them. (This abstracts from problems about making different numbers by rearranging numerals from the date and about different ways of lining up at the door.)

- **Number sentence problems**
  Make up number sentences that equal \( N \). (Implicit, also, are the range of operations admitted in the number sentence. In addition, the domain of numbers used might vary.)

The researcher’s analysis also includes a list of four common features for these problems.

1. Each of them presents a problem with **multiple solutions**, and thereby allows a natural hierarchy of increasingly ambitious questions that can be posed.
2. Each of them is really one of a family of tasks that involves **parameters**, parameters whose variation affects the **complexity** and difficulty of the various tasks. At one extreme, the tasks are approachable by hands on, essentially empirical exploration. When the complexity is increased their treatment requires the construction of more conceptual or analytical tools.
3. Each of them can be modeled, or represented, with **different contexts**. The features of the contexts can (and do) influence the mathematical work, and understanding of the tasks.
4. Each of them **can be directly engaged** by young students, using only common experience and rudimentary mathematical knowledge, yet it can lead the kids quickly and naturally into substantial and interesting mathematical explorations. (020322TskMem)

The first of these could be about the instructional advantage of having “a natural hierarchy of increasingly ambitious questions” to pose, but it also seems to be about a mathematical impulse to identify the terrain suggested by a problem, as part of the mathematical work to be done. In particular, the researcher elaborated a set of inquiries seen as implicit in the problems.

- Does a solution exist?
- Is a solution unique?
- What are all of the solutions?
- What is a natural organizing structure for the “solution space”??
In a related way, the fourth feature, about accessibility, also seems to be about locating problems in a landscape of increasing complexity and difficulty. In doing mathematical research, a “rudimentary” problem is potentially interesting primarily because of ways in which it informs and fits into a structure of more substantial problems. Likewise, work on a substantial problem often progresses by identifying a more rudimentary one or by situating it in a context where experience can inform the work. Part of the consideration is about the difficulty of the task in the sense of what its solution requires mathematically and about how the difficulties of the different problems are related. Does solving one solve another, or help in the solution? An understanding of the difficulty provides a basis for creating a hierarchical organization.

These aspects can be thought of as features of the tasks, but they can also be seen as suggesting a distinctive mathematical lens for observing tasks in practice. It may be that the particular set of problems in this instance led to the particular formulation of a landscape conceived in terms of existence, uniqueness, completeness, and structure, but the impetus is one of identifying, in general, an organizing structure for a problem; such a structure motivates and directs the mathematical work to be done. This structure is informed by locating the problem in a larger, organized set of related problems. This activity is common in mathematical research, where problems typically sit within some kind of structure that both motivates the work and structures it. One can imagine similarly situating problems encountered in school in a larger family of problems or work, at least when the instruction contains some reasonably engaging exchange about the mathematical issues being taught and learned. And, the larger family or structure might not be one of existence, uniqueness, completeness, and structure. For instance, such a theoretical, disciplinary orientation applied to instruction about multi-digit subtraction, depending on the classroom dynamics, might examine questions of algorithm specification, efficiency, and justification. These three issues are pertinent to any mathematical examination of algorithms. The analysis might also create a hierarchy or structure for any of these three features of algorithms, with analogies to domains or aspects of the discipline that address such questions. For instance, algorithm efficiency is a distinctive sort of study in mathematics, with its distinctive concepts and techniques (called “complexity theory”). Similarly, such an analysis of instruction on negative integers, generated with specific reference to things being said and done in instruction, might reveal a great deal more about key aspects of negative numbers that are currently missing from observations and treatments generated by curriculum designers, mathematics education researchers, or professional mathematicians who express their opinions but are not engaged in such an analysis of practice.
Analyzing answers. Another set of low-level categories is about answers. Along with examining the tasks and problems that arise in instruction, in a mathematical analysis one might examine the answers, partial answers, and hints of answers that arise. These fold into the structural analysis of problems described above.

- Noticing answers
- Expressing answers in terms of their basic mathematical structure or mathematically salient features
- Noticing when there are multiple solutions and the mathematical structure of the solution space
- Noticing when solutions to a problem have a similar mathematical structure with solutions to some other problem that occurs earlier, later, or that might have occurred
- Noticing mathematical moves that advance the correct answer

Answers are common currency in mathematics classrooms. However, attentive listening for answers from a disciplinary perspective suggests more, mathematically speaking, than judging whether answers are right or wrong. It is about listening for potential answers, for characterizing those answers in more general terms, and for moves that hold promise for advancing a viable solution. It provides a complementary way to develop a sense of the structure of mathematics at play in a lesson. Similar to the work on identifying tasks, a disciplinary analysis of answers is concerned with the structure of answers, salient features of answers, and the structure into which those answers fit. For instance, in the three coin problem, listening for solutions students offer, translating their responses into a canonical form, such as \(12 = 2p + d\), and tracing these throughout the lesson can begin to develop ways of representing the solution space. The formulations developed move freely into disciplinary forms of language and representation, but constructed from the interactions of instruction. This grounding in instruction puts attention on activities surrounding the content. The disciplinary perspective, then, is not only of the content; it is also of the activity surrounding that content.

This focus on problems and answers as part of a disciplinary perspective is strategic in the sense that such a focus, because it is analogous to the problem-doing and answer-generating work that goes on in instruction, provides a version of the discipline that is closer to the work of teaching. It complements and informs consideration of the instructional tasks being used. For instance, in a set of project meeting notes, the meeting recorder wrote that, having observed a high school classroom where the students are graphing quadratic equations and discussing them, one of the researchers:

Tried to describe the task developmentally. Starts with “sketching” (Sally line 2) and then went to something called “comparing.” (Sally line one from the bottom of first page).
Later Sally says, “There’s a purpose… hang on” (which makes me think that there is another task coming). Then she says, “Let’s logically go through.”

Parts of the task:
1. sketching there are a lot of ways of sketching it
2. agree or disagree
3. next sketch
4. who disagrees with his sketch? Some sketches disagree with this one
5. comparing
6. putting ideas up… will get to the “but” part in a minute
7. “There’s a purpose… hang on”
8. “Let’s logically go through these”
9. “This is something you might see on an SAT test”
10. understand the skills (010613PrjMtg)

Across the notes in this section, a set of mathematics problems suggested by different comments and actions on the part of students and the teacher seems to shape the analysis.

- Sketch a graph of an equation.
- Sketch the graph in a different way.
- Is this sketch an accurate representation of this equation?
- Are two different sketches both accurate?
- What are similarities and differences among sketches?
- How might this set of sketches be logically organized?

One can see here the use of a mathematical perspective that begins by listening attentively to instruction and drawing out a collection of mathematical tasks. Continuing down this path could lead to formulating a more coherent set of mathematical problems and identifying an organizing structure and set of salient features that landscape the mathematical work of graphing quadratic equations in ways that might well inform the work of teaching.

4.1.3 Solutions and Explanations: Windows on Establishing Disciplinary Knowledge

In addition to the analysis of problems and answers, a disciplinary analysis of developing (and developed) solutions and explanations can contribute to the study of practice. Central to the discipline of mathematics is the solving of mathematical problems and the building of mathematical explanations or justifications for those solutions or claims. In many ways, these processes work in tandem; attempts to understand and explain lead to solutions, and solutions, if they are to make sense and be convincing, require explanation and justification. That these two features are central aspects of mathematics is straightforward enough. That they are key to a disciplinary analysis of practice follows because they are not only central to the discipline but also central to the knowledge-building activities of teaching and learning mathematics.
This is not to say that the disciplinary practices of solving and explaining are the same as the school-based practices. Explanation in school, because it is about learning knowledge that is already established, is often more about being able to follow a line of thinking or a sequence of steps assumed to be correct than it is about establishing a mathematical justification in order to determine whether something is true or false. Likewise, solving in school is often about applying routines or algorithms to known types of problems, whereas in mathematics solving is a generative act that involves creativity, intuition, and exploration of the truly unknown. Making mathematics sensible in these two environments is different. For one thing, authority often has a different emphasis in school, where the teacher is an authority figure. Second, making sense in school is oriented toward demonstrating proficiency with known ideas and procedures whereas making sense in the discipline is more oriented toward gaining leverage on deductive arguments for establishing the validity of claims. Even though these differences are significant, solutions and explanations differ more by degree than by nature in the two contexts. Part of making sense of mathematics in school is, indeed, about having a mathematical rationale for ideas and procedures. As Ball and Bass (2003a) argue, mathematical reasoning constitutes a basic skill in school mathematics because it is essential to understanding mathematics, to using mathematics in meaningful and effective ways, and to reconstructing faded knowledge (p. 28). Similarly, in the discipline, the acceptance of a mathematical argument is to an extent determined by its review and approval by reputable colleagues. In essence, established colleagues are authority figures and have an authority similar to that of a teacher. In both contexts, the acceptance of solutions and explanations is a collective, negotiated process among less and more expert colleagues, including the author.

How, then, might a disciplinary lens on solutions and explanations in the classroom be helpful in the analysis of practice? What is a disciplinary view of solutions and explanations and does a disciplinary consideration provide a helpful view of mathematics teaching and learning in schools? From my analysis of project documents, it seems as if, in examining classroom interactions, attending to the twists, turns and emerging possibilities of generating solutions and explanations and framing classroom solutions and explanations in terms of structures and conventions that typify the discipline constitute a disciplinary perspective that fits with the activities of teaching and learning. Formulated in this way, a disciplinary perspective on the solving of problems and on explaining and justifying can provide an important bridge between the discipline and school mathematics.

Analyzing solutions. I found that in project documents many of the comments about mathematics that seemed productive to the research endeavor were about potential seeds and
steps that might lead to solutions. Three low-level categories, while different in their focus, involved a kind of alertness to possibilities that might advance a solution.

- Noticing ideas that might lead to promising claims or solutions
- Noticing unconventional (within the discipline) ideas and approaches
- Noticing activities that constitute a plausible approach to working on the problem or that furthers the work on the problem (generating examples, developing a recording system, summing up current progress or an issue at hand, …)

The mucking-around search for solutions in mathematics seems to call for keeping a steady eye out for possible solutions and intermediate claims. Non-standard ideas and approaches require even an additional level of attention and appraisal, precisely because, being non-standard, their potential may be more readily overlooked. This attention includes an eye for activities that have the potential to open up new insights and novel approaches.

A disciplinary attention to solutions is both similar to and different from the attention given to student thinking in mathematics education. Theories of learning are framed in terms of bridging from prior understanding to new understanding, and attention to such prior understanding is recognized as an important feature of effective instruction. A significant body of literature identifies student misconceptions and the examination of student work and student thinking is seen as an important skill for teachers. A mathematical analysis of practice, however, is not concerned with identifying student misconceptions or student thinking per se, but instead is concerned with the potential of students’ ideas (or the teacher’s) to suggest a solution, additional solutions, or greater understanding. It examines student thinking, not for the purpose of identifying the platform from which instruction must start as it builds a bridge to content knowledge, but to identify the mathematical elements that can be mined in order to find a solution or construct an understanding.

For instance, when Sean claims that six is odd because it is three groups of two, in a mathematical analysis one is less concerned with what is actually in Sean’s head or with what might be done to help him advance his thinking than one is with potential claims implied by the contrast of being three groups of two as well as two groups of three (which is Sean’s basis for saying that six is even). One potential claim, based on Mei’s generalization of Sean’s idea to numbers that are an odd number of groups of two, is that these are numbers with 2 left over when divided by 4. (These were already studied by the ancient Greeks, and observed to be those numbers not expressible as a difference of two squares.) Another potential claim is that it can be split into three equal groups, rather than two; this might be about three equal even-sized groups or not. Here, the situation is about making a claim rather than solving a problem, but the analysis is
similar. Also, the mathematical analysis is not unrelated to the pedagogical one. Seeing the two potential claims might well be part of figuring out what the student might be thinking. However, the focus and purpose are different, and this difference matters for what the analysis brings to light.

*Analyzing explanations.* In addition to listening for possible ways of solving a problem is listening for aspects of reasoning that arises in the exchanges amongst the teacher and students. Nine of the low-level codes were about explanations or justifications and the accompanying decisions about the validity of mathematical claims.

- Noticing explanations, especially new or novel lines of reasoning
- Evaluating the correctness and adequacy of reasoning
- Expressing explanations in terms of their basic mathematical structure or mathematically salient features
- Inferring from problematic explanations when they suggest a line of reasoning (especially a novel one) that can be seen to be made to work (inferring both a correct line of thinking and what caused the person to give an incorrect explanation)
- Noticing the grounds for reasoning and its relation to accepted grounds in the profession
- Noticing proof standards (criteria, conventions, models, challenges, authority)
- Noticing how conflicting explanations and challenges to explanations are negotiated
- Noticing sensibilities or judgments about what is true, but that are more instinctive than reasoned
- Identifying and evaluating positions, grounds for them, and shifts in positions

These nine codes can be parsed into three broader foci for characterizing the analysis: (i) examination of particular reasoning for particular content; (ii) examination of the implications for making progress on solving problems and establishing claims; and (iii) consideration, at a meta-level, of general aspects of the structure, grounds, and standards for mathematical explanation.

The first involves analysis of the absence and presence of explanation, of its correctness, completeness and structure, of the basis from which it is reasoned, and of the instincts and intuitions implicit in it. The overarching question is what it suggests about the particular topics and ideas at hand. For instance, as teachers provide instruction on the subtraction algorithm and as students do subtraction problems and talk about their solutions, a mathematical analysis notices whether and when explanations are given. It attends to the character and logical structure of explanations and the basis on which explanations are built.

These features are evident in an annotated transcript of a classroom episode where the class is discussing their work over a couple of days on writing number sentences that equal ten. Lucy proposes that one hundred divided by ten equals ten. When asked why, the children struggle to say why. Mei says she agrees because she had it on her paper too. Lucy says she
believes it because Mei’s mother said so. When pressed for another reason, Lucy offers other examples:

Um, well um, I have um, a lot of other ones and like, if I had 20 divided by two, Mei said it was right, so then, I said that she said it was right, so if I, if I ten divided, ten divided by one equals ten and then I have it, then I have like 50 divided by five equals ten and Mei said that it was right so, if like you have 100 divided by ten it would be right because, if you were like to divide by five then it would be five more. And then cause 50 divided less then 100 so, so it would be ten cause five divided by—50 divided by 5 it would equal. (980501TrnAnal)

Betsy says that she thinks she knows what Lucy is saying. “She just plus 50 plus 50 and made it into 100. Five plus five, made it into a ten and then she divided and put it into ten, made it ten.” Mei then adds that, “Those kind of problems are really the opposite of times, and like if, ten times five it would be 50, so 50 divided in five will be ten.” When the teacher asks what Mei means by “ten times five is fifty,” students are not able to say what this means. The teacher then suggests putting the decision about the division examples in brackets for the time being because the class is not able to “prove” it as they have for other problems.

The next example proposed is 200 – 190 = 10. Again the teacher asks how the class can be sure that 200 – 190 equals ten, and again the class has an extended discussion about whether this number sentence is true and why. Central to the discussion is Betsy’s argument that it cannot be 10 because, in writing the subtraction problem vertically and trying to carrying out the subtraction as she has seen in other problems (but without the need to regroup the number), she treats the 0 – 9 in the tens column as resulting in nine, “because zero minus nine still equals nine.” The students have resources for establishing the veracity of this number sentence, but the algorithm, which raises unfamiliar and unexamined challenges, carries weight independent of whether it makes sense or is substantiated.

Annotations of this transcript note every occasion in which an explanation is requested, suggested, or given, often reformulating the reasoning in conventional mathematical terms. For instance, when Mei says, “And like if, ten times five it would be 50; so, if 50 divided in five will be ten,” an annotation briefly remarks, “Mei illustrates: 10 x 5 = 50 ==> 50/5 = 10.” The point of this observation does not appear to be an accurate characterization of Mei’s language, or even of her thinking and logic, but instead to associate her reasoning with conventional forms in the discipline. An observation that follows a few exchanges later leads to noticing a not-so-obvious relationship between the two kinds of examples being discussed.
The examples here are of the form, \( N \times 10 / N \). Note that these are exactly the multiplicative analogue of the examples, \( N - N + 10 \), that Betsy’s group comes up with later. (980501TrnAnal)

Noticing and naming the explanations arising in instruction, both their content and their logic, provides a distinctive lens for interpreting explanations given in the classroom.

Annotations also comment on the basis of each explanation — because Mei’s mom said so, because it does or does not fit with an intuitive estimate, or because the formal ritual of an algorithm carries authority — and on the correctness and completeness of explanations. For instance, when Lisa reiterates Betsy’s explanation, an annotation comments:

**Authority:** is given by Lisa to the procedure (formal ritual) of the algorithm, without questioning its correctness. The formal procedure carries the weight of authority. (980501TrnAnal)

When Betsy becomes persuaded by other students’ arguments that if you subtract 190 from 200 you should not still have 190 left, an annotation comments:

Betsy is persuaded by the unreasonableness of her answer, but still says the difference probably would not go all the way down to 10. (980501TrnAnal)

A little later, when Riba restates Betsy’s “because zero minus nine still equals zero” as “zero take away nine — you can’t take away nine”:

Riba adds one more grain of detail to Betsy’s solution, and reaffirms the authority of Betsy’s procedure. (980501TrnAnal)

When Betsy adopts the approach of thinking about how big the result should be, an annotation comments:

**Speculation:** Though Betsy gets the details tangled and confused, she is perhaps trying to illustrate, by the example, 190 – 100 = 90, that the difference between large numbers should be large, thus casting doubt that one could have 200 – 190 = 10. (980501TrnAnal)

Each of these attends to explanations being given, notes the logic, correctness and completeness of the explanation, and considers intuitions associated with it.

A second feature of a disciplinary analysis of explanations is the examination of the implications for making progress on solving problems and establishing claims. This includes contributions from the examination of faulty reasoning, of what an explanation makes apparent, and of the locus and movement of conviction within the group. Explanations in mathematics are in the service of finding answers, determining the validity of claims, and ultimately understanding
more. Some mathematical arguments do not illuminate, but the hope for understanding is typically there. The improvement of explanations is often through what they make apparent and what they explain, which clarifies both what is explained and how it is explained. Even before this, faulty explanations can be examined for what went wrong, why they didn’t work, which can then be used to home in on fixing that which went wrong. Andrew Wiles recalled about his work to prove Fermat’s last theorem:

I was sitting at my desk one Monday morning … examining the Kolyvagin-Flach method. It wasn’t that I believed I could make it work, but I thought that at least I could explain why it didn’t work. I thought I was clutching at straws, but I wanted to reassure myself. Suddenly, totally unexpectedly, I had this incredible revelation. (Singh 1997, p. 297)

It was through an examination of why his use of the Kolyvagin-Flach method did not work that Wiles gained the understanding he needed to eventually solve the problem.

Related to this dynamic of examining flawed reasoning is another of reassessing what one believes about what is true and where efforts should be directed. In research mathematics, whether a conjectured claim is true or false is suspected, but until it is proven, it remains ultimately unknown. This sizing up can be about one’s own convictions, or it can be about conviction in the community. And it unfolds in the development of an explanation, in challenges raised, and in negotiation among competing and conflicting explanations. In analyzing instruction, attention to the position both of individual students and groups of students and to the unfolding competition among explanations can inform an understanding of mathematical progress.

For instance, in the example described above, when Riba reiterates Betsy’s argument, the teacher says, “I didn’t understand what you meant when you said, zero take away nine, you can’t do it so you have nine.” An initial annotation says that the teacher “seizes on the extra opening that Riba has exposed.” The point here seems to be that not being able to take nine away from zero would seem to expose the flaw in Betsy’s argument that 200 minus 190 is 190. The observation continues a few exchanges later where it says that the teacher “has now moved the discussion into focus on the trouble spot in Betsy’s procedure.” This attention to error and to identifying a trouble spot in student thinking is not at all new in mathematics education. My point here, however, is that this (the claim that the teacher seizes on the extra opening that Riba has exposed) is part of the analysis, not because it is conceived in terms of individual cognition or a particular view of learning, but because it (spotting openings that are exposed) is a disciplinary practice, such as Wiles’ efforts to pinpoint and repair his flawed reasoning, a disciplinary practice
that usefully complements activities of teaching and learning mathematics in the classroom. Here, the teacher identifies the flawed reasoning, both modeling this profoundly mathematical skill and moving the mathematical work forward. And again, the interpretation that the teacher is modeling a skill that is key to advancing the mathematical work is an interpretation from a perspective of disciplinary practice — one that complements interpretations from cognitive science, socio-cultural psychology, or other common perspectives used to analyze teaching and learning.

Likewise, the collection of annotations described above begin to provide an overall picture of ways in which explanations unfold in the process of solving problems and establishing claims. The observation that, “Betsy is persuaded by the unreasonableness of her answer, but still says the difference probably would not go all the way down to 10,” takes note of her thinking, but also notes a change in her thinking, how her thinking sits in relation to the thinking of others, and contributes to a story of progressing mathematical work in the classroom.

The third feature of a disciplinary analysis of explanations is the consideration, at a meta-level, of general aspects of the structure, grounds, and standards for mathematical explanation. Across the comments described above there seems to be a background attention to disciplinary standards and conventions — for accepted grounds of reasoning, for models and criteria, and for ways of challenging arguments made. A mathematical analysis of practice would seem to be sensitive both to the correctness of statements and to the basis for those statements. In analyzing the episode above, annotations are made about the authority invoked and the presence, absence and quality of the mathematical basis of answers and claims. For instance, when the teacher tables the proposal of 50/5 as a number sentence equal to 10, an annotation says:

The teacher rules the multiplication/division examples out of order until the students have the means to pay the logical price of admission. (980501TrnAnal)

It is this sense of what is in or “out of order” and of the “logical price of admission” that characterizes a disciplinary concern for standards for answers and claims that are distinctive of the discipline.

Ball and Bass (2003) write:

As we began to develop and use mathematical lenses to look in on third graders, we noticed them working empirically and generating conjectures from that work. We saw them constructing arguments intended to make their classmates believe that what they had noticed was true. We even saw them confronting the very nature and challenge of mathematical proof. (p. 196)
These observations seem to grow from a steady consideration of the unfolding story of explanations in the classroom as they fit and did not fit with accepted notions of mathematical explanation and justification as practiced in the profession. What are the role and the limits of empirical work in the profession and when and how do these play out in classrooms? What counts as mathematical explanation and what are accepted ways of challenging explanations? What is the mathematical price of admission? Asking and answering these questions in the context of the dynamics of instruction is an important part of a mathematical analysis of practice.

4.1.4 Expression and Language: Windows on the Creation and Development of Ideas

The importance of talk in teaching and learning is also well recognized. Spoken and written language is the medium of instruction and is inextricably linked to thinking and learning. Kieran, Forman, and Sfard (2003) argue that a discursive, or communicational, research framework has developed in mathematics education because of the “inherently social nature of human thought” (p. 5). With a similar concern for language as central to both thought and instruction, Pimm (1987) uses language as a metaphor for mathematics and uses linguistic constructs as a means to interpret mathematics and mathematical thought. One important linguistic construct is that of metaphor. He claims that, “metaphor is as central to the expression of mathematical meaning, as it is to the expression of meaning in everyday language” (pp. 10-11); in other words, that the “process of metaphor,” of using a metaphor as a way to think oneself into an idea or make sense of an idea, is key to the construction of meaning in mathematics. Lakoff and Nunez (2000) make a similar argument about the centrality of metaphor. In their effort to launch a cognitive science of mathematics, they conclude that, “we have discovered that a great many of the most fundamental mathematical ideas are inherently metaphorical in nature” (p. xvi).

Each of these researchers is arguing for the power of using ideas from linguistics to analyze mathematics. My aim, however, is different. My question is the opposite. What can we learn about mathematics teaching and learning, in particular about the mathematical talk of teaching and learning, by using the discipline of mathematics as a lens? Mathematics, like any human endeavor, is very much about language. Language plays an important role in mathematical thought itself and is essential to communicating and sanctioning the products of that thought. In addition, the language of mathematics is distinctive. Indeed, mathematics is often described as a distinct language and the learning of mathematics is likened to the learning of a foreign language. But, what is the language work of mathematics? What are the language practices of the discipline? What is the nature and role of language in the conduct of mathematical work?
Unfortunately, these questions are not well understood or studied. For the most part, the philosophy of mathematics has focused on language issues as they relate to issues of ontology and epistemology, such as questions about whether mathematical ideas exist in some Platonic sense, independent of the thought and language of humans. Language, nevertheless, plays an important role in the professional practice of doing mathematics. Likewise, it seems important in a disciplinary analysis of the language-intensive mathematical activities of teaching and learning mathematics.

In analyzing project documents, I found that many of the comments about mathematics that seemed productive to the research endeavor were about how ideas were expressed in the classroom. Of the low-level codes, ten seemed related to an attention and analysis of linguistic expressions and the ideas implied by them.

- Noticing the combined linguistic and conceptual challenge of articulating emergent reasoning
- Inferring from problematic expressions when the problems are due to inadequate or underdeveloped language
- Noticing imprecision in the use of language
- Noticing inconsistencies
- Noticing the need for, and introduction of, definitions and language to support the mathematical work
- Formalizing questions, answers and explanations given by the teacher and by students
- Noticing distinctions among different formulations or expressions
- Noticing violations of conventions and common understanding
- Noticing the demand for consistency
- Connecting different representations of ideas (e.g., symbolic and geometric)

In reflecting on these codes and in examining them across the work, I propose that a mathematical analysis of teaching and learning involves an examination of expressions, both as incipient ideas and as definitive formulations, as these are related to the conduct of mathematical work. In particular, it involves a focus on six key features of how ideas are expressed and how expressions of ideas are exercised.

- Difficulty in expressing ideas (empathy, what might be, need for more language)
- Need for terms and definitions
- Move to the precise and formal
- Differences and consistency among expressions (logical and conventional)
- Connecting and mapping among different expressions
- Explicit demand for definition, precision and consistency

In the remainder of this section, I begin by discussing the first of these in some detail, using Bryer’s (2007) notion of ambiguity in mathematics to frame a mathematically motivated form of intellectual generosity important in the creative phases of generating mathematical questions,
answers, and explanations. I then describe the dynamic between this first feature and the next two features, which represent responses to the initial difficulties. I end by briefly characterizing the last three features as additional aspects of creating greater clarity, precision, and canonical expressions of ideas.

The language of generative mathematical work. In a mathematical analysis of classroom teaching and learning, one attends to occasions when, because of the nature of the mathematical work, ideas may be hard to express. As ideas are forming, expressions of emergent thinking can be tangled and convoluted. For instance, when the teacher asks Sean how he could know that 200 minus 190 is 10, Sean haltingly says:

You could — Okay, like, oh look 200, 200 take away 190 um, 190 is about ten less than 200, so the um, 190 is ten less then 200 so that, so 200 take away 190 we, we um, we, equal ten, equal ten there and so —

An annotation describes Sean as “groping for the right words “ (980501TrnAnal). A few minutes later, Mei says:

I think I agree with Sean. See, ‘cause 100--if you take away 200, if it was 200 and you take away 200, then there would only be 200 left, and take, then take away ninety—

The annotation for this says that Mei “gets tangled in her explanation but seems to be trying to say that 200 – 100 = 100, and then 100 – 90 leaves 10” (980501TrnAnal). (This presumes that Mei meant 100 instead of 200 the last two times she said them.) The degree of “reading into” the thinking is striking. It is as if in these cases the children are seen as groping to develop and express new ideas for determining 200 – 190 in ways that are similar to the ways in which mathematicians experience the challenge of getting a hold on a novel idea. The children are seen as having potential ideas and are, correspondingly, granted leeway in the initial expression of the ideas.

What I mean to point at here is not the same thing as the leeway teachers often give children because children are learners. For instance, Ball (1997) gives three challenges of listening to children: (i) listening across differences in age, culture, and experience; (ii) listening through the multiple influences of context (with sensitivity for the uncertain, context-dependent nature of understanding); and (iii) listening with and through one’s desire for students to learn and to be successful. In describing these challenges, Ball is pointing at a kind of generous listening teachers need to do if they are to support children’s growing understanding of mathematics. The point here is that the basis for Ball’s notion of generous listening lies in key
features of teaching and learning — that children are different from teachers, that learning and understanding are complex, and that teachers want children to be right. In contrast, my claim that a mathematical analysis involves a kind of generous listening is based on an impulse in mathematics to listen for potential ideas when that is what is called for by the mathematical work at hand.

These two kinds of generous listening, the pedagogically motivated listening described by Ball and the mathematically motivated listening I am describing, have much in common, but are distinct precisely because the different motivations imply different judgments about when and how they matter. For instance, both can be quite contrary to the disciplinary concern for precision and correctness, but the way in which they interact with this latter concern is different. A pedagogical attention might persist to question a student whose words seem precise and correct when experience suggests that those words often obscure significant misunderstandings, or a pedagogical attention might grant extra leeway to a student’s performance when the task is ambitious and the student’s self-esteem are in jeopardy. In contrast, a mathematical attention is likely to value precision and correctness when ideas have been developed and the work is focused on establishing a concluding justification and to respond in an intellectually generous way when students are exploring a novel idea or working to generate potential approaches to solving an unfamiliar problem.

The mathematically motivated generous listening that I am describing here is related to Byers’ (2007) notion of ambiguity in mathematics.

It is true that mathematicians are motivated to understand, that is, to move toward clarity, but if they wish to be creative then they must continually go back to the ambiguous, to the unclear, to the problematic, for that is where new mathematics comes from. Thus ambiguity, contradiction, and paradox and their consequences—conflict, crises, and the problematic—cannot be excised from mathematics. They are its living heart. (p. 24)

For Byers, tangled, convoluted expressions may indicate occasions of creative work distinctive of the discipline. I briefly describe his examination of ambiguity in mathematics because it provides a way of thinking about the purpose and approach of a mathematical analysis of student and teacher talk. In other words, I am arguing that an important aspect of a mathematical analysis of practice that addresses language issues attends to how ideas are expressed, but does so with sensitivity to nature of the work, in particular to times in the work that are about generating ideas and times that are about formalizing those ideas. Byers notion of ambiguity informs the first of these.
Drawing on Arthur Koestler’s work on creativity, Byers defines ambiguity as involving “a single situation or idea that is perceived in two self-consistent but mutually incompatible frames of references” (p. 28). Byers uses his definition of ambiguity to interpret numerous ambiguity-laden situations in mathematics. For instance, the Greeks perceived the square root of two in two self-consistent contexts (pp. 35-39). As a consequence of the Pythagorean Theorem, it was a geometric object, the length of the hypotenuse of a right-angled triangle with sides of unit length. At the same time, they proved that this geometric number could not be expressed as the ratio of two integers. These formed two self-consistent but mutually incompatible frames of reference — one geometric, one arithmetic. This, in turn, led to a crisis, which was the genesis of new ideas and a frame that incorporated and reconciled the two initially incompatible frames.

Ambiguity, Byers argues, is the driving force in mathematics; from ambiguity, ideas grow, and ideas are the organizing principles that help us answer the fundamental question of what is going on mathematically, namely, that help us to understand mathematics. He claims, “One can feel when ‘something is going on,’ and good mathematicians are very sensitive to that feeling.” (p. 196) It is this feeling for when something is going on mathematically which is central to a mathematical analysis of teaching and learning.

Looking across the project data, Byers’ notion of ambiguity as involving two self-consistent but mutually incompatible frames of references can be used to interpret many of the classroom episodes in ways that constitute distinctive and productive mathematical analyses. For instance, when Sean announces that six is both even and odd, there are two frames, one in which six is even and another in which six has something odd about it. Out of initially confusing and problematic classroom talk grows the idea of some numbers being an odd number of groups of two, which later on are dubbed Sean numbers. Similarly, when Riba says you cannot subtract 9 from 0, there are two frames, one in which it makes no sense to “take away” from a group with nothing in it and another in which it is part of subtracting 190 from 200. Out of this apparent paradox, the class invests the next month in developing the notion of negative integers, which then helps to clarify the nature of the regrouping needed in the subtraction algorithm. When Keith presents his initial thinking that 4/4 is equal to 4/8, there are two frames, one in which 4/4 is two times as large as 4/8 and another in which 4/4 and 4/8 are the same size — when the representation of 4/8 is twice as long and the pieces are the same size as those for 4/4. Out of this grows a clearer language and clearer thinking about the “whole” or “unit” implicit in a fraction. In a mathematical analysis, one notices these ambiguities, these situations in which an idea is perceived in two self-consistent but mutually incompatible frames of references and the tangled, convoluted talk that often accompanies them; one notices when students or teacher seem to be
groping for words and when the things being said contain ambiguity, and one pauses to consider the potential such expressions have for generating useful mathematical ideas.

The language of formalizing mathematical work. These situations involve a fumbling for words that often leads to further language work, such as clarifying what is being said, introducing new terms, and defining terms being used. In a mathematical analysis, one also watches for these activities — both their presence and their absence, or rather, their possibility. These elements were evident in the analysis that led to distinguishing the different definitions for even and odd numbers discussed in Chapter 3. They are also evident in a paper where Ball, Lewis, and Thames (2008) posit three elements that undergird the doing of mathematical work in the classroom: naming and using names; making and interpreting claims; and evaluating claims. For example, analyzing a six-minute episode, they observe the following nine claims about zero in an ambiguity-rife discussion of whether zero is even, or odd, or “special.”

<table>
<thead>
<tr>
<th>Mathematical claims about zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>zero is not even or odd</td>
</tr>
<tr>
<td>zero could be even</td>
</tr>
<tr>
<td>zero is not odd</td>
</tr>
<tr>
<td>zero has to be an even</td>
</tr>
<tr>
<td>zero is not an even number</td>
</tr>
<tr>
<td>zero is always going to be an even number</td>
</tr>
<tr>
<td>zero is not always going to be an even number</td>
</tr>
<tr>
<td>zero is even</td>
</tr>
<tr>
<td>zero is special</td>
</tr>
</tbody>
</table>

They go on to describe the nature of the mathematical language work involved in clarifying these claims.

This list highlights two central features of mathematical claims — the importance of precise language and the need to carefully clarify the meaning of claims. More than in other disciplines, mathematics demands precision. Precision is one of its hallmarks. Reading through the list of claims about zero, asking what is meant by each of these different claims, one can see why that precision is important. Differences are subtle, yet significant. “Is,” “could be,” “has to be,” “is always going to be,” “is not always going to be” — each of these means different things. Each involves different lines of argument. When Sheena says, “It could be even because of the ones on each side is odd, so that couldn’t be odd,” what is she claiming? Is she saying zero is even, or that it is not odd, or does she see these as the same? And what is the role of the “could” and “couldn’t” in her statement? To argue that zero could be even, one might focus on
consistency (i.e., considering zero to be even fits with other agreements already made). To argue that zero is even, one might return to the definition for even numbers. Quantification and negation play prominent roles in formal mathematical logic as well as in less-formal mathematical reasoning. And, as we see in these exchanges, the students are exploring this important mathematical terrain. (p. 27-28)

Noticing these expressions, exploring the ways in which they fumble with mathematical ambiguity, examining the ideas that might emerge from them, and considering the work involved in clarifying and developing precise language for them are important aspects of a mathematical analysis of practice.

Ball, Lewis, and Thames go on to describe the interplay between generating ideas and making those ideas formal and precise.

With these claims about zero, we see that precision shapes and is shaped by reasoning. Precision matters, even in third grade. In turn, the need for precision heightens the need to clarify claims. And we see that as the students in this class make claims they seem to engage in a process of clarifying those claims, whether on their own or with the support of the teacher. The major chunks of student talk in this six-minute segment are, in large part, occasions of making and clarifying claims. Sean asks Sheena for clarification. Ball, along with other students, asks Nathan for clarification. And several members of the class ask Sean for clarification. Much of the mathematical work we see going on in the segment (a kind of work that the teacher scaffolds from the first day of class) is about clarifying, testing, and revising (further clarifying) the claims being made. (p. 28)

Each of the requests for clarification referred to above represents a disciplinary press for precision and formal expression of ideas. As Byers writes, mathematical work involves a process of moving between two poles — one of ambiguity and incompatibility, the other of precision and formality.

