FLAT FORMS IN BANACH SPACES

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2009

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To Juha Heinonen.

ACKNOWLEDGEMENTS

I am grateful to the many individuals who have contributed to my mathematical education and to this work. Thanks to the University of Michigan mathematics department for fostering a stimulating research environment and to Tara McQueen, Nathalie Fiveland, Warren Noone, Bert Ortiz, and Anjie Ridenour for their professionalism, assistance, and patience with my administrative questions.

I am grateful for the generous fellowship support I received from the National Physical Science Consortium from 2003-2009.

I warmly thank Robert Hardt and the mathematics department at Rice University for hosting me as a visiting graduate student for the winter semester of 2007.

Over the past few years, I have had many productive discussions with the following individuals, whom I thank for their comments, interest, and suggestions: Thierry De Pauw, Kai Rajala, Leonid Kovalev, Stefan Wenger, Steven Keith, Urs Lang, Laurent Moonens, Karen Smith, Lesley Ward, Marshall Williams, Jasun Gong, Kevin Wildrick, Ben Weiss, Paul Kessenich, Ryan Kinser, Jesse Metcalf-Burton, Sue Sierra, Hester Graves, Wei Zhu, Eric Samansky, and Ryan Dunning.

I thank the members of my committee for their time and expertise: Alexander Barvinok, Mario Bonk, Pekka Pankka, Ralf Spatzier, and Martin Strauss.

Pekka Pankka's careful reading, insightful comments, and many hours of discussion greatly improved this document. Kiitos!

I am much indebted to my advisor, Mario Bonk, for his generosity with both time and mathematical knowledge. I am impressed by the clarity, precision, insight, and enthusiasm with which he approaches mathematics, and I thank him for sharing these with me. I am also grateful for his support and encouragement.

Finally, I wish to acknowledge my late advisor, Juha Heinonen, whose guidance and vision were instrumental in this project. Juha was an exemplary mathematician, teacher, and mentor, and is sorely missed. He remains an inspiration for my future mathematical endeavors.

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CHAPTER I

Introduction

For Euclidean spaces, Whitney [Whi57] developed geometric integration theory to integrate "functions" over "sets" in such a way that the integral depends on how the set is positioned in $\mathbb{R}^n$. In this theory, the sets over which one integrates are flat chains (limits of polyhedral chains under the so-called flat norm), and the functions one integrates over these chains are flat forms. Flat forms are L^∞-differential forms with L^∞-exterior derivatives. The flat norm of such a form is the maximum of the L^∞-norm of the form and that of its derivative. Since flat chains can be thought of as L^1-functions on $\mathbb{R}^n$, it is natural to consider flat forms as dual to flat chains, where the action of a form on a chain is given by integration. Moreover, this duality respects the norms: the space of flat forms endowed with the flat norm is isometric to the space of flat cochains with the norm dual to the flat norm on chains. This fundamental result was proven by Wolfe in 1948 (see [Whi57]).

In [Ada08], Adams extended the theory of flat chains from $\mathbb{R}^n$ to Banach spaces. In this thesis, we define *flat partial forms* in a Banach space, and prove that the space of these forms is the dual to Adams's space of flat chains. We consider only Banach spaces over $\mathbb{R}$.

A similar duality result is true in the case of sharp forms (forms with Lipschitz continuous coefficient functions) and sharp cochains (linear functionals on so-called sharp chains); namely, the space of sharp forms is dual to the space of sharp chains. The proof of this fact in the Euclidean case can be found in Whitney [Whi57]. In [Nol86], Noltie extended this result to Banach spaces.

The current paper is part of the recent efforts to generalize geometric measure theory beyond Euclidean space. In particular, currents in metric spaces have been the subject of recent investigation. Roughly speaking, currents in Euclidean space are linear functionals on differential forms. In 2000, Ambrosio and Kirchheim [AK00] developed a theory of currents in metric spaces as linear functionals on "tuples" of Lipschitz functions. Following this, Lang [Lan] developed a variant of this theory, and Wenger [Wen05], [Wen07] generalized the isoperimetric inequality found in [AK00] and studied convergence properties of metric currents.

Generalizations of chains, which can be regarded as "pre-dual" to forms, have also been investigated. In addition to Adams's paper on chains in Banach spaces, De Pauw and Hardt have defined and studied rectifiable and flat chains in metric spaces (see [DPH]).

Our goal is to identify the dual space to the space of flat chains with a space of suitably defined differential forms. The key idea in our definition of a partial form is the following. Instead of defining a k-form as a function that maps points in a Banach space V to k-covectors (alternating k-linear maps from $[V]^k$ to $\mathbb{R}$) as in [Car70], we view a form as a function from a subset of the product space $V \times [V]^k$ to $\mathbb{R}$. We require that such a function, or *measurable partial k-form*, be measurable when restricted to any horizontal or vertical slice in the product space, and that it satisfies some multi-linearity conditions on finite-dimensional affine subspaces of V (see Definition IV.4). This relaxation of the standard definition allows us to consider forms which may not be smooth and to define an equivalence relation between partial forms.

In this work we consider only global measurable partial forms. Using Fubini's theorem, one can show that a measurable partial form on a finite-dimensional space is also a classical measurable differential form on that space (see Section 4.2).

We extend many standard geometric concepts to this setting: we define the wedge product of two partial forms (see 4.6) and an exterior differentiation operator d on (locally integrable) partial k-forms mapping partial k-forms to partial $(k+1)$-forms. Our exterior d is defined in a weak sense so that Stokes's Theorem holds when the forms act by integration on polyhedral chains (see 4.5). Finally, we define an L^∞-type norm on partial k-forms by

$$\|F\|_\infty := \sup_{W^k \subseteq V} \left\{ \operatorname*{ess\,sup}_{p \in W^k} \{|F_p(\nu_{W^k})|\} \right\},$$

where the supremum is taken over all k-dimensional affine subspaces W^k of the Banach space V and ν_{W^k} denotes the unit mass* k-vector (see 2.3) whose components span W^k. Equipped with this norm, we define the flat norm on the space of partial forms as in the Euclidean setting to be $\|F\|_\flat := \max\{\|F\|_\infty, \|dF\|_\infty\}$. The space of flat partial forms consists of those partial forms with finite flat norm. Here, even though our L^∞-type norm does not agree with the L^∞-norm in $\mathbb{R}^n$, the resulting space of flat partial forms in $\mathbb{R}^n$ agrees with classical flat forms in $\mathbb{R}^n$.

Our main result is that the space of flat partial forms is the dual space to Adams's space of flat chains.

Theorem I.1. *The space of flat partial k-forms endowed with the flat norm is isometric to the space of bounded linear functionals on flat k-chains with the norm dual to the flat norm on chains.*

A priori, a partial form need not be defined for as many points in the product space $V \times [V]^k$ as a classical differential form. However, using Theorem I.1 and techniques from [BL00], we prove in Chapter VII that a *flat* partial form in a separable Banach space V is defined for "many" points in the product space, and hence is not too far from a classical form (i.e., a function from V to $\mathrm{Hom}(\Lambda_k V, \mathbb{R})$).

Theorem I.2. *Let F be a flat partial form in a separable Banach space V. There exists an Aronszajn-null set N in V so that for all $p \in V \setminus N$ and all $\nu \in \Lambda_k V$, the point (p, ν) is in the domain of F.*

In the Euclidean setting, flat chains and forms are invariant under bi-Lipschitz maps. Our final result (see Chapter VIII) is in this direction.

Theorem I.3. *In a Banach space, flat partial k-forms pull back to flat partial k-forms under Lipschitz maps.*

To prove Theorem I.3, we show that polyhedral chains push forward to flat chains under Lipshitz maps.

A long-term goal of this project is to apply it to the question of the bi-Lipschitz embeddability of a general metric space in $\mathbb{R}^k$ (see [HS02], [HK], and [HPR]); Theorem I.3 indicates that flat forms are stable under Lipschitz mappings and hence are natural objects to consider in this study.

Another interesting open question is the relationship between partial differential forms and the metric chains and currents defined in [AK00], [Lan] and [DPH] in a Banach space.

CHAPTER II

Preliminaries

2.1 The spaces $\Lambda_k V$ and $\Lambda^k V$

Given a Banach space V, we denote the k-fold product of V by $[V]^k = V \times \cdots \times V$. For a set $X \subset V$, $\chi_X \colon V \to \{0,1\}$ is the characteristic function of X, defined by $\chi_X(x) = 1$ for $x \in X$ and $\chi_X(x) = 0$ otherwise. Let $C([V]^k)$ denote the free vector space over $[V]^k$, i.e., $C([V]^k) := \operatorname{span}_{\mathbb{R}}\{\chi_{\{(v_1,\ldots,v_k)\}} : (v_1,\ldots,v_k) \in [V]^k\}$. The vectors

$$(y_1, y_2, \ldots, y_k) - \lambda(y_1, \ldots, x_1^i, \ldots, y_k) - \mu(y_1, \ldots, x_2^i, \ldots, y_k),$$

where $y_i = \lambda x_1^i + \mu x_2^i$ for $i = 1, \ldots, k$, together with the vectors

$$(y_1, y_2, \ldots, y_k)$$

where $y_i = y_j$ for some $i \neq j$ generate a subspace $G \subset C([V]^k)$. The space of k-vectors on a Banach space V, denoted $\Lambda_k V$, is the quotient space $C([V]^k)/G$. Denote the equivalence class of the k-tuple $(v_1, \ldots, v_k)$ in this quotient space by $v_1 \wedge \cdots \wedge v_k$. We define a function $\mu \colon [V]^k \to \Lambda_k V$ that maps a k-tuple in $[V]^k$ to its equivalence class in the quotient space $\Lambda_k V$:

$$\mu(v_1, \ldots, v_k) := v_1 \wedge \cdots \wedge v_k \tag{2.1}$$

for all $(v_1, \ldots, v_k)$ in $[V]^k$. A *simple k-vector* is an element of $\Lambda_k V$ that lies in the image of the map μ. The space $\Lambda_k V$, together with the map μ, has the following universal property. If $f\colon [V]^k \to \mathbb{R}$ is an alternating k-linear map, then there is a unique linear map $\omega\colon \Lambda_k V \to \mathbb{R}$ with the property that $f = \omega \circ \mu$.

Towards defining the dual space to $\Lambda_k V$, we endow $\Lambda_k V$ with the quotient topology τ induced by μ. In other words, U of $\Lambda_k V$ is in τ if and only if $\mu^{-1}(U)$ is open in $[V]^k$ under the product topology.

Suppose that $f\colon [V]^k \to \mathbb{R}$ is an alternating k-linear map. As in [Car71], the continuity of f with respect to the product topology on $[V]^k$ is equivalent to its boundedness with respect to the norm $\|\cdot\|_{\text{Cartan}}$, where

$$\|f\|_{\text{Cartan}} := \sup\{|f(x_1, \ldots, x_k)| : x_i \in V,\ |x_i| \leq 1,\ i = 1, \ldots, k\}.$$

Lemma II.1. *Let $f\colon [V]^k \to \mathbb{R}$ be an alternating k-linear map, and let $\omega\colon \Lambda_k V \to \mathbb{R}$ be the map for which $f = \omega \circ \mu$. The continuity of f with respect to the product topology on $[V]^k$ is equivalent to the continuity of ω with respect to the topology τ on $\Lambda_k V$.*

Proof: The proof immediately follows from the definition of the topology τ. The function f is continuous if and only if for each open $U \subset \mathbb{R}$, $f^{-1}(U)$ is open in the product topology. Since $f^{-1}(U) = \mu^{-1} \circ \omega^{-1}(U)$, ω is continuous. □

Definition II.2. The space of k-covectors on V, denoted $\Lambda^k V$, is the space of linear maps from $\Lambda_k V$ to $\mathbb{R}$ that are continuous with respect to τ.

By the universal property of $\Lambda_k V$ and Lemma II.1, $\Lambda^k V$ is isomorphic to the space of alternating k-linear functions $f\colon [V]^k \to \mathbb{R}$ that are continuous under the product topology on $[V]^k$.

For $\varphi \in \Lambda_k V$ and $\omega \in \Lambda^k V$, we use the notation $\omega(\varphi) = \langle \omega, \varphi \rangle = \langle \varphi, \omega \rangle$.

General references on the spaces $\Lambda_k V$ and $\Lambda^k V$ are [Gre78], [Fed69], and [Car71].

2.2 Norms on $\Lambda_k V$ and $\Lambda^k V$

In a Banach space V, the space of k-vectors of elements in the dual space V^* can be viewed as a subspace of the space of k-covectors in V in the following way. Let $f_1, \ldots, f_k$ be elements of V^*, and define the action of $f_1 \wedge \cdots \wedge f_k$ on the simple k-vector $v_1 \wedge \ldots \wedge v_k$ by

$$\langle f_1 \wedge \cdots \wedge f_k, v_1 \wedge \ldots \wedge v_k \rangle := \det(\langle f_i, v_j \rangle).$$

This action is well-defined because the determinant is alternating and multilinear. We then have a natural dual pairing $\Lambda_k(V^*) \times \Lambda_k V \to \mathbb{R}$ given by

$$\Big\langle \sum_i f_1^i \wedge \cdots \wedge f_k^i, \sum_j v_1^j \wedge \ldots \wedge v_k^j \Big\rangle = \sum_i \sum_j \langle f_1^i \wedge \cdots \wedge f_k^i, v_1^j \wedge \ldots \wedge v_k^j \rangle.$$

In [Gro83], Gromov defines two norms, mass and mass*, on the space of simple k-vectors. In this work we are more interested in the mass* norm, but since the mass norm is used to construct the mass* norm, we review both definitions in this section.

Essentially, the mass of a simple k-vector is the infimum over all its simple representations of the product of the lengths of its components. The precise definition follows.

Let V be a Banach space and suppose that $\nu = v_1 \wedge \cdots \wedge v_k \in \Lambda_k V$ is a nonzero simple k-vector in $\Lambda_k V$. One can show (see Lemma II.6) that if $0 \neq \nu \in \Lambda_k V$ and $\nu = u_1 \wedge \cdots \wedge u_k = w_1 \wedge \cdots \wedge w_k$, then $\operatorname{span}(u_1, \ldots, u_k) = \operatorname{span}(w_1, \ldots, w_k)$. We denote the unique (oriented) subspace spanned by the components of ν by $T_\nu := \operatorname{span}(v_1, \ldots, v_k)$.

If $\nu \in \Lambda_k V$ is simple and $L : T_\nu \to T_\nu$ is any linear transformation, it induces a linear transformation on the one-dimensional space $\Lambda_k T_\nu$ given by $v_1 \wedge \cdots \wedge v_k \mapsto L(v_1) \wedge \cdots \wedge L(v_k)$. This induced transformation is just multiplication by a scalar, defined to be the *determinant of* L, $\det(L)$. Thus, for all $v_1 \wedge \cdots \wedge v_k \in \Lambda_k V$, we have $L(v_1) \wedge \cdots \wedge L(v_k) = \det(L) \cdot v_1 \wedge \cdots \wedge v_k$.

Definition II.3. Let ν be a simple k-vector. If $\nu \neq 0$ then the *mass* of ν is given by the following infimum:

$$\|\nu\|_{\mathrm{m}} := \inf\left\{\prod_{i=1}^{k} |Lv_i|_V : L\colon T_\nu \to T_\nu \text{ linear, } \det L = 1\right\}.$$

If $\nu = 0$, define $\|\nu\|_{\mathrm{m}} := 0$.

Gromov uses mass in the dual space V^* to define the mass* of a simple k-vector. The following characterization of mass* comes from [ÁPT04].

Definition II.4. Let V be a Banach space and ν a simple k-vector in $\Lambda_k V$. The *mass** of ν is

$$\|\nu\|_{\mathrm{m}^*} := \sup\{\langle \nu, \xi\rangle : \xi \in \Lambda_k V^* \text{ simple}, \|\xi\|_{\mathrm{m}} \leq 1\}. \tag{2.2}$$

For any simple k-vector ν, Gromov proves [Gro83, p. 30] that

$$\|\nu\|_{\mathrm{m}} \leq \|\nu\|_{\mathrm{m}^*} \leq c(k)\|\nu\|_{\mathrm{m}}, \tag{2.3}$$

where $c(k) = k^{k/2}$.

Gromov's mass* extends to a norm on the entire space $\Lambda_k V$ of k-vectors (see [ÁPT04]). We note that in fact, that one can actually extend mass* to all k-vectors using equation (2.2).

Lemma II.5. *The quantity* $\|\nu\|_{m^*} := \sup\{\langle \nu, \xi\rangle : \xi \in \Lambda_k V^*$ *simple*, $\|\xi\|_m \leq 1\}$ *defines a norm on the space* $\Lambda_k V$ *of* k*-vectors on* V*.*

Proof: That $\|c\nu\|_{\mathrm{m}^*} = |c| \cdot \|\nu\|_{\mathrm{m}^*}$ for any $c \in \mathbb{R}$ follows immediately from the fact that $\Lambda_k V^*$ is a subspace of $\Lambda^k V$. For the triangle inequality, let ν, $\mu \in \Lambda_k V$. Then

$$\begin{aligned}
\|\nu+\mu\|_{\mathrm{m}^*} &= \sup\{\langle \nu+\mu, \xi\rangle : \xi \in \Lambda_k V^* \text{ simple, } \|\xi\|_{\mathrm{m}} \le 1\} \\
&= \sup\{\langle \nu, \xi\rangle + \langle \mu, \xi\rangle : \xi \in \Lambda_k V^* \text{ simple, } \|\xi\|_{\mathrm{m}} \le 1\} \\
&\le \sup\{\langle \nu, \xi\rangle : \xi \in \Lambda_k V^* \text{ simple, } \|\xi\|_{\mathrm{m}} \le 1\} \\
&\quad + \sup\{\langle \mu, \xi\rangle : \xi \in \Lambda_k V^* \text{ simple, } \|\xi\|_{\mathrm{m}} \le 1\} \\
&= \|\nu\|_{\mathrm{m}^*} + \|\mu\|_{\mathrm{m}^*}.
\end{aligned}$$

To see that $\|\cdot\|_{\mathrm{m}^*}$ is finite, let $\nu \in \Lambda_k V$. Then $\nu = \sum_{i=1}^N \nu^i$, where ν^i is simple for i from 1 to N. For any simple $\xi \in \Lambda_k V^*$ and all $i = 1, \ldots, N$, $\langle \nu^i, \xi\rangle \le \|\nu^i\|_{\mathrm{m}^*}$. Thus, $\langle \nu, \xi\rangle \le \sum_i \|\nu^i\|_{\mathrm{m}^*}$, so $\|\nu\|_{\mathrm{m}^*}$ is finite.

Let $\nu = \sum_{i=1}^N v_1^i \wedge \cdots \wedge v_k^i$ be a nonzero k-vector in $\Lambda_k V$. Let $W = \operatorname{span}\{v_j^i\}$. Then $\nu \in \Lambda_k W$. Since $\nu \ne 0$ and W is finite dimensional, there exists a simple covector $\xi = \xi_1 \wedge \cdots \wedge \xi_k \in \Lambda^k W^*$ so that $\langle \nu, \xi\rangle > 0$. We may assume that $|\xi_i|_{W^*} = 1$ for $i = 1, \ldots, k$. Extend each ξ_i by the Hahn-Banach theorem to a functional $\overline{\xi_i}$ in V^* with norm one and set $\overline{\xi} := \overline{\xi_1} \wedge \cdots \wedge \overline{\xi_k}$. Then $\|\overline{\xi}\|_{\mathrm{m}} \le |\overline{\xi_1}| \cdots |\overline{\xi_k}| = 1$ and $\langle \nu, \overline{\xi}\rangle > 0$, so $\|\nu\|_{\mathrm{m}^*} > 0$. □

We can then define the comass* of a k-covector $\omega \in \Lambda^k V$ to be the dual to the mass* norm:

$$\|\omega\|_{\mathrm{c}^*} := \sup\{\langle \omega, \nu\rangle : \nu \in \Lambda_k V, \|\nu\|_{\mathrm{m}^*} \le 1\}.$$

2.3 k-directions

We normalize the space of k-vectors using the mass* norm. A simple k-vector whose mass* is one is called a *k-direction.*

We note that each oriented k-dimensional vector subspace P of V (i.e., each element of the oriented Grassmannian of V) corresponds to a unique k-direction

ν_P in the following way. An orientation of P consists of a choice of a basis and an ordering of the basis elements. By definition, two ordered bases $(v_1, \ldots, v_k)$ and $(w_1, \ldots, w_k)$ of P induce the same orientation if and only if $v_1 \wedge \cdots \wedge v_k = \lambda w_1 \wedge \cdots \wedge w_k$ for some $\lambda > 0$. Thus, if the orientation of P is given by $(v_1, \ldots, v_k)$, then we define ν_P, *the k-direction of P*, to be

$$\nu_P := \frac{v_1 \wedge \cdots \wedge v_k}{\|v_1 \wedge \cdots \wedge v_k\|_{\mathrm{m}^*}}.$$

For an affine oriented k-plane P, let P_0 be the subspace parallel to P that passes through the origin, i.e., if $p \in P$, then $P_0 = P - p$. In this case, define $\nu_P := \nu_{P_0}$.

Given a k-direction $\nu = v_1 \wedge \cdots \wedge v_k$, we say that an affine subspace W of V is a *ν-superplane* if $T_\nu \subset W_0$, where W_0 is parallel to W and contains the origin.

Lemma II.6. *Given a k-vector $\nu \in \Lambda_k V$, there is a unique minimal subspace $W_{\nu,min}$ of V for which there exists a representation $\nu = \sum_{i=1}^N \nu^i$ so that $W_{\nu,min}$ is a ν^i-superplane for all $1 \leq i \leq N$.*

The subspace $W_{\nu,\mathrm{min}}$ is called the *envelope of ν in V*.

Proof: Suppose that W and U are both subspaces of minimal dimension as in the statement of the lemma. Thus, there exist representations

$$\nu = \sum_i w_1^i \wedge \cdots \wedge w_k^i = \sum_j u_1^j \wedge \cdots \wedge u_k^j,$$

where $\mathrm{span}\{w_m^i\} = W$ and $\mathrm{span}\{u_m^i\} = U$. Let $d = \dim W \leq \dim U = D$ and let $\ell = \dim W \cap U$. We will show that $\ell = d$.

Choose a basis $v_1, \ldots, v_\ell$ of $W \cap U$. Then choose $d - \ell$ additional linearly independent vectors $\{\widetilde{w}_1, \ldots, \widetilde{w}_{d-\ell}\} \subset W \backslash U$ so that $\mathcal{B}_W = \{\widetilde{w}_1, \ldots, \widetilde{w}_{d-\ell}, v_1, \ldots, v_\ell\}$ is a basis of W. Similarly, choose $\{\widetilde{u}_1, \ldots, \widetilde{u}_{D-\ell}\} \subset U \backslash W$ so that $\mathcal{B}_U = \{\widetilde{u}_1, \ldots, \widetilde{u}_{D-\ell}, v_1, \ldots, v_\ell\}$ is a basis of U. Note that by construction, the vectors $\{\widetilde{w}_i\}$ and $\{\widetilde{u}_i\}$ are linearly independent. We assume the basis vectors all have norm one.

Thus,

$$\mathcal{B}_{W+U} = \{\widetilde{w}_1, \ldots, \widetilde{w}_{d-\ell}, \widetilde{u}_1, \ldots, \widetilde{u}_{D-\ell}, v_1, \ldots, v_\ell\}$$

is a basis of $W + U$, and we have the dual basis

$$\{\widetilde{w}_1^*, \ldots, \widetilde{w}_{d-\ell}^*, \widetilde{u}_1^*, \ldots, \widetilde{u}_{D-\ell}^*, v_1^*, \ldots, v_\ell^*\}$$

of the space $(W + U)^*$, where as usual $\widetilde{w}_i^*(\widetilde{w}_i) = 1$ and $\widetilde{w}_i^*(x) = 0$ if x is any other basis element, etc.

Renaming $v_1 = \widetilde{w}_{d-\ell+1}, \ldots, v_k = \widetilde{w}_d$, we have the following basis for $\Lambda_k W$:

$$\{\widetilde{w}_I = \widetilde{w}_{i_1} \wedge \cdots \wedge \widetilde{w}_{i_k} : I = \{1 \leq i_1 < \cdots < i_k \leq d\}\}.$$

We rewrite ν in terms of this basis:

$$\nu = \sum_i w_{i_1} \wedge \cdots \wedge w_{i_k} = \sum_I \lambda_I \widetilde{w}_I,$$

where $\lambda_I \in \mathbb{R}$.

To reach a contradiction, suppose that $\ell \lneq d$. Since W was a minimal subspace, there exists a multi-index $I' = \{1 \leq i_1' < \cdots < i_k' \leq d\}$ with $i_1' = 1$ where $\lambda_{I'} \neq 0$. In other words, the dual element $\widetilde{w}_{I'}^* = \widetilde{w}_{i_1'}^* \wedge \cdots \wedge \widetilde{w}_{i_k'}^*$ has the property that $\langle \widetilde{w}_{I'}^*, \nu \rangle = \lambda_{I'} \neq 0$. On the other hand, by construction $\widetilde{w}_1^*(x) = 0$ for all $x \in U$, so

$$\begin{aligned} \langle \widetilde{w}_{I'}^*, \nu \rangle &= \left\langle \widetilde{w}_{I'}^*, \sum_j u_1^j \wedge \cdots \wedge u_k^j \right\rangle \\ &= \sum_j \langle \widetilde{w}_{I'}^*, u_1^j \wedge \cdots \wedge u_k^j \rangle \\ &= 0, \end{aligned}$$

a contradiction. □

Remark II.7. We define the mass* of $\vec{v} = (v_1, \ldots, v_k) \in [V]^k$ by

$$\|\vec{v}\|_{\mathrm{m}^*} := \|\mu(\vec{v})\|_{\mathrm{m}^*} = \|v_1 \wedge \cdots \wedge v_k\|_{\mathrm{m}^*}.$$

2.4 John's theorem

In the remaining chapters, we will often use the fact that up to a factor of $\sqrt{k}$, the norm of any k-dimensional Banach space is comparable to the Euclidean norm in $\mathbb{R}^k$.

