The Geometry and Combinatorics of Ehrhart δ-Vectors

by

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TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER I

Introduction

It is well-known that there is an intimate connection between the geometry of toric varieties and the combinatorics of polytopes. More precisely, toric varieties are determined by collections of cones called *fans*, and given a d-dimensional rational polytope $P \subseteq \mathbb{R}^d$ containing the origin in its interior, one can associate two fans $\triangle$ and Σ, such that the corresponding toric varieties $X(\triangle)$ and $Y(\Sigma)$ are projective. Here $\triangle$ denotes the fan over the faces of P while Σ denotes the normal fan of P. Let us recall the following application of this correspondence.

Consider a d-dimensional polytope $P \subseteq \mathbb{R}^d$ and let f_i denote the number of i-dimensional faces of P, for $-1 \leq i \leq d-1$, where we set $f_{-1} = 1$. The vector $(f_{-1}, f_0, \ldots, f_{d-1})$ is called the *f-vector* of P. If we write $\sum_{i=-1}^{d-1} f_i t^{i+1}(1-t)^{d-1-i} = \sum_{i=0}^{d} h_i t^i$, then $(h_0, h_1, \ldots, h_d)$ is called the *h-vector* of P, and one easily verifies that the f-vector and h-vector record equivalent information. It is a long-standing unsolved problem to characterize all possible f-vectors (or, equivalently, h-vectors) of polytopes. In the special case when P is *simple*, i.e. when every vertex lies in exactly d codimension 1 faces, the following beautiful answer was conjectured by McMullen [46] and proved by Stanley [57] and Billera and Lee [11]: any vector $(h_0, \ldots, h_d)$ of non-negative integers with $h_0 = 1$, $h_i = h_{d-i}$, for $1 \leq i \leq d$, and satisfying certain inequalities (the so-called *g-inequalities*) is the h-vector of a simple polytope and, conversely, the h-vector of a simple polytope satisfies these conditions.

We will briefly recall Stanley's proof that the h-vector of a simple polytope satisfies these conditions. Firstly, one can translate P so that it contains the origin in its interior and then deform P to a polytope with rational vertices without changing its f-vector. If $\triangle$ denotes the fan over the faces of P, then the corresponding projective, complex toric variety $X = X(\triangle)$ has no odd cohomology and satisfies $h_i = \dim H^{2i}(X;\mathbb{Q})$, for $1 \leq i \leq d$ (see, for example, [25]). Poincaré duality implies $h_i = h_{d-i}$, for $1 \leq i \leq d$, and the Hard Lefschetz Theorem implies that the h-vector of P satisfies the g-inequalities.

We remark the same relations hold for a general polytope P if one considers the *generalized h-vector* of Stanley [54]. In the case when P has rational vertices, this is proved in a similar way, by interpreting the coefficients of the generalized h-vector as dimensions of intersection cohomology groups of a projective toric variety and applying Poincaré duality and the Hard Lefschetz Theorem [54], while the general case is due to Karu [38].

In this thesis, we will consider the more subtle but analogous problem of counting the number of lattice points in all dilations of a lattice polytope. More specifically, fix a d-dimensional polytope P with vertices in $\mathbb{Z}^d$, and, for each positive integer m, let $f_P(m) := \#\left(mP \cap \mathbb{Z}^d\right)$ denote the number of lattice points in the m^{th} dilate of P. A famous theorem of Ehrhart [19] asserts that $f_P(m)$ is a polynomial in m of degree d, called the *Ehrhart polynomial* of P, and that $(-1)^d f_P(-m)$ equals the number of interior lattice points in mP, for every positive integer m, a result known as *Ehrhart Reciprocity*. In fact, if Σ denotes the normal fan to P, then $f_P(m)$ can be interpreted as the Hilbert polynomial of an ample line bundle on the toric variety

$Y(\Sigma)$ and it can be shown that Ehrhart reciprocity is a consequence of Serre duality [25]. We remark that the Hilbert polynomial of any ample line bundle on a projective toric variety can be interpreted as the Ehrhart polynomial of an associated lattice polytope.

The generating series of $f_P(m)$ can be written in the form

$$\frac{\delta_P(t)}{(1-t)^{d+1}} = \sum_{m \geq 0} f_P(m)\, t^m \,,$$

where $\delta_P(t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d$ is a polynomial of degree at most d with integer coefficients, called the *δ-polynomial* of P. We call $(\delta_0, \delta_1, \ldots, \delta_d)$ the *(Ehrhart) δ-vector* of P; alternative names in the literature include *Ehrhart h-vector* and *h^*-vector* of P. Stanley proved that the coefficients δ_i are non-negative by interpreting them as the dimensions of the graded pieces of the quotient of a graded Cohen-Macauley ring by a regular sequence [56]. We remark that δ-vectors naturally occur in many different situations. For example, Danilov and Khovanskii computed the dimensions of the pieces of the mixed Hodge structure of a hypersurface in a torus, that is non-degenerate with respect to P, in terms of the δ-vector of P and the poset of faces of P [16].

Viewing the coefficients of the polynomial $f_P(m)$ as an analogue of the f-vector of P and the δ-vector as an analogue of the h-vector of P, we have the following longstanding unsolved problem:

Question I.1. *Can one characterize those vectors which occur as the δ-vector of a lattice polytope (equivalently, those polynomials which occur as the Ehrhart polynomial of a lattice polytope)?*

Although we do not characterize δ-vectors, the goal of this thesis is to use techniques of toric geometry and combinatorics to describe new properties of the δ-vector of a lattice polytope.

Let us return to the analogy between h-vectors and δ-vectors. Firstly, unlike in the case of h-vectors, the δ-vector of a lattice polytope is not symmetric in general. In fact, Hibi proved that the δ-vector of P is symmetric if and only if P is isomorphic to one of a small class of polytopes known as *reflexive polytopes* [33]. In the case of reflexive polytopes, though, results of Batyrev, Dais [7] and Yasuda [67] imply that our analogy works quite well: one can associate to P a simplicial, complex toric variety $X = X(\triangle)$ as before, and then the *orbifold cohomology groups* $H^j_{\mathrm{orb}}(X;\mathbb{Q})$ of X [14] vanish in odd degree and satisfy $\delta_i = \dim H^{2i}_{\mathrm{orb}}(X;\mathbb{Q})$, for $1 \leq i \leq d$. Moreover, Poincaré duality for orbifold cohomology implies $\delta_i = \delta_{d-i}$, for $1 \leq i \leq d$. On the other hand, there is no general version of the Hard Lefschetz Theorem for orbifold cohomology [24], and in fact, the coefficients of the δ-vector of a reflexive polytope are not necessarily *unimodal* i.e. $\delta_0 \leq \delta_1 \leq \cdots \leq \delta_{\lceil \frac{d}{2} \rceil}$ [49].

In the remainder of the introduction, we will outline the results of this thesis. Firstly, in order to extend the analogy between h-vectors and δ-vectors further, we need to introduce some sort of 'generalized' δ-vector. We will assume that P contains the origin in its interior, and refer the reader to Chapter II for the general case. We assign a *weight* $w(v) \in (-1, 0]$ to each lattice point $v \in \mathbb{Z}^d$ as follows: if m is the smallest rational number such that $v \in mP$, then $w(v) = m - \lceil m \rceil$, where $\lceil m \rceil$ is the round-up of m. For each $k \in (-1, 0]$, we define the *weighted Ehrhart polynomial* to be the function $f_k(m) := \#\{v \in mP \cap \mathbb{Z}^d \mid w(v) = k\}$, for $m \in \mathbb{Z}_{>0}$, so that, by definition, $\sum_k f_k(m) = f_P(m)$. One can prove that $f_k(m)$ is either identically zero or a polynomial of degree d (Theorem II.20), and hence its generating series has the form $\sum_{m \geq 0} f_k(m)\, t^m = \frac{\delta_k(t)}{(1-t)^{d+1}}$, where $\delta_k(t)$ is a polynomial of degree at most d with integer coefficients. The *weighted δ-polynomial* of P is defined by $\delta^0(t) := \sum_k \delta_k(t)t^k$, and one verifies that $\delta^0(t)$ is a well-defined element of $\mathbb{Z}[t^{1/N}]$, for some positive integer N (see Section 2.1). It follows from the definitions that δ_i equals the sum of the coefficients of t^j in $\delta^0(t)$, for $i-1 < j \leq i$, i.e. we recover $\delta_P(t)$ from $\delta^0(t)$ by rounding up the exponents. In Chapter II, we show that $\delta^0(t) = t^d\delta^0(t^{-1})$ has non-negative, symmetric coefficients (Corollary II.12, Theorem II.31). Hibi's characterization of polytopes with symmetric δ-vectors is a simple consequence (Corollary II.24), as well as some other famous results from Ehrhart theory. More specifically, the fact that the Ehrhart function of a rational polytope is a quasi-polynomial as well as Ehrhart Reciprocity for rational polytopes follow easily (Corollary II.21, Remark II.23).

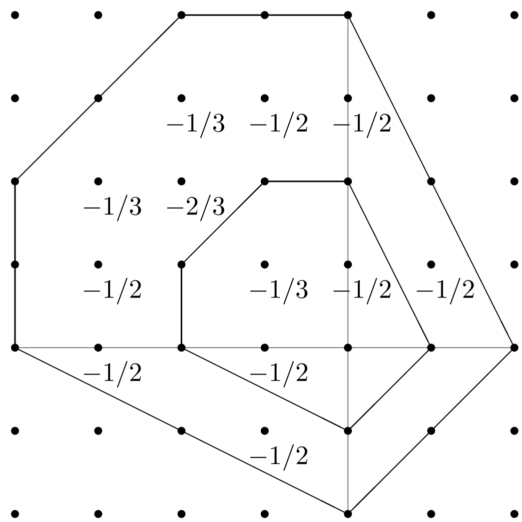
Example I.2. If $P \subseteq \mathbb{Z}^2$ is the lattice polytope with vertices $(1,0)$, $(0,2)$, $(-1,2)$, $(-2,1)$, $(-2,0)$ and $(0,-1)$, then we calculate:

$$\delta^0(t) = t^2 + 2t^{3/2} + t^{4/3} + 4t + t^{2/3} + 2t^{1/2} + 1$$

$$\delta_P(t) = 4t^2 + 7t + 1.$$

In Figure 1.1, we indicate the non-zero weights of lattice points in $2P$.

We now consider the geometric side of the story. Choose a simplicial fan $\triangle$ refining the fan over the faces of P and with rays through the vertices $\{b_i\}$ of P. The data of the fan $\triangle$ as well as a distinguished lattice point b_i on each ray is called a *stacky fan*, and one can associate to it a corresponding *toric stack* $\mathcal{X} = \mathcal{X}(\triangle, \{b_i\})$, with coarse moduli space $X = X(\triangle)$ [12]. The theory of orbifold cohomology, developed by Chen and Ruan [13, 14], associates to $\mathcal{X}$ a finite-dimensional $\mathbb{Q}$-algebra $H^*_{\text{orb}}(\mathcal{X}, \mathbb{Q})$, graded by $\mathbb{Q}$. An explicit presentation of the orbifold cohomology ring of a toric stack was given by Borisov, Chen and Smith [12]. We have the following combinatorial interpretation of the dimensions of the graded pieces (Theorem II.31):



$f_0(m) = 3m^2 + 3m + 1$

$f_{-1/2}(m) = 2m^2$

$f_{-1/3}(m) = m(m+1)/2$

$f_{-2/3}(m) = m(m-1)/2$

$f_P(m) = 6m^2 + 3m + 1$

Figure 1.1: Weighted Ehrhart theory for polytope P

Theorem I.3. *If P is a lattice polytope containing the origin in its interior, then*

$$\delta^0(t) = \sum_j \dim_{\mathbb{Q}} H^{2j}_{\mathrm{orb}}(\mathcal{X}, \mathbb{Q}) t^j.$$

In particular, the coefficient δ_i of the δ-vector of P is a sum of dimensions of orbifold cohomology groups,

$$\delta_i = \sum_{i-1<j\leq i} \dim_{\mathbb{Q}} H^{2j}_{\mathrm{orb}}(\mathcal{X}, \mathbb{Q}).$$

In particular, the above theorem and Poincaré duality for orbifold cohomology implies the symmetry of the coefficients of $\delta^0(t)$. We conclude that the coefficients of the weighted δ-polynomial are a natural analogue of the generalized h-vector of a polytope.

The above theorem suggests a geometric proof of a theorem of Stanley which states that if $Q \subseteq P$ is an inclusion of lattice polytopes, then the coefficient $\delta_{i,Q}$ of t^i in $\delta_Q(t)$ is at most the coefficient $\delta_{i,P}$ of t^i in $\delta_P(t)$ [59]. More specifically, Abramovich, Graber and Vistoli introduced the orbifold Chow ring $A^*(\mathcal{Y}, \mathbb{Q})$ of a Deligne-Mumford stack $\mathcal{Y}$ in [1] as an algebraic analogue of the orbifold cohomology ring. In Section 2.4, we consider a toric stack Y and a closed substack Y', such that $\delta_{i,Q} = \dim_{\mathbb{Q}} A^{2i}(Y', \mathbb{Q})$ and $\delta_{i,P} = \dim_{\mathbb{Q}} A^{2i}(Y, \mathbb{Q})$. Stanley's result then follows from the observation that the inclusion $Y' \hookrightarrow Y$ induces a surjective, graded ring homomorphism $A^*(Y, \mathbb{Q}) \to A^*(Y', \mathbb{Q})$.

If P is reflexive, then $\delta^0(t) = \delta_P(t)$ and we recover the results mentioned at the end of page 3. Let us outline the original proof that $\delta_i = \dim_{\mathbb{Q}} H^{2i}_{\text{orb}}(X(\triangle), \mathbb{Q})$ in this case. To any complex variety Y, one can associate its corresponding arc space $J_\infty(Y)$, which parameterizes all morphisms $\operatorname{Spec} \mathbb{C}[[t]] \to Y$. The geometry of $J_\infty(Y)$ encodes a lot of information about the birational geometry of Y and has been the subject of much attention in recent times (see, for example, [22, 23, 47]). Kontsevich introduced the theory of *motivic integration* in [42], which assigns a measure to subsets of $J_\infty(Y)$ and allows one to compare invariants on birationally equivalent varieties (see, for example, [5, 17, 18]). To any pair (Y, D) consisting of a normal variety Y and a $\mathbb{Q}$-divisor D such that (Y, D) has relatively mild (more specifically, *Kawamata log-terminal*) singularities, Batyrev attached via motivic integration an invariant $E_{\text{st}}(Y, D; u, v)$, called the *stringy E-function* of (Y, D), which is a rational function in $u^{1/N}$ and $v^{1/N}$, for some positive integer N. If P is reflexive, then the corresponding pair $(X(\triangle), 0)$ is Kawamata log terminal and Batyrev and Dais proved that $E_{\text{st}}(X(\triangle), 0; u, v) = \delta_P(uv)$ [7].

Motivated by Abramovich and Vistoli's theory of *twisted curves* [2], Yasuda extended the theory of arc spaces and motivic integration to Deligne-Mumford stacks in [66] and [67]. More specifically, to any smooth Deligne-Mumford stack $\mathcal{Y}$ he associated the space $|\mathcal{J}_\infty \mathcal{Y}|$ of *twisted arcs* of $\mathcal{Y}$, which parametrizes all representable morphisms $[\operatorname{Spec} \mathbb{C}[[t]]/\mu_l] \to \mathcal{Y}$, where μ_l is the group of l^{th} roots of unity acting on $\operatorname{Spec} \mathbb{C}[[t]]$ by scaling t, and $[\operatorname{Spec} \mathbb{C}[[t]]/\mu_l]$ is the corresponding quotient stack. Analogous to Batyrev's stringy functions, given a divisor $\mathcal{D}$ on

$\mathcal{Y}$ such that $(\mathcal{Y}, \mathcal{D})$ is *Kawamata log terminal*, Yasuda defined a motivic integral $\Gamma(\mathcal{Y}, \mathcal{D})(u, v)$. For example, the pair $(\mathcal{Y}, 0)$ is always Kawamata log terminal and $\Gamma(\mathcal{Y}, 0)(u, v) = \sum_j \dim_{\mathbb{Q}} H^{2j}_{\mathrm{orb}}(\mathcal{Y}, \mathbb{Q})(uv)^j$. In the case when P is reflexive, $X = X(\triangle)$ has a canonical stack structure and Yasuda's results imply that $E_{\mathrm{st}}(X(\triangle), D; u, v) = \Gamma(X(\triangle), D)(u, v)$, for every Kawamata log terminal pair (X, D). We conclude that $\delta_P(uv) = E_{\mathrm{st}}(X(\triangle), 0; u, v) = \Gamma(X(\triangle), 0)(u, v) = \sum_j \dim_{\mathbb{Q}} H^{2j}_{\mathrm{orb}}(X(\triangle), \mathbb{Q})(uv)^j$, as desired.

This suggests a natural generalization of our theory. Recall that to a lattice polytope P containing the origin in its interior, we associate a stacky fan $(\triangle, \{b_i\})$ and a toric stack $\mathcal{X}$. In Chapter III, we give an explicit presentation of Yasuda's theory of twisted arcs in the case of toric stacks and present a decomposition of the space of twisted arcs of $\mathcal{X}$ into locally closed subsets indexed by the lattice points in $\triangle$ (Theorem III.5). This decomposition is analogous to Ishii's decomposition of the arc space of a toric variety [36]. It allows one to compute the motivic integrals $\Gamma(\mathcal{X}, \mathcal{E})(u, v)$ of all Kawamata log terminal pairs $(\mathcal{X}, \mathcal{E})$, where $\mathcal{E}$ is a torus-invariant $\mathbb{Q}$-divisor on $\mathcal{X}$.

Let us explain the relation with lattice point enumeration. There is a bijective correspondence between piecewise $\mathbb{Q}$-linear functions $\lambda : \mathbb{R}^n \to \mathbb{R}$ with respect to $\triangle$ satisfying $\lambda(b_i) > -1$ for every vertex b_i of P, and torus-invariant $\mathbb{Q}$-divisors $\mathcal{E}$ on $\mathcal{X}$ such that the pair $(\mathcal{X}, \mathcal{E})$ has Kawamata log terminal singularities. Fixing λ as above, we associate a *weight* $w_\lambda(v) := w(v) + \lambda(v)$ to each lattice point $v \in \mathbb{Z}^d$. Recall that if m is the smallest rational number such that $v \in mP$, then $w(v) = m - \lceil m \rceil$. In Chapter II, we associate functions $f_k^\lambda(m)$ recording the number of lattice points of λ-weight k in mP, and define a rational function $\delta^\lambda(t) \in \mathbb{Q}((t^{1/N}))$, for some positive integer N, that specializes to $\delta^0(t)$ when $\lambda \equiv 0$. Generalizing our previous results we have the following theorem (Theorem II.12).