Logic moves in one direction, the direction of clarity, coherence, and structure. Ambiguity moves in the other direction, that of fluidity, openness, and release. Mathematics moves back and forth between these two poles. (p. 78)

In a mathematical analysis, one is attentive to both of these poles. One is attentive to expressions that grope in ways that might hold the seeds of new ideas and one is attentive to the necessary work of clarifying and making precise. On the one hand, one is sympathetic to the combined linguistic and conceptual challenge of articulating emergent reasoning and one is sensitive to the potential that ambiguous expressions hold. On the other hand, one is sensitive to mathematical
precision, correctness, and convention and one examines the presence and absence of these features in classroom practice. In addition, one is attentive to the work between the poles — to activities of mathematical play, questioning, and creativity that draw potential ambiguities to the fore and to activities of introducing terms, definitions, and expressions that move in the direction of clarity, coherence and structure. And, one is attentive to which kind of work is needed to make progress mathematically.

*Additional features of mathematical rigor in the expression of ideas.* Beyond the more immediate work of defining terms and clarifying ideas, several additional aspects of this move in the direction of logic, clarity, coherence, and structure are worth mentioning. A first is an examination of inconsistencies between what is said and other things being said. For instance, when Mei argues that 200 minus 190 is 100 (because zero minus zero is zero; zero minus nine is zero; and two minus one is one) several students ask her what she did with the nine. An annotation here says, “It’s interesting that they [students] ask this here, but were content to let Betsy say that 0 – 9 = 9.” (980501TrnAnal) In this case, the analysis notices that asking what was done with the nine seems to be meaningful in the context of 0 – 9 = 0, but not in the context of 0 – 9 = 9. At first, this might not seem to be a language issue, but on reflection one can see that it is deeply related to language. As with the claims about zero given above, the question, “What did you do with the nine?” is a linguistic expression that can be interpreted in many different ways, is dependent on the context in which it is spoken, and is playing an important role in the process of clarifying the meaning of what is being expressed in the classroom.

In a mathematical analysis, one examines, not only the consistency within an expression, but also consistency among different things being said, as well as consistency between an expression and conventions of the discipline. One observes that the word *odd* has a conventional meaning and that Sean’s co-opting of the word to describe the property he has noticed is at odds with disciplinary convention. One notices when different definitions are being used, when the meaning of a term shifts, and when different wordings of a claim or problem have different meanings. One also notices when different expressions are non-trivially equivalent and in what sense they are equivalent. This work extends into making connections among distinct expressions, including systematically mapping between these distinct linguistic representations of ideas.

A final feature of a mathematical analysis of expressions and language used in the practice of teaching and learning is attention to meta-comments about the nature of mathematical language and its demand for definitions, precision and consistency. For instance, in the discussion of 200 minus 190, the teacher says:
An annotation notes that the teacher affirms that this mathematical conflict is unacceptable and that mathematics demands reconciliation of conflicting opinions about what the expression 200 minus 190 equals. (980501TrnAnal) A demand for consistency may seem odd as a language issue, but it is central to making sense of ideas being expressed and central to the language work being done to clarify those ideas and to make them precise. Comments about this demand, as expressed in the language work of the classroom — about when a definition is needed, or about the need for and the nature of precision in mathematics — are worth noting and worth framing in relation to how these issues are perceived in the discipline.

4.1.5 Character(s) of the Work: Windows on Mathematical Work as a Human Endeavor

Above, I have discussed three features attended to in a mathematical analysis of practice: problems posed and answers given; solutions and explanations generated; and the nature of expressions and language use. Each of these features focuses on a particular component or slice of mathematical work. This last feature considers the character of mathematical work as a whole and the mathematical characters who carry out that work. By characters, I mean the people, but considered as idealized people doing mathematics. Mathematics is a human endeavor. Thus, one way of characterizing the doing of mathematics is to characterize the roles people play as they work. It is the ways people carry out the work that gives mathematical work its character. In other words, this last component of a mathematical analysis of practice sets its sights on more globally characterizing the unfolding mathematical work that goes on in teaching and learning, both overall characteristics of the work and the mathematical characters who do that work.

Mathematical work does not have a clear beginning and end. Instead, it spirally progresses from formulating definitions and problems, to searching for answers, to proving or revising those solutions and generating new questions. And, each of these activities has certain activities that characterize it. Mathematical work can also have a distinctive feel to it, times when it is slow and methodical, times when it is nimble and jumps quickly from place to place, and times when it requires perseverance. Along with this, it is carried out by people who play different roles in a collective effort and different roles at different times in their individual efforts, exhibiting this or that disposition to the detriment or benefit of the work. One mathematician may be particularly good at generating new ideas or approaches, in seeing previously unrecognized connections. Another may be good at careful review of proposed arguments and identifying missteps or false presumptions. Every mathematician needs to develop these skills for
him or herself, but there are also important ways in which mathematicians function as a community, relying on the individual and collective strengths of others.

In contrast, the work of school mathematics is often limited to a routine of demonstration, done publicly, and extensive practice, typically done individually. Its character sometimes shares elements in common with the character of work in the discipline, but this is more likely the exception, not the rule. There are two reasons for attending to the character of mathematical work in a mathematical analysis of practice. First, reform agendas have lobbied for more ambitious instructional goals and for alternative instructional formats, and the discipline offers a sensible source of ideas for instructional approaches that are likely to support ambitious goals. Returning to consider the bottom line, mathematics instruction in school aims to produce learning of canonical knowledge that supports a wide range of personal, social and economic goals. Central to effective instruction is engaging students in substantive time on task, in other words in the mathematical work of doing and learning mathematics. Thus, attention to the nature of mathematical work in teaching and learning might provide insight into the improvement of teaching and learning. Second, the notion of mathematical work focuses attention on the nature of activity in doing mathematics. This focus on activity provides a useful intersection between the discipline (focused on doing mathematical work) and teaching and learning (themselves activities, in part mathematical activities, to be carried out). In other words, an activity-focused conception of mathematics is likely to inform the designing and orchestrating of activity that typifies the work of teaching.

Analysis of project documents suggests three aspects of a disciplinary examination of mathematical work that may be high leverage for examining mathematics teaching and learning and the mathematical demands they create. These include an examination of (i) the organization of mathematical work; (ii) the collective nature of mathematical work; and (iii) the people doing that work (as people filling particular roles in doing mathematics). These three aspects do not exist in isolation from features of tasks, explanations, and language already discussed, but they cut across these and provide additional texture.

Each of these aspects of mathematical work is central to identifying work involved in doing research in mathematics, but is also central to shaping instruction in classrooms. Instructional design is about selecting instructional tasks, but also about sequencing and structuring work on those tasks. And, this happens in the context of classrooms, where groups of people must function in close proximity. As Jackson (1968) points out, classrooms are one of the most perpetually crowded places in our society. In addition, teaching, in contrast with tutoring, is in part about leveraging the group for the purpose of learning. Hence, a focus on the collective
work of mathematics, a prominent aspect of work in the discipline often overshadowed by images of solitary contemplation and individual accomplishment, yet central to the doing of mathematics as well as to one’s sense of what to do individually and to the valuing of questions and the acceptance of results, seems strategic for informing an analysis of teaching and learning. Last, the mathematical personalities that conduct the work represents a useful intersection between views of work in the discipline and the orchestration that teachers need to do of children doing work in the classroom.

*The organization of mathematical work.* I found that in project documents many of the comments about mathematics that seemed productive to the research endeavor were about the mathematical work being done. Two of the low-level categories were about beginnings and ends of particular chunks of mathematical work.

- Noticing when major chunks of the work begin and end (posing of the problem, search for solution, presentation and review of solution, agreed acceptance of a solution)
- Noticing the absence of beginning and ends, or rather, when instructional starts and stops do not match mathematical ones (for instance when instructional work seems to begin before a clear problem has been put forward or when class “closes” without finishing mathematical work on the problem)

A disciplinary attention to the structure and sequence of mathematical work in the classroom is both similar to and different from the kind of attention given to lesson structure typical in mathematics education. The latter is concerned with lesson structure that fits in planned blocks of time. It is concerned with issues of orchestrating group activity within those constraints and with making connections across boundaries consonant with what is known about learning, such as the value of accessing prior knowledge and reviewing what has been learned.

For instance, in the Connected Mathematics Project (CMP, 2008), the authors identify a three-phase instructional model: launch, explore, and summarize. Quoting from their descriptions, the launch “involves helping students understand the problem setting, the mathematical context, and the challenge.” In it, “the teacher introduces new ideas, clarifies definitions, reviews old concepts, and connects the problem to past experiences of the students,” being “careful to not tell too much and consequently lower the challenge of the task to something routine.” In the explore phase, students “gather data, share ideas, look for patterns, make conjectures, and develop problem-solving strategies.” The teacher’s role is “to move about the classroom, observing individual performance, encouraging on-task behavior, and to help students persevere in their work by asking appropriate questions and providing confirmation and redirection where needed.” In the summarize phase, “the teacher guides students to reach the
mathematical goals of the problem and to connect their new understanding to prior mathematical goals and problems in the unit."

My point in describing this instructional model is to draw attention to its framing in terms of cognitive science, learning theory, and pedagogical practice. It suggests activating prior knowledge by linking the new problem to past experiences; it addresses the need for students to understand what they are learning and to fit it into a larger set of mathematical ideas so that they are aware of and retain what they are learning; it recommends that the teacher steer the students’ work by confirming aspects that are correct and redirecting when they go astray. The one exception is perhaps the description of students’ activity during the explore phase, which might be thought of as including mathematical activities, but the listed activities seem to be formulated from a pedagogical perspective, just one that includes the content being taught. In contrast, a mathematical perspective attends to a logical sequence of activities that accomplish mathematical work. Among those I identified in project analyses are: (i) posing of the problem; (ii) search for a solution; (iii) presentation and review of solution; and (iv) acceptance of a solution by the group. These are not necessarily linear, but play out in cycles that loop back at a variety of junctures depending on circumstance. It is interesting to consider a potential relationship between these four disciplinary practices and the phases of the instructional model described above. The launch phase might be seen to include the posing of a problem; the explore phase might be seen to correspond to searching for a solution; and the summarize phase might be seen as encompassing the presentation, review, and acceptance of a solution. Note, however, that the purpose and the overall guiding structure are different. CMP’s instructional model is focused on creating instructional blocks to fit into instructional periods of time toward the engagement of students in predetermined content according to understood theories about how people learn. The three blocks of activity are framed in terms of an instructional sequence that engages prior knowledge, creates cognitive dissonance, provides for individual work time, focuses attention on canonical solutions, and brings the instructional undertaking to closure. In a mathematical analysis, the four activities are dictated by accomplishing the mathematical work of formulating and answering mathematically interesting questions. The four activities indicate choices about where to focus efforts in order to make progress on producing mathematical knowledge claims.

I also do not mean to overstate this point about the differences between instructional work and mathematical work. There can be useful intersections between the two: indeed, this is why an understanding of mathematical work is important for understanding and improving teaching and learning. The orientations are distinct, however, and offer different insights into what is happening in classrooms. In fact, both matches and mismatches between the structure of
instructional work and the structure of mathematical work can inform the interpretation of teaching and learning.

The collective nature of mathematical work. The second aspect of a disciplinary examination of mathematical work attends to the collective nature of mathematical work. The doing of mathematics is often thought of as a distinctively individual endeavor — best done by eccentrics in the deep recesses of beautiful minds. However, mathematics is also clearly a human endeavor carried out collectively within societal institutions. The notion of “standing on the shoulders of giants,” quite popular among mathematicians, suggests both the exceptional role certain individuals play and the reality of the ways in which those individuals stand on collectively built foundations.

For instance, the notions of proof and what it means to prove a claim are typically thought of in relation to constructing a sequence of deductive steps from first premises using accepted rules of inference. The inadequacies of such an image are generally acknowledge (Thurston, 1994). But, what then does it mean to prove a statement? An alternative image is of a mathematician, fervently writing at the blackboard, waving his or her hands, and anxiously glancing over the shoulder to see whether or not other mathematicians are convinced. This, too, is inadequate. Instead, it would seem that proving a claim is about engaging in socially practiced, disciplined forms of reasoning, as described by Ball and Bass (2003).

The desire to know and to understand has led people to develop disciplined means of reasoning, of exploring and verifying, of hypothesizing and justifying, in many arenas of human activity. Historians reason about evidence from the past, physicians reason about patients’ symptoms, chefs reason about composing ingredients under particular conditions, and pilots reason about instrument readings. In none of these examples do individuals make sense in whatever ways they choose. Instead, in each of these arenas, people have developed methods of reliable thinking that afford inspection, analysis, judgment, and conclusions. These methods of reasoning are the particular means of constructing and evaluating knowledge in a domain. (p. 29)

Such disciplined reasoning lies somewhere between an idealization of Euclid and individuals making sense in whatever way they choose.

Thurston (1994) provides a personal narrative of learning the distinctive character of disciplined reasoning in mathematics.

When I started as a graduate student at Berkeley, I had trouble imagining how I could “prove” a new and interesting mathematical theorem. I didn’t really understand what a “proof” was.
By going to seminars, reading papers, and talking to other graduate students, I gradually began to catch on. Within any field, there are certain theorems and certain techniques that are generally known and generally accepted. When you write a paper, you refer to these without proof. You look at other papers in the field, and you see what facts they quote without proof, and what they cite in their bibliography. You learn from other people some idea of the proofs. Then you’re free to quote the same theorem and cite the same citations. You don’t necessarily have to read the full papers or books that are in your bibliography. Many of the things that are generally known are things for which there may be no known written source. As long as people in the field are comfortable that the idea works, it doesn’t need to have a formal written source.

At first I was highly suspicious of this process. I would doubt whether a certain idea was really established. But I found that I could ask people, and they could produce explanations and proofs, or else refer me to other people or to written sources that would give explanations and proofs. There were published theorems that were generally known to be false, or where the proofs were generally known to be incomplete. Mathematical knowledge and understanding were embedded in the minds and in the social fabric of the community of people thinking about a particular topic. This knowledge was supported by written documents, but the written documents were not really primary.

(p. 8-9)

Disciplined reasoning in mathematics, then, is a collective act, where, in the terminology of Ball and Bass (2003a), the determination of a base of common knowledge and collectively acceptable inferences are held and upheld by the group according to disciplined approaches.

One foundation is a body of public knowledge on which to stand as a point of departure and that defines the granularity of acceptable mathematical reasoning within a given community. The second foundation of mathematical reasoning is language — symbols, terms, and other representations and their definitions — and rules of logic and syntax for their meaningful use in formulating claims and the networks of relationships used to justify them. (p. 30)

While talking and working together may be advantageous for learning or may be the essential nature of learning, they are also the basis of mathematical reasoning and mathematical work.

In analyzing project documents, I found abundant instances of three specific kinds of comments related to the collective nature of mathematical work.

- Noticing public accessibility of and engagement with ideas
- Noticing standards for individual work and their contribution to the collective work
• Noticing standards about collective responsibility for determining the correctness of mathematical results

Several issues arise as mathematical work shifts from individual and private space to collective and public space and back again. When is it best to work privately and when does it become important to engage with others in exploring ideas, reviewing proposals, or establishing results? In what forms should work be shared? Who should be involved and to what end? Along with these questions, are issues regarding collective responsibility for sanctioning approaches, reasoning, and results.

What happens in schools is often quite remote from a disciplinary image of the collective mathematical work, but a disciplinary perspective is on the lookout for when and how methods and knowledge are established. It is concerned with the disciplinary basis for this process. When a teacher presents a mathematical idea or argument, a mathematical perspective attends to the mathematical reasoning used to justify the idea or argument. Because mathematical reasoning is fundamentally collective in nature, this reasoning must be available for public review and its review requires the active engagement of the group’s membership. For instance, a mathematical analysis attends to occasions when a teacher blocks from view key elements of an explanation or when students are remiss in questioning faulty or incomplete reasoning.

**The characters of mathematical work.** Related to the collective nature of mathematical work, are the ways in which people populate the work — the roles they take and the personalities associated with those roles. Certain dispositions help one do mathematics. Confidence, perseverance, flexibility, and curiosity can be important attributes for accomplishing work in a number of fields, but they are particularly important in mathematics and there are distinct versions of these attributes specific to mathematics. Likewise, although labeling people as good or bad at mathematics is commonplace, a disciplinary perspective is concerned with identifying particular contributions stemming from particular personal strengths and weaknesses. Such a characterization informs what and how people can contribute to collective work, when they should be heeded or ignored, and in the end, such a characterization defines what it means to do mathematics, individually or collectively.

In analyzing project documents, I found two specific kinds of comments related to identifying mathematical characters engaged in the work.

• Characterizing people as particular kinds of mathematical thinkers
• Noticing exhibition of important mathematical dispositions

For instance, when Betsy argues that 200 minus 190 is 190, an annotation says that, “though Betsy is a good abstract thinker, and very confident, she often loses control, to computational or
A disciplinary perspective is concerned with characterizing people because such characterizations help to read people when listening to them, help to inform whom to turn to, or expect from, as collective work progresses, and help to understand the individual’s engagement in mathematical work. It looks for who is involved in, contributing to, and shaping the outcome of mathematical activity. Such observations are small and of the moment, but over time they contribute to a sense of the players, of the community and how it works, and of the unfolding dynamics that shape the nature of the mathematics being done.

Together, examinations of the organization of mathematical work, of the collective nature of mathematical work, and of the personalities doing that work round out the substantive examinations of problems and answers, solutions and explanations, and expressions and language. These four themes, together, function in combination to produce a vision of the mathematical work unfolding in classrooms. To illustrate their combined leverage, I next use them to analyze five minutes of classroom instruction.

4.1.6 Example: A Mathematical Analysis of Five Minutes of Instruction

I illustrate these four themes in a mathematical analysis of practice. In doing so, I argue that the view of instruction provided by a disciplinary lens is distinctive and offers different insights into what is happening and into the specific content and nature of the mathematical demands of teaching. This five-minute video segment from instruction with fourth and fifth graders is not meant to illustrate all aspects of the four themes equally. Some of the issues just discussed, such as identifying the beginnings and ends of chunks of work or characterizations of the mathematical skills, dispositions, and personalities of particular children are better viewed by examining more expansive windows on instruction. This short segment, however, provides an illustration of the nature of a mathematical analysis and of the insights it can afford.

The class has been studying fractions. On the board is a list labeled “working ideas” about fractions. It enumerates three points:

- identify the whole
- equal parts
- how many parts of the whole

The children were given the following diagram and have been working individually on the three questions and recording their work in their notebooks (where the triangle is shaded blue and the small rectangle in the lower left corner is shaded green).
The teacher asks for someone to explain his or her thinking for the first question and she waits until she has gotten a number of children to volunteer. Mahluli says it is one half. When asked to explain, he says:

Because they both equal—they both equal—and one, one, half of it is shaded in and the other half is not.

He then goes to the board to show what he means. As Mahluli makes his way to the front, the teacher tells the class that they should be listening and thinking about his explanation and she asks Doran to repeat what Mahluli said.

He said he's looking at the squ-rectangle, and he's saying it's one half of the rectangle not, not just, he just— he's not looking at the whole... he's just looking at the whole, he just-

The teacher interrupts him, asking him not to go on and explain it yet, and she turns back to Mahluli to have him explain his thinking using a large poster of the figure that is stuck to the blackboard. Mahluli repeats his explanation, quickly pointing to the two triangles and saying that they are equal, so the shaded one is one half.

The teacher then suggests looking back at the working ideas the class has agreed on, asking Mahluli what he is calling the whole and has him run his finger around the part he is calling the whole. Mahluli indicates that he is using the upper right rectangle as the whole. Following the working ideas, the teacher has him indicate the equal parts and say how many of the parts are shaded. She, then, works with another student to reiterate Mahluli’s explanation, checking with him that they are understanding him correctly and then asks the class, “If Mahluli calls this the whole, is he right that that’s one half?” Getting affirmatives, she then says:

Now the question asks you something a little bit different. So who can tell everybody what question we’re trying to answer? What Mahluli did is right, but he used something different to be the whole.
Referring back to the first question, the teacher asks Avery what she thinks is meant by “the big rectangle.” Avery ventures, “The whole rectangle?” “What whole rectangle? You want to come up and show us? Mahluli, are you watching?” Avery uses her finger to trace around the outside of the full figure. The teacher then reiterates the first question, emphasizing the intended whole: “If you use the whole big rectangle to be the whole, how much is shaded blue?” Before inviting students to answer the intended question, she checks in with Mahluli, “Do you see the difference between the question you answered and this question?”

Mahluli: You gotta try to figure out out of the whole square.
Teacher: Out of the whole rectangle. And you used what?
Mahluli: And I did half of the rectangle.
Teacher: You did a smaller part of the rectangle. Okay?

The class goes on to discuss answers why the answer is one eighth.

This is only a short excerpt from a longer lesson, but it still offers much to consider. I first use each of the four features identified above in isolation, then synthesize them and consider their implications for identifying mathematical knowledge for teaching.

Problems and answers. In the section on problems and answers, I argued that a mathematical analysis of problems and answers includes analyzing the mathematical structure of problems and answers and the role of problems and answers in the conduct of mathematical work in the classroom. What, then, are the mathematical problems and answers in this short excerpt? What is their structure, and to what kinds of problems in the discipline are they similar? Below, I provide three observations developed from a mathematical analysis of problems in this video segment and then briefly consider the structure of answers.

The three questions (what fraction is shaded blue, shaded green, and shaded altogether) are questions about fractions as area and the combining of such fractions. As such, the issue of representation of fractions plays a key role. In contrast to arithmetic questions where it is expected that the quantities will be treated arithmetically, questions such as which is larger $\frac{3}{16}$ or $\frac{1}{4}$ and what is their sum, the questions in this lesson are framed geometrically, both making and requiring a link between the arithmetic and the geometry. The identification of problems that relate issues and ideas across different mathematical sub-domains is an important theme in research mathematics. For instance, analytic geometry, such as the study of conic sections, grows out of a marriage between the analysis of equations and the geometry of curves that is made possible by the conception of a Cartesian plane. It relates a question about the solution to the equation $Ax^2 + By^2 = C$ to a question about characterizing oval shapes. Problems in one domain are often “translated” into analogous versions in another domain. A shift of this kind can provide
new insights and open up the possibility of transforming ideas, techniques, and ways of thinking from one domain of mathematics into another.

This sort of analog thinking also happens within a specific domain of mathematics. For instance, the three questions in this lesson, framed in terms of area, might be seen as analogous to basic questions of commensurability posed in Euclid, where they are framed in terms of lengths. The question at the heart of commensurability is co-measuring two lengths. In other words, for any two lengths, does there exist a third length such that integer multiples of it are equal to each of the two original lengths. The three questions in this lesson might be viewed as asking on what basis can you compare, or co-measure, the blue and green areas, and what does this imply for adding them? Related to this, the issue might be seen as one of scissors congruence, where one figure is cut into a finite number of pieces and rearranged into another of equal area. It is known that any polygon can be cut into a finite number of pieces and rearranged to form any other polygon having the same area. The point here is not that these issues be part of the instructional work in the classroom, but that the questions posed here can be viewed as suggesting an analogy between problems that are often viewed arithmetically (comparing or adding fractions by finding common denominators) and geometric issues (comparing or adding fractions by finding co-measures or making geometric arguments about areas).

A second observation is that the questions are typical of problems in mathematics that involve direct reasoning about definitions. The question of what fraction of the big rectangle is shaded blue (or green) is, at its core, a question of definition. It requires returning to a definition, mapping between the definition and the particular situation, and extending the partitioning so that the definition can be applied. Problems that, at their core, involve reasoning about and from definitions are common in the discipline; or rather, returning to a definition is a recurring move in addressing mathematical problems and constructing mathematical proofs. In these particular problems, the class’ “working ideas” distinguish key definitional issues and can be used in ways analogous to the use of a definition. For instance, the first question involves: (i) identifying the big rectangle as the whole; (ii) identifying equal parts chosen in such a way that they provide an integral measure of both the blue triangle and the big rectangle; and finally, (iii) determining how those equal parts can be measured by the blue triangle. The second step is potentially challenging. In this case the blue triangle subdivides the big rectangle into eight equal regions, so the blue triangle is one eighth, but this requires seeing and filling in the missing subdivision. All three of these problems involve more than simply attaching names to objects, because in order to make the situation fit with the definition an appropriate subdivision of the big rectangle needs to be identified. Instead, they require a more systematic consideration of the requirements of the
definition, which then lead to reasoning about appropriate names — in this case, reasoning that each of the pieces are one eighth and that the two shaded pieces are the same size and thus have the same fractional name.

The mix of two apparently unrelated shapes makes the tasks of seeing their equivalence and of adding them more complex. Dealing with the two shapes requires an argument establishing that the shapes are of equal area. This can be done either directly (by transforming one shape into another) or indirectly (establishing that each is the same fraction of the whole). Again, regardless of the approach, the fundamental nature of the reasoning and the check on a final answer refers back to the definitional issues of identifying the whole and the equal parts of that whole and of establishing the number of those parts comprising the quantity of interest.

A third observation, implicit in the first two, is that the three problems build on each other mathematically. The first two combine to support answering the third. This notion of a *hierarchy of progressively compelling problems* — both of a problem being part of a larger one and of its being decomposed into smaller, more manageable ones — is a common feature of mathematical problems and the conduct of mathematical work. The third problem could have been given on its own, without the first two. What makes it interesting is the way it sits inside of larger, more ambitious questions about relating and adding fractions involving (un)equal parts (unequal as shapes yet equal as areas), about comparing and adding fractions as abstracted real numbers, and about the area-equivalence of distinct shapes. These larger issues provide important mathematical motivation for the problem and suggest subsequent mathematical work that might be done. Likewise, one approach to solving the problem is to ask the first two questions about the fraction shaded blue and the fraction shaded green. These are not the only problems that could be used to decompose the work of solving the third problem, but they represent a mathematically sensible approach.

This analysis of a hierarchy of problems as I am describing it here is a disciplinary analysis suggested by a perspective that grows from the discipline and from the nature of doing mathematics research. It is parallel to, yet distinct from, an analysis that might be done from a perspective concerned with curriculum or the design of instruction. Nothing I have said above is framed in terms of learning. The issue of motivation is about what about the character of mathematical motivation for problems and for working on those problems as it is distinctive in the discipline. Likewise, the sequencing of problems has a different flavor, and potentially different outcome, when viewed from the discipline as when viewed from the perspective of curriculum or instruction. At the same time, the value of such a disciplinary analysis is the ways
in which it can directly inform the related instructional issues of instructional motivation and sequencing.

Turning from problems to answers, it is interesting to notice that this brief segment involves the giving of the single answer of $\frac{1}{2}$. The teacher took up the answer and the class engaged in significant mathematical work with this answer. Even if the teacher had not pursued Mahluli’s idea, a mathematical analysis, on hearing $\frac{1}{2}$ as an answer, even if only murmured, would be listening for answers given, would ask what line of reasoning supports the answer, and would use the answer to consider a larger space of answers. This work suggests a range of answers that arise from varying the choice of the unit or whole. For instance, the area of the triangle that is shaded blue might be seen as $\frac{1}{4}$ of one of the intermediate rectangles in the figure. Likewise, an arbitrary unit could be given and then the area of the triangle might be of any value. This suggests a different yet related set of problems that reverse the specification: if the area of the triangle is $\frac{1}{3}$, give the area of the big rectangle, or construct a rectangle (or square) of area 1. The structure of answers to this problem, however, are similar to the structure of the problems in that they are fundamentally definitional in nature, where the issues of whole, equal parts, and the number of equal parts are central.

Furthermore, it is the three features of the definition that can be seen to structure the work being done in the classroom. In this case, the teacher explicitly uses the “working ideas” as an organizing tool for the mathematical work to be done. In other words, it is the sequence of issues given in the “working ideas” that Mahluli is asked to address.

Solutions and explanations. In the section on solutions and explanations, I argued that a mathematical analysis of solutions and explanations involves attending to the twists, turns and emerging solutions and explanations and involves framing classroom solutions and explanations in terms of structures and conventions common in the discipline. In particular, I suggested three foci for characterizing the analysis: (i) examination of particular reasoning for particular content; (ii) examination of the implications for making progress on solving problems and establishing claims; and (iii) consideration, at a meta-level, of general aspects of the structure, grounds, and standards for mathematical explanation. The discussion above about definitions and their use in mathematical reasoning has already addressed much about the nature and the structure of the reasoning in this short video clip. In addition, because the segment is short, there is much about the unfolding nature of approaches and explanations that requires an examination across the full lesson. Still, though, a mathematical analysis of this short clip might examine particular instances of reasoning and might consider implicit aspects of the structure, grounds, and standards for mathematical explanation.
When the teacher asks Mahluli to explain why he answered $\frac{1}{2}$, Mahluli says: “

Because they both equal-they both equal-and one, one, half of it is shaded in and the other half is not.

Mahluli seems to be saying that the two pieces are equal, which is important, but seems to be merely stating that half is shaded and half not without providing any basis for this. Simply restating what is to be shown does not constitute mathematical justification. Mahluli’s explanation is, thus, incomplete.

This type of examination of proposed explanations is similar to the kind of writing and reviewing of arguments central to professional work in mathematics. The practice of spotting and filling in gaps in mathematical arguments is commonplace. Mahluli’s explanation is mathematically inadequate. What happens to this explanation? What would make it complete? When he goes to the board to “point and show” what he means, he again says:

They both equal, and half of it is shaded in. So that makes it one, one half.

Accompanying his words he points briefly to the upper right corner of a large poster of the shaded figure. At one moment, he uses his thumb and index finger to indicate the two regions that are equal. This addition, however, addresses the first half of his statement, that the two triangles are equal. It again leaves unexplained the jump to the shaded triangle being $\frac{1}{2}$. To explain this part of his statement, he needs to connect his comment about the drawing back to question being asked. In other words, he needs to say something about the meaning of $\frac{1}{2}$ that makes it a reasoned answer. For instance, he might add that the shaded triangle is $\frac{1}{2}$ because it is one of the two equal pieces that make the rectangle. The mathematical move here is to argue that the shaded triangle is $\frac{1}{2}$ by referencing the definition of fractions (in this case for $\frac{1}{2}$) and verifying that the situation fits the definition.

This, indeed, is the move the teacher makes by suggesting the class look at the class’ working ideas about fractions.

So let's look at our working ideas about fractions that we were doing earlier today.

Mahluli’s explanation needs to return to core features of an agreed-on interpretation or definition of fractions. The working ideas in this case include identifying the whole, establishing that parts are equal, and identifying how many parts of the whole are included. His explanation has done the second of these. To complete his explanation, he needs to identify the whole and stipulate that the shaded triangle is one of the two equal parts. What is more, the first part, of identifying
the whole, is the source of both the problem with his answer and key clarification that makes his answer reasonable. The class’ work ideas constitute a basic mathematical structure for constructing an explanation.

The observations I am making here may seem obvious or even tedious. However, it is important to recognize that the source of the observations is a systematic, disciplinary analysis of practice that includes noticing the correctness, adequacy, potential and trajectory of explanations. These tools were applied to the lone statement given by Mahluli and would lead to the same analysis regardless of whether the class had a set of working ideas about fractions or had pursued Mahluli’s explanation. In addition, we see that the observations generated by this analysis provide a powerful tool for interpreting, and even informing, subsequent instruction. In this excerpt from the lesson, the teacher asked Mahluli three questions:

- Can you put your finger around the part you are calling the whole?
- Okay. And where are the equal parts? Can you show us the equal parts?
- Okay. And how many parts are shaded?

These questions led to a mathematically correct and adequate explanation based on the class’ working ideas about fractions.

Expressions and language. Turning our attention to a mathematical consideration of the language evident in this five-minute segment, we can see the interplay between ambiguity-laden expressions and the press for precision playing out even in this brief episode. When the teacher first asks Mahluli to explain why he answered \( \frac{1}{2} \), he says: “

Because they both equal—they both equal—and one, one, half of it is shaded in and the other half is not.

Reading this as text is not so easy. Although it might be easier to interpret as it is spoken in the flow of the classroom, interpreting it is still non-trivial. Returning to Ball’s (1997) challenges for pedagogically motivated listening, it is perhaps worth noting that the differences in background, the context-dependent nature of understanding, and the desire for Mahluli to be right are important for being able to “hear,” but they are only part of the picture of what makes it hard. In addition, what Mahluli says creates mathematical ambiguity (two self-consistent but mutually incompatible frames of references). Interpreting it requires the deeply mathematical work of pursuing the mathematical ambiguity. Again, I do not mean this as a claim about what should happen instructionally. Instead, I am saying that a mathematical analysis of practice is about mathematically motivated listening that ponders the distinctively mathematical ambiguity of seeming to have two apparently reasonable but contradictory claims and seeks out the
mathematical ambiguity of the word *rectangle* as it is used in the question and the idea of the *whole* as it is used in the idea of fraction.

I have discussed at length above the core mathematical issues about the ideas about rectangle, whole, and fractions. It is important to see here that an additional way into these issues is through mathematically motivated listening that attends to the words and ideas being expressed in a framework of the interplay between ambiguity and precision. One interpretation of what is happening in this episode is that Mahluli is mistaken and that the teacher engages in maintaining his respectability by developing the reasonableness of his answer. One might see the episode as one of making sure that Mahluli (and any others who share his confusion) understands the question. The mathematical analysis of expressions and language that I am proposing here would view Mahluli as having precipitated (however inadvertently) important mathematical work of exposing ambiguity and establishing precision. In this case, it did not lead to a major mathematical breakthrough, but it is, nevertheless, in line with the distinctive nature of routine mathematical work.

Another term that stands out in this segment is that of *square*. It is used twice by Mahluli to refer to the small rectangle around the triangle — in other words, in ways that are incorrect, or at least inconsistent with standard usage. The second time, the teacher reiterates what he has said, but replaces the word square with rectangle. It is not used in a substantive way in the reasoning, so does not signal a particular ambiguity and is not significant in advancing the mathematical work of the class. A mathematical analysis would probably note the incorrect use of the term, but would be unlikely to pursue it in any depth. Not pursuing far in the analysis would be due to its minor relevance to the work, not due its minor role in the class.

The third issue arising with the analysis of expressions and language in this segment is a meta-issue about the language work of mathematics, one explicitly mentioned by the teacher in this segment. In mathematics, listening to the exact expression of ideas is an important part of the work. From listening, one gets a sense of what needs to happen next to advance the work. Throughout this brief segment students are expected to listen to the words, consider what they might mean as reasoned mathematical ideas, and refine the expression of ideas in ways that are precise and agreed on in the group.

The teacher makes much of this explicit. After Mahluli’s initial explanation, she says

> Did everyone hear what Mahluli said? You should be thinking already about his reason. Who can repeat what Mahluli said? Okay. Well if you're listening carefully, you should always be able to tell what someone just said.
Listening is not just about politeness; it is central to the collective aspect of mathematical work. And listening is about making mathematical sense of the terms and ideas; there is a distinctive mathematical demand for precision in such listening. It is not clear, for instance, that repeating what someone else has just said is a necessary activity in a lesson on history or on interpreting literature — necessary beyond the usual demands of effective communication. In mathematics, that demand seems to be heightened and specifically attentive to precision and consistency — explicitly identified and agreed on terms and meanings. For instance, a few minutes later, when Mahluli goes to the board to explain his thinking, the teacher tells the class to look up at where he is pointing in order to understand his explanation. The expectation of being able to repeat what someone has said and of attending to exactly what the referents are for the words being used is foundational to a disciplinary practice of “peer review” and reflects a disciplinary language practice of making expressions precise and of agreeing on the definitions of terms.

Another example comes a few minutes later when the teacher asks for someone to read the problem aloud and, “say what we are supposed to interpret the whole to be from that question.” This is an explicit request to make the meaning of the whole mathematically precise. She then concludes

Okay, now we want to talk about all of this (pointing to the large rectangle). The question asks, if you use the whole big rectangle to be the whole, how much is shaded blue? Mahluli, do you see the difference between the question you answered and this question?