We may identify a k-dimensional Banach space V^k with $\mathbb{R}^k$ equipped with a different norm than the standard Euclidean norm $|\cdot|_2^k$. Denoting this norm by $|\cdot|_{V^k}$, we have $V^k = (\mathbb{R}^k, |\cdot|_{V^k})$. The closed unit ball in V^k is then $\mathbb{B}_{V^k} := \{x \in \mathbb{R}^n : |x|_{V^k} \le 1\}$. Since $\mathbb{B}_{V^k}$ is a closed symmetric convex set with nonempty interior in $\mathbb{R}^k$, John's theorem implies (see [Bal97]) that if $\mathcal{E}$ is the maximal ellipsoid contained in $\mathbb{B}_{V^k}$, then

$$\mathcal{E} \subset \mathbb{B}_{V^k} \subset \sqrt{k}\mathcal{E}. \tag{2.4}$$

This property of the maximal ellipsoid inside $\mathbb{B}_{V^k}$, due to Fritz John, implies the following theorem.

Theorem II.8. *Given any k-dimensional Banach space V^k, there exists a $\sqrt{k}$-bi-Lipschitz linear isomorphism $L\colon \mathbb{R}^k \to V^k$.*

Proof: Let $|\cdot|_{\mathcal{E}}$ denote the norm on $\mathbb{R}^k$ whose unit ball is the ellipsoid $\mathcal{E}$ given by John's theorem. The identity map $I\colon (\mathbb{R}^k, |\cdot|_{V^k}) \to (\mathbb{R}^k, |\cdot|_{\mathcal{E}})$ is then a $\sqrt{k}$-bi-Lipschitz isomorphism. A change of coordinates φ mapping the (orthonormal) axes of $\mathcal{E}$ to the standard Euclidean basis vectors will map $\mathcal{E}$ to the unit sphere. Under this change of coordinates, $(\mathbb{R}^k, |\cdot|_{\mathcal{E}})$ is $\mathbb{R}^k$ with the Euclidean norm. Take L to be the map $\varphi \circ I$. $\square$

CHAPTER III

Flat Chains in Banach Spaces

In this section, we review a few definitions and facts about flat chains in Banach spaces from [Ada08] and prove some lemmas that are needed for our study of partial forms.

3.1 Polyhedral chains

We start by defining a k-dimensional simplex in a Banach space V.

Definition III.1. A k-dimensional simplex σ in V is the convex hull of $k+1$ affinely independent vectors $v_0, \ldots, v_k$ in V, together with an orientation. We call this convex hull the support of σ. The vectors $v_0, \ldots, v_k$ are the *vertices* of σ. An orientation of σ is a choice of orientation of the affine k-plane containing σ.

Any (non-degenerate) oriented k-simplex σ in V lies in a unique oriented affine k-plane P_σ which we call the *k-plane of* σ. The k-direction of the k-plane P_σ is the *k-direction of* σ, denoted ν_σ. We note that the k-direction of σ describes the orientation of σ.

We denote by $[v_0, \ldots, v_k]$ the simplex with vertices $v_0, \ldots, v_k$ and orientation

$$\nu_\sigma = \frac{(v_1 - v_0) \wedge \cdots \wedge (v_k - v_0)}{\|(v_1 - v_0) \wedge \cdots \wedge (v_k - v_0)\|_{\mathrm{m}^*}}.$$

The simplex with the same vertices but the opposite orientation is denoted by $-[v_0, \ldots, v_k]$. A *refinement* of the simplex σ is a finite formal sum of simplexes $\sum \sigma_i$ that satisfies the following properties. First, the union of the supports of the simplexes $\{\sigma_i\}$ must equal the support of σ. Second, the orientation of each simplex σ_i must match the orientation of σ. Finally, the simplexes $\{\sigma_i\}$ must be pairwise disjoint except on sets of k-measure zero.

We can now define the space of polyhedral k-chains in V.

Definition III.2. Consider the free real vector space $\mathcal{S}_k$ generated by k-dimensional oriented simplexes in V. Let G_k be the subspace of $\mathcal{S}_k$ generated by elements of the form $\sigma + (-\sigma)$ and $\sigma - \sum \sigma_i$, where $\sum \sigma_i$ is a refinement of σ. The space of polyhedral k-chains in V, denoted $\mathcal{P}_k(V)$, is the quotient $\mathcal{S}_k/G_k$.

The *boundary* $\partial\sigma$ of a k-simplex $\sigma = [v_0, \ldots, v_k]$ is the $(k-1)$-chain

$$\partial\sigma := \sum_{i=0}^{k} (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k].$$

Extend the boundary operator is linearly to $\mathcal{S}_k$ so that the boundary of the polyhedral k-chain $P = \sum_{i=1}^{N} \lambda_i \sigma_i$ is

$$\partial P := \sum_{i=1}^{N} \lambda_i \partial\sigma_i.$$

As usual, $\partial\partial\sigma$ is equivalent to the zero $(k-2)$-chain.

The boundary operator descends to an operator $\partial\colon \mathcal{P}_k(V) \to \mathcal{P}_{k-1}(V)$ on the quotient space $\mathcal{P}_k(V)$.

A *simple polyhedral k-chain* A is an element of S_k/G_k that has convex support and can be represented as a sum of consistently oriented simplexes with pairwise disjoint interiors, i.e., A is a convex, oriented polyhedron.

The following definition of k-dimensional mass, denoted $|\cdot|_k$, on simple polyhedral k-chains is from [Ada08, p. 3].

Definition III.3. The *mass* of a 0-dimensional polyhedral chain A in a Banach space V is $|A|_0 := \inf\{\sum|\lambda_i| : A = \sum \lambda_i \sigma_i\}$. If A is a simple k-dimensional polyhedral chain for $k > 0$, the mass of A is inductively defined to be

$$|A|_k := \sup_{\substack{f \in V^* \\ \|f\| \leq 1}} \int_{\mathbb{R}} |A \cap f^{-1}(x)|_{k-1}\, dx. \tag{3.1}$$

Here, $A \cap f^{-1}(x)$ is the *slice of* A *by* f *at* x as defined in [Ada08]. For a simple k-chain A, these slices are again simple chains of dimension k or lower. If the dimension of a slice is k, then we define the $(k-1)$-mass of the slice to be ∞. In this case all other slices will have mass zero, so $\int_{\mathbb{R}} |A \cap f^{-1}(x)|_{k-1}\, dx = 0$. If the dimension of a slice is less than $(k-1)$, we define its $(k-1)$-dimensional mass to be *zero*. When the dimension k is clear we will simply write $|A|$ instead of $|A|_k$.

We note that the quantity $|A|$ does not change if the supremum in equation (3.1) is instead taken over the smaller set of maps $f \in V^*$ with $\|f\| = 1$. For every $f \in V^*$ with $0 < \|f\| < 1$, the map $\tilde{f} := \frac{1}{\|f\|} f$ has unit norm and $|A \cap f^{-1}(x)| = |A \cap \tilde{f}^{-1}(\frac{x}{\|f\|})|$. Hence, $\int_{\mathbb{R}} |A \cap f^{-1}(x)|\, dx \leq \int_{\mathbb{R}} |A \cap \tilde{f}^{-1}(x)|\, dx$, so

$$|A| = \sup_{\substack{f \in V^* \\ \|f\| = 1}} \int_{\mathbb{R}} |A \cap f^{-1}(x)|\, dx.$$

Adams proved that the 1-dimensional mass of a simplex $[a, b]$ is its length, $|[a, b]| = |b - a|_V$.

We extend mass to a norm on all polyhedral k chains in the following way. The mass of a polyhedral k-chain A is the following infimum, taken over all representations of A as finite sums of simple chains:

$$|A| := \inf\{\sum_{i=1}^{n} |\lambda_i||\sigma_i| : A = \sum_{i=1}^{n} \lambda_i \sigma_i\}. \tag{3.2}$$

3.2 Haar measure

For a polyhedral chain A, denote by $A + p$ the translation of A by $p \in V$. We show below that Adams's mass is translation invariant.

Lemma III.4. *For all $k \geq 0$, k-dimensional mass is translation invariant on simple polyhedral chains.*

Proof: We use induction on k. Clearly mass is translation invariant for simple 0-chains. Let $k \geq 1$ and let A be a simple polyhedral chain. For any $p \in V$, $f \in V^*$, and $y \in \mathbb{R}$, the slice of $(A+p)$ by f at $(y + f(p))$ is the translation (by p) of the slice of A by f at y. By the induction hypothesis,

$$\int_{\mathbb{R}} |A \cap f^{-1}(x)|_{k-1}\, dx = \int_{\mathbb{R}} |(A+p) \cap f^{-1}(x)|_{k-1}\, dx.$$

Taking the supremum of both sides of the previous equation over all functionals $f \in V^*$ with $\|f\| \leq 1$ gives $|A|_k = |A+p|_k$, as desired. □

In fact, any sequence of functionals which approaches the supremum for A in equation (3.1) will also approach the supremum for any translate $(A+p)$ of A.

Corollary III.5. *Let $k \geq 1$ and A be a simple polyhedral k-chain. Suppose that $(f_i) \subset V^*$ is a sequence of linear functionals with $\|f_i\| = 1$ for all i and that*

$$\lim_{i\to\infty} \int_{\mathbb{R}} |A \cap f_i^{-1}(x)|_{k-1}\, dx = |A|.$$

Then

$$\lim_{i\to\infty} \int_{\mathbb{R}} |(A+p) \cap f_i^{-1}(x)|_{k-1}\, dx = |A+p|.$$

Proof: Suppose that $(f_i) \subset V^*$ is a sequence of linear functionals with $\|f_i\| = 1$ so that

$$\lim_{i\to\infty} \int_{\mathbb{R}} |A \cap f_i^{-1}(x)|_{k-1}\, dx = |A|.$$

Then

$$\begin{aligned} |A+p| &= \sup_{\substack{g\in V^* \\ \|g\|\le 1}} \int_{\mathbb{R}} |(A+p)\cap g^{-1}(x)|_{k-1}\,dx \\ &= \sup_{\substack{g\in V^* \\ \|g\|\le 1}} \int_{\mathbb{R}} |A\cap g^{-1}(x)|_{k-1}\,dx \\ &= \lim_{i\to\infty} \int_{\mathbb{R}} |A\cap f_i^{-1}(x)|_{k-1}\,dx \\ &= \lim_{i\to\infty} \int_{\mathbb{R}} |(A+p)\cap f_i^{-1}(x)|_{k-1}\,dx. \end{aligned}$$

□

The following is a corollary to Adams's Scaling Lemma ([Ada08, Lemma 2.4]).

Lemma III.6. *Let P be an oriented, affine k-plane, $k\ge 1$, and fix a basis $\{v_1,\ldots,v_k\}$ of P. Let Q be the parallelepiped with support $\{\sum_{i=1}^k \lambda_i v_i : 0\le\lambda\le 1\}$ and orientation $v_1\wedge\ldots\wedge v_k$. Let $(f_i)\subset V^*$ be a sequence of linear functionals with $\|f_i\|=1$ for all i so that*

$$\lim_{i\to\infty}\int_{\mathbb{R}} |Q\cap f_i^{-1}(x)|_{k-1}\,dx = |Q|.$$

For $q\in\mathbb{R}$ with $q>0$, let Q_q be the parallelepiped with support $\{q\sum_{i=1}^k \lambda_i v_i : 0\le\lambda\le 1\}$ and orientation $v_1\wedge\ldots\wedge v_k$. Then $|Q_q| = q^k|Q|$ and

$$\lim_{i\to\infty}\int_{\mathbb{R}} |Q_q\cap f_i^{-1}(x)|_{k-1}\,dx = |Q_q|.$$

Proof: Let P, Q, Q_q, and $(f_i)\subset V^*$ be as in the lemma statement. By the proof of the Scaling Lemma in [Ada08, p. 5], $|Q_q| = q^k|Q|$, and for each $f\in V^*$ with $\|f\|=1$,

$$\int_{\mathbb{R}} |Q_q\cap f^{-1}(x)|_{k-1}\,dx = q^k\int_{\mathbb{R}} |Q\cap f^{-1}(x)|_{k-1}\,dx.$$

Hence,

$$\lim_{i\to\infty}\int_{\mathbb{R}} |Q_q\cap f_i^{-1}(x)|_{k-1}\,dx = |Q_q|_k.$$

□

Proposition III.7. *(Construction of Haar measure from mass.) Let P be an oriented, affine k-plane, $k \geq 1$, and fix a basis $\{v_1, \ldots, v_k\}$ of P. Let Q be the parallelepiped with support $\{\sum_{i=1}^{k} \lambda_i v_i : 0 \leq \lambda \leq 1\}$ and orientation $v_1 \wedge \ldots \wedge v_k$. Let $(f_i) \subset V^*$ be a sequence of linear functionals with $\|f_i\| = 1$ for all i so that*

$$\lim_{i \to \infty} \int_{\mathbb{R}} |Q \cap f_i^{-1}(x)|_{k-1}\, dx = |Q|_k.$$

If A is a simple polyhedral k-chain in P, then the sequence (f_i) also approaches the supremum of the mass integral for A:

$$\lim_{i \to \infty} \int_{\mathbb{R}} |A \cap f_i^{-1}(x)|_{k-1}\, dx = |A|_k.$$

Moreover, there exists a Haar measure $\mathcal{M}^k$ on P for which $\mathcal{M}^k(A) = |A|_k$.

Proof: We use induction on k. For $k = 1$, the lemma follows from the fact that the Adams mass of a simple polyhedral chain is its length.

For $k > 1$, let P, Q, and $(f_i) \subset V^*$ be as in the lemma statement. For $q \in \mathbb{Q}$ with $q > 0$, let Q_q be the parallelepiped from Corollary III.6, and let $\mathcal{B}$ denote the countable collection of parallelepipeds

$$\mathcal{B} := \{Q_q + r_1 v_1 + \cdots + r_k v_k : q, r_1, \ldots, r_k \in \mathbb{Q}\}.$$

Let A be a simple polyhedral k-chain in P. We may represent the interior of A (denoted $\operatorname{int} A$) as a countable union of elements of $\mathcal{B}$, so that $\operatorname{int} A = \bigcup_{j=1}^{\infty} B_j$ where $B_j \in \mathcal{B}$, $B_j \subset \operatorname{int} A$, and the sets B_j are pairwise disjoint except on sets of k-measure zero.

By Fubini's theorem and the existence of $(k-1)$-dimensional Haar measure, for any $f \in V^*$ with $\|f\| = 1$ and almost every $x \in \mathbb{R}$,

$$|A \cap f^{-1}(x)|_{k-1} = \Big|\Big(\bigcup_j B_j\Big) \cap f^{-1}(x)\Big|_{k-1} = \sum_j |B_j \cap f^{-1}(x)|_{k-1}. \tag{3.3}$$

Thus,

$$
\begin{aligned}
|A|_k &= \sup_{\substack{f\in V*\\ \|f\|=1}} \int_{\mathbb{R}} |A\cap f^{-1}(x)|_{k-1}\,dx \\
&= \sup_{\substack{f\in V*\\ \|f\|=1}} \int_{\mathbb{R}} \sum_j |B_j\cap f^{-1}(x)|_{k-1}\,dx \\
&= \sup_{\substack{f\in V*\\ \|f\|=1}} \sum_j \int_{\mathbb{R}} |B_j\cap f^{-1}(x)|_{k-1}\,dx \qquad (3.4)\\
&= \lim_{i\to\infty} \sum_j \int_{\mathbb{R}} |B_j\cap f_i^{-1}(x)|_{k-1}\,dx \qquad (3.5)\\
&= \sum_j \lim_{i\to\infty} \int_{\mathbb{R}} |B_j\cap f_i^{-1}(x)|_{k-1}\,dx \qquad (3.6)\\
&= \sum_j |B_j|. \qquad (3.7)
\end{aligned}
$$

Equations (3.5) and (3.7) follow from Corollary III.6, and equations (3.4) and (3.6) follow from the Lebesgue Dominated Convergence Theorem.

Also, by equation (3.3),

$$
\begin{aligned}
\sum_j |B_j|_k &= \lim_{i\to\infty} \sum_j \int_{\mathbb{R}} |B_j\cap f_i^{-1}(x)|_{k-1}\,dx \\
&= \lim_{i\to\infty} \int_{\mathbb{R}} \Big|\Big(\bigcup_j B_j\Big)\cap f^{-1}(x)\Big|_{k-1}\,dx \\
&= \lim_{i\to\infty} \int_{\mathbb{R}} |A\cap f_i^{-1}(x)|_{k-1}\,dx.
\end{aligned}
$$

We construct the Haar measure $\mathcal{M}^k$ on P by Caratheodory's approach (see [Roy88]), using the parallelepipeds $\mathcal{B}$ instead of the Euclidean rectangles used to construct Lebesgue measure on $\mathbb{R}^n$. For any set $E\subset P$, we define the outer measure $(\mathcal{M}^k)^*$ by

$$(\mathcal{M}^k)^*(E) := \inf\{\sum_{j\in\mathbb{N}} |B_j| : B_j\in\mathcal{B}, E\subset \bigcup_j B_j\}.$$

A set $E\subset P$ is $\mathcal{M}^k$-measurable if for every $S\subset P$,

$$(\mathcal{M}^k)^*(S) = (\mathcal{M}^k)^*(S\cup E) + (\mathcal{M}^k)^*(S\setminus E).$$

In this case, $(\mathcal{M}^k)(E) := (\mathcal{M}^k)^*(E)$. By construction, the measure $\mathcal{M}^k$ assigns to any simple polyhedral k-chain a measure equal to the Adams k-dimensional mass of the chain. □

By the uniqueness of the Haar measure, the measure $\mathcal{M}^k$ (defined in Proposition III.7) on any fixed k-plane P is equal to a constant multiple κ_P of the Hausdorff measure $\mathcal{H}^k$ on P. By Corollary III.5, if the k-planes P and R are parallel, $\kappa_P = \kappa_R$, otherwise the constants κ_P and κ_R may be distinct. Since parallel k-planes have the same spanning k-direction ν, κ_P depends only on ν, so we refer to this constant as κ_ν.

The remarks above yield another the interpretation of the mass of an arbitrary polyhedral chain. If A is a polyhedral chain with representation $A = \sum_{i=1}^n \lambda_i \sigma_i$ as a weighted sum of disjoint simple polyhedral chains σ_i, A is naturally associated with a function $f\colon V \to \mathbb{R}$ where f is the corresponding weighted sum of characteristic functions of the simple chains σ_i,

$$f_A = \sum \lambda_i \chi(\sigma_i).$$

Then we have

$$|A| = \int f_A \, d\mathcal{M}^k = \sum_{i=1}^n |\lambda_i| \cdot |\sigma_i|, \tag{3.8}$$

showing that the infimum in equation (3.2) is not needed if one considers only disjoint representations of A.

S. Wenger pointed out to the author that the volume norm on a k-dimensional affine plane in V induced by Gromov's mass* norm is equal to the volume norm $\mathcal{M}^k$ induced by Adams's mass norm.

Lemma III.8. *(Wenger's Lemma) Let $\nu = v_1 \wedge \cdots \wedge v_k$ be a simple k-vector in $\Lambda_k V$, and let $A = A_\nu = \{\sum_{i=1}^k t_i v_i : 0 \le t_i \le 1\}$ be the parallelepiped spanned by* *the components of ν with the same orientation as ν. Then the mass* of ν equals the Adams mass of A, that is,*

$$\|\nu\|_{m^*} = |A|.$$

Proof: Let $\nu = v_1 \wedge \cdots \wedge v_k$ be a simple k-vector in $\Lambda_k V$, and let $A = A_\nu$ be the parallelepiped spanned by the components $\{v_1, \ldots, v_k\}$ of ν. By the definition of Adams mass,

$$\begin{aligned} |A| &= \sup_{\substack{f_1 \in V^* \\ \|f_1\|=1}} \int_{\mathbb{R}} |A \cap f_1^{-1}(x_1)|\, dx_1 \\ &= \sup_{\substack{f_1 \in V^* \\ \|f_1\|=1}} \int_{\mathbb{R}} \sup_{\substack{f_2 \in V^* \\ \|f_2\|=1}} \int_{\mathbb{R}} |A \cap f_1^{-1}(x_1) \cap f_2^{-1}(x_2)|\, dx_2 dx_1 \end{aligned}$$

A priori, the choice of f_2 depends on the point $x_1 \in \mathbb{R}$. Thus

$$\begin{aligned} &\sup_{\substack{f_1 \in V^* \\ \|f_1\|=1}} \int_{\mathbb{R}} \sup_{\substack{f_2 \in V^* \\ \|f_2\|=1}} \int_{\mathbb{R}} |A \cap f_1^{-1}(x_1) \cap f_2^{-1}(x_2)|\, dx_2 dx_1 \\ &\qquad \geq \sup_{\substack{f_1 \in V^* \\ \|f_1\|=1}} \sup_{\substack{f_2 \in V^* \\ \|f_2\|=1}} \int_{\mathbb{R}} \int_{\mathbb{R}} |A \cap f_1^{-1}(x_1) \cap f_2^{-1}(x_2)|\, dx_2 dx_1. \end{aligned} \tag{3.9}$$

Let P_A be the oriented k-plane containing A, and fix a functional $f_1 \in V^*$ with $\|f_1\| = 1$. By Proposition III.7 we have equality in (3.9).

Thus,

$$\begin{aligned}
|A| &= \sup_{\substack{f_1 \in V^* \\ \|f_1\|=1}} \int_{\mathbb{R}} \cdots \sup_{\substack{f_k \in V^* \\ \|f_k\|=1}} \int_{\mathbb{R}} |A \cap f_1^{-1}(x_1) \cap \ldots \cap f_k^{-1}(x_k)| \, dx_1 \ldots dx_k \\
&= \sup_{\substack{f_1 \in V^* \\ \|f_1\|=1}} \cdots \sup_{\substack{f_k \in V^* \\ \|f_k\|=1}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |A \cap f_1^{-1}(x_1) \cap \ldots \cap f_k^{-1}(x_k)| \, dx_1 \ldots dx_k \\
&= \sup_{\substack{f_i \in V^* \\ \|f_i\|=1 \\ 1 \le i \le k}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} |A \cap f_1^{-1}(x_1) \cap \ldots \cap f_k^{-1}(x_k)| \, dx_1 \ldots dx_k \\
&= \sup_{\substack{F: V \to \mathbb{R}^k \\ F=(f_1,\ldots,f_k) \\ f_i \in V^*, \|f_i\|=1}} \int_{\mathbb{R}^k} |A \cap F^{-1}(p)| \, dp \\
&= \sup_{\substack{F: V \to \mathbb{R}^k \\ F=(f_1,\ldots,f_k) \\ f_i \in V^*, \|f_i\|=1}} \int_{\mathbb{R}^k} \chi_{F(A)} \, dp \\
&= \sup_{\substack{F: V \to \mathbb{R}^k \\ F=(f_1,\ldots,f_k) \\ f_i \in V^*, \|f_i\|=1}} \mathcal{L}^k(F(A)).
\end{aligned}$$

In the last equation above, $\mathcal{L}^k$ is Lebesgue k-measure on $\mathbb{R}^k$.

For a linear map $F : V \to \mathbb{R}^k$, $F(A)$ is the parallelepiped determined by the vectors $F(v_1), \ldots, F(v_k)$. Hence the Lebesgue measure of $F(A)$ is given by the determinant of the matrix whose column vectors are $F(v_i)$ for $i = 1, \ldots, k$, i.e., $\mathcal{L}^k(F(A)) = \det(f_j(v_i))$.

Thus,

$$\begin{aligned}
|A| &= \sup\{\mathcal{L}^k(F(A))\} \\
&= \sup_{\substack{f_i \in V^* \\ \|f_i\|=1}} \{\det(f_j(v_i))\} \\
&= \sup\{\langle \nu, \xi \rangle : \xi = f_1 \wedge \cdots \wedge f_k, f_j \in V^*, ||f_j|| = 1, 1 \le j \le k\} \\
&= \|\nu\|_{\mathrm{m}^*}.
\end{aligned}$$

□

In other words, Wenger's Lemma says that the mass* of a simple k-vector ν is equal to the mass of the oriented parallelepiped defined by the components of ν, where the parallelepiped is viewed as a k-chain whose mass is defined as in [Ada08].

3.3 Flat chains

Given a polyhedral k-chain σ, we define the quantity $|\sigma|_\flat$ as the following infimum, taken over all polyhedral $(k+1)$-chains τ:

$$|\sigma|_\flat := \inf_{\tau \in \mathcal{P}_{k+1}(V)} \{|\tau| + |\sigma - \partial\tau|\}.$$

Adams proved (see [Ada08, p. 13]) that if $|P_i - P|_\flat \to 0$, then $|P| \leq \liminf |P_i|$. This lower semicontinuity property is then used to prove that $|\cdot|_\flat$ defines a norm, called the *flat norm*, on the space of polyhedral k-chains.