Theorem I.4. *With the notation above,* $\delta^\lambda(t) = t^d \delta^\lambda(t^{-1})$.

In conclusion, the following theorem shows the relationship between lattice point counting on polytopes and motivic integrals on toric stacks (Theorem III.13).

Theorem I.5. *If* $(\mathcal{X}, \mathcal{E})$ *is the Kawamata log terminal pair corresponding to the piecewise* $\mathbb{Q}$*-linear function* λ *on* $\triangle$*, then* $\delta^\lambda(uv) = \Gamma(\mathcal{X}, \mathcal{E})(u, v)$.

One of the main tools in motivic integration is a change of variables formula which allows one to compare motivic integrals on birational varieties. Using Yasuda's generalization to Deligne-Mumford stacks [67] and the above theorem, in Section 3.6 we deduce a change of variables formula (Proposition II.13) comparing the invariants $\delta^\lambda(t)$ on different polytopes. We also provide a combinatorial proof in Section 2.1.

In Chapter IV, we consider a purely combinatorial approach to Question I.1. More specifically, we address the question of which linear inequalities hold between the coefficients of the Ehrhart δ-vector of a lattice polytope P. Let us first summarize the previous state of knowledge.

The degree s of $\delta_P(t)$ is called the *degree* of P and $l = d + 1 - s$ is the *codegree* of P. It is a consequence of Ehrhart Reciprocity (Corollary II.21) that l is the smallest positive integer such that lP contains a lattice point in its relative interior (see, for example, [32]). It follows from the definition that $\delta_0 = 1$ and $\delta_1 = f_P(1) - (d+1) = |P \cap N| - (d+1)$, and it can be deduced from Ehrhart Reciprocity that δ_d is the number of lattice points in the relative interior of P (see, for example, [32]). Since P has at least $d+1$ vertices, we have the inequality $\delta_1 \geq \delta_d$. We list the previously known inequalities satisfied by the Ehrhart δ-vector (cf. [8]).

$$\delta_1 \geq \delta_d \tag{1.1}$$

$$\delta_0 + \delta_1 + \cdots + \delta_{i+1} \geq \delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \text{ for } i = 0, \ldots, \lfloor d/2 \rfloor - 1, \tag{1.2}$$

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i} \text{ for } i = 0, \ldots, \lfloor (s-1)/2 \rfloor, \tag{1.3}$$

$$\text{if } \delta_d \neq 0 \text{ then } 1 \leq \delta_1 \leq \delta_i \text{ for } i = 2, \ldots, d-1. \tag{1.4}$$

Inequalities (1.2) and (1.3) were proved by Hibi in [30] and Stanley in [58] respectively. Both proofs are based on commutative algebra. Inequality (1.4) was proved by Hibi in [34] using combinatorial methods. Recently, Henk and Tagami [29] produced examples showing that the analogue of (1.4) when $\delta_d = 0$ is false. That is, it is not true that $\delta_1 \leq \delta_i$ for $i = 2, \ldots, s-1$. An explicit example is provided in Example IV.4 below.

We improve upon these inequalities by proving the following result, and remark that the proof is purely combinatorial. We set $\delta_i = 0$ for $i < 0$ and $i > d$.

Theorem I.6. *If P is a d-dimensional lattice polytope of degree s and codegree l, then its Ehrhart δ-vector $(\delta_0, \ldots, \delta_d)$ satisfies the following inequalities:*

$$\delta_1 \geq \delta_d, \tag{1.5}$$

$$\delta_2 + \cdots + \delta_{i+1} \geq \delta_{d-1} + \cdots + \delta_{d-i} \textit{ for } i = 0, \ldots, \lfloor d/2 \rfloor - 1, \tag{1.6}$$

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i} \textit{ for } i = 0, \ldots, \lfloor (s-1)/2 \rfloor, \tag{1.7}$$

$$\delta_{2-l} + \cdots + \delta_0 + \delta_1 \leq \delta_i + \delta_{i-1} + \cdots + \delta_{i-l+1} \textit{ for } i = 2, \ldots, d-1. \tag{1.8}$$

Remark I.7. Equality can be achieved in all the inequalities in the above theorem. For example, let N be a lattice with basis $e_1, \ldots, e_d$ and let P be the regular simplex with vertices $0, e_1, \ldots, e_d$. In this case, $\delta_P(t) = 1$ and each inequality above is an equality.

Remark I.8. Observe that (1.5) and (1.6) imply that

$$\delta_1 + \cdots + \delta_i + \delta_{i+1} \geq \delta_d + \delta_{d-1} + \cdots + \delta_{d-i}, \tag{1.9}$$

for $i = 0, \ldots, \lfloor d/2 \rfloor - 1$. Since $\delta_0 = 1$, we conclude that (1.2) is always a strict inequality. We note that inequality (1.6) was suggested, without proof, by Hibi in [34].

Remark I.9. We can view the above result as providing, in particular, a combinatorial proof of Stanley's inequality (1.7).

Remark I.10. We claim that inequality (1.8) provides the correct generalisation of Hibi's inequality (1.4). Our contribution is to prove the case when $l > 1$ and we refer the reader to [34] for a proof of (1.4). In fact, in the proof of the above theorem we show that (1.8) can be deduced from (1.4), (1.6) and (1.7).

In order to prove this result, we consider the polynomial

$$\bar{\delta}_P(t) = (1 + t + \cdots + t^{l-1})\delta_P(t)$$

and use a result of Payne (Theorem 2 in [51]) to establish the following decomposition theorem (Theorem IV.14). In the case when $l = 1$, this decomposition is originally due to Betke and McMullen (Theorem 5 in [10]), while it also follows immediately from Theorem I.3 and Poincaré duality for orbifold cohomology (Remark II.18).

Theorem I.11. *The polynomial $\bar{\delta}_P(t)$ has a unique decomposition*

$$\bar{\delta}_P(t) = a(t) + t^l b(t),$$

where $a(t)$ and $b(t)$ are polynomials with integer coefficients satisfying $a(t) = t^d a(t^{-1})$ and $b(t) = t^{d-l} b(t^{-1})$. Moreover, the coefficients of $b(t)$ are non-negative and, if a_i denotes the coefficient of t^i in $a(t)$, then

$$1 = a_0 \leq a_1 \leq a_i,$$

for $i = 2, \ldots, d-1$.

Using some elementary arguments we show that our desired inequalities are equivalent to certain conditions on the coefficients of $\bar{\delta}_P(t)$, $a(t)$ and $b(t)$ (Lemma IV.5) and hence are a consequence of the above theorem.

We also consider the following result of Stanley (Theorem 4.4 [55]), which was proved using commutative algebra. In the case when $l = 1$, this result is Hibi's characterisation of symmetric δ-polynomials.

Theorem I.12. *If P is a lattice polytope of degree s and codegree l, then $\delta_P(t) = t^s\delta_P(t^{-1})$ if and only if lP is a translate of a reflexive polytope.*

We show that this result is a consequence of the above decomposition of $\bar{\delta}_P(t)$, thus providing a combinatorial proof of Stanley's theorem (Corollary IV.18).

Recent work of Athanasiadis [3, 4] relates the existence of certain triangulations of a lattice polytope P to inequalities satisfied by its δ-vector.

Theorem I.13 (Theorem 1.3 [3]). *Let P be a d-dimensional lattice polytope. If P admits a regular unimodular lattice triangulation, then*

$$\delta_{i+1} \geq \delta_{d-i} \text{ for } i = 0, \ldots, \lfloor d/2 \rfloor - 1, \tag{1.10}$$

$$\delta_{\lfloor (d+1)/2 \rfloor} \geq \cdots \geq \delta_{d-1} \geq \delta_d, \tag{1.11}$$

$$\delta_i \leq \binom{\delta_1 + i - 1}{i} \text{ for } i = 0, \ldots, d. \tag{1.12}$$

As a corollary of our decomposition of $\bar{\delta}_P(t)$, we deduce the following theorem (Theorem IV.20).

Theorem I.14. *Let P be a d-dimensional lattice polytope. If the boundary of P admits a regular unimodular lattice triangulation, then*

$$\delta_{i+1} \geq \delta_{d-i} \tag{1.13}$$

$$\delta_0 + \cdots + \delta_{i+1} \leq \delta_d + \cdots + \delta_{d-i} + \binom{\delta_1 - \delta_d + i + 1}{i+1}, \tag{1.14}$$

for $i = 0, \ldots, \lfloor d/2 \rfloor - 1$.

We note that (1.13) provides a generalisation of (1.10) and that (1.14) may be viewed as an analogue of (1.12). We remark that the method of proof is different to that of Athanasiadis.

The contents of Chapter II appear in [63], with the exception of Section 2.4. The contents of Chapter III appear in [62] and the contents of Chapter IV appear in [61].

CHAPTER II

Weighted Ehrhart Theory

2.1 Weighted Ehrhart theory

The goal of this section is to define the basic combinatorial objects we will use, and study some of their properties. We will fix the following notation throughout the thesis, and remark that our setup will be slightly more general than that in the introduction (cf. Remark II.3). We refer the reader to [25] for the relevant background on toric varieties. We will always work over $\mathbb{C}$ and will often identify schemes with their $\mathbb{C}$-valued points.

Let N be a lattice of rank d and set $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$. Let Σ be a simplicial, rational, d-dimensional fan in $N_{\mathbb{R}}$, with convex support $|\Sigma|$. Recall that Σ is *complete* if $|\Sigma| = N_{\mathbb{R}}$. Let $\rho_1, \ldots, \rho_r$ denote the rays of Σ, with primitive integer generators $v_1, \ldots, v_r$ in N. Fix elements $b_1, \ldots, b_r$ in N such that $b_i = a_i v_i$ for some positive integer a_i, for $i = 1, \ldots, r$. The data $\mathbf{\Sigma} = (N, \Sigma, \{b_i\})$ is called a *stacky fan* [12]. Let $\psi : |\Sigma| \to \mathbb{R}$ be the function that is $\mathbb{Q}$-linear on each cone of Σ and satisfies $\psi(b_i) = 1$, for $i = 1, \ldots, r$. We define

$$Q = Q_{\mathbf{\Sigma}} = \{v \in |\Sigma| \mid \psi(v) \leq 1\}.$$

Observe that Q need not be convex and that the union of the facets in the boundary ∂Q of Q not containing 0 is given by $\{v \in N_{\mathbb{R}} \mid \psi(v) = 1\}$. Hence ψ depends only on Q, not on the corresponding stacky fan.

Remark II.1. A *pure lattice complex* of dimension d is a simplicial complex such that all maximal faces are lattice polytopes of dimension d [10]. For any cone τ in Σ, $Q \cap \tau$ is a lattice simplex containing the origin as a vertex. It follows that Q has the structure of a pure lattice complex. Conversely, let K be a pure lattice complex of dimension d embedded in $N_{\mathbb{R}}$ such that the smallest cone containing K is convex. If every maximal simplex in K contains the origin as a vertex, then $K = Q_{\mathbf{\Sigma}}$ for some stacky fan $\mathbf{\Sigma}$ as above. More specifically, we take Σ the cone over the faces in K not containing the origin and make an appropriate choice of $\{b_i\}$.

For each positive integer m, let $f_Q(m)$ be the number of lattice points in mQ. Then $f_Q(m)$ is a polynomial in m of degree d, called the *Ehrhart polynomial* of Q [10]. We write

$$f_Q(m) = c_d m^d + c_{d-1} m^{d-1} + \cdots + c_0. \tag{2.1}$$

The generating series of the Ehrhart polynomial can be written in the form

$$\sum_{m \geq 0} f_Q(m) t^m = \delta_Q(t)/(1-t)^{d+1},$$

where $\delta_Q(t)$ is a polynomial of degree less than or equal to d with non-negative integer coefficients, called the *Ehrhart δ-polynomial* of Q [56]. If we write

$$\delta_Q(t) = \delta_0 + \delta_1 t + \cdots + \delta_d t^d, \tag{2.2}$$

then $(\delta_0, \delta_1, \ldots, \delta_d)$ is called the *Ehrhart δ-vector* of Q.

Remark II.2. Consider a triple $(N, \triangle, \{b_i\})$ as above but without the assumption that $\triangle$ is simplicial. Suppose there exists a piecewise $\mathbb{Q}$-linear function $\psi : |\triangle| \to \mathbb{R}$ satisfying $\psi(b_i) = 1$, and set $Q = \{v \in |\triangle| \mid \psi(v) \leq 1\}$. There exists a simplicial fan

Σ refining $\triangle$ and with the same rays as $\triangle$ (see, for example, [50]). If $\mathbf{\Sigma}$ denotes the stacky fan $(N, \Sigma, \{b_i\})$, then $Q = Q_{\mathbf{\Sigma}}$.

Remark II.3. We consider the following important example. Let P be a lattice polytope and let α be a lattice point in P. After translating, we may assume that α is the origin. Let $\triangle$ be the fan over the faces of P not containing the origin. As in Remark II.2, if Σ is a simplicial fan refining $\triangle$ and with the same rays as $\triangle$, then, with the appropriate choice of $\{b_i\}$, $P = Q_{\mathbf{\Sigma}}$. In the introduction, we stated the main results of the thesis in this context.

Fix a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \{b_i\})$, and let $\lambda : |\Sigma| \to \mathbb{R}$ be an arbitrary piecewise $\mathbb{Q}$-linear function with respect to Σ. We introduce the 'weight function' w_λ on $|\Sigma| \cap N$,

$$w_\lambda : |\Sigma| \cap N \to \mathbb{Q}$$

$$w_\lambda(v) = \psi(v) - \lceil \psi(v) \rceil + \lambda(v).$$

Note that when $\lambda \equiv 0$, the corresponding weight function w_0 is determined by Q. Think of w_λ as assigning a weight to every lattice point in $|\Sigma|$. For every rational number k and for every non-negative integer m, denote by $f_k^\lambda(m)$ the number of lattice points of weight k in mQ. Note that the Ehrhart polynomial of Q can be recovered as $f_Q(m) = \sum_{k \in \mathbb{Q}} f_k^\lambda(m)$. For every rational number k, consider the power series

$$\delta_k^\lambda(t) := (1-t)^{d+1} \sum_{m \geq 0} f_k^\lambda(m) t^m.$$

If $\lambda(v) \geq 0$ for all v in $|\Sigma|$, we will show that $\delta_k^\lambda(t)$ is a polynomial in t with integer coefficients (Corollary II.9). The Ehrhart δ-polynomial of Q decomposes as $\delta_Q(t) = \sum_{k \in \mathbb{Q}} \delta_k^\lambda(t)$. We define the *weighted δ-power series* of Q by

$$\delta^\lambda(s,t) := \sum_{k \in \mathbb{Q}} \delta_k^\lambda(t) s^k.$$

It follows from the definition that the weighted δ-power series can be written as

$$\delta^{\lambda}(s,t) = (1-t)^{d+1} \sum_{m \geq 0} (\sum_{v \in mQ \cap N} s^{w_{\lambda}(v)}) t^m,$$

and hence $\delta^{\lambda}(s,t)$ is a well-defined element of $\mathbb{Z}[s^q \mid q \in \mathbb{Q}]\,[[t]]$. Note that when $s = 1$, we recover the Ehrhart δ-polynomial $\delta^{\lambda}(1,t) = \delta_Q(t)$. Later, as in the introduction, we will consider the case when $\delta^{\lambda}(t,t)$ lies in $\mathbb{Z}\,[[t^{1/N}]]$, for some positive integer N. Our first aim is to express $\delta^{\lambda}(s,t)$ as a rational function in $\mathbb{Q}(s^{1/N}, t)$, for some positive integer N (Proposition II.6).

For each cone τ in Σ, let Σ_{τ} be the simplicial fan in $(N/N_{\tau})_{\mathbb{R}}$ with cones given by the projections of the cones in Σ containing τ. If τ is not contained in the boundary of $|\Sigma|$, then Σ_{τ} is complete. The h-vector of Σ_{τ} is given by

$$h_{\tau}(t) := \sum_{\tau \subseteq \sigma} t^{\dim \sigma - \dim \tau} (1-t)^{\operatorname{codim} \sigma}.$$

We will sometimes write $h_{\Sigma}(t)$ for $h_{\{0\}}(t)$. We will use the following standard lemma. For a combinatorial proof, we refer the reader to [60] and Lemma 1.3 [34]. We provide a geometric proof, deducing the result as a corollary of Lemma II.29, which is proved independently.