She introduces the new phrase “the whole big rectangle” to linguistically point at the intended whole and she checks with Mahluli to be sure that the class has agreement on this whole. Throughout this segment students themselves are expected first to listen in mathematically generous ways to Mahluli and then both Mahluli and the rest of the class are expected to agree on the whole to be used and language to specify it.

Character(s) of the work. In the section on the character of mathematical work, I argued that a mathematical analysis considers three features of mathematical work: its organization, its collective nature, and its agents. The segment itself is a chunk of mathematical work with a beginning and end and it fits within a larger chunk of work on the first of the three problems (and work on all three problems) that extends beyond the segment. Inside of these larger chunks of work, Mahluli offers an answer of ½. Others in the group have also been working on the problem, and many have a different answer. Whether Mahluli’s answer is surprising or not, an answer calls for a reasoned explanation, and an explanation needs to be understood and evaluated by the group. In this case, Mahluli’s explanation calls for clarification and agreement on the
meaning of the question. This clarification marks an end to the chunk of mathematical work set in motion by his answer. Because his answer is surprising and does not lead to an accepted solution to the problem, it sits as a smaller cycle within the larger context and the work loops back to consider other solutions. Its end positions the group to return to the unfinished work of answering the first question as intended and eventually the set of three questions.

Throughout this segment, much is communicated about the nature of collective work and the work of individuals inside that effort. At the start of the segment, the teacher says:

I'd like to see some hands from people I haven't seen speaking in whole group yet today. Who thinks that they could explain their thinking for question one? Still waiting to see a few more hands. Lots of people have work done in their notebooks. Okay Mahluli. What do you think about question one?

The class has been working individually on the problem and is now convening to decide on the collectively accepted answer. In a sense, this might be seen as analogous to mathematicians coming together at a conference to review one another’s work collectively. As might be the case at a conference, the teacher’s comments seem to convey an expectation that everyone is participating in this phase of the work, that the work to be done is to present and explain solutions that have been generated. When Mahluli gives his answer of $\frac{1}{2}$, he is expected to explain it, and the teacher asks him to go to the board to present it publicly, so it can be reviewed by everyone.

As he makes his way to the board, she says to the class:

You should be thinking already about his reason. Who can repeat what Mahluli said? Okay. Well if you're listening carefully, you should always be able to tell what someone just said. Doran, what did he say?

This is a fascinating statement about the individual responsibility of audience members during the presentation of a claim. Thinking about reasons for a proposed answer, on one’s own, even before it is publicly presented, communicates something very much in tune with a disposition one would find in the profession — not in all cases, but as an inclination. The second point about listening carefully to explanations and being able to repeat them, especially when one is responsible for reviewing and sanctioning work, is also in fitting with what is involved in the professional role of reviewer.

Repeatedly, the teacher communicates to students that their role is to first understand and then evaluate Mahluli’s idea. They are to do this individually (as a focus of their individual attention and thinking) and collectively (in the sense that they need to reach agreement on what is
being said and on its evaluation). Before Mahuli goes to the board to explain his thinking, the teacher says:

   Everyone should be looking up at where Mahluli is pointing, otherwise you won't understand his explanation.

After he explains his thinking, the teacher asks:

   Who knows what Mahluli did to get his answer of one half? I don't want to hear how you agree or disagree. I just want you to tell me what did he do.

After reiterating his explanation, the teacher then asks the class:

   So if Mahluli calls this the whole, is he right that that's one half?

This set of exchanges articulates central tasks in the collective conduct of mathematical work, in particular of presenting and reviewing claims.

   The requests to listen to classmates, to repeat what is said, and to decide on statements may have pedagogical motivations as well. However, from a disciplinary perspective, they also point to important characteristics of mathematical work.

   Hinted at, but not directly addressed in the analysis above, is the fact that the mathematical issues arising from this analysis of practice have important implications for an understanding of mathematical knowledge for teaching. Also evident in this video segment are a host of pedagogical issues. Later in this chapter, after specifying key features of a pedagogical analysis, I return to this episode with Mahluli. However, before turning to the pedagogical analysis of practice, I conclude this specification of a mathematical analysis by summarizing each of the four themes, the focus of each, and some illustrative examples of what each yields.

4.1.7 Summarizing the Composition of a Mathematical Perspective

   The four themes described in this section are not the only themes possible. Other themes might be added and the issues addressed by these themes might be formulated in different ways. However, the set of themes proposed here have two important characteristics. First, the issues addressed by the four themes are arguably key issues for any mathematical analysis of practice. It is hard to imagine any set of foci that does not provide serious treatment of the problems and tasks used, of mathematical reasoning and knowledge building, and of mathematical language. Each of these is essential to making sense of what is happening mathematically in classrooms interactions. The fourth focus on mathematical activity simply addresses these issues from a more holistic vantage point. Second, the four themes are formulated in ways that incorporate
pedagogical concerns in important ways, key to assuring their relevance to teaching and learning. Throughout this section, I have pointed to ways in which each focus represents an important intersection between the two distinct perspectives of mathematics and pedagogy. The problem I identified at the outset was the need for a mathematical analysis, grounded in the discipline, yet maintaining relevance to issues of teaching and learning. In this section I have elaborated the many ways in which these four themes, as formulated, are attuned to practice. Alternative proposals for the focus of a mathematical analysis need to be considered with this crucial feature in mind.

The figures below summarizes the four themes, or foci: (i) the nature of the link each creates between the discipline and teaching, (ii) the unique contribution that a disciplinary version of the theme affords in the analysis of practice, (iii) key features to focus on as part of an analysis, and (iv) examples of the kind of results yielded by the focus. Looking across the example results for the four foci, a feature that seems to stand out is the emphasis on mathematical work in a mathematical analysis of practice. For instance, the example results for focusing on problems and answers include the mathematical motivation for problems, genres of problems, and hierarchies and sequencing of problems. This collection of issues might be seen as helping to situate problems and answers in a disciplinary image of mathematical work. Or, rather, they might be seen as developing a notion of “mathematical work” that serves to bridge between the discipline and activities of teaching and learning. This is a point to which I return in the third section of this chapter, where I examine the coordination of mathematical and pedagogical perspectives in the analysis of practice.
<table>
<thead>
<tr>
<th><strong>Problems and Answers</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Analysis of the mathematical structure of problems and answers and of the role of problems and answers in the conduct of mathematical work in the classroom.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Nature of Intersection Between the Discipline and Teaching</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>The notions of “problem” and “answer” are central in the discipline and in school. Including “mathematical tasks” as a kind of sub-problem arising in the doing of mathematics further expands the intersection.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Unique Contribution</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>A mathematical eye on the statement of problems, subtle issues of wording in classroom talk, underlying mathematical structure of problems and answers, and relationships among problems can inform the selection, creation, and use of problems and tasks in classroom teaching.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Where to Focus</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>• Notice tasks—what the tasks are, how they shift, and how they are related.</td>
</tr>
<tr>
<td>• Attend to changes or refinements in conditions that might clarify or fundamentally alter tasks.</td>
</tr>
<tr>
<td>• Express tasks in terms of their basic mathematical structure or mathematically salient features.</td>
</tr>
<tr>
<td>• Evaluate the mathematical difficulty of a task (not for children, but what it takes mathematically) and where it sits in a mathematical landscape.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th><strong>Example Results</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>Identify <em>genres of tasks</em> with disciplinary properties abstracted from particulars of the classroom, such as:</td>
</tr>
<tr>
<td>• <em>Multiple solutions</em>, associated with a hierarchy of increasingly ambitious questions.</td>
</tr>
<tr>
<td>• Family of <em>parameterized tasks</em>, whose variation affects the complexity and difficulty of the tasks.</td>
</tr>
<tr>
<td>• Modeled, or represented, with <em>different contexts</em>.</td>
</tr>
<tr>
<td>• Directly engaged using <em>only common experience and rudimentary mathematical knowledge</em>, yet leading to substantial mathematics.</td>
</tr>
<tr>
<td>Or:</td>
</tr>
<tr>
<td>• Does a solution <em>exist</em>?</td>
</tr>
<tr>
<td>• Is a solution <em>unique</em>?</td>
</tr>
<tr>
<td>• What are <em>all</em> of the solutions?</td>
</tr>
<tr>
<td>• What is a natural organizing structure for the <em>solution space</em>?</td>
</tr>
</tbody>
</table>

Develop *hierarchies of problems* — both of a problem being part of a larger one and of its being decomposed into smaller, more manageable ones. Identify *problems that relate issues and ideas across different mathematical sub-domains*. Articulate *mathematical sequencing of problems* as viewed from the discipline (instead of from the perspective of curriculum or instruction). Notice which problems, at their core, *involve reasoning about and from definitions* — requiring the return to a definition to solve or prove. Identify the character of the *mathematical motivation for problems* and for working on those problems.

Figure 4.3. Summary of *Problems and Answers* theme for a mathematical analysis of practice.
**Solutions and Explanations**

Analysis of twists and turns in solving and explaining in the classroom, particularly in terms of disciplinary structures and conventions.

**Nature of Intersection Between the Discipline and Teaching**

Solving and explaining are not only central activities of the discipline but also central to the knowledge-building activities of teaching and learning mathematics.

**Unique Contribution**

A mathematical eye on the basis for conviction in the classroom can inform mathematical motivation and sense making in classroom teaching and learning.

**Where to Focus**

- Notice, name and examine *particular reasoning* for particular content, including its absence and presence, correctness, completeness and structure, basis, and the instincts and intuitions implicit in it.
- Examine correct and faulty reasoning for what they imply about *making progress* on solving problems and establishing claims, including what they make apparent and the locus and movement of conviction in the group.
- Consider, at a *meta-level*, general aspects of the structure, grounds, and standards for mathematical explanation.

**Example Results**

Identify the *structure and the basis of particular arguments*, such as:

<table>
<thead>
<tr>
<th></th>
<th>Sean</th>
<th>Sheena</th>
</tr>
</thead>
<tbody>
<tr>
<td>Claim</td>
<td>Zero is not even (but not that it is odd).</td>
<td>Zero is even.</td>
</tr>
<tr>
<td>Definition</td>
<td>An even number is a number that can be made up of two (equal) things.</td>
<td>Even and odd numbers alternate on the number line.</td>
</tr>
<tr>
<td>Prior knowledge</td>
<td>1 and –1 are odd numbers (fourth graders said so).</td>
<td></td>
</tr>
<tr>
<td>Argument</td>
<td>What two things can make it (zero)?</td>
<td>Zero is situated between negative one and one.</td>
</tr>
</tbody>
</table>

Identify the *base of public knowledge* available for public use — as a point of departure for mathematical reasoning in the classroom.

Examine the *nature and role of definitions* in classroom mathematical work.

Interpret the *mathematical character of breakdowns* in mathematical reasoning in the classroom, such as recognizing that Sean and Sheena are reasoning from different definitions in the exchange above.

Identify and classify *types of arguments*, such as *generic proof*:

[Generic proof] refers to a situation where an argument is formulated around a representation of a strategically chosen example of a claim, intended to support one’s intuition while following the reasoning. But the argument itself appeals only to generic, rather than idiosyncratic, features of the example.  (Bass, 2009)

Figure 4.4. Summary of *Solutions and Explanations* theme for a mathematical analysis of practice.
Expressions and Language
Analysis of expressions that fumble with mathematical ambiguity, including the ideas that might emerge from them and the work involved in clarifying them and making them precise.

Nature of Intersection Between the Discipline and Teaching
Language plays an important role in mathematical thought and is essential to sanctioning and communicating products. It is also central to the talk-intensive activities of teaching and learning.

Unique Contribution
A mathematical eye on expressions, both as incipient ideas and as rigorous formulations, with distinctive roles in the conduct of mathematical work, can inform the interpretation of classroom events and the development of student thinking and of collective mathematical work in the classroom.

Where to Focus
- Notice the language of generative mathematical work, especially problematic expressions, with an eye for what might be gleaned and with attention to the need for more language.
- Notice the language of formalizing mathematical work, including the need for and introduction of terms, definitions and more precise and formal language.
- Consider additional features of mathematical rigor in the expression of ideas, such as logical consistency, the fit with convention, connections and mappings among different expressions, and explicit demand for definition, precision and consistency.

Example Results
Identify ambiguity and use the notion of two self-consistent but mutually incompatible frames of references to interpret classroom episodes, pausing to consider the potential such expressions have for generating useful mathematical ideas.
- When Sean announces that six is both even and odd, there are two frames, one in which six is even and another in which six has something odd about it. This leads to the idea of some numbers being an odd number of groups of two, which later on are dubbed Sean numbers.
- When Riba says you cannot subtract 9 from 0, there are two frames, one in which it makes no sense to “take away” from a group with nothing in it and another in which it is part of subtracting 190 from 200. This leads to developing the notion of negative integers, which then helps to clarify the nature of the regrouping needed in the subtraction algorithm.
- When Keith presents his initial thinking that 4/4 is equal to 4/8, there are two frames, one in which 4/4 is two times as large as 4/8 and another in which 4/4 and 4/8 are the same size. This leads to clearer language and clearer thinking about the “whole” or “unit” implicit in a fraction.

Examine the expression of claims, such as:

<table>
<thead>
<tr>
<th>Mathematical claims about zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>zero is not even or odd</td>
</tr>
<tr>
<td>zero could be even</td>
</tr>
<tr>
<td>zero is not odd</td>
</tr>
<tr>
<td>zero has to be an even</td>
</tr>
<tr>
<td>zero is not an even number</td>
</tr>
<tr>
<td>zero is always going to be an even number</td>
</tr>
<tr>
<td>zero is not always going to be an even number</td>
</tr>
<tr>
<td>zero is even</td>
</tr>
<tr>
<td>zero is special</td>
</tr>
</tbody>
</table>

And explore the process of naming and using names; of making and interpreting claims; and of evaluating claims. (Ball, Lewis, & Thames, 2008).

Figure 4.5. Summary of Expressions and Language theme for a mathematical analysis of practice.
Character(s) of the Work
Analysis of features of the mathematical work that goes on in classrooms: the organization of the work, the collective nature of the work, and the characters who do the work (people viewed as mathematical actors with dispositions and roles who populate the workspace).

Nature of Intersection Between the Discipline and Teaching
The notion of character(s) of mathematical work provides a useful intersection between the discipline (focused on doing mathematical work) and teaching (focused on engaging students in time on task, in other words in the mathematical work of doing and learning mathematics).

Unique Contribution
A mathematical eye on mathematical work in classrooms can inform the sequencing and structuring of work on instructional tasks, in particular the collective work as engaged by students learning to carry out specific tasks and roles in doing mathematics.

Where to Focus
• Notice the organization of mathematical work, including beginnings, transitions, and ends that make up a logical sequence of activities that accomplish mathematical work.
• Notice the collective nature of mathematical work, including public accessibility of and engagement with ideas and standards for individual work and collective responsibility for sanctioning approaches, reasoning, and results.
• Notice the people doing the mathematical work (as people with particular dispositions, being particular kinds of thinkers, and filling particular roles in the doing of mathematics).

Example Results
Identify a logical sequence of mathematical activities for accomplishing (collective) mathematical work. One example addresses more global activities:
• Posing of a problem
• Search for a solution
• Presentation and review of solutions
• Acceptance of a solution by the group.
A second example addresses the review of mathematical explanations:
• Think about reasons for a proposed answer, on one’s own, perhaps even before it is publicly presented.
• Listen well enough to be able to reproduce explanations that are given.
• Consider what explanations might mean as reasoned mathematical ideas.
• Refine the expression of ideas in ways that are precise and agreed on in the group.
Identify practices of collective mathematical work, such as:

<table>
<thead>
<tr>
<th>Establishing Common Understanding</th>
<th>Producing and Using Each Other’s Ideas</th>
<th>Arriving at Agreed Upon Solutions and Methods</th>
</tr>
</thead>
<tbody>
<tr>
<td>• Establishing common understanding of the problem and of the work</td>
<td>• Studying and conferring</td>
<td>• Establishing and debating methods</td>
</tr>
<tr>
<td>• Establishing common language</td>
<td>• Presenting ideas publicly</td>
<td>• Agreeing on the use and meaning of representations</td>
</tr>
<tr>
<td>• Establishing common knowledge</td>
<td>• Attending to and responding to ideas</td>
<td>• Revising</td>
</tr>
<tr>
<td></td>
<td>• Contributing and receiving parts of a whole (possibly discrepant)</td>
<td>• Maintaining consistency</td>
</tr>
<tr>
<td></td>
<td>• Recording collective mathematical work</td>
<td>• Agreeing on solutions</td>
</tr>
<tr>
<td></td>
<td>• Turning individual contributions into sites for collective work</td>
<td></td>
</tr>
</tbody>
</table>

Table of practices of collective mathematical work. (020328BkChp)

Figure 4.6. Summary of Character(s) of the Work theme for a mathematical analysis of practice.
4.2 Specifying a Pedagogical Perspective

The central inspiration guiding the overall program of research on mathematical knowledge for teaching at the University of Michigan has been the combining of attention to content knowledge with attention to the work of teaching. In the previous section, I examined what might be involved in carrying out a mathematical analysis of practice with the purpose of identifying mathematical knowledge for teaching. My goal was to pull forward aspects of the research endeavor regarding a mathematical analysis of practice that have been implicit in past work of the University of Michigan research team. In this section, I examine what might be involved in carrying out the kind of pedagogical analysis of practice that the group has engaged in, one attentive to the work of teaching and likewise concerned with the purpose of identifying mathematical knowledge for teaching.

In the previous section, part of my claim for the value of investigating the composition and nature of a disciplinary analysis was that no such theoretical lens currently exists. My argument in this section is similar. As with mathematics, analysis of the defining character of teaching is under-theorized. Research on teaching has been an active field of study for a number of decades. Despite this, research on teaching has only very recently begun to theorize teaching. As Grossman and McDonald (2008) argue:

> [M]ore than a quarter of a century after Lortie (1975) depicted the lack of a common technical vocabulary with which to describe the work of teaching, the field of research on teaching still lacks powerful ways of parsing teaching that provide us with the analytic tools to describe, analyze, and improve teaching. (p. 185)

It is this parsing of the work of teaching — of naming its components and its underlying grammar — that would characterize a pedagogical perspective as I mean it here and as has been implicit, yet central, to research conducted by the University of Michigan group. As Grossman and McDonald argue, the development of such a pedagogical perspective, with a common technical vocabulary, could support the analysis (and improvement) of practice in a variety of important ways beyond just the study of mathematical knowledge for teaching.

In other words, my goal in this section is twofold: to articulate more fully what is involved in the kind of pedagogical analysis of practice that the research group has been using in its study of mathematical knowledge for teaching; and simultaneously, to contribute to a broader conversation in the field about conceptions of teaching and the pedagogical analysis of practice. Although research on teaching has only very recently begun to theorize teaching, this appears to
be a growing interest among a number of researchers. Senior researchers such as Pamela Grossman, Megan Franke, Magdalene Lampert, Gaea Lindhardt, Deborah Ball, Peg Smith, David Cohen, Mary Kay Stein, and others have turned significant attention and resources to the parsing and theorizing of teaching, especially as it relates to and supports teacher education. My goal in this section is to draw from ideas of these researchers and from an examination of project documents to develop a characterization of a pedagogical analysis designed to inform the study of the mathematical demands of teaching. It is not my goal to provide a full-fledged framework for characterizing teaching. I want, however, to characterize strategic features of a pedagogical analysis and to offer examples of parsings of teaching that illustrate these features. These proposals are generated out of the specific context of studying mathematical knowledge for teaching and may have limited reach beyond this context, but at the conclusion of the section I discuss their potentially broader application.

This section, then, attempts to answer the following questions:

- What might it mean to use a pedagogical perspective for the purpose of identifying the mathematical demands of teaching?
- What constitutes such an analysis and what light might it shed on a study of practice that aims to identify mathematical knowledge for teaching?

As with the work in the previous section, several issues make this endeavor both interesting and challenging. Most of the current research that seeks to theorize teaching has been developed for the purpose of improving teaching and teacher education. This leads to the question of whether and how well ideas from this research apply to efforts to identify mathematical knowledge for teaching. For instance, research focused on developing a curriculum for teacher education is concerned with identifying “teachable components” (Grossman and McDonald, 2008; Ball et al., 2009). It is not clear that a parsing of teaching designed around teachable components is necessary or well suited for a study of mathematical knowledge of teaching. Part of the challenge is identifying which ideas about the parsing of teaching from the current literature are likely to be important for the study of mathematical knowledge for teaching and which are not.

A second issue that makes this investigation both interesting and challenging is analogous to the issue discussed in the previous section — that mathematics and teaching are very different species of activity and that their mutual engagement may often be at odds. For instance, differences in the nature of knowledge in these two domains may mean that a parsing of teaching that maintains the integrity of teaching may make it difficult to articulate the kind of propositional knowledge most readily identified with articulations of knowledge in the discipline. My central
goal in this section is to specify a pedagogical analysis of practice that is useful in a study that aims to identify mathematical knowledge for teaching. The goal of identifying mathematical knowledge out of, at least in part, a pedagogical analysis places a very different, and potentially conflicting demand of such an analysis. This returns us again to the initial issue of the extent to which this specific purpose leads to unique features and requirements or whether its features and requirements are in the end common.

A third issue, regarding the generality and specificity of a parsing of teaching, is evident in current speculations about the nature and composition of such a parsing. For instance, Grossman and McDonald (2008) argue that a framework for teaching would require:

… a careful parsing of the domain, an effort to identify the underlying grammar of practice, and the development of a common language for naming its constituent parts. A framework for teaching could identify the key components of teaching, both those that are common across grade levels, subject areas, students, and school context and those that are particular to specific subject matters, to specific kinds of learners, such as English-language learners, or to particular teaching contexts. A framework for the field would also need to be agnostic with respect to various models of teaching; it must work equally well to describe components common to both direct instruction and more inquiry-oriented teaching while offering the flexibility required to recognize the significant differences in how such components might be enacted. This effort to parse teaching would need to respect the difficulty of breaking apart such a complex system of activity and the dangers of doing irreparable harm to the integrity of the whole by making incisions at the wrong places. Such a framework could inform both research on teaching and the improvement of professional education (p. 186)

They consider the generality and specificity of components of teaching. They suggest that the components identified need to be agnostic with respect to different models or methods. And, they point out that the decomposition of teaching needs to respect the integrity of the whole, of teaching as a coherent, comprehensive activity, implying, among other things, that the decomposition needs to be re-composable into the whole.

Similar features are evident in Ball et al. (2009). When describing criteria to help identify high-leverage practices for teacher education, they write:

Two goals shaped our considerations. First was a concern for generalizability and usefulness. High-leverage practices are ones needed for teaching in any setting, regardless of variations such as curricula or “teaching style.” Second was “teachability.” We sought to choose practices that lend themselves to careful
instruction in the context of our teacher education courses. Although the specific practices identified as high-leverage might vary with content and context, the criteria for high-leverage practices—in particular, the overarching concerns of generalizability, usefulness, and teachability—may, in fact, be useful across subject matter and teacher education settings. (p. 461)

These researchers echo the notion of a decomposition being agnostic, but characterize this idea as one of generalizability and usefulness. In this, they also include curricula but then say that specific practices might vary with content and context. It would seem that, given their focus on teacher education, they want to prepare students to teach across different curricula and teaching styles, and presumably across different student populations and school contexts, but that decompositions may vary with respect to the circumstances of a particular teacher education program and that versions of a particular practice might vary across subject areas or different sites in which it is situated in the program. At the same time, however, they do not seem to mean that across or within programs highly contextualized, idiosyncratic decompositions of teaching are appropriate. Elsewhere they go to great lengths to explain the collective nature of the endeavor and the need to develop shared knowledge and a common technical vocabulary reminiscent of Lortie (1975).

These two formulations (by Grossman and McDonald and by Ball et al.) of the nature of a useful decomposition of teaching and of what is involved in a pedagogical analysis of teaching characterize important aspects of the endeavor. In this section, I argue for a different formulation, one that aims to characterize a set of underlying features that may be related, but are formulated in different terms. In doing so, I see this formulation as complementing and clarifying the two described above, or rather, of developing foundational features that would inform them, not replace them. Among other things, I argue that in the formulations given above the visions about generality and specificity are hard to disentangle, and that the formulation I develop in this section helps to disentangle them.

After I develop a characterization of a pedagogical analysis suited for the study of mathematical knowledge for teaching, I will return to the three issues outlined above: (i) what the purpose of identifying mathematical knowledge for teaching implies for a pedagogical analysis; (ii) ways in which the identification of work of teaching may be at odds with identifying mathematical knowledge; and (iii) the need to disentangle generality and specificity in parsing teaching.

To address the research questions outlined earlier, I generated some initial hypotheses and then worked iteratively back and forth between project documents and revised hypotheses.
As mentioned before, I read through a representative sample of project documents, considering each sentence as a means of commenting on something, of drawing something to the fore in the research. For each sentence, I asked: *What is this a comment about?* I wrote short descriptions for each and then reviewed these and created categories. Five broad foci emerged as salient.

- Mathematics
- Students
- Teacher
- Instruction
- Research

I then looked more closely at documents and parts of documents engaged in the analysis of practice. Regarding comments about students, the teacher, and instruction, I asked the question: *What about teaching is the comment about?* Using these descriptions and existing ideas about the characterization of teaching I created a small collection of themes, which I iteratively reapplied to the data and revised. This yielded the following three features or themes.

- Attention to the work entailed in teaching
- Identifying potential pedagogical purpose in practice
- Layering tasks in the specification of teaching

If taking of a pedagogical perspective is about parsing the work of teaching into components that capture its underlying grammar, then I am claiming here that part of what is involved is looking at practice with a pedagogical eye, but in ways that are consonant with these three themes. It is about identifying tasks of teaching that have generality across different approaches to teaching and that are centrally about designing and managing instructional interactions. Focusing on core tasks of teaching means that the mathematical knowledge associated with these tasks will have broad applicability and will be framed in ways that support use of the knowledge in practice. A second important theme is a focus on making more explicit the potential pedagogical purposes associated with specific activities. This affords perspective that clarifies mathematical demands and that supports judgments about the knowledge needed as well as about its use. In the context of studying mathematical knowledge for teaching, explicitly identifying the kinds of purposes that might be in play at a particular moment in teaching can help to orient the mathematical analysis of practice, keeping it focused on mathematical issues that bear on teaching. And, last, a pedagogical analysis of practice can better support the study of mathematical knowledge for teaching, if it frames the work of teaching at a range of levels that
provide specificity and support the examination of a full range of mathematical demands associated with the work.

In contrast to the four themes identified for a mathematical analysis, the themes identified here are orientations or principles to guide the conduct of a pedagogical analysis. Throughout the section, I give examples of particular foci that might be used in a pedagogical analysis, but the section is organized around these three broader themes.

<table>
<thead>
<tr>
<th>Attention to the Work Entailed in Teaching</th>
</tr>
</thead>
<tbody>
<tr>
<td>The <em>work entailed in teaching</em> is that which is there to be done to achieve the goal of student learning as it combines with the many competing goals that society and local communities have for schools. Key to identifying and testing candidate tasks of teaching is reference back to the definition of teaching as the design and management of instructional interactions.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Identifying Potential Pedagogical Purpose in Practice</th>
</tr>
</thead>
<tbody>
<tr>
<td>The specification of a task entailed in teaching can be supported by identifying potential <em>pedagogical purpose in practice</em>: in particular, by: (i) identifying an act of teaching; (ii) exploring potential purposes, alternative actions, the nature of conditions relevant to the action, priorities associated with it, competing goals, and analogous actions in the context of varied teaching situations, such as different grade levels, teaching styles, or topics or subject matter; and (iii) using these explorations to formulate a version of the task that is further from being an idiosyncratic teaching act and closer to being about central work entailed in teaching.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Layering Tasks in the Specification of Teaching</th>
</tr>
</thead>
<tbody>
<tr>
<td>For the purpose of identifying mathematical knowledge for teaching, tasks need to be <em>layered at multiple levels</em>. They need to be: (i) identified at a variety of levels; (ii) elaborated into collections of sub-tasks; and (iii) situated within the context of more general tasks.</td>
</tr>
</tbody>
</table>

Figure 4.7. Overview of three themes (principles) of a pedagogical analysis.

The next three subsections define and illustrate these themes.

4.2.1 *Attention to the Work Entailed in Teaching*

As a starting point, let me begin with a few brief comments about *work* as I mean it here. I do not mean the more contemporary notion of paid employment, but instead the general notion of something done or something to be done, often in reference to results or directed to a definite end. Thus, the meaning of the *work of teaching*, or of any occupation, is a referent to that which is there to be done to achieve the goal, which in the case of teaching includes a goal of student learning — as it combines with the many competing goals that society and local communities have for schools. Interestingly, in the Bible, *works* are moral actions, usually contrasted with *faith* or *grace*. In this context, qualified by *of*, the term expresses the moral quality of action, as in *works of charity* or *works of darkness*. The implication of teaching as moral action is
interesting in its own right, but more relevant here is the contrast with faith and grace, where the work of teaching then becomes deliberative action that takes responsibility for the goals set for it. A specification of the work of teaching then becomes the development of a common technical language that decomposes teaching into component tasks constituting that deliberative action.

Returning again to Grossman and McDonald (2008), they claim that a decomposition of teaching “needs to be agnostic with respect to various models of teaching.” More than being ecumenical, which would suggest a friendliness among different ideological positions, being agnostic with respect to models of teaching implies taking a position that there is no one correct model or method of teaching. Ball et al. (2009) write:

… high-leverage practices are ones needed for teaching in any setting, regardless of variations such as curricula or ‘teaching style’ … (p. 461)

These authors seem to extend the notion of being agnostic with respect to teaching styles to being agnostic with respect to curricula, with respect to different groups of students, and with respect to school setting. From what is written in their paper, it is not clear exactly what variation is allowable and what variation is not, but the variation they discuss includes more than just variation in models of teaching.

They also situate this notion of being agnostic within a notion of generalizability, which they combine with the notion of usefulness. In other words, well beyond the notion of being ideologically friendly, their concern appears to include a concern for applicability. They are concerned with identifying teaching practices that specify a curriculum for teacher education. This means that the practices they seek to identify would have practical utility in teacher education, and ultimately in improving teaching and learning in schools. Hand in hand with the notions of generalizability and usefulness they use language that both defines these terms but also suggests other related concerns — ubiquity, bearing, mathematical integrity, professional responsibility, leverage, and so forth. For instance, they identify four criteria base on examination of the work of mathematics teaching:

- Supports work that is central to mathematics
- Helps to improve the learning and achievement of all students
- Is done frequently when teaching mathematics
- Applies across different approaches to teaching mathematics

Generalizability and usefulness capture some of what seems to be important here, but not everything suggested by these criteria and by the language and phrasing used throughout the paper.
Returning to Grossman and McDonald, they turn to clinical psychology for a characterization of the kind of common teaching practices, or components, that they have in mind.

Such components might be comparable to what clinical psychology refers to as the “common factors” of therapeutic practice. Such common factors, including most famously the ability to establish a therapeutic alliance with a client, have been shown to be critical to the success of any particular therapeutic approach—be it cognitive-behavioral therapy (cf. Beck, 1979) or a Rogerian client-centered approach (cf. Rogers, 1961). Research in clinical psychology suggests that if a therapeutic alliance exists, a number of approaches can be successful; without such an alliance in place, any approach will be less likely to work (Grencavage & Norcross, 1990; Horvath, 1988; Stevens, Hynan, & Allen, 2000). One direction for research on teaching would be to continue the search to identify such “common factors” in teaching that are critical for success. (p. 187)

In this, we again see the notion of generalizability in a way that goes beyond being agnostic. Something else, however, comes through in these comments — something about activities that are critical to success, something related to essential functionality of the endeavor as distinct from particular approaches. The notion of generalizability suggests that a way to identify the kind of decomposition of teaching that we want would be to examine different approaches and identify what is common to them. Although this makes sense and can contribute to developing a decomposition of teaching, even one that seeks to identify mathematical knowledge for teaching, a more distinctive feature of the decomposition of teaching that we seek is that it focus on what is entailed in the endeavor of teaching due to its underlying purpose, clientele, and setting. After clarifying what I mean by this phrase, I then discuss implications for the analysis of practice.

The meaning of “entailed in the endeavor.” First, I explain what I mean by “entailed in.” Then I expand to look at what I mean by “entailed in the endeavor of teaching due to its underlying purpose, clientele, and setting”. Third, I discuss the relationship between this notion of “entailed in” and the ideas of agnosticism and generalizability described above.

The word “entail” derives from the Old French taillier meaning to cut or shape, hence to limit. In legal terminology a “tail” referred to restrictions on an inheritance according to conditions set by the donor, typically on who could inherit an estate, and “entail” became the establishing of such restrictions (OED, 1989). In the 1800s, it came to mean to bring on by way of necessary consequence, to involve logically, or necessitate (OED, 1989). Later still, it came to mean to have as a necessary accompaniment (American Heritage, 2000). It is this later meaning that I have in mind. Thus, by the work entailed in teaching, I mean that work which is a
necessary accompaniment of teaching, namely the work implicated by the aim of designing and managing instruction.

The phrase “entailed in” may sound awkward to some, but an electronic scanning of its usage seems to fit well with what I mean. The phrase “entailed by” is particularly common in philosophy and logic, where the synonyms entailment and implication are used technically to refer to particular deductive processes. The use of “entailed by” in computer science, law, and the popular press seems to emphasize this technical, deductive character. The phrase “entailed in” is used in a broader range of settings, with less deductive character and more of the sense of involving — for instance, entailed in installing a satellite dish, entailed in managing a health care system, or entailed in generating economic growth. Where “entailed by” focuses on the sense of necessary consequence, “entailed in” focuses on necessary accompaniment, or on what is involved in an activity or issue.

Likewise, the former phrase indicates dependence on the presence or absence of conditions while the latter presumes the presence of the condition and indicates how the aspect grows out of those conditions. For instance, if A implies B, and A is true, then it is said that B is entailed by A. Here, A indicates a condition which when met implies that B is true. The phrase “B is entailed in A” presumes A, and focuses instead on the way in which the seed for B lies in A. In this sense, when we refer to “the risks entailed in skydiving,” the risks are a consequence stemming from inherent aspects of the activity, not from a logical relationship among propositions. The word involve is indeed a close synonym, but its root is the notion of enfolding or enveloping and it conveys a sense of something being “put into” something else, rather than being derived from it.

A moment ago I said that the phrase “entailed in” focuses on the necessary accompaniments of an activity. The word “necessary” here needs clarification. In the shift from necessary consequence to necessary accompaniment, the meaning of “necessary” shifts as well — from meaning inevitable to meaning intimately connected. It is the nature of this intimate connection that is hard to put a finger on. My suggestion is that it is about identifying what is basic or inherent in the endeavor, what is intimately connected to a practice of teaching that is attentive to its distinctive or defining features as an endeavor.

In other words, when I argue that a decomposition of teaching needs to focus on identifying what is “entailed in the endeavor of teaching due to its underlying purpose, clientele, and setting,” I mean to draw attention to the nature of the relationship between the components identified and the character of teaching as a whole: that the components need to be necessary accompaniments of the endeavor of designing and managing the interactions of instruction.
defined by Cohen, Raudenbush, and Ball (2002) as described in Chapter 1. This formulation makes two important contributions. First, it clarifies the character of the relationship — that the components need to be part of the very nature of teaching, permanently characteristic of it or necessarily involved in it. Second, it clarifies that the judgment about the quality of the decomposition is made in reference to shaping the dynamics of instruction.