It follows from the definition of the flat norm that for any polyhedral chain σ, $|\sigma|_\flat \leq |\sigma|$.

Lemma III.9. *The boundary operator $\partial \colon \mathcal{P}_k(V) \to \mathcal{P}_{k-1}(V)$ is bounded in the flat norm, and $\|\partial\| \leq 1$.*

Proof: For any polyhedral k-chain σ, $|\partial\sigma|_\flat \leq |\sigma|_\flat$. To see this, choose $\varepsilon > 0$ and find a polyhedral $(k+1)$-chain τ so that

$$|\sigma|_\flat \geq |\tau| + |\sigma - \partial\tau| - \varepsilon.$$

Then

$$\begin{aligned} |\partial\sigma|_\flat &\leq |\sigma - \partial\tau| + |\partial\sigma - \partial(\sigma - \partial\tau)| \\ &\leq |\sigma|_\flat + \varepsilon. \end{aligned}$$

Since ε was arbitrary, $|\partial\sigma|_\flat \leq |\sigma|_\flat$. □

The completion of the space of polyhedral k-chains under the flat norm, denoted $\mathcal{F}_k(V)$, is the *space of flat k-chains.* Since the space of polyhedral chains is a linear space, $\mathcal{F}_k(V)$ equipped with the flat norm is a Banach space.

Since the boundary operator ∂ is bounded in the flat norm on polyhedral chains, it can be uniquely extended to all flat chains.

3.4 Flat cochains

The space of flat k-cochains, denoted $\mathcal{F}^k(V)$, is the dual space to the Banach space $\mathcal{F}_k(V)$. Since dual spaces are always complete, one may also consider the space of flat k-cochains as the space of bounded linear functionals on the space of polyhedral k-chains. If X is a flat k-cochain and A is a flat k-chain, we denote evaluation of the functional X on the chain A by $\langle X, A\rangle := X(A)$.

The flat norm $|\cdot|_\flat$ on the space of flat k-cochains is the dual norm to the flat norm on flat k-chains:

$$|X|_\flat := \sup\{\langle X, \sigma\rangle : \sigma \in \mathcal{F}_k(V), |\sigma|_\flat \leq 1\}.$$

There is a natural "coboundary" operator, denoted "d," on cochains which is the adjoint of the boundary operator on chains. More specifically, if X is a k-cochain, then the cochain dX is the $(k+1)$-cochain whose action on any $(k+1)$-chain τ is

$$\langle dX, \tau\rangle := \langle X, \partial\tau\rangle.$$

Proposition III.10. *The coboundary operator is a bounded operator $d : \mathcal{F}^k(V) \to \mathcal{F}^{k+1}(V)$ with $|dX|_\flat \leq |X|_\flat$.*

Proof: Let $X \in \mathcal{F}^k(V)$. For any $\tau \in \mathcal{P}_{k+1}(V)$ with $|\tau|_\flat \leq 1$, we have

$$\langle dX, \tau\rangle = \langle X, \partial\tau\rangle \leq |X|_\flat \cdot |\partial\tau|_\flat \leq |X|_\flat \cdot |\tau|_\flat \leq |X|_\flat.$$

Hence $|dX|_\flat \leq |X|_\flat$, so the coboundary operator $d : \mathcal{F}^k(V) \to \mathcal{F}^{k+1}(V)$ is bounded.

□

One can also define comass as the dual norm to mass on polyhedral chains.

Definition III.11. The comass of a cochain X, denoted by $|X|$, is

$$|X| := \sup\{\langle X, \sigma\rangle : \sigma \in \mathcal{P}_k, |\sigma| \leq 1\}.$$

This definition yields another characterization of the flat norm on flat cochains:

Lemma III.12. *If X is a flat cochain, then $|X|_\flat = \max\{|X|, |dX|\}$.*

Proof: Let $X \in \mathcal{F}^k(V)$.

We first show that $|X|_\flat \leq \max\{|X|, |dX|\}$. Suppose $\sigma \in \mathcal{P}_k(V)$. Then for all $\tau \in \mathcal{P}_{k+1}(V)$,

$$\begin{aligned}
\langle X, \sigma\rangle &= \langle X, \sigma - \partial\tau\rangle + \langle X, \partial\tau\rangle \\
&\leq \langle X, \sigma - \partial\tau\rangle + \langle dX, \tau\rangle \\
&\leq |X| \cdot |\sigma - \partial\tau| + |dX| \cdot |\tau| \\
&\leq \max\{|X|, |dX|\}(|\sigma - \partial\tau| + |\tau|).
\end{aligned}$$

Taking the infimum over all such τ, we have $\langle X, \sigma\rangle \leq \max\{|X|, |dX|\}|\sigma|_\flat$, so $|X|_\flat \leq \max\{|X|, |dX|\}$.

To show the opposite inequality, note that for all $\varepsilon > 0$, there exists $\sigma \in \mathcal{P}_k(V)$ with $|\sigma| \leq 1$ so that

$$|\langle X, \sigma\rangle| > |X| - \varepsilon.$$

Since $|\sigma|_\flat \leq 1$, we also have

$$|\langle X, \sigma\rangle| \leq |X|_\flat.$$

Hence, $|X| - \varepsilon < |X|_\flat$, and so $|X| \leq |X|_\flat$.

Similarly, for all $\varepsilon > 0$, there exists $\tau \in \mathcal{P}_{k+1}(V)$ with $|\tau| \le 1$ so that

$$|\langle X, \partial\tau\rangle| = |\langle dX, \tau\rangle| > |dX| - \varepsilon.$$

Since $|\partial\tau|_\flat \le |\tau| \le 1$, we also have

$$|\langle X, \partial\tau\rangle| \le |X|_\flat.$$

Hence, $|dX| - \varepsilon < |X|_\flat$, and so $|dX| \le |X|_\flat$. This shows that $\max\{|X|, |dX|\} \le |X|_\flat$.

□

3.5 Mass distortion under linear maps

Simplexes push forward in a natural way under (injective) linear maps. Let $L : V \to W$ be a linear map between Banach spaces and let $\sigma = [v_0, \ldots, v_k]$ be a simplex in V. If L maps the affine plane P_σ into W, then $[L(v_0), \ldots, L(v_k)]$ is a simplex in W and we define $L(\sigma) := [L(v_0), \ldots, L(v_k)]$. If L is not injective on P_σ, then the image $L(\sigma)$, when considered as a polyhedral k-chain, is equivalent to the zero k-chain. We then define the pushforward of a polyhedral k-chain $A = \sum_{i=1 \to N} \lambda_i \sigma_i$ by $L(A) := \sum_{i=1 \to N} \lambda_i L(\sigma_i)$.

Moreover, if L is an isomorphism, then $L(\partial\sigma_i) = \partial L(\sigma)$ for all i, so $L(\partial A) = \partial L(A)$.

The following lemma shows that the mass of a simple polyhedral chain P increases by a controlled amount under a linear map.

Lemma III.13. *Let $f \colon V \to W$ be a linear map between k-dimensional Banach spaces with operator norm $\|f\|$ and let σ be a simple polyhedral k-chain in V. Then $|f\sigma| \le \|f\|^k |\sigma|$.*

Proof: We use induction on k. The case $k = 0$ is trivial. For $k > 0$, we assume that f is injective since otherwise, the k-dimensional mass of $f\sigma$ would be zero. Now assume the lemma holds in dimension $k-1$. Given $f\colon V \to W$, we associate to every linear map $\widetilde{g} \in W^*$ with $\|\widetilde{g}\| = 1$ a map $g \in V^*$ that has $\|g\| \leq 1$:

$$\widetilde{g} \mapsto g := \frac{\widetilde{g} \circ f}{\|f\|}.$$

By the definition of mass,

$$\begin{aligned} |f\sigma| &= \sup_{\substack{\widetilde{g}\in W^* \\ \|\widetilde{g}\|=1}} \int_{\mathbb{R}} |f\sigma \cap \widetilde{g}^{-1}(x)|\, dx \\ &= \sup_{\substack{\widetilde{g}\in W^* \\ \|\widetilde{g}\|=1}} \int_{\mathbb{R}} \left| f\sigma \cap f \circ g^{-1}\left(\frac{x}{\|f\|}\right)\right| dx. \end{aligned}$$

By the change of variables $z = \frac{x}{\|f\|}$, and the fact that $f\sigma \cap (f \circ g^{-1})(z) = f(\sigma \cap g^{-1}(z))$, we have

$$\begin{aligned} |f\sigma| &= \sup_{\substack{\widetilde{g}\in W^* \\ \|\widetilde{g}\|=1}} \|f\| \int_{\mathbb{R}} |f\sigma \cap (f\circ g^{-1})(z)|\, dz \\ &= \sup_{\substack{\widetilde{g}\in W^* \\ \|\widetilde{g}\|=1}} \|f\| \int_{\mathbb{R}} |f(\sigma \cap g^{-1}(z))|\, dz. \end{aligned}$$

Since each $\widetilde{g}$ with norm one corresponds to a g with norm at most one, the supremum in the previous equation gets larger if we take it over all g in V^* with norm one:

$$\sup_{\substack{\widetilde{g}\in W^* \\ \|\widetilde{g}\|=1}} \|f\| \int_{\mathbb{R}} |f(\sigma \cap g^{-1}(z))|\, dz \leq \sup_{\substack{g\in V^* \\ \|g\|=1}} \|f\| \int_{\mathbb{R}} |f(\sigma \cap g^{-1}(z))|\, dz$$

Each slice $f(g^{-1}(z) \cap \sigma)$ is a simple polyhedral $(k-1)$-chain, so we apply the induction hypothesis to obtain

$$|f\sigma| \leq \sup_{\substack{g\in V^* \\ \|g\|=1}} \|f\| \int_{\mathbb{R}} \|f\|^{k-1} |\sigma \cap g^{-1}(z)|\, dz = \|f\|^k \cdot |\sigma|.$$

□

Let $P = \sum \lambda_i \sigma_i \in \mathcal{P}_k(V)$, where the simple polyhedral chains $\{\sigma_i\}$ have disjoint interiors. Since $|P| = \sum |\lambda_i| \cdot |\sigma_i|$, we can extend the result of Lemma III.13 to all polyhedral chains.

Definition III.14. The *fullness* of a simplex σ, denoted $\Theta(\sigma)$, is the ratio

$$\Theta(\sigma) := \frac{|\sigma|}{\operatorname{diam}(\sigma)^k}.$$

Given this definition of fullness, Lemma III.13 yields the following corollary.

Corollary III.15. *Let $f\colon V \to W$ be a linear isomorphism between k-dimensional Banach spaces and let σ be a k-simplex in V. Then*

$$\Theta(f\sigma) \geq C\Theta(\sigma),$$

where $C = C(\|f\|, \|f^{-1}\|, k) > 0$ depends only on $\|f\|$, $\|f^{-1}\|$, and k.

Lemma III.16. *Given a linear isomorphism $f\colon V \to W$ between n-dimensional Banach spaces and a k-dimensional polyhedral chain $P \subset \mathcal{P}_k(V)$, f does not increase the flat norm of P by more than a constant factor:*

$$|fP|_\flat \leq C(\|f\|, k)|P|_\flat.$$

Proof: Let $\varepsilon > 0$. There exists a $(k+1)$-chain $\tilde{Q} \in \mathcal{P}^{k+1}(V)$ such that $|\tilde{Q}| + |\partial\tilde{Q} - P| \leq |P|_\flat + \varepsilon$. By Lemma III.13,

$$|f\tilde{Q}| \leq \|f\|^{k+1}|\tilde{Q}|$$

and

$$|f(\partial\tilde{Q} - P)| \leq \|f\|^k|\partial\tilde{Q} - P|.$$

Since f and the boundary operator ∂ are both linear and f is an isomorphism, $f(\partial\tilde{Q} - P) = \partial(f\tilde{Q}) - fP$. Thus,

$$|\partial(f\tilde{Q}) - fP| \leq C(\|f\|, k)|\partial\tilde{Q} - P|.$$

Putting these inequalities together gives

$$\begin{aligned} |fP|_\flat &\le |f\tilde{Q}| + |\partial(f\tilde{Q}) - fP| \\ &\le \max\{\|f\|^k, \|f\|^{k+1}\} \cdot (|\tilde{Q}| + |\partial\tilde{Q} - P|) \\ &\le C'(\|f\|, k)(|P|_\flat + \varepsilon). \end{aligned}$$

Since ε was arbitrary, $|fP|_\flat \le C'(\|f\|, k)|P|_\flat$. □

3.6 The mass of a flat chain

So far we have only used mass for *polyhedral* chains. In [Ada08], Adams defines the mass of a flat chain A to be the quantity

$$|A| := \liminf_{P_j \to_\flat A} |P_j|.$$

In other words, $|A|$ is the smallest number with the property that there exists a sequence (P_j) of polyhedral chains converging to A in the flat norm whose masses converge to $|A|$.

By the lower-semicontinuity of mass proven in [Ada08], this definition of mass agrees with Definition III.3 on polyhedral chains.

CHAPTER IV

Partial Forms

4.1 Motivational remarks

With an eye toward generalization to the Banach space setting, we begin with a discussion of Wolfe's duality theorem in Euclidean space. We first recall the definition of a flat differential form.

Definition IV.1. Let $\omega\colon \mathbb{R}^n \to \Lambda^k\mathbb{R}^n$ be a measurable differential k-form, i.e., a differential form that can be written as $\omega(x) = \sum a_{i_1,\ldots,i_k}(x)dx_{i_1}\cdots dx_{i_k}$, where the sum is taken over all increasing sequences $1 \le i_1 < \cdots < i_k \le n$ and the coefficient functions $a_{i_1,\ldots,i_k}(x)$ are measurable. If the coefficients are also locally integrable, one defines the distributional exterior derivative $d\omega$ of ω to be the (unique) measurable $(k+1)$-form that satisfies

$$\int_\Omega d\omega \wedge \eta = (-1)^{k+1}\int_\Omega \omega \wedge d\eta$$

for every smooth compactly supported $(n-k-1)$-form η, if such a form exists. We say that ω is *flat* if both $\|\omega\|_\infty < \infty$ and $\|d\omega\|_\infty < \infty$, where

$$\|\omega\|_\infty := \operatorname*{ess\,sup}_{x\in\mathbb{R}^n} \|\omega(x)\|_{\text{comass}}.$$

In this case, the *flat norm of* ω is

$$\|\omega\|_\flat := \max\{\|\omega\|_\infty, \|d\omega\|_\infty\}.$$

In the preceding definition, two forms ω and ω' are equivalent if and only if $\|\omega - \omega'\|_\infty = 0$. For more details, see [Whi57] and [Hei05]. Whitney's proof of this fact uses a technique similar to Lebesgue differentiation to produce the differential form associated to a given flat k-cochain.

Let X be a flat k-cochain, and fix a point $p \in \mathbb{R}^n$ and a k-direction $\nu \in \Lambda_k\mathbb{R}^n$. Let W be the ν-superplane containing p. Consider sequences (σ_i) of oriented, η-full simplexes in W that have p as a vertex, carry the same orientation as W, and whose diameters decrease toward zero. Let $D_X(p,\nu)$ be the following limit, if it exists for *every* such sequence (σ_i):

$$D_X(p,\nu) := \lim_{i\to\infty} \frac{\langle X, \sigma_i\rangle}{|\sigma_i|}, \tag{4.1}$$

Whitney proves that one can obtain a flat differential form from these limits. We restate this result (Theorem 5A from [Whi57, p. 261]) below.

Theorem IV.2. *(Whitney) Let X be a flat k-cochain in $\mathbb{R}^d$, $d \geq k$. Then there is a set Q of full measure in $\mathbb{R}^d$ such that for each $p \in Q$, $D_X(p,\nu)$ is defined for all k-directions ν, and is extendable to all k-directions ν, giving a k-covector $D_X(p)$. The function $D_X : \mathbb{R}^d \to \Lambda^k\mathbb{R}^d$ is a bounded, measurable k-form in $\mathbb{R}^d$. Furthermore, for any k-simplex σ in $\mathbb{R}^d$, D_X is a measurable k-form on the affine k-plane P_σ, and $\langle X, \sigma\rangle = \int_\sigma D_X$.*

The same facts are true for the cochain dX. Also, $\|D_X\|_\infty = |X|$ and $\|D_{dX}\|_\infty = |dX|$.

Wolfe's theorem (Theorem 7a, 7b, and 7c from [Whi57, pp. 263–5]) asserts that we can *isometrically* identify the space of flat k-forms with the space of flat k-cochains in $\mathbb{R}^n$:

Theorem IV.3. *(Wolfe) Suppose that ω is a flat k-form in $\mathbb{R}^n$. Then there exists a flat differential k-form $\widetilde{\omega}$ such that $\|\omega - \widetilde{\omega}\|_\infty = 0$ and there exists a unique flat k-cochain X so that for every simplex $\sigma \subset \mathbb{R}^n$,*

$$\int_\sigma \widetilde{\omega} = \langle X, \sigma \rangle.$$

Furthermore, the flat norms of X and ω are isometric, i.e., $\|\omega\|_\flat = \|\widetilde{\omega}\|_\flat = |X|_\flat$.

In order to generalize Wolfe's theorem to the Banach space setting, we must first define a differential k-form on a Banach space V. One natural definition (see [Fed69, p. 17]) would be to define a form to be a map $\omega \colon V \to \mathrm{Hom}(\Lambda_k V, \mathbb{R})$. However, in order to define a flat form, we desire some notion of equivalence classes of differential forms, and in a general Banach space there is no natural measure on V that we can use to construct equivalence classes. More importantly, we would like to use the Lebesgue differentiation approach outlined above to define the form associated with a k-cochain. However, this approach only gives a well defined action almost everywhere on every affine k-plane *in the direction of that k-plane.* In finite dimensions one can use Fubini's theorem to conclude that almost everywhere the action is defined in *every* direction (thus proving Theorem IV.2). However, if $\dim V = \infty$, this approach would not prove even for a single p in V that $\omega(p)(\nu)$ is defined for *every* k-vector ν, which is necessary for all points p in the domain of ω.

4.2 Partial forms

For a classical differential form ω on a Banach space V, given a point $p \in V$, $\omega(p)$ is a linear map from $\Lambda_k V$ to $\mathbb{R}$. By the universal property discussed in Chapter II, this linear map corresponds to a unique alternating k-linear map on the product space $[V]^k$. We overcome the difficulties mentioned in the preceding section by defining a partial form to be a certain type of function on a subset of the product space $V \times [V]^k$.

Let V be a Banach space, and $F\colon U \to \mathbb{R}$ a function on a subset U of the product space $V \times [V]^k$. For a point $\vec{v} = (v_1, \ldots, v_k) \in [V]^k$, define the set $U_{\vec{v}} \subset V$ by $U_{\vec{v}} := \{p \in V : (p, \vec{v}) \in U\}$ and the *horizontal slice* function by $F_{\vec{v}} := F(\cdot, \vec{v}) : U_{\vec{v}} \to \mathbb{R}$. For a point $p \in V$, define the set $U_p \subset [V]^k$ by $U_p := \{\nu \in [V]^k : (p, \nu) \in U\}$. The *vertical slice* function is $F_p := F(p, \cdot)\colon U_p \to \mathbb{R}$.

Definition IV.4. Let V be a Banach space, and $F\colon U \to \mathbb{R}$ a function on a subset U of the product space $V \times [V]^k$. The function F is a *measurable partial k-form* on V if it satisfies the following three properties.

(i) For each $\vec{v} \in [V]^k$, $F_{\vec{v}}$ is a Borel function, i.e., preimages of open sets are Borel.

(ii) For all $p \in V$, F_p is Borel.

(iii) (Multilinearity conditions) For any d-dimensional affine subspace $W^d \subset V$ with $d \geq k$ and almost every point $p \in W^d$, the set $[W_0^d]^k$ is in U_p and the restriction of F_p to $[W_0^d]^k$ is alternating and k-linear.

Notation IV.5. Let F be a measurable partial k-form, and $p, v_1, \ldots, v_k \in V$. Denote $T = \operatorname{span}\{v_1, v_2, \ldots, v_k\}$. If $[T]^k$ is in U_p and the restriction of F_p to $[T]^k$ is alternating and k-linear, then for any $x_1, \ldots, x_k$ and $y_1, \ldots, y_k$ in T with $x_1 \wedge \ldots \wedge x_k = y_1 \wedge \ldots \wedge y_k = \nu \in \Lambda_k T$,

$$F(p, (x_1, \ldots, x_k)) = F(p, (y_1, \ldots, y_k)),$$

so we write

$$F(p, \nu) := F(p, (x_1, \ldots, x_k)).$$

In the case that $V = \mathbb{R}^n$, the space of measurable partial k-forms coincides with the space of classical measurable k-forms.

Remark IV.6. Every partial form can be regarded as a collection of differential forms on finite-dimensional affine subspaces of V that agree almost everywhere on the overlap of the subspaces.

Let F be a partial k-form on V and let $W \subset V$ be a d-dimensional affine subspace of V with $d \geq k$. There exists a Borel function $\omega_W : W \to \Lambda^k W_0$ with the following property. At almost every point $x \in W$, the k-covector $\omega_W(x)$ satisfies $\langle \omega_W(x), \alpha \rangle = F(x, \alpha)$ for all $\alpha \in \Lambda_k W_0$. The function ω_W is a differential form on the subspace W. The collection $\{\omega_W : W \subset V\}$ of these forms has the property that if W and W' are affine subspaces with $\dim(W \cap W') = r \geq k$, then $\omega_{W \cap W'} = \omega_W$ $\mathcal{H}^r$-a.e. on $W \cap W'$.

However, given such a collection of forms on all finite-dimensional affine subspaces of V, it is not clear whether one can produce a single function which satisfies all the conditions of a partial form and agrees with each original form almost everywhere on the domain of the original form.

Given a partial form $F : U \to \mathbb{R}$ in a Banach space V and an affine subspace W of V, let $U' = U \cap (W \times [W_0]^k)$. We define the *restriction of F to W* to be the function $F_W : U' \to \mathbb{R}$ given by $F_W(p, \vec{v}) = F(p, \vec{v})$ for all $(p, \vec{v}) \in U'$. We note that almost everywhere in W, $(F_W)_p$ is equal to the form $\omega_W(p)$ from Remark IV.6.

4.3 The comass norm

In this section we endow the space of partial forms with the so-called comass norm; this norm will be used to define equivalence classes of partial forms and, in Section 4.7, to define the flat norm on partial forms.

We define the comass, or L^∞-norm on partial forms using the k-dimensional affine subspaces of V.

Let F be a partial k-form. Fix an oriented k-dimensional affine subspace $W \subset V$. By property (iii) of a partial form, for almost every point $p \in W$, F_p is alternating and multilinear on $[W]^k$. Hence, we may define

$$\|F\|_{\infty,W} := \operatorname*{ess\,sup}_{p\in W} |F_p((v_1,\ldots,v_k))|, \tag{4.2}$$

where $v_1 \wedge \cdots \wedge v_k = \nu_W$ is any representation of the k-direction of W^k.

The comass of the partial k-form F is then the following, where the supremum is taken over all k-dimensional affine subspaces of V:

$$\|F\|_\infty := \sup_{W^k \subset V} \|F\|_{\infty,W^k}. \tag{4.3}$$

Remark IV.7. The norm $\|\cdot\|_\infty$ from equation (4.3) does not match the standard definition of the comass in Euclidean space. Classically, the comass of a k-form $\omega\colon \mathbb{R}^n \to \Lambda^k\mathbb{R}^n$ is defined to be

$$\|\omega\|_\infty := \operatorname*{ess\,sup}_{p\in\mathbb{R}^n} \|\omega(p)\|_{\text{comass}}.$$

Here $\|\omega(p)\|_{\text{comass}}$ represents the Euclidean comass of the k-covector $\omega(p)$ as defined in Appendix A. We note that the classical comass of a k-form in $\mathbb{R}^n$ can be strictly less than the comass of such a form with our definition, as shown in the following example.

Example IV.8. Let $\omega\colon \mathbb{R}^2 \to \Lambda^1\mathbb{R}^2$ be the 1-form

$$\omega((x,y)) = \chi_{\{0\}}(x)\,dy,$$

where $(x,y) \in \mathbb{R}^2$ The partial 1-form F associated to ω (i.e., the partial form F for which $\omega_p(v) = F(p,v)$) is given by

$$F((x,y),(a,b)) = b \cdot \chi_0(x),$$

where (x, y) and $(a, b) \in \mathbb{R}^2$. As a Euclidean 1-form, $\|\omega\|_\infty = 0$, but as a partial form, $\|F\|_\infty = 1$.

If $F\colon U_F \to \mathbb{R}$ and $G\colon U_G \to \mathbb{R}$ are partial k-forms, let $U := U_F \cap U_G$ and define $F + G\colon U \to \mathbb{R}$ by

$$(F + G)(p, \vec{v}) := F(p, \vec{v}) + G(p, \vec{v})$$

for all pairs $(p, \vec{v}) \in U$. The function $F + G$ can easily be seen to satisfy conditions (i)–(iii) of a partial form. Furthermore, if $a \in \mathbb{R}$, we define the function $aF\colon U_F \to \mathbb{R}$ by

$$(aF)(p, \vec{v}) := a \cdot F(p, \vec{v}).$$

The function aF is also a partial form. We define an equivalence relation on partial forms that associates two partial k-forms F and G if and only if $\|F - G\|_\infty$ is zero:

$$F \sim G \qquad \Leftrightarrow \qquad \|F - G\|_\infty = 0.$$

Since the comass given by equation (4.3) can be larger than the classical L^∞-norm of a form in $\mathbb{R}^n$, the equivalence classes determined by equation (4.3) are smaller than those obtained classically.