Lemma II.4. *For each cone τ in Σ, $h_{\tau}(t)$ is a polynomial of degree at most* $\operatorname{codim} \tau$ *with non-negative integer coefficients. If τ is not contained in the boundary of Σ, then $h_{\tau}(t) = t^{\operatorname{codim} \tau} h_{\tau}(t^{-1})$ and the coefficients of $h_{\tau}(t)$ are positive integers.*

Proof. It follows from the definition that $h_{\tau}(t)$ is a polynomial of degree at most $\operatorname{codim} \tau$. Consider the simplicial toric variety $X = X(\Sigma_{\tau})$. By Lemma II.29, we can interpret the coefficient of t^i in $h_{\tau}(t)$ as the dimension of the $2i^{\text{th}}$ cohomology group of X. In particular, each coefficient is non-negative. If τ is not contained in the boundary of Σ, then X is complete. In this case, $h_{\tau}(t) = t^{\operatorname{codim} \tau} h_{\tau}(t^{-1})$ follows from Poincaré duality on X and the coefficients of $h_{\tau}(t)$ are positive by Lemma II.29. $\square$

We define the *weighted h-vector* $h^{\lambda}_{\tau}(s,t)$ of Σ_{τ},

$$(2.3)\quad h^{\lambda}_{\tau}(s,t) := \sum_{\tau\subseteq\sigma} s^{\sum_{\rho_i\subseteq\sigma\smallsetminus\tau}\lambda(b_i)} t^{\dim\sigma-\dim\tau}(1-t)^{\operatorname{codim}\sigma}\prod_{\rho_i\subseteq\sigma\smallsetminus\tau}(1-t)/(1-s^{\lambda(b_i)}t).$$

The weighted h-vector is a rational function in $\mathbb{Q}(s^{1/N'},t)$, for some positive integer N'. Note that if $\lambda\equiv 0$ or if we set $s=1$, then $h^{\lambda}_{\tau}(s,t)$ is equal to the usual h-vector $h_{\tau}(t)$ of Σ_{τ}. By expanding and collecting terms, we have the following equality,

$$(2.4)\qquad t^{\operatorname{codim}\tau}h^{\lambda}_{\tau}(s^{-1},t^{-1}) = \sum_{\tau\subseteq\sigma}(t-1)^{\operatorname{codim}\sigma}\prod_{\rho_i\subseteq\sigma\smallsetminus\tau}(t-1)/(s^{\lambda(b_i)}t-1).$$

Lemma II.5. *Suppose that τ is a cone not contained in the boundary of Σ. Then*

$$h^{\lambda}_{\tau}(s,t) = t^{\operatorname{codim}\tau}h^{\lambda}_{\tau}(s^{-1},t^{-1}).$$

Proof. This is an application of Möbius inversion (see, for example, [60]). Let $\mathcal{P}$ be the poset consisting of the cones in Σ_{τ} and a maximal element $\{\Sigma_{\tau}\}$. By Lemma II.4, $h_{\tau}(t) = t^{\operatorname{codim}\tau}h_{\tau}(t^{-1})$. Substituting $t=0$ gives $\sum_{\tau\subseteq\sigma}(-1)^{\operatorname{codim}\sigma}=1$. It follows that we can compute the Möbius function of $\mathcal{P}$ inductively. Möbius inversion says that if $f:\mathcal{P}\to A$ is a function, for some abelian group A, then

$$(2.5)\qquad f(\{\Sigma_{\tau}\}) = g(\{\Sigma_{\tau}\}) + \sum_{\tau\subseteq\sigma}(-1)^{\operatorname{codim}\sigma+1}g(\sigma),$$

where, for each p in $\mathcal{P}$, $g(p)=\sum_{q\leq p}f(q)$. If we set $f(\{\Sigma_{\tau}\})=0$ and $f(\sigma) = (t-1)^{\operatorname{codim}\sigma}\prod_{\rho_i\subseteq\sigma\smallsetminus\tau}(t-1)/(s^{\lambda(b_i)}t-1)$, then $g(\{\Sigma_{\tau}\}) = t^{\operatorname{codim}\tau}h^{\lambda}_{\tau}(s^{-1},t^{-1})$ by (2.4) and we calculate

$$\begin{aligned}
g(\sigma) &= \sum_{\tau\subseteq\sigma'\subseteq\sigma} f(\sigma')\\
&= (t-1)^{\operatorname{codim}\tau}\sum_{\tau\subseteq\sigma'\subseteq\sigma}\ \prod_{\rho_i\subseteq\sigma'\smallsetminus\tau}1/(s^{\lambda(b_i)}t-1)\\
&= (t-1)^{\operatorname{codim}\tau}\prod_{\rho_i\subseteq\sigma\smallsetminus\tau}(1+1/(s^{\lambda(b_i)}t-1))\\
&= s^{\sum_{\rho_i\in\sigma\smallsetminus\tau}\lambda(b_i)}t^{\dim\sigma-\dim\tau}f(\sigma).
\end{aligned}$$

By (2.3) and (2.5),

$$h_\tau^\lambda(s,t) = \sum_{\tau\subseteq\sigma}(-1)^{\operatorname{codim}\sigma} g(\sigma) = g(\{\Sigma_\tau\}) = t^{\operatorname{codim}\tau} h_\tau^\lambda(s^{-1}, t^{-1}).$$

□

As in [12], for each non-zero cone τ of Σ, set

$$\text{Box}(\boldsymbol{\tau}) = \{v \in N \mid v = \sum_{\rho_i \subseteq \tau} q_i b_i \text{ for some } 0 < q_i < 1\}. \tag{2.6}$$

We set $\text{Box}(\mathbf{\{0\}}) = \{0\}$ and $\text{Box}(\boldsymbol{\Sigma}) = \cup_{\tau\in\Sigma}\text{Box}(\boldsymbol{\tau})$. Given any v in $|\Sigma| \cap N$, let $\sigma(v)$ be the cone in Σ containing v in its relative interior. As in [49, p.7], v has a unique decomposition

$$v = \{v\} + v' + \sum_{\rho_i \subseteq \sigma(v) \smallsetminus \tau} b_i, \tag{2.7}$$

where $\{v\}$ lies in $\text{Box}(\boldsymbol{\tau})$ for some $\tau \subseteq \sigma(v)$ and v' is a linear combination of the $\{b_i \,|\, \rho_i \subseteq \sigma(v)\}$ with non-negative integer coefficients. We think of $\{v\}$ as the 'fractional part' of v. Using this decomposition, we compute a local formula for the weighted δ-power series of Q. The method of proof is the same as that of Theorem 1.3 in [49] and Theorem 1.2 in [51]. In fact, Proposition II.6 can be deduced from Theorem 1.2 in [51].

Proposition II.6. *The weighted δ-power series $\delta^\lambda(s,t)$ is a rational function in $\mathbb{Q}(s^{1/N}, t)$, for some positive integer N, and has the form*

$$\delta^\lambda(s,t) = \sum_{\tau\in\Sigma} h_\tau^\lambda(s,t) \sum_{v\in\text{Box}(\boldsymbol{\tau})} s^{w_\lambda(v)} t^{\lceil\psi(v)\rceil} \prod_{\rho_i\subseteq\tau}(t-1)/(s^{\lambda(b_i)}t-1).$$

Proof.

$$\begin{aligned}\delta^\lambda(s,t) &= (1-t)^{d+1}\sum_{m\geq 0}\sum_{v\in mQ\cap N} s^{w_\lambda(v)}t^m \\ &= (1-t)^{d+1}\sum_{v\in|\Sigma|\cap N}\sum_{m\geq\psi(v)} s^{w_\lambda(v)}t^m \\ &= (1-t)^{d}\sum_{v\in|\Sigma|\cap N} s^{w_\lambda(v)}t^{\lceil\psi(v)\rceil}.\end{aligned}$$

Now consider the decomposition (2.7). We have

$$w_\lambda(v) = w_\lambda(\{v\}) + w_\lambda(v') + \sum_{\rho_i\subseteq\sigma(v)\smallsetminus\tau}\lambda(b_i).$$

Also

$$\lceil\psi(v)\rceil = \lceil\psi(\{v\})\rceil + \psi(v') + \dim\sigma(v) - \dim\tau.$$

We obtain the following expression for $\delta^\lambda(t)$

$$(1-t)^d\sum_{\substack{\tau\in\Sigma\\ v\in\mathrm{Box}(\boldsymbol{\tau})}} s^{w_\lambda(v)}t^{\lceil\psi(v)\rceil}\sum_{\tau\subseteq\sigma} s^{\sum_{\rho_i\subseteq\sigma\smallsetminus\tau}\lambda(b_i)}t^{\dim\sigma-\dim\tau}\prod_{\rho_i\subseteq\sigma}1/(1-s^{\lambda(b_i)}t).$$

Substituting in (2.3) and rearranging gives the result. □

Example II.7. If $\lambda\equiv 0$, then

$$\delta^0(s,t) = \sum_{\substack{\tau\in\Sigma\\ v\in\mathrm{Box}(\boldsymbol{\tau})}} s^{\psi(v)-\lceil\psi(v)\rceil}t^{\lceil\psi(v)\rceil}h_\tau(t),$$

where $h_\tau(t)$ is the usual h-vector of Σ_τ. In this case, $\delta^0(s,t)\in\mathbb{Z}[s^{1/N},s^{-1/N},t]$, for some positive integer N. By Lemma II.4, the coefficients of $h_\tau(t)$ are non-negative integers. Hence the coefficients of $\delta^0(s,t)$ are non-negative integers. Recall that, for any rational number $-1<k\leq 0$, we can recover the polynomial $\delta^0_k(t)$ as the coefficient of s^k in $\delta^0(s,t)$. Note that the degree of $h_\tau(t)$ is at most $\operatorname{codim}\tau$ and $\lceil\psi(v)\rceil\leq\dim\tau$ for any v in $\mathrm{Box}(\boldsymbol{\tau})$. It follows that $\delta^0_k(t)$ is a polynomial in $\mathbb{Z}[t]$ of degree less than or equal to d, with non-negative coefficients.

Example II.8. If we set $s = 1$, then we recover the Ehrhart δ-polynomial of Q as $\delta_Q(t) = \delta^\lambda(1,t)$ and Proposition II.6 gives the local formula of Betke and McMullen for $\delta_Q(t)$ (Theorem 1 in [10]),

$$\delta_Q(t) = \sum_{\substack{\tau\in\Sigma \\ v\in\mathrm{Box}(\boldsymbol{\tau})}} t^{\lceil\psi(v)\rceil} h_\tau(t).$$

Note that the origin in N corresponds to a contribution of $h_\Sigma(t)$ in the above sum. It follows that $\delta_Q(t)$ is a polynomial of degree less than or equal to d with non-negative coefficients and constant term 1 [10]. If Σ is complete, then Lemma II.4 implies that $\delta_Q(t)$ is a polynomial of degree d with positive integer coefficients [10].

Corollary II.9. *If λ satisfies the condition*

$$\lambda(v) \geq 0, \tag{2.8}$$

for every v in $|\Sigma|$, then for every rational number k, $\delta_k^\lambda(t)$ is a polynomial in $\mathbb{Z}[t]$.

Proof. We know that $\delta_k^\lambda(t)$ is the coefficient of s^k in $\delta^\lambda(s,t)$ and is a power series in $\mathbb{Z}[[t]]$. We will show that it has bounded degree. By Proposition II.6,

$$\delta^\lambda(s,t) = \sum_{\tau\in\Sigma} h_\tau^\lambda(s,t) \sum_{v\in\mathrm{Box}(\boldsymbol{\tau})} s^{w_\lambda(v)} t^{\lceil\psi(v)\rceil} \prod_{\rho_i\subseteq\tau} (t-1)/(s^{\lambda(b_i)}t - 1).$$

Expanding the right hand side gives

$$(1-t)^d \sum_{\substack{\tau\in\Sigma \\ v\in\mathrm{Box}(\boldsymbol{\tau})}} s^{w_\lambda(v)} t^{\lceil\psi(v)\rceil} \sum_{\tau\subseteq\sigma} s^{\sum_{\rho_i\subseteq\sigma\smallsetminus\tau}\lambda(b_i)} t^{\dim\sigma - \dim\tau} \prod_{\rho_i\subseteq\sigma} 1/(1 - s^{\lambda(b_i)}t).$$

If a monomial $t^l s^k$ appears in the expansion of this expression, then k must have the form

$$k = w_\lambda(v) + \sum_{\rho_i\subseteq\sigma\smallsetminus\tau} \lambda(b_i) + \sum_{i=1}^{r} \alpha_i \lambda(b_i), \tag{2.9}$$

for some v in $\mathrm{Box}(\mathbf{\Sigma})$ and α_i non-negative integers that are equal to zero if $\lambda(b_i) = 0$, and such that

$$l \le \lceil \psi(v) \rceil + 2d + \sum_{i=1}^{r} \alpha_i.$$

It follows from Condition (2.8) that for a fixed k, there are only finitely many possibilities for α_i such that (2.9) holds. Therefore l is bounded. □

By a standard argument, this is equivalent to the following corollary.

Corollary II.10. *If λ satisfies the condition $\lambda(v) \ge 0$, for every v in $|\Sigma|$, then for every rational number k and for every m sufficiently large (depending on k), $f_k^{\lambda}(m)$ is a polynomial function in m, of degree less than or equal to d.*

Proof. Fix a rational number k. By Corollary II.9, we can write $F_k^{\lambda}(t) = P_k(t)/(1 - t)^{d+1} + Q_k(t)$, where $P_k(t) = p_{0,k} + p_{1,k}t + \cdots + p_{d,k}t^d$ and $Q_k(t)$ are polynomials. Expanding the right hand side gives

$$F_k^{\lambda}(t) = \sum_{j=0}^{d} p_{j,k} \sum_{m \ge 0} \binom{m+d}{d} t^{m+j} + Q_k(t).$$

Hence, for $m > \deg Q_k(t)$,

$$f_k^{\lambda}(m) = \sum_{j=0}^{d} p_{j,k} \binom{m+d-j}{d}.$$

□

Example II.11. As in Example II.7, suppose that $\lambda \equiv 0$. We have seen that for every rational number k, $\delta_k^0(t)$ is a polynomial in $\mathbb{Z}[t]$ with non-negative coefficients, of degree less than or equal to d. We claim that $f_k^0(m)$ is either identically zero or a polynomial of degree d in $\mathbb{Q}[t]$ with positive leading coefficient. This follows from the above proof, which shows that $f_k^0(m)$ is a polynomial of degree less than or equal to d, and the observation that the coefficient of t^d is $\sum_{j=0}^{d} p_{j,k}/d!$, where the $p_{j,k}$ are the non-negative coefficients of $\delta_k^0(t)$.

Suppose that λ satisfies the condition

$$\lambda(b_i) > -1 \text{ for } i = 1, \ldots, r. \tag{2.10}$$

In this case we define

$$h_\tau^\lambda(t) := h_\tau^\lambda(t,t),$$

$$\delta^\lambda(t) := \delta^\lambda(t,t).$$

By Proposition II.6, we have the following expression for $\delta^\lambda(t)$,

$$\delta^\lambda(t) = \sum_{\tau \in \Sigma} h_\tau^\lambda(t) \sum_{v \in \mathrm{Box}(\boldsymbol{\tau})} t^{\psi(v)+\lambda(v)} \prod_{\rho_i \subseteq \tau} (t-1)/(t^{\lambda(b_i)+1} - 1). \tag{2.11}$$

It follows from (2.10) and (2.3) that there is a positive integer N' such that $h_\tau^\lambda(t)$ lies in $\mathbb{Z}\,[[t^{1/N'}]]$ for all τ in Σ. By (2.10), $w_\lambda(v) + \lceil \psi(v) \rceil = \psi(v) + \lambda(v)$ is non-negative for all v in $|\Sigma| \cap N$. Hence (2.11) implies that $\delta^\lambda(t) \in \mathbb{Z}\,[[t^{1/N}]]$, for some positive integer N. In this case, we will abuse notation and call $\delta^\lambda(t)$ the *weighted δ-power series* associated to $\boldsymbol{\Sigma}$ and λ.

We now describe a well-known involution ι on $|\Sigma| \cap N$. Consider a cone τ in Σ and v in $\mathrm{Box}(\boldsymbol{\tau})$. Then v can be uniquely written in the form $v = \sum_{\rho_i \subseteq \tau} q_i b_i$, for some $0 < q_i < 1$. We define

$$\iota = \iota_{\boldsymbol{\Sigma}} : \mathrm{Box}(\boldsymbol{\tau}) \to \mathrm{Box}(\boldsymbol{\tau}) \tag{2.12}$$

$$\iota(v) = \sum_{\rho_i \subseteq \tau} (1 - q_i) b_i.$$

As in (2.7), every v in $|\Sigma| \cap N$ can be uniquely written in the form $v = \{v\} + \tilde{v}$, where $\{v\}$ is in $\mathrm{Box}(\boldsymbol{\tau})$ for some $\tau \subseteq \sigma(v)$ and $\tilde{v}$ is in $N_{\sigma(v)}$. Here $\sigma(v)$ is the cone of Σ containing v in its relative interior. Then ι extends to an involution on $|\Sigma| \cap N$

$$\iota = \iota_{\boldsymbol{\Sigma}} : |\Sigma| \cap N \to |\Sigma| \cap N$$

$$\iota(v) = \iota(\{v\}) + \tilde{v}.$$

Using (2.11) and after rearranging and collecting terms, we can write $t^d\delta^\lambda(t^{-1})$ as

$$\sum_{\substack{\tau\in\Sigma \\ v\in\mathrm{Box}(\boldsymbol{\tau})}} t^{\operatorname{codim}\tau} h_\tau^\lambda(t^{-1}) t^{\sum_{\rho_i\subseteq\tau}\lambda(b_i)+\dim\tau-\psi(v)-\lambda(v)} \prod_{\rho_i\subseteq\tau}(t-1)/(t^{\lambda(b_i)+1}-1).$$

Note that for any v in $\mathrm{Box}(\boldsymbol{\tau})$,

$$\sum_{\rho_i\subseteq\tau}\lambda(b_i) + \dim\tau - \psi(v) - \lambda(v) = \psi(\iota(v)) + \lambda(\iota(v)),$$

where ι is the involution (2.12). Hence

$$t^d\delta^\lambda(t^{-1}) = \sum_{\tau\in\Sigma} t^{\operatorname{codim}\tau} h_\tau^\lambda(t^{-1}) \sum_{v\in\mathrm{Box}(\boldsymbol{\tau})} t^{\psi(v)+\lambda(v)} \prod_{\rho_i\subseteq\tau}(t-1)/(t^{\lambda(b_i)+1}-1). \tag{2.13}$$

Corollary II.12. *If Σ is a complete fan and $\lambda(b_i) > -1$ for $i = 1, \ldots, r$, then*

$$\delta^\lambda(t) = t^d\delta^\lambda(t^{-1}).$$

Proof. By Lemma II.5, for any cone τ in Σ, $h_\tau^\lambda(t) = t^{\operatorname{codim}\tau}h_\tau^\lambda(t^{-1})$. The result follows by comparing expressions (2.11) and (2.13). □

We have the following change of variables formula for weighted δ-power series. A geometric proof involving motivic integration is given in Section 3.6. We write $\psi = \psi_\Sigma$ and $\delta^\lambda(t) = \delta^\lambda_\Sigma(t)$.