In project documents, the term “entail” is only used in reference to mathematics entailed in the work of teaching. Its use in project documents fits with the discussion above. In this section, however, I am using the term to clarify the phrase “work of teaching.” To specify the work of teaching, and its constituent tasks, is to identify what is *entailed in* the endeavor of teaching due to its underlying purpose, clientele, and setting, as described above. In other words, it is to identify tasks that are the necessary accompaniment of designing and managing instruction in school — tasks that are to teaching as risk is to skydiving. Such an analogy might seem jarring because tasks of teaching are about the decomposition of an activity and risk is about the character of the activity. It might be more natural to say that these tasks are to teaching as diagnosis is to medical practice, but my point is about the nature of the relationship between the two terms, not the type of object. I am arguing that medical diagnosis and figuring out what students think are professionally responsible activities associated with each respective profession; they are not logical deductions, but are basic activities, in the sense that they are associated with basic realities of the endeavor, as risk is associated with skydiving. For instance, Lortie (1975) argues that teaching is characterized by uncertainty and he traces that uncertainty into the unavoidable, intrinsic features of the activity. Uncertainty, then, is entailed in teaching. It is this relationship that I mean to identify, but with a focus on tasks of the activity instead of characteristics of the endeavor.

Returning, now, to the issues of agnosticism and generalizability, the notion of “entailed in” does not replace these, but frames them in a particular light. It suggests that being agnostic with respect to models of teaching is more than a political move: thinking across forms of teaching provides a useful tool for figuring out the work entailed in teaching as I have described it above. Likewise for generalizability: attention to the work entailed in teaching suggests that the concern is less about broad use across varied content and context and more about use that has broad application because it is specific to the issues of underlying purpose, clientele, and setting as framed by the instructional interactions represented in the instructional triangle of Figure 1.2.

This helps to clarify the potential confusion described earlier about ways in which components of teaching need to be applicable across approaches, curricula, content, students, and settings (generalized), yet at the same time be tailored to particular content, students, or settings.
(specific). Instead, components need to be general in the sense that they need to be about inherent aspects of designing and managing instructional dynamics, and they need to be specific to the extent that the particulars of content, students, and settings matter in the work. I realize that this does not resolve the problem of determining when a component practice should be specified in general terms and when it should be specified in specific terms, but it clarifies the kind of thinking that needs to go into such a determination. For instance, it clarifies that being agnostic or identifying highly generalized practices are not the ultimate goals in the search for a decomposition of teaching that could improve teaching and learning. Instead, they are tools for clarifying and testing proposed components of teaching. The goal is whether the component bears significantly on the work of designing and managing instructional dynamics. Significantly, it suggests, for instance, that aspects of teaching that use features of the content to accomplish the design and management of instructional dynamics are likely to be content specific or have content-specific versions. Aspects of teaching that arise from structural features of the interactions among teacher and students around content can probably be formulated in more general terms. This proposal means that, because the goal is the design and management of instructional dynamics, it is this issue that serves as the final arbitrator.

To clarify what I mean in saying that a pedagogical analysis needs to examine the work entailed in teaching, I next consider the selection and analysis of data that would support an investigation of the work entailed in teaching. This discussion helps to illustrate the ideas presented above and to create a more complete picture of what is involved in conducting such a pedagogical analysis of practice.

Data selection and analysis for identifying the work entailed in teaching. There are different agendas one might have in selecting samples of teaching practice to study. One might aim to change the nature of what gets taught or how it gets taught. One might aim to illuminate what goes into exceptional teaching. In contrast, one might aim to improve modal practice. Or, in an effort to raise the floor for mathematics teaching and learning, one might aim to determine a basis for minimal teaching. Depending on ones goals, the practices one seeks to illuminate are different:

- Practices of existing, or typical, teaching
- Practices of specific kinds of teaching (“this kind of teaching” or “ambitious teaching”)
- More challenging practices of expert teaching
- Basic practices of the work entailed in teaching

Each of these choices implies something different for data that would be best to study. The first calls for sampling typical practice and engaging in a process of documenting what
teachers actually do. The second suggests a need for finding instances of the particular kind of teaching and then describing the practices that enact such teaching. Existence proofs would likely be important and descriptive analysis of them would probably be the mainstay of such work. Other instances of teaching might be used to draw contrasts and to sharpen what is meant, but the endeavor would need to focus on positive instances of the kind of teaching of interest. Data selection and analysis for elaborating expert practice would be similar in the sense that it would involve the identification of the specific practice or practices of interest and the study, primarily, of these exceptional instances.

The goal I want to identify with here is definitely not the first of these — the description of existing practice. Instead, the goal is primarily the last, with occasional excursions into the second and third. Although many reformers and researchers approach the problem as one of achieving more ambitious goals for mathematics education than the goals of the past, I do not. In Chapter 1, I argued that student achievement is a problem, independent of new and ambitious reform goals, and that the problem of improving student achievement amounts to improving instructional dynamics, which in turn amounts to improving the design, implementation and management of those instructional interactions. Here, I also suggest that the wholesale improvement of mathematics teaching in this country will not be achieved by developing a particular approach or style of teaching and lobbying for its broad adoption. The circumstances of strong individualism, local control, uncertainty in the work, and the weak nature of research evidence make this impractical. Nor do I think that we will achieve the needed improvement by better understanding expert practice and trying to remake the massive population of elementary school teachers into uniformly expert practitioners. This, too, is impractical. Instead, I identify the problem as one of building a professional knowledge base for teaching, as called for by Lortie (1975), and as argued by Grossman and McDonald (2008). Key to such a knowledge base would be an understanding of the work entailed in teaching, construed as practices central to designing, implementing, and managing instructional dynamics.

The sampling of practice, then, needed to study the work entailed in teaching, would be any cases of practice where, in some form, the design and management of instruction is being carried out. The goal would be a kind of “job analysis,” as with the goal articulated in Ball and Bass (2003b) for investigating the mathematical work of teaching. The aim, as I am formulating it here, is to characterize the demands created by the endeavor of teaching due to its underlying purpose, clientele, and setting. This suggests that, at least in theory, any instance of instruction could be mined for the demands it makes, regardless of how well the teacher contends with those demands, including a wide spectrum of instruction that occurs both inside of and outside of
schools. However, practically speaking, different instances of practice support the study of the work entailed in teaching to a lesser or greater extent.

An important issue is that much of the work of teaching is invisible. As Lewis (2006) points out, in addition to the work of teaching that is literally invisible, such as teacher cognition or behind the scenes work, much of what is present for all to see goes unseen. Lewis identifies a range of reasons for this, including problems with observation generally, with distinctive features of the work of teaching, and with the low status of the profession. For instance, much of the relational and transactional work important for teaching may go unnoticed because it is informal, less explicitly valued, and underspecified in current thinking. Without designed analytic tools and methods, analysis that relies on observation may overlook the very phenomena it seeks to study.

Three specific concerns constrain the selection of instances of practice that are well suited for studying the work entailed in teaching, especially with video records. First is the need for interactions among the teacher and students around content to be available for analysis. Watching a classroom in which everyone sits silently is unlikely to yield much opportunity to examine the work entailed in teaching, at least for initial work in this area. Similarly, abundant talk, but not about the content, is unlikely to be useful. This is not to say that teaching and learning do not occur in settings that are relatively quiet or to say that good teaching does or does not encourage abundant talk, but it is to say that access to key interactions is an important consideration for selecting data. In particular, it seems important to select instances of practice that provide access to student thinking about the content in either spoken or written form and to the comments, directives and feedback that teachers provide to students concerning content.

Second, is the need for instances of instruction to afford access to teacher reasoning about the design and management of instruction. Such access might be gained from lesson plans, teacher journals, or pre- and post-interviews, but these sources vary in their reliability. It is important that some access to teacher reasoning be available from the observation of practice as it is expressed directly in teacher talk and indirectly in teacher action. Even beyond the reasoning teachers provide to students about what they are doing and why, talk that directly serves pedagogical purposes, some teachers engage in a teaching practice in which they explicitly narrate, or make visible, their practice for others. For instance, some teachers who routinely have outside observers in their classroom, in particular observers they mentor or with whom they collaborate, may develop unobtrusive ways of helping others notice what is going on and why. This is not a well-understood or developed activity, but some practitioners have invested significant effort in making their practice study-able by others, in particular by narrating more of
what they do and why they do it than most teachers typically do. It is not that the narration is correct or that it provides a true picture of what the teacher is doing, but analogous to the way in which substantive student and teacher talk in the midst of instruction provides more access to the interactions that constitute instruction, substantive teacher narration of practice during instruction can provide more access to the engagement of teachers in designing and managing the dynamics of instruction.

A third factor that can be important in selecting instances of practice to study has to do with the situated nature of the data. In analyzing the work entailed in teaching due to its underlying purpose, clientele, and setting, it can become important to gain insight into the context in order to judge purpose, clientele, and setting. For instance, when observing a video of instruction, access to curriculum materials from which the teacher is drawing can help immensely in making sense of what a teacher is doing and why. Likewise, there are aspects of the work entailed in teaching, such as the establishment of classroom culture or of a shared sense of what counts as a good explanation, which require analysis of what came before or what happened next. The point here is not that such material and additional layers of data need be prominent, but that there are times in instruction and aspects of teaching where further access to the context becomes essential. Indeed, in showing video of instruction in classes and at academic presentations with the expressed purpose of engaging audience members in an analysis of the work of teaching, at least some people feel a need to know a great deal about the context before they are able to comment meaningfully on instruction. This seems mistaken; a great deal of analysis can be done with very little additional knowledge about the larger context, and the context can, at times, even distract from attention to the interactions that are central. Still, however, making sense of the interactions and of the work entailed in shaping interactions can, in some important ways, be significantly informed by examining specific aspects of the context that bear on the situation.

These three criteria are not meant to privilege any particular approach to teaching. For instance, the requirement of access to student thinking does not mean that the only kind of teaching that is important to analyze is teaching where discussion is the primary instructional format. Instead, it means that: (i) certain approaches to teaching might be a useful initial source of ideas that can then be tested more broadly; (ii) widely varied pedagogical approaches to getting a read on student thinking need to be considered when gathering data (because all teaching has ways of getting or presuming this information); and (iii) in instruction where the teacher’s approach to reading student thinking is not evident in extant records of practice, additional techniques, such as student interviews or teacher interviews might need to be developed. In brief, instances of instruction that provide easy access to instructional exchanges, to teacher reasoning,
and to additional layers of the context can be valuable data sources for research that seeks to
develop initial hypotheses about comprehensive decompositions of the work entailed in teaching
as characterized by Grossman and McDonald (2008).

In focusing on the work entailed in teaching, it is also important to recognize the ways in
which the analysis is about the work of teaching independent from the specifics of the people
doing the work, the particular ways in which it is being done, and the idiosyncratic features of the
context. The goal of the analysis is not to characterize what a particular teacher does, what
teachers in the country currently do, or even what they should do. It is more a matter of
entertaining what a teacher might responsibly do given the situation and what the range of things
teachers might do suggests about the nature of the work. Thus, it is also important to note that a
full range of practice, from highly competent to either novice or problematic teaching might be
used productively to analyze the work entailed in teaching. One might be tempted to say that
both the presence and absence of effective teaching practices afford opportunities for seeing the
demands of the work, just in different ways. High-quality teaching can offer insights into the
demands and the possibilities of the work by what it helps to draw attention toward, while low-
quality teaching can expose what is missing and essential. In saying this, though, it would be
important not to get confused and let effective practice become the object of study. Instead, it is
as if, together, the presence and absence of effective practices are the negative space of the object
of study, which is the demands that effective and ineffective practices respond to, either well or
poorly. It is the characterization of work to be done as defined by the boundary between the
demands of instructional situations and the rather wide range of teaching moves that might
responsibly be deployed in response to those demands. The selection and analysis of data needs
to be carried out with this central issue in mind.

This discussion sheds light, as well, on three commitments that the University of
Michigan research group frequently claimed were characteristic of the classrooms they presumed
were widely desired: (i) classrooms in which mathematics was treated with integrity; (ii) where
students’ mathematical thinking was taken seriously; and (iii) where students and their teacher
worked collectively on mathematics. These commitments might be read as being about a
commitment to a certain approach or a particular kind of teaching, but they might also be read as
being about foundational features of designing and managing the instructional dynamics in the
classroom. My point here is that the commitments seem to be, in part, an expression of what is
meant by the work entailed in teaching due to its purpose. This is not to deny that the formulation
of the commitments were also an attempt to characterize research with a secondary agenda of
studying a specific kind of teaching with specific goals, broadly conceived.
In conclusion, then, data useful for developing a decomposition of teaching in ways that lay bare its underlying structure and grammar need to provide access to instructional interactions, teacher reasoning, and context and the pedagogical analysis of practice needs to maintain a focus on these basic elements of designing and managing instruction. Next, I consider the importance of explicitly identifying pedagogical purpose in the analysis of practice.

4.2.2 Identifying Potential Pedagogical Purpose in Practice

The second theme that supports the pedagogical analysis of practice in a study that seeks to identify mathematical knowledge for teaching is an attention to potential purposes associated with tasks of teaching that are identified. I argue that, in addition to maintaining an orientation to the work entailed in teaching as described above, the process of specifying a task entailed in teaching can be supported by identifying an action, then exploring potential purposes, alternative actions, the nature of conditions relevant to the action, and priorities associated with it, and finally using these to formulate a version of the task that is further from being an idiosyncratic teaching act and closer to being about central work entailed in teaching. This process supports the formulation of a task entailed in teaching in ways that are at least sensitive to, if not expressive of, the multiple purposes and contingent character of teaching acts. It also supports the identification of mathematical knowledge for teaching because it includes orienting information that supports judgments about and formulation of mathematical knowledge for teaching. To elaborate both what I mean and why it is significant, I begin by looking back at project documents where discussions of pedagogical purpose arose.

Discussions of potential pedagogical purposes were common in the work of the research team. For instance, the group watched video of a class in which many of the children’s hands were raised while the teacher was responding to a student. The teacher asks, “Why are your hands up?” As soon as she asks the question, all of the students promptly put their hands down. The teacher follows by asking, “Why are your hands down?” Notes from a project meeting discussion of this interaction record the following possible purposes for this move.

[The teacher] wants students to be listening when people are talking, and if kids’ hands are up when others are talking, waiting to say something, this statement could intervene on that tendency; this could be a lead in to wanting to broach to having hands.

When kids’ hands went down when she asked “Why are hands down?” There could be a couple of purposes here. This move could be to get kids to think for themselves by challenging their quick hands-down when she had asked “Why are your hands up?”
Maybe the teacher has a sense of humor.

There might be very different reasons for asking “Why are your hands up?” versus “Why are your hands up?” Could be about challenging existing practices – e.g., not doing your own independent thinking.

[R1] remembers a time when kids had their hands up and she didn’t know why their hands were up and she asked, “Why are your hands up?” Why were kids having their hands up? Does the teacher know? Maybe this is a genuine question.

Wanting to consider each answer, rather than getting a whole collection of answers.

There might be some different reasons that kids had their hands up and she might have been trying to figure out why.

Deliberations such as these explore the nature of interactions between the teacher and students in ways that expand one’s sense of the action. This is a relatively minor and isolated action on the teacher’s part and did not lead directly to the development of a robust teaching task, but even here seeds for tasks can be seen. For instance, the first observation above suggests that a task of teaching might be to teach students what to listen to and how to listen in order to make productive use of instruction. An aspect of this work might include intervening on tendencies that routinely distract students from maintaining active mental attention to key instructional productions. Similarly, the observation that perhaps the teacher has a sense of humor might be read as a reminder that not all acts are intentional, but it might also be read as proposing a potential task of maintaining rapport with students and the potential role of humor in doing so. The last observation might lead to considering a potential task of teaching that is about reading students and gaining a sense of what they are doing and why. These could each be developed further and they are but a small sampling of the kind of tasks that might be suggested by this brief exchange. My point here is that each of these adds to the other a richer sense of possibility associated with the conduct of any one of them. In combination, each is captured with reminders that any task functions in a context of other goals that must be juggled.

In project work, deliberations like these, about observed actions, potential tasks, and the relationship between them, would lead to identifying potential tasks that typically served multiple goals, or at least were framed with sensitivity to multiple goals, and addressed basic, instrumental goals of instruction. These discussions would also explore alternative actions and hypothetical conditions that might influence the action. There were discussions of what-if questions, such as:
What if no one answered? Or, how might that be done if a teacher were concerned about the
c Child’s self-esteem? As discussions about alternatives unfolded, competing goals and priorities
among goals were often considered. In addition, as tasks were discussed, they were considered
and tested in the context of other, varied teaching scenarios, such as at different grade levels, with
different teaching styles, or in teaching a different topic or even subject matter.

Such discussions often lead to identifying potential tasks of teaching. Here is a list
generated from an examination of six project documents.

- Using routines to manage complexity and multiple goals
- Valuing by rules, praise, and time
- Getting everyone on board
- Presenting self as confused about student thinking
- Using language to distance child from problem
- Use of introductions
- Honoring as a way of labeling claims
- Co-construction of curriculum and student thinking
- Serving the group while attending to the individual
- Using tasks and questions to structure mathematical work
- Teaching kids how to do the work of instruction and learning
- Establishing social and intellectual habits
- Establishing classroom culture and expectations
- Making things in the instructional space usable by students

Interestingly, these tasks seem both generic and instrumentally related to basic elements of
managing instructional interactions. My point here is not about whether or not these are good
tasks, but about the character of them and what they might be doing in the analysis of teaching.
Indeed they are rough and uneven — intermediate by-products of research being done. However,
they all seem reasonable initial candidates as components of teaching that are relatively agnostic
and generalizable. They can be considered in terms of rather basic elements of designing,
enacting, and managing instructional dynamics. Part of the way in which they have become so,
or are so, is because they fold in and have been tempered by the consideration of important
elements of pedagogical purpose.

To illustrate what I mean, consider the first on the list. In a document in which one of the
researchers, as part of a project “homework” assignment, set out to attend to pedagogical issues in
a lesson from a third-grade classroom toward the beginning of the year. The researcher asked the
question: What structures are in play in the decision-making that the teacher does?

What I want to bring to the group at this time is some of what I
found with regard to structure. Below I propose several
structures that seem to be at play in building routine in the
classroom, routine that might carry some of the burden for
pedagogical decision making in the moment on the part of the teacher and might convey to students both a way of working and a vision of the intellectual substance of the class. An example of this might be "boardwork problems". These create a routine for students coming into class, but they also create a routine for planning a class—planning involves, in part, finding a boardwork problem; they provide a routine for enacting a class and leading a discussion. In addition, they are a powerful choice for carrying out multiple objectives in the teaching, such as, they shift the vision of what it means to do math, they emphasize understanding and a different set of skills, they support a different kind of classroom discourse, they support learning in particular ways, and so on. (980501PrjAss)

The use of “boardwork” problems might well be seen as idiosyncratic to this classroom. The specific goals might be seen as idiosyncratic to a particular kind of teaching or a particular set of instructional goals. However, what also can be seen happening here is the consideration of multiple goals, of similar goals that might be swapped in or out, and of a framing of the task in ways that are basic to designing and managing instructional dynamics. For instance, consideration of classroom routine as a means to manage the daunting array of decisions (of action or inaction) that a teacher must make creates a link between an event in a classroom and an event formulated in ways that make it a candidate for a potentially basic component of teaching.

The piece cited above goes on to explore, from classroom observation, what might be involved in establishing a routine and what further purposes such a routine might play. Among other things, it considers the practice of having children write the boardwork problem in their notebooks at the start of class and the ways in which this: (i) helps children take responsibility for getting to work; (ii) supports student learning (because it “familiarizes the student with the problem, provides a title for the work they do, reminds them as they work what it is that they are working on”); and (iii) creates a record of student work for the teacher, the parents, and the student.

From this example, we can begin to see a kind of investigation that might generate the kind of candidate high-leverage tasks of teaching that might constitute a decomposition of teaching. For instance, what does the work of establishing a routine look like, what kinds of purposes is it suited to serve, are there commonalities across different routines, both in the work of establishing them and in their purposes, and might the task of establishing a routine then be a high-leverage practice? Or, might there be specific routines for recording work; are there common features of such routines; what place does the recording of student work hold in the design and management of instructional interactions; and might routines for recording then
constitute a high-leverage practice? Central to these deliberations is the consideration of multiple, competing, contingent purposes related to activities of teaching.

The process of identifying potential pedagogical purpose seems important in the broader endeavor of decomposing teaching in an effort to create a curriculum for teacher development, but it is also important specific to research on mathematical knowledge for teaching. In arguing this I offer two examples. The two tasks, analyzing student errors and rescaling mathematics problems, have both been developed into tasks with significant mathematical demands that have been elaborated by the group in a variety of settings. Each of these, however, has involved extensive elaboration of the task of teaching with highly developed notions of instructional purpose. Furthermore, these elaborations have played an important role in shaping the identification and specific formulation of the mathematical knowledge for teaching entailed in each.

One result of the research group has been identifying the teaching task of analyzing student errors. In examining practice, student errors occurred frequently and engaged the research group in conversations about the nature of errors in classrooms and about pedagogical decisions in treating error. The more the group noticed and examined errors in the classroom, the more error seemed fundamental to the work of teaching. It seemed as if, because the goal of teaching is learning, student errors often serve as beacons that light the way to what to do to bring about that learning. In this way, they seem to have a role at the heart of teaching. Exploring a variety of situations, the group experimented with developing a typology of the kind of errors that arise in the classroom, as well as the different circumstances in which they arise, such as in homework or tests, in small-group work, in a whole class discussion, or in written work at the blackboard. The group also explored the moral dimensions of error. A paper prepared for the 2000 annual meeting of the American Education Research Association and titled, The moral work in teaching: The treatment of right and wrong answers in mathematics class, considered the nature of error in mathematics and doing mathematics, the nature of error in social life of the classroom, potential tensions between imperatives for mathematical correctness and concern for the intellectual life and emotional well-being of students, and implications these together have for teaching.

The group also examined the mathematical demands of treating error (Ball, Bass, & Hill, 2004; Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008). They argued that beyond simply being able to do a problem oneself and to determine whether a student’s answer is right or wrong, teachers need to be able to analyze what is happening in students’ error — is there a pattern, what reasoning might lead to it, and what ideas might constructively challenge it? In this work the
researchers identified mathematical knowledge and skill central to teaching tasks that involve analyzing and treating student error.

In the previous section, on the mathematical analysis of practice, I argued that a disciplinary perspective examines student errors for ideas about ways in which those errors might inform and advance mathematical work by clarifying the definition of terms, exposing the crux of a problem, or potentially inspiring a novel approach. The point here is that the mathematical questions raised in the list above — is there a pattern, what reasoning might lead to it, and what ideas might constructively challenge it — go beyond the kinds of questions that a mathematical analysis would raise on its own. Indeed, these questions, and their potential importance as mathematical knowledge for teaching, are the result of having surfaced many different and at times competing purposes a teacher needs to juggle in treating error in the classroom and of having used those potential purposes to continue the mathematical analysis into a range of more focused and purposeful mathematical skills and associated components of knowledge needed for analyzing and treating error.

Not only does the extended analysis of pedagogical issues raise additional grist for mathematical analysis, it can also provide sensibility for gauging that analysis. The analysis of error in teaching is similar to the analysis of error in the discipline, but its purpose is different and that difference matters for shaping the particular character of the work and the specific knowledge and skill needed for it. A second example illustrates this further.

The group also analyzed the mathematical teaching task of rescaling mathematics problems — of generating easier and harder variations of problems to use with students. This idea about the mathematical work of teaching arose in a project meeting from our observation and discussion of practice. It made sense to everyone as something teachers need to do as a matter of course in their work and as something that demanded mathematical knowledge and skill. To explore the idea further the group gave itself an assignment of generating rescaled versions of what it called the Cookie Jar Problem. This initial assignment was developed into an activity used in a number of different contexts, with different groups of people, such as in workshops for mathematicians and in teacher education courses.

Original cookie Jar Problem

There was a jar of cookies on the table. Amanda was hungry because she hadn't had breakfast, so she ate half the cookies. Then Eva came along and noticed the cookies. She thought they looked good, so she ate a third of what was left in the jar. Heather came by and decided to take a fourth of the remaining cookies with her to her next class. Then Chantel came dashing
up and took a cookie to munch on. When Kristi looked at the cookie jar, she saw that there were two cookies left. "How many cookies were there in the jar to begin with?" she asked.

This problem lends itself to many different approaches, from trial and error, to the use of algebra, to geometric representations. Also, thinking can proceed from the two cookies back through successive stages to the initial jar on the table or progressing from the full set of cookies in the jar at the outset to the end point of two remaining cookies. The potential confusion between what is taken and what is left at successive stages requires some care in notation, language and thinking. This problem helps to make salient the essential but often implicit reference to a whole when using fractions.

Before examining ways in which a pedagogical elaboration of the task of rescaling problems (both of the work entailed in the task and of the purposes it serves) provides sensibilities that guide and gauge the analysis of mathematical knowledge for teaching, let me first briefly describe what an analysis looks like in the absence of such a pedagogical elaboration. A disciplinary perspective on generating similar problems is guided by notions about breaking hard problems into easier ones as a means of eventually solving the hard problems and about generalizing a problem into one that helps to reveal basic mathematical structure and have broader reach. Consider the following three problems, each similar to the original in some way.

1. The local train from Philadelphia to New York stopped first at Princeton, half way there. The next stop, at New Brunswick, was 2/3 the remaining distance. Metuchen was the stop 2/5 the distance from New Brunswick to NY. Finally, 3/7 of the distance from Metuchen to NY, the train arrived at Newark, only 20 miles from NY. How far is the train ride from Philadelphia to NY?

2. Donna made brownies with a pound of chocolate and one ounce of nuts. Herman asked her to quadruple the amount of nuts. Donna obliged, but also added 8 oz. of chocolate. Herman, when her back was turned, tripled the amount of nuts. What proportion of the resulting brownies was made of nuts?

3. One can change the (whole) numbers involved the original Cookie Jar Problem. In this case a random choice of the numbers will not lead to an integer solution; you end up with a rational number of cookies! What conditions must be imposed on the integer data of the problem so that the solution is again an integer? (adapted from 981110PrjAss)
In writing, from a disciplinary perspective, about generating “similar problems,” one researcher wrote:

Some problems involve fairly explicit numerical parameters, and variation of these can vary the technical complexity of the versions of the task. Moreover, one can escalate the theoretical level, and even view these parameters as abstract variables, and try to solve the generic versions of the problem in symbolic form. (981110PrjAss)

The first and third rescaled problems above are examples of this. The first uses slightly larger numbers and slightly messier fractions than the original problem. The third can be thought of as one way of escalating the theoretical level of the problem, where a class of problems with integral solutions is sought.

A second notion of problems being “mathematically similar” from a disciplinary perspective has to do with framing different problems so that they can all be seen as being variations of the same problem. The second rescaled problem above is in a sense structurally “isomorphic” to the original problem. Instead of having the related amounts of “what is taken” and “what remains” that together comprise the whole, it has chocolate and nuts as the two related components that make up the whole. And, instead of progressively subtracting from the whole, it progressively adds to it. One could continue to generate a variety of problems, all of which can be framed in a way that views them as being a common type of problem.

These ideas are mathematically interesting and contribute to notions of mathematical knowledge for teaching. For instance, the idea of identifying a set of parameters of a problem and exploring what happens to the problem as each parameter is varied provides a teacher with a tool for systematically generating variations on problems with some control over the process (instead of haphazardly) and with some view toward mathematical generality. In no way do I mean to discredit important contributions to the study of mathematical knowledge for teaching that such notions contribute.

However, the three problems given above might also be viewed, from a pedagogical perspective, as slightly perverse. Increasing technical complexity, as in the first rescaled problem above, has a place in instruction, but it can also lead to tedium. The second problem, although arguably “isomorphic” to the original, leads to very different kinds of thinking and work than the original problem, in ways that are likely to make students’ work on it feel rather remote from work on the original. The third, while providing interesting mathematical perspective on the original problem, would again lead to significantly different mathematical work and a different set of instructional goals than the goals one would likely have in using the original problem. In
other words, each of these rescaled problems might be used in instruction, but their relevance is limited. The first might be used as practice and to establish or test the robustness of students understanding by requiring they manage the greater technical complexity. The last two might be used somewhere in the same unit, but for different instructional goals or as part of a larger set of instructional goals that sequenced these somewhat remotely related tasks in some sensible way.

My point here is that further elaboration of pedagogical purpose and the work of teaching suggests different kinds of rescaled problems and exposes different mathematical knowledge for teaching. For instance, from such conversations, the group articulate three situations in which teachers need to resize mathematics problems.

1. A teacher wants to give her or his class another problem like the one that students have already done, perhaps as practice, or to see whether they understand the problem.

2. A teacher wants to give her or his class a problem that is basically the same, but somehow a bit less challenging.

3. A teacher wants to give her or his class a problem that is basically the same, but somehow a bit more challenging.

These were then used in a workshop, where the following rescaled problems were generated.

<table>
<thead>
<tr>
<th>Easier</th>
<th>Similar</th>
<th>Harder</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. A takes 1/3 of the cookies. B takes 1/4 of those left. 9 remain.</td>
<td>5. Several friends left on a road trip. Albert drove 1/2 of the distance, then Bertha took the wheel, driving 1/3 of the remaining distance. Clarence drove next for 1/4 of the remaining distance. Doris drove two hundred miles, and Ellis drove the last 100 miles. How long was the trip?</td>
<td>8. There was a jar of cookies on the table. Amanda was hungry because she hadn’t had breakfast, so she ate half the cookies. Then Eva came along and noticed the cookies. She though they looked good, so she ate a third of what was left in the jar. Heather came by and decided to take a fourth of the remaining cookies with her to her next class. Then Chantel came dashing up and took a cookie to munch on. When Kristi looked at the cookie jar, she saw a whole number of cookies. How many cookies could Kristi</td>
</tr>
<tr>
<td>3. A ate 1/4, B took 1/3 of remainder, C took 1/2 of what was left. D found 3 cookies. What was in jar originally?</td>
<td>6. Josie had some money. She spent 1/2 on pants and 1/3 of the remainder on lunch. If she spends 5/6 on a gift, she has $3 left over for parking. How much money did she start with?</td>
<td></td>
</tr>
</tbody>
</table>

(020112TffHnd)
A challenge created in the original problem arises from the problem having enough steps that one needs a system for handling the multiple steps, instead of just trying to hold it all in one’s head. This, in combination with the first step of Amanda eating half, where the half eaten and the half remaining are equal, lulls one into not clearly differentiating the quantities of what is eaten and what remains, both in this step of solving the problem and in subsequent steps. The three easier problems given above (#1, #2, and #3) play with the possibility that decreasing the number of steps or reordering the different fractional parts being removed will make the problem more tractable. Preserving other surface features of the problem makes it more likely that students will see the relationship between the simpler problem and the original and will thus be able to use work on the simpler problem to understand the original problem more fully. Notice that in this case the act of making a simpler problem is not quite identical to the disciplinary inclination to consider a simpler problem as a means of solving the original problem. Someone trying to solve the original problem is not in the same position as the teacher is and would not be in a position to know to try these particular simplifications. A teacher who has carefully thought through the variety of ways in which this problem might be solved and the challenges associated with each is specifically looking for a version of the problem that bears on key issues that arise in solving the problem.

Likewise, consider the two harder versions (#7 and #8). These might be used to solidify or test the robustness of students’ understanding once they have developed and discussed solutions to the original problem. They do so by removing some of the props present in the original problem that can support working through to a solution. The first problem (#7) might be seen as simply adding technical complexity as did the first problem given in the discussion of problems generated from a disciplinary perspective on similarity (about traveling from Philadelphia to New York). Although this problem (#7) replaces the integral amount found at the end with a mixed fractional amount, what is happening here has less to do with parameterization or increased technical complexity and more to do with the removal of easy-to-use intellectual supports for solving the problem. Using measures of flour, which are more continuous in nature than the discrete notion of cookies, together with the slightly cumbersome one and one fourth makes it harder to work backwards mentally. Instead, the intuition behind the problem seems to be that students may be encouraged to use the mathematical tools they have developed in their
work on the original problem to handle this ever-so-slightly-less-well supported situation. The last problem (#8) also removes the easy-to-use concrete image of one remaining cookie at the end, but by generalizing to any whole number of cookies. Here students are not only being asked to solve the original problem, but to directly use what they have been learning to contend with the additional problem of finding a set of numbers that creates situations that are isomorphic to the original problem.

Some differences exist between the mathematical knowledge for teaching implicated by the work of generating these problems and the mathematical knowledge for teaching implicated by the work of generating the three problems resulting from a disciplinary focus. Generating these problems requires more attention to the variety of ways in which the original problem and the rescaled problems might be solved. Second, generating these problems is less about attending to mathematical structure when that structure is not closely related to the ways in which the problems might be solved. And, these are only a few examples of the kind of problems that might be generated. As the research group explored a variety of instructional purposes for rescaling problems, they created a purposed, contingent, flexible understanding of the mathematical work of rescaling problems which contributed to identifying a distinctive collection of specific mathematical issues important for teaching.

This discussion is about the mathematical knowledge that arises from the elaboration of the task of rescaling problems. The explicit identification of pedagogical purpose and the ongoing refinement of the pedagogical insides and surrounds of the task allow a more refined notion of the mathematical nature of the task in ways that help direct and gauge the identification and specification of mathematical knowledge for rescaling problems. These two examples, then, one of analyzing error and the other of rescaling problems, suggest the nature and role of identifying potential pedagogical purpose in the analysis of practice.

4.2.3 Layering Tasks in the Specification of Teaching

In addition to the theme of identifying pedagogical purpose for tasks of teaching is a related theme of layering tasks in the specification of teaching. Tasks entailed in teaching can be identified in broad and general terms, such as planning instruction, or in narrow and focused terms, such as, for a given mathematics problem, generate a similar practice problem. In this section I argue that, at least for the purpose of identifying mathematical knowledge for teaching, it is prudent to identify tasks at a variety of levels and also to situate tasks at lower levels within the context of more general tasks, creating layers of nested tasks. I first argue that specific, focused tasks are particularly helpful for identifying mathematical knowledge for teaching, and then explain the value of situating these focused tasks in a larger web of layered tasks.
In trying to identify and specify mathematical knowledge for teaching, the more focused the task is, the clearer it often becomes what mathematics is entailed in the task. For instance, teachers have to evaluate student productions. To say this, though, says little about what mathematics might be useful in doing so. To go on to focus specifically on evaluating student explanations begins to suggest some mathematical knowledge that might be useful. For instance, understanding what counts as an explanation in mathematics, being familiar with different types of mathematical explanation, and having language to describe components and characteristics of explanations would all be potentially useful in evaluating student explanations. These all seem important, but they also seem general in ways that make it hard to know what to include and exclude. And, depending on how they were treated, it is not clear that teachers would know how to use the knowledge in practice. The problem exemplified by Ms. Daniels, as I sketched in Chapter 1, remains. It is as if specifying the task of evaluating student explanations opens up mathematical territory that would seem to bear on teaching, but in general and unspecified ways. To go further in specifying the task, specifically focusing it on determining whether a student’s non-standard approach or method would work in general provides even greater clarity about the mathematical knowledge and skill that would be needed and about when, where, and how this knowledge would be used.

Another advantage of specific and focused tasks is that they provide a basis for analytic distinctions that support the refinement of ideas about what constitutes mathematical knowledge for teaching and ways in which it is structured. For instance, in arguing four distinct categories of mathematical knowledge for teaching, Ball, Thames, and Phelps (2008) describe the way in which small changes in the framing of questions shift the nature of the knowledge implicated. With the subtraction problem, 307–168, being able to do the problem oneself and recognize whether an answer is right or wrong is common content knowledge (CCK), whereas analyzing student errors on the problem, sizing up non-standard approaches, and justifying the standard subtraction algorithm are specialized content knowledge (SCK). Shifting the focus of the work of teaching again shifts the mathematics entailed in the work. Knowing which errors students are likely to make, which non-standard approaches are common, and what sense students are likely to make of different explanations are each examples of pedagogical content knowledge (PCK), or more specifically, knowledge of students and content (KSC). Shifting the focus again provides examples of a different kind of pedagogical content knowledge — knowledge of teaching and content (KTC). Consider, for example, remediating a particular kind of error, deciding whether and how to leverage a non-standard approach for teaching and learning, and developing
explanations of the standard subtraction algorithm in ways that support student understanding and are sensitive to the difficulties students tend to have.