The following lemma shows that the space of measurable partial k-forms on V equipped with this equivalence relation is a linear space.

Lemma IV.9. *Let $F\colon U_F \to \mathbb{R}$ and $G\colon U_G \to \mathbb{R}$ be partial k-forms. Suppose that $\widetilde{F}\colon U_{\widetilde{F}} \to \mathbb{R}$ and $\widetilde{G}\colon U_{\widetilde{G}} \to \mathbb{R}$ are partial k-forms such that $\widetilde{F} \in [F]$ and $\widetilde{G} \in [G]$. Then the partial form $\widetilde{F} + \widetilde{G}$ is in the equivalence class $[F + G]$.*

Proof: Define the set $U \subset V \times [V]^k$ by $U := U_F \cap U_G \cap U_{\widetilde{F}} \cap U_{\widetilde{G}}$. For almost every point $p \in W$ and every $\vec{v} \in [W_0]^k$, $(p, \vec{v}) \in U$, and F_p, G_p, $\widetilde{F}_p$, and $\widetilde{G}_p$ are alternating and multilinear in $\vec{v}$ by property (iii) of Definition IV.4. Let $H = ((F+G)-(\widetilde{F}+\widetilde{G}))$:

$U \to \mathbb{R}$. For any $(p, \vec{v}) \in U$,

$$\begin{aligned} H(p, \vec{v}) &= (F(p, \vec{v}) + G(p, \vec{v})) - (\widetilde{F}(p, \vec{v}) + \widetilde{G}(p, \vec{v})) \\ &= F(p, \vec{v}) - \widetilde{F}(p, \vec{v}) + G(p, \vec{v}) - \widetilde{G}(p, \vec{v}) \end{aligned}$$

Let W be a k-dimensional affine subspace of V. Since $\widetilde{F} \in [F]$, we have $\|F - \widetilde{F}\|_\infty = 0$, so

$$\operatorname*{ess\,sup}_{p \in W} \{F(p, \nu_W) - \widetilde{F}(p, \nu_W)\} = 0.$$

Similarly,

$$\operatorname*{ess\,sup}_{p \in W} \{G(p, \nu_W) - \widetilde{G}(p, \nu_W)\} = 0.$$

Hence,

$$\operatorname*{ess\,sup}_{p \in W} \{H(p, \nu_W)\} = \operatorname*{ess\,sup}_{p \in W} \{F(p, \nu_W) - \widetilde{F}(p, \nu_W) + G(p, \nu_W) - \widetilde{G}(p, \nu_W)\} = 0.$$

□

We now show that, equipped with the equivalence relation given above, $\|\cdot\|_\infty$ is a norm.

Lemma IV.10. $\|\cdot\|_\infty$ *is a norm on the space of finite-comass partial k-forms.*

Proof: Clearly, $\|\lambda F\|_\infty = |\lambda| \cdot \|F\|_\infty$ for any $\lambda \in \mathbb{R}$ and any partial form F. It is also clear from the definition of the equivalence relation on partial forms that if $\|F\|_\infty = 0$ then F is equivalent to the zero form $\underline{0} : V \times [V]^k \to \mathbb{R}$ that maps $(p, \vec{v}) \mapsto 0$. To show the triangle inequality, suppose F and G are partial k-forms. For $p \in V$, denote as usual by U_p the subset of $[V]^k$ for which $(F + G)_p$ is defined. Similarly, let U'_p and U''_p denote the domains of the functions F_p and G_p, respectively. For all p and all $\vec{v} \in U_p$,

$$(F + G)(p, \vec{v}) := \omega(p, \vec{v}) + \eta(p, \vec{v}).$$

Thus, for a fixed k-dimensional subspace W^k in V with k-direction $\nu_W = v_1 \wedge \cdots \wedge v_k$, set $\vec{v}_W = (v_1, \ldots, v_k)$ so that

$$\begin{aligned} \|F+G\|_{\infty,W^k} &= \operatorname*{ess\,sup}_{p\in W^k} \{(F+G)_p(\vec{v}_W) : \vec{v}_W \in U_p\} \\ &\leq \operatorname*{ess\,sup}_{p\in W^k} \left\{F_p(\vec{v}_W) : \vec{v}_W \in U'_p\right\} + \operatorname*{ess\,sup}_{p\in W^k} \left\{G_p(\vec{v}_W) : \vec{v}_W \in U''_p\right\} \\ &= \|F\|_{\infty,W^k} + \|G\|_{\infty,W^k}. \end{aligned}$$

We then have

$$\|F+G\|_\infty := \sup_{W^k \subset V} \|F+G\|_{\infty,W^k} \tag{4.4}$$

$$\leq \sup_{W^k \subset V} \|F\|_{\infty,W^n} + \sup_{W^k \subset V} \|G\|_{\infty,W^n} \tag{4.5}$$

$$= \|F\|_\infty + \|G\|_\infty. \tag{4.6}$$

This shows that $\|\cdot\|_\infty$ satisfies the triangle inequality and is a norm on the space of partial forms that have finite comass. $\square$

The comass of a partial form F bounds the comass of the partial form restricted to any finite dimensional subspace W^n of V: $\|F_{W^n}\|_\infty \leq \|F\|_\infty$.

4.4 Locally integrable partial forms act on polyhedral chains

In order to integrate a partial forms over polyhedral chains, we consider pullbacks of partial forms under affine transformations.

Definition IV.11. Given a partial k-form F and an affine k-plane P, let L be an orientation-preserving affine isomorphism from $\mathbb{R}^k$ to P. Then $F(x, \nu_P)$ is well-defined for almost every $x \in P$. The pullback L^*F is a top-dimensional differential form on $\mathbb{R}^k$ given by the formula

$$L^*F(y, e_1 \wedge \cdots \wedge e_k) := F(L(y), \nu_P).$$

The pullback L^*F is defined on almost every $y \in \mathbb{R}^k$.

Suppose that $\widetilde{L}$ is another orientation-preserving affine isomorphism from $\mathbb{R}^k$ to P. The pullback $\widetilde{L}^*F$ is also a top-dimensional differential form on $\mathbb{R}^k$, and can be again pulled back under the map $T = \widetilde{L}^{-1} \circ L$. Then we have

$$T^*(\widetilde{L}^*F)(y) = \widetilde{L}^*F(T(y)) = F((\widetilde{L} \circ T)(y), \nu_P) = F(L(y), \nu_P) = L^*F(y).$$

Thus, if L^*F is locally integrable, so is $\widetilde{L}^*F$. We use this fact to define a locally integrable partial k-form.

Definition IV.12. A partial k-form F is *locally integrable* if for every affine k-plane P there exists an affine isomorphism L_P from $\mathbb{R}^k$ to P so that the pullback L_P^*F is locally integrable.

Hence, a locally integrable partial k-form has an intrinsic integral over simple polyhedral chains σ in V given by these pullbacks. In particular, if P is the k-plane containing σ, then

$$\int_\sigma F := \int_{L_P^{-1}\sigma} L^*F,$$

where the integral on the right-hand side is taken with respect to Lebesgue k-measure $\mathcal{L}^k$ on $\mathbb{R}^k$. This integral is well defined by the preceding remarks.

Hence, a locally integrable partial k-form F on a Banach space V acts in a natural way by integration on k-dimensional simplexes σ in V:

$$\langle F, \sigma \rangle := \int_\sigma F < \infty.$$

In the following lemma we show that one can actually regard the action of F on σ as integration in V with respect to Adams's mass measure (Gromov's mass*-measure) on k-dimensional planes.

Lemma IV.13. *If F is a locally integrable partial k-form and σ is a k-simplex in V, choose $\vec{v}_\sigma = (v_1, \dots, v_k) \in [V]^k$ so that $\nu_\sigma = v_1 \wedge \cdots \wedge v_k$. Then*

$$\langle F, \sigma \rangle = \int_\sigma F(p, \vec{v}_\sigma)\, d\mathcal{M}^k(p),$$

where $\mathcal{M}^k$ is the k-dimensional mass measure on the affine k-plane containing σ.*

Proof: Fix an affine isomorphism L from $\mathbb{R}^k$ to the affine k-plane P containing σ. We first use Wenger's Lemma to calculate the pushforward of the measure $\mathcal{H}^k$ under L; i.e., we calculate the constant C for which $L_*\mathcal{H}^k = C\mathcal{M}^k$. (Note that C depends on the k-direction ν_σ of σ.) Let $(e_1, \dots, e_k)$ be the standard orthonormal basis of $\mathbb{R}^k$. Then the measure of the k-simplex $[0, e_1, \dots, e_k]$ is

$$\frac{1}{k!} = \mathcal{H}^k[0, e_1, \dots, e_k] = L_*\mathcal{H}^k[L(e_1), \dots, L(e_k)].$$

On the other hand, by Wenger's Lemma, the Adams mass of the simplex $[L(0), L(e_1), \dots, L(e_k)]$ is

$$|[L(0), L(e_1), \dots, L(e_k)]| = \frac{1}{k!}\|L(e_1) \wedge \cdots \wedge L(e_k)\|_{\mathrm{m}^*}. \tag{4.7}$$

Since $L_*\mathcal{H}^k[L(e_1), \dots, L(e_k)] = C|[L(e_1), \dots, L(e_k)]|$, we conclude that

$$C = \frac{1}{\|L(e_1) \wedge \cdots \wedge L(e_k)\|_{\mathrm{m}^*}}.$$

Then

$$\begin{aligned}
\langle F, \sigma \rangle &= \int_{L^{-1}\sigma} L^*F \\
&= \int_{L^{-1}\sigma} L^*F(q, (e_1, \dots, e_k))\, d\mathcal{H}^k(q) \\
&= \int_\sigma F(p, L_*(e_1, \dots, e_k))\, d(L_*\mathcal{H}^k)(p) \\
&= \int_\sigma F(p, (L(e_1), \dots, L(e_k)))\, d(L_*\mathcal{H}^k)(p).
\end{aligned}$$

However, for almost all $p \in \sigma$,

$$F(p, (L(e_1), \ldots, L(e_k))) = \|(L(e_1), \ldots, L(e_k))\|_{\mathrm{m}^*} \cdot F(p, \vec{v}_\sigma).$$

Thus

$$\begin{aligned}\int_\sigma F(p, (L(e_1), \ldots, L(e_k)))\, d(L_*\mathcal{H}^k)(p) &= \|L(e_1) \wedge \cdots \wedge L(e_k)\|_{\mathrm{m}^*} \int_\sigma F\left(p, \vec{v}_\sigma\right)\, d(L_*\mathcal{H}^k)(p) \\ &= \int_\sigma F(p, \vec{v}_\sigma)\, d\mathcal{M}^k(p),\end{aligned}$$

completing the proof. □

Corollary IV.14. *For any polyhedral k-chain A,*

$$|\langle F, A\rangle| \leq |A| \cdot \|F\|_\infty. \tag{4.8}$$

Proof: The statement follows from Lemma IV.13 equation (3.8) □

The following lemma shows that if two locally integrable partial forms act identically (by integration) on all polyhedral chains, the forms are equivalent.

Lemma IV.15. *Suppose that F_1 and F_2 are locally integrable k-forms such that for all polyhedral k-chains τ, $\langle F_1, \tau\rangle = \langle F_2, \tau\rangle$. Then $F_1 \sim F_2$, i.e., $\|F_1 - F_2\|_\infty = 0$.*

Proof: Let P be a k-plane with k-direction $\nu = \nu_P$ and let $L\colon \mathbb{R}^k \to P$ denote the $\sqrt{k}$-bi-Lipschitz linear isomorphism given by Theorem II.8. As in Definition IV.11, the pullbacks L^*F_1 and L^*F_2 are measurable, locally integrable functions on $\mathbb{R}^k$. As classical top-dimensional forms (i.e., functions on $\mathbb{R}^k$), $d(L^*F_1) = d(L^*F_2) = 0$.

By assumption, for any polyhedral k-chain τ in $\mathbb{R}^k$,

$$\int_\tau L^*F_1 = \int_{L(\tau)} F_1 = \int_{L(\tau)} F_2 = \int_\tau L^*F_2.$$

The Euclidean flat form $(L^*F_1 - L^*F_2)$ is identified with the zero cochain via Wolfe's theorem (see Section 4.1), since for any τ,

$$\langle L^*F_1 - L^*F_2, \tau\rangle := \int_\tau L^*F_1 - L^*F_2 = 0.$$

Again by Wolfe's theorem, the flat norm of $L^*F_1 - L^*F_2$ as a flat differential form is zero, and so

$$\|L^*F_1 - L^*F_2\|_\infty := \operatorname*{ess\,sup}_{x \in \mathbb{R}^k}\{(L^*F_1 - L^*F_2)(x)\} = 0. \tag{4.9}$$

Equation (4.9) implies that $(F_1)_\nu - (F_2)_\nu = 0$ almost everywhere on the affine k-plane P. Since P was arbitrary, $\|(F_1 - F_2)\|_\infty = 0$, so $F_1 \sim F_2$. □

4.5 The exterior-d operator

In this section we define the exterior derivative of a locally integrable partial form. Not every locally integrable partial form will have a derivative, but if the derivative exists, it is defined (up to equivalence class, see Lemma IV.15) by its action on polyhedral chains of the appropriate dimension.

Definition IV.16. Suppose that F is a locally integrable k-form. Then dF (if it exists) is the locally integrable $(k+1)$-form such that for all polyhedral $(k+1)$-chains τ,

$$\langle dF, \tau \rangle = \int_\tau dF = \int_{\partial\tau} F = \langle F, \partial\tau \rangle. \tag{4.10}$$

To show that the exterior-d operator is well defined, suppose that F_1 and F_2 are equivalent partial forms (i.e., $\|F_1 - F_2\|_\infty = 0$) and that dF_1 exists. If τ is a $(k+1)$-chain, then $F_1 = F_2$ almost everywhere on $\partial\tau$, so $\langle dF_1, \tau\rangle = \langle F_1, \partial\tau\rangle = \langle F_2, \partial\tau\rangle$, so dF_1 is the exterior derivative of F_2.

Suppose F and G are locally integrable partial k-forms for which the $(k+1)$-forms dF and dG exist. Since the action of forms on polyhedra is linear, the partial $(k+1)$-form $d(F+G)$ exists and equals the partial $(k+1)$-form $dF + dG$, since for all polyhedral $(k+1)$-chains τ,

$$\begin{aligned}\langle d(F+G),\tau\rangle &:= \langle F+G,\partial\tau\rangle \\ &= \langle F,\partial\tau\rangle + \langle G,\partial\tau\rangle \\ &= \langle dF,\tau\rangle + \langle dG,\tau\rangle \\ &= \langle dF+dG,\tau\rangle.\end{aligned}$$

Thus, the exterior d operator is linear. Furthermore, equation (4.10) implies that $d^2F = d(dF) \equiv 0$, because $\partial^2\tau = \partial(\partial\tau) \equiv 0$ for any polyhedral $(k+1)$-chain τ.

Remark IV.17. Our notion of an exterior-d operator is motivated by the concept of a weak derivative. Recall that for $u, v \in L^1(\mathbb{R})$, v is the weak derivative of u if for every continuously differentiable, compactly supported *test function* φ, $\int_{\mathbb{R}} u(t)\varphi'(t)\,dt = -\int_{\mathbb{R}} v(t)\varphi(t)\,dt$.

In the setting of partial forms, we require equation (4.10), that the exterior derivative satisfy Stokes's theorem for all *test polyhedra.*

4.6 The wedge product of partial forms

Let S_n be the symmetric group consisting of all permutations of the set $\{1, 2, \ldots, n\}$. If σ is a permutation in S_n, we denote the sign of σ by $\operatorname{sgn}(\sigma)$.

Suppose that $F\colon U_F \to \mathbb{R}$ is a measurable partial k-form and $G\colon U_G \to \mathbb{R}$ is a measurable partial ℓ-form according to Definition IV.4.

Towards defining the wedge product of the forms F and G, we first define a set $U \subset V \times [V]^{k+\ell}$ which will be the domain of this wedge product. A pair $(p, (v_1, \ldots, v_{k+\ell}))$ is in U if and only if for all permutations $\sigma \in S_{k+\ell}$,

$$(p, (v_{\sigma(1)}, \ldots, v_{\sigma(k)})) \in U_F \quad \text{and} \quad (p, (v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)})) \in U_G.$$

The function $F \wedge G \colon U \to \mathbb{R}$ is then defined by the formula

$$(4.11)\quad (F \wedge G)(p, (v_1, \dots, v_{k+\ell})) := \frac{1}{(k+\ell)!} \sum_{\sigma \in S_{k+\ell}} \operatorname{sgn}(\sigma) \cdot F(p, (v_{\sigma(1)}, \dots, v_{\sigma(k)})) \cdot G(p, (v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})),$$

where $p, v_1, \dots, v_{k+\ell} \in V$.

We now show that $F \wedge G$ satisfies the conditions for a partial form. Property (i) is clear; for any $\vec{v} = (v_1, \dots, v_{k+\ell}) \in V^{k+\ell}$, $(F \wedge G)_{\vec{v}}$ is a Borel function. To see property (ii), let $p \in V$. Since both F_p and G_p are Borel, the function $(F \wedge G)_p$ is Borel.

To show property (iii), fix an affine space W^d with $d \geq k+\ell$. If $\operatorname{span} v_1, \dots, v_{k+\ell} \subset W_0^d$ (i.e. W^d is a $\vec{v}$-superplane) then for all permutations $\sigma \in S_{k+\ell}$, W^d is both a $(v_{\sigma(1)}, \dots, v_{\sigma(k)})$-superplane and a $(v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)})$-superplane. Hence by property (iii) for F and G, for almost every $p \in W^d$ and every $\vec{v} \in [V]^{k+\ell}$, the values $F(p, (v_{\sigma(1)}, \dots, v_{\sigma(k)}))$ and $G(p, (v_{\sigma(k+1)}, \dots, v_{\sigma(k+\ell)}))$ are defined for all permutations $\sigma \in S_{k+\ell}$, so $\mathcal{H}^d(W^d \setminus U_{\vec{v}}) = 0$. Furthermore, by construction, at almost every point $p \in W^d$, $(F \wedge G)_p$ is alternating and $(k+\ell)$-linear on $[W_0^d]^{k+\ell}$.

4.7 The flat norm on the space of partial forms

We are now ready to define the space of flat partial k-forms, denoted $\mathbb{F}^k(V)$.

Definition IV.18. Let F be a partial form for which the exterior derivative dF exists. The form F is a *flat partial form* if and only if $\|F\|_\infty$ and $\|dF\|_\infty$ are both finite.

We note that the definition above implies that flat forms are locally integrable.

Given a flat partial k-form $F \in \mathbb{F}^k(V)$, we set

$$\|F\|_\flat := \max\{\|F\|_\infty, \|dF\|_\infty\}.$$

By the linearity of the exterior-d operator and the fact that comass is a norm on locally integrable partial forms, $\|\cdot\|_\flat$ is a norm (called the *flat norm*) on the space of flat partial k-forms.

The following lemma discusses restrictions of flat partial forms in a Banach space to finite-dimensional subspaces.

Lemma IV.19. *Let F be a flat partial k-form on the Banach space V and suppose W is a finite-dimensional subspace with dimension at least $(k+1)$. Then both F_W and $(dF)_W$ can be regarded as classical flat forms $F_W\colon W \to \Lambda^k W$ and $(dF)_W\colon W \to \Lambda^{k+1}W$. Furthermore, the exterior derivative (in the sense of distributions) of F_W is the form $(dF)_W$, i.e., $d(F_W) = (dF)_W$.*

Proof: That F_W and $(dF)_W$ can be regarded as classical flat forms $F_W\colon W \to \Lambda^k W$ and $(dF)_W\colon W \to \Lambda^{k+1}W$ follows from property (iii) of Definition IV.4.

Let d be the dimension of W. By pulling back to $\mathbb{R}^d$ and applying Wolfe's theorem (Theorem IV.3), F_W has a well defined trace on every k-dimensional affine subspace of W and is associated with a unique k-cochain X for which $\langle X, \sigma\rangle = \int_\sigma F_W$. The exterior derivative of F in the sense of distributions, denoted $d(F_W)$, is a function $d(F_W)\colon W \to \Lambda^{k+1}W$ that satisfies Stokes's theorem for any polyhedral $(k+1)$-chain τ in W:

$$\int_\tau d(F_W) = \int_{\partial\tau} F_W$$

By Wolfe's theorem (again pulling back to Euclidean space), $d(F_W)$ corresponds to a unique flat cochain Y with

$$\begin{aligned} \langle Y, \tau \rangle &= \int_\tau d(F_W) \\ &= \int_{\partial\tau} F_W \\ &= \int_{\partial\tau} F \\ &= \int_\tau dF \\ &= \int_\tau (dF)_W \end{aligned}$$

This shows that $d(F_W) = (dF)_W$ as cochains; hence they are equivalent as $(k+1)$-dimensional differential forms.

□

Corollary IV.20. *Let F be a flat partial k-form on the Banach space V and suppose W is a finite-dimensional subspace with dimension at least $(k+1)$. Then $\|F_{W^n}\|_\flat \leq \|F\|_\flat$.*

Proof: By definition, $\|F_W\|_\infty \leq \|F\|_\infty$ and $\|(dF)_W\|_\infty \leq \|dF\|_\infty$. Thus,

$$\begin{aligned} \|F_W\|_\flat &= \max\{\|F_W\|_\infty, \|d(F_W)\|_\infty\} \\ &\leq \max\{\|F_W\|_\infty, \|(dF)_W\|_\infty\} \\ &\leq \max\{\|F\|_\infty, \|dF\|_\infty\} \\ &= \|F\|_\flat \end{aligned}$$

□

In Proposition IV.23 below, we show that the wedge product of flat partial forms is a flat partial form, a fact which gives the set of flat forms the structure of a graded algebra. To do this we need the following two lemmas.

Lemma IV.21. *Let F and G be partial k- and ℓ-forms, respectively. Then*

$$\|F \wedge G\|_\infty \leq C(k,\ell)\|F\|_\infty \|G\|_\infty,$$

Proof: It suffices to show that for any $(k+\ell)$-dimensional affine subspace W in V, $\|F \wedge G\|_{\infty,W}$ is bounded above by a constant that is independent of W.

Let $W \subset V$ be an affine subspace of dimension $(k+\ell)$. For almost every $p \in W^{k+\ell}$,

$$\sup\{F_p(\mu) : \mu \in \Lambda_k W_0, \|\mu\|_{\mathrm{m}^*} = 1\} \leq \|F\|_{\infty,W},$$

$$\sup\{G_p(\gamma) : \gamma \in \lambda_\ell W_0, \|\gamma\|_{\mathrm{m}^*} = 1\} \leq \|G\|_{\infty,W},$$

and $[W_0]^{k+\ell}$ is contained in the domain of $(F \wedge G)_p$.

Let p be such a point and let ν be a $(k+\ell)$-direction. Fix $v_1, \cdots, v_{k+\ell}$ in V so that $|v_i| = 1$ for $i = 1, \ldots, k+\ell$ and so that $\nu = v_1 \wedge \cdots \wedge v_{k+\ell}$. Then for every permutation $\sigma \in S_{k+\ell}$, $v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)}$ is a k-direction and $v_{\sigma(k+1)} \wedge \cdots \wedge v_{\sigma(k+\ell)}$ is an ℓ-direction.

Hence,

$$\begin{aligned}
&(F \wedge G)_p(v_1, \cdots, v_{k+\ell}) \\
&\qquad = \frac{1}{(k+\ell)!} \sum_\sigma \mathrm{sgn}(\sigma) \cdot F_p(v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(k)}) \cdot G_p(v_{\sigma(k+1)} \wedge \ldots \wedge v_{\sigma(k+\ell)}).
\end{aligned}$$

Because the partial forms F and G both have finite comass, for almost every p in W, $(F \wedge G)_p$ is defined for every $\nu \in \Lambda_{k+\ell} W$ and

$$\begin{aligned}
&(F \wedge G)_p(v_1 \wedge \cdots \wedge v_{k+\ell}) \\
&\qquad = \frac{1}{(k+\ell)!} \sum_\sigma \mathrm{sgn}(\sigma) \cdot F_p(v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(k)}) \cdot G_p(v_{\sigma(k+1)} \wedge \ldots \wedge v_{\sigma(k+\ell)}) \\
&\qquad \leq \frac{1}{(k+\ell)!} C(k,\ell) \sum_\sigma \|F_p\|_\infty \cdot \|G_p\|_\infty \\
&\qquad = C(k,\ell)\|F_p\|_\infty \cdot \|G_p\|_\infty
\end{aligned}$$

for every $(k+\ell)$-direction $v_1 \wedge \cdots \wedge v_{k+\ell}$ in $\Lambda_{k+\ell}W$.