Proposition II.13. *Let N be a lattice of rank d, and let $\boldsymbol{\Sigma} = (N, \Sigma, \{b_i\})$ and $\triangle = (N, \triangle, \{b'_j\})$ be stacky fans such that $|\Sigma| = |\triangle|$. Let λ be a piecewise $\mathbb{Q}$-linear function with respect to Σ satisfying $\lambda(b_i) > -1$ for every b_i, and set*

$$\lambda' = \lambda + \psi_\Sigma - \psi_\triangle.$$

If λ' is piecewise $\mathbb{Q}$-linear with respect to $\triangle$ and satisfies $\lambda'(b'_j) > -1$ for every b'_j, then $\delta^\lambda_\Sigma(t) = \delta^{\lambda'}_\triangle(t)$.

Proof. By letting $s = t$ in the first calculation in the proof of Proposition II.6, we see that

$$\delta_{\Sigma}^{\lambda}(t) = (1-t)^d \sum_{v \in |\Sigma| \cap N} t^{\psi_{\Sigma}(v)+\lambda(v)},$$

which gives,

$$\delta_{\triangle}^{\lambda'}(t) = (1-t)^d \sum_{v \in |\triangle| \cap N} t^{\psi_{\triangle}(v)+\lambda'(v)} = \delta_{\Sigma}^{\lambda}(t).$$

□

2.2 Weighted Ehrhart Reciprocity

The goal of this section is to investigate the case when $\lambda \equiv 0$. In this case, $\delta^0(t)$ is a polynomial of degree at d with rational powers and non-negative integer coefficients, and we can recover the Ehrhart δ-polynomial $\delta_Q(t)$ from $\delta^0(t)$. While $\delta_Q(t)$ is not symmetric in its coefficients, Corollary II.12 implies that if Σ is complete, then $\delta^0(t) = t^d \delta^0(t^{-1})$. The main point is that we can exploit this symmetry to deduce facts about the Ehrhart δ-polynomial.

Note that the weight function $w_0(v) = \psi(v) - \lceil \psi(v) \rceil$ takes values between -1 and 0. By Example II.7, for each rational number $-1 < k \leq 0$, we can write

$$\delta_k^0(t) = \delta_{0,k} + \delta_{1,k}t + \cdots + \delta_{d,k}t^d,$$

for some non-negative integers $\delta_{i,k}$. Since the Ehrhart δ-polynomial decomposes as $\delta_Q(t) = \sum_{k \in (-1,0]} \delta_k^0(t)$, we have, with the notation of (2.2),

$$\delta_i = \sum_{k \in (-1,0]} \delta_{i,k}, \tag{2.14}$$

Throughout this section we will set $s = t$, so that the weighted δ-polynomial is given by

$$\delta^0(t) = \sum_{k \in (-1,0]} \delta_k^0(t) t^k = \sum_{k \in (-1,0]} \sum_{i=0}^{d} \delta_{i,k} t^{i+k}.$$

By (2.14), δ_i is the sum of the coefficients of t^j in $\delta^0(t)$ for $i-1 < j \leq i$. By Example II.7,

$$\delta^0(t) = \sum_{\substack{\tau \in \Sigma \\ v \in \mathrm{Box}(\boldsymbol{\tau})}} t^{\psi(v)} h_\tau(t). \tag{2.15}$$

Later we will see that the coefficients of $\delta^0(t)$ are dimensions of orbifold Chow groups of a toric stack (Theorem II.31).

Remark II.14. The weight function w_0 and hence the weighted δ-polynomial $\delta^0(t)$ are determined by the underlying space of the lattice complex Q. Recall, from Remark II.3, that any lattice polytope P, after translation, has the form $P = Q_{\boldsymbol{\Sigma}}$, for some stacky fan $\boldsymbol{\Sigma}$. In this case, $\delta^0(t)$ is determined by P and the choice of a lattice point α in P.

Remark II.15. Note that the non-zero lattice points of weight 0 are those that lie in a facet of $\partial(mQ)$ not containing 0, for some positive integer m. Let ∂Q_0 denote the union of the facets in ∂Q not containing the origin. Consider the lattice $N \times \mathbb{Z}$ and the lattice complex

$$K_0 = \{(v, \mu) \in (N \times \mathbb{Z})_{\mathbb{R}} \mid 0 < \mu \leq 1,\, v \in \partial(\mu Q)_0\} \cup \{0\}.$$

We can interpret $f_0^0(m)$ as the number of lattice points in mK_0 and $\delta_0^0(t)$ as the δ-polynomial associated to K_0 (cf. [61]).

Example II.16. Let P be a lattice d-simplex containing the origin in its interior. Let Σ be the fan over the faces of the boundary of P, with the $\{b_i\}$ given by the vertices of P. One can verify that $h_\tau(t) = 1 + t + \cdots + t^{\operatorname{codim} \tau}$ for any cone τ in Σ. Hence

$$\delta^0(t) = \sum_{\substack{\tau \in \Sigma \\ v \in \mathrm{Box}(\boldsymbol{\tau})}} t^{\psi(v)} (1 + t + \cdots + t^{\operatorname{codim} \tau}).$$

Corollary II.17. *If Σ is a complete fan, then $\delta_0^0(t)$ has degree d and positive integer coefficients, and satisfies $\delta_0^0(t) = t^d\delta_0^0(t^{-1})$. Moreover, for $-1 < k < 0$, if we write $\delta_k^0(t) = t\tilde{\delta}_k(t)$, then $\tilde{\delta}_k(t) = t^{d-1}\tilde{\delta}_{-1-k}(t^{-1})$.*

Proof. By considering the contribution of $0 \in \mathrm{Box}\{0\}$ in (2.15) and using Lemma II.4, we see that $\delta_0^0(t)$ has degree d and positive integer coefficients. By (2.15), $\delta_k^0(t)$ has no constant term for $-1 < k < 0$. By Corollary II.12, we have $\delta^0(t) = t^d\delta^0(t^{-1})$, and we can write

$$\delta^0(t) = \sum_{k\in(-1,0]} \delta_k^0(t)t^k.$$

$$\begin{aligned} t^d\delta^0(t^{-1}) &= \sum_{k\in(-1,0]} t^d\delta_k^0(t^{-1})t^{-k} \\ &= t^d\delta_0^0(t^{-1}) + \sum_{k\in(-1,0)} t^{d+1}\delta_k^0(t^{-1})t^{-1-k} \\ &= t^d\delta_0^0(t^{-1}) + \sum_{k\in(-1,0)} t^{d+1}\delta_{-1-k}^0(t^{-1})t^k. \end{aligned}$$

Comparing the expressions above yields $\delta_0^0(t) = t^d\delta_0^0(t^{-1})$ and $\delta_k^0(t) = t^{d+1}\delta_{-1-k}^0(t^{-1})$, for $-1 < k < 0$. Since each $\delta_k^0(t)$ is a polynomial of degree less than or equal to d, the corollary follows. □

Remark II.18. Suppose Σ is complete and set $a(t) = \delta_0^0(t)$ and $b(t) = \sum_{k\in(-1,0)} \tilde{\delta}_k(t)$. Since $\delta_Q(t) = \sum_{-1<k\leq 0} \delta_k^0(t)$, we have a decomposition

$$\delta_Q(t) = a(t) + tb(t),$$

where $a(t) = a(t^{-1})$ is a polynomial of degree d with positive integer coefficients and $b(t) = t^{d-1}b(t^{-1})$ is a polynomial of degree at most $d-1$ with non-negative coefficients. This decomposition is originally due to Betke and McMullen (Theorem

5 [10]). We give an analogous result in the case when Σ is not necessarily complete in Chapter IV.

Remark II.19. We can exploit the symmetry properties of the above decomposition to translate inequalities between the coefficients of $\delta_Q(t)$ into inequalities between the coefficients of $a(t)$ and $b(t)$. This is explained in Chapter IV in a more general setting.

We give a reformulation of Corollary II.17. We note that the $k = 0$ case can be deduced from Ehrhart Reciprocity (see Remark II.22).

Theorem II.20 (Weighted Ehrhart Reciprocity)**.** *If Σ is a complete fan, then, for every rational number $-1 < k \leq 0$, $f_k^0(m)$ is either identically zero or a polynomial of degree d in $\mathbb{Q}[t]$ with positive leading coefficient, and, for any positive integer m,*

$$f_k^0(-m) = \begin{cases} (-1)^d f_k^0(m-1) & \textit{if } k = 0 \\ (-1)^d f_{-1-k}^0(m) & \textit{if } -1 < k < 0. \end{cases}$$

Proof. Fix a rational number k. The first statement follows from Example II.11. We see from the proof of Corollary II.10 that

$$f_k^0(m) = \sum_{j=0}^{d} \delta_{j,k} \binom{m+d-j}{d}.$$

Hence

$$f_k^0(-m) = (-1)^d \sum_{j=0}^{d} \delta_{j,k} \binom{m+j-1}{d} \text{ for } m \geq 1.$$

We define

$$F_k(t) = \sum_{m \geq 0} f_k^0(m) t^m = \delta_k^0(t)/(1-t)^{d+1},\ \tilde{F}_k(t) = \sum_{m \geq 1} f_k^0(-m) t^m,$$

and compute,

$$\begin{aligned}
-F_k(t^{-1}) &= -\delta_k^0(t^{-1})/(1-t^{-1})^{d+1} \\
&= (-1)^d t^{d+1}\delta_k^0(t^{-1})/(1-t)^{d+1} \\
&= (-1)^d \sum_{j=0}^{d}\sum_{m\geq 0} \delta_{j,k}\binom{m+d}{d} t^{m+d+1-j} \\
&= \sum_{j=0}^{d}\sum_{m'\geq d+1-j} (-1)^d \delta_{j,k}\binom{m'+j-1}{d} t^{m'} \\
&= \sum_{m'\geq 1}\sum_{j=0}^{d} (-1)^d \delta_{j,k}\binom{m'+j-1}{d} t^{m'} \\
&= \tilde{F}_k(t).
\end{aligned}$$

After substituting in definitions, Corollary II.17 implies that $tF_0(t) = (-1)^{d+1}F_0(t^{-1})$ and $F_k(t) = (-1)^{d+1}F_{-1-k}(t^{-1})$, for $-1 < k < 0$. The result follows by comparing coefficients of the resulting expressions $\tilde{F}_0(t) = (-1)^d tF_0(t)$ and $\tilde{F}_k(t) = (-1)^d F_{-1-k}(t)$, for $-1 < k < 0$. $\square$

The above result should be viewed as the weighted version of Ehrhart Reciprocity, a classical result due to Ehrhart [20]. In particular, we verify below that Ehrhart Reciprocity is a consequence. We emphasize that this proof can be interpreted as a reformulation of previous proofs of Ehrhart's theorem.

Corollary II.21 (Ehrhart Reciprocity [20])**.** *For every positive integer m, $(-1)^d f_Q(-m)$ is equal to the number of lattice points in the interior of mQ.*

Proof. It follows from the definitions that

$$(-1)^d f_Q(-m) = \sum_{k\in(-1,0]\cap\mathbb{Q}} (-1)^d f_k^0(-m) \tag{2.16}$$

$$|\operatorname{Int}(mQ)\cap N| = f_0^0(m-1) + \sum_{k\in(-1,0)\cap\mathbb{Q}} f_k^0(m). \tag{2.17}$$

The fan Σ induces a lattice triangulation $\mathcal{T}$ of ∂Q. There exists a positive integer n and a lattice point α in the interior of nQ such that, after translating α to the origin, the collection of cones over the faces of $\mathcal{T}$ form a simplicial fan Σ'. It follows from (2.16), (2.17) and Example II.7 that the functions $|\,\mathrm{Int}(mQ) \cap N|$ and $(-1)^d f_Q(-m)$ are polynomials in m. Hence, after replacing Q by a multiple and replacing Σ by Σ', we may assume that Σ is complete. The result now follows by applying Theorem II.20 to (2.16) in order to obtain (2.17).

$\square$

Remark II.22. This result was conjectured by Ehrhart in 1959 and proved by him in [20]. Another proof was given by MacDonald in [45]. If Σ is complete, then

$$f_Q(m) - (-1)^d f_Q(-m) = f_0^0(m) - f_0^0(m-1), \tag{2.18}$$

since both sides are equal to the number of lattice points in $\partial(mQ)$. With the notation of (2.1), it follows that c_{d-1} is half the surface area of Q, normalised with respect to the sublattice on each facet of Q. Note that c_d is the normalised volume of Q in N. These facts were established in [21] and [44]. Using (2.18) and applying a short induction, Ehrhart Reciprocity implies that $f_0^0(-m) = (-1)^d f_0^0(m-1)$ for any positive integer m, which was proved in Theorem II.20 (cf. [61]).

Remark II.23. In a similar way, one can show that Weighted Ehrhart Reciprocity implies Ehrhart Reciprocity for rational polytopes (see, for example, [60]). More specifically, fix a positive integer r and set $Q' = (1/r)Q$. For $0 \leq l < r$, one verifies from the definitions that

$$f_{Q'}(l + mr) = \sum_{l/r-1<k\leq 0} f_k(m) + \sum_{-1<k\leq l/r-1} f_k(m+1).$$

It follows that $f_{Q'}(m)$ is a quasipolynomial with period dividing r [60]. As in the proof of Corollary II.21, after replacing Q by sQ for some positive integer s coprime to r, we can apply Theorem II.20 to deduce that, for any positive integer m, $(-1)^d f_{Q'}(-m)$ is equal to the number of lattice points in the interior of mQ'.

Let P be a rational polytope and fix a positive integer r such that rP is a lattice polytope. The function $f_P(m)$ is a quasipolynomial with period dividing r [60]. After replacing P by sP for some positive integer s coprime to r and translating by a lattice point, we may assume that P contains the origin. Setting $Q' = P$ and $Q = rP$, we deduce that $(-1)^d f_P(-m)$ is equal to the number of lattice points in the interior of mP.

The following result of Hibi was originally shown to be a consequence of Ehrhart Reciprocity [33]. In a similar way, it follows from Theorem II.20. Recall that ψ is the piecewise $\mathbb{Q}$-linear function satisfying $\psi(b_i) = 1$, for $i = 1, \ldots, r$.

Corollary II.24 ([33]). *If Σ is complete, then $\delta_Q(t) = t^d \delta_Q(t^{-1})$ if and only if ψ is a piecewise linear function.*

Proof. Observe that ψ is piecewise linear if and only if $w_0 \equiv 0$. If $w_0 \equiv 0$, then by Corollary II.17, $\delta_Q(t) = \delta^0(t) = t^d \delta^0(t^{-1}) = t^d \delta_Q(t^{-1})$. Conversely, suppose that $\delta_Q(t) = t^d \delta_Q(t^{-1})$. Assume w_0 is not identically 0. Choose $j+1$ minimal such that $\delta_{j+1,k} > 0$ for some $-1 < k < 0$. By Corollary II.17 and (2.14), $\delta_j = \delta_{j,0} + \sum_{k \in (-1,0)} \delta_{j,k} = \delta_{j,0}$ and $\delta_{d-j} = \delta_{d-j,0} + \sum_{k \in (-1,0)} \delta_{d-j,k} = \delta_{j,0} + \sum_{k \in (-1,0)} \delta_{j+1,k} > \delta_j$. This is a contradiction. $\square$

Remark II.25. If P is a lattice polytope then δ_0 is 1 and δ_d is the number of interior lattice points of P. A lattice polytope P is *reflexive* if it contains the origin in its interior and ψ is piecewise linear, and hence Corollary II.24 says that $\delta_P(t) = t^d \delta_P(t^{-1})$ if and only if P is the translate of a reflexive polytope [33].

We conclude this section by proving a lemma which allows us to compute examples when $d = 2$ and Σ is complete, and then presenting a corresponding example. Recall that ∂Q_0 denotes the union of the facets of ∂Q not containing the origin.

Lemma II.26. *The weighted δ-polynomial $\delta^0(t)$ is a polynomial of degree less than or equal to d with rational powers and non-negative integer coefficients, satisfying:*

1. *If Σ is complete, then $\delta^0(t) = t^d \delta^0(t^{-1})$.*
2. *$\delta^0(1) = d! \operatorname{vol}_d(Q)$.*
3. *The constant coefficient in $\delta^0(t)$ is 1.*
4. *For $0 < l < 1$, the coefficient of t^l in $\delta^0(t)$ is $|\{v \in Q \cap N \mid \psi(v) = l\}|$.*
5. *The coefficient of t in $\delta^0(t)$ is $|\partial Q_0 \cap N| - d$.*

Proof. We established the initial claims in Corollary II.10 and Corollary II.17. We showed in the proof of Corollary II.10 that $f_k^0(m)$ is a polynomial of degree d with leading coefficient $\sum_{j=0}^{d} \delta_{j,k}/d!$, and hence $d! \operatorname{vol}_d(Q) = \sum_{k \in (-1,0]} \sum_{j=0}^{d} \delta_{j,k} = \delta^0(1)$. For the other claims we compare both sides of the expression

$$\delta^0(t) = (1-t)^{d+1} \sum_{m \geq 0} \sum_{v \in mQ \cap N} t^{w_0(v)+m}.$$

The constant coefficient on the right hand side is 1. For $0 < l < 1$, the coefficient on the right hand side is $|\{v \in Q \cap N \mid w_0(v) + 1 = l\}|$. Note that if v lies in $Q \cap N$ and $w_0(v) \neq 0$ then $\lceil \psi(v) \rceil = 1$. Finally, the coefficient of t on the right hand side is $|\{v \in Q \cap N \mid w_0(v) = 0\}| - (d+1)$. The only elements of $Q \cap N$ of weight zero are the origin and the elements of $\partial Q_0 \cap N$. $\square$

Remark II.27. When $d = 2$ and Σ is complete, one can show that

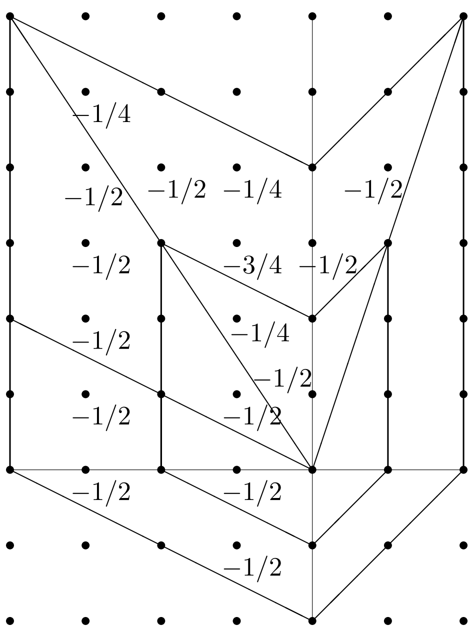
$$f_0^0(m) = |\partial Q \cap N| m(m+1)/2 + 1,$$

$$f_k^0(m) = (f_k^0(1) + f_{-1-k}^0(1))m^2/2 + (f_k^0(1) - f_{-1-k}^0(1))m/2, \text{ for } k \neq 0.$$

Example II.28. Let $N = \mathbb{Z}^2$ and let Σ be the complete fan with primitive integer vectors $(1,0)$,$(1,3)$, $(0,1)$, $(-2,3)$,$(-2,1)$,$(-1,0)$ and $(0,-1)$, and set a_i to be $1,1,2,1,1,2$ and 1 respectively. Since Σ is a complete fan, weighted Ehrhart Reciprocity holds and the weighted δ-polynomial $\delta^0(t)$ is symmetric. The example is illustrated in Figure 2.1 below, in which we have marked the lattice points of non-zero weight in $2Q$. The computations were made using Lemma II.26 and Remark II.27. Observe that removing the ray through $(-2,1)$ does not affect the results below.

$$\delta^0(t) = t^2 + 3t^{3/2} + t^{5/4} + 8t + t^{3/4} + 3t^{1/2} + 1$$

$$\delta_Q(t) = 5t^2 + 12t + 1$$



$$f_0^0(m) = 5m^2 + 5m + 1$$

$$f_{-1/2}^0(m) = 3m^2$$

$$f_{-1/4}^0(m) = m(m+1)/2$$

$$f_{-3/4}^0(m) = m(m-1)/2$$

$$f_Q(m) = 9m^2 + 5m + 1$$

Figure 2.1: Weighted Ehrhart theory for non-convex Q

2.3 Orbifold Cohomology

We will now prove our geometric interpretation of the coefficients of the Ehrhart δ-polynomial. Recall that Σ is a simplicial, d-dimensional fan with convex support.