For each of the categories (CCK, SCK, KCS, and KCT), the examples above range across three task domains. One is in relation to the errors students produce. A second is about approaches used to solve a problem. The third is associated with understanding of the standard algorithm. The nature of the task associated with each domain shifts across the four categories of knowledge. These are represented in the figure below.

<table>
<thead>
<tr>
<th></th>
<th>Error</th>
<th>Approach</th>
<th>Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>CCK</td>
<td>Recognizing errors</td>
<td>Doing the problem oneself</td>
<td>Tacit knowledge for doing the subtraction algorithm (a la Polyani’s example of riding a bicycle)</td>
</tr>
<tr>
<td>SCK</td>
<td>Analyzing student errors</td>
<td>Sizing up students non-standard approaches</td>
<td>Justifying the standard algorithm</td>
</tr>
<tr>
<td>KSC</td>
<td>Knowing which errors</td>
<td>Recognizing students use of non-standard</td>
<td>Anticipating what sense students are likely to make of different explanations</td>
</tr>
<tr>
<td></td>
<td>students are likely to</td>
<td>approaches</td>
<td></td>
</tr>
<tr>
<td>KTC</td>
<td>Remediating an error</td>
<td>Avoiding and using non-standard approaches in instruction</td>
<td>Designing instruction to develop student understanding of the subtraction algorithm</td>
</tr>
</tbody>
</table>

Figure 4.9. Shifts in tasks of teaching that support the identification of useful categories of mathematical knowledge for teaching.

Thus, the four categories, in effect, reflect shifts in the kind of task of teaching being highlighted. What is significant about these categories is that, although the shifts in the framing of the work of teaching may seem small and even artificial, the analytic distinctions among the four categories can have important implications for the mathematical knowledge associated with each and for programs and institutions designed to address the improvement of teaching. Ball, Thames, and Phelps offer three contributions the four knowledge categories might make.

First, in studying the relationships between teachers’ content knowledge and their students’ achievement, it would be useful to ascertain whether there are aspects of teachers content knowledge that predict student achievement more than others. If, for instance, teachers’ specialized content knowledge is the greatest predictor of students’ achievement, this might direct our efforts in ways different than if advanced content knowledge has the largest effect. However, such studies are sorely missing. Second, it could be useful to study whether and how different
approaches to teacher development have different effects on particular aspects of teachers’ pedagogical content knowledge. Third, and closely related, a clearer sense of the categories of content knowledge for teaching might inform the design of support materials for teachers as well as teacher education and professional development. Indeed, it might clarify a curriculum for the content preparation of teachers that is professionally based—both distinctive, substantial and fundamentally tied to professional practice and to the knowledge and skill demanded by the work. (p. 45)

Another important issue not directly mentioned here is that the categories also inform conversations about who is responsible for what in the mathematical preparation of teachers. For instance, CCK and SCK are legitimate types of knowledge to address in a mathematics content course taught in a mathematics department by a mathematician. Specific knowledge about students and about teaching is knowledge that teacher educators should know.

My point here, however, is that identifying categories such as these requires more specific and focused tasks of teaching. The more specific tasks add precision to terms and meaning, and they decompose mathematical activities occurring in the work of teaching in ways that support the identification of common elements arising across contexts of the work. As tasks of teaching, they maintain a clear sense of relevance to teaching, but they also support a recomposing of mathematical knowledge for teaching into frameworks that relate better to other needs and to existing institutions.

Notice also that the shifts in tasks identified in the figure above represent a kind of layering of tasks from simpler to more complex. In each column, moving down the column, the tasks become broader in scope, comprising more multifaceted actions, and incorporating more complex notions of purpose. This layering of tasks provides a way of addressing the important issue of purpose discussed in the previous section. Take, for instance, the second column regarding approach. The CCK task of doing the problem oneself sits, for teachers, inside the kind of purposes suggested by tasks further down the column. Thus, the spirit of doing the problem oneself, as a teacher, is also about doing the problem in many different ways, in anticipation of the different ways it might be done and of potential relationships among the different approaches. Likewise, sizing up different approaches takes on particular meaning when considered within the context of spotting, evaluating, and deciding what to do with students’ non-standard approaches. The development of layers like this, around a common issue for teaching, such as dealing with student approaches, helps to provide meaning for the mathematical knowledge and skill being identified. Thus, it serves both to guide the identification of mathematical knowledge for teaching and to provide tools for capturing and communicating what that knowledge is.
The generation of layers of related tasks also offers a basis for determining what mathematical knowledge for teaching is central and what knowledge is idiosyncratic and peripheral. As tasks are considered in light of other tasks and in light of layers of related tasks, it becomes clearer whether a task is central to core domains of the work of teaching, what constitutes the task, and what the task buys you vis-à-vis other related work. Priorities for focused tasks get associated with priorities for broader tasks, and broader tasks get unpacked into focused tasks key to carrying them out. The result is a system of layered tasks that provide flexible support for considering potential formulations and decompositions of the work of teaching as well as for exploring mathematical knowledge for teaching.

Returning to a project memo on tasks of teaching discussed briefly in Chapter 3, we can see what this work might be like, looking from the top level down. The memo proposes several core task domains of teaching and elaborates the work inside each domain.

5. Setting and using instructional tasks
   a. Selecting tasks
   b. Remodeling tasks
   c. Setting tasks
   d. Enacting tasks
   e. “Making something of the task” toward an instructional goal
   f. Making a transition from one task to the next

6. Establishing ways of working, talking, and behaving in the classroom in ways that support learning
   a. Creating and maintaining basic standards for behavior in the classroom
   b. Establishing and maintaining routines
   c. Establishing and managing how knowledge is to be viewed and treated
   d. Establishing and managing the exchange of knowledge

7. Figuring out whether students are “getting it”
   a. Assessing what students can do prior to instruction
   b. Anticipating what students may say or do in response to an assignment, question, task
   c. Listening to students’ responses during a lesson, interpreting what they are saying
   d. Correcting student papers
   e. Listening to what kids say or reading their work and deciphering it, comparing it with goals;
   f. Reading, interpreting, and evaluating students’ answers to a quiz
   g. Sizing up what the group (not the individuals within the class) knows and can do
   h. Mapping kids’ progress against objectives, goals, curricula

8. Other
   (Extracted from 981129TskMem, with detailed text for each item omitted)

The specification of domains and tasks given in the memo is incomplete, but represents an attempt to grapple with the overall framing of the work entailed teaching. The domains and their formulation are conceived in relation to designing and managing instruction in broadly applicable
terms. For instance, in reference to the domain of “setting and using instructional tasks,” the memo says:

This task domain concerns the selection, posing, enacting, and using instructional materials and tasks. Several major subtasks comprise this domain; each can be done in a wide variety of ways depending on views of instruction, knowledge, and learning, as well as context, content, and the approach and skill of the teacher. Some of these subtasks may be done concurrently; some may be done briefly or left to be done entirely by students. (981129TskMem)

As this says, the tasks are formulated to allow for a wide variety of ways of enacting them. Further, these tasks can be elaborated in terms of subtasks that are also formulated to allow for different approaches to teaching.

The memo also specifies each of the tasks identified. For instance, the first task in the domain of “setting and using instructional tasks” is “selecting tasks.” The memo characterizes the variety of ways it might be carried out and it suggests subtasks that constitute any professionally responsible enactment.

Selecting tasks: The genesis of an instructional task varies. A task may be selected on the basis of some goal, because it comes next in the textbook, because the teacher has the task and thinks it looks worth doing, or the teacher may make up the task herself. Selecting tasks includes subtasks such as sizing up what a task is about; situating it in a larger view of the curriculum; assessing what is entailed in doing it; evaluating the task’s appropriateness for the curriculum or for particular students. (981129TskMem)

A second subtask, “remodeling tasks,” provides a similar description and also describes the overlapping nature of its subtasks with other tasks.

Remodeling tasks: Teachers also change tasks. They may scale a task up or down, seeking to increase its challenge or to simplify it. They may seek to tailor it to a particular goal or group of learners. They may devise a similar task for homework, or they may create an extension for further development. They may multiply a task to make others like it. Remodeling a task includes subtasks that overlap those involved in selecting tasks, but also entails creation of tasks – writing, selecting contexts or texts, preparing a task for student use. (981129TskMem)
Seeing focused tasks of teaching in the context of core tasks offers a basis for evaluating the significance of specific tasks. It clarifies the critical (or non-critical) nature of a focused task and it provides a sense of the reach of its utility across the full spectrum of the work.

These points about the specification of more and more detailed subtasks is related to the notion of elaboration, as discussed by Cohen and Ball (2007) and by Ball et al. (2009). In writing about the problem of scaling up a reform, Cohen and Ball argue that elaboration of the innovation is essential: “to illuminate an innovation’s requirements for use, to alert designers and implementers to work to be done, and to reveal potential problems” (p. 8). The detail of focused tasks constitutes a similar form of elaboration that seeks to articulate the demands of the work in ways that enable enactment rather than overly constrain or limit it. In writing about the efforts to develop a professional curriculum for teacher education, Ball et al. argue that the kind of elaboration they have in mind is not meant “to preempt the interactive work of teaching, but to provide common structures and tools for the interactive work” (p. 467). The layers of increasingly focused tasks that I am describing here represent a form of elaboration of the work entailed in teaching in ways that provide structures and tools for teaching and for making sense of teaching.

My argument here is that the elaboration of general, complex tasks simultaneously with the contextualizing of specific, focused tasks in a layered structure of related tasks is useful to the work of identifying mathematical knowledge for teaching for three reasons. First, in the context of not yet having a well-developed decomposition of the work of teaching, collections of related layered tasks provide a means of having detail without losing site of the whole and of making considered judgments about priorities. They also capture some of the sense of purpose that serves to guide the mathematical analysis of practice and the identification of mathematical knowledge for teaching as described in the previous section. And last, the nested character helps to convey the meaning and the conduct of tasks.

In conclusion, then, these three themes, attention to the work entailed in teaching, explicitly identifying pedagogical purpose, and developing layers of related tasks in specifying teaching contribute in important ways to parsing teaching in ways that support an investigation of mathematical knowledge for teaching. Next I illustrate these themes by using them to reanalyze the five-minute segment in which Mahluli explains his solution of one half.

4.2.4 Example: A Pedagogical Analysis of Five Minutes of Instruction

The teaching issues that come up in this segment are boundless. Some might seem quite remote, such issues of developing relationships with parents or of considering the role of this lesson for covering the curriculum or preparing for state assessments. Although perhaps
apparently remote, these issues are latent even in this brief segment and might have relevance depending on circumstances not apparent to an outside observer. Another set of issues arises deep inside the details of what is happening. For instance, every pause could be mined for its potential relevance for the work of teaching. In this analysis I do not try to address the full range of issues that might be considered. Instead, I highlight a few issues suggested rather immediately by what is said and done and I use these to illustrate the three themes presented earlier in this section. To develop this analysis, I begin by briefly sketching some of the tasks of teaching evident in the first minute of this segment and then provide an overview of tasks of teaching that emerge across the full five minutes and elaborate a few tasks in more detail.

The segment starts with the teacher waiting for more students to volunteer to explain their answer to the question of what fraction of the big rectangle is shaded blue (the triangle).

**Mathematical task, grade 4-5**

- What fraction of the big rectangle is shaded blue?
- What fraction of the big rectangle is shaded green?
- What fraction of the big rectangle is shaded altogether?

Punctuated with pauses, she says:

I'd like to see some hands from people I haven't seen speaking in whole group yet today. Who thinks that they could explain their thinking for question one? Still waiting to see a few more hands. Lots of people have work done in their notebooks. Okay Mahluli. What do you think about question one?

A half of a minute into this segment and many tasks of teaching are already quite apparent. Where did this task come from? How is it being used? What answers and explanations might students have? How are the three different problems related and how will the presentations develop across the three problems? In short, there is a host of work entailed in selecting, preparing, and using a mathematics problem like this in instruction. Second, these few seconds suggest the work of enlisting student participation and attention. Who should be called on? How can students be motivated to participate, to take an interest, to pay attention, and to use this time to learn mathematics? What specifically should students be asked to do? Just give an answer?
Explain? To what end? What would draw in students who are less inclined or able to participate? How is calling on a particular student likely to shape the work? Shape the social dynamics of the classroom? Contribute to the student’s sense of themselves as someone who does and learns mathematics? And, each of these questions implies a task of teaching, or several.

In response, Mahluli says, “Question one, I say it’s one half.” This brings several different tasks of teaching to the fore. Teachers must listen to, interpret, and respond to students’ contributions. They must also decide how to handle the errors students make, both entertaining that the student might be right and maintaining the feeling of safety. Where did the \( \frac{1}{2} \) come from? What options are there for responding? What are others perceiving Mahluli’s response? The teacher goes on to ask Mahluli to explain his answer, and he says, “Because they both equal — they both equal — and one, one, half of it is shaded in and other half is not.” Again, this suggests the challenge of interpreting student contributions. Student talk is often halting and incomplete. Students are struggling to grapple with ideas they are only learning.

The teacher then asks Mahluli to go up to the board to explain his answer. While he makes his way to the board, she continues to talk to the rest of the class.

Did everyone hear what Mahluli said? You should be thinking already about his reason. Who can repeat what Mahluli said? Okay. Well if you're listening carefully, you should always be able to tell what someone just said. Doran, what did he say?

She then engages Doran in explaining what he understands Mahluli to have said.

These interactions suggest a number of other tasks of teaching. It raises questions about how to support a student in explaining their thinking both to him or herself and to the teacher. What does Mahluli understand about giving an explanation? What would help make his explanation clearer? It also raises questions about how this time can be made productive for other students. A moment ago, I mentioned that the work of teaching includes motivating to participate, to take an interest, to pay attention, and to use this time to learn mathematics. Here we see that this permeates the work of teaching, moment to moment, and that a version of it is generating ideas about what to do within the pauses that occur in doing something already decided on.

Continuing across the segment and highlighting some of the aspects of the work into which this video provides insight might lead to a list such as the following.

- Selecting/designing tasks
- Enlisting student participation and attention
• Identifying and working toward the mathematical goal of the lesson
• Listening to and interpreting students’ responses
• Teaching students what counts as “mathematics” and mathematical practice
• Maintaining a safe environment
• Making error a fruitful site for mathematical work
• Attending to ambiguity in the expression “big rectangle”
• Deciding what to clarify, what to make more precise, what to leave in student’s own language

This list is not exhaustive, but more like a sampling. For instance, it does not include explicit attention to the pedagogical work of addressing issues of equity or to making students feel comfortable in the space. The discussion so far might seem, indeed, somewhat haphazard and overwhelming. It raises questions about what to pursue and what to ignore in a pedagogical analysis of practice. Unfortunately, we do not yet have a fully developed decomposition of teaching that specifies what to attend to in a pedagogical analysis.

Earlier, I discussed a project memo on tasks of teaching that proposed three core domains:

• Setting and using instructional tasks
• Establishing ways of working, talking, and behaving in the classroom in ways that support learning
• Figuring out whether students are “getting it”

Each of these is core to designing and managing instruction in the sense that each addresses a prominent component of instructional dynamics. Returning to the notion that instruction is the set of interactions among teacher and students around content, the first might be thought of as focusing on the formulation and enactment of content in instruction; the second might be thought of as focusing on the interactions among teacher and students, with content still in the picture, but in the background; and the third might be thought of as focusing on the key issue of what students are making of instruction and what they are learning. The teacher is responsible for designing and managing these interactions. As such, it makes sense that a direct focus on his or her own role in the dynamics of instruction is less prominent — as though the material that a teacher needs to focus deliberatively on shaping are the other elements of instruction, beyond self. This is not to say that attention to self is not important in teaching; it is. However, its treatment might be more punctuated and occur in different ways than the distinctively challenging work of managing contributions not fully or readily under a teacher’s immediate control. These three, then, cover important bases of the work entailed in teaching.
Using this structure to examine the work of teaching in this segment, might lead to organizing the list as follows.

1. Setting and using instructional tasks
2. Selecting/designing tasks
3. Enlisting student participation and attention
4. Making error a fruitful site for mathematical work
5. Establishing ways of working, talking, and behaving in the classroom in ways that support learning
6. Teaching students what counts as “mathematics” and mathematical practice
7. Maintaining a safe environment
8. Deciding what to clarify, what to make more precise, what to leave in student’s own language
9. Figuring out whether students are “getting it”
10. Listening to and interpreting students’ responses
11. Identifying and working toward the mathematical goal of the lesson
12. Attending to ambiguity in the expression “big rectangle”

Doing so, provides some organization, but also casts each of the tasks in a larger picture, giving a fuller sense of the task and helping to link it back to foundational notions of the work of teaching as designing and managing the core interactions of instruction. Some of this linking occurs in the choices made in naming specific tasks (largely left invisible in this discussion) and part of it occurs in this process of situating a task as an activity entailed in foundational demands of teaching.

In this large, complex space, two other issues help to guide a pedagogical analysis of this segment. One is that it is situated in the context of studying mathematical knowledge for teaching. In other words, the choices made about what to highlight and what to ignore reflect, in part, sensibilities about what tasks are likely to have significant mathematical demands associated with them. The second issue that propels the analysis is the elaboration of particular tasks. To illustrate this, I continue the analysis of the task of enlisting student participation and attention, further developing it and mining it for ideas about mathematical knowledge for teaching.

*Enlisting student participation and attention.* There are several different ways in which a teacher might motivate students to work in school. Motivation has been extensively studied and theories exist regarding different types of motivation and their role in learning. In addition, Dewey (1964/1904) drew attention to the potential disparity between the trappings of paying attention and actual minds-on activity. In this segment, one can see a particular form of enlisting student participation and attention in the many tasks and questions the teacher gives to students to do and to answer. In response to Mahluli’s initial explanation, the teacher says:
Okay. Can you come up to the board and point and show us what you're looking at? Just- there's a diagram right there. Can you come up and show? Did everyone hear what Mahluli said? You should be thinking already about his reason. Who can repeat what Mahluli said? Okay. Well if you're listening carefully, you should always be able to tell what someone just said. Doran, what did he say?

In these few seconds, the teacher creates intellectually compelling work for students to do and questions to answer. By instructing Mahluli to “point and show” on the diagram, she is asking him to take the reasoning he generated for himself and situate it in the context of the diagram, mapping between the diagram and his explanation. As he does this, she lays before the other students a number of tasks that they should be doing. In asking whether everyone heard Mahluli, she implies that they should have been listening to him. In saying that they should be thinking already about his reason, she implies that they should be doing thinking about his answer and his explanation in ways that anticipate what he will be saying when he goes to board to use the diagram to explain his thinking. In asking who can repeat it, she implies that they should be listening well enough (with sufficient detail and attention to meaning) that they can repeat what he said. Each of these gives students something to be doing that requires thought about the mathematics at hand. This ongoing creation of mathematical work that fills the moment-to-moment of instruction suggests that related to the task of enlisting participation and attention is a task of creating such work.

The point here is not what this teacher does, but that what we see this teacher doing and a consideration of why she might be doing it together suggest that part of the work of engaging students, because it is about engaging them in learning mathematics, is about being able to articulate the insides of mathematical work in the form of questions and tasks that can be readily called on in the flow of instruction. This suggests that articulating the insides of mathematical work is a task of teaching, and one that is well aligned with foundational issues of instruction. In addition, the task of articulating the insides of mathematical work has direct implications for mathematical knowledge for teaching. The fact that this discussion arises in the context of examining the task of enlisting student participation and attention helps to ground a further examination of what it is about the nature of an articulation of mathematical work that is needed.

Next, I conclude this specification of a pedagogical analysis by summarizing the three themes described in this section and some illustrative examples of what they together yield.
4.2.5 Summarizing the Composition of a Pedagogical Perspective

In contrast to the themes identified for a mathematical analysis, the three themes described in this section are more like orientations or principles for developing a pedagogical analysis. The figure below summarizes the points made in this section.

**Summary of Themes Identified for a Pedagogical Analysis**

<table>
<thead>
<tr>
<th>The goal is to identify basic practices of the work of teaching, not modal practice, expert teaching, or specific kinds of teaching. The work of teaching is that which is there to be done to achieve the goal of student learning as it combines with the many competing goals that society and local communities have for schools. A specification of the work of teaching then becomes the development of a common technical language that decomposes teaching into component tasks constituting the deliberative design and management of instructional interactions. The tasks entailed in teaching are the necessary accompaniment of designing and managing instruction in school — tasks that are to teaching as uncertainty is to teaching and as risk is to skydiving, in the sense that they are basic activities associated with basic realities of the endeavor. The fundamental goal of designing and managing instructional dynamics is the whetstone for honing the technical language and the formulation of component tasks of teaching. Thus, tasks need to be general in the sense that they need to be about inherent aspects of designing and managing instructional dynamics, and they need to be specific to the extent that the particulars of content, students, and settings matter for this work. Important data sources for research that seeks to develop decompositions of the work entailed in teaching are instances of instruction that provide easy access: (i) to instructional exchanges, (ii) to teacher reasoning, and (iii) to additional layers of the context. Both “good teaching” and “problematic teaching” can be used productively to analyze the work entailed in teaching. Together, the presence and absence of effective practice is the negative space of the object of study, which is the demands that effective and ineffective practices respond to, either well or poorly. The process of specifying a task entailed in teaching can be supported by:</th>
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<tbody>
<tr>
<td>• Identifying an action or possible action. • Exploring potential purposes, alternative actions, the nature of conditions relevant to the action, priorities associated with it, competing goals, and analogous actions in the context of varied teaching situations, such as different grade levels, teaching styles, or topics or subject matter. • Using these explorations to formulate a version of the task that is further from being an idiosyncratic teaching act and closer to being about central work entailed in teaching. Deliberations about observed actions, potential tasks, and the relationship between them, can lead to identifying potential tasks that are framed with sensitivity to multiple, instrumental goals of instruction. For example, observing a teacher who has children write the “problem of the day” in their notebooks at the start of class might lead to identifying potential purposes:</td>
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<td>• Helps children take responsibility for getting to work. • Familiarizes the student with the problem. • Provides a title for the work students do in their notebooks. • Reminds students as they work what it is that they are working on. • Creates a usable record of student work for the teacher, the parents, and the student. Turning back to the notion of designing and managing instructional interactions might then lead to developing candidates for a set of basic tasks of teaching.</td>
</tr>
<tr>
<td>• Teaching students how to record mathematical work.</td>
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</tbody>
</table>

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• Creating usable records of student thinking to support meta-cognition.
• Setting-up an instructional task.

For the purpose of identifying mathematical knowledge for teaching, tasks need to be *layered at multiple levels*. They need to be: (i) identified at a variety of levels; (ii) elaborated into collections of sub-tasks; and (iii) situated within the context of more general tasks.

Figure 4.10. Summary of the themes identified for a pedagogical analysis.

Using these principles to identify candidate tasks of teaching leads to tasks that are grounded in core issues of teaching, articulated in relation to other tasks, and useful in identifying the mathematical demands of teaching. The next figure provides examples of layered tasks of teaching that convey pedagogical purpose and coordinate perspectives.

<table>
<thead>
<tr>
<th>Task of Teaching</th>
<th>Rationale</th>
<th>Dimensions of the Task</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sizing-up and selecting mathematics problems and tasks</td>
<td>Mathematics problems are the central means for capturing the subject matter and for designing instructional activities.</td>
<td>• Size-up what the problem is about.</td>
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<tr>
<td></td>
<td></td>
<td>• Situate the problem in a larger view of the curriculum.</td>
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<td></td>
<td></td>
<td>• Assess what is involved in doing the problem.</td>
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<td></td>
<td></td>
<td>• Evaluate the appropriateness of the problem for the curriculum or for the particular students.</td>
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<tr>
<td>Rescaling mathematics problems</td>
<td>The selection and use of problems is central to teaching and being able to modify problems is central to fine tuning them for instruction.</td>
<td>• Develop a problem like an original problem, perhaps as practice or to check understanding.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Develop a problem like an original problem but somehow a bit less (or more) challenging.</td>
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<tr>
<td></td>
<td></td>
<td>• Examine the relationship between problems in regard to how the problems are solved.</td>
</tr>
<tr>
<td>Developing justifications for standard algorithms</td>
<td>Mathematical justification can provide reason and understanding for algorithms in ways that support meaningful use.</td>
<td>• Present clear and concise mathematical arguments.</td>
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<td></td>
<td></td>
<td>• Respond to “why” questions from students.</td>
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<td></td>
<td></td>
<td>• Anticipate what sense students are likely to make of different explanations.</td>
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<tr>
<td></td>
<td></td>
<td>• Establish students’ sense of mathematical conviction and ability to use and reconstruct algorithms.</td>
</tr>
<tr>
<td>Sizing-up student productions</td>
<td>The process of eliciting, sizing-up, and responding to verbal and written student productions is a key feedback loop informing teaching and learning.</td>
<td>• Anticipate alternative approaches and ways of thinking about problems and ideas.</td>
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<tr>
<td></td>
<td></td>
<td>• Decide whether a student’s non-standard approach would work in general.</td>
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<td></td>
<td></td>
<td>• Decide whether to take up or to pass on.</td>
</tr>
<tr>
<td>Analyzing student errors</td>
<td>Student errors often represent the crux of what there is to be learned and reveal much about what to do to support learning.</td>
<td>• Identify pattern in errors.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Figure out what reasoning might have led to it.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>• Develop ideas about what might constructively challenge error and make it a site for fruitful mathematical work.</td>
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</tbody>
</table>

Figure 4.11. Examples of layered tasks of teaching that convey pedagogical purpose and coordinate perspectives.

At the outset of this section, I identified three issues that make the specification of a pedagogical perspective both interesting and challenging: (i) what the purpose of identifying...
mathematical knowledge for teaching implies for a pedagogical analysis; (ii) ways in which the identification of work of teaching may be at odds with identifying mathematical knowledge; and (iii) the need to disentangle generality and specificity in parsing teaching. Before closing this section, I discuss these briefly.

Other efforts to decompose teaching have been mostly concerned with teacher learning. As such, the decompositions for which they argue and the criteria they propose are tailored for that agenda. The ultimate agenda here is also concerned with teacher learning and the improvement of teaching, but it is specifically focused in an agenda of identifying mathematical knowledge for teaching. It would be good if the tasks generated from the kind of pedagogical analysis I describe here were likewise tailored for teachers learning, but this is a step of the process that is beyond the immediate agendas of decomposing teaching for the purpose of identifying its mathematical demands, and secondarily, of decomposing teaching in such a way that it can accommodate the somewhat foreign mathematical analysis of practice. For these agendas, explicit discussion of potential pedagogical purpose becomes more important because it provides important orientation for further mathematical analysis. Also, the layering of tasks, in addition to conveying purpose, expands the range of tasks available for mathematical analysis.

Similarly, the specification of layered tasks of teaching, together with clarification of potential purposes associated with those tasks, provides a picture of the instruction being observed that avoids the problems that arose in early research in the University of Michigan group where mathematicians’ commentaries on practice, written with little guidance, tended to sprawl off into issues remote from teaching and even at odds with basic pedagogical sensibilities and with practical experience. A central lesson from this early work is that pedagogical sensibility and practical experience need to have an explicit role in the work, on par with the mathematical analyses that initially inspired the research. The themes identified in this section provide a role for a pedagogical analysis that can provide precisely this grounding.

Finally, the discussions above about the work entailed in teaching and the grounding of this work in the dynamics of instruction provide an important addition to the notions of generality and specificity described at the beginning of this section. Decisions about whether tasks need to be made more specific or more general and ways in which this might be done are directly informed by consideration of the relationship of a task to the interactions among teacher and students around content. I now turn to the final issue taken up in this chapter, the coordination of a mathematical and a pedagogical perspective in the analysis of practice.
4.3 Coordinating Perspectives in the Analysis of Practice

As described both at the outset of this study and in the analysis of the three global approaches developed in Chapter 3, a significant challenge for the line of research on mathematical knowledge for teaching that is grounded in an empirical analysis of practice is the coordination of the two critical perspectives being brought to bear in the analysis — a disciplinary perspective and a pedagogical perspective. In Chapter 2, I described some of the underlying tensions between these two perspectives. In Chapter 3, I described specific examples of these tensions and the challenges of reconciling the two perspectives. Having characterized more fully the two perspectives, I now consider ways in which these perspectives might be coordinated in research of this kind. To some extent, this section does not bring significantly new ideas to the table. Instead, it might be seen as reviewing approaches that have already been described, but foregrounding and making more explicit ways in which these approaches coordinate the two central perspectives used in this research program.

Before describing the nature of this coordination, I first want to pause to consider the word coordinate itself. Etymologically, there appear to be two possible histories for the word. The first, interestingly, is as a combining form, co-ordain, literally to ordain together, where the meaning of ordain has to do with establishing the status of something, submitting it to order, or bringing it into agreement. A second possible history is that it was developed as a parallel to subordinate, still with the same root, but focused on the notion of being equally placed (in contrast to being subordinate). Both of these possibilities are reflected in the different definitions of coordinate. Scanning several dictionaries, four distinct meanings are identifiable.

- To place in the same order, rank, or division.
- To place in proper position relative to each other and to the system of which they form parts.
- To work together for the production of a particular result.
- To regulate and combine so as to produce harmonious action.

The third meaning is the one that informed my initial examination of ways in which the two perspectives might be coordinated — made to work together for the purpose of identifying mathematical knowledge for teaching.

The first two meanings raise a different question: What is the relative status of the two perspectives in the conduct of the research? Although this was not the focus of my examination, the question of relative status is to some extent implicit in the question of how these two perspectives can be made to work together. Early in the study, I entertained the idea that the
pedagogical perspective might serve as the final arbitrator in reconciling the two perspectives. This idea is implicit in Section 3.1.5, which concludes the discussion of the first of the three major approaches identified in that chapter. That discussion borders on suggesting that to deal with the challenges of an analysis combining these two very different perspectives, it may be the case that the pedagogical perspective should be allowed to censor and overrule the mathematical perspective when differences seem irreconcilable. In this section, a different view will be put forward, one in which the two perspectives function on more equal footing and where other mechanisms for reconciling differences are put forward. I do not mean to take a clear stand on the issue of their relative status, but instead mean to suggest that this is not a particularly useful question. The idea that one perspective can trump the other may be, indeed, more of a distraction than a solution to the problem of coordination. I will return briefly to this issue later, but this section will primarily focus on productive functioning of the perspectives and not on their relative status.

Returning to the four definitions, the third and fourth meanings might be seen as rather similar. However, the third meaning can be read to include a range of interpersonal or even emotional issues, such as people getting along or the simplicity and ease of the interaction, whereas the fourth meaning seems to focus more narrowly on establishing structures and mechanisms that combine the perspectives and regulate their interaction. It is this latter image that dominates the results of my examination of the coordination of the perspectives.

In using project documents to investigate ways in which the coordination of perspectives might happen, I generated some initial hypotheses and then worked iteratively back and forth between the documents and revised hypotheses. As mentioned before, I read through a representative sample of project documents, considering each sentence as a means of commenting on something, of drawing something to the fore in the research. For each sentence, I asked: What is this a comment about? I wrote short descriptions for each and then reviewed these and created categories. I then looked more closely at documents and parts of documents engaged in the analysis of practice. Next, I recoded comments in documents, asking the question: In what way, if any, does the comment coordinate a mathematical perspective and a pedagogical perspective? Using these descriptions and existing ideas about the nature of such coordination, I created a small collection of approaches to the coordination of perspectives, which I iteratively reapplied to the data and revised.

Below I propose five approaches for coordinating mathematical and pedagogical perspectives in the analysis of practice in the study of mathematical knowledge for teaching. The first two suggest ways of further extending the two perspectives in the service of the other. The
third section argues that places where the two perspectives share common concern can provide a productive focus for analysis. It examines two examples:

- Establishing and using a base of common, or public, knowledge
- Remembering mathematical work

It also argues that identifying and exploring places where the two perspectives seem to be in tension, when knowingly approached as sites of tension and when the tension itself becomes the object of study, can yield productive observations about mathematical knowledge for teaching.

The fourth section considers four specific foci that, in different ways, provide analytic structures that help coordinate the two perspectives.

- Examining student mathematical practices in instruction
- Identifying mathematical practices critical to teaching
- Examining mathematical work in instruction
- Identifying mathematical work and mathematical tasks of teaching

These first four sections identify conceptual or analytic tools for coordinating perspectives. The final section describes three additional research practices that seem important in coordinating mathematical and pedagogical perspectives.

- Collectively and flexibly engaging in the analysis
- Empirically grounding the analysis
- Systematically documenting the analysis

The section ends with a synthesis of these assorted approaches to coordinating the twin perspectives of mathematics and pedagogy.

### 4.3.1 Aligning Mathematical Perspective with the Pedagogical

A second way of managing the challenges of coordinating these two very different perspectives is to formulate the mathematical analysis in a way that emphasizes disciplinary features that parallel related features of instruction. For instance, by focusing on problems, on explanations, on the specific language used to express ideas, and on the structure and character of the work, a mathematical analysis is addressing features that matter for teaching and learning. Of course, this does not resolve all tensions between the two perspectives, but I argue that it reduces their likelihood. Many of the tensions I described at the end of the first section of Chapter 3, where I discussed challenges for the first approach to the analysis of practice, would be avoided in an analysis that focused on the four themes identified above. In that section, I asked: “But how could a mathematician decide what is pertinent without establishing interpretations and opinions
about the unfolding instruction?” Part of the answer is that the four features assure a degree of pertinence so that the mathematical perspective does not have to overstep its sphere of expertise.

As an example of this, one of the mathematical commentaries written early in the life of the project raised an issue that became an extended, and irreconcilable, point of conflict. In the classroom situation being analyzed, the class is discussing whether zero is even or odd. Nathan argues that zero is special. He says:

Um, first I said that um, zero was even but then I guess I revised so that zero, I think, is special because um, I-- um, even numbers, like they they make even numbers; like two, um, two makes four, and four is an even number; and four makes eight; eight is an even number; and um, like that. And, and go on like that and like one plus one and go on adding the same numbers with the same numbers. And so I, I think zero's special.

In a mathematical commentary, one of the researchers, R1, wrote that powers of two is “the concept which he [Nathan] is beginning to generate.” R1 described the process Nathan introduces as “iterated doubling, starting first with two, and then even with one” and pointed out that this process generates exactly the powers of two. In arguing that this process is closely connected with Nathan’s thinking, he wrote that “if one agrees that Nathan has explicitly described the equivalent of doubling, with the starting point one, then he has in fact given a recursive definition of powers of two.” In response to the reaction of another researcher, R2, that Nathan is talking about addition, not multiplication, R1 responded, “But, at this level, what else is multiplication but iterated addition: in particular, multiplication by two is doubling.” He went on to say that, even though the process was not named, Nathan should not deprived credit for the underlying idea.

To reinforce his claim that Nathan has multiplication and powers of two in mind, R1 turned to Nathan’s claim that zero is special. He wrote, “What is the mysterious logic that leads Nathan at the end to say, ‘And so, I, I think zero’s special.” He argued that if Nathan were thinking about adding he would have more naturally generated 0, 2, 4, 6, 8, … by successively adding two. R1 argued that the fact that Nathan generates 1, 2, 4, 8, … is evidence that Nathan is focused on, and thinking about, “doubling, the equivalent of multiplying by two, and it is by doubling that he understands that even numbers make even numbers.” R1 wrote:

Now I think that he has in mind here, make new even numbers. If we start with zero, then doubling produces nothing new, and so zero is special in this regard.
Using his interpretation that by “make” Nathan means “by doubling,” R1 went on to comment on what the teacher says and does. He wrote that her second question to Nathan, whether he is saying that “all even numbers are made up of even numbers,” amounts to asking whether all even numbers are of the form $2n + 2n$, but that Nathan “always returns to the same sequence, 1, 2, 4, 8, …, and he never discusses doubling of any other even number.” For R1, this is evidence that Nathan’s process was restricted to numbers generated from the single starting point, namely powers of two, and would not include numbers such as 12 or 20, which are doubles of even numbers that are not powers of two, 6 and 10 respectively.