This shows that $\|F \wedge G\|_{\infty, W^{k+\ell}} \leq \|F\|_{\infty,W} \cdot \|G\|_{\infty,W}$ for every $(k+\ell)$-dimensional affine space in V, which proves the lemma. □

Lemma IV.22. *(Leibniz rule for differential forms.) If F is a flat partial k-form and G is a flat partial ℓ-form then*

$$d(F \wedge G) = dF \wedge G + (-1)^k F \wedge dG.$$

Proof: The proof proceeds by reducing to the Euclidean case where the Leibniz rule holds. Let τ be a $(k+\ell+1)$-dimensional simplex and W an affine subspace containing τ. Then $(F \wedge G)_W = F_W \wedge G_W$, and by Lemma IV.19, $d(F_W) = (dF)_W$ and $d(G_W) = (dG)_W$. We also have the usual equation

$$d(F \wedge G)_W = d(F_W) \wedge G_W + (-1)^k F_W \wedge d(G_W).$$

Thus,

$$\langle (d(F_W) \wedge G_W + (-1)^k F_W \wedge d(G_W)), \tau \rangle = \langle (F \wedge G)_W, \partial\tau \rangle.$$

Hence,

$$\langle (dF \wedge G + (-1)^k F \wedge dG), \tau \rangle = \langle (F \wedge G), \partial\tau \rangle,$$

and so $d(F \wedge G) = dF \wedge G + (-1)^k F \wedge dG$. □

Proposition IV.23. *Given a flat k-form F and a flat ℓ-form G, the $(k+\ell)$-form $F \wedge G$ is flat.*

Proof: By Lemma IV.21, $\|F \wedge G\|_\infty < \infty$. By Lemma IV.22 and the triangle inequality, $\|d(F \wedge G)\|_\infty \leq \|dF \wedge G\|_\infty + \|F \wedge dG\|_\infty$. Since both F and G are flat, all four of the following quantities are finite: $\|F\|_\infty$, $\|G\|_\infty$, $\|dF\|_\infty$, $\|dG\|_\infty$. Thus, by again applying Lemma IV.21, $\|d(F \wedge G)\|_\infty < \infty$, and hence $F \wedge G$ is flat. □

Remark IV.24. Proposition IV.23 provides a rich source of examples of flat partial k-forms. Any bounded Lipschitz function $f\colon V \to \mathbb{R}$ is a flat 0-form, and for any such f, df is a flat 1-form. Hence, if $f_1, \ldots, f_k$ are Lipschitz functions, then $df_1 \wedge \cdots \wedge df_k$ is a flat partial k-form.

CHAPTER V

Embedding the Space of Flat Partial Forms into the Space of Flat Cochains

In this section we will define a linear isometric embedding from the space $\mathbb{F}^k(V)$ of flat partial k-forms to the space $\mathcal{F}^k(V)$ of flat k-cochains.

We first note that a locally integrable partial k-form F induces a linear functional X_F on the space $\mathcal{P}_k V$ of polyhedral k-chains by

$$\langle X_F, \sigma \rangle = \int_\sigma F = \int_\sigma F(p, \vec{v}_\sigma)\, d\mathcal{M}(p)$$

for every polyhedral k-chain σ. If F is a flat partial k-form then the induced linear functional X_F is bounded in the flat norm. We define the map $\Psi : \mathbb{F}^k(V) \to \mathcal{F}^k(V)$ by $F \mapsto X_F$. Note that for $\tau \in \mathcal{P}_{k+1}$,

$$\langle X_{dF}, \tau \rangle = \int_\tau dF = \int_{\partial\tau} F = \langle X_F, \partial\tau \rangle = \langle dX_F, \tau \rangle,$$

so $X_{dF} = d(X_F)$.

The map Ψ is clearly a linear map, since the integration action of a form on a polyhedral chain is linear. We now show that Ψ is an isometry, i.e., that $\|F\|_\flat = |X_F|_\flat$.

Theorem V.1. *The map $\Psi : \mathbb{F}^k(V) \to \mathcal{F}^k(V)$ defined above is a linear isometric embedding.*

The proof follows directly from Lemma V.2 and Lemma V.4.

Lemma V.2. *If F is a flat partial k-form, then $|X_F|_\flat \leq \|F\|_\flat$.*

Proof: First, we note that for arbitrary polyhedral chains $\sigma \in \mathcal{P}_k$ and $\tau \in \mathcal{P}_{k+1}$,

$$\begin{aligned}
\langle X_F, \sigma \rangle &= \langle X_F, \sigma - \partial\tau \rangle + \langle X_F, \partial\tau \rangle \\
&= \int_{\sigma - \partial\tau} F + \int_{\partial\tau} F \\
&= \int_{\sigma - \partial\tau} F + \int_{\tau} dF \\
&\leq \|F\|_\infty \cdot |\sigma - \partial\tau| + \|dF\|_\infty \cdot |\tau| \\
&\leq \max\{\|F\|_\infty, \|dF\|_\infty\} \cdot (|\sigma - \partial\tau| + |\tau|).
\end{aligned}$$

Since this is true for all $\tau \in \mathcal{P}_{k+1}$,

$$\begin{aligned}
\langle F, \sigma \rangle &\leq \max\{\|F\|_\infty, \|dF\|_\infty\} \cdot \inf_\tau (|\sigma - \partial\tau| + |\tau|) \\
&\leq \|F\|_\flat \cdot |\sigma|_\flat.
\end{aligned}$$

□

In order to prove the other inequality ($|X_F|_\flat \geq \|F\|_\flat$) we first prove some intermediate results.

Lemma V.3. *Let F be a flat partial k-form. Then*

$$\sup_{W^k \subset V} \|F\|_{\infty, W^k} \leq |X_F|.$$

Proof: By Definition III.11, $|X_F| = \sup\{\langle X_F, \sigma \rangle : \sigma \in \mathcal{P}_k, |\sigma| \leq 1\}$. If we consider only k-simplexes σ in V, the supremum decreases, so

$$\begin{aligned}
|X_F| &\geq \sup\{\langle X_F, \sigma \rangle : \sigma \text{ a simplex}, |\sigma| \leq 1\} \\
&= \sup\left\{ \frac{\langle X_F, \sigma \rangle}{|\sigma|} : \sigma \text{ a simplex} \right\} \\
&\geq \sup_{W^k \subset V} \sup_{x \in W^k} \left\{ \limsup_{i \to \infty} \left\{ \frac{\langle X_F, \sigma_i \rangle}{|\sigma_i|} : \sigma_i \subset W^k, \sigma_i \searrow x \right\} \right\}. \qquad (5.1)
\end{aligned}$$

In inequality (5.1), the lim sup is taken over all sequences of simplexes (σ_i) in W^k containing x as a vertex, with fullness bounded away from zero, and whose diameters decrease toward zero. This quantity decreases still more if the second supremum is replaced with an essential supremum:

$$(5.2) \qquad |X_F| \geq \sup_{W^k \subset V} \operatorname*{ess\,sup}_{x \in W^k} \left\{ \limsup_{i \to \infty} \left\{ \frac{\langle X_F, \sigma_i \rangle}{|\sigma_i|} : \sigma_i \subset W^k, \sigma_i \searrow x \right\} \right\}.$$

As noted in Section 4.7, F_W is a top-dimensional form on $W = W^k$ with

$$\|F_W\|_\infty = \|F\|_{\infty,W}.$$

We equip W with a Euclidean norm by identifying W with $\mathbb{R}^k$. We denote k-dimensional Lebesgue measure on W by $|\cdot|_E := \mathcal{L}^k(\cdot)$. Fix a mass*-$k$-direction $\nu \in \Lambda_k W$ and normalize ν by its Euclidean mass $|\nu|_2$ to obtain a Euclidean k-direction $\frac{\nu}{|\nu|_2}$. Define a (top-dimensional) Euclidean k-form $F_W^E : W \to \mathbb{R}$ on W by

$$F_W^E(x) := F_W\left(x, \frac{\nu_W}{|\nu_W|_2}\right).$$

By construction, $\|F_W^E\|_\infty = \frac{1}{|\nu_W|_E}\|F_W\|_\infty < \infty$. Hence, F_W^E is a Euclidean flat form on W (with the Euclidean norm), so by Wolfe's theorem (Theorem 7c, [Whi57, p. 265]) F_W^E is equivalent to the form $\widetilde{F}_W^E$ defined by

$$\widetilde{F}_W^E(x) := \lim_{i \to \infty} \left\{ \frac{\langle X_F, \sigma_i \rangle}{|\sigma_i|_E} : \sigma_i \subset W^k, \sigma_i \searrow x \right\},$$

where the limit, which exists for almost every $x \in W$, is taken over all sequences of simplexes (σ_i) in W containing x as a vertex, with fullness bounded away from zero, and whose diameters decrease toward zero. This shows that $\|F_W^E\|_\infty = \|\widetilde{F}_W^E\|_\infty$.

Since Wenger's Lemma (Lemma III.8) implies that $|\sigma_i|_E = |\nu_W|_E|\sigma_i|$, the function $\widetilde{F}_W$ on W defined by

$$\widetilde{F}_W(x) := \lim_{i \to \infty} \left\{ \frac{\langle X_F, \sigma_i \rangle}{|\sigma_i|} : \sigma_i \subset W^k, \sigma_i \searrow x \right\} = (|\nu_W|_E)\widetilde{F}_W^E(x)$$

is defined for almost every $p \in W$.

Hence,

$$\begin{aligned} \|F_W\|_\infty &= |\nu_W|_E \cdot \|F_W^E\|_\infty \\ &= |\nu_W|_E \cdot \|\widetilde{F}_W^E\|_\infty \\ &= \|\widetilde{F}_W\|_\infty. \end{aligned}$$

Since by equation (5.2), $|X_F| \geq \|\widetilde{F}_W\|_\infty = \|F_W\|_\infty = \|F\|_{\infty,W}$,

$$|X_F| \geq \sup_{W^k \subset V} \|F\|_{\infty,W^k} = \|F\|_\infty.$$

□

Lemma V.4. $\|F\|_\flat \leq \|X_F\|_\flat$.

Proof: By the definition of the comass norm $\|\cdot\|_\infty$, we have the equality

$$\|F\|_\flat = \max\left\{ \sup_{W^k \subset V} \|F\|_{\infty,\, W^k}, \sup_{W^{k+1} \subset V} \|dF\|_{\infty,\, W^{k+1}} \right\}. \tag{5.3}$$

The terms on the right-hand side of equation (5.3) are bounded above by Lemma V.3:

$$\sup_{W^k} \|F\|_{\infty,\, W^k} \leq |X_F|,$$

and

$$\sup_{W^{k+1}} \|dF\|_{\infty,\, W^{k+1}} \leq |X_{dF}| = |dX_F|.$$

Combining the two preceding inequalities with equation (5.3) and applying Lemma III.12, we have

$$\|F\|_\flat \leq \max\{|X_F|, |dX_F|\} = |X_F|_\flat,$$

as desired. □

CHAPTER VI

Wolfe's Theorem in a Banach Space

In the last chapter we showed that there is a linear isometric embedding Ψ from the space $\mathbb{F}^k(V)$ of flat partial k-forms to the space $\mathcal{F}^k(V)$ of flat k-cochains.

The main result of this chapter is the following theorem.

Theorem VI.1. *The map* $\Psi : \mathbb{F}^k(V) \to \mathcal{F}^k(V)$ *is surjective.*

Theorem I.1, the Banach space analog to Wolfe's theorem, follows from Theorem VI.1 and the results of the previous chapter.

We will show that Ψ is a surjective map by finding, for each flat k-cochain X, a flat partial k-form F_X with the property that $\Psi(F_X) = X$.

Let X be a flat k-cochain, and fix a point $p \in V$ and a k-direction $\nu \in \Lambda_k V$. Let W be the ν-superplane containing p. Consider sequences (σ_i) of oriented simplexes in W that satisfy the following three properties:

1. for all i, σ_i has one vertex at the point p and has the same orientation as W,
2. as $i \to \infty$, $|\sigma_i| \to 0$,
3. there exists $\eta > 0$ so that $\Theta(\sigma_i) > 0$ for all i.

Let $D_X(p, \nu)$ be the following limit, if it exists for *every* such sequence (σ_i) of simplexes:

$$D_X(p,\nu) := \lim_{i\to\infty} \frac{\langle X, \sigma_i\rangle}{|\sigma_i|}. \tag{6.1}$$

In the case that ν is a nonzero simple k-vector we define $D_X(p,\nu) := |\nu| D_X\big(p, \frac{\nu}{|\nu|}\big)$; if ν is the zero vector, we set $D_X(p,\nu) = 0$. This construction is analogous to that of Whitney in the Euclidean case; see Section 4.1.

We now define a set $U \subset V \times [V]^k$ and a function $F_X : U \times V^k \to \mathbb{R}$, which will turn out to be the partial form we seek. The set U consists of those pairs $(p, \vec{v})$ for which the limit $D_X(p, \mu(\vec{v}))$ exists and is finite. (Recall that $\mu(v_1, \ldots, v_k) = v_1 \wedge \cdots \wedge v_k$.) The function F_X is given by

$$F_X(p, \vec{v}) := D_X(p, \mu(\vec{v})).$$

Theorem VI.2. *F_X is a partial k-form.*

We begin by proving that the horizontal slices of F_X are Borel.

Proposition VI.3. *Suppose that X is a flat cochain in a Banach space V. Then for all $(v_1, \ldots, v_k) = \vec{v} \in V^k$ the restrictions $(F_X)_{\vec{v}} : U_{\vec{v}} \to \mathbb{R}$ defined by $(F_X)_{\vec{v}}(p) := F_X(p, \vec{v}) = D_X(p, v_1 \wedge \cdots \wedge v_k)$ are Borel functions.*

Proof: We will first show that the set $U_{\vec{v}}$ is Borel. Let $\vec{v} = (v_1, \ldots, v_k)$ be a k-tuple such that $\nu = v_1 \wedge \cdots \wedge v_k$ has unit mass. Let $\sigma \subset \operatorname{span}\{v_1, \ldots, v_k\}$ be a nondegenerate simplex containing the origin. Now, for $p \in V$, let $T_p\sigma$ denote the simplex σ translated by p. In particular, $T_p\sigma$ has fullness $\Theta(\sigma)$ and one vertex at the point p.

Define the map $g_\sigma : V \to \mathbb{R}$ by

$$g_\sigma(p) := \langle X, T_p\sigma \rangle,$$

and define

$$\widetilde{g}_\sigma(p) = \frac{1}{|\sigma|} g_\sigma(p).$$

For any choice of ν and any subsequent choice of σ, we show that the map g_σ is continuous, and hence, that $\widetilde{g}_\sigma$ is continuous.

For a unit vector $\vec{u} \in V$ and $\delta > 0$, set $p' := p + \delta\vec{u}$. Denote by τ the (possibly degenerate) $(k+1)$-dimensional prism that is the convex hull of $T_p\sigma$ and $T_{p'}\sigma$, and orient τ to match the orientations of $-T_p\sigma$ and $T_{p'}\sigma$ on those simplexes. We can bound the flat norm of $T_p\sigma - T_{p'}\sigma$ using the $(k+1)$-chain τ:

$$\begin{aligned} |T_p\sigma - T_{p'}\sigma|_\flat &\leq |(T_p\sigma - T_{p'}\sigma) - \partial\tau| + |\tau| \\ &\leq C_k[\delta(k+1)\operatorname{diam}(\sigma)^{k-1} + \delta|\sigma|]. \end{aligned}$$

Thus,

$$\begin{aligned} |g_\sigma(p) - g_\sigma(p')| &= |\langle X, T_p\sigma\rangle - \langle X, T_{p'}\sigma\rangle| \\ &= |\langle X, T_p\sigma - T_{p'}\sigma\rangle| \\ &\leq |X|_\flat \cdot |T_p\sigma - T_{p'}\sigma|_\flat \\ &\leq |X|_\flat \cdot C_k((k+1)\operatorname{diam}(\sigma)^{k-1} + |\sigma|) \cdot \delta, \end{aligned}$$

so $|g_\sigma(p) - g_\sigma(p')| \to 0$ as $\delta \to 0$.

This shows that g_σ is continuous. Since $|\sigma|$ is constant, the normalized function $\widetilde{g}_\sigma = \frac{1}{|\sigma|} g_\sigma$ is also continuous, and hence a Borel function.

As a k-dimensional linear subspace of V, P_ν is separable. Fix a countable dense set Q of points in P_ν and consider the set of ordered k-tuples of elements in Q. Define the set $\mathcal{G}_\nu$ to consist of oriented k-simplexes with vertices $0, x_1, \ldots, x_k$, where $(x_1, \ldots, x_k) \in [Q]^k$. Note that each oriented simplex with one vertex at zero corresponds in a natural way to an equivalence class of k-tuples. Here, two k-tuples

$(v_1, \dots, v_k)$ and $(w_1, \dots, w_k)$ are in the same equivalence class if they contain the same set of k points and if there exists a permutation σ of sign $+1$ so that $v_i = w_{\sigma(i)}$ for all i. Since the set of k-tuples is dense, the (countable) set $\mathcal{G}_\nu$ of simplexes generated by the set of ordered k-tuples chosen above is also dense (with respect to the mass norm) in the set $\{[0, v_1, \dots, v_k] : v_1, \dots, v_k \in P_\nu\}$ of simplexes in P_ν with one vertex at 0. To see this, let $S = [0, v_1, \dots, v_k]$ be a simplex with $v_1, \dots, v_k \in P_\nu$ and choose sequences $(x_i^1), \dots, (x_i^k)$ in Q with $x_i^j \to v_j$ for $1 \leq j \leq k$. The sequence $(S_i) = ([0, x_i^1, \dots, x_i^k])$ of simplexes approaches S in Hausdorff measure and hence in the mass measure.

For $\eta \in \mathbb{R}^+$, let $\mathcal{G}_{\nu,\eta} := \{\sigma \in \mathcal{G}_\nu : \Theta(\sigma) \geq \eta\}$ and define the function $\underline{D}_X^\eta \colon V \times \Lambda_k V \to \mathbb{R}$ by

$$\underline{D}_X^\eta(p, \nu) := \liminf_{m \to \infty} \inf_{\substack{\sigma \in \mathcal{G}_{\nu,\eta} \\ |\sigma| < 1/m}} \widetilde{g}_\sigma(p). \tag{6.2}$$

Similarly, define $\overline{D}_X^\eta \colon V \times \Lambda_k V \to \mathbb{R}$ by

$$\overline{D}_X^\eta(p, \nu) := \limsup_{m \to \infty} \sup_{\substack{\sigma \in \mathcal{G}_{\nu,\eta} \\ |\sigma| < 1/m}} \widetilde{g}_\sigma(p). \tag{6.3}$$

For fixed ν and η, the functions $\underline{D}_X^\eta$ and $\overline{D}_X^\eta$ are Borel functions of p. We see that the subset of V where $\underline{D}_X^\eta(\cdot, \nu) < \infty$ is Borel, so the set where $\underline{D}_X^\eta(\cdot, \nu) = \infty$ is also Borel. Similarly, the subset of V where $\underline{D}_X^\eta(\cdot, \nu) > -\infty$ is Borel, so the set where $\underline{D}_X^\eta(\cdot, \nu) = -\infty$ is Borel. In addition, the set in V where $\underline{D}_X^\eta(\cdot, \nu) = \overline{D}_X^\eta(\cdot, \nu)$ is Borel. Thus, the subset

$$\mathcal{S}_{\nu,\eta} := \{p \in V : -\infty < \underline{D}_X^\eta(p, \nu) = \overline{D}_X^\eta(p, \nu) < \infty\}$$

is Borel.

The set $U_{\vec{v}}$ (the domain of the function $F_{\vec{v}}$) is clearly contained in $\mathcal{S}_{\nu,\eta}$ for all

$\eta > 0$, so

$$U_{\vec{v}} \subset \bigcap_{\ell=1}^{\infty} \mathcal{S}_{\nu, \frac{1}{\ell}}.$$

On the other hand, suppose $p \in \bigcap_{\ell=1}^{\infty} \mathcal{S}_{\nu,\frac{1}{\ell}}$, and let (τ_i) be a sequence of simplexes in P_ν whose diameters shrink to zero and such that each simplex τ_i contains the point p and has fullness greater than η. Approximating each simplex τ_i in the mass norm by a sequence of simplexes σ_j^i and using a diagonalization argument, one can show that the limit $D_X(p,\nu)$ exists and equals the limit $\underline{D}_X^\eta(p,\nu) = \overline{D}_X^\eta(p,\nu)$. Thus, $U_{\vec{v}} \supset \bigcap_{\ell=1}^{\infty} \mathcal{S}_{\nu,\frac{1}{\ell}}$, and so

$$U_{\vec{v}} = \bigcap_{\ell=1}^{\infty} \mathcal{S}_{\nu, \frac{1}{\ell}}.$$

This shows that $U_{\vec{v}}$ is Borel.

To see that the function $F_{\vec{v}}$ is Borel, note that at any point p in $U_{\vec{v}}$, $\underline{D}_X^\eta(p,\nu) = \overline{D}_X^\nu(p,\eta)$ for every fixed η. Hence, we may define the function $D_X^\eta \colon E_\nu \times \Lambda_k V \to \mathbb{R}$ by

$$D_X^\eta(p,\nu) := \lim_{m\to\infty} \sup_{\substack{\sigma\in\mathcal{G}_{\vec{v}} \\ \Theta(\sigma)\geq\eta \\ |\sigma|_{\text{mass}}<1/m}} \widetilde{g}_\sigma(p). \tag{6.4}$$

Since $D_X^\eta(p,\nu)$ is Borel for the fixed k-direction ν and fullness η,

$$(F_X)_{\vec{v}}(p) = D_X(p,\nu) = \lim_{\ell\to\infty} D_X^{1/\ell}(p,\nu)$$

is also a Borel function. It follows that $(F_X)_{\vec{v}}$ is a Borel function for all $\vec{v}$ in V^k. $\square$

Towards proving Theorem VI.2, we prove the following proposition, which will be used to prove that F_X satisfies properties (ii) and (iii) of a partial form. The proposition, whose proof relies on Theorem II.8, essentially says that if two k-directions α and β are sufficiently close in the mass* norm, then they can be represented as $\alpha = u_1 \wedge \cdots \wedge u_k$ and $\beta = v_1 \wedge \cdots \wedge v_k$ where the components of α are "almost orthogonal," as are the components of β, and where for each i, u_i is close to v_i.

Proposition VI.4. *For $k > 0$, there exists a constant $B = B(k) > 0$ depending only on k so that the following is true. Let α, β be k-directions in $\Lambda_k V$ such that $\|\alpha - \beta\|_{m^*} \leq B$. Then there exist $u_1, \dots, u_k$ and $v_1, \dots, v_k$ in V so that*

(i) $\alpha = u_1 \wedge \cdots \wedge u_k$ *and* $\beta = v_1 \wedge \cdots \wedge v_k$,

(ii) for all $\lambda_1, \dots, \lambda_k \in \mathbb{R}$,

$$|\lambda_1 u_1 + \cdots + \lambda_k u_k|_V \simeq \left(\sum_{i=1}^{k} \lambda_i^2\right)^{1/2} \text{ and } |\lambda_1 v_1 + \cdots + \lambda_k v_k|_V \simeq \left(\sum_{i=1}^{k} \lambda_i^2\right)^{1/2},$$

with similarity constants depending only on k,

(iii) for all i in $\{1, \dots, k\}$, $|u_i - v_i|_V < c(k)\|\alpha - \beta\|_{m^*}$.

Let $\nu = v_1 \wedge \cdots \wedge v_k$ be a simple k vector in $\Lambda_k V$ and $L : W \to V$ an injective linear map between finite-dimensional Banach spaces. We denote the vector $L(\nu) := L(v_1) \wedge \cdots \wedge L(v_k)$. If $\nu \in \Lambda_k V$ is an arbitrary k-vector with representation $\nu = \sum \lambda_i \nu_i$ where each ν_i is simple, let $L(\nu) := \sum \lambda_i L(\nu_i)$. Furthermore, suppose $\zeta \in W^*$. We define $L\zeta \in V^*$ to be the composition $L\zeta := \zeta \circ L^{-1}$.

The following lemma is the first step in proving Proposition VI.4.

Lemma VI.5. *Let α and β be k-directions in $\Lambda_k V$, $P_\alpha := \operatorname{span}\{u_i : i = 1, \dots, k\}$, and $P_\beta := \operatorname{span}\{v_i : i = 1, \dots, k\}$. Let $V^{\alpha\beta} = \operatorname{span}\{P_\alpha, P_\beta\}$ and $d = \dim V^{\alpha\beta}$. Let $L \colon \mathbb{R}^d \to V^{\alpha\beta}$ be the $\sqrt{d}$-bi-Lipschitz John map from Theorem II.8. Then there exists a constant C depending only on k so that*

(i) $\frac{1}{C}\|\alpha\|_{m^*} \leq \|L^{-1}(\alpha)\|_{m^*} \leq C\|\alpha\|_{m^*}$,

(ii) $\frac{1}{C}\|\alpha - \beta\|_{m^*} \leq \|L^{-1}(\alpha - \beta)\|_{m^*} \leq C\|\alpha - \beta\|_{m^*}$.

Proof: By definition

$$\|\alpha\|_{\mathrm{m}^*} = \sup\{\langle \alpha, \xi \rangle : \xi = \xi_1 \wedge \cdots \wedge \xi_k, \xi_i \in V^*, |\xi_i|_{V^*} \leq 1\}.$$

Suppose that $\zeta_i \in (\mathbb{R}^d)^*$ for $i = 1, \ldots, k$ all have $|\zeta_i|_{(\mathbb{R}^d)^*} \leq 1$. It suffices to show that $|\langle L^{-1}(\alpha), \zeta_1 \wedge \cdots \wedge \zeta_k \rangle| \leq c(k)\|\alpha\|_{\mathrm{m}^*}$.