In this case, we have the following combinatorial description of the Betti numbers of the corresponding toric variety $X = X(\Sigma)$.

Lemma II.29. *If Σ is a simplicial, d-dimensional fan with convex support and $X = X(\Sigma)$ is the corresponding toric variety, then X has no odd cohomology over $\mathbb{Q}$ and $\dim H^{2i}(X, \mathbb{Q})$ is equal to the coefficient of t^i in the h-vector $h_\Sigma(t)$ of Σ. Moreover, if Σ is complete, then $\dim H^{2i}(X, \mathbb{Q}) > 0$, for $i = 0, \ldots, d$.*

Proof. The case when Σ is complete is due to Danilov (Theorem 10.8 [15]). Suppose Σ is not complete. Let ρ be a ray in the interior of $-|\Sigma|$ and let $\triangle$ be the fan with cones given by the cones of Σ as well as the cones generated by ρ and a face of Σ contained in $\partial|\Sigma|$. The toric variety $Y = Y(\triangle)$ is simplicial and complete and if $D = D(\triangle_\rho)$ denotes the $\mathbb{Q}$-Cartier torus-invariant divisor corresponding to ρ, then D is simplicial and complete and satisfies $Y \smallsetminus D = X$. By considering the long exact sequence of cohomology with compact supports, we have a diagram,

$$\begin{tikzcd}
 & A^i(Y,\mathbb{Q}) \arrow[r, "\alpha"] \arrow[d] & A^i(D,\mathbb{Q}) \arrow[r] \arrow[d] & 0 & \\
0 \arrow[r] H_c^{2i}(X,\mathbb{Q}) \arrow[r] & H_c^{2i}(Y,\mathbb{Q}) \arrow[r] & H_c^{2i}(D,\mathbb{Q}) \arrow[r] & H_c^{2i+1}(X,\mathbb{Q}) \arrow[r] & 0.
\end{tikzcd}$$

Since a simplicial, complete toric variety has no odd cohomology, the bottom row is exact. The vertical maps take a cycle to its corresponding rational cohomology class and both maps are isomorphisms by Theorem 10.8 of [15]. The map α 'restricts' cycles of Y to cycles on the $\mathbb{Q}$-Cartier divisor D (p33 [26]). Let γ be a cone in $\partial|\Sigma|$ corresponding to a T-invariant subvariety $V(\gamma)$ of Y. If we set $\sigma = \gamma + \rho$, then σ corresponds to a T-invariant subvariety $V_\rho(\sigma)$ of D and the set-theoretic intersection of $V(\gamma)$ and D is $V_\rho(\sigma)$. It follows that $\alpha([V(\gamma)])$ is a positive multiple of $[V_\rho(\sigma)]$ [25]. By Proposition 10.3 of [15], $A^*(D, \mathbb{Q})$ is generated by classes of the form $[V_\rho(\sigma)]$ and hence α is surjective and the diagram above has exact rows. We conclude that

$H_{\mathrm{c}}^{2i+1}(X,\mathbb{Q}) = 0$ and $\dim H_{\mathrm{c}}^{2i}(X,\mathbb{Q})$ is equal to the i^{th} coefficient of

$$t^d h_{\triangle}(t^{-1}) - t^d h_{\triangle_\rho}(t^{-1}) = \sum_{\tau \in \Sigma} (t-1)^{\operatorname{codim} \tau} = t^d h_{\Sigma}(t^{-1}).$$

By Poincaré duality, $\dim H^{2i}(X,\mathbb{Q})$ is equal to the coefficient of t^i in $h_\Sigma(t)$. $\square$

We can associate to the stacky fan $\mathbf{\Sigma} = (N, \Sigma, \{b_i\})$ a Deligne-Mumford toric stack $\mathcal{X} = \mathcal{X}(\mathbf{\Sigma})$ over $\mathbb{C}$ with coarse moduli space $X = X(\Sigma)$ [12]. We refer the reader to Section 3.1 for more details. Let $\mathcal{Y}_1, \ldots, \mathcal{Y}_t$ denote the connected components of the corresponding inertia stack $\mathcal{IX}$ (see Section 3.2). There is a degree shifting function

$$\iota_{\mathcal{X}} : \mathcal{IX} \to \mathbb{Q},$$

which is constant on connected components. Let $\overline{\mathcal{Y}}_i$ denote the coarse moduli space of $\mathcal{Y}_i$. For $i \in \mathbb{Q}$, Chen and Ruan [14] defined the i^{th} orbifold cohomology group of $\mathcal{X}$ by

$$H^i_{\mathrm{orb}}(\mathcal{X},\mathbb{Q}) = \bigoplus_{j=1}^{t} H^{i-2\iota_{\mathcal{X}}(\mathcal{Y}_j)}(\overline{\mathcal{Y}}_j,\mathbb{Q}).$$

Similarly, we can consider the orbifold cohomology $H^*_{\mathrm{orb,c}}(\mathcal{X},\mathbb{Q})$ of $\mathcal{X}$ with compact support. Chen and Ruan gave $H^*_{\mathrm{orb}}(\mathcal{X},\mathbb{Q})$ a ring structure and established Poincaré duality between $H^*_{\mathrm{orb}}(\mathcal{X},\mathbb{Q})$ and $H^*_{\mathrm{orb,c}}(\mathcal{X},\mathbb{Q})$ (Proposition 3.3.1 [14]).

Borisov, Chen and Smith (Proposition 4.7 [12]) show that the connected components of $\mathcal{IX}$ are indexed by the elements of $\mathrm{Box}(\mathbf{\Sigma})$. Moreover, if v in $\mathrm{Box}(\boldsymbol{\tau})$ corresponds to the connected component $\mathcal{Y}_v$, then $\iota_{\mathcal{X}}(\mathcal{Y}_v) = \psi(v)$ and $\overline{\mathcal{Y}}_v = X(\Sigma_\tau)$. Hence

$$H^{2i}_{\mathrm{orb}}(\mathcal{X},\mathbb{Q}) = \bigoplus_{\tau \in \Sigma} \bigoplus_{v \in \mathrm{Box}(\boldsymbol{\tau})} H^{2(i-\psi(v))}(X(\Sigma_\tau),\mathbb{Q}). \tag{2.19}$$

Remark II.30. Similarly, Abramovich, Graber and Vistoli [1] defined the *orbifold Chow ring* $A^*_{\mathrm{orb}}(\mathcal{X},\mathbb{Q})$ of $\mathcal{X}$. For $i \in \mathbb{Q}$,

$$A^i_{\mathrm{orb}}(\mathcal{X},\mathbb{Q}) = \bigoplus_{j=1}^{t} A^{i-\iota_{\mathcal{X}}(\mathcal{Y}_j)}(\overline{\mathcal{Y}}_j,\mathbb{Q}).$$

The cohomology ring with rational coefficients and Chow ring with rational coefficients of a simplicial, complete toric variety are isomorphic (Theorem 10.8 [15]). Hence, if Σ is complete, then $A^i_{\mathrm{orb}}(\mathcal{X}, \mathbb{Q}) \cong H^{2i}_{\mathrm{orb}}(\mathcal{X}, \mathbb{Q})$.

We finally arrive at the main result of this section. The case when $w_0 \equiv 0$ is proved in [49].

Theorem II.31. *We have the following geometric interpretation of the Ehrhart δ-polynomial:*

$$\delta^0(t) = \sum_j \dim_{\mathbb{Q}} H^{2j}_{\mathrm{orb}}(\mathcal{X}(\mathbf{\Sigma}), \mathbb{Q}) t^j.$$

Moreover, the coefficient δ_i of t^i in the δ-polynomial $\delta_Q(t)$ is a sum of dimensions of orbifold cohomology groups,

$$\delta_i = \sum_{i-1<j\leq i} \dim_{\mathbb{Q}} H^{2j}_{\mathrm{orb}}(\mathcal{X}(\mathbf{\Sigma}), \mathbb{Q}).$$

Proof. By (2.15),

$$\delta^0(t) = \sum_{\tau\in\Sigma} \sum_{v\in \mathrm{Box}(\boldsymbol{\tau})} h_\tau(t) t^{\psi(v)}.$$

The first assertion now follows from (2.19) and Lemma II.29, and the second statement then follows from (2.14). □

Remark II.32. By Poincaré duality, the coefficient of t^j in $t^d\delta^0(t^{-1})$ is equal to the dimension of the $2j$'th orbifold cohomology group of $\mathcal{X}$ with compact support.

Remark II.33. When Σ is complete, we showed in Corollary II.17 that

$$\delta^0(t) = t^d \delta^0(t^{-1}).$$

By Theorem II.31, we can interpret this symmetry as a consequence of Poincaré duality for orbifold cohomology. In particular, since Weighted Ehrhart Reciprocity (Theorem II.20) is equivalent to Corollary II.17, this provides a geometric proof of Weighted Ehrhart Reciprocity.

A corollary of Theorem II.31 is the following result which interprets the coefficients of the Ehrhart δ-polynomial of a lattice polytope as dimensions of orbifold cohomology groups of a $(d+1)$-dimensional orbifold (cf. [39, Section 1]). More specifically, let P be a d-dimensional lattice polytope in N and fix a lattice triangulation $\mathcal{T}$ of P. If σ denotes the cone over $P \times \{1\}$ in $(N \times \mathbb{Z})_{\mathbb{R}}$, then $\mathcal{T}$ determines a simplicial fan refinement $\triangle$ of σ. The corresponding toric variety $Y = Y(\triangle)$ has a canonical stack structure: if $w_1, \ldots, w_s$ are the primitive integer vectors of the rays of $\triangle$, then the corresponding stacky fan is $(N \times \mathbb{Z}, \triangle, \{w_i\})$. We will write $H^{2i}_{\mathrm{orb}}(Y, \mathbb{Q})$ for the $2i^{\mathrm{th}}$ orbifold cohomology group of the canonical stack associate to Y.

Theorem II.34. *Let P be a d-dimensional lattice polytope and let $\mathcal{T}$ be a lattice triangulation of P corresponding to a $(d+1)$-dimensional toric variety Y as above. The Ehrhart δ-polynomial of P has the form*

$$\delta_P(t) = \sum_{i=0}^{d} \dim_{\mathbb{Q}} H^{2i}_{\mathrm{orb}}(Y, \mathbb{Q}) t^i.$$

Proof. With the notation of the previous discussion, let $\mathcal{Y}$ denote the toric stack associated to the stacky fan $(N \times \mathbb{Z}, \triangle, \{w_i\})$. In this case, $\psi : |\triangle| \to \mathbb{R}$ is the restriction of the projection $N_{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$ to $|\triangle|$, and hence $Q = \{v \in |\triangle| \mid \psi(v) \leq 1\}$ is the convex hull of $P \times \{1\}$ and the origin, called the *pyramid* over P. Since the weight function $w_0(v) = \psi(v) - \lceil \psi(v) \rceil$ is identically zero on $|\triangle| \cap (N \times \mathbb{Z})$, the weighted δ-polynomial $\delta^0(t)$ is just the usual δ-polynomial $\delta_Q(t)$. It is a standard fact that $\delta_P(t) = \delta_Q(t)$ (see, for example, Remark 2.6 [6]). On the other hand, Theorem II.31 implies that $\delta_P(t) = \delta_Q(t) = \delta^0(t) = \sum_{i=0}^{d} \dim_{\mathbb{Q}} H^{2i}_{\mathrm{orb}}(\mathcal{Y}, \mathbb{Q}) t^i$. $\square$

Remark II.35. If $\mathcal{T}$ is a unimodular triangulation of P, then Y is smooth and $H^{2i}_{\mathrm{orb}}(Y, \mathbb{Q}) = H^{2i}(Y, \mathbb{Q})$. In this case, the above theorem and Lemma II.29 imply the well-known fact that $\delta_P(t)$ is equal to the h-vector of $\mathcal{T}$ [31].

2.4 Stanley's Monotonicity Theorem

A classical result of Stanley states that δ-polynomials of lattice polytopes satisfy the following monotonicity property [59, Theorem 3.3]. If $f(t) = \sum_i f_i t^i$ and $g(t) = \sum_i g_i t^i$ are polynomials with integer coefficients, then we write $f(t) \leq g(t)$ if $f_i \leq g_i$ for all $i \geq 0$.

Theorem II.36 (Stanley's Monotonicity Theorem)**.** *If $Q \subseteq P$ are lattice polytopes, then $\delta_Q(t) \leq \delta_P(t)$.*

In this section, we provide a geometric proof by showing that Stanley's result is a consequence of Theorem II.34. An alternative combinatorial proof of this theorem was given by Beck and Sottile in [9].

We fix a lattice polytope P of dimension d in N and let $Q \subseteq P$ be a lattice polytope of possibly smaller dimension. One verifies that we may choose a regular lattice triangulation $\mathcal{T}$ of P which restricts to a regular lattice triangulation of Q. More specifically, if $v_1, \ldots, v_r$ denote the vertices of P not in Q, then we will inductively define a regular lattice triangulation of the convex hull Q_i of Q and $\{v_1, \ldots, v_i\}$. First choose a regular lattice triangulation $\mathcal{T}_0$ of $Q = Q_0$, and then define $\mathcal{T}_i$ to be the triangulation of Q_i with faces given by the faces of $\mathcal{T}_{i-1}$ together with the convex hulls F' of any (possibly empty) face F of $\mathcal{T}_{i-1}$ and x_i, provided $F' \cap Q_{i-1} = F$.

If σ denotes the cone over $P \times \{1\}$ in $N_{\mathbb{R}} \times \mathbb{R}$, then the triangulation $\mathcal{T}$ induces a simplicial fan refinement $\triangle$ of σ, with cones given by the cones over the faces of $\mathcal{T}$, and we may consider the $(d+1)$-dimensional, simplicial toric variety $Y = Y(\triangle)$ associated to $\triangle$. Recall that the triangulation $\mathcal{T}$ is *regular* if and only if $Y = Y(\triangle)$ is quasi-projective. If N' denotes the intersection of N with the affine span of Q and σ' denotes the cone over $Q \times \{1\}$ in $(N')_{\mathbb{R}} \times \mathbb{R}$, then the fan $\triangle$ refining σ

restricts to a fan Σ refining σ' and we may consider the simplicial toric variety $Y' = Y'(\Sigma)$. The inclusion of N' in N induces a locally closed immersion $Y' \hookrightarrow Y$. More specifically, Y' embeds as a closed subvariety of an open subset of Y isomorphic to $Y' \times (\mathbb{C}^*)^{\dim P - \dim Q}$ [25].

The toric varieties Y and Y' are *semi-projective* in the sense that they contain a torus-fixed point and are projective over the affine toric varieties U_σ and $U_{\sigma'}$, corresponding to σ and σ', respectively [28]. The cohomology ring $H^*(X, \mathbb{Q})$ of a semi-projective, simplicial toric variety X was computed by Hausel and Sturmfels in [28], and it was observed by Jiang and Tseng [37, Lemma 2.7] that Hausel and Sturmfel's proof, along with the results in [25, Section 5.1], imply that $H^*(X, \mathbb{Q})$ is isomorphic to the Chow ring $A^*(X, \mathbb{Q})$.

As in Theorem II.34, Y and Y' have a canonical orbifold structure, corresponding to the choice of the primitive integer vectors on the rays of $\triangle$ and Σ as the lattice points in the associated stacky fans. If we consider the corresponding orbifold Chow rings $A^*_{\mathrm{orb}}(Y, \mathbb{Q})$ and $A^*_{\mathrm{orb}}(Y', \mathbb{Q})$ (cf. Remark II.30), then Theorem II.34, together with the above remarks, imply that $\delta_P(t) = \sum_{i \in \mathbb{Q}} \dim_{\mathbb{Q}} A^i_{\mathrm{orb}}(Y, \mathbb{Q}) t^i$ and $\delta_Q(t) = \sum_{i \in \mathbb{Q}} \dim_{\mathbb{Q}} A^i_{\mathrm{orb}}(Y', \mathbb{Q}) t^i$. We conclude that Stanley's theorem follows from the lemma below.

Lemma II.37. *The morphism $Y' \hookrightarrow Y$ induces a surjective graded ring homomorphism $A^*_{\mathrm{orb}}(Y, \mathbb{Q}) \to A^*_{\mathrm{orb}}(Y', \mathbb{Q})$.*

Proof. Let $\mathcal{Y}$ and $\mathcal{Y}'$ denote the canonical toric stacks with coarse moduli spaces Y and Y' respectively. The inclusion of N' in N induces an inclusion of $\mathcal{Y}'$ as a closed substack of $\mathcal{Y}' \times (\mathbb{C}^*)^{\dim P - \dim Q}$ and an inclusion of $\mathcal{Y}' \times (\mathbb{C}^*)^{\dim P - \dim Q}$ as an open substack of $\mathcal{Y}$. These inclusions induce a corresponding restriction map $\iota : A^*_{\mathrm{orb}}(\mathcal{Y}, \mathbb{Q}) \to A^*_{\mathrm{orb}}(\mathcal{Y}', \mathbb{Q})$, which we describe below.