To finish squaring his interpretation, that Nathan had in mind powers of two, against an interpretation that Nathan is referring to even multiples of two, R1 wrote that 6 is the first even number exception in either case. Six is the first even number that is not a power of two, and it is the first even number that is an odd multiple of two. (I am not sure why he did not consider 2 to be the first odd multiple of two, but it may have had to do with 2 being a special case.) Thus, a few minutes later in the class, when Betsy introduces six as an exception, Nathan’s response that he is not talking about six fits with either interpretation.

R1 ended his commentary by stepping back and asking why it would be of interest to know whether Nathan’s thinking was “focused close to the idea of powers of two.”

Well, this is a very important mathematical notion, and sensing its proximity presents the teacher with options to consider. Is this something that could or should be pursued with the class at that moment? That is a decision that depends on many factors, and I am not urging necessarily that it happen. I only note that there are ways that it seems like a feasible possibility. One could ask for the numbers that Nathan’s procedure leads to, listing the first several, say up to 16, or 32. Then one could list the even numbers and make observations about how rapidly the “Nathan numbers” (powers of two) grow among them, missing most of them. This could be a brief excursion, that gives the students a glimpse of things to come — multiplicative structure, exponential growth, …

R1’s early statements about this situation, though, were challenged by other researchers, because the statements seemed to overstate what one could definitively know about what Nathan was thinking. At first blush, the difference seemed to be one of how to say the point rather than what the point was. Certainly Nathan’s comments suggest the notion of powers of two; clearly though we cannot know what Nathan was thinking. As the exchanges continued, though, the difference in perspectives seemed more and more significant. R1 felt that Nathan was thinking about the concept of powers of two and that this observation was significant for decisions about
what should happen instructionally. Some of the other researchers in the group felt that Nathan was not thinking about a concept which would appropriately be called powers of two and that pursuing the notion of powers of two in this classroom on this day would be unwise without significantly more clarity regarding what it is about powers of two one would pursue and why.

This commentary points to mathematical issues relevant to instruction. In addition to noticing that the numbers Nathan mentions are all powers of two, it is also interesting to consider the analogous nature between even numbers and powers of two (one based on successively adding two and the other, as R1 points out, based on multiplying by two), as well as R1’s notion of powers of two as in the proximity of the mathematics that arises on this day and foreshadows mathematics that will come in subsequent years. However, this commentary also highlights difficulties that may arise in having mathematicians’ produce commentaries on classroom interactions without any guidance or structure for the analysis, even when those commentaries are produced in interaction with teachers and those more familiar with teaching.

My point here is that a mathematical analysis structured around the four features identified in the first section of this chapter would avoid, or at least frame in significantly different ways, this emphasis on powers of two as an important idea for the teacher to pursue. Instead, the analysis would focus on the mathematical problems being asked and on explanations and the solutions they imply and to which they lead. The analysis might consider ways in which ideas are being expressed, but it would be oriented toward issues seen as bearing on a coherent picture of mathematical work being conducted in the classroom and less likely to pursue what, in this light, becomes a flight of fancy in the sense that it does not readily fit into a mathematical analysis as framed above.

In the first section of this chapter, I provided arguments for ways in which each of the four disciplinary themes is parallel to a central concern of teaching and learning. I will not repeat those here. Instead, I want to point to an additional mechanism that can be used to align a mathematical perspective with the pedagogical. I want to argue that other analytic features seen as important from a pedagogical perspective can be similarly explored from a mathematical perspective by considering disciplinary analogies and using these to examine practice. I briefly sketch four examples of this technique.

For instance, participation structures and turn taking are often thought to be important in instruction and in the work of teaching. Turning this issue back into the context of disciplinary work, one can ask what “turns” look like in disciplinary exchanges. To give this more focus, consider turns that make up the work of giving explanations. What does turn taking in explanation work look like in the discipline? To develop a sense of this, one might consider both
formal and informal settings in which explanations are given and engaged, such as professional meetings where polished results are presented and intimate settings where colleagues are working together either to develop an emerging explanation or to test a previously developed explanation. Having a sense of a range of different kinds of turn taking that occur around the giving of explanations, one could then examine practice to see what one notices about the turn taking occurring in classrooms when explanations are being given.

Similarly, a common elementary school teaching practice is to personalize academic content in some way. This can be done, for instance, by situating problems in contexts that are familiar to students, by using children’s names in word problems, or by associating ideas or results with particular children. Personalizing seems to serve a number of purposes, including motivating learners, creating a sense of immediacy, and providing context to support thinking about the content. What gets personalized in mathematics and why? Questions? Answers? Particular ideas or approaches? What role do these play in the personalizing mathematical work for mathematicians and how does such personalization support them in their work? One analog in the discipline might be associating people’s names with particular ideas and results and attributing or crediting of ideas. With a sense of issues such as these, one could then return to the classroom to see what one notices about practices of personalizing in teaching and learning, including the presence and absence of disciplinary practices and similarities and differences between disciplinary and pedagogical practices.

A third example has to do with recording work and shaping public space. An important question teachers routinely face is about what to record and where. What gets put on the board? What gets erased and when? Should children make records of their work and how might these be used? What needs to be viewed by others and how is this viewing important for advancing the doing and learning of mathematics? One could consider related issues in the discipline. What do private and the public recording practices look like in the discipline? What gets recorded and by whom, and how is it used? Ideas about these issues in the discipline could then be used to examine teaching and learning to see what one notices when attending to these issues. Such an analysis might well reveal mathematical knowledge and skill that would inform and support teaching.

For a final example, consider the work teachers do to establish and use routines in the classroom. Looking at the discipline, are there common routines for engaging in mathematical work for instance? What are structures for doing mathematical work? When does one work alone and when with others? When does one sit quietly thinking, when does one write, when does one talk with others, and are there routines for such activities? With a set of ideas about
these practices in the discipline, one might examine practice with an eye for: whether versions of these routines occur in classrooms, occasions in instruction when they might be used, and potential implications and repercussions for advancing the mathematical work at hand.

The four components, or themes, of a mathematical perspective presented in Section 4.1 were developed inductively from past research of the group and from a consideration of the discipline and the nature of disciplinary work. In other words, each of the themes arose from a formulation of the discipline, but was selected because of its bearing on teaching and learning. In the discussion above, I have argued for a systematic way to generate additional themes that are also formulated in terms of the discipline, but are likely to have relevance to teaching in ways that coordinate the two perspectives in research on identifying mathematical knowledge for teaching from an analysis of practice. In contrast to the four themes identified in Section 4.1, the examples above are not attempting to identify key themes or to be comprehensive in characterizing a mathematical perspective. Nor am I claiming that the pedagogical issues in the examples above are of equal or major importance. Instead, I am arguing that one way to coordinate mathematical and pedagogical perspectives in the analysis of practice is to select a pedagogical issue, consider ways in which that issue may have analogues in the discipline, develop an understanding of disciplinary practices of those analogues, and then use this understanding to analyze teaching and learning. In the next section, I reverse these ideas to ones about extending the nature of a pedagogical analysis of teaching and learning in ways that are coordinated with mathematical perspective.

4.3.2 Aligning Pedagogical Perspective with the Mathematical

Similar to the extending of a mathematical analysis described above, one can expand specific aspects of a pedagogical analysis to incorporate features of a mathematical analysis in ways that coordinate the two perspectives. The characterization of a pedagogical perspective given in Section 4.2 was not as specific in nature as the characterization of a mathematical analysis in Section 4.1. Correspondingly, it did not provide arguments about important parallels between the two concerns that were important to leverage. Instead, it argued that a pedagogical analysis needs to be formulated in such a way as to convey a nuanced sense of the work — by characterizations grounded in a theory of instructional interactions, by explicit identification of purpose, and by a layering of tasks that both specify and contextualize specific formulations of the work. My argument was that such a characterization of the work of teaching can orient the mathematical analysis of practice in ways that better assure its bearing on teaching. In this section, I propose an additional way of aligning the pedagogical analysis of teaching and learning with a mathematical perspective, but by shifting to a different technique. I argue that, having
developed some sense of what constitutes a mathematical analysis, its analytic features can be incorporated into a pedagogical analysis.

To convey what I mean here, let me first illustrate this at a more personal level. A version of what this might mean was suggested from past work, where I noticed that people with little training in mathematics, who routinely analyzed practice from a pedagogical perspective informed by years of teaching experience, over time began to incorporate into their own pedagogical analysis of practice the attentions of a mathematical perspective. In other words, these people began watching for mathematics problems, for instance, independent from the instructional task being given and with an eye for the mathematical structure and relatedness of tasks. One researcher, who typically engaged in sensitive pedagogical analyses of practice, commented:

I was struck by something that they were being asked to do that was never named. They are being asked to work on a small component of a problem that had come up before, and are working on that little piece. 0–9 grew from 300–190. And Mei is trying to put it back into the context of that bigger problem. (010625PrjMtg)

I mean to suggest that in this example the researcher is engaged in a pedagogical analysis of the work of the teacher in the instruction, but that in doing so, the researcher has incorporated a tool of mathematical analysis — looking for problems, noticing components of problems, and situating problems in the context of larger problems and mathematical work. Although this comment could have been made simply from the perspective of an experienced teacher as part of a pedagogical analysis of the work of teaching, my sense is that it reflects a development of the pedagogical perspective used by the group that adopted elements from mathematical analysis being done by the group. In saying this, I am not trying to make a claim, however, about the history of the research group. Instead, I mean to suggest that borrowing of specific foci of a mathematical perspective into pedagogical analyses of practice is a promising technique for further coordination of the two perspectives in the analysis of practice. It also suggests ways in which being explicit about a framework for the mathematical analysis of practice can support more coordinated conduct of perspectives in the analysis of practice.

In other words, it suggests that, in the midst of carrying out a pedagogical analysis — identifying purposed work and layers of related tasks entailed in teaching — one might consider framing that work in terms of key disciplinary features, such as problems, explanations, language, or mathematical work. Beyond simply an observation about the conduct of past research in the University of Michigan research group, this proposal raises more fully the idea of using a
disciplinary perspective to frame the work of teaching so that it is developed from an analysis of practice and can be used to analyze practice, especially in the context of studying mathematical knowledge for teaching.

For instance, a mathematical perspective on mathematics problems as described above in Section 4.1 suggests that mathematics problems can be thought of as sitting in constellations of problems — easier problems that provide a manageable way to work on harder problems and harder problems that provide motivation and direction for easier ones. Another disciplinary issue for problems has to do with the sub-problems that constitute work on the problem. These and other notions of how mathematics problems are considered and engaged in the discipline could then be used in a pedagogical analysis to frame the work of teaching, and then this framing of the work of teaching could be engaged and elaborated in the context of analyzing practice.

As an example of this, consider the ways in which a pedagogical analysis of mathematics problems given in the classroom might examine sets of related problems shaping instruction either implicitly or explicitly and the potential implications of these for the work of teaching. The related problems might be like some of the examples reported in Section 4.1 as results of mathematical analyses of teaching and learning. In other words, I am arguing that, inspired by some of these, one could consider their use in the work of teaching and analyze this potential in the analysis of practice. Thus, imagine a textbook gives the following problem to explore experimentally.

A bag has 10 red chips and 10 blue chips. If you draw two chips from the bag simultaneously, what is the probability that they are the same color?

One can consider several different sets of mathematics problems related to this problem, some of which decompose or simplify it and some of which situate or generalize it. One could parameterize the numbers of chips of each color or the number of chips drawn. A potentially interesting case of this might be the a series of problems with: 1 red and 1 blue chip; 1 red and 2 blue chips; 2 red and 2 blue chips; and 2 red and 3 blue chips. One could also attempt to characterize the sequence of tasks that need to be done to complete the task, such as: generate a hypothesis about how likely the outcome is; run the experiment a few times; check the results against the hypothesis; revise as necessary; appraise the stability of the results to decide how long to continue the experiment; and seek s reason why the observed results make sense. Or, one could identify the conditions of the problem that serve as evidence that a proposed answer is a solution. Further, one could systematically vary the selection procedure or the outcome sought.
Along with ideas about these aspects of the mathematics problem, one could consider tasks of teaching that these ideas might inform.

• Create an easier problem, or a sequence of problems, to scaffold students’ work.
• Generate an example to use at the outset to give students a sense of the larger problem.
• Prepare questions to ask of students to help them figure out what to do to make progress.
• Review the goal with students and develop ways to check that you have reached it.

These tasks could then be used to analyze practice — refining, elaborating, and prioritizing the tasks. This would include determining how robust they are, in what ways they are entailed in teaching, what range of purposes they might serve, and their relationship to other tasks and to the overall work of teaching. My claim is that a process such as this, focused on work of teaching that is formulated in relation to a feature deemed significant for mathematical analysis, can yield important insights into the mathematical nature of mathematics teaching and into mathematical knowledge for teaching. And, the process does so in a manner that coordinates the two perspectives and could be developed for any of the different features emphasized in a mathematical analysis — explanations, expression of ideas and use of language, character of the work, and so forth.

4.3.3 Focused Concepts for Regulating and Combining Perspectives in the Study of MKT

In earlier sections of this chapter, I developed specifications for a mathematical perspective and for a pedagogical perspective. For each, I developed the perspective with the other perspective in mind, focusing on aspects that would work in harmony with the outlook and goals of the other perspective. In the sections immediately above, I took this a step further by considering elements of each perspective that might be adopted and adapted for the other, again creating coordination between the two perspectives. Throughout, my point has been that the coordination of perspectives requires significantly more than finding ways for people operating from the two perspectives to reach agreement: It is about developing analytic tools, as stated in the fourth definition of coordinate above, that “regulate and combine so as to produce harmonious action.” In other words, it is about building conceptual tools that pre-design much of the structured coordination that needs to be done between perspectives.

In this section, I revisit some of these same devices, but with a view of them as conceptual tools generated from the analysis of practice and designed to coordinate the two perspectives. I do not mean to argue that these constructs are fundamentally different from those already presented. Instead, I want to argue that focused concepts, such as these, that are able to
carry much of the burden of coordinating, can be created precisely for that purpose through collective analysis of the issue. In this sense, they are not really the sole purview of either perspective, but are a product of the analysis of practice, a product that is also useful in the analysis of practice. I offer two examples of concepts from past work of the research group that were products of the work and, as formulated, contribute to the coordination of the twin perspectives: establishing and using a base of common, or public, knowledge and remembering mathematical work. I briefly describe the construct and then examine ways in which it coordinates the two perspectives in an analysis of practice for the purpose of studying mathematical knowledge for teaching.

Establishing a base of common/public knowledge. Ball and Bass (2000b) argue that reasoning of justification in mathematics rest on two foundations: a base of public knowledge and mathematical language. The first of these is:

… a body of public knowledge on which to stand as a point of departure, and which defines the “granularity” of acceptable mathematical reasoning with a given context or community. (p. 201)

… mathematical knowledge that is available for public use by a particular community in constructing mathematical claims and for seeking to justify them to others. (p. 201)

… that knowledge that can comfortably be assumed and used publicly without additional explanation. (p. 202)

They go on to say that it is defined relative to a particular community of reasoners and to say that, for professional mathematicians, it might consist of an axiom system for some mathematical structure plus a body of previously developed and publicly accepted mathematical knowledge derived from those axioms. It is, then, foundational to mathematical reasoning because it is the literal foundation, the Ursprung.

A process of reasoning typically consists of a sequence of steps, each of which has the form of justifying one claim by invocation of another, to which the first claim is logically reduced. This process, which merely transforms one claim into another, is not a vicious circle because the reduced claim is typically of a more elementary or accessible nature, and, in a finite number of steps, one arrives at a claim which requires no further warrant, because it is part of the base of public knowledge, and is therefore universally persuasive within a particular community of reasoners. (p. 203-4)
Thus, because it identifies a critical aspect of the dynamic of deductive mathematical reasoning, Ball and Bass argue that it is useful in understanding instruction in elementary school as well.

This is not a term or concept, however, whose impetus lies solely in the articulation of the discipline for the sake of articulating the doing of mathematics. It is also not one whose impetus grows directly from an attempt to articulate a pedagogical perspective of teaching and learning. As I have described for other examples, it represents a marriage of the two perspectives, but its impetus lies in collaborative attempts to understand what is going on in instruction, considered jointly from the two perspectives. One approach to the analysis of practice is to choose an issue important to both perspectives and to coordinate the two perspectives within the interactive exchange between the two perspectives. In general, this occurs with different people holding different perspectives in discussions about episodes of instruction, but it might as well occur in an internal dialog with a single person. In describing the research in this way, I am harking back to descriptions of research method given by Ball (1999) and Ball and Bass (2000a) as interweaving of interdisciplinary perspectives, but with little specification of what that means and how it is conducted. My point here, however, is to argue that a specific focus of such research, whatever is involved in doing it, can be to produce focused concepts which can then be used as analytic tools for analyzing practice and that such tools have, pre-designed into them, structure that coordinates the twin perspectives.

Ball and Bass then go on to use the concept to examine episodes of instruction where one can see the teacher and students using and adding to the base of public knowledge in their classroom. They also explore some of the work of teaching that arises in connection to establishing and using a base of public knowledge. One can see different ways in which teachers go about establishing a base of public knowledge. For instance, university instructors set prerequisites for courses and use the introductory chapter of textbooks to establish what will be comfortably assumed and used; they also word assigned problems in ways that communicate what can be taken as given and what cannot. Elementary teachers may pause in the process of solving a problem to ask, “And what do we know? We know that….” They may also establish a norm that ideas presented in a lesson or written in the text of a lesson being used can be taken as known when solving problems assigned in the lesson. In addition, the work of teaching involves a host of activities around the use of the base of public knowledge in the reasoning. Teachers have to teach students how to reason and they are responsible for maintaining standards for reasoning. They have to troubleshoot breakdowns in reasoning and scaffold students’ engagement in reasoning.
Thus, the notion of a base of public knowledge and its role in mathematical reasoning provides a tool that can be then used to enhance or expand either a mathematical analysis of practice or a pedagogical analysis of practice. In the same ways that a mathematical analysis systematically examines reasoning being given in the classroom, it could note the implied base for the reasoning, the ground on which it stands. Or, it might analyze practice in an attempt to identify the specific body of knowledge that might serve as a base of public knowledge for the given community. Because the notion of a base of common knowledge has been developed with pedagogical concerns in mind, these mathematical analyses are likely to be relevant to teaching and to be aligned with the orientation and goals of a pedagogical perspective. Likewise, a pedagogical analysis might analyze practice with an eye on developing a set of teaching tasks entailed in establishing and using a base of public knowledge in the classroom. It might articulate the multiple purposes these tasks might serve and create layers of related tasks to convey the full range of the work of teaching related to establishing and using a base of public knowledge in instruction. Again, because the notion of a base of public knowledge has been developed with mathematical concerns in mind, these pedagogical analyses are likely to treat mathematics with integrity and to be aligned with the orientations and sensibilities of a mathematical perspective.

*Remembering mathematical work.* As a second example, I consider a focused concept developed in project meetings, but never written up and published. This example contrasts the first in that, where a base of common knowledge might seem, at first glance, to be part and parcel of a mathematical perspective, the notion of remembering might seem to be part and parcel of a pedagogical perspective. Both of these, I am arguing, are, in their very conception, coordinating concepts. They are products of coordinated analyses and both offer additional tools for analysis involving something more than just taking one of the two perspectives and formulating it with sensitivity to the other perspective.

In examining what third graders might need to learn to do to study and learn mathematics in the classroom, members of the research group were struck by how much remembering children would need to do. This remembering was likened to historicizing, including both constructing a history of the class’ mathematics work and making use of it. To participate fully in the mathematical work of the classroom and to use classroom interactions to learn mathematics, it seemed that children would need to remember specific solutions, other people’s ideas, particular explanations, what got written in notebooks, who said certain things, and so forth. The activity of remembering mathematical work arose on different occasions and in different ways in the analysis of practice. In these discussions, remembering in the classroom seemed related to a number of different purposes as well.
Memory is central to learning; as is metacognition — the ability to predict one’s performance on tasks and to monitor one’s level of understanding (Bransford, Brown, & Cocking, 2000). Remembering is a key activity for both. It is often argued the effective teaching is explicit about what is to be learned, what is being learned, and what has been learned. Naming the experience and clearly stating goals and outcomes seem to prepare people for learning and solidify what they take away — including their ability to transfer what they have learned to other contexts (Bransford et al., 2000). Likewise, as several researchers of teaching have argued, for students to learn, they have to know how to use instruction for learning (see, for instance, Lampert, 2001). Learning how to use instruction well requires, among other things, a kind of tracking on what is going on. For these reasons, and others, remembering is an important activity of instruction.

The research group has not written or published papers that address this idea of remembering mathematical work, so what I write here is my own attempt to express this notion, as best I can, as it was discussed in the group. It is the activity of recalling and representing central features of the mathematical work being done within and by the group. First, this involves recall of the problems, ideas, and explanations that constitute mathematical work and the casting of those into a story of the work. This includes both tracking on what individuals are doing and contributing and tracking on the collective activity, decisions, and story. Second, remembering mathematical work is about expanding the nature of the work and gaining ownership of it. By this I mean that in remembering the mathematical work, one re-presents the problem, idea, or explanation. The act of re-presenting puts one in the position of, at least momentarily, being the presenter. In this, it creates some demand that one figure out where one stands in relation to whatever is being identified. This re-presenting of some element of the work can also become an act of creating group ownership. Furthermore, the act of re-presenting can be seen as an act of creating one’s own representation for whatever is being identified. Such a representation becomes yet a thing of its own and can become an additional contribution to the mathematical work.

Remembering mathematical work involves, or is at least closely related to, several other issues. Choices about what to record, both individually and publicly, shape the supports for remembering, as does opportunity and skill in using these records. Choices about naming and labeling elements of the work also matter. Having a name can make it easier to refer to. Associating a student’s name with a mathematical contribution can personalize it and make it easier to remember (as in: Lisa’s idea, Fasil’s question, or Sahara’s explanation). Likewise, the need to remember requires listening and paying attention. The work of teaching is centrally about
helping students remember mathematical work and issues related to recording, naming, and
listening are important in that work.

The activity of remembering mathematical work is also central within the discipline, as
are recording, naming, and paying attention. Keeping track of where one is in the flow of
mathematical work, having ways of filing and recalling mathematical questions, methods, and
results that bear on a line of work, and gaining ownership over these elements are all important in
the profession.

The point here, however, is not that this activity is an important one, but that, as
something to focus on, it grows out of a jointly conducted pedagogical and mathematical analysis
of practice and is an encapsulation of the coordination managed in that analysis. As such, it
becomes a possible tool for the structured coordination of perspectives in future analyses of
practice. A pedagogical perspective tells us that remembering is important, but might not of its
own accord select out the elements of mathematical work as key. Indeed, the learning of
mathematics is more often thought of in terms of remembering knowledge than doing
mathematical work. Similarly, a disciplinary perspective is unlikely to single out the
remembering of mathematical work as a key practice. Instead, this notion is a coordinating
concept for this research.

Although this may sound self-referential — that a tool for coordinating the analysis of
practice is produced from a coordinated analysis of practice — I am arguing that, indeed, an
important source for structures that can help produce analyses that more fully coordinate these
two key perspectives can be results from those successful attempts to coordinate the
perspectives — by whatever means available. I am also not arguing that the goal of mathematics
teaching should be the teaching of central practices of doing mathematics rather than
propositional knowledge and procedural skill of mathematics. I fully concur with opinions such
as that conveyed in Adding It Up that see the goal of mathematics education to be proficiency that
rests on several distinct strands, including both aspects of knowledge of concepts and procedures
and skills associated with these. Instead, my focus here on remembering mathematical work has
to do with the ways it serves to instantiate a disciplinary concern for the mathematical integrity of
what is said and done in the class and a pedagogical concern for activities that support learning.

Thus, although not necessarily a chosen expression of either perspective alone, the notion
of remembering mathematical work can be used to enhance or expand either a mathematical
analysis of practice or a pedagogical analysis of practice. Parallel to an analysis of the character
of mathematical work, a mathematical analysis of practice might examine the remembering of
that work. What needs to be remembered? How is the work remembered? Where do
breakdowns in remembering the work of the past cause current work to suffer? Or it might consider the nature of recording, naming, and listening in the discipline, and then use these ideas to examine teaching and learning. For instance, what elements are typically named in mathematical work and how? For what elements and under what conditions are people’s names attached? Results from these could then be used as a lens to examine these issues in practice. Because the notion of remembering mathematical work has been developed with pedagogical concerns in mind, these mathematical analyses are likely to be relevant to teaching and to be aligned with the orientation and goals of a pedagogical perspective. Likewise, a pedagogical analysis might analyze practice with an eye on developing a set of teaching tasks entailed in remembering mathematical work in the classroom. It might articulate the multiple purposes these tasks might serve and create layers of related tasks to convey the full range of the work of teaching related to remembering mathematical work in instruction. Again, because the notion of remembering mathematical work has been developed with mathematical concerns in mind, these pedagogical analyses are likely to treat mathematics with integrity and to be aligned with the orientations and sensibilities of a mathematical perspective.

My point in this section has been to that there may be important tools for coordinating mathematical and pedagogical perspectives that would probably not emerge from work that seeks to develop either of the two perspectives, even with significant regard for the other perspective, but instead emerge from conversations between the two perspectives as they engage in the analysis of practice. The distinction I make here may not be very important conceptually, but might have practical implications for thinking about how to go about creating tools for coordinating the analysis of practice. Earlier ideas suggested the development of one of the two perspectives, but done in a fashion that considered the other perspective, that sought out and emphasized intersections with the other perspective, and that framed its notions with sensibility for the other perspective. This section suggests approaching this same place but from a different angle — keeping an eye on what emerges as fertile ground in the mutual analysis of practice and developing these emerging intersections in their own right, as coordinating tools in further analyses. Here is a short list of candidate ideas for similar tools. These are significantly underdeveloped, but they provide an image of the kind and range of tools that might be developed.

- Clarifying of problems. Before beginning to work on a problem, it is helpful to have a clear sense of what the problem is asking. For instance, at least with certain types or problems, it may be helpful to do a few examples, mentally mapping them back onto the problem statement. This may be a common step in the professional work that is also key work in the context of teaching and learning mathematics.
• Checking answers against conditions. As with the first, this may be a common practice in professional mathematics that, while taken for granted or taken as obvious and thus slightly invisible, may be an important aspect to emphasis in the context of teaching and learning.

• Public and private recording. I discussed this issue earlier as an important concern of pedagogical practice — that teaching is concerned with what to have written down and how such records are made usable. Here I suggest that there may be aspects of this that emerge as particularly important for mathematics teaching and learning.

• Basis of authority for claims. In mathematics and in mathematics classrooms, claims are made and arguments given. These might rest on mathematical reasoning, but they also might rest on the authority held or conferred on particular people or communities. Developing a notion of this that combines the sensibilities of both groups might provide a useful tool for analyzing practice.

These notions, produced from the analysis of practice, may fit my descriptions here of focused concepts that stand with some independence from either individual perspective.

4.3.4 Broad Domains for Regulating and Combining Perspectives in the Study of MKT

In the previous section I argued for developing focused conceptual tools that could be used to analyze practice and that, by their design, support the coordination of perspectives. In this section, I step back to take a big-picture view of the analysis of practice and to argue that there are four broad domains strategic for studying mathematical knowledge for teaching, in part because, as with the focused concepts just discussed, they provide a structured approach to the coordination of perspectives in the analysis of practice.

• Examining student mathematical practices in instruction
• Identifying mathematical practices critical to teaching
• Examining mathematical work in instruction
• Identifying mathematical work and mathematical tasks of teaching

Again, these have emerged as central themes throughout this dissertation. Each provides a distinctive intersection of mathematical and pedagogical issues and each is central in shaping the dynamics of instruction and the quality of teaching and learning of mathematics. In this section, for each domain, I characterize the theme and argue that it is a core domain for an analysis of mathematical knowledge for teaching precisely because it is central to doing mathematics, central to the work of teaching, and formulated in a way that coordinates the two perspectives.

Examining Student Mathematical Practice in Instruction. In the study of mathematical knowledge for teaching, it perhaps makes some sense to back the question up into the space of the work of teaching, but does it make sense to back it up even further, into the space of the work of learning that students do, and, in particular, into the mathematical practices of students in the classroom? There are several reasons why it might. First, an understanding of the activities of
students in the classroom is key to interpreting what is going on in the classroom. Because students are engaged in doing and learning mathematics, part of that activity is likely to be a form of mathematical practice. For instance, as they learn mathematics, students often engage in mathematical reasoning, in using mathematical language, and in creating and using representations to communicate their ideas to themselves and others. As such, students’ emerging mathematical practices are central components of instruction and are key to interpreting, designing and managing instruction.

This is not to say that the goal of instruction should be to teach mathematical practices to students. Instead, it is to say that mathematical practices are present in important ways in doing and learning mathematics in the classroom and thus become an important aspect of instruction, of teaching, and of an analysis of practice for the purpose of identifying mathematical knowledge for teaching. Ball and Bass (2003b) write:

A recurrent dispute about school mathematics learning has been whether children should function as “little mathematicians.” Our analysis does not take a position on whether or not they should function as such, but suggests that when their interactions and work are viewed from a disciplinary perspective, particular features of the construction of mathematical knowledge often unseen become visible. (p. 218)

These authors seem to be saying that disciplinary practices regarding the construction of mathematical knowledge can shed valuable light on student practices regarding the construction of knowledge. Disciplinary practices for constructing knowledge and students’ practices for constructing knowledge are not isomorphic, but they are enough related that the disciplinary lens can be used productively to illuminate student practices as they shape the dynamics of instruction.

I argue that this is a particularly ripe domain for analysis because it is an important domain of teaching that often gets overlooked and because the mathematical knowledge for teaching demanded in this domain is currently underspecified. In addition, it is an important domain because it analytically combines mathematical and pedagogical perspectives in its formulation. A focus on mathematical practices is consonant with a mathematical perspective, but the particular focus on students’ mathematical practices focuses that attention on a particular concern of teaching, namely students, and it orients the consideration of mathematical practices into a space that allows for a broader, more expansive, and more inclusive conception of mathematical practice. Likewise, a focus on students and on students’ practices is consonant with a pedagogical perspective. Combining this with the specific attention to students’ mathematical
practices focuses that attention onto mathematically laden aspects of instruction and the work of teaching.

Identifying mathematical practices critical to teaching. A second broad domain that I argue is strategic for studying mathematical knowledge for teaching because it is both an important domain and because it analytically combines the twin perspectives of mathematics and pedagogy in a way that coordinates them is a focus on mathematical practices that have a significant role in the work of teaching. Above, I argued that mathematical practices are central to learning. This, of course, makes them important for teachers to understand. My point here, however, is that certain mathematical practices are particularly prominent in the work teachers have to do. Teaching is not the same kind of endeavor as is learning. It seems worthwhile to consider more closely which mathematical practices are central to the work of teaching. In broad terms, teachers need facility with explanation, with selecting and using representations, and with the use of mathematical language. Although facility with all mathematical practices would be desirable, it seems likely that these might be important candidates as components of teaching. In contrast, it might be argued that generating new and interesting mathematical questions is not as important for teachers because it is fine for them to turn to other sources for these. It would still be important for them to be able to recognize a mathematically interesting problem when they see one and it would be nice if they were good at generating them, but this latter skill might not be as important.

As with the notion of students’ mathematical practices, focusing on teachers’ mathematical practices creates coordination between the two perspectives. Examining practice with an eye on the mathematical practices that teachers are engaged in, or rather that seem entailed in their work, coordinates the notion of a disciplinary practice with pedagogical work that teachers do. What exactly do teachers need to do with representations? In what ways is their use of representations similar to and different from practice in the discipline? What would a disciplinary characterization of teachers’ use of representation look like? Each of these questions puts a particular spin on the notion in a way that regulates and combines the two perspectives. Similarly, a focus on mathematical practices of teaching provides a pedagogical analysis of teaching and learning with a means of keeping the content squarely in the picture.

Examining Mathematical Work in Instruction. The last two domains shift the framing from distinguished mathematical practices to the composition of mathematical work. In making this distinction, I mean to distinguish a practice, as an activity involved in getting work done, from work, as the overall endeavor. Thus, the examination of mathematical work shifts the focus to a bigger picture that includes attention to the specific ends.
In a paper entitled, *Making Mathematics Work in School*, Ball, Lewis, and Thames (2008) give four meanings that their title is meant to evoke. The first meaning they describe as follows.

In one sense, the title refers to what teachers give students to *do* so that they will learn mathematics. Teachers define and *make the mathematical work* in which students engage: tasks, activities, and questions (Doyle, 1983). We investigate what that “work” is: What do teachers asks students to do that we would call the “mathematics work” of a lesson or a sequence of lessons? Certainly it is more than the assigned tasks (Arendt, 2000). Surely it includes the time, the materials, and what is done with them. It also includes the *talk* that carries and surrounds the tasks. That talk, and the work that constitutes it, is the focus of this chapter. (p. 16)

In pointing to the tasks, activities, and questions, and in referencing Doyle (1983), which draws attention to the important ways that these shape instruction, these authors focus attention on a unit of analysis that serves to frame the overall endeavor. As just described, if work is that which is there to be done to achieve a particular goal, then mathematical work is that which is there to be done to solve specific mathematics problems or accomplish specific mathematical goals.

In contrasting *practice* and *work* in this way and in attending to both, I mean to call attention to two important foci: the latter, which keeps its eye on the overall endeavor, and the former, which isolates particular components of that endeavor so that they may be more deeply examined and understood. Related to this contrast of scope is also a shift in attention from the individual to the group. In examining students and the mathematical practices in which they engage or called on to engage, attention is turned to a particular practice as it is carried out by a particular person. In examining mathematical work in instruction, attention is shifted to the class as a whole, or at least to the largest unit or group in which the work is being carried out. When work is being carried out by individuals alone, there may be little or no distinction here, but as work is conducted in small groups or by the class as a whole, then this larger setting is central in the analysis, even when much is done individually inside of this overall frame. Individual contributions might well be considered in examining mathematical work in instruction, but they would be considered in the context of the overall picture of what is being done. So, this third proposed domain is about the overall endeavor of accomplishing that which is there to be done to address mathematical problems and tasks orchestrated by the teacher in classroom instruction. Building on Doyle’s argument for the centrality of tasks (and what is done with them) in instruction and in the work of teaching, the notion of mathematical work provides a lens that is readily combines sensibilities of both disciplinary and pedagogical perspectives.
Identifying mathematical work and mathematical tasks of teaching. Shifting the focus one final time, the last domain important for the study of mathematical knowledge for teaching takes the notion of mathematical work and considers specialized forms of it that occur in teaching. Thus, in the previous section, the meaning of mathematical work was drawn primarily from the discipline, where it was that which is done to accomplish mathematical goals. With the phrase mathematical work of teaching, I mean to refer to work of teaching as might be conceived from a pedagogical perspective, but that work of teaching that is distinctively mathematical in nature. This distinction was discussed in more detail in the previous chapter where I characterized the third approach extracted from past work of the research group (Section 3.3). My point here is that this third approach uses a focus on the domain of mathematical work of teaching as a means of coordinating mathematical and pedagogical perspectives. It is the use of this notion that makes the third approach so successful in identifying important domains of mathematical knowledge for teaching.

The notion of mathematical work of teaching combines the two perspectives and regulates their use and interaction. The focus on work of teaching is consonant with a pedagogical perspective, but the particular focus on the mathematical activities that occur within that work, or on mathematical dimensions of that work, combines disciplinary conceptions into formulations of the work of teaching. Combining the specific attention to mathematics in formulations of the work of teaching focuses attention onto mathematically laden aspects of teaching in ways that effectively coordinate the two perspectives.