Since L is a $\sqrt{d}$-bi-Lipschitz map,

$$|L\zeta_i|_{(V^{\alpha\beta})^*} \leq \sqrt{d}|\zeta_i|_{(\mathbb{R}^d)^*} \leq \sqrt{d}.$$

By the Hahn-Banach theorem, we may extend each $L\zeta_i$ to a linear functional $\widehat{L\zeta_i}$ on V with $|\widehat{L\zeta_i}|_{V^*} \leq \sqrt{d}$. Let $\xi = \frac{\widehat{L\zeta_1}}{\sqrt{d}} \wedge \cdots \wedge \frac{\widehat{L\zeta_k}}{\sqrt{d}} = d^{-k/2}(\widehat{L\zeta_1} \wedge \cdots \wedge \widehat{L\zeta_k})$. Then the mass of ξ is bounded by 1:

$$d^{-k/2}|\widehat{L\zeta_1} \wedge \cdots \wedge \widehat{L\zeta_k}|_{\mathrm{m}} \leq d^{-k/2}|\widehat{L\zeta_1}| \cdots |\widehat{L\zeta_k}| \leq 1.$$

Thus we have the following estimate:

$$\begin{aligned} |\langle L^{-1}(\alpha), \zeta_1 \wedge \cdots \wedge \zeta_k \rangle| &= |\langle \alpha, L\zeta_1 \wedge \cdots \wedge L\zeta_k \rangle| \\ &= |\langle \alpha, \widehat{L\zeta_1} \wedge \cdots \wedge \widehat{L\zeta_k} \rangle| \\ &\leq d^{k/2}|\langle \alpha, \xi \rangle| \\ &\leq d^{k/2}\|\alpha\|_{\mathrm{m}^*}. \end{aligned}$$

Since $k \leq d \leq 2k$, let $C = (2k)^{k/2}$. Then we have $\|L^{-1}(\alpha)\|_{\mathrm{m}^*} \leq C\|\alpha\|_{\mathrm{m}^*}$.

A similar argument using $\zeta_i \in (V^{\alpha\beta})^*$ with $|\zeta_i|_{(V^{\alpha\beta})^*} \leq 1$ for $i = 1, \ldots, k$ shows that for $C = (2k)^{k/2}$, $\|\alpha\|_{\mathrm{m}^*} \leq C\|L^{-1}(\alpha)\|_{\mathrm{m}^*}$. This proves part (i) of the lemma.

Part (ii) is proven in the same manner. □

Lemma VI.6. *The norms $\|\cdot\|_{m^*}$ and $|\cdot|_2$ on $\Lambda_k\mathbb{R}^d$, $d \geq k$ are equivalent up to a factor that depends only on k and d.*

Here, the norm $|\cdot|_2$ refers to the Euclidean mass norm on k-vectors described in Appendix A.

Proof: Let $\{e_1, \ldots, e_d\}$ be the standard orthonormal basis of $\mathbb{R}^d$. Then $\mathcal{B} = \{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < \cdots < i_k \leq d\}$ is an orthonormal basis of $\Lambda_k \mathbb{R}^d$ (with respect to the norm $|\cdot|_2$) and for every $b \in \mathcal{B}$, $|b|_2 = \|b\|_{\mathrm{m}^*} = 1$. Let $|\cdot|_1$ denote the L^1 norm on $\Lambda_k \mathbb{R}^d$, and let $|\cdot|_\infty$ denote the L^∞ norm on $\Lambda_k \mathbb{R}^d$. By the triangle inequality, for any $\nu \in \Lambda_k \mathbb{R}^d$,

$$|\nu|_1 \geq \|\nu\|_{\mathrm{m}^*}.$$

After uniquely representing ν as $\sum \lambda_I b_I$ for $b_I \in \mathcal{B}$ and applying the definition of the mass* norm we have

$$\|\nu\|_{\mathrm{m}^*} \geq \max_I \{\lambda_I\} = |\nu|_\infty.$$

Thus, by John's theorem,

$$\frac{1}{\sqrt{C}} |\nu|_2 \leq |\nu|_\infty \leq \|\nu\|_{\mathrm{m}^*} \leq |\nu|_1 \leq \sqrt{C} |\nu|_2,$$

where $C = \binom{d}{k}$. □

Proof of Proposition VI.4: Throughout the proof, we abuse notation and use $c(k)$ to denote a constant depending only on k, although the specific constant referred to may be different in different instances.

As in the Lemma VI.5, let $V^{\alpha\beta} = \mathrm{span}\{P_\alpha, P_\beta\}$. Let $d \leq 2k$ denote the dimension of $V^{\alpha\beta}$. Let $L \colon \mathbb{R}^d \to V^{\alpha\beta}$ be the $\sqrt{d}$-bi-Lipschitz John map from Theorem II.8. Denote the simple k-vectors $L^{-1}(\alpha)$ and $L^{-1}(\beta)$ by $L^{-1}(\alpha) = \alpha'$ and $L^{-1}(\beta) = \beta'$, and let the associated k-directions be $\alpha'_0 := \frac{\alpha'}{\|\alpha'\|_{\mathrm{m}^*}}$ and $\beta'_0 := \frac{\beta'}{\|\beta'\|_{\mathrm{m}^*}}$. By Lemma VI.5,

$$\|\alpha' - \beta'\|_{\mathrm{m}^*} = \|L^{-1}(\alpha - \beta)\|_{\mathrm{m}^*} \leq c(k) \|\alpha - \beta\|_{\mathrm{m}^*}.$$

Combined with Lemma VI.6, this implies that

$$|\alpha' - \beta'|_2 \leq c(k) \|\alpha - \beta\|_{\mathrm{m}^*}.$$

Choose the constant B so that

$$|\alpha_0' - \beta_0'|_2 \leq 1 \tag{6.5}$$

whenever $\|\alpha - \beta\|_{\mathrm{m}^*} \leq B$. Thus, for α and β with $\|\alpha - \beta\|_{\mathrm{m}^*} \leq B$, the orthogonal projection mapping from the plane determined by the k-direction α_0' to the plane determined by β_0' is an isomorphism. Moreover, by [Whi57, I.15], inequality (6.5) implies that the angle θ between the k-planes $L^{-1}P_\beta$ and $L^{-1}P_\alpha$ satisfies $\cos\theta \geq 1/2$.

We now choose specific representations of the k-directions α_0' and β_0'. To do this, choose an orthonormal basis $e_1, \ldots, e_k$ of the (Euclidean) plane $L^{-1}P_\alpha = P_{\alpha'}$ spanned by the components of α'. (Choose this basis so that $e_1 \wedge \cdots \wedge e_k$ has the same orientation as α'.) Since Wenger's Lemma implies that $\|e_1 \wedge \cdots \wedge e_k\|_{\mathrm{m}^*} = 1$,

$$e := e_1 \wedge \cdots \wedge e_k = \alpha_0'.$$

Let p be the Euclidean projection of $\mathbb{R}^d$ onto the plane $L^{-1}P_\beta = P_{\beta'}$, and denote $x_i := p(e_i)$ for $i = 1, \ldots, k$. The vectors $x_1, \ldots x_k$ form a basis of $P_{L^{-1}\beta}$, which we normalize as follows:

$$\overline{x_i} := \frac{x_i}{(\|x_1 \wedge \cdots \wedge x_k\|_{\mathrm{m}^*})^{1/k}},$$

so that $\|\overline{x_1} \wedge \cdots \wedge \overline{x_k}\|_{\mathrm{m}^*} = 1$. Then

$$\overline{x} := \overline{x_1} \wedge \cdots \wedge \overline{x_k} = \beta_0'.$$

Inequality (6.5) allows us to bound the quantity $\|x_1 \wedge \cdots \wedge x_k\|_{\mathrm{m}^*}$; by [Whi57, I.15] and Wenger's lemma,

$$\|x_1 \wedge \cdots \wedge x_k\|_{\mathrm{m}^*} = \|p(e_1) \wedge \cdots \wedge p(e_k)\|_{\mathrm{m}^*} \geq \frac{1}{2}\|e_1 \wedge \cdots \wedge e_k\|_{\mathrm{m}^*} = \frac{1}{2}. \tag{6.6}$$

We push the vectors $e_1, \ldots, e_k$ and $\overline{x_1}, \ldots, \overline{x_k}$ forward by the map L and renormalize to obtain the vectors $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ from part (i) of the statement of this proposition: Set $u_i = \frac{Le_i}{\sqrt[k]{\|L(e)\|_{\mathrm{m}^*}}}$ and $v_i = \frac{L\overline{x_i}}{\sqrt[k]{\|L(\overline{x})\|_{\mathrm{m}^*}}}$ for $i = 1, \ldots, k$.

By construction, $\|u_1\wedge\cdots\wedge u_k\|_{\mathrm{m}^*}=1$, so $\alpha=u_1\wedge\cdots\wedge u_k$. Also, for $\lambda_1,\ldots\lambda_k\in\mathbb{R}$,

$$\begin{aligned}|\lambda_1 u_1+\cdots+\lambda_k u_k|_V &\simeq_k |L^{-1}(\lambda_1 u_1+\cdots+\lambda_k u_k)|_{\mathbb{R}^d}\\ &= |\lambda_1 L^{-1}(u_1)+\cdots+\lambda_k L^{-1}(u_k)|_{\mathbb{R}^d}\\ &= \left|\lambda_1\frac{e_1}{\sqrt[k]{\|L(e)\|_{\mathrm{m}^*}}}+\cdots+\lambda_k\frac{e_k}{\sqrt[k]{\|L(e)\|_{\mathrm{m}^*}}}\right|_{\mathbb{R}^d}\\ &\simeq_k |\lambda_1 e_1+\cdots+\lambda_k e_k|_{\mathbb{R}^d}\\ &= c(k)\left(\sum\lambda_i^2\right)^{1/2}.\end{aligned}$$

In the preceding sequence, (6.7) follows from Lemma VI.5.

Since $v_1\wedge\cdots\wedge v_k$ has unit mass*, $v_1\wedge\cdots\wedge v_k=\pm\beta$, and since α and β are close in mass*, $v_1\wedge\cdots\wedge v_k=\beta$. Also, since $\overline{x}$ is a k-direction in $\mathbb{R}^d$ and L is $\sqrt{d}$-bi-Lipschitz, Lemma VI.5 implies that $\|L(\overline{x})\|_{\mathrm{m}^*}\simeq c(k)\|\overline{x}\|_{\mathrm{m}^*}=c(k)$. Thus, for $\lambda_1,\ldots\lambda_k\in\mathbb{R}$,

$$\begin{aligned}|\lambda_i v_i|_V &\simeq_k \left|L^{-1}\sum\lambda_i v_i\right|_{\mathbb{R}^d}\\ &= \left|\sum\lambda_i\frac{\overline{x_i}}{\sqrt[k]{\|L(\overline{x})\|_{\mathrm{m}^*}}}\right|_{\mathbb{R}^d}\\ &\simeq_k \left|\sum\lambda_i\overline{x_i}\right|_{\mathbb{R}^d}.\end{aligned}$$

Furthermore, by inequality (6.6), $\sqrt[k]{\|x_1\wedge\cdots\wedge x_k\|_{\mathrm{m}^*}}\simeq 1$, so

$$\begin{aligned}c(k)\left|\sum\lambda_i\overline{x_i}\right|_{\mathbb{R}^d} &= c(k)\left|\sum\lambda_i\frac{x_i}{\sqrt[k]{\|x_1\wedge\cdots\wedge x_k\|_{\mathrm{m}^*}}}\right|_{\mathbb{R}^d}\\ &\simeq_k \left|\sum\lambda_i p(e_i)\right|_{\mathbb{R}^d}\\ &\simeq_k \left|\sum\lambda_i e_i\right|_{\mathbb{R}^d}\\ &= \left(\sum\lambda_i^2\right)^{1/2}.\end{aligned}$$

This proves part (ii) of the proposition.

Finally, for each i, we have

$$\begin{aligned}
|u_i - v_i|_V &= \left| \frac{Le_i}{\sqrt[k]{\|L(e)\|_{\mathrm{m}^*}}} - \frac{L\overline{x_i}}{\sqrt[k]{\|L(\overline{x})\|_{\mathrm{m}^*}}} \right|_V \\
&\simeq_k |Le_i - L\overline{x}_i|_V \qquad (6.7) \\
&\simeq_k |e_i - \overline{x_i}|_{\mathbb{R}^d} \\
&= \left| e_i - \frac{p(e_i)}{\sqrt[k]{\|x_1 \wedge \cdots \wedge x_k\|_{\mathrm{m}^*}}} \right|_{\mathbb{R}^d} \\
&\simeq_k |e_i - p(e_i)|_{\mathbb{R}^d} \qquad (6.8) \\
&\leq c(k)\|\alpha' - \beta'\|_{\mathrm{m}^*} \\
&\leq c(k)\|\alpha - \beta\|_{\mathrm{m}^*}.
\end{aligned}$$

Equation (6.7) holds because by Lemma VI.5, both $\|L(e)\|_{\mathrm{m}^*}$ and $\|L(\overline{x})\|_{\mathrm{m}^*}$ are comparable to 1 with comparability constants depending only on k. Equation (6.8) again uses the fact that $\sqrt[k]{\|x_1 \wedge \cdots \wedge x_k\|_{\mathrm{m}^*}} \simeq 1$. This shows that

$$|u_i - v_i|_V < c(k)\|\alpha - \beta\|_{\mathrm{m}^*}.$$

□

In the situation of Proposition VI.4, we can define a linear map π from P_α onto the plane P_β by the following formula:

$$\pi\left(\sum_{i=1}^{k} \lambda_i u_i\right) := \sum_{i=1}^{k} \lambda_i v_i. \qquad (6.9)$$

By Proposition VI.4,

$$|x - \pi(x)|_V < c(k)|x| \cdot \|\alpha - \beta\|_{\mathrm{m}^*} \qquad (6.10)$$

for all $x \in P_\alpha$.

Lemma VI.7. *Let α and β be k-directions in $\Lambda_k V$ so that $\|\alpha - \beta\|_{m^*}$ is less than the bound in Proposition VI.4. Suppose that $\sigma \subset P_\alpha$ is an η-full simplex of diameter* *at most 1 and one vertex at* 0. *Let* π *be the map given in equation* (6.9). *Then*

$$\left|\frac{\pi(\sigma)}{|\pi(\sigma)|} - \frac{\sigma}{|\sigma|}\right|_\flat \le C_{k,\eta} \cdot \|\alpha - \beta\|_{m^*}, \tag{6.11}$$

where, in particular, $C_{k,\eta}$ *does not depend on the mass of* σ.

Proof: By the linearity of the map π,

$$\begin{aligned}
\left|\frac{\pi\sigma}{|\pi\sigma|} - \frac{\sigma}{|\sigma|}\right|_\flat &\le \left|\frac{\pi\sigma}{|\pi\sigma|} - \frac{\pi\sigma}{|\sigma|} + \frac{\pi\sigma}{|\sigma|} - \frac{\sigma}{|\sigma|}\right|_\flat \\
&\le |\pi\sigma|_\flat \cdot \frac{||\sigma| - |\pi\sigma||}{|\pi\sigma|\cdot|\sigma|} + \frac{|\pi\sigma - \sigma|_\flat}{|\sigma|} \\
&\le |\pi\sigma| \cdot \frac{||\sigma| - |\pi\sigma||}{|\pi\sigma|\cdot|\sigma|} + \frac{|\pi\sigma - \sigma|_\flat}{|\sigma|} \\
&= \frac{||\sigma| - |\pi\sigma||}{|\sigma|} + \frac{|\pi\sigma - \sigma|_\flat}{|\sigma|}.
\end{aligned}$$

Hence, assuming $\|\alpha - \beta\|_{\mathrm{m}^*}$ is sufficiently small, it suffices to show that $\frac{||\sigma|-|\pi\sigma||}{|\sigma|}$ and $\frac{|\pi\sigma-\sigma|_\flat}{|\sigma|}$ are each bounded by a constant multiple of $\|\alpha - \beta\|_{\mathrm{m}^*}$ depending only on k and η.

Part I: We first show that $\frac{||\sigma|-|\pi\sigma||}{|\sigma|}$ is bounded above up to a constant factor by $\|\alpha - \beta\|_{\mathrm{m}^*}$. Let $\sigma = [0, x_1, \ldots, x_k]$ and $\pi\sigma = [0, \pi x_1, \ldots, \pi x_k]$. One may choose "normalized simplexes" σ' and $\pi\sigma'$ so that $|\sigma'| = 1$ and $||\sigma'| - |\pi\sigma'|| = \frac{||\sigma|-|\pi\sigma||}{|\sigma|}$ as follows. By the scaling lemma for mass from [Ada08], $|\lambda\sigma| = \lambda^k|\sigma|$ for any positive real scalar λ. Thus,

$$1 = \left|\frac{\sigma}{|\sigma|}\right| = \left|[0, |\sigma|^{-1/k}x_1, \ldots, |\sigma|^{-1/k}x_k]\right|,$$

and

$$\frac{|\pi\sigma|}{|\sigma|} = \left|\frac{\pi\sigma}{|\sigma|}\right| = \left|[0, |\sigma|^{-1/k}\pi x_1, \ldots, |\sigma|^{-1/k}\pi x_k]\right|.$$

We denote the vertices of σ' by $\{x_i' : i = 0 \ldots k\}$, where $x_0' = 0$ and $x_i' = |\sigma|^{-1/k}\pi x_i$.

Since

$$\frac{||\sigma| - |\pi\sigma||}{|\sigma|} = \left|\frac{|\sigma|}{|\sigma|} - \frac{|\pi\sigma|}{|\sigma|}\right|,$$

it suffices to show that the normalized simplexes $\sigma' := [0, |\sigma|^{-1/k}x_1, \ldots, |\sigma|^{-1/k}x_k]$ and $\pi\sigma' := [0, |\sigma|^{-1/k}\pi x_1, \ldots, |\sigma|^{-1/k}\pi x_k]$ are close in mass, which we will accomplish by bounding the Lipschitz constant of the map π and applying Lemma III.13.

By inequality 6.10,

$$1 - c(k)\|\alpha - \beta\|_{\mathrm{m}^*} < |\pi(z)| < 1 + c(k)\|\alpha - \beta\|_{\mathrm{m}^*}$$

for all z in $\partial B(0,1) \cap P_\alpha$. Thus, we can bound the operator norms of $\pi : P_\alpha \to P_\beta$ and $\pi^{-1} : P_\beta \to P_\alpha$ by

$$\|\pi\| := \sup\{|\pi(z)| : z \in \partial B(0,1) \cap P_\alpha\} < 1 + c(k)\|\alpha - \beta\|_{\mathrm{m}^*}$$

and

$$\|\pi^{-1}\| = \frac{1}{\inf\{|\pi(z)| : z \in \partial B(0,1) \cap P_\alpha\}} < \frac{1}{1 - c(k)\|\alpha - \beta\|_{\mathrm{m}^*}}.$$

First applying Lemma III.13 to the map π, we have

$$|\pi\sigma'| \leq (1 + c(k)\|\alpha - \beta\|_{\mathrm{m}^*})^k.$$

On the other hand, we apply Lemma III.13 to the map π^{-1} to conclude that

$$|\pi^{-1}(\pi\sigma')| \leq \|\pi^{-1}\|^k |\pi\sigma'|$$

and hence that

$$|\pi\sigma'| \geq \frac{1}{\|\pi^{-1}\|^k} > (1 - c(k)\|\alpha - \beta\|_{\mathrm{m}^*})^k.$$

Thus, $||\pi\sigma'| - 1| \leq C(k) \cdot \|\alpha - \beta\|_{\mathrm{m}^*}$, so

$$\begin{aligned} \frac{||\sigma| - |\pi\sigma||}{|\sigma|} &= ||\sigma'| - |\pi\sigma'|| \\ &= |1 - |\pi\sigma'|| \\ &\leq C(k) \cdot \|\alpha - \beta\|_{\mathrm{m}^*}. \end{aligned}$$

This completes Part I of the proof.

Part II: For the k-simplex $\sigma = [p_0, \ldots, p_k]$ and the projection π, let $H(\sigma)$ be the $(k+1)$-chain (see [Whi57, p. 257]):

$$H(\sigma) := \sum_{i=0}^{k} (-1)^i \tau_i,$$

where τ_i is the $(k+1)$-simplex $[p_0, \ldots, p_i, \pi(p_i), \ldots, \pi(p_k)]$. By construction, $H(\sigma)$ can be triangulated using $(k+1)$ simplexes of dimension $(k+1)$, each of which has mass at most $c(k)\frac{\|\alpha-\beta\|_{\mathrm{m}^*}\cdot(\operatorname{diam}\sigma)^{k+1}}{(k+1)!}$, by [Ada08, p. 7] and Wenger's Lemma.

Similarly, if $\partial\sigma = \sum_{j=1}^{k+1} \rho_j$, where $\{\rho_j\}$ is a disjoint set of $(k-1)$-simplexes, we define the k-chain $H(\partial\sigma) := \sum H(\rho_j)$, where for each j, $H(\rho_j)$ can be triangulated using k simplexes of dimension k, each of which has mass at most $c(k)\frac{\|\alpha-\beta\|_{\mathrm{m}^*}\cdot(\operatorname{diam}\sigma)^{k}}{(k)!}$.

Since $\sigma - \pi\sigma = \partial H(\sigma) + H(\partial\sigma)$, we have

$$\begin{aligned}
|\sigma - \pi\sigma|_\flat &= |\partial H(\sigma) + H(\partial\sigma)|_\flat \\
&\leq |\partial H(\sigma)|_\flat + |H(\partial\sigma)|_\flat \\
&\leq |H(\sigma)| + |H(\partial\sigma)| \\
&\leq c(k)\frac{\|\alpha-\beta\|_{\mathrm{m}^*}\cdot(\operatorname{diam}\sigma)^{k+1}}{k!} + (k+1)c(k)\frac{\|\alpha-\beta\|_{\mathrm{m}^*}\cdot(\operatorname{diam}\sigma)^{k}}{(k-1)!} \\
&\leq c(k)\|\alpha-\beta\|_{\mathrm{m}^*}\frac{|\sigma|}{\eta} \\
&\leq c(k,\eta)\|\alpha-\beta\|_{\mathrm{m}^*}|\sigma|,
\end{aligned}$$

as desired. □

Using Lemma VI.7, we can now show that the vertical slices of F_X are Borel.

Proposition VI.8. *Suppose that X is a flat cochain in a Banach space V. Then for all $p \in V$ the induced maps $(F_X)_p\colon U_p \subset V^k \to \mathbb{R}$ defined by $(F_X)_p(\vec{v}) := F_X(p,\vec{v}) = D_X(p, v_1 \wedge \cdots \wedge v_k)$ are Borel functions.*

Proof: We will show $(F_X)_p$ is continuous with respect to the product topology on

$[V]^k$. Equivalently, we show that $D_X(p,\nu)$ is continuous with respect to ν in the mass* norm.

Fix $p \in V$. Suppose that α and β are k-directions in $\Lambda_k V$ for which both $D_X(p,\alpha)$ and $D_X(p,\beta)$ exist and for which $\|\alpha-\beta\|_{\mathrm{m}^*}$ is less than the bound B from Proposition VI.4. Let π be the projection defined in equation (6.9).

Note that by Lemma III.13, if σ is an η-full simplex lying in P_β, then $\pi(\sigma)$ is a $c(k)\eta$-full simplex in P_α.

Now let (σ_i) be a sequence of $(\frac{1}{c_k}\eta)$-full simplexes in P_β that contain the point p and whose diameters decrease to zero. Then the corresponding sequence $(\pi(\sigma_i))$ consists of η-full simplexes in P_α containing the point p with diameters shrinking to zero.

By Lemma VI.7, for all $i \in \mathbb{N}$,

$$\left|\frac{\pi(\sigma_i)}{|\pi(\sigma_i)|} - \frac{\sigma_i}{|\sigma_i|}\right|_\flat \leq C_{k,\eta} \cdot \|\alpha-\beta\|_{\mathrm{m}^*}.$$

Since $D_X(p,\alpha)$ and $D_X(p,\beta)$ exist, we can evaluate each limit using the sequences (σ_i) and $(\pi\sigma_i)$:

$$\begin{aligned}
|D_X(p,\alpha) - D_X(p,\beta)| &= \left|\lim_{|\pi\sigma_i|\searrow 0} \frac{\langle X, \pi\sigma_i\rangle}{|\pi\sigma_i|} - \lim_{|\sigma_i|\searrow 0} \frac{\langle X, \sigma_i\rangle}{|\sigma_i|}\right| \\
&= \lim_{i\to\infty} \left|\left\langle X, \frac{\pi\sigma_i}{|\pi\sigma_i|} - \frac{\sigma_i}{|\sigma_i|}\right\rangle\right| \\
&\leq \limsup_{i\to\infty} |X|_\flat \cdot \left|\frac{\pi\sigma_i}{|\pi\sigma_i|} - \frac{\sigma_i}{|\sigma_i|}\right|_\flat \\
&\leq |X|_\flat \cdot C_{k,\eta}\|\alpha-\beta\|_{\mathrm{m}^*}.
\end{aligned}$$

Since $D_X(p,\nu) := c \cdot D_X(p,\nu/c)$, the limit $D_X(p,\cdot)$ is continuous in all simple k-vectors, hence $(F_X)_p$ is continuous on U_p.