If F is a face of $\mathcal{T}$ and σ_F denotes the cone over F, then let $\mathrm{Box}(F) := \mathrm{Box}(\sigma_F)$. Here we set $\sigma_F = \{0\}$ if F is the empty face. Borisov, Chen and Smith [12] showed that the inertia stack of $\mathcal{Y}$ decomposes into connected components as $\mathcal{I}\mathcal{Y} = \coprod_{F\in\mathcal{T}}\coprod_{w\in\mathrm{Box}(F)}\mathcal{Y}_w$, where, if $w \in \mathrm{Box}(F)$, then $|\mathcal{Y}_w|$ is isomorphic to the torus-invariant subvariety $V(\sigma_F)$ of Y. Moreover, if $\psi : N_{\mathbb{R}} \times \mathbb{R} \to \mathbb{R}$ denotes projection onto the second co-ordinate, then the age of $\mathcal{Y}_w$ is $\psi(w) \in \mathbb{Z}$. Similarly, if $\mathcal{T}|_Q$ denotes the restriction of $\mathcal{T}$ to Q, then the inertia stack of $\mathcal{Y}'$ decomposes into connected components as $\mathcal{I}\mathcal{Y}' = \coprod_{F\in\mathcal{T}|_Q}\coprod_{w\in\mathrm{Box}(F)}\mathcal{Y}'_w$, where, if $w \in \mathrm{Box}(F)$, then the age of $\mathcal{Y}'_w$ is $\psi(w)$ and $|\mathcal{Y}'_w|$ is isomorphic to the torus-invariant subvariety $V(\sigma_F)'$ of Y'.

For each face $F \in \mathcal{T}|_Q$, the inclusion of N' in N induces a closed immersion $V(F)' \hookrightarrow V(F)' \times (\mathbb{C}^*)^{\dim P - \dim Q}$ and an open immersion $V(F)' \times (\mathbb{C}^*)^{\dim P - \dim Q} \hookrightarrow V(F)$. The corresponding restriction map $\nu_F : A^*(V(F), \mathbb{Q}) \to A^*(V(F)', \mathbb{Q})$ is surjective since if W' is an irreducible closed subvariety of $V(F)'$ and W denotes the closure of $W' \times (\mathbb{C}^*)^{\dim P - \dim Q}$ in $V(F)$, then $\nu_F([W]) = [W']$. The restriction map $\iota : A^*_{\mathrm{orb}}(\mathcal{Y}, \mathbb{Q}) \to A^*_{\mathrm{orb}}(\mathcal{Y}', \mathbb{Q})$ has the form

$$\iota : \coprod_{F\in\mathcal{T}}\ \coprod_{w\in\mathrm{Box}(F)\cap(N\times\mathbb{Z})} A^*(|\mathcal{Y}_w|, \mathbb{Q})[\psi(w)] \to \coprod_{F\in\mathcal{T}|_Q}\ \coprod_{w\in\mathrm{Box}(F)\cap(N\times\mathbb{Z})} A^*(|\mathcal{Y}'_w|, \mathbb{Q})[\psi(w)],$$

where for each $F \in \mathcal{T}$ and $w \in \mathrm{Box}(F)$, ι restricts to ν_F (with a grading shift) on $A^*(|\mathcal{Y}_w|, \mathbb{Q})[\psi(w)]$ if $F \subseteq Q$ and restricts to zero otherwise. One can verify from the description of the ring structure of an orbifold Chow ring in [1] that ι is a ring homomorphism. We conclude that ι is a surjective ring homomorphism. $\square$

Remark II.38. If we regard the empty face as a face of the triangulation $\mathcal{T}$ of dimension -1, then the h-vector of $\mathcal{T}$ is defined by $h_{\mathcal{T}}(t) = \sum_F t^{\dim F+1}(1-t)^{d-\dim F}$, where the sum ranges over all faces F in $\mathcal{T}$. It is a well known fact that $0 \leq h_{\mathcal{T}}(t) \leq \delta_P(t)$ and $h_{\mathcal{T}}(t) = \delta_P(t)$ if and only if $\mathcal{T}$ is a unimodular triangulation [10, 51]. We have the following geometric interpretation of this result.

It follows from the definition of the orbifold Chow ring (see Section 2) that $A^*(Y,\mathbb{Q})$ is a direct summand of $A^*_{\mathrm{orb}}(Y,\mathbb{Q})$ and $A^*(Y,\mathbb{Q}) = A^*_{\mathrm{orb}}(Y,\mathbb{Q})$ if and only if Y is smooth. The result now follows from the fact that $h_{\mathcal{T}}(t) = \sum_{i\geq 0} \dim_{\mathbb{Q}} A^i(Y,\mathbb{Q})t^i$ [28, Corollary 2.12] and the fact that Y is smooth if and only if $\mathcal{T}$ is a unimodular triangulation.

Remark II.39. The dimensions of the graded pieces of $A^*(V(F),\mathbb{Q})$ are equal to the coefficients of an h-vector of a fan [28, Corollary 2.12]. The analogous combinatorial proof of Stanley's theorem goes as follows: one can express $\delta_P(t)$ and $\delta_Q(t)$ as sums of shifted h-vectors [10, 51], and then apply Stanley's monotonicity theorem for h-vectors [59] to conclude the result.

Remark II.40. Consider the *deformed group ring* $\mathbb{Q}[N\times\mathbb{Z}]^{\triangle} := \oplus_{v\in\sigma\cap(N\times\mathbb{Z})}\mathbb{Q}\cdot y^v$, with ring structure defined by

$$y^v \cdot y^w = \begin{cases} y^{v+w} & \text{if there exists a cone } \tau\in\triangle \text{ with } v,w\in\tau \\ 0 & \text{otherwise.}\end{cases}$$

If $v_1,\ldots,v_t$ denote the vertices of $\mathcal{T}$ and $M = \mathrm{Hom}_{\mathbb{Z}}(N,\mathbb{Z})$, then Jiang and Tseng [37, Theorem 1.1] showed that there is an isomorphism of rings

$$A^*_{\mathrm{orb}}(Y,\mathbb{Q}) \cong \frac{\mathbb{Q}[N\times\mathbb{Z}]^{\triangle}}{\{\sum_{i=1}^t \langle (v_i,1),u\rangle y^{(v_i,1)} \mid u\in M\times\mathbb{Z}\}}.$$

Similarly, if $v_1,\ldots,v_s$ are the vertices of $\mathcal{T}|_Q$ and $M' = \mathrm{Hom}_{\mathbb{Z}}(N',\mathbb{Z})$, then

$$A^*_{\mathrm{orb}}(Y',\mathbb{Q}) \cong \frac{\mathbb{Q}[N'\times\mathbb{Z}]^{\Sigma}}{\{\sum_{i=1}^s \langle (v_i,1),u\rangle y^{(v_i,1)} \mid u\in M'\times\mathbb{Z}\}}.$$

Consider the surjective ring homomorphism $j : \mathbb{Q}[N\times\mathbb{Z}]^{\triangle} \to \mathbb{Q}[N'\times\mathbb{Z}]^{\Sigma}$ satisfying $j(y^v) = y^v$ if $v\in\Sigma$ and $j(y^v) = 0$ if $v\notin\Sigma$. The induced ring homomorphism

$$\frac{\mathbb{Q}[N\times\mathbb{Z}]^{\triangle}}{\{\sum_{i=1}^t \langle (v_i,1),u\rangle y^{(v_i,1)} \mid u\in M\times\mathbb{Z}\}} \longrightarrow \frac{\mathbb{Q}[N'\times\mathbb{Z}]^{\Sigma}}{\{\sum_{i=1}^s \langle (v_i,1),u\rangle y^{(v_i,1)} \mid u\in M'\times\mathbb{Z}\}}$$

corresponds to the restriction map $\iota : A^*_{\text{orb}}(Y, \mathbb{Q}) \to A^*_{\text{orb}}(Y', \mathbb{Q})$ under the above isomorphisms. The existence of such a ring homomorphism was used by Stanley in his original commutative algebra proof of Theorem II.36 [59].

2.5 A Toric Proof of Weighted Ehrhart Reciprocity

We have provided a combinatorial proof of Weighted Ehrhart Reciprocity (Theorem II.20) as well as a geometric proof via orbifold cohomology (Remark II.33). In this section, we give a third proof in the case when Q is a lattice polytope. More specifically, we show that Weighted Ehrhart Reciprocity can be deduced from Serre Duality as well as some vanishing theorems for ample divisors on toric varieties due to Mustaţă [48]. This proof generalises the toric proof of Ehrhart Reciprocity given in Section 4.4 of [25].

Throughout this section we will assume that Σ is a complete fan and $P := Q = \{v \in N_{\mathbb{R}} \mid \psi(v) \leq 1\}$ is a lattice polytope (containing the origin in its interior). By definition, for every rational number $-1 < k \leq 0$ and for any positive integer m,

$$f^0_k(m) - f^0_k(m-1) = |\partial(m+k)P \cap N|. \tag{2.20}$$

Also, $f^0_0(0) = 1$ and $f^0_k(0) = 0$ for $-1 < k < 0$. Our goal is to provide a toric proof of the following version of Weighted Ehrhart Reciprocity.

Theorem II.41. *Let P be a d-dimensional lattice polytope containing the origin in its interior. For every rational number $-1 < k \leq 0$, $f^0_k(m)$ is a polynomial in m of degree at most d and, for any positive integer m,*

$$f^0_k(-m) = \begin{cases} (-1)^d f^0_{-1-k}(m) & \text{if } -1 < k < 0 \\ (-1)^d f^0_k(m-1) & \text{if } k = 0. \end{cases}$$

We first recall some facts about toric varieties and refer the reader to [25] for the relevant details. A d-dimensional lattice polytope P in N, containing the origin in its interior, determines a d-dimensional, projective toric variety Y over $\mathbb{C}$ and an effective ample torus-invariant divisor D on Y. If $M = \operatorname{Hom}(N, \mathbb{Z})$ denotes the dual lattice to N, then the normal fan to P in $M_{\mathbb{R}}$ determines the toric variety Y. Let $u_1, \ldots, u_s$ denote the primitive integer vectors along the rays of the normal fan, corresponding to the torus-invariant prime divisors $D_1, \ldots, D_s$ of Y. If we write $D = \sum_{i=1}^{s} a_i D_i$, then $a_i = -\min_{v \in P \cap N} \langle u_i, v \rangle \in \mathbb{Z}_{>0}$. Given a torus-invariant $\mathbb{Q}$-divisor $E = \sum_{i=1}^{s} b_i D_i$, consider the (possibly empty) polytope

$$P_E := \{v \in N_{\mathbb{R}} \mid \langle u_i, v \rangle + b_i \geq 0 \text{ for } i = 1, \ldots, s\}.$$

It can be verified that $\lambda P = P_{\lambda D}$ for any rational number $\lambda > 0$. Every lattice point v in N corresponds to a character χ^v on the torus contained in Y. In particular, we may view χ^v as a rational function on Y. If we let $\lfloor E \rfloor = \sum_{i=1}^{s} \lfloor b_i \rfloor D_i$ denote the round down of E, then the global sections of E are given by

$$H^0(Y, \mathcal{O}(\lfloor E \rfloor)) = \bigoplus_{v \in P_E \cap N} \mathbb{C}\chi^v. \tag{2.21}$$

In particular, $\dim H^0(Y, \mathcal{O}(\lfloor E \rfloor)) = |P_E \cap N|$. We identify the canonical divisor of Y with $K_Y = -\sum_{i=1}^{s} D_i$ and write $\lceil E \rceil = \sum_{i=1}^{s} \lceil b_i \rceil D_i$ for the round up of E.

Lemma II.42. *If $E = \sum_{i=1}^{s} b_i D_i$ is a torus-invariant $\mathbb{Q}$-divisor on Y then*

$$H^0(Y, \mathcal{O}(K_Y + \lceil E \rceil)) = \bigoplus_{v \in \operatorname{Int} P_E \cap N} \mathbb{C}\chi^v.$$

In particular, for each $-1 < k \leq 0$ and for every positive integer m,

$$f_k^0(m) - f_k^0(m-1) = \dim H^0(Y, \mathcal{O}(\lfloor (m+k)D \rfloor)) - \dim H^0(Y, \mathcal{O}(K_Y + \lceil (m+k)D \rceil)).$$

Proof. Observe that v in N lies in the interior of P_E if and only if

$$\langle u_i, v \rangle + b_i > 0 \text{ for } i = 1, \ldots, s.$$

This condition holds if and only if

$$\langle u_i, v\rangle + \lceil b_i \rceil - 1 \geq 0 \text{ for } i = 1, \dots, s.$$

We conclude that $\operatorname{Int} P_E \cap N = P_{K_Y + \lceil E \rceil} \cap N$ and the first statement follows from (2.21). By (2.20), for each $-1 < k \leq 0$ and for every positive integer m,

$$\begin{aligned} f_k^0(m) - f_k^0(m-1) &= |(m+k)P \cap N| - |\operatorname{Int}(m+k)P \cap N| \\ &= \dim H^0(Y, \mathcal{O}(\lfloor (m+k)D \rfloor)) - \dim H^0(Y, \mathcal{O}(K_Y + \lceil (m+k)D \rceil)). \end{aligned}$$

□

We recall the following vanishing theorem due to Mustaţă. A $\mathbb{Q}$-divisor E on Y is ample if mE is an ample divisor for some positive integer m.

Theorem II.43 (Corollary 2.5 [48])**.** *Let Y be a projective toric variety and let E be an ample torus-invariant $\mathbb{Q}$-divisor on Y. For $i > 0$,*

1. $H^i(Y, \mathcal{O}(K_Y + \lceil E \rceil)) = 0$ *(Kawamata-Viehweg vanishing)*
2. $H^i(Y, \mathcal{O}(\lfloor E \rfloor)) = 0$.

For any divisor D' on Y, let $\chi(Y, D') = \sum_{i \geq 0} (-1)^i \dim H^i(Y, \mathcal{O}(D'))$ denote the Euler characteristic of D'. By the above vanishing theorem and Lemma II.42, for any positive integer m,

$$f_k^0(m) - f_k^0(m-1) = \chi(Y, \mathcal{O}(\lfloor (m+k)D \rfloor)) - \chi(Y, \mathcal{O}(K_Y + \lceil (m+k)D \rceil)). \tag{2.22}$$

The following fact is due to Snapper.

Theorem II.44 ([52])**.** *Let X be a complete variety of dimension d over an algebraically closed field k. If $\mathcal{F}$ is a coherent sheaf on X and $\mathcal{L}$ is a line bundle on X, then there is a polynomial $Q(t)$ of degree at most d such that $Q(m) = \chi(X, \mathcal{F} \otimes \mathcal{L}^m)$ for every integer m.*

If we apply the above result to the coherent sheaf $\mathcal{O}(\lfloor kD \rfloor)$ and the line bundle $\mathcal{O}(D)$ on Y, we deduce that there is a polynomial $Q_1(t)$ of degree at most d such that $Q_1(m) = \chi(Y, \mathcal{O}(\lfloor (m+k)D \rfloor))$ for each integer m. Moreover, the coefficient of t^d in $Q_1(t)$ is the intersection number $\mathrm{rk}(\mathcal{O}(\lfloor kD \rfloor)) \cdot D^d/d! = D^d/d! > 0$ [41]. Similarly, there is a polynomial $Q_2(t)$ of degree d with leading term $D^d/d!$ such that $Q_2(m) = \chi(Y, \mathcal{O}(K_Y + \lceil (m+k)D \rceil))$ for each integer m. By (2.22), for any positive integer m, $f_k^0(m) - f_k^0(m-1) = Q_1(m) - Q_2(m)$. It now follows from standard arguments (see, for example, p. 49 [27]) that $f_k^0(m)$ is a polynomial in m of degree at most d.

We have the following application of Serre Duality.

Lemma II.45. *If* $S_k(m) = f_k^0(m) - f_k^0(m-1)$ *then*

$$(-1)^{d+1} S_k(-m) = \begin{cases} S_{-1-k}(m+1) & \text{if } -1 < k < 0 \\ S_k(m) & \text{if } k = 0. \end{cases}$$

Proof. Since both sides of (2.22) are polynomials in m, $(-1)^{d+1} S_k(-m)$ is equal to

$$-(-1)^d \chi(Y, \mathcal{O}(\lfloor (-m+k)D \rfloor)) + (-1)^d \chi(Y, \mathcal{O}(K_Y + \lceil (-m+k)D \rceil)).$$

By Serre Duality (see, for example, Corollary 3.7.7 [27]), this is equal to

$$-\chi(Y, \mathcal{O}(K_Y - \lfloor (-m+k)D \rfloor)) + \chi(Y, \mathcal{O}(-\lceil (-m+k)D \rceil)).$$

Since for any real number a, $-\lfloor -a \rfloor = \lceil a \rceil$,

$$(-1)^{d+1} S_k(-m) = -\chi(Y, \mathcal{O}(K_Y + \lceil (m-k)D \rceil)) + \chi(Y, \mathcal{O}(\lfloor (m-k)D \rfloor)).$$

When $k = 0$, we get $(-1)^{d+1} S_0(-m) = S_0(m)$. For $-1 < k \leq 0$, after writing $m - k = m + 1 + (-1-k)$, we see that $(-1)^{d+1} S_k(-m) = S_{-1-k}(m+1)$. $\square$

We will now complete our proof of Theorem II.41 by induction on m.