4.3.5 Establishing a Research Culture for Coordinating Perspectives

The sections above identified conceptual or analytic tools for coordinating perspectives. This section turns its attention to some features of the culture or practice of research that seeks to coordinate mathematical and pedagogical perspective. Three features seem important. The first has to do with engaging multiple points of view and in flexibly working across them. The demands in this regard lend themselves to collective work but also require the fostering of sensibilities for working across perspectives. The second is about the need to keep the analysis grounded in empirical evidence. The third identifies the role of documenting the observations and ideas generated in ways that support public review.

Collectively and flexibly engaging in the analysis. Part of what is involved in coordinating the twin perspectives of mathematics and pedagogy is that both perspectives need to be engaged and that the thinking and analysis needs to be done across the two. Such an analysis requires expertise that spans distinct communities. It requires somewhat singular attention to one of the two perspectives, while also listening to and incorporating the other perspective.
Additionally, a pedagogical analysis, itself, is aided by a degree of flexibility and engagement in multiple views. Although analysis might be carried out by a single individual, collective engagement in the analysis has certain advantages.

Collective work on analysis of practice includes both the participation of different people with different perspectives and a particular stance toward the work. When observing teaching and learning, there is much one can attend to and having multiple eyes and ears can be helpful. Beyond this, resources for taking and holding fundamentally different perspectives can be augmented by including people with expertise in, experience with, and commitment to three distinct arenas: (i) to the discipline of mathematics; (ii) to the practice of teaching; and (iii) to education research. The first two represent the two primary perspectives, and the third can provide expertise in conducting such research. However, in suggesting these three groups of people, I do not mean to imply that the approach is about the people. Indeed, it should not be presumed that in asking mathematicians to engage in a mathematical analysis of practice that they will naturally do so. It is often the case that in asking people, including research mathematicians, to observe instruction with a mathematical eye they can be more inclined to attend to other aspects of what is happening. However, engaging different people in analyzing practice from the perspective that is distinctive of their expertise can help to contribute to building the kind of resources that might break with past characterizations of mathematics for teachers and unearth distinctively disciplinary aspects of classroom instruction that are simultaneously relevant to teaching.

In addition to including people with diverse training, it becomes important for people to learn to take and hold perspectives while also developing the capacity to hear and engage with different perspectives. The work requires simultaneously seeking to understand another commentator’s perspective on its own terms while also asking in what ways it might conflict with one’s own. For instance, a person primarily focused on observing practice from a pedagogical perspective might need to work to understand a mathematical issue being raised by someone else and what the person found mathematically interesting about it, at the same time, asking whether the proposed notion resonated with pedagogical sensibilities and with a sense of what might be helpful when teaching. For instance, a mathematician might attribute advanced understanding to a child that a teacher, from experience with children, might interpret as “overhearing” the child, of reading more into what is said than is actually there. Likewise, a mathematician may see a particular mathematical idea as foundational to a lesson in ways that are at odds with a teacher’s sense of what is instructionally useful. This might happen, in particular, when formal development of a topic obscures developmental aspects important for learning. The point here is
not that pedagogical sensibilities trump mathematical ones, but that such situations require a coordination of perspectives in the collective analysis of practice.

*Empirically grounding the analysis.* A second feature of the research culture and practice is that the coordination of perspectives is aided by routinely and explicitly supporting observations and claims in the particulars of a specific instance of practice. This grounding in evidence can take two forms depending on whether the ideas are about what actually happened in the classroom or about what might happen, either in a hypothetical future or in other classrooms. In the first case, the researchers insist that claims about classroom events be routinely backed up by evidence from the data. Experiences showing video of classroom instruction to people—prospective teachers, education researchers, mathematicians, teachers, policymakers—suggest that people often leap to evaluating teaching, make sweeping claims about what the teacher should or should not have done, and talk in general and abstract terms that often reflect vague thinking and are readily misunderstood. Insisting that people refer back to what they heard or saw that led them to react as they did and that they explain how this evidence supports the claim they are making generates more productive commentary and makes both the motives and the substance of ideas more apparent to others. The other form that this feature might take arises when people make statements about an alternative move on the teacher’s part and what would have subsequently happened or about what happens in other classrooms across the country or about what happened in some classroom where none of the other people in the conversation was present. In cases of this type, it is helpful to routinely insist on detail and concrete examples. For instance, if a person wants to argue the relevance of a specific idea to a lesson, in addition to direct reference to what was said and done in the lesson, providing detailed examples of what might have been said and done instead is more likely to communicate the proposed idea effectively and to open it to constructive review.

In part, a central assumption of the approach being discussed in this study is that it be practice-based—that consideration of the mathematics that teachers need to know be grounded back in an empirical study of practice that examines the mathematical demands that seem to arise in teaching. However, my point here is that the use of records of practice and disciplined reference to them is a key resource for coordinating perspectives. As observations and claims are put forward, immediate and specific reference to particular moments in a video or particular statements in a transcript can play an important role in supporting comprehension across perspectives in collective work as well as coordination of the two perspectives.

*Systematically documenting the analysis.* Analogous to the ways in which the specifics contained in records of practice provide common references around which to communicate, test
ideas, and coordinate mathematical and pedagogical perspectives, creating written documentation of the interpretations, ideas, and interactions arising during analysis can provide valuable resources for coordinating perspectives. In discussing the documents used in this study, I described three types: planning documents (e.g., proposals, work plans, and planning memos), primary working documents (e.g., analyses, analytic exercises, analytic memos, and meeting notes), and products and uses (e.g., reports, papers, presentations, and materials for courses). Of course, planning documents play a role in creating a common vision for the research and in helping people understand and be able to carry out their role. Clear articulation of research design can help with the coordination of perspectives. Perhaps less obvious, though, is that certain working documents and intermediate products can be particularly valuable in supporting the coordination of perspectives.

In this dissertation I have made use of the detailed annotations of records of practice, layered sets of commentaries and reactions, project meeting notes, slides from presentations, and other similar documents. They have provided valuable insight into the nature and conduct of the research. However, my point here is that such documents are also an important resource for conducting the research in the first place, in particular for coordinating perspectives in the work. Recording interpretations, ideas, and interactions opens them up for public review and focused response. In the same way that video and transcript allow people to pause the video and point to exact wordings and link ideas to empirical evidence, well-designed research documents allow researchers to slow down the research conversation, checking their interpretations of different perspectives against exact wordings, and probe, test, and refine their interpretations of the perspectives.

I do not mean to suggest that the documents used by the University of Michigan research group are the best or only ones. Instead, I only mean to argue that creating detailed records of analyses as they are produced, discussed, and refined can aid in the coordination of mathematical and pedagogical perspectives. Given other dynamics discussed in this chapter, three principles seem worth considering. First, it may be useful to build documents on top of original records of practice or documents that contain realistic instructional interactions. This might include annotated transcripts, but could also include video annotations using newer multi-media software or annotations of student work. They might also include commentaries or animated videos of instruction. Second, it seems important to document mathematical and pedagogical analyses in forms that support reaction and commentaries by others and to document the exchanges about practice among people with expertise and background both in mathematics and in pedagogy. Third, it can be useful to collectively represent ideas being developed by a group of researchers,
such as for presentations or reports. This may work better with more informal or preliminary
documents and activities, than with formal publications both because they occur earlier in the
process when ideas are still being developed and because they may allow for greater flexibility.

4.3.6 Summarizing the Coordination of Perspectives

In Chapter 1, I argued that a significant challenge for practice-based and disciplinary-
grounded approaches to studying practice is the coordination of the mathematical perspective and
the pedagogical perspective. Already visible in the previous sections that specified a
mathematical perspective and a pedagogical perspective is the notion that each perspective can be
formulated in ways that are attuned to the other perspective. Put another way, coordination of the
perspectives can be pre-designed into the concepts or analytic tools used in the analysis of
practice. Such pre-design work serves to regulate and combine the two perspectives to produce a
harmonious analysis of practice.

The proposals in this section complement, yet go beyond, the notion of “border crossers”
or “ambassadors” (Ball and Lampert, 1999) and beyond a focus on the dialogue and exchange
among researchers coming from different perspectives (Ball, 1999). Instead, they emphasize
systematic approaches to the coordination of perspectives that are based on analytic distinctions
that are vital to notions of mathematical knowledge for teaching and to the study of mathematical
knowledge for teaching. The first two approaches in the table below extend the ideas already
visible in the discussions above. The next two single a set of focused concepts and a set of broad
concepts that stand on their own. The last gives three research practices that matter for the
coordination of perspectives.

<table>
<thead>
<tr>
<th>Nature of Coordination</th>
<th>Examples</th>
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<tbody>
<tr>
<td>Pre-structures pedagogical relevance into a disciplinary perspective.</td>
<td>• Disciplinary formulations of turn taking in explanation work can be used to analyze turn taking occurring in classrooms when explanations are being given.</td>
</tr>
<tr>
<td>(This does not guarantee relevance, but it increases its likelihood.)</td>
<td>• Disciplinary practices of personalizing mathematical work can be used to analyze the potential for personalizing mathematical work in teaching and learning.</td>
</tr>
<tr>
<td></td>
<td>• Disciplinary practices of private and public recording can be used to analyze the recording of mathematical work and the shaping of public space in teaching and learning.</td>
</tr>
<tr>
<td></td>
<td>• Disciplinary practices for doing mathematical work can be used to analyze routines in classrooms around the mathematical work (such as when to work alone, or when to think and when to write).</td>
</tr>
</tbody>
</table>
Aligning Pedagogical Perspective with the Mathematical

Analytic features developed for or growing from a mathematical perspective can be explored from a pedagogical perspective and then pedagogical formulations, such as particular tasks of teaching, can be further elaborated in an examination of practice.

Nature of Coordination

Approaches the study of teaching by pre-structuring certain disciplinary conceptions and sensibilities into formulations of the work of teaching and into a pedagogical perspective then used to study practice.

An Example

Using a disciplinary view of both the decomposition of a problem into simpler problems and sub-problems and the situating of a problem in a larger mathematical landscape of related problems, one might generate tasks of teaching such as:

- Create an easier problem, or a sequence of problems, to scaffold students’ work.
- Generate an example to use at the outset to give students a sense of the larger problem.
- Prepare questions to ask of students to help them figure out what to do to make progress.
- Review the goal with students and develop ways to check that you have reached it.

Focused Concepts for Regulating and Combining Perspectives

Analogous to the processes described above, conceptual tools can be developed that are attune to both mathematical and pedagogical perspectives and precisely for the purpose of coordinating the perspectives.

Nature of Coordination

As described above, such conceptual tools pre-design coordination into the conceptual frames used to analyze practice.

Examples

- *Establishing a base of public knowledge* (“knowledge that can comfortably be assumed and used publicly without additional explanation,” Ball and Bass, 2000b, p 202). The impetus for this notions lies in collaborative attempts to understand what is going on in instruction, considered jointly from the two perspectives, and it illuminates significant mathematical demands of maintaining standards for mathematical reasoning, teaching students how to reason, and troubleshooting breakdowns in classroom talk.
- *Remembering mathematical work* (the activity of recalling problems, ideas, and explanations and recasting these into a story of the work being done within and by the class). This notion integrates a disciplinary concern for the mathematical integrity of what is said and done in the class and a pedagogical concern for activities that support memory, meta-cognition, and learning, and it illuminates significant mathematical demands of deciding what needs to be remembered, shaping how mathematical work is remembered, and considering the nature of what gets recorded, named, and listened to in the support of learning.

Broad Domains for Regulating and Combining Perspectives

In disciplinary-grounded analyses of practice, a few broad categories emerge as major points of intersection between mathematical and pedagogical perspectives.

Nature of Coordination

These domains, as with the concepts described above, are formulated in ways that pre-design mathematical and pedagogical perspectives.

Four Domains with Examples

- Examining student mathematical practices in instruction.
- Reading mathematical text and notation
- Giving explanations
- Making claims
- Using representations
- Identifying mathematical practices critical to teaching.
- Giving and evaluating explanations
Choosing and using representations
• Using mathematical language
• Examining mathematical work in instruction.
• Organization of mathematical work
• Nature of collective mathematical work
• People doing mathematical work
• Identifying mathematical work and mathematical tasks of teaching.
• Analyzing errors
• Re-scaling a problem
• Choosing an example
• Responding to “why” questions

Establishing a Research Culture for Coordinating Perspectives
Features of the culture or practice of research can matter for the coordination of mathematical and pedagogical perspectives.

Nature of Coordination
These practices increase resources for taking and holding fundamentally different perspectives and provide a basis for negotiating meaning and decisions across perspectives.

Examples
• Collective and flexible engagement. When observing teaching and learning, multiple eyes and ears can be helpful. Beyond this, resources for taking and holding fundamentally different perspectives can be augmented by including people with expertise in, experience with, and commitment to three distinct arenas: (i) to the discipline of mathematics; (ii) to the practice of teaching; and (iii) to education research. In addition, it is important for people to learn to take and hold perspectives while also developing capacity to hear and engage with different perspectives. The work requires simultaneously seeking to understand another commentator’s perspective on its own terms while also asking in what ways it might conflict with one’s own.

• Empirical grounding. Routinely and explicitly supporting observations and claims in the particulars of a specific instance of practice provides common references around which to communicate, test ideas, and coordinate perspectives.

• Systematical documentation. Recording interpretations, ideas, and interactions opens these up for public review and focused response. In the same way that video and transcript allow people to pause the video and point to exact wordings and link ideas to empirical evidence, well-designed research documents allow researchers to slow down the research conversation, checking their interpretations of different perspectives against exact wordings, and probe, test, and refine their interpretations of the perspectives. Three principles are worth considering: (i) build documents on top of original records of practice; (ii) document mathematical and pedagogical analyses in forms that support reaction and commentaries by others and document the exchanges about practice among people with expertise and background both in mathematics and in pedagogy; and (iii) collectively represent and discuss ideas being developed by a group of researchers, such as for presentations or reports.

Figure 4.12. Summary of approaches to coordinating mathematical and pedagogical perspectives in the analysis of practice.

In this chapter, I have argued that the notion of a practice-based and discipline-grounded approach to studying MKT takes seriously the twin commitments of being based in and relevant to practice and of being grounded in concerns and sensibilities of the discipline. The two underlying perspectives do not “talk” easily or comfortably with one another, and approaches that rely on dialogue and exchange among researchers face an losing battle in keeping both
perspectives squarely in play; discussions of mathematics are apt to lose relevance to teaching and discussions of teaching are apt to lose track of key mathematical issues. I propose that the use of analytic devices that pre-structure the coordination of the perspectives into concepts then used to analyze practice is important for coordinating perspectives. For instance, the notion of a mathematical task of teaching combines the two perspectives and regulates their place in the analysis of practice. Such an analysis can be driven by strong mathematical impulses, but must frame those impulses in terms of work of teaching; likewise, it can be driven by strong pedagogical impulses, but must then consider the mathematical dimensions of those impulses. Concepts that pre-structure the coordination of perspectives provide some assurance of informing the study of MKT. Indeed, I argue that it is this principle that lies at the heart of, and at the success of, the notion of mathematical knowledge for teaching and its predecessor, pedagogical content knowledge.
CHAPTER 5: FINDINGS AND LIMITATIONS

In recent years, concern about teachers’ mathematical knowledge has shifted to concern about identifying *mathematics that matters for teaching*. It turns out, however, that this is not straightforward. Central to the issue of conceptualizing such knowledge is the issue of how to investigate it; in other words, notions of what it is and how to study it are deeply intertwined. This study takes up the latter issue, but the former is never far off.

I set out in this study to bring to the surface a research design for a practice-based, discipline-grounded approach to studying practice in order to develop notions of mathematical knowledge for teaching. In particular, I cited Ball’s (1999) claim that such study would require the development of new research methods:

\[T\]his work also precipitates a need for new developments in method, in the cultivation of something we might be able to call “mathematical analyses” of teaching and learning. In complement to this, this research will need to continue to consider the development of methods for intertwining pedagogical and mathematical perspectives where the perspectives draw on different orientations and even different disciplinary training. (p. 36)

She calls for the development of new methods to support this novel approach to determining mathematical knowledge for teaching, and this dissertation has sought to respond to this call and to the challenges that it raises.

The primary motivation for pursuing these issues is that research at the University of Michigan on a practice-based theory of mathematical knowledge for teaching has made significant contributions to the field, but its underlying methods and design have been largely under-specified. In other words, the intention was to provide a fuller specification of research design issues so that the work might be engaged with greater clarity and so that other groups wanting to build on the work would have some basis for doing so. For instance, a goal of this study has been to support researchers and developers wanting to explore content knowledge for teaching in other subject areas and mathematical knowledge for teaching at the secondary or collegiate levels. Likewise, a goal has been to support more systematic production of instruments for measuring mathematical knowledge for teaching and the development of courses and materials for teaching mathematical knowledge for teaching. With these goals in mind, this
conclusion is structured as a draft technical memo that could be used to communicate to a wide
variety of audiences about the findings of this study, followed by a three hypothetical emails from
prospective “clients” together with responses that clarify what this study might imply for these
different lines of work— both what other people might use from this study and what they want
but that is not provided. Although the artifacts developed in this conclusion seem to framed as a
set of guidelines, or even a how-to manual, they discuss the conceptual results of this study,
including the ways in which issues of how to study mathematical knowledge for teaching
contribute to an understanding of what it is and how best to conceptualize it.

5.1 Developing a Practice-Based and Discipline-Grounded Approach to
Studying MKT: A Technical Memo

In using this technique of including a draft technical memo, email inquiries about it, and
my responses, my goal is to situate the expressions of the ideas in this dissertation in the context
of a potential audience for the work. I have written these as standing on their own and I provide
only minimal additional introduction or commentary. (Because this memo represents in some
ways the actual text of my summary of this work, I include it as text rather than as a figure, but I
use a line along the left margin to indicate the scope of the memo.)

Developing a Practice-Based and Discipline-Grounded Approach
to Studying MKT: A Technical Memo on Research Design

Ball, Thames, and Phelps (2008) define mathematical knowledge for teaching (MKT) as
the mathematical knowledge, skills, and sensibilities needed to perform the recurrent tasks of
teaching mathematics to students. This notion has garnered significant attention in recent years,
as has the approach to studying MKT taken by these researchers and their colleagues. The
approach involves examining practice in ways that are attuned to the discipline in order to unearth
the knowledge demands of teaching, but as Ball (1999) argues, the research design and methods
of the approach are underspecified.

This work also precipitates a need for new developments in
method, in the cultivation of something we might be able to call
"mathematical analyses" of teaching and learning. In
complement to this, this research will need to continue to
consider the development of methods for intertwining
pedagogical and mathematical perspectives where the
perspectives draw on different orientations and even different
disciplinary training. (p. 36)

The purpose of this memo is to summarize guidelines for the design of a practice-based and
discipline-grounded approach to studying practice with the purpose of identifying MKT.
First, let me clarify the problem. The central research question is: What mathematical
knowledge is entailed in the work of teaching mathematics? The approach, though, involves
three distinct issues: (i) mathematical analysis of practice (namely, one grounded in a disciplinary perspective); (ii) pedagogical analysis of practice (grounded in a pedagogical perspective); and (iii) ways of coordinating these two sets of analyses and the distinct perspectives on which they are based. This third strand is particularly important because the two perspectives, while not necessarily at odds, are based on distinct experiences, views, and allegiances and often seem unable to “talk” to one another. In the three sections that follow, I clarify how these two perspectives can be conceived so as to produce plausible and useful analyses of practice and explain how an understanding of coordination of the two perspectives is central to the very conception of MKT.

**Designing a mathematical analysis of practice.** A mathematical analysis of practice needs to address issues central to the discipline that are also significant pedagogically. A set of four foci developed from a recent analysis of work done at the University of Michigan includes: (i) problems and answers; (ii) solutions and explanations; (iii) expressions and language; and (iv) character(s) of the work. These foci are not mutually exclusive, exhaustive, or the only ones possible; others might be added and these four might be formulated in different ways.

<table>
<thead>
<tr>
<th>Four Foci for a Mathematical Analysis</th>
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<tbody>
<tr>
<td><strong>Problems &amp; Answers:</strong> Analysis of the mathematical structure of problems and answers and of the role of problems and answers in the conduct of mathematical work in the classroom.</td>
</tr>
<tr>
<td><strong>Rationale:</strong> The notions of “problem” and “answer” are central in the discipline and in school. Including “mathematical tasks” as a kind of sub-problem arising in the doing of mathematics further expands the intersection. A mathematical eye on the statement of problems, the subtlety of wordings in classroom talk, the underlying mathematical structure of problems and answers, and relationships among problems can inform the selection, creation, and use of problems and tasks in classroom teaching.</td>
</tr>
<tr>
<td><strong>Example:</strong> Identify <em>genres of tasks</em> with disciplinary properties abstracted from particulars of the classroom, such as: a hierarchy of increasingly ambitious questions; a family of parameterized tasks; “isomorphic” tasks modeled or represented with different contexts; and tasks directly engaged using only common experience and rudimentary knowledge, yet leading to substantial mathematics.</td>
</tr>
</tbody>
</table>

| **Solutions & Explanations:** Analysis of twists and turns in solving and explaining in the classroom, particularly in terms of disciplinary structures and conventions. |
| **Rationale:** Solving and explaining are central activities of the discipline and key knowledge-building activities of teaching and learning mathematics. A mathematical eye on the basis for both inspiration and conviction in the classroom can inform mathematical motivation, the generation of ideas, and sense making in classroom teaching and learning. |
| **Example:** Identify the *structure and the basis of particular arguments*, such as: |
| **Claim** | Zero is not even (but not that it is odd). |
| **Sheena** | Zero is even. |
| **Definition** | An even number is a number that can be made up of two (equal) things. |
| **Even and odd numbers alternate on the number line.** |
| **Prior knowledge** | 1 and –1 are odd numbers (fourth graders said so). |
| **Argument** | What two things can make it (zero)? |
| **Zero is situated between negative one and one.** |
| **Conclusion** | Imply: no two things can make it. |
| **Therefore zero is even.** |

Analysis of student thinking taken from Ball, Lewis, and Thames, 2008.

| **Expressions & Language:** Analysis of expressions, including: (i) the ambiguous language of generative mathematical work, especially problematic expressions; (ii) the language of formalizing mathematical work, including the need for and introduction of terms, definitions and precision; and (iii) additional features of mathematical rigor, such as logical consistency, the fit with convention, connections and mappings among different expressions, and explicit demand for definition, precision and consistency. |
| **Rationale:** Language plays an important role in mathematical thought and is essential to sanctioning and communicating ideas. It is also central to the talk-intensive activities of teaching and learning. A mathematical eye on expressions, both as incipient ideas and as rigorous formulations, with distinctive roles in the conduct of mathematical work, can inform the interpretation of teaching and learning. |
| **Example:** Examine the expression of claims (such as: zero is not even or odd; zero could be even; zero is not odd; and zero has to be an even) and explore the process of naming and using names; of making and interpreting claims; and of evaluating claims. (Ball, Lewis, and Thames, 2008). |
**Character(s) of the Work:** Analysis of features of the mathematical work that goes on in classrooms: the organization of the work, the collective nature of the work, and the characters who do the work (people viewed as mathematical actors with dispositions and roles who populate the workspace).

**Rationale:** The notion of character(s) of mathematical work provides a useful intersection between the discipline (focused on doing mathematical work) and teaching (focused on engaging students in the mathematical work of doing and learning mathematics). A mathematical eye on mathematical work in classrooms, including its organization and collective nature, can inform the sequencing and structuring of work on instructional tasks, in particular the collective work as engaged by students learning to carry out specific tasks and roles in doing mathematics.

**Example:** Identify a logical sequence of mathematical activities for accomplishing (collective) mathematical work. One example addresses more global activities, such as: problem posing; search for solutions; presentation and review of solutions; acceptance of solutions by the group. A second example addresses the review of mathematical explanations: think about reasons for a proposed answer; be able to reproduce explanations; consider what explanations might mean as reasoned mathematical ideas; refine the expression of ideas in ways that are precise and agreed on in the group.

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**Designing a pedagogical analysis of practice.** Central to the problem of identifying the mathematical demands of teaching is an articulation of the work of teaching. However, as Grossman and McDonald (2008) argue, “research on teaching still lacks powerful ways of parsing teaching that provide us with the analytic tools to describe, analyze, and improve teaching” (p. 185). It is this parsing of teaching — of naming its components and its underlying grammar — that is an essential ingredient for the study of MKT. My concern here is about a pedagogical analysis that informs the study of MKT, but the ideas may have broader application.

The phrase **work of teaching** refers to that which is **there to be done** to achieve the goal of student learning (as it combines with the many competing goals that society and local communities have for schools). The notion of “there to be done” focuses attention on activities fundamental to the endeavor instead of focusing on what is done or what ought to be done. The desired parsing, then, is not about describing modal practice, expert teaching, or specific kinds of teaching. Instead, it is about identifying tasks **entailed in designing and managing the interactions of instruction.** Two points are worth making here. First, the phrase **entailed in** refers to this notion of “there to be done”; namely, tasks of teaching are ones associated with basic demands of the endeavor. Second, the association of a task with basic demands of the endeavor is determined by reference to its role in the design and management of “the interactions among teachers, students, and content in various environments” (Cohen, Raudenbush, & Ball, 2002, p. 88). This means that tasks of teaching need to be formulated in ways that provide leverage across related instances of designing and managing instruction. This complements but goes beyond notions about being “agnostic with respect to various models of teaching” (Grossman & McDonald, 2008, p. 186) and “generalizability” (Ball, Sleep, Boerst, & Bass, 2009, p. 461).

The **process** of specifying a task entailed in teaching involves:

- Identifying, from the observation of practice, an action or possible action.
- Exploring potential purposes, alternative actions, the nature of conditions relevant to the action, priorities associated with it, competing goals, and analogous actions in the context of varied teaching situations, such as different grade levels, teaching styles, or topics or subject matter.
- Using these explorations to formulate a version of the task that is further from being an idiosyncratic teaching act and closer to being about central work entailed in teaching.

Deliberations about a teacher’s action, potential tasks, and the relationship between them can lead to identifying potential tasks that are **framed with sensitivity to multiple, instrumental goals of instruction.** For example, observing a teacher who has children keep notebooks to record their work and has them write the “problem of the day” in their notebooks at the start of class might lead to speculating about purposes, such as:

- Helps children take responsibility for getting to work and working independently
- Familiarizes the student with the problem
• Provides a title for the work students do in their notebooks
• Reminds students as they work what it is that they are working on
• Creates a usable record of student work for the teacher, the parents, and the student.

Turning back to the notion of designing and managing instructional interactions could then lead to a range of candidate tasks of teaching that capture more fundamental entailments of the work.

• Teaching students how to record mathematical work
• Creating usable records of student thinking to support meta-cognition
• Setting-up an instructional task

Such candidate tasks of teaching can then be used to analyze other instances of instruction and in these analyses the tasks can be further tested and refined (as expressions of what is “there to be done” in teaching).

In addition, for the purpose of identifying mathematical knowledge for teaching, tasks of teaching need to be **layered at multiple levels**. That is, they need to be identified at a variety of levels, elaborated into collections of sub-tasks, and situated within the context of more general tasks. Layering helps to capture meaning and purpose and to support flexible use.

<table>
<thead>
<tr>
<th>Task</th>
<th>Rationale for the Task</th>
<th>Dimensions of the Task</th>
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</table>
| Sizing-up and selecting mathematics problems | Mathematics problems are the central means for capturing the subject matter and for designing instructional activities. | • Sizing-up what the problem is about.  
• Situating problems in a larger view of the curriculum.  
• Determining what is involved in doing the problem.  
• Evaluating the appropriateness of the problem for the curriculum or for the particular students. |
| Rescaling mathematics problems             | The selection and use of problems is central to teaching and being able to modify problems is central to fine tuning them for instruction. | • Developing a problem similar to an original problem, perhaps as practice or to check understanding.  
• Developing a problem similar to an original problem but somehow a bit less (or more) challenging.  
• Examining the relationship between problems in regard to how the problems are solved. |
| Establishing justifications for standard algorithms | Mathematical justification can provide reason and understanding for algorithms in ways that support meaningful use. | • Presenting clear and concise mathematical arguments.  
• Responding to “why” questions from students.  
• Anticipating what sense students are likely to make of different explanations.  
• Developing a sense of mathematical conviction and ability to use and reconstruct algorithms. |
| Sizing-up student productions              | The process of eliciting, sizing-up, and responding to verbal and written student productions is a key feedback loop informing teaching and learning. | • Having anticipated alternative approaches and ways of thinking about problems and ideas.  
• Deciding whether a student's non-standard approach would work in general.  
• Deciding whether to take up or to pass on. |
| Analyzing student errors                   | Student errors often represent the crux of what there is to be learned and reveal much about what to do to support learning. | • Identifying pattern in errors.  
• Figuring out what reasoning might have led to it.  
• Developing ideas about what might constructively challenge error and make it a site for fruitful mathematical work. |

Research that seeks to develop decompositions of the work of teaching requires data sources that provide easy access to: (i) instructional exchanges, (ii) teacher reasoning (about the teaching, either explicit or inferred), and (iii) additional layers of the context (such as curriculum materials or student work). It is important to remember that the goal of this research is not about what a particular teacher does, what teachers in the country currently do, or even what they should do. **It is about what a teacher might responsibly do given the situation and about what the**
range of things a teacher might responsibly do and reasons for doing them suggests about the nature of the work. Thus, it worth noting that a full range of practice — from highly competent to either novice or problematic teaching — could profitably be used to study the work of teaching.

**Coordinating perspectives in the analysis of practice.** A significant challenge for practice-based and disciplinary-grounded approaches to studying practice is the coordination of the mathematical perspective and the pedagogical perspective. Already visible in the descriptions given above is the notion that each perspective is formulated in ways that are attuned to the other perspective. Put another way, structures can be set up in advance to occasion, or pre-design, the coordination of the two perspectives. These pre-designed concepts or analytic tools serve to regulate and combine the two perspectives.

This proposal complements, yet is distinct from Ball and Lampert’s (1999) notions of “border crossers” or “ambassadors” and from Ball’s (1999, p. 30-31) focus on the dialogue and exchange among researchers coming from different perspectives. Instead, it emphasizes the systematic coordination of perspectives based on analytic distinctions vital to conceptualizing and studying MKT. The table below describes five such techniques. The first two extend the ideas already visible in the discussions above. The next two identify a set of focused concepts and a set of broad concepts that stand on their own. The last gives three research practices that matter for the coordination of perspectives.

<table>
<thead>
<tr>
<th>Five Techniques for Coordinating Perspectives in the Analysis of Practice</th>
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<tbody>
<tr>
<td><strong>Aligning Mathematical Perspective with the Pedagogical:</strong></td>
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<tr>
<td>Analytic features seen as important from a pedagogical perspective can be explored from a mathematical perspective and then disciplinary formulations can be used to examine practice.</td>
</tr>
<tr>
<td><strong>Rationale:</strong> This coordinates the two perspectives by pre-structuring pedagogical relevance into a disciplinary perspective. (Pre-structuring does not guarantee relevance, but it increases its likelihood.)</td>
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<tr>
<td><strong>Examples:</strong></td>
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<tr>
<td>• Disciplinary formulations of turn taking in explanation work can be used to analyze turn taking occurring in classrooms when explanations are being given.</td>
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<tr>
<td>• Disciplinary practices of personalizing mathematical work can be used to analyze the potential for personalizing mathematical work in teaching and learning.</td>
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<tr>
<td>• Disciplinary practices of private and public recording can be used to analyze the recording of mathematical work and the shaping of public space in teaching and learning.</td>
</tr>
<tr>
<td>• Disciplinary practices for doing mathematical work can be used to analyze routines in classrooms around the mathematical work (such as when to work alone, or when to think and when to write).</td>
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<tr>
<td><strong>Aligning Pedagogical Perspective with the Mathematical:</strong></td>
</tr>
<tr>
<td>Analytic features developed for or growing from a mathematical analysis can be explored from a pedagogical perspective and then a pedagogical formulation, such as a particular task of teaching pertinent to that feature, can be further developed and elaborated in an examination of practice.</td>
</tr>
<tr>
<td><strong>Rationale:</strong> This coordinates the two perspectives by pre-structuring disciplinary conceptions and sensibilities into formulations of the work of teaching and into a pedagogical perspective, which can then be used to study practice.</td>
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<tr>
<td><strong>Example:</strong> Generate tasks of teaching by adopting and adapting a disciplinary view of the decomposition of a problem into simpler problems and sub-problems and of the situating of a problem in a larger mathematical landscape of related problems, such as:</td>
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<tr>
<td>• Create an easier problem, or a sequence of problems, to scaffold students’ work.</td>
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<tr>
<td>• Generate an example to use at the outset to give students a sense of the larger problem.</td>
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<tr>
<td>• Prepare questions to ask of students to help them figure out what to do to make progress.</td>
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<tr>
<td>• Review the goal with students and develop ways to check that you have reached it.</td>
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<tr>
<td><strong>Focused Concepts for Regulating and Combining Perspectives:</strong></td>
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<tr>
<td>Analogous to the processes described above, conceptual tools can be developed that are attuned to both mathematical and pedagogical perspectives and precisely for the purpose of coordinating the perspectives. Often, these seem to stand on their own, independent of either individual perspective.</td>
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<tr>
<td><strong>Rationale:</strong> As described above, such conceptual tools pre-design coordination into the conceptual frames used to analyze practice.</td>
</tr>
<tr>
<td><strong>Examples:</strong></td>
</tr>
</tbody>
</table>
| • *Establishing a base of public knowledge* ("knowledge that can comfortably be assumed and used publicly without additional explanation," Ball & Bass, 2000b, p 202). The impetus of this notion lies in collaborative attempts to understand what is going on in instruction, considered jointly from the two perspectives, and it illuminates significant mathematical demands of maintaining standards for
mathematical reasoning, teaching students how to reason, and troubleshooting breakdowns in classroom talk.

- Remembering mathematical work (the activity of recalling problems, ideas, and explanations and recasting these into a story of the work being done within and by the class). This notion integrates a disciplinary concern for the mathematical integrity of what is said and done in the class and a pedagogical concern for activities that support memory, meta-cognition, and learning, and it illuminates significant mathematical demands of deciding what needs to be remembered, shaping how mathematical work is remembered, and considering the nature of what gets recorded, named, and listened to in the support of learning.

### Broad Concepts for Regulating and Combining Perspectives:

In practice-based and disciplinary-grounded analyses of practice, a few broad categories emerge as major points of intersection between mathematical and pedagogical perspectives.

**Rationale:** These domains, as with the concepts described above, are formulated in ways that pre-design mathematical and pedagogical perspectives.

**Four Broad Concepts with Examples:**
- Examining "student mathematical practices" in instruction.
  - (Such as: reading mathematical text and notation, giving explanations, making claims, and using representations.)
- Identifying "mathematical practices" critical to teaching.
  - (Such as: giving and evaluating explanations, choosing and using representations, or using mathematical language.)
- Examining "mathematical work" in instruction.
  - (Such as: organization of mathematical work, nature of collective mathematical work, or people doing mathematical work.)
- Identifying the "mathematical work/tasks of teaching."
  - (Such as: analyzing errors, re-scaling a problem, choosing an example, or responding to "why" questions.)

### Establishing a Research Culture for Coordinating Perspectives:

Features of the culture or practice of research can matter for the coordination of mathematical and pedagogical perspectives.

**Rationale:** These features increase resources for taking and holding fundamentally different perspectives and provide a basis for negotiating meaning and decisions across perspectives.

**Examples:**
- **Collective and flexible engagement.** When observing teaching and learning, multiple eyes and ears can be helpful. Beyond this, resources for taking and holding fundamentally different perspectives can be augmented by including people with expertise in, experience with, and commitment to three distinct arenas: (i) to the discipline of mathematics; (ii) to the practice of teaching; and (iii) to education research. In addition, it is important for people to learn to take and hold perspectives while also developing capacity to hear and engage with different perspectives. The work requires simultaneously seeking to understand another commentator's perspective on its own terms while also asking in what ways it might conflict with one's own.