If the set U_p is also Borel, it follows that $(F_X)_p$ is a Borel function. We show an even stronger result, namely, the set U_p is closed. Let $\vec{a} := (a_1,\ldots,a_k)$ be a k-tuple in V^k which is not in U_p and let $\alpha := a_1 \wedge \cdots \wedge a_k$. In particular, the limit $D_X(p, \alpha)$ does not exist.

We show that there exists ε so that for all k-directions β with $\|\alpha - \beta\|_{\mathrm{m}^*} < \varepsilon$, $D_X(p, \beta)$ is not defined. Now, for some $0 < \eta < 1$, there must be two sequences (σ_i) and (τ_i) of η-full simplexes in P_α containing p whose diameters decrease to zero such that

$$\frac{\langle X, \sigma_i \rangle}{|\sigma_i|} \le c < d \le \frac{\langle X, \tau_i \rangle}{|\tau_i|}.$$

Otherwise, since the flatness of X rules out the possibility that all sequences diverge to ∞, every such sequence would have to converge to $c = d$, which contradicts our assumption that $D_X(p, \alpha)$ does not exist.

Let $C_{k,\eta}$ be the constant from Lemma VI.7 and B the constant from Lemma VI.4. Let $\epsilon < \frac{1}{k} \max\{|d - c|/2, C_{k,\eta}|X|_\flat B\}$, suppose that $\vec{b} := (b_1, \ldots, b_k)$ is a k-tuple with associated k-vector $\beta := b_1 \wedge \cdots \wedge b_k$ such that $\|\alpha - \beta\|_{\mathrm{m}^*} < \epsilon/(C_{k,\eta}|X|_\flat)$. The map π is again the projection defined in equation (6.9). Thus, the sequences $(\pi(\sigma_i))$ and $(\pi(\tau_i))$ are η'-full sequences, where $\eta' > 0$. Then applying Lemma VI.7, we have

$$\left| \frac{\pi(\sigma_i)}{|\pi(\sigma_i)|} - \frac{\sigma_i}{|\sigma_i|} \right|_\flat \le C_{k,\eta} \cdot \|\alpha - \beta\|_{\mathrm{m}^*},$$

and hence

$$\left| \frac{\langle X, \pi(\sigma_i) \rangle}{|\pi(\sigma_i)|} - \frac{\langle X, \sigma_i \rangle}{|\sigma_i|} \right| \le C_{k,\eta} \cdot \|\alpha - \beta\|_{\mathrm{m}^*} \cdot |X|_\flat < \epsilon.$$

Similarly,

$$\left| \frac{\langle X, \pi(\tau_i) \rangle}{|\pi(\tau_i)|} - \frac{\langle X, \tau_i \rangle}{|\tau_i|} \right| < \epsilon.$$

Combining the previous two inequalities, we have

$$\left| \frac{\langle X, \pi(\sigma_i) \rangle}{|\pi(\sigma_i)|} \right| \le c + \epsilon < d - \epsilon \le \left| \frac{\langle X, \pi(\tau_i) \rangle}{|\pi(\tau_i)|} \right|.$$

Thus, the limit $D_X(p, \beta)$ does not exist, and hence $(U_p)^c$ is open. $\square$

Proof of Theorem VI.2: Let X be a flat cochain in a Banach space V.

Propositions VI.3 and VI.8 show that F_X satisfies properties (i) and (ii) of a partial form.

For $d \geq k$, fix an affine subspace W^d. Since X is a flat k-cochain on V, it can be restricted to a cochain X_W on W^d that acts on $\sigma \in \mathcal{P}_k(W)$ by $\langle X_W, \sigma \rangle := \langle X, \sigma \rangle$. We equip $\Lambda_k W^d$ with the Euclidean norm, $| \cdot |_2$, i.e., we identify W^d with $\mathbb{R}^d$. We denote k-dimensional Lebesgue measure on W^d by $| \cdot |_E$.

Fix a mass*-k-direction $\nu \in \Lambda_k W_0^d$. Let λ denote the factor by which the Euclidean mass $|\nu|_2$ of ν differs from $\|\nu\|_{\mathrm{m}^*}$:

$$\lambda = \frac{|\nu|_2}{\|\nu\|_{\mathrm{m}^*}} = |\nu|_2.$$

Let μ be the Euclidean k-direction $\mu := \frac{\nu}{|\nu|_2}$. Now, for any representation of μ as $\mu = u_1 \wedge \cdots \wedge u_k$, the simplex σ_μ with vertices $\{u_1, \ldots, u_k\}$ has Euclidean volume $|\sigma_\mu|_E = \frac{1}{k!}$. By Wenger's Lemma, for any representation of ν, the corresponding simplex σ_ν has Adams mass $|\sigma_\nu| = \frac{1}{k!}$. This implies that for any k-simplex σ parallel to ν (i.e., any simplex lying in a k-dimensional ν-superplane), we have

$$|\sigma|_E = \lambda |\sigma|.$$

The cochain X_W on $\mathbb{R}^d (= W^d)$ is flat, so by Wolfe's theorem, it can be identified with a (Euclidean) flat k-form β_{X_W} on W^d. The form β_{X_W} is obtained by a similar limit process as in equation (6.1), as follows. For a *Euclidean* k-direction on W, we first define the limit $D^E_{X_W}(p, \nu)$ by

$$D^E_{X_W}(p, \nu) := \lim_{i \to \infty} \frac{\langle X, \sigma_i \rangle}{|\sigma_i|_E},$$

wherever it exists. Here, each simplex σ_i must contain p and the sequence (σ_i) must have fullness bounded away from zero and diameters decreasing to zero. The superscript E is used to keep track of the fact that this limit normalizes by the Euclidean mass of each simplex σ_i. If $0 \neq |\nu|_2 \neq 1$, let

$$D^E_{X_W}(p,\nu) := |\nu|_2 D^E_{X_W}(p, \frac{\nu}{|\nu|_2}).$$

For almost every point $p \in W$ we have that $\beta_{X_W} = D^E_{X_W}$. By Theorem IV.2, for almost every point $p \in W^d$, for *every* simple k-vector $\alpha \in \Lambda_k W^d_0$,

$$\begin{aligned}
D^E_{X_W}(p,\alpha) &= |\alpha|_2 \cdot D^E_{X_W}\left(p, \frac{\alpha}{|\alpha|_2}\right) \\
&= \lambda \|\alpha\|_{\mathrm{m}^*} \cdot D^E_{X_W}\left(p, \frac{\alpha}{|\alpha|_2}\right) \\
&= \|\alpha\|_{\mathrm{m}^*} \lim_{i\to\infty} \frac{\lambda\langle X, \sigma_i\rangle}{|\sigma_i|_E} \\
&= \|\alpha\|_{\mathrm{m}^*} \lim_{i\to\infty} \frac{\langle X, \sigma_i\rangle}{|\sigma_i|} \\
&= \|\alpha\|_{\mathrm{m}^*} \cdot D_X\left(p, \frac{\alpha}{\|\alpha\|_{\mathrm{m}^*}}\right) \\
&= D_X(p,\alpha).
\end{aligned}$$

In the above sequence of equations, $\lambda = \frac{|\alpha|_2}{\|\alpha\|_{\mathrm{m}^*}}$

By Theorem IV.2, $D^E_{X_W}$ has the property that for almost every point p in W^d, the limit $D^E_{X_W}(p,\nu)$ is extendable to a linear function on k-vectors ν in $\Lambda_k W^d_0$. Hence, F_X satisfies property (iii) of a partial form. □

In the next lemma we prove that the comass of F_X is bounded by $|X|_\flat$.

Lemma VI.9. *Let X be a flat cochain. Then $\|F_X\|_\infty \leq |X|_\flat$.*

Proof: For all k-chains $\sigma \in V$,

$$\frac{\langle X, \sigma\rangle}{|\sigma|} \leq \frac{|X|_\flat \cdot |\sigma|_\flat}{|\sigma|} \leq |X|_\flat.$$

Thus when defined, $D_X(p,\nu) \leq |X|_\flat$ for all k-directions ν. Let $W = W^k \subset V$ be a k-dimensional affine subspace of V. At almost every $p \in W^k$, for all $(v_1, \ldots, v_k) \in$

$[W_0]^k$,

$$\begin{aligned} F_X(p,(v_1,\ldots,v_k)) &= \|v_1\wedge\cdots\wedge v_k\|_{\mathrm{m}^*} D_X\left(p,\nu_W\right) \\ &\leq \|v_1\wedge\cdots\wedge v_k\|_{\mathrm{m}^*}|X|_\flat. \end{aligned}$$

Since

$$\|F_X\|_{\infty,W^k} = \operatorname*{ess\,sup}_{p\in W^k} \frac{(F_X)_p(v_1,\ldots,v_k)}{\|v_1\wedge\cdots\wedge v_k\|_{\mathrm{m}^*}},$$

it follows that $\|F_X\|_{\infty,W^k} \leq |X|_\flat$, and hence that $\|F_X\|_\infty \leq |X|_\flat$. □

By Theorem IV.2 and the proof of Theorem VI.2, for any simplex $\sigma\subset V$, the action of X on σ is given by integration of the partial form F_X:

$$\langle X,\sigma\rangle = \int_\sigma F_X. \tag{6.12}$$

This action can be extended naturally to all polyhedral chains $P\in\mathcal{P}_k(V)$.

We now show that F_X is flat.

Theorem VI.10. *F_X is a flat partial k-form.*

Proof: By Proposition III.10, $|dX|_\flat \leq |X|_\flat < \infty$, so the cochain dX is flat. Applying Theorem VI.2 to the flat cochain dX, we conclude that F_{dX} is a partial $(k+1)$-form. Applying Lemma VI.9 to the cochain dX, we have $\|F_{dX}\|_\infty \leq |dX|_\flat$. Finally, by equation (6.12), the integral of F_{dX} over a polyhedral $(k+1)$-chain τ is given by $\int_\tau F_{dX} = \langle dX,\tau\rangle$, so for all such τ,

$$\int_\tau F_{dX} = \langle dX,\tau\rangle = \langle X,\partial\tau\rangle = \int_{\partial\tau} F_X.$$

Thus, $F_{dX} = dF_X$.

Since $\max\{\|F_X\|_\infty, \|dF_X\|_\infty\} \leq |X|_\flat$, F_X is a flat partial k-form. □

By equation (6.12), $\Psi(F_X) = X$, so we conclude that Ψ is surjective, proving Theorem VI.1.

CHAPTER VII

Classical Differential Forms

If we start with a flat k-cochain X, then by the results of the previous section, X can be associated with the partial differential form F_X, where $F_X(p, \vec{v}) = D_X(p, \nu)$ for $\vec{v} := (v_1, \ldots, v_k)$ and $\nu := v_1 \wedge \cdots \wedge v_k$.

The domain of F_X is the set $U_X \subset V \times V^k$, which is given by

$$U_X := \{(p, \vec{v}) : D_X(p, \nu) \text{ exists}\}.$$

For a given point $p \in V$, we would like to know when $D_X(p, \nu)$ exists for all simple k-vectors ν. We define the set $\Upsilon_X \subset V$ by

$$\Upsilon_X := \{p \in V : D_X(p, \nu) \text{ exists for all simple } \nu\}.$$

In Υ_X, F_X extends to a differential form differential form

$$\omega_X \colon \Upsilon_X \to \Lambda^k V.$$

We will show that if V is separable, then the set Υ_X is large, in the sense that its complement is an Aronszajn null set.

For convenience, we recall the definition of an Aronszajn null set (see [BL00]). For a Banach space V and a nonzero vector $y \in V$, we define the sets

$$\mathcal{A}(y) := \{A \subset V : A \text{ is Borel and } \mathcal{H}^1(A \cap L) = 0 \text{ for all lines } L \text{ parallel to } y\}.$$

If $\{x_n\} \subset V$ is a countable (finite or infinite) set of nonzero vectors, then we define

$$\mathcal{A}(\{x_n\}) := \{A \subset V : A \text{ is Borel, } A = \cup A_n, \text{ where } A_n \in \mathcal{A}(x_n) \text{ for all } n\}.$$

Note that if $\{x_n\} \subseteq \{y_n\}$ then $\mathcal{A}(\{x_n\}) \subseteq \mathcal{A}(\{y_n\})$.

Definition VII.1. A set A is *Aronszajn null* if $A \in \cap \mathcal{A}(\{x_n\})$, where the intersection is taken over all sequences whose span is dense in V.

Theorem VII.2. *Given a flat k-cochain X on a separable Banach space V, the set $V \setminus \Upsilon_X$ is Aronszajn null.*

The proof is similar to the proof that the set of Gateaux non-differentiability of a Lipschitz map from a separable Banach space to a space with the Radon-Nikodym Property is Aronszajn null (see Theorem 6.42 in [BL00]).

Proof: If $\vec{v} = (v_1, \dots, v_k) \in V^k$, we define $D_X(p, \vec{v}) := D_X(p, v_1 \wedge \cdots \wedge v_k)$. Suppose $\{x_n\}$ is any sequence of vectors whose span is dense in V. For $j \in \mathbb{N}$, define the subspace

$$V_j := \text{span}\{x_1, \dots, x_j\}.$$

The subset $D_j \subset V$ is the "good set" for V_j:

$$D_j := \{p \in V : D_X(p, \nu) \text{ is linear on } \Lambda_k V_j\}.$$

Note that for all j, $V_j \subset V_{j+1}$ and $D_j \supset D_{j+1}$.

For $y \in V$, consider the subset of V_j obtained by translating the "bad set" for V_j by y:

$$[(V \setminus D_j) + y] \cap V_j =: S_j.$$

If $p \in [(V \setminus D_j) + y]$, then $p - y \in V \setminus D_j$. Thus there exists $\nu = v_1 \wedge \cdots \wedge v_k \in \Lambda_k V$ with the property that the limit $D_X(p - y, \nu)$ does not exist (or the function is not k-linear in the components of ν). Consider the cochain $(X - y)$ defined by $\langle X - y, \sigma\rangle := \langle X, \sigma + y\rangle$. S_j is the set where $D_{(X-y)}(p, \nu)$ doesn't exist for some $\nu \in \Lambda_k V_j$ or where $D_{(X-y)}(p, \nu)$ is not linear in ν. Since $(X - y)$ restricted to V_j is a flat k-cochain on the finite dimensional space V_j, Whitney's results [Whi57, p. 254] show that there exists a flat form ω that is defined almost everywhere in V_j and is equivalent (equal almost everywhere) to the restriction of $D_{(X-y)}$ on V_j. Hence we conclude that S_j has Lebesgue measure zero in V_j. By [BL00, Proposition 6.29],

$$V \setminus D_j \in \mathcal{A}(\{x_1, \ldots, x_j\}) \subset \mathcal{A}(\{x_n\}).$$

Then $V \setminus (\cap_n D_n) \in \mathcal{A}(\{x_n\})$; as $\{x_n\}$ was arbitrary, $V \setminus (\cap_n D_n)$ is Aronszajn null.

The fact that both $V \setminus D_j$ and $V \setminus (\cap_n D_n)$ are Borel (which is necessary for these sets to be in $\mathcal{A}(\{x_n\})$) follows from the fact that D_j is a Borel set for all $j \in \mathbb{N}$. (If $j < k$, D_j is the empty set.) We note that by definition,

$$D_j = \bigcap_{\vec{v}\in[V_j]^k} U_{\vec{v}},$$

where $U_{\vec{v}} \subset V$ is the set from Definition IV.4. Hence $D_j \subset \bigcap_i U_{\vec{w}_i}$. Let $\{\vec{w}_i\}$ be a countable dense set in $[V_j]^k$. By the proof of Theorem VI.2, the set U_p is closed for all $p \in V$. If a point $p \in V$ is in $\bigcap_{i=1}^{\infty} U_{\vec{w}_i}$, then the sequence $\{\vec{w}_i\}$ is a subset of U_p. Since U_p is closed and $\{\vec{w}_i\}$ is dense in $[V_j]^k$, $U_p \supset [V_j]^k$. This shows that $D_j \supset \bigcap_i U_{\vec{w}_i}$, so $D_j = \bigcap_i U_{\vec{w}_i}$. Since each $U_{\vec{w}_i}$ is Borel (Proposition VI.3), D_j is also Borel.

We will show that $V \setminus \Upsilon_X$ is an Aronszajn null set by proving that $V \setminus \Upsilon_X = V \setminus (\cap_n D_n)$, or equivalently, that $\Upsilon_X = (\cap_n D_n)$. If a point is in Υ_X it must be in every "good set" D_n, so $\Upsilon_X \subset (\cap_n D_n)$.

Assume that $p \in (\cap_n D_n)$. The limit $D_X(p, \nu)$ exists for all k-vectors $\nu = v_1 \wedge \cdots \wedge v_k$ with $v_1, \ldots, v_k \subset \text{span}\{x_n\}$. In other words $(v_1, \ldots, v_k)$ is in U_p for all

$v_1, \ldots, v_k \subset \text{span}\{x_n\}$. Since U_p is closed and $\text{span}\{x_n\}$ is dense in V^k, $U_p = [V]^k$. Thus, $p \in \Upsilon_X$, as desired. $\square$

CHAPTER VIII

Invariance Under Lipschitz Maps

8.1 Pushing polyhedral chains forward under Lipschitz maps

In order to show that flat partial forms can be pulled back under Lipschitz maps, we first show that polyhedral chains can be pushed forward to flat chains.

Since every polyhedral k-chain A can be represented as a sum of simple k-chains, it is enough to show that we can push a simple chain forward. Since we can push chains forward under linear maps, we approximate our Lipschitz map by a sequence of appropriately chosen piecewise affine maps. We then show that the sequence of (polyhedral) images of a simple chain under such a sequence of piecewise affine approximations converges in the flat norm, and hence defines a flat chain. This yields a construction of the pushforward of a simple chain which we then extend to arbitrary chains.

We denote the pushforward of a simplex σ under a linear map by $f(\sigma)$, or $f_*(\sigma)$.

Definition VIII.1. Let P be a simple polyhedral k-chain in a Banach space. A *simplicial subdivision of* P is a refinement of P consisting of simplexes that are pairwise disjoint except on their boundaries. A *polyhedral subdivision of* P is a refinement of P consisting of simple polyhedral chains that are pairwise disjoint except on their boundaries. The vertices of a subdivision are the vertices of the simplexes (or polyhedra) in the refinement. We say that a subdivision has *fullness* $\eta > 0$ if each simplex (or polyhedron) in the subdivision has fullness at least η. The *mesh* of a subdivision is the maximum of the diameters of its simplexes (or polyhedra).

We mostly use simplicial subdivisions in this chapter; when the type of subdivision is not specified it is assumed to be simplicial.

Definition VIII.2. Let V and W be Banach spaces, and $P \in \mathcal{P}$ be a polyhedral chain. A map $f\colon P \to W$ is *piecewise affine* if there exists a polyhedral subdivision of P so that f is affine on any polyhedron in the subdivision.

Definition VIII.3. Suppose that $f\colon \mathbb{R}^k \to V$ is a Lipschitz map and σ is a k-dimensional simplex in $\mathbb{R}^k$. The *affine approximation of f with respect to σ* is the unique affine map g from $\mathbb{R}^k$ to V that agrees with f on the vertices of σ. If $\mathfrak{S}$ is a simplicial subdivision of the simplex σ, then the *affine approximation to f with respect to $\mathfrak{S}$* is the piecewise-affine map $g\colon \sigma \to V$ that agrees with f on the vertices of $\mathfrak{S}$ and is affine on each simplex in the subdivision $\mathfrak{S}$.

Lemma VIII.4. *Let σ be a k-simplex in V with an η-full subdivision $\mathfrak{S}$ for some $\eta > 0$. Suppose that f is a Lipschitz map from σ to W, and that g is the affine approximation to f with respect to $\mathfrak{S}$. Then*

$$\mathrm{Lip}(g) \leq C(k)\frac{\mathrm{Lip}(f)}{\eta},$$

where $C(k)$ is a constant depending only on k.

Proof: The proof proceeds as in [Whi57, p. 290], using [Ada08, Lemma 3.3] in place of [Whi57, IV, 15.3]. □

The following lemma states that Lipschitz maps on simplexes are limits of piecewise linear maps under the topology of uniform convergence.

Lemma VIII.5. *Let σ be a k-simplex in V and let f be an L-Lipschitz map from V to W. Then there exist $\eta > 0$ depending only on the fullness of σ and an η-full sequence of subdivisions $\mathfrak{S}_i$ of σ and a sequence of maps $f_i\colon \sigma \to W$ that satisfy the following properties:*

(i) f_i *is affine on each subsimplex of* $\mathfrak{S}_i$*,*

(ii) f_i *is* $(c(k,\eta)L)$*-Lipschitz,*

(iii) $\|f_i - f\|_\infty \to 0$ *as* $i \to \infty$*.*

Proof: Let $(\mathfrak{S}_i)$ be the sequence of standard subdivisions described in [Ada08, p. 9]. By construction, each subsimplex of each subdivision has diameter less than $1/i$ and fullness greater than η for some positive η that depends only on $\Theta(\sigma)$. Let f_i be the affine approximation to f with respect to $\mathfrak{S}_i$. Property (i) is holds by definition. Property (ii) follows from Lemma VIII.4. To show property (iii), let $x \in \sigma$. Fix $i \in \mathbb{N}$ and let y be a vertex of $\mathfrak{S}_i$ that has minimal distance from x. Then by the triangle inequality, $|f_i(x) - f(x)| \leq |f_i(x) - f_i(y)| + |f_i(y) - f(x)|$. Since y is a vertex of $\mathfrak{S}_i$, $f_i(y) = f(y)$, so $|f_i(x) - f(x)| \leq |f_i(x) - f_i(y)| + |f(y) - f(x)|$. Since f and f_i are Lipschitz, we conclude that $|f_i(x) - f(x)| \leq \mathrm{Lip}(f_i)|x-y| + \mathrm{Lip}(f)|x-y| \leq (\mathrm{Lip}(f_i) + \mathrm{Lip}(f))(1/i)$. This proves (iii). □

We recall the definition of Gâteaux differentiability (see [BL00]).

Definition VIII.6. Let X and Y be Banach spaces and $f\colon X \to Y$ a function. Then f is *Gâteaux differentiable* at a point $x_0 \in X$ if there is a bounded linear operator $T\colon X \to Y$ such that

$$T(u) = \lim_{t \to 0} \frac{f(x_0 + tu) - f(x_0)}{t}$$

for every $u \in X$.

For fixed $x_0, u \in X$, the limit $\lim_{t\to 0} \frac{f(x_0+tu)-f(x_0)}{t}$, if it exists, is defined to be the directional derivative of f at x_0 in the direction u, denoted $D_f(x_0, u)$.

We now define the *Jacobian* of a map from Euclidean space to a Banach space:

Definition VIII.7. Let V be a Banach space and $f\colon \mathbb{R}^k \to V$ a function that is Gâteaux differentiable at $p \in \mathbb{R}^k$. The *Jacobian* $J_f(p)$ of f at p is the k-vector

$$J_f(p) := D_f(p, e_1) \wedge \cdots \wedge D_f(p, e_k),$$

where the vectors $e_1, \ldots, e_k$ constitute the standard (positively oriented) orthonormal basis of $\mathbb{R}^k$.

The Euclidean version of the following lemma is [Whi57, Lemma X.5a, p. 295].

Lemma VIII.8. *Let $A = \Sigma_{i=1}^N \sigma_i^k$ be a polyhedral k-chain in $\mathbb{R}^k$ with boundary $\partial A = \Sigma_{j=1}^M \tau_j^{k-1}$, where σ_i and τ_j are simple k- and $(k-1)$-chains, respectively. Let f and h be Lipschitz mappings from A to V which are affine on each simplex σ_i^k and τ_j^{k-1}. Let $L = \max\{\mathrm{Lip}(f), \mathrm{Lip}(h)\}$. Then*

$$|f(A) - h(A)|_\flat \leq c(k)\|f - h\|_\infty \cdot (L^k|A| + L^{k-1}|\partial A|),$$

where $c(k)$ depends only on k.

Recall that simple polyhedral chains need not be simplexes!

Proof: Let I denote the unit interval $[0, 1]$, and let $I \times A := \sum I \times \sigma_i$, where $I \times A$ is the Cartesian product as defined in [Whi57, p. 365]. Let $F : I \times A \to V$ be the following homotopy between f and h:

$$F(t, p) = (1 - t)f(p) + t \cdot h(p), \qquad 0 \leq t \leq 1, \qquad p \in A.$$

Since the maps f and h are affine on each simplex σ_i, F is affine on $I \times \sigma_i$ for each i. Then $F(\{0\} \times A) = f(A)$ and $F(\{1\} \times A) = h(A)$. Thus, we may rewrite the chain

$h(A) - f(A)$ as

$$h(A) - f(A) = F(I \times \partial A) + \partial F(I \times A).$$

Suppose that the point $p \in A$ is in the interior of one of the simplexes σ_i^k. Let $\{e_1, \dots, e_k\}$ be an orthonormal basis of $\mathbb{R}^k$ and let e_0 be the unit vector along I. Thus, $\{e_0, e_1, \dots, e_k\}$ is an orthonormal basis of $I \times \sigma_i^k$. Since f is affine in σ_i^k,

$$\left| \lim_{h \to 0^+} \frac{f(p + he_r) - f(p)}{h} \right|_V \leq L$$

for all $r \in \{1, \dots k\}$. In other words, the directional derivatives of f at p are bounded above by L:

$$|Df(p)(e_r)|_V \leq L$$

for $r \in \{1, \dots k\}$.