Proof. We have seen that for each $-1 < k \leq 0$, $f_k^0(m)$ is a polynomial in m of degree at most d. We will first prove Theorem II.41 in the case when $m = 1$. Recall that $f_0^0(0) = 1$ and $f_k^0(0) = 0$ for $-1 < k < 0$. By (2.22) and Serre Duality,

$$f_0^0(0) - f_0^0(-1) = \chi(Y, \mathcal{O}_Y) - \chi(Y, \mathcal{O}(K_Y)) = (1 - (-1)^d)\chi(Y, \mathcal{O}_Y).$$

It follows from Theorem II.43 that $\chi(Y, \mathcal{O}_Y) = 1$ and we conclude that $f_0^0(-1) = (-1)^d = (-1)^d f_0^0(0)$ as desired. When $-1 < k < 0$, Lemma II.45 implies that $(-1)^{d+1} S_k(0) = S_{-1-k}(1)$. That is,

$$(-1)^{d+1}(f_k^0(0) - f_k^0(-1)) = f_{-1-k}^0(1) - f_{-1-k}^0(0),$$

and hence $(-1)^d f_k^0(-1) = f_{-1-k}^0(1)$. This completes the proof when $m = 1$.

Now consider the case when $m > 1$. By Lemma II.45 and induction on m,

$$\begin{aligned}(-1)^d f_0^0(-m) &= (-1)^{d+1} S_0(-m+1) + (-1)^d f_0^0(-(m-1)) \\ &= S_0(m-1) + f_0^0(m-2) \\ &= f_0^0(m-1).\end{aligned}$$

Similarly, when $-1 < k < 0$,

$$\begin{aligned}(-1)^d f_k^0(-m) &= (-1)^{d+1} S_k(-m+1) + (-1)^d f_k^0(-(m-1)) \\ &= S_{-1-k}(m) + f_{-1-k}^0(m-1) \\ &= f_{-1-k}^0(m).\end{aligned}$$

□

CHAPTER III

Motivic Integration on Toric Stacks

3.1 Toric Stacks

We continue with the notation from Chapter II. Associated to a stacky fan $\mathbf{\Sigma} = (N, \Sigma, \{b_i\})$, there is a smooth Deligne-Mumford toric stack $\mathcal{X} = \mathcal{X}(\mathbf{\Sigma})$ over $\mathbb{C}$ with coarse moduli space X [12]. Each cone σ in Σ corresponds to an open substack $\mathcal{X}(\boldsymbol{\sigma})$ of $\mathcal{X}$. We identify $\mathcal{X}(\{\mathbf{0}\})$ with the torus T of X. The open substacks $\{\mathcal{X}(\boldsymbol{\sigma}) \mid \dim \sigma = d\}$ give an open covering of $\mathcal{X}$. We will give an explicit description of $\mathcal{X}(\boldsymbol{\sigma})$, for a fixed cone σ of dimension d.

If N_σ denotes the sublattice of N generated by $\{b_i \mid \rho_i \subseteq \sigma\}$, then $N(\sigma) = N/N_\sigma$ is a finite group with elements in bijective correspondence with $\coprod_{\tau \subseteq \sigma} \operatorname{Box}(\boldsymbol{\tau})$. Let M_σ be the dual lattice of N_σ and let M be the dual lattice of N. If σ' denotes the cone in N_σ generated by $\{b_i \mid \rho_i \subseteq \sigma\}$, with corresponding dual cone $(\sigma')^\vee$ in M_σ, then we will make the identifications

$$\operatorname{Spec} \mathbb{C}[(\sigma')^\vee \cap M_\sigma] \cong \mathbb{A}^d, \; \operatorname{Hom}_{\mathbb{Z}}(M_\sigma, \mathbb{C}^*) \cong (\mathbb{C}^*)^d.$$

If we regard $\mathbb{Q}/\mathbb{Z}$ as a subgroup of $\mathbb{C}^*$ by sending p to $\exp(2\pi\sqrt{-1}p)$, then we have a natural isomorphism

$$N(\sigma) \cong \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Hom}_{\mathbb{Z}}(N(\sigma), \mathbb{Q}/\mathbb{Z}), \mathbb{C}^*) = \operatorname{Hom}_{\mathbb{Z}}(\operatorname{Ext}^1_{\mathbb{Z}}(N(\sigma), \mathbb{Z}), \mathbb{C}^*). \tag{3.1}$$

We apply the functor $\mathrm{Hom}_{\mathbb{Z}}(\ ,\mathbb{Z})$ to the exact sequence

$$0 \to N_\sigma \to N \to N(\sigma) \to 0,$$

to get

$$0 \to M \to M_\sigma \to \mathrm{Ext}^1_{\mathbb{Z}}(N(\sigma),\mathbb{Z}) \to 0.$$

Applying $\mathrm{Hom}(\ ,\mathbb{C}^*)$ and the natural isomorphism (3.1), gives an injection $N(\sigma) \to (\mathbb{C}^*)^d$. The natural action of $(\mathbb{C}^*)^d$ on $\mathbb{A}^d$ induces an action of $N(\sigma)$ on $\mathbb{A}^d$. We identify $\mathcal{X}(\boldsymbol{\sigma})$ with the quotient stack $[\mathbb{A}^d/N(\sigma)]$, with corresponding coarse moduli space the open subscheme $U_\sigma = \mathbb{A}^d/N(\sigma)$ of X [12].

Consider an element g in $N(\sigma)$ of order l corresponding to v in $\mathrm{Box}(\boldsymbol{\sigma(v)})$, where $\sigma(v)$ denotes the cone containing v in its relative interior. We can write $v = \sum_{i=1}^d q_i b_i$, for some $0 \leq q_i < 1$. Note that $q_i \neq 0$ if and only if $\rho_i \subseteq \sigma(v)$. If $x_1,\ldots,x_d$ are the coordinates on $\mathbb{A}^d = \mathrm{Spec}\,\mathbb{C}[(\sigma')^\vee \cap M_\sigma]$ and $\zeta_l = \exp(2\pi\sqrt{-1}/l)$, then one can verify that the action of $N(\sigma)$ on $\mathbb{A}^d$ is given by

$$g \cdot (x_1,\ldots,x_d) = (\zeta_l^{lq_1} x_1,\ldots,\zeta_l^{lq_d} x_d), \tag{3.2}$$

and the *age* of g (see Subsection 7.1 [1]) is equal to

$$\mathrm{age}(g) = (1/l)\sum_{i=1}^d lq_i = \psi(v). \tag{3.3}$$

3.2 Twisted Jets of Toric Stacks

We use the discussion of twisted jets of Deligne-Mumford stacks in [67] to give an explicit description of the toric case. More specifically, for any non-negative integer n, we describe the stack of twisted n-jets $\mathcal{J}_n\mathcal{X}$ of the toric stack $\mathcal{X}$.

Fix an affine scheme $S = \mathrm{Spec}\,R$ over $\mathbb{C}$ and let $D_{n,S}$ denote the affine scheme $\mathrm{Spec}\,R[t]/(t^{n+1})$. If we fix a positive integer l and consider the group μ_l of lth roots of unity with generator $\zeta_l = \exp(2\pi\sqrt{-1}/l)$, then μ_l acts on $D_{nl,S}$ via the morphism $p : D_{nl,S} \times \mu_l \to D_{nl,S}$, corresponding to the ring homomorphism $R[t]/(t^{nl+1}) \to R[t]/(t^{nl+1}) \otimes \mathbb{C}[x]/(x^l - 1), t \mapsto t \otimes x$. That is, μ_l acts on $D_{nl,S}$ by scaling t. If $\mathcal{D}^l_{n,S}$ denotes the quotient stack $[D_{nl,S}/\mu_l]$, then we have morphisms

$$D_{nl,S} \xrightarrow{\pi} \mathcal{D}^l_{n,S} \to D_{n,S},$$

such that π is an atlas for $\mathcal{D}^l_{n,S}$ and $D_{n,S}$ is the coarse moduli space of $\mathcal{D}^l_{n,S}$. The composition of the two maps is the quotient of $D_{nl,S}$ by μ_l and corresponds to the ring homomorphism $R[t]/(t^{n+1}) \to R[t]/(t^{nl+1})$, $t \mapsto t^l$. The atlas π corresponds to the object α of $\mathcal{D}^l_{n,S}$ over $D_{nl,S}$

$$\begin{tikzcd} D_{nl,S} \times \mu_l \arrow[r, "p"] \arrow[d, "pr_1"] & D_{nl,S} \\ D_{nl,S} & \end{tikzcd}$$

and every object in $\mathcal{D}^l_{n,S}$ is locally a pullback of α. Consider the automorphism

$$\theta = \zeta_l \times \zeta_l^{-1} : D_{nl,S} \times \mu_l \to D_{nl,S} \times \mu_l$$

over $\zeta_l : D_{nl,S} \to D_{nl,S}$. Every automorphism of α is a power of θ and hence every object and automorphism in $\mathcal{D}^l_{n,S}$ is locally determined by a pullback of α and a power of θ.

A *twisted n-jet of order l* of $\mathcal{X}$ over S is a representable morphism $\mathcal{D}^l_{n,S} \to \mathcal{X}$. By the above discussion, a twisted jet is determined by the images of α and θ. Yasuda defined the stack $\mathcal{J}^l_n\mathcal{X}$ of twisted n-jets of order l of $\mathcal{X}$. An object of $\mathcal{J}^l_n\mathcal{X}$ over S is a twisted n-jet $\gamma : \mathcal{D}^l_{n,S} \to \mathcal{X}$ of order l. Suppose $\gamma' : \mathcal{D}^l_{n,T} \to \mathcal{X}$ is another twisted n-jet of order l. If $f : S \to T$ is a morphism, there is an induced morphism $f' : \mathcal{D}^l_{n,S} \to \mathcal{D}^l_{n,T}$. A morphism in $\mathcal{J}^l_n\mathcal{X}$ from γ to γ' over $f : S \to T$ is a 2-morphism from γ to $\gamma' \circ f'$. The stack $\mathcal{J}_n\mathcal{X}$ of twisted n-jets is the disjoint union of the stacks

$\mathcal{J}_n^l\mathcal{X}$ as l varies over the positive integers. Both $\mathcal{J}_n\mathcal{X}$ and the $\mathcal{J}_n^l\mathcal{X}$ are smooth Deligne-Mumford stacks (Theorem 2.9 [67]). The stack $\mathcal{J}_0\mathcal{X}$ is identified with the *inertia stack* $\mathcal{I}(\mathcal{X})$ of $\mathcal{X}$. That is, an object of $\mathcal{J}_0\mathcal{X}$ over S is determined by a pair (x, α), where x is an object of $\mathcal{X}$ over S and α is an automorphism of x. We identify the stack $\mathcal{J}_0^1\mathcal{X}$ with $\mathcal{X}$. For any $m \geq n$ and for each $l > 0$, the truncation map $R[t]/(t^{ml+1}) \to R[t]/(t^{nl+1})$ induces a morphism $\mathcal{D}_{n,S}^l \to \mathcal{D}_{m,S}^l$. Via composition, we get a natural affine morphism (Theorem 2.9 [67])

$$\pi_n^m : \mathcal{J}_m\mathcal{X} \to \mathcal{J}_n\mathcal{X}.$$

The projective system $\{\mathcal{J}_n\mathcal{X}\}_n$ has a projective limit with a projection morphism (p15 [67])

$$\mathcal{J}_\infty\mathcal{X} = \lim_{\leftarrow} \mathcal{J}_n\mathcal{X}.$$

$$\pi_n : \mathcal{J}_\infty\mathcal{X} \to \mathcal{J}_n\mathcal{X}.$$

Remark III.1. The open covering $\{\mathcal{X}(\boldsymbol{\sigma}) \mid \dim \sigma = d\}$ of $\mathcal{X}$ induces an open covering $\{\mathcal{J}_n\mathcal{X}(\boldsymbol{\sigma}) \mid \dim \sigma = d\}$ of $\mathcal{J}_n\mathcal{X}$, for $0 \leq n \leq \infty$.

Recall that if $\mathcal{Y}$ is a stack over $\mathbb{C}$, then we can consider the set of *points* $|\mathcal{Y}|$ of $\mathcal{Y}$ over $\mathbb{C}$ [43]. Its elements are equivalence classes of morphisms from $\operatorname{Spec}\mathbb{C}$ to $\mathcal{Y}$. Two morphisms $\psi, \psi' : \operatorname{Spec}\mathbb{C} \to \mathcal{Y}$ are equivalent if there is a 2-morphism from ψ to ψ'. It has a Zariski topology; for every open substack $\mathcal{Y}'$ of $\mathcal{Y}$, $|\mathcal{Y}'|$ is an open subset of $|\mathcal{Y}|$. If $\mathcal{Y}$ has a coarse moduli space Y, then $|\mathcal{Y}|$ is homeomorphic to $Y(\mathbb{C})$. We will often identify Y with $Y(\mathbb{C})$.

Let $D_{\infty,\mathbb{C}} = \operatorname{Spec}\mathbb{C}[[t]]$ and $\mathcal{D}_{\infty,\mathbb{C}}^l = [D_{\infty,\mathbb{C}}/\mu_l]$. A *twisted arc* of order l of $\mathcal{X}$ over $\mathbb{C}$ is a representable morphism from $\mathcal{D}_{\infty,\mathbb{C}}^l$ to $\mathcal{X}$. Two twisted arcs $\alpha, \alpha' : \mathcal{D}_{\infty,\mathbb{C}}^l \to \mathcal{X}$ of order l are equivalent if there is a 2-morphism from α to α'. The set of equivalence classes of twisted arcs over $\mathcal{X}$ is identified with $|\mathcal{J}_\infty\mathcal{X}|$ (p16 [67]), which we will call the *space of twisted arcs* of $\mathcal{X}$.

For $0 \leq n \leq \infty$ and a positive integer l, the scheme $D_{nl,\mathbb{C}}$ has a unique closed point. A (not necessarily representable) morphism $\gamma : \mathcal{D}^l_{n,\mathbb{C}} \to \mathcal{X}$ induces an object $\bar{\gamma}$ of $\mathcal{X}$ over $\mathbb{C}$,

$$\bar{\gamma} : \operatorname{Spec}\mathbb{C} \to D_{nl,\mathbb{C}} \to \mathcal{D}^l_{n,\mathbb{C}} \to \mathcal{X}.$$

The automorphism group of the object in $\mathcal{D}^l_{n,\mathbb{C}}$ corresponding to the morphism $\operatorname{Spec}\mathbb{C} \to \mathcal{D}^l_{n,\mathbb{C}}$ is μ_l. Hence we get a homomorphism $\phi : \mu_l \to \operatorname{Aut}(\bar{\gamma})$. Since every automorphism in $\mathcal{D}^l_{n,\mathbb{C}}$ is locally induced by θ, ϕ is injective if and only if $\operatorname{Aut}(\chi) \to \operatorname{Aut}(\gamma(\chi))$ is injective for all objects χ in $\mathcal{D}^l_{n,\mathbb{C}}$. This holds if and only if γ is representable [43]. We conclude that γ is representable if and only if ϕ is injective.

By considering coarse moduli spaces we have a commutative diagram

$$\begin{tikzcd} \mathcal{D}^l_{n,\mathbb{C}} \arrow[r, "\gamma"] \arrow[d] & \mathcal{X} \arrow[d] \\ D_{n,\mathbb{C}} \arrow[r, "\gamma'"] & X. \end{tikzcd}$$

We will consider the nth *jet scheme* $J_n(X) = \operatorname{Hom}(D_{n,\mathbb{C}}, X)$ of X and identify jet schemes with their $\mathbb{C}$-valued points. When $n = \infty$, $J_\infty(X)$ is called the *arc space* of X. We have a map

$$\tilde{\pi}_n : |\mathcal{J}_n\mathcal{X}| \to J_n(X) \tag{3.4}$$

$$\tilde{\pi}_n(\gamma) = \gamma'.$$

Fix a d-dimensional cone σ of Σ and consider the open substack $\mathcal{X}(\boldsymbol{\sigma}) = [\mathbb{A}^d/N(\sigma)]$ of $\mathcal{X}$. A twisted n-jet $\mathcal{D}^l_{n,S} \to \mathcal{X}(\boldsymbol{\sigma})$ can be lifted to a morphism between atlases, $D_{nl,S} \to \mathbb{A}^d$. Yasuda used these lifts to describe the stack $\mathcal{J}^l_n\mathcal{X}$ (Proposition 2.8 [67]). We will present his result and a sketch of the proof in our situation. We first fix some notation. The action of μ_l on $D_{nl,\mathbb{C}}$ extends to an action on $J_{nl}(\mathbb{A}^d)$. The action of $N(\sigma)$ on $\mathbb{A}^d$ also extends to an action on $J_{nl}(\mathbb{A}^d)$. If g is an element of $N(\sigma)$ of order l, let $J_{nl}^{(g)}(\mathbb{A}^d)$ be the closed subscheme of $J_{nl}(\mathbb{A}^d)$ on which the actions of ζ_l in μ_l and g in $N(\sigma)$ agree.

Lemma III.2 ([67])**.** *For $0 \le n \le \infty$, we have a homeomorphism*

$$|\mathcal{J}_n\mathcal{X}(\boldsymbol{\sigma})| \cong \coprod_{g\in N(\sigma)} J_{nl}^{(g)}(\mathbb{A}^d)/N(\sigma).$$

Proof. We will only show that there is a bijection between the two sets. We have noted that a representable morphism $\gamma : \mathcal{D}^l_{n,\mathbb{C}} \to [\mathbb{A}^d/N(\sigma)]$ is determined by the images of α and θ. These images have the form

$$\begin{tikzcd}
 & & \mathbb{A}^d \\
D_{nl,\mathbb{C}} \times N(\sigma) \arrow[r, "\zeta_l \times g^{-1}"'] \arrow[urr, "(v,\lambda)\mapsto\nu(v)\cdot\lambda"] \arrow[d, "pr_1"] & D_{nl,\mathbb{C}} \times N(\sigma) \arrow[ur, "(v,\lambda)\mapsto\nu(v)\cdot\lambda"'] \arrow[d, "pr_1"] & \\
D_{nl,\mathbb{C}} \arrow[r, "\zeta_l"] & D_{nl,\mathbb{C}} &
\end{tikzcd}$$

for some g in $N(\sigma)$ of order l and some nl-jet $\nu : D_{nl,\mathbb{C}} \to \mathbb{A}^d$. Conversely, given such a diagram, we can construct a representable morphism. The diagram is determined by any choice of g and ν satisfying

$$\begin{tikzcd}
D_{nl,\mathbb{C}} \arrow[r, "\nu"] \arrow[d, "\zeta_l"] & \mathbb{A}^d \arrow[d, "g"] \\
D_{nl,\mathbb{C}} \arrow[r, "\nu"] & \mathbb{A}^d.
\end{tikzcd}$$

That is, γ is determined by the pair (g, ν), where g has order l and ν lies in $J_{nl}^{(g)}(\mathbb{A}^d)$. Suppose γ' is a twisted n-jet of order l determined by the pair (g', ν'). A 2-morphism from γ to γ' is determined by a morphism $\beta : \gamma(\alpha) \to \gamma'(\alpha)$ in $[\mathbb{A}^d/N(\sigma)]$ over the identity morphism on $D_{nl,\mathbb{C}}$, such that the following diagram commutes

$$\begin{tikzcd} \gamma(\alpha) \arrow[r, "\beta"] \arrow[d, "\gamma(\theta)"] & \gamma'(\alpha) \arrow[d, "\gamma'(\theta)"] \\ \gamma(\alpha) \arrow[r, "\beta"] & \gamma'(\alpha). \end{tikzcd}$$

One verifies that the morphism β is determined by an element h in $N(\sigma)$ such that $\nu = \nu' \cdot h$ and that the diagram above is commutative if and only if $g = g'$. Hence the equivalence class of γ in $|\mathcal{J}_n\mathcal{X}(\boldsymbol{\sigma})|$ corresponds to a point in $J_{nl}^{(g)}(\mathbb{A}^d)/N(\sigma)$. This gives our desired bijection.