- **Empirical grounding.** Routinely and explicitly supporting observations and claims in the particulars of a specific instance of practice provides common references around which to communicate, test ideas, and coordinate perspectives.

- **Systematical documentation.** Recording interpretations, ideas, and interactions opens these up for public review and focused response. In the same way that video and transcript allow people to pause the video and point to exact wordings and link ideas to empirical evidence, well-designed research documents allow researchers to slow down the research conversation, checking their interpretations of different perspectives against exact wordings, and probe, test, and refine their interpretations of the perspectives. Three principles are worth considering: (i) build documents on top of original records of practice; (ii) document mathematical and pedagogical analyses in forms that support reaction and commentaries by others and document the exchanges about practice among people with expertise and background both in mathematics and in pedagogy; and (iii) collectively represent and discuss ideas being developed by a group of researchers, such as for presentations or reports.

The notion of a **practice-based and discipline-grounded** approach to studying MKT takes seriously the twin commitments of being based in and relevant to practice and of being grounded in concerns and sensibilities of the discipline. The two underlying perspectives do not “talk” easily
or comfortably with one another. Approaches that rely on dialogue and exchange among researchers face a losing battle in keeping both perspectives squarely in play; discussions of mathematics are apt to lose relevance to teaching and discussions of teaching are apt to lose track of key mathematical issues. This memo proposes the use of analytic devices that pre-structure the coordination of the perspectives into concepts then used to analyze practice. For instance, the notion of a mathematical task of teaching combines the two perspectives and regulates their place in the analysis of practice. Such an analysis can be driven by strong mathematical impulses, but must frame those impulses in terms of the work of teaching; likewise, it can be driven by strong pedagogical impulses, but must then consider the mathematical dimensions of those impulses. Concepts that pre-structure the coordination of perspectives provide some assurance of informing the study of MKT. Indeed, it is this principle that lies at the heart of, and at the success of, the notion of mathematical knowledge for teaching and its predecessor, pedagogical content knowledge.

5.2 Developing a Practice-Based and Discipline-Grounded Approach to Studying MKT: Responding to Inquiries

This section includes three potential “users” of the memo presented above. One is an inquiry about implications of the work for the development of measures of mathematical knowledge for teaching. The second considers adapting the proposed design to the study of mathematical knowledge for teaching at the collegiate level. And the third explores implications for research on teaching not directly focused on identifying mathematical knowledge for teaching, but instead concerned with issues of teacher education.

5.2.1 From a Department of Education Officer: Designing Measures of MKT

Below is a fictitious email from a director of assessment in a stated department of education. (Because these emails and responses represent in some ways the actual text of my discussion of implications of this work, I include them as text rather than figures, but I use single spacing and a line along the left margin to indicate the scope of each email.)

Dear Dr. Thames,

Your name was given to me as someone who might help me think about developing measures of teacher content knowledge at scale for use in our state. I also read your technical memo, Developing a Practice-Based and Discipline-Grounded Approach to Studying MKT, which I found informative. It addressed research, not test development, but I wondered if we might adapt it to produce items to measure MKT. Do you think this is realistic? How would you propose doing so?

To give you a little background, our state legislature has allocated significant funds for the development of measures of teacher content knowledge to evaluate content-specific professional development initiatives and to provide system-level feedback to inform policy decisions. I have
been given responsibility here at the State DOE for developing a set of measures similar to those available from the LMT project but at greater scale and specific to high school. I need to have a set of pilot forms for Algebra I (3-5 forms of 30 items each?), Algebra II (3-5 forms of 30 items each?), and geometry (3-5 forms of 30 items each?) within a year, a set of operational forms at the end of two years, and the capacity for ongoing production by two staff members and a modest consultant budget after that.

For item and test development, I have six staff with recent experience writing items for teacher content knowledge based on examples from a variety of sources, including LMT, and I plan to contract another dozen consultants, mathematicians, master teachers, and specialists, to help write and review items. As resources, I have:

- 150 piloted, secure, low-stakes items that vary in quality but could be improved
- 400 draft items to use as seed ideas
- A detailed state curriculum guide and copies of common textbooks used in the state
- 1500 videos of high school lessons from a teacher mentoring program

My current plan is to have pairs, one mathematician and one practitioner, use the resources above to identify a mathematical issue and a pedagogical context as the basis for drafting an item. Items written by one team will be reviewed by other teams and then either revised by the reviewers or returned to the writers to revise.

Does this plan make sense? Do you have suggestions or recommendations?

Thank you,
Malcolm Chang
Director of Assessment
Division of Teacher Quality

Here is a fictitious reply.

Dear Director Chang,

Thank you for your interest and for investing in developing measures of MKT. Your proposed project seems ambitious, but realistic if it attends adequately to two specific issues: (i) that you help your staff develop notions and skills for holding mathematics and pedagogy together in ways that have integrity; and (ii) that you find ways to inject the work with new ideas of MKT to be measured. In short, I think your current plan is sensible, but needs some small modifications to turn it into a successful project that would make major contributions to the field. Unless it finds ways to address these two issues above, though, it runs a risk of struggling to produce enough items and of producing very mediocre items leading to weak instruments.

I have some particular things in mind for each of the two points above and see them as having important implications for the design of your assessment work. My proposal for your work is to make greater and more systematic use of the video archive to develop people’s skills in keeping both perspectives present and coordinated in the items and to inject new ideas into the mix. In addition, I recommend that you focus on identifying mathematical tasks of teaching as a way to capture and represent the MKT that gets identified. By engaging your staff in identifying mathematical tasks of teaching you will help structure their exchanges and work in ways that will
keep it on track for identifying and assessing MKT. To explain what I mean, I'll discuss the two points above in more detail.

Much of my memo argues for designs that structure the coordination of the perspectives beyond simply encouraging dialogue between people who come from the two perspectives. In your case, I think having people work in pairs is a move in the right direction, but just putting them together will not be sufficient. They will need to learn to work together and to coordinate their different perspectives and they will need to develop shared notions about what counts as MKT.

I suggest thinking about including occasions, especially at the start, where the group works together to analyze video of teaching and to identify mathematical tasks of teaching. You might engage the group in watching short segments of video (pausing about every 5 minutes to discuss) to discuss: (i) what strikes them mathematically about what is happening; (ii) what is happening pedagogically; and (iii) candidate mathematical tasks of teaching. For item writing, I think you will want to focus on the third of these, but starting with the first two and occasionally diving back into them will give members of the team a chance to get to know each other and learn to talk together and it will help people learn to track on and coordinate the two underlying perspectives needed for writing MKT items.

One approach to facilitating this collective work is to let a focus emerge in the discussion, but then to interject questions from the perspective that seems to have slipped from view. For instance, after having discussed a mathematical issue arising in the discussion of the video, one might ask how the issue is related to the work of teaching, why it is significant, or what it implies about what teachers need to do. This move can add another dimension to the discussion, can help to direct it in ways that are more likely to make it relevant to unearthing MKT, and can be used to resolve conflicts or simply bring people back to a common page. Discussions that veer into pedagogical issues can be punctuated by questions about the mathematical demands of such work. This same process applies in discussions of mathematical tasks of teaching: as people invest in one of the perspectives in ways that loose touch with the other perspective, this dynamic can be monitored, shifted, and narrated, possibly by a facilitator, but eventually by everyone. The goal of these sessions would be to develop skill in bringing the two perspectives together in ways that are important for identifying MKT and for writing items that address MKT.

You might want to bring someone in to do training along these lines in a few initial workshops.

The second concern I have is that relying on existing ideas for items (from existing items, curriculum specifications, or the knowledge that individuals bring to the work from their past experiences) will not be adequate. MKT is knowledge that still needs unearthing. It is knowledge not currently well known by any particular group. The idea of having people watch video either as a group or in pairs may seem too great an investment of time for producing items at scale, but I think it is essential, not just to the quality of items, but also essential to producing the number of items you have in mind. I think with some initial training focused on identifying mathematical tasks of teaching from the observation of practice, your staff will then be able to work in pairs to identify such tasks, to clarify both what the mathematics is and where in the work of teaching it arises, and to write items that address important mathematical issues and do so in ways that have integrity in relation to the work of teaching.

I hope these comments are helpful. This is very interesting work you are proposing and I would be happy to help in any way. Please keep me posted.

Sincerely,
Mark Thames
Dear Dr. Thames,

I am a faculty member in the education department at our state research university and am working with a colleague in the mathematics department to investigate "mathematical knowledge for teaching" for undergraduate abstract algebra courses. Our secondary education students take abstract algebra along with other students completing a major or minor in mathematics. Most of the students are successful in the courses required for the major up to this point in the program, but struggle to pass this course.

The university and the mathematics department are concerned about improving this course and are committed to long-term work that gets at core issues. We have one year of internal funding to videotape and analyze classes. Two of our colleagues have agreed to teach the course next Fall and again next Spring and their have classes videotaped and studied. In addition, we developing a proposal for three years of in-depth analysis and reporting with the primary aim of uncovering mathematical knowledge critical for teaching the course.

We are familiar with the research on mathematical knowledge for teaching at the University of Michigan and have read your technical memo: Developing a Practice-Based and Discipline-Grounded Approach to Studying MKT. I am writing with specific questions about the design of our studies.

1) The research at the University of Michigan addresses elementary school teaching. It seems as if the demands of secondary and collegiate teaching might be quite different, and perhaps much more mathematically substantial. Does your approach apply to the secondary or collegiate levels, or are there aspects that need to be altered?

2) Although you emphasize the need to combine mathematical and pedagogical perspectives, you seem to separate the mathematical analysis and the pedagogical analysis into two very different activities. Why? Do you recommend that we have two different research teams (for example, one composed of mathematicians to do the mathematical analysis and one composed of teachers to do the pedagogical analysis), or should we try to combine work into a joint mathematical-and-pedagogical analysis?

3) You propose a number of different ways to coordinate mathematical and pedagogical perspectives. Do you have recommendations about which of these would be most appropriate for our study?

4) My colleague and I have read your memo carefully, but are still unsure how to actually conduct these analyses. Do you have an extended report that could serve as a “how-to” manual?

Thank you for your help with these matters. Any other advice would be welcome as well.

Sincerely,
Wendy Horowitz

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Here is a fictitious reply.
Dear Wendy Horowitz,

Thank you for your interest in our work and for sharing your upcoming projects. The work you are launching seems fascinating, and I look forward to hearing about your progress over the coming months and years.

Your questions are very interesting, and I have responded to each below. I apologize for the length of my response, but your questions pointed to many issues central to the work that I felt couldn’t be addressed with a quick reply.

1. The research at the University of Michigan addresses elementary school teaching. It seems as if the demands of secondary and collegiate teaching might be quite different, and perhaps much more mathematically substantial. Does your approach apply to the secondary or collegiate levels, or are there aspects that need to be altered?

The nature of the differences in MKT at the elementary, secondary, and collegiate levels (and implications for how it is conceptualized and studied) is an open question (which is one of the reasons your work is so interesting!). However, throughout our work, I have been increasingly struck by the applicability of MKT across different contexts. Therefore, although I have not studied this issue and thus have no solid evidence to back it up, I am inclined to think that the significant differences between mathematics for secondary and collegiate teaching and mathematics for elementary teaching are nearly only about differences in the topics being taught. Thus, the short answer to your first question is that I suspect the approach I describe in the memo applies equally well to collegiate teaching. With that said, let me consider some potential differences that might be worth keeping in mind as you conduct your study.

In many ways, the identification of MKT can be thought of as a practical act, or even a political one. By this I mean that what gets pursued, developed, and emphasized as MKT is significantly determined by what seems strategic for the improvement of teaching and learning, and this is related to the specific circumstances and current needs in a particular situation. Two issues for your study occur to me in this regard.

First, collegiate teaching may make different assumptions about students’ responsibilities for learning. A central goal in elementary school is to teach students how to “do school,” as well as how to study and direct their own learning. For instance, in first grade, a central part of the curriculum is often teaching students how to do “seatwork.” Similar examples exist up through the grades, for example, students learning to conduct a library search or complete a larger project. Examples specific to mathematics also arise: students learn what is expected of them in math class, how to check their work, and the form that answers are to take. My sense is that one of the reasons a lecture format is more likely to occur in college than in elementary school is that it seems reasonable to assume college students are more able to work independently than are first graders. More generally, the mathematical work of orchestrating mathematical talk, of teaching students to reason mathematically, or of scaffolding student work might be quite different in the two contexts, which would then have implications for MKT. Thus, although I suspect that the main differences in MKT across these two realms are related to topical differences, there may in fact be differences in emphasis due to differences in the work of teaching.

Related to this may be issues of the normative role of the discipline in shaping the character both of curriculum and of teaching in an abstract algebra course. Disciplinary sensibilities might play strong roles in the selection of a textbook, in the attention given to student thinking within the context of instruction, and in types of activities in which students are engaged. This has implications for the work of teaching both arising in and relevant to the abstract algebra course.
Thus, the normative role of the discipline and the distinctive character of instruction at the undergraduate level, and in the abstract algebra course in particular, might be indicative of important disjunctions between the work of teaching and between the mathematical demands of teaching in the two contexts of elementary school and the university.

Having made these comments, though, I want to caution against assumptions that work of collegiate and elementary teaching is necessarily so different. For example, it may be that the breakdowns you describe in the abstract algebra course are in part due to inadequate attention to supporting students to work independently or due to ineffective design and use of coursework for learning. Furthermore, there may be reasons why this comes to a head in the abstract algebra course. Thus, it is possible that prominent aspects of the work of teaching elementary school might be helpful for thinking about what may be missing from the teaching of the abstract algebra course, perhaps some of the very differences I mentioned above. For instance, students may need to be taught more about how to read a mathematics textbook or common instructional practice in the course may need improved designs for attending to student thinking. Some of these important aspects of the work of teaching may be missing from existing practice in this course and/or across the departmental offerings more generally.

In the technical memo, I mentioned that the framework of “work of teaching” as a lens for examining practice is not about work as described by modal practice, expert teaching, or specific kinds of teaching. Instead, it is a conception of teaching developed in reference to notions of the fundamental character of designing and managing instructional interactions — in other words, as tested and refined in reference to the instructional triangle. This is perhaps an extreme position, but one I feel is worth pushing forward in the conversation. Clearly, context matters. The “work of teaching” that makes substantial use of “hands-on” materials and the mathematical demands of such work are different from the “work of teaching” that is designed as a lecture series for 500 students. In the end, my inclination is to focus on more generally applicable formulations of the work of teaching as high leverage for improving professional practice and to leave method-specific aspects of the work of teaching to later or more local development. Thus, with your move to adapt our approach to the context of an abstract algebra course, I think the approach is fully appropriate, but may require significant work that diligently questions and developed legitimate and useful formulations of the work of teaching that receives attention. Our elementary school research may provide important guidance and may require important additions.

Second, the very fact that something seems to be going awry in this course suggests that there may be specific pedagogical issues that are not being adequately addressed and need more focused attention in a study of mathematical knowledge for teaching abstract algebra. For instance, perhaps there is something distinctive about the content that raises specific aspects of its teaching to the fore. The fact that students seem to be struggling in the course suggests that paying particularly close attention to students and what is happening for them may need to be brought more explicitly to the surface in your study than in other studies of MKT. The study of students and of student learning is central to the work of teaching, and thus to any study of MKT, but given your situation, you might want to focus more attention on tasks of teaching that address this key domain of the work of teaching.

With this in mind, why did you decide to address the problem you face as a study of MKT rather than as a study of students and student learning? Have you thought about designing a study with two strands: one focused on the work of teaching and MKT, and another focused on students and student learning? In asking these questions, I don’t mean to discourage your emphasis on MKT; your impetus here seems right. However, given the situation, a study of students, situated into a design that addresses teaching and its knowledge demands, might be key.

2. Although you emphasize the need to combine mathematical and pedagogical perspectives, you seem to separate the mathematical analysis and the pedagogical analysis into two very different activities. Why? Do you recommend that we have two different research teams (for example, one composed of mathematicians to do the mathematical analysis and one composed
of teachers to do the pedagogical analysis), or should we try to combine work into a joint mathematical-and-pedagogical analysis?

This is an important question that I have not explicitly discussed in the memo. Although I have strong inclinations here, I am not confident that I have thought this through as carefully as I should, so please take my response as a tentative answer.

Part of the reason I talk about coordinating perspectives but not about a coordinated perspective is that I think it may be important to maintain the two perspectives as distinct. It is also important to distinguish a particular perspective from the people who might take such a perspective. In other words, I don’t mean to attribute a perspective as something intrinsic to a person or a group of people.

In the memo, I talk about two key perspectives for the study of practice: a mathematical perspective and a pedagogical perspective. By mathematical perspective, I mean a perspective that is grounded specifically in the discipline, as opposed to one that takes “mathematical” more broadly, such as including non-disciplinary aspects of school mathematics, other mathematically intensive professions, or mathematical activities or objects common in the larger society. All of my work has focused on the subject area of mathematics, but I often use the word discipline to suggest that the ideas may be relevant across content areas and because I sometimes want to refer to things uniquely associated with the discipline of mathematics as opposed to the potentially broader interpretation of the word mathematical.

By pedagogical perspective, I mean a perspective that focuses on teaching and emerges from the study of teaching. A pedagogical perspective can used to analyze practice, as well as to support the improvement of teaching and learning. For me, an important community involved in developing and taking a pedagogical perspective are those engaged in research on teaching. Although this is not a well-defined community with a less well-defined perspective (nor does not have a recognized sole claim in defining such a perspective), it is an important one for this perspective. As you mentioned, another community to draw from for this perspective is teachers. This has certain appeal, but also certain limitations. Not to be disparaging of teachers, but if our aim is to improve teaching, then teachers may not always be the best source of expertise. The idea of drawing on expert teachers is problematic as well, as there is no consensus about who is expert and no reason to assume that those who are good at teaching are able to express what it is that they do. On the other hand, taking Dewey’s notion that teaching is at its core about studying students, studying subject matter, and studying how these come together, teaching is deeply about studying teaching. Therefore, teachers can be seen as involved in a kind of ongoing research on teaching and might be an important community involved in developing and taking a pedagogical perspective. The pedagogical perspective I have in mind considers and investigates teaching in its own right, is under development as a perspective, and its current best versions reside unevenly across communities attending to teaching, including researchers and practitioners.

My sense is that both perspectives have much to offer to discussions about teaching and learning. This issue has a long history in the United States and beyond, harking back to debates in the late 1800s and the rise of schools of education nationally. The debates are fundamentally about the role of universities in public education, the role of disciplinary departments in the specification, preparation, and evaluation of schools and of teaching, and the role, more broadly, of subject matter in pedagogy. Without diving into these issues, let me just say that I think teacher content knowledge matters for teaching, that mathematics departments have an important role to play in the education of teachers, and that mathematicians have an important role to play in better understanding teaching and learning. This is my bias anyway.

More specific to the issue of studying practice to identify MKT is that a serious challenge, at least at this stage in the work, is understanding and holding onto each of the perspectives, especially in the face of trying, at the same time, to bring the other perspective into play. Thus, my inclination
is to say that developing two distinct lines of work—a mathematical analysis of practice and a pedagogical analysis of practice—may be important because it creates space for developing each perspective in its own right and because there may be something valuable in letting each perspective pursue its inclinations without being overly constrained by the other. However, these distinct lines of work need also to talk with each other.

Put another way, the approach laid out in the memo is about maintaining a balance between valuing each perspective as having something important to contribute to the analysis of practice, while also structuring ways that allow the two to communicate with each other. Engaging in two distinct analyses, one mathematical and the other pedagogical, provides some confidence that new and important mathematical insights will surface and that a fuller articulation of the work of mathematics teaching will emerge. The pursuit of MKT is in part about unearthing mathematical knowledge that matters for teaching but that currently is not clearly recognized, examined, or taught, and its pursuit depends on a reasonable understanding of the work of teaching. The two distinct analyses seem like important, distinct ingredients in indentifying MKT.

At a practical level, this suggests maintaining at least some vision and space for a mathematical analysis as distinct from a pedagogical analysis and vice versa, but not in separating them to the extent that they do not have opportunities to bump up against each other in significant ways. If you have two non-overlapping research teams, they might grow independently and not learn how to talk across the divide between perspectives. If you combine the work into a single mathematical-and-pedagogical analysis, you may find that one or both of the perspectives flounders, both in its development as a perspective to use to analyze practice and in the contributions it makes to the study of MKT. In our work at the University of Michigan, we have mostly had a single team (with occasional subgroups and transitions) and this group, in its different configurations, has taken on different activities, both at a point in time and over time, with different emphasis with respect to engagement in a mathematical analysis and a pedagogical analysis.

In the memo, I try to argue that, in addition to having dialogue and exchange among researchers coming from the two perspectives, it is important to structure sensibilities of each perspective into the other and to structure the exchange between researchers by focusing on certain analytic issues that pre-structure a place and role for each of the perspectives in the exchange. The goal here is not just to have people with different perspectives get along or agree, but to emphasize that MKT is a kind of coordinated knowledge. In other words, coordination is important for engaging in the study of practice, but it is also important for expressing the lessons learned from such study.

Having said all of this about maintaining distinct lines of work associated with the two perspectives, I feel compelled to add that I am somewhat unsure about the need to maintain their separation. In the memo, I was trying to put forward an image of two separate types of analysis, along with proposals for sensitizing and engaging each with the other. However, one might read the memo as suggesting the need to build a new, integrated perspective with new concepts that fully integrates the two underlying perspectives. Perhaps at this point in time, because such a perspective does not adequately exist in its own right, we need to leverage the other two, engage them with each other in whatever ways we can, and use the results from this work to build a unified perspective that is up to the task of an integrated analysis of practice.

[I am reminded here of a claim made by W.E.B. DuBois (1903) in The Souls of Black Folks: Essays and Sketches, that in the world as it should be, becoming a better black man (as he learned and grew in the black community) should contribute to his becoming a better American (as he learned and grew in the “American” society). These represented two parts of his identity and his experience. He viewed participation in such communities as vehicles for becoming human — different windows on the same world, offering different tools for insight — and as such, although initially or at times working in different ways or providing conflicting views of reality, that they ought to converge in a
greater vision being human. Of course, in the end, his sadness was that his experience of these two parts of his life failed to converge in this way.

I do not mean to say that the issue DuBois writes about is analogous to the use of mathematical and pedagogical perspectives in the analysis of practice. I only mean to share, at a more personal level, a question I have about larger issues that might bear on this question about forming a unified perspective. Although I am inclined to lean toward maintaining the two distinct perspectives, while continuing to build the bridges that allow them to talk to one another, I cannot help but wonder both whether ideally, over time, these two perspectives might converge in a unified perspective, perhaps distinctive of mathematics education, and whether there might be some deep roots to a schism between the two. These seem like open questions.

Returning to the specifics of your study, I recommend two options. A first option is that you have a single research team that engages in collective analyses of practice, but that you analytically structure different lines of work. These different analyses might be conducted at different times, moving back and forth between the two perspectives, or they might be conducted in parallel (e.g., a single project meeting fluidly moving between lines of work, yet with some careful attention and explicit narrating of this movement either in the meeting or in meeting notes). A second, or complementary, option is to have separate teams: one for the mathematical analysis and another for the pedagogical analysis. However, it would be important that some people participate in both groups and that you occasionally bring the two teams together to inform each other about the distinct lines of work being done and to engage in some collective analysis of practice to promote the cross-fertilization of ideas and perspectives.

[The mention above to structuring different lines of work seems to assume the two primary types of analyses that have been discussed, mathematical and pedagogical ones, but they also might involve ones focused on one or more of the coordinating concepts, such as establishing and using a base of common knowledge, student mathematical practices, or the composition of mathematical work. Across the project, it seems to me that you will want a way to keep track of and further develop mathematical insights that surface as well as pedagogical ones. My inclination is to do these in the context of perspective specific work, but they might also be folded into the study of one of these coordinating concepts.]

3. You propose a number of different ways to coordinate mathematical and pedagogical perspectives. Do you have recommendations about which of these would be most appropriate for our study?

I have provided several images above of how you might coordinate mathematical and pedagogical perspectives in your work, but I will try to make some more specific recommendations here. I imagine the researchers involved in your study will be quite different in their backgrounds, some with very limited experience with education research and some with limited perspective on the mathematical content of the course. Given this, I suggest three strands of analysis of records of practice:

- A broadly conceived mathematical analysis of practice.
- A broadly conceived pedagogical analysis of practice.
- The identification of mathematical tasks of teaching arising from the first two strands.

Initially, it seems that a central agenda will need to be the development of the group and people’s ongoing learning of what is involved in this kind of work. Engaging the group in watching short segments of video (pausing about every 5 minutes to discuss) and discussing what strikes them mathematically about what is happening might help the group learn to talk about mathematics together and might lead over time to a sense of key mathematical issues that arise in the teaching of the course. One approach to facilitating these discussions is to maintain a primary
focus on having these be mathematically interesting, engaging discussions, but periodically return them to teaching — how they are related to the work of teaching the course, why they are significant, and what they imply about what instructors need to do. This move can add another dimension to the discussion, can help to direct it in ways that are more likely to make it relevant, and can be used to resolve conflicts or simply bring people back to a common page.

Second, you might again engage the group in watching short video segments, but this time discuss what strikes them about what is happening pedagogically and what this suggests for the work of teaching the course. A central goal of these discussions would be to identify work of teaching more generally, as well specific to teaching this course, both in the sense of what seems central to the work being done and what seems to be undone in ways that might be leading to breakdowns in instruction. These discussions can begin to develop understanding across the group about what is involved in teaching and how to attend to and talk about teaching. Again, these discussions can be punctuated by questions about mathematical dimensions of the work of teaching being identified and about the mathematical demands of such work.

The third strand, focusing on identifying mathematical tasks of teaching, might be introduced after the other two strands begin to get established. Whereas the first two strands give team members a chance to get to know each other and learn how to talk together, introducing this third strand can focus the group on bringing the two perspectives together analytically in ways that will be important for identifying MKT. Indeed, because the goal of your project is to identify MKT for an abstract algebra course, my proposal includes a focus on mathematical tasks of teaching as a means of capturing the knowledge the group is generating in a form that is well suited for representing MKT. It does not need to be the only form you use, but it might help the group focus its efforts and accumulate a body of “results.”

My proposal here is mostly focused on a way to get started, to get a feel for team members, individually and as a group, and to build capacity for doing the work. After roughly a year of this work, you could then make decisions about specific foci to pursue and ways to pursue them.

Your question about how to go about coordinating perspectives also makes me think about specific issues related to coordinating perspectives around the teaching and learning of abstract algebra, issues which might also provide important motivation for your study of MKT related to the course. One way to view the problem with the current abstract algebra course is to see it as a problem simply of getting better at communicating ideas in the field to a broader public. The science community seems better at communicating sophisticated scientific ideas to non-technical audiences: it engages people in substantive ideas without requiring that they spend hours in the laboratory or that they plow through a mountain of empirical evidence. The mathematics community could stand to improve in the way it conveys key ideas to people who are not going to become mathematicians, in particular significant, unfamiliar ideas. Part of the problem in teaching abstract algebra may be about a need to get better at such communication. For some students, abstract algebra is the first undergraduate course in which problems are not routinely about following templates for solving a class of problems. Instead, it is a course where students have to deal with new levels of abstraction and must routinely generate formal proofs, and of a distinctive type that they are not particularly familiar. In this sense, it is not altogether unlike high school algebra, where the shift from arithmetic to algebra constitutes a new way of thinking. The shift from the closely related worlds of college algebra, calculus, and analysis to the new and unusual area of abstract algebra may constitute a similar shift — one that causes students excessive difficulty. The point, then, is that major shifts in thinking such as these might highlight the need for better ways of communicating substantial collections of ideas to a broader, non-technical audience. In this sense, the teaching and learning of abstract algebra provides a context for the issue of public communication of mathematical ideas.

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11 The ideas in the next few paragraphs grew from a conversation with my committee, whose ideas and suggestions I have used here. Responsibility, of course, especially for the particular expression of ideas here, is my own.
If any of this is right, then my suggestion here is that the coordination of disciplinary and pedagogical perspectives might be well suited for making important contributions to the development of better ways of communicating key ideas, and ultimately, improving the teaching and learning of abstract algebra. Part of my suggestion is that beyond coordinating a mathematical perspective and a lay perspective, the particular coordination of mathematical and pedagogical perspectives in the context of a study of the teaching and learning of abstract algebra might be distinctly valuable in creating better ways of communicating these ideas inside and outside the classroom. Taking this thinking one step further, it might be the case that the notion of rendering sophisticated ideas for broader audiences constitutes an important form of mathematical knowledge for teaching. In other words, not only does the teaching and learning of abstract algebra provide a context for the issue of communicating mathematical ideas, the opposite may also hold — exploration of the communication of mathematical ideas in the sense implied here may be an important dimension of the teaching and learning of abstract algebra with important knowledge demands that your study might unearth. So, a secondary question to ask is about what we, or teachers, need to know, and need to know and be able to do mathematically, in order to communicate sophisticated mathematical ideas to broader, non-technical audiences.

Thus, one reason for studying MKT in the context of the abstract algebra course might be that it involves a set of ideas for which the discipline has not developed good ways of communicating with a non-technical audience and that a study of classroom teaching and learning that coordinates these two key perspectives might be effective in developing better ways of communicating central ideas of the course. With this, I do not mean to suggest that you change the focus or design of your study, but to suggest that thinking about the situation in this light might afford an additional way to engage in discussions about what is happening in classrooms and might help you see important additional products emerging from your discussions about teaching and learning.

4. My colleague and I have read your memo carefully, but are still unsure how to actually conduct these analyses. Do you have an extended report that could serve as a “how-to” manual?

My dissertation work and the technical memo represent an attempt to consider the design of research that studies practice with the purpose of identifying MKT. I have not tried to specify units of analysis for the work or specific procedures for carrying it out; I have instead provided a set of ideas in which these can be considered. Perhaps my comments above provide more of a sense of how you might proceed with your analyses, but the development of methods is still unfinished work. As you conduct your analyses, I encourage you to consider including the development of methods as one of the goals of your project and to document the methods you develop and use.

I’m happy to discuss these matters further if you would like.

Best,
Mark

5.2.3 From a Teacher Education Researcher: Analyzing Teaching and Learning

Below is a fictitious email from a mathematics education researcher engaged with a mathematician in the study of mathematical knowledge for teaching an abstract algebra course.
Dear Dr. Thames,

I am a graduate student in teacher education and am interested in research on teaching. I'm not in mathematics education, but my advisor gave me a copy of your memo, *Developing a Practice-Based and Discipline-Grounded Approach to Studying MKT*. I am interested in analyzing practice across content areas (most likely math, language arts, and social studies) in an attempt to identify high-leverage teaching practices to be included in an elementary teacher education curriculum. In particular, I have been synthesizing the literature with regard to proposed high-leverage practices for teacher education and want to test these against the demands of novice teaching. I'm planning to analyze videotape of four second-year teachers: the first two weeks of the school year, one week from the middle of the year, and one week from a month before the end of the year, working both inductively from the data to identify candidate tasks and using the data to test existing candidates. I'm still sorting out the exact formulation of my research questions and the design of the study, but it will be something along these lines.

My understanding is that you developed your guidelines for analyzing pedagogy from your dissertation work, which was about the analysis of practice for the specific purpose of identifying MKT. Given that my purpose is different, what should I take from your study? I want to keep content knowledge in the picture, but will not be able to delve deeply into each content area. Do you have suggestions for ways in which I might handle this in my study?

Thank you,
Hassan Cole

Here is a fictitious reply.

Dear Hassan,

Thank you for your interest in my memo. My degree is also in teacher education, and the work you propose is of great interest to me. You raise some interesting questions about the relevance of my work to the study of teaching more broadly, about practical issues regarding implications beyond the study of teacher content knowledge, and about limitations of the design I have proposed. I'll offer brief comments on each of these.

*Relevance of my work to the study of teaching more broadly.* The study of MKT, as conceived in our work at the University of Michigan, requires getting a reasonable handle on the work of teaching. I have made arguments for a particular conception of the work of teaching and for particular approaches for studying the work of teaching. In particular, I argue that the study of the work of teaching needs to be carried out in reference to designing and managing the interactions of instruction. Doing so helps discern the ways in which tasks need to be specific and general and helps assure that tasks are high leverage, in the sense that they have broad applicability across particulars of teaching without ignoring essential particulars. I think these arguments apply to identifying the work of teaching beyond the study of MKT.

In addition, I argue that tasks need to be framed with sensitivity to multiple, instrumental goals of instruction; identified at a variety of levels; elaborated into collections of sub-tasks; and situated within the context of more general tasks. My initial hypothesis was that the layering of tasks was important for supporting people (mathematicians in particular) in mathematical analysis of practice, as it can be easy to lose track of pedagogical concerns. I have become less convinced of this. Increasingly, it seems that the layering of tasks plays an important role in representing
teaching. I have not explored this carefully, but it intrigues me. For instance, in our teacher education courses, when we work with students on a particular task of teaching, our investigations typically involve situating a particular task within a larger domain of work and then decomposing it into constituent tasks. It may be that this is an important part of rendering the task meaningful, learnable, and usable. I would need to go back and look more carefully, but it seems that part of the power of the representation of teaching that Lampert (2001) offers in *Teaching Problems and the Problems of Teaching* may be in the ways she, in effect, creates layered tasks. These musing suggest that much of the design for a pedagogical analysis might have applicability beyond the study of MKT.

*Practical issues and implications beyond a focus on teacher content knowledge.* Your project is about identifying high-leverage practices for teacher education. You also want to study subject-specific features of teaching, without delving deeply into each subject area, and you ask about ways to handle this in your study. My first suggestion is that you consider using coordinating concepts that keep both the content issues and pedagogical issues immediately in play. I think that using the notion of content tasks of teaching in a systematic way in your analysis of practice would be a powerful way to keep content in the picture as you analyze teaching. Let me know if you would like more detail on what I mean by content tasks of teaching or how you could use them to analyze practice. Other techniques seem beyond the scope of the work you would be in a position to take on within your study.

There is one additional point I would like to raise here. It does not bear so directly on your proposed study, but you might find it interesting. This is not an issue I have addressed in my work, but it cropped up from time to time as I was working. In the memo, I talk about the importance of coordinating perspectives in the analysis of practice. It sees to me that an important content-knowledge issue for teaching is, itself, the coordination of perspectives. In other words, I wonder if the coordination of perspectives might itself be a high-leverage practice. It might be interesting to analyze practice with an eye on occasions when, and ways in which, the work of teaching is about coordinating perspectives. This is just a thought, though.

*Limitations of my proposed design.* I find your proposed ideas very interesting and realize you are still sorting out the exact set of questions and research design, but one small concern arises for me that you might want to keep in mind as you finalize your study. You say you want to test proposed high-leverage practices against the demands of novice teaching. This is a nice idea. However, the research design I proposed in my memo is for generating new ideas about potential work of teaching and the content knowledge demands associated with that work. Asking about the relative leverage of different tasks for beginning teachers is a different goal and may require a different sort of design and methods. You might be able to adapt or add to my proposals, but you may find serious limitations with the ideas I have proposed depending on your final research questions and on whether you can figure out ways to adapt the design I have proposed.

I hope these comments are helpful. Please keep me informed about your work as it progresses. And let me know if there is any way I can be helpful.

Regards,
Mark

Looking back over these three inquiries and my responses to them, one issue that stands out is that in all three cases I emphasized mathematical tasks of teaching in my response. Reflecting on this, I realize that this is not because the other techniques are unimportant. Instead,
it is that all three of these cases involve people with little prior experience with this work and who are engaged in adapting the design beyond the context of studying mathematical knowledge for teaching elementary school. The other techniques seem to me more pertinent to continuing and deepening the study of mathematical knowledge for teaching elementary school. As such, they are perhaps more for internal use in future work at the University of Michigan. This conclusion has focused on considering implications, limits, and communication for a broader audience.
BIBLIOGRAPHY