The previous inequality is also true for the map h, so by linearity, we can also bound the norm of $DF(t,p)(e_r)$ by L:

$$|DF(t,p)(e_r)|_V = |(1-t)Df(p)(e_r) + tDh(p)(e_r)|_V \leq L.$$

For the derivative in the direction of e_0, we have

$$\begin{aligned} |DF(t,p)(e_0)|_V &= \left| \lim_{h \to 0^+} \frac{F(t+h,p) - F(t,p)}{h} \right|_V \\ &= |h(p) - f(p)|_V \\ &\leq \|f - h\|_\infty. \end{aligned}$$

Hence, by inequality (2.3) and Definition II.3,

$$\|J_F(t,p)\|_{\mathrm{m}^*} \leq k^{k/2} \|f - h\|_\infty L^k \tag{8.1}$$

almost everywhere in $I \times A$.

Since F is affine on $I \times \sigma_i$ for each i,

$$|F(I \times \sigma_i^k)| = \int_{I \times \sigma_i^k} \|J_F(x)\|_{\mathrm{m}^*} \, d\mathcal{H}^{k+1}(x). \tag{8.2}$$

For each τ_j in ∂A, let F_j denote the restriction of F to $I \times \tau_j$. Then

$$|F(I \times \tau_j^{k-1})| = \int_{I \times \tau_j^{k-1}} \|J_{F_j}(x)\|_{\mathrm{m}^*} \, d\mathcal{H}^k(x). \tag{8.3}$$

Combining equation (8.2) with inequality (8.1), we have

$$\begin{aligned} |F(I \times \sigma_i^k)| &\leq \int_{I \times \sigma_i^k} (c(k)\|f-h\|_\infty L^k) \, d\mathcal{H}^{k+1}(p) \\ &= c(k)\|f-h\|_\infty L^k |\sigma_i^k|. \end{aligned}$$

Thus,

$$\begin{aligned} |F(I \times A)| &\leq \sum_{i=1}^{M} |F(I \times \sigma_i^k)| \\ &\leq c(k)\|f-h\|_\infty L^k |A|. \end{aligned}$$

Analogously, we can combine equation (8.3) with inequality (8.1) to conclude that

$$|F(I \times \partial A)| \leq c(k)\|f-h\|_\infty L^{k-1} |\partial A|.$$

Since $F(0,p) = f(p)$ and $F(1,p) = h(p)$, we have

$$\begin{aligned} |h(A) - f(A)|_\flat &= |F(I \times \partial A) + \partial F(I \times A)|_\flat \\ &\leq |F(I \times \partial A)|_\flat + |\partial F(I \times A)|_\flat \\ &\leq |F(I \times \partial A)| + |F(I \times A)| \\ &\leq c(k)\|f-h\|_\infty L^{k-1}|\partial A| + c(k)\|f-h\|_\infty L^k |A| \\ &= c(k)\|f-h\|_\infty (L^{k-1}|\partial A| + L^k|A|). \end{aligned}$$

□

Proposition VIII.9. *Let A be a simplex in $\mathbb{R}^k$ and let $f\colon A \to V$ be L-Lipschitz. Suppose that (g_i) and (h_i) are sequences of piecewise affine $\widetilde{L}$-Lipschitz maps from A to V such that $\|g_i - f\|_\infty \to 0$ and $\|h_i - f\|_\infty \to 0$ as $i \to \infty$. Then* *(i) The sequence $(g_i(A))$ is Cauchy in the flat norm, and*

(ii) $|g_i(A) - h_i(A)|_\flat \to 0$ *as* $i \to \infty$.

Proof: Part (i) of the Proposition follows immediately from Lemma VIII.8. Part (ii) follows from Lemma VIII.8 and the triangle inequality (which implies that $\|g_i - h_i\|_\infty \to 0$ as $i \to 0$). □

Definition VIII.10. Let $\sigma \subset \mathbb{R}^k$ be a simple polyhedral k-chain and $f \colon \mathbb{R}^k \to V$ an L-Lipschitz map. As in [Ada08, p. 11], there exists a $C(k)$-full simplex S containing σ. Let $(\mathfrak{S}_i)$ be a sequence of η-full subdivisions of S whose mesh sizes shrink to zero as $i \to \infty$, where $\eta > 0$ depends only on k. Let $f_i \colon \sigma \to V$ be the affine approximations to f determined by $\mathfrak{S}_i$. We define the *pushforward of* σ *by* f by the following limit with respect to the flat norm

$$f_*(\sigma) := \lim_{i \to \infty} f_i(\sigma).$$

Lemma VIII.5 and Proposition VIII.9 show that this limit exists and is unique; hence $f_*(\sigma)$ is well-defined.

Next, we use isomorphisms between finite-dimensional Banach spaces and Euclidean space to show that simple polyhedral chains push forward under Lipschitz maps between Banach spaces.

Lemma VIII.11. *Let* $f : W \to V$ *be a Lipschitz map between Banach spaces and* σ *be a simple* k*-chain in* $\mathcal{P}_k(W)$*. Let* $L_1, L_2 \colon \mathbb{R}^k \to W$ *be injective linear maps so that* $\sigma \subset L_1(\mathbb{R}^n) \cap L_2(\mathbb{R}^k)$*. Then*

$$(f \circ L_1)_*(L_1^{-1}(\sigma)) = (f \circ L_2)_*(L_2^{-1}(\sigma)).$$

Proof: Let $L := \mathrm{Lip}(f)$, $K_1 := \mathrm{Lip}(L_1)$ and $K_2 := \mathrm{Lip}(L_2)$, and let $\varphi \colon \mathbb{R}^k \to \mathbb{R}^k$

be $\varphi = L_1^{-1} \circ L_2$. Then $\mathrm{Lip}(f \circ L_2) \leq LK_2$, so the polyhedral chain $L_2^{-1}(\sigma)$ pushes forward under $f \circ L_2$ as in Definition VIII.10.

There exists a $C(k)$-full simplex $S \subset \mathbb{R}^K$ containing $L_2^{-1}(\sigma)$ and a sequence $(\mathfrak{S}_i)$ of η-full subdivisions of S whose mesh sizes shrink to zero. We then let $(g_i \colon S \to V)$ be the sequence of affine approximations of $f \circ L_2$ with respect to $\mathfrak{S}_i$. Thus,

$$g_i(L_2^{-1}(\sigma)) \to (f \circ L_2)_*(L_2^{-1}(\sigma))$$

in the flat norm.

Similarly, the chain $L_1^{-1}(\sigma)$ pushes forward to V under $f \circ L_1$. We will show that the sequence $(g_i(L_2^{-1}(\sigma)))$ gives a polyhedral approximation of this chain, and therefore that $(f \circ L_1)_*(L_1^{-1}(\sigma)) = (f \circ L_2)_*(L_2^{-1}(\sigma))$.

For each i, $\varphi(\mathfrak{S}_i)$ is an η'-full subdivision of $L_1^{-1}(\sigma)$, where η' is independent of i. Thus,

$$h_i(L_1^{-1}(\sigma)) \to f \circ L_1(L_1^{-1}(\sigma))$$

where (h_i) is the sequence of piecewise affine approximations of $f \circ L_1$ with respect to the subdivisions $(\varphi(\mathfrak{S}_i))$. By construction, $g_i = h_i \circ \varphi$, so

$$g_i(L_2^{-1}(\sigma)) = h_i \circ (L_1^{-1} \circ L_2)(L_2^{-1}(\sigma)) = h_i(L_1^{-1}(\sigma)),$$

as desired. □

In the proof above, we note that for each i, $g_i(L_2^{-1}(\sigma)) = f_i(\sigma)$, where f_i is the piecewise affine approximation of f with respect to the (full) subdivision $L_2(\mathfrak{S}_i)$. Thus we can define the pushforward of a simple chain in a Banach space as in Definition VIII.10.

Definition VIII.12. Let $\sigma \subset W$ be a simple polyhedral k-chain and $f \colon W \to V$ an L-Lipschitz map between Banach spaces. As in [Ada08, p. 11], there exists a $C(k)$-full simplex S containing σ. Let $(\mathfrak{S}_i)$ be a sequence of η-full subdivisions of S whose mesh sizes shrink to zero as $i \to \infty$, where $\eta > 0$ depends only on k. Let $f_i \colon \sigma \to V$ be the affine approximations to f determined by $\mathfrak{S}_i$. We define the *pushforward of σ by f* by the following limit with respect to the flat norm

$$f_*(\sigma) := \lim_{i\to\infty} f_i(\sigma).$$

The following lemma will allow us to extend this notion of a pushforward to arbitrary polyhedral chains.

Lemma VIII.13. *Let $f \colon W \to V$ be a Lipschitz map, σ a simple polyhedral chain in $\mathcal{P}_k(W)$, and $\sum \tau_j$ a refinement of σ. Then*

(i) $f_*(\sigma) + f_*(-\sigma) = 0$,

(ii) $f_*(\sigma) - \sum f_*(\tau_j) = 0$

Proof: Let $(\mathfrak{S}_i)$ be a full sequence of subdivisions of σ with mesh size approaching zero. By definition, $f_*(\sigma) = \lim f_i(\sigma)$, where $f_i(\sigma)$ is the approximation to f with respect to $\mathfrak{S}_i$. Since $(\mathfrak{S}_i)$ is also a full sequence of subdivisions of $-\sigma$, we have $f_*(-\sigma) = \lim f_i(-\sigma)$, and since f_i is piecewise affine, $f_i(-\sigma) = -f_i(\sigma)$. Thus $f_*(\sigma) + f_*(-\sigma) = 0$.

For a refinement $\sum \tau_j$ of σ, let $(\mathfrak{T}_i)$ be a full sequence of subdivisions of $\sum \tau_j$ with mesh size approaching zero. This subdivision is also a subdivision of σ, so for the affine approximations g_i of f with respect to $\mathfrak{T}_i$ we have

$$f_*(\sigma) = \lim f_i(\sigma) = \lim \sum f_i(\tau_j) = \lim f_i(\sum \tau_j) = f_*(\sum \tau_j).$$

□

For a Lipschitz map $f \colon W \to V$ and a polyhedral k-chain $A = \sum \lambda_i \sigma_i \in \mathcal{P}_k(V)$, we define

$$f_*(A) := \sum \lambda_i f_*(\sigma_i).$$

Lemma VIII.13 shows that this is well-defined.

Lemma VIII.14. *Let $f : W \to V$ be a Lipschitz map and τ a polyhedral k-chain in W. Then*

$$f_*(\partial\tau) = \partial(f_*(\tau)).$$

Proof: Write $\tau = \sum_{j=1}^{n} \sigma_j$, where σ_j is a simple polyhedral k-chain. Let $(\mathfrak{S}_i)$ be a sequence of η-full subdivisions of τ whose mesh sizes approach zero. This subdivision induces a sequence of subdivisions $(\mathfrak{T}_i)$ of $\partial\tau$. The sequence $(\mathfrak{T}_i)$ has fullness bounded away from zero. This holds because it is true in the case that W is Euclidean. We push each of the simple summands σ_j forward to $\mathbb{R}^k$ by a $\sqrt{k}$-bi-Lipschitz linear isomorphism $L : \mathbb{R}^k \to W_{\sigma_j}$ as in Theorem II.8 so that by Corollary III.15 the pushforward $(L^{-1})_*(\mathfrak{S}_i)$ has fullness $C = C(k, \eta)$ and conclude that the pushforward $(L^{-1})_*(\mathfrak{S}_i)$ induces a full sequence of subdivisions on the boundary $\partial((L^{-1})_*(\sigma_j))$. Pushing forward again, this time by the map L and applying Corollary III.15, we have that $(L_*((L^{-1})_*(\mathfrak{S}_i))) = (\mathfrak{S}_i)$ induces a full sequence of subdivisions on the boundary $\partial\sigma_j$, and hence on $\partial\tau$.

Let f_i be the sequence of piecewise affine approximations to f with respect to $(\mathfrak{S}_i)$. Then

$$\begin{aligned}
f_*(\partial\tau) &= \lim f_i(\partial\tau) \\
&= \lim \partial(f_i(\tau)) \qquad (8.4)\\
&= \partial(\lim f_i(\tau)) \qquad (8.5)\\
&= \partial(f_*(\tau)).
\end{aligned}$$

In the preceding sequence, equation (8.4) holds because f_i is piecewise affine and equation (8.5) is a result of the extension of the boundary operator to flat chains.

□

Next, we prove a lemma that bounds the flat norm of $f_*(A)$ by a multiple of the flat norm of A.

Lemma VIII.15. *Let A be a polyhedral chain in W and let $f\colon W \to V$ be a Lipschitz map with Lipschitz constant L. Then $|f_*(A)|_\flat \leq C(L,k)|A|_\flat$, where the constant $C(L,k)$ depends only on L and k.*

Proof: Fix $\varepsilon > 0$, and choose a polyhedral $(k+1)$-chain D so that

$$|A - \partial D| + |D| \leq |A|_\flat + \varepsilon.$$

Then

$$\begin{aligned} |f_*(A)|_\flat &= |f_*(A - \partial D + \partial D)|_\flat \\ &= |f_*(A - \partial D) + f_*(\partial D)|_\flat \\ &\leq |f_*(A - \partial D)|_\flat + |f_*(\partial D)|_\flat \\ &\leq |f_*(A - \partial D)| + |f_*(D)|. \end{aligned} \tag{8.6}$$

The k-chain $A - \partial D$ may be represented by $\sum_j \lambda_j \sigma_j$, where $\lambda_j \in \mathbb{R}$ and σ_j is a simple polyhedral k-chain. By [Ada08, p. 11], each σ_j lies in an $C(k)$-full simplex which we call α_j. For each j, let $(\mathfrak{S}_i^j)$ denote a standard sequence of subdivisions of α_j whose mesh sizes approach zero. Each subdivision has fullness greater than a constant depending only on k. As before, we denote by f_i^j the piecewise affine map that is constant on each simplex of the i-th subdivision of σ_j and agrees with f on the vertices of the subdivision.

Similarly, the $(k+1)$-chain D may be represented by $\sum_\ell \mu_\ell \tau_\ell$, where $\mu_\ell \in \mathbb{R}$ and τ_ℓ is a simple polyhedral k-chain. By [Ada08, p. 11], each σ_ℓ lies in an $C(k+1)$-full simplex which we call β_ℓ. For each ℓ, let $(\mathfrak{T}_i^\ell)$ denote a standard sequence of subdivisions of β_ℓ whose mesh sizes approach zero. Each subdivision has fullness greater than a constant depending only on k. We denote by f_i^ℓ the piecewise affine map that is constant on each simplex of the i-th subdivision of τ_ℓ and agrees with f on the vertices of the subdivision. Then we have

$$
\begin{aligned}
|f_*(A-\partial D)|+|f_*(D)| &\leq |f_*(\sum_j \lambda_j\sigma_j)|+|f_*(\sum_\ell \mu_\ell\tau_\ell)| \\
(8.7) \qquad &\leq \sum_j \liminf_i \lambda_j |f_i^j(\sigma_j)| + \sum_\ell \liminf_i \mu_\ell |f_i^\ell(\tau_\ell)| \\
(8.8) \qquad &\leq \sum_j \liminf_i \lambda_j (\mathrm{Lip}(f_i^j))^k|\sigma_j| + \sum_\ell \liminf_i \mu_\ell (\mathrm{Lip}(f_i^\ell))^{k+1}|\tau_\ell| \\
(8.9) \qquad &\leq c(k)\max\{(\mathrm{Lip}(f))^k,(\mathrm{Lip}(f))^{k+1}\}(\sum_j \lambda_j|\sigma_j|+\sum_\ell \mu_\ell|\tau_\ell|) \\
&\leq c(k)\max\{(\mathrm{Lip}(f))^k,(\mathrm{Lip}(f))^{k+1}\}(|A-\partial D|+|D|) \\
&\leq c(k)\max\{(\mathrm{Lip}(f))^k,(\mathrm{Lip}(f))^{k+1}\}(|A|_\flat+\varepsilon).
\end{aligned}
$$

Inequality (8.7) is a result of the lower-semicontinuity of the mass norm on polyhedral chains proven in [Ada08]. Inequality (8.8) is a consequence of Lemma III.13, and inequality (8.9) follows from the fact that the Lipschitz constants of the maps f_i^j and f_i^ℓ can be uniformly bounded by the Lipschitz constant of f (Lemma VIII.4) up to a factor depending only on k. □

Remark VIII.16. The pushforward operation is a functor from the category of Banach spaces to the category of polyhedral/flat chains in Banach spaces. Specifically, given a Lipschitz map $f\colon W \to V$ between Banach spaces:

$$f_*\colon \mathcal{P}_k(W) \to \mathcal{F}_k(V),$$

and if $g\colon V \to X$ is another Lipschitz map and $\tau \in \mathcal{P}_k(V)$,

$$(g\circ f)_*(\tau) = g_*(f_*(\tau)).$$

8.2 Pulling flat partial forms back under Lipschitz maps

We have shown that polyhedral chains in a Banach space W push forward under Lipschitz maps $f\colon W \to V$.

For such a map f, given a flat partial k-form F on the space V, we define the pullback f^*F as follows. Consider the cochain $X_F = \Psi(F)$ associated to F as in Section V. This cochain pulls back to a flat cochain f^*X_F whose action on a polyhedral chain P is given by

$$\langle f^*X_F, P\rangle := \langle X_F, f_*(P)\rangle.$$

By Lemma VIII.15, $|f_*(P)|_\flat \leq C|P|_\flat$, so f^*X_F is a bounded operator on polyhedral chains. Hence f^*X_F has a unique extension to the completion $\mathcal{F}_k(V)$, so f^*X_F is a flat cochain. By Theorem I.1, f^*X_F corresponds to a flat partial form, which we define to be the pullback of F under f and denote by f^*F.

Let $f\colon W \to V$ and $g\colon V \to X$ be Lipschitz maps and let $F \in \mathbb{F}^k(X)$ be a flat partial k-form. Then by the functoriality of pushforwards (Remark VIII.16),

$$(g \circ f)^*F = f^*(g^*F)$$

and the pullback operation defines a contravariant functor from the category of Banach spaces to the category of flat forms on Banach spaces.

Finally, we note that the pullback operation commutes with exterior differentiation.

Lemma VIII.17. *Let $f : W \to V$ be a Lipschitz map and X a flat k-cochain in $\mathcal{F}^kV$. Then*

$$f^*(dX) = d(f^*(X)).$$

Proof: Let τ be a polyhedral $(k+1)$-chain in $\mathcal{P}_k W$. Then

$$
\begin{aligned}
\langle df^*X, \tau\rangle &= \langle f^*X, \partial\tau\rangle \\
&= \langle X, f_*(\partial\tau)\rangle \\
&= \langle X, \partial(f_*(\tau))\rangle \qquad (8.10) \\
&= \langle dX, f_*(\tau)\rangle \\
&= \langle f^*(dX), \tau\rangle,
\end{aligned}
$$

where equation (8.10) follows from Lemma VIII.14. Since this holds for all τ, the cochains $f^*(dX)$ and $d(f^*(X))$ are equal. □

Corollary VIII.18. *Let $f : W \to V$ be a Lipschitz map and ω a flat partial k-form in $\mathbb{F}^k V$. Then*

$$f^*(dF) = d(f^*(F)).$$

APPENDICES

APPENDIX A

Norms on k-vectors and k-covectors in Euclidean Space

A.1 Mass and Comass in $\Lambda_k\mathbb{R}^n$ and $\Lambda^k\mathbb{R}^n$

This section contains the Euclidean definitions of mass and comass on the spaces of k-vectors and k-covectors, respectively.

The space $\Lambda_k\mathbb{R}^n$ naturally inherits an inner product from the inner product on $\mathbb{R}^n$ by

$$\langle a_1 \wedge \cdots \wedge a_k, b_1 \wedge \cdots \wedge b_k\rangle = \det\left(\langle a_i, b_j\rangle\right).$$

Specifically, if $\{e_1, \ldots, e_k\}$ is an orthonormal basis of $\mathbb{R}^n$, then the $\binom{n}{k}$ elements $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ form an orthonormal basis of $\Lambda_k\mathbb{R}^n$. Denote the norm induced by this inner product on $\Lambda_k\mathbb{R}^n$ by $|\cdot|_2$.

In this setting, the comass of a k-covector $\omega : \Lambda^k\mathbb{R}^n$ is defined [Fed69, p. 38–39] to be

$$\|\omega\|_{\text{comass}} := \sup\{\langle \omega, \varphi\rangle : \varphi \in \Lambda_k\mathbb{R}^n \text{ simple}, |\varphi|_2 \le 1\}.$$

The *mass* of an arbitrary k-vector ν is then defined to be the dual norm to the comass norm

$$\|\nu\|_{\text{mass}} := \sup\{\langle \omega, \nu\rangle : \omega \in \Lambda^k\mathbb{R}^n, |\omega|_{\text{comass}} \le 1\}.$$

A.2 Whitney's mass and Gromov's mass* on $\Lambda_k\mathbb{R}^n$

By Wenger's Lemma, in Euclidean space, the mass* of a simple k-vector (as defined in 2.2) is equal to the mass of a k-vector as defined by Whitney [Whi57, p. 51]. Whitney, however, extends mass (denoted $|\cdot|_0$) to all k-vectors by the formula:

$$|\nu|_0 := \inf\{\sum_i \|\nu_i\|_{\mathrm{m}^*} : \nu = \sum \nu_i, \nu_i \text{ simple}\}.$$

One has $|\nu|_0 \leq \|\nu\|_{\mathrm{m}^*}$ for non-simple ν, but since the space of k-vectors in Euclidean space is finite dimensional, the norms $|\nu|_0$ and $\|\nu\|_{\mathrm{m}^*}$ are comparable up to a constant depending only on dimension.

BIBLIOGRAPHY

BIBLIOGRAPHY

[Ada08] Tarn Adams, *Flat chains in Banach spaces*, J. Geom. Anal. **18** (2008), no. 1, 1–28.

[AK00] Luigi Ambrosio and Bernd Kirchheim, *Currents in metric spaces*, Acta Math. **185** (2000), no. 1, 1–80.

[ÁPT04] J. C. Álvarez Paiva and A. C. Thompson, *Volumes on normed and Finsler spaces*, A sampler of Riemann-Finsler geometry, Math. Sci. Res. Inst. Publ., vol. 50, Cambridge Univ. Press, Cambridge, 2004, pp. 1–48.

[Bal97] Keith Ball, *An elementary introduction to modern convex geometry*, Flavors of geometry, Math. Sci. Res. Inst. Publ., vol. 31, Cambridge Univ. Press, Cambridge, 1997, pp. 1–58.

[BL00] Y. Benyamini and J. Lindenstrauss, *Geometric nonlinear functional analysis, volume i*, Colloquium Publications, vol. 48, Amer. Math. Soc., 2000.

[Car70] Henri Cartan, *Differential forms*, Houghton Mifflin Co., Boston, Mass, 1970.

[Car71] ______, *Differential calculus*, Hermann, Paris, 1971.

[DPH] Thierry De Pauw and Robert Hardt, *Rectifiable and flat G-chains in metric spaces*, Work in preparation.

[Fed69] H. Federer, *Geometric measure theory*, Die Grundlehren der mathematischen Wissenschaften, vol. 153, Springer-Verlag, New York, 1969.

[Gre78] Werner Greub, *Multilinear algebra*, second ed., Springer-Verlag, New York, 1978.

[Gro83] Mikhael Gromov, *Filling Riemannian manifolds*, J. Differential Geom. **18** (1983), no. 1, 1–147.

[Hei05] Juha Heinonen, *Lectures on Lipschitz analysis*, Report. University of Jyväskylä Department of Mathematics and Statistics, vol. 100, University of Jyväskylä, Jyväskylä, 2005.

[HK] Juha Heinonen and Stephen Keith, *Flat forms, bi-Lipschitz parametrizations and smoothability of manifolds*, Work in preparation.

[HPR] J. Heinonen, P. Pankka, and K. Rajala, *Quasiconformal frames*, Preprint 359, Department of Mathematics and Statistics, Jyväskylä, 2007.

[HS02] Juha Heinonen and Dennis Sullivan, *On the locally branched Euclidean metric gauge*, Duke Math. J. **114** (2002), no. 1, 15–41.

[Lan] Urs Lang, *Local currents in metric spaces*, Lecture notes from the 17th Jyväskylä Summer School.

[Nol86] Stephen V. Noltie, *Chains in Banach spaces*, Real and stochastic analysis, Wiley Ser. Probab. Math. Statist. Probab. Math. Statist., Wiley, New York, 1986, pp. 211–248.

[Roy88] H. L. Royden, *Real analysis*, third ed., Macmillan Publ. Co., 1988.

[Wen05] S. Wenger, *Isoperimetric inequalities of Euclidean type in metric spaces*, Geom. Funct. Anal. **15** (2005), no. 2, 534–554.

[Wen07] Stefan Wenger, *Flat convergence for integral currents in metric spaces*, Calc. Var. Partial Differential Equations **28** (2007), no. 2, 139–160.

[Whi57] Hassler Whitney, *Geometric integration theory*, Princeton University Press, Princeton, N. J., 1957.