$\square$

Remark III.3. Borisov, Chen and Smith gave a decomposition of $|\mathcal{I}\mathcal{X}|$ into connected components indexed by $\mathrm{Box}(\boldsymbol{\Sigma})$ (Proposition 4.7 [12]). Given v in $\mathrm{Box}(\boldsymbol{\Sigma})$, let $\sigma(v)$ be the cone containing v in its relative interior and let $\Sigma_{\sigma(v)}$ be the simplicial fan in $(N/N_{\sigma(v)})_{\mathbb{R}}$ with cones given by the projections of the cones in Σ containing $\sigma(v)$. The associated toric variety $X(\Sigma_{\sigma(v)})$ is a T-invariant closed subvariety of X. The connected component of $|\mathcal{I}\mathcal{X}|$ corresponding to v is homeomorphic to the simplicial toric variety $X(\Sigma_{\sigma(v)})$. By taking $n = 0$ in Lemma III.2, we recover this decomposition for $|\mathcal{I}\mathcal{X}(\boldsymbol{\sigma})|$.

Consider an element g in $N(\sigma)$ of order l corresponding to v in $\mathrm{Box}(\boldsymbol{\tau})$, for some cone τ contained in σ. We will give an explicit description of $J_{nl}^{(g)}(\mathbb{A}^d)$. If we write $v = \sum_{i=1}^d q_i b_i$, for some $0 \leq q_i < 1$, then recall from (3.2) that the action of g on $\mathbb{A}^d$ is given by

$$g \cdot (x_1, \ldots, x_d) = (\zeta_l^{lq_1} x_1, \ldots, \zeta_l^{lq_d} x_d).$$

An element ν of $J_{nl}(\mathbb{A}^d)$ can be written in the form

$$\nu = (\sum_{j=0}^{nl} \alpha_{1,j} t^j, \ldots, \sum_{j=0}^{nl} \alpha_{d,j} t^j),$$

for some $\alpha_{i,j}$ in $\mathbb{C}$, $1 \leq i \leq d$, $0 \leq j \leq nl$. We have

$$g \cdot \nu = (\, \zeta_l^{lq_1} \sum_{j=0}^{nl} \alpha_{1,j} t^j \,, \ldots ,\, \zeta_l^{lq_d} \sum_{j=0}^{nl} \alpha_{d,j} t^j \,)$$

$$\zeta_l \cdot \nu = (\, \sum_{j=0}^{nl} \alpha_{1,j} \zeta_l^j t^j \,, \ldots ,\, \sum_{j=0}^{nl} \alpha_{d,j} \zeta_l^j t^j \,).$$

Hence v lies in $J_{nl}^{(g)}(\mathbb{A}^d)$ if and only if $\alpha_{i,j} = 0$ whenever $j \not\equiv lq_i \pmod{l}$. We conclude that

$$(3.5) \qquad J_{nl}^{(g)}(\mathbb{A}^d) = \{ (\, \sum_{j=0}^{nl} \alpha_{1,j} t^j \,, \ldots ,\, \sum_{j=0}^{nl} \alpha_{d,j} t^j \,) \mid \alpha_{i,j} = 0 \text{ if } j \not\equiv lq_i \pmod{l} \}.$$

3.3 Twisted Arcs of Toric Stacks

In this section we describe an action of $J_\infty(T)$ on an open, dense subset of $|\mathcal{J}_\infty \mathcal{X}|$ and compute the corresponding orbits and orbit closures.

We first recall Ishii's decomposition of the arc space of the toric variety X [36]. Recall that $D_1, \ldots, D_r$ denote the T-invariant prime divisors of X and let $J_\infty(X)' = J_\infty(X) \smallsetminus \cup_i J_\infty(D_i)$. The action of T on X extends to an action of $J_\infty(T)$ on $J_\infty(X)'$ (Proposition 2.6 [36]). Given an arc γ in $J_\infty(X)'$, we can find a d-dimensional cone σ such that $\gamma : \operatorname{Spec} \mathbb{C}\,[[t]] \to U_\sigma \subseteq X$ corresponds to the ring homomorphism $\gamma^\# : \mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}\,[[t]]$. We have an induced semigroup morphism $\sigma^\vee \cap M \to \mathbb{N}$, $u \mapsto \operatorname{ord}_t \gamma^\#(\chi^u)$, which extends to homomorphism from M to $\mathbb{Z}$, necessarily of the form $\langle \,, v \rangle$, for some v in $\sigma \cap N$. Consider the arc

$$\gamma_v : \operatorname{Spec} \mathbb{C}\,[[t]] \to U_\sigma \subseteq X$$

corresponding to the ring homomorphism

$$\gamma_v^\# : \mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}\,[[t]]$$

$$\chi^u \mapsto t^{\langle u, v \rangle}.$$

If ϕ denotes the arc in $J_\infty(T)$ corresponding to the ring homomorphism $\phi^\# : \mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[[t]]$, $\chi^u \mapsto \gamma^\#(\chi^u)/t^{\langle u,v\rangle}$, then $\gamma = \gamma_v \cdot \phi$ and both v in $|\Sigma| \cap N$ and ϕ in $J_\infty(T)$ are uniquely determined. We have shown the first part of the following theorem.

Theorem III.4 ([36]). *We have a decomposition of $J_\infty(X)'$ into $J_\infty(T)$-orbits*

$$J_\infty(X)' = \coprod_{v \in |\Sigma| \cap N} \gamma_v \cdot J_\infty(T).$$

Moreover, $\gamma_w \cdot J_\infty(T) \subseteq \overline{\gamma_v \cdot J_\infty(T)}$ if and only if $w - v$ lies in some cone σ containing v and w.

We will give a similar decomposition of the space $|\mathcal{J}_\infty \mathcal{X}|$ of twisted arcs of $\mathcal{X}$. For $i = 1, \ldots, r$, we first describe a closed substack $\mathcal{D}_i$ of $\mathcal{X}$ with coarse moduli space D_i [12]. Fix a maximal cone σ. If ρ_i is not in σ then $\mathcal{D}_i$ is disjoint from $\mathcal{X}(\boldsymbol{\sigma})$. Suppose ρ_i is a ray of σ and consider the projection $p_i : N \to N(\rho_i) = N/N_{\rho_i}$, where N_{ρ_i} is the sublattice of N generated by b_i. If σ_i denotes the cone $p_i(\sigma)$ in $N(\rho_i)$, then $(N(\rho_i), \sigma_i, \{p_i(b_j)\}_{\rho_j \subseteq \sigma})$ is the stacky fan corresponding to $\mathcal{D}_i \cap \mathcal{X}(\boldsymbol{\sigma})$[1]. Note that p_i induces an isomorphism between $N(\sigma) = N/N_\sigma$ and $N(\rho_i)(\sigma_i) = N(\rho_i)/N(\rho_i)_{\sigma_i}$. We have an inclusion of $\mathbb{A}^{d-1}$ into $\mathbb{A}^d$ by setting the coordinate corresponding to ρ_i to be zero. We conclude that $\mathcal{D}_i \cap \mathcal{X}(\boldsymbol{\sigma}) \cong [\mathbb{A}^{d-1}/N(\sigma)]$ and the inclusion of $\mathbb{A}^{d-1}$ into $\mathbb{A}^d$ induces the inclusion of $\mathcal{D}_i \cap \mathcal{X}(\boldsymbol{\sigma})$ into $\mathcal{X}(\boldsymbol{\sigma})$. Moreover, by Lemma III.2, if g is an element of $N(\sigma)$, then we have an induced inclusion

$$J_\infty^{(g)}(\mathbb{A}^{d-1})/N(\sigma) \hookrightarrow J_\infty^{(g)}(\mathbb{A}^d)/N(\sigma), \tag{3.6}$$

corresponding to the closed inclusion $|\mathcal{J}_\infty(\mathcal{D}_i \cap \mathcal{X}(\boldsymbol{\sigma}))| \hookrightarrow |\mathcal{J}_\infty \mathcal{X}(\boldsymbol{\sigma})|$. We define

[1]Note that $N(\rho_i)$ may have torsion and so σ_i is a cone in the image of $N(\rho_i)$ in $N(\rho_i)_\mathbb{Q}$. There are no difficulties in generalising to this situation. See Section 3.7 for a discussion of this issue. This is the level of generality used in [12].

$|\mathcal{J}_\infty \mathcal{X}|'$ to be the open subset

$$|\mathcal{J}_\infty \mathcal{X}|' = |\mathcal{J}_\infty \mathcal{X}| \smallsetminus \cup_{i=1}^r |\mathcal{J}_\infty \mathcal{D}_i|.$$

Similarly, we consider $|\mathcal{J}_\infty \mathcal{X}(\boldsymbol{\sigma})|'$ and let $J_\infty(\mathbb{A}^d)'$ be the open locus of arcs of $\mathbb{A}^d$ that are not contained in a coordinate hyperplane. Setting $J_\infty^{(g)}(\mathbb{A}^d)' = J_\infty^{(g)}(\mathbb{A}^d) \cap J_\infty(\mathbb{A}^d)'$, it follows from Lemma III.2 and (3.6) that

$$|\mathcal{J}_\infty \mathcal{X}(\boldsymbol{\sigma})|' \cong \coprod_{g \in N(\sigma)} J_\infty^{(g)}(\mathbb{A}^d)'/N(\sigma). \tag{3.7}$$

We will often make this identification.

For a fixed positive integer l, the open embedding of T in $\mathcal{X}(\boldsymbol{\sigma})$ extends to an open embedding of $J_\infty(T) \cong \mathcal{J}_\infty^l T$ in $\mathcal{J}_\infty^l \mathcal{X}(\boldsymbol{\sigma})$. We obtain an action of $J_\infty(T)$ on $|\mathcal{J}_\infty^l \mathcal{X}(\boldsymbol{\sigma})|$, which restricts to an action on $|\mathcal{J}_\infty^l \mathcal{X}(\boldsymbol{\sigma})|'$. More specifically, we identify $J_\infty(T) \cong J_\infty^{(l)}(T)/N(\sigma)$, where $J_\infty^{(l)}(T)$ is the closed subscheme of $J_\infty(T)$ fixed by the action of μ_l. In coordinates,

$$J_\infty^{(l)}(T) = (\sum_{j=0}^\infty \beta_{1,j} t^{lj}, \dots, \sum_{j=0}^\infty \beta_{d,j} t^{lj}) \mid \beta_{i,0} \neq 0 \text{ for } i = 1, \dots, d\}.$$

Fix an element g in $N(\sigma)$ of order l and recall from (3.5) that

$$J_\infty^{(g)}(\mathbb{A}^d) = \{(\sum_{j=0}^\infty \alpha_{1,j} t^j, \dots, \sum_{j=0}^\infty \alpha_{d,j} t^j) \mid \alpha_{i,j} = 0 \text{ if } j \not\equiv lq_i \pmod{l}\}.$$

The elements in $J_\infty^{(g)}(\mathbb{A}^d)'$ also satisfy the property that for each $i = 1, \dots, d$, there exists a non-negative integer j such that $\alpha_{i,j} \neq 0$. Componentwise multiplication of power series gives an action,

$$J_\infty^{(l)}(T)/N(\sigma) \times J_\infty^{(g)}(\mathbb{A}^d)'/N(\sigma) \to J_\infty^{(g)}(\mathbb{A}^d)'/N(\sigma).$$

By (3.7), this induces an action of $J_\infty(T)$ on $|\mathcal{J}_\infty^l \mathcal{X}(\boldsymbol{\sigma})|'$. By varying l, we obtain an action of $J_\infty(T)$ on $|\mathcal{J}_\infty \mathcal{X}|'$. We would like to describe the $J_\infty(T)$-orbits.

Let w be an element of $\sigma \cap N$. There is a unique decomposition $w = v + \sum_{i=1}^{d} \lambda_i b_i$, where v lies in Box($\boldsymbol{\tau}$), for some $\tau \subseteq \sigma$, and the λ_i are non-negative integers. We will use the notation $\{w\} = v$. Suppose v corresponds to an element g in $N(\sigma)$ of order l. If we write $v = \sum_{i=1}^{d} q_i b_i$, for some $0 \leq q_i < 1$, then $w = \sum_{i=1}^{d} w_i b_i = \sum_{i=1}^{d} (\lambda_i + q_i) b_i$, where $w_i = \lambda_i + q_i$. We define $\tilde{\gamma}_w \in |\mathcal{J}_{\infty}^{l} \mathcal{X}(\boldsymbol{\sigma})|' \subseteq |\mathcal{J}_{\infty} \mathcal{X}|'$ to be the equivalence class of twisted arcs corresponding to the element

$$(t^{l(\lambda_1 + q_1)}, \ldots, t^{l(\lambda_d + q_d)}) = (t^{lw_1}, \ldots, t^{lw_d}) \tag{3.8}$$

of $J_{\infty}^{(g)}(\mathbb{A}^d)'$, under the isomorphism (3.7).

Theorem III.5. *We have a decomposition of $|\mathcal{J}_{\infty}\mathcal{X}|'$ into $J_{\infty}(T)$-orbits*

$$|\mathcal{J}_{\infty}\mathcal{X}|' = \coprod_{v \in |\Sigma| \cap N} \tilde{\gamma}_v \cdot J_{\infty}(T).$$

Moreover, $\tilde{\gamma}_w \cdot J_{\infty}(T) \subseteq \overline{\tilde{\gamma}_v \cdot J_{\infty}(T)}$ if and only if $w - v = \sum_{\rho_i \subseteq \sigma} \lambda_i b_i$ for some non-negative integers λ_i and some cone σ containing v and w. With the notation of Theorem III.4 and (3.4),

$$\tilde{\pi}_{\infty} : |\mathcal{J}_{\infty}\mathcal{X}|' \rightarrow J_{\infty}(X)'$$

is a $J_{\infty}(T)$-equivariant bijection satisfying

$$\tilde{\pi}_{\infty}(\tilde{\gamma}_v) = \gamma_v.$$

Proof. Let σ be a d-dimensional cone in Σ and let g be an element in $N(\sigma)$ of order l corresponding to an element v in Box($\boldsymbol{\tau}$), for some $\tau \subseteq \sigma$. By (3.5), and with the notation of the previous discussion, we have a decomposition

$$J_{\infty}^{(g)}(\mathbb{A}^d)' = \coprod_{\substack{w \in \sigma \cap N \\ \{w\} = v}} (t^{lw_1}, \ldots, t^{lw_d}) \cdot J_{\infty}^{(l)}(T),$$

and hence

$$J_\infty^{(g)}(\mathbb{A}^d)'/N(\sigma) \cong \coprod_{\substack{w\in\sigma\cap N\\ \{w\}=v}} \tilde{\gamma}_w \cdot J_\infty(T).$$

Also, $\tilde{\gamma}_w \cdot J_\infty(T) \subseteq \overline{\tilde{\gamma}_{w'} \cdot J_\infty(T)}$ in $J_\infty^{(g)}(\mathbb{A}^d)'/N(\sigma)$ if and only if $w_i \geq w'_i$ for $i = 1, \ldots, d$. We conclude that

$$|\mathcal{J}_\infty \mathcal{X}(\boldsymbol{\sigma})|' = \coprod_{w\in\sigma\cap N} \tilde{\gamma}_w \cdot J_\infty(T),$$

and $\tilde{\gamma}_w \cdot J_\infty(T) \subseteq \overline{\tilde{\gamma}_{w'} \cdot J_\infty(T)}$ in $|\mathcal{J}_\infty \mathcal{X}(\boldsymbol{\sigma})|'$ if and only if $\{w\} = \{w'\}$ and $w_i \geq w'_i$ for $i = 1, \ldots, d$. This is equivalent to $w - w' = \sum_{i=1}^d \lambda_i b_i$ for some non-negative integers λ_i. Since the open subsets $|\mathcal{J}_\infty \mathcal{X}(\boldsymbol{\sigma})|'$ cover $|\mathcal{J}_\infty \mathcal{X}|'$ as σ varies over all maximal cones, we obtain the desired decomposition and closure relations.

Consider the sublattice $N_\sigma \subseteq N$ and recall that σ' is the cone in N_σ generated by $b_1, \ldots, b_d$. If H denotes the quotient of N by the sublattice generated by $v_1, \ldots, v_d$, then we have a pairing $\langle\ ,\ \rangle : N \times M_\sigma \to \mathbb{Q}$, $\langle v_i, u_j \rangle = \delta_{i,j}|H|$, where $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise. If $u_1, \ldots, u_d$ are the primitive integer generators of $\sigma^\vee$ in M, then $u_1/a_1|H|, \ldots, u_d/a_d|H|$ are the primitive integer generators of $(\sigma')^\vee$ in M_σ. We have made an identification throughout that $\operatorname{Spec} \mathbb{C}[(\sigma')^\vee \cap M_\sigma] \cong \mathbb{A}^d$. Consider an element γ in $J_\infty^{(g)}(\mathbb{A}^d)'$, for some g in $N(\sigma)$ of order l, corresponding to an element of $|\mathcal{J}_\infty \mathcal{X}(\boldsymbol{\sigma})|'$. Applying $\tilde{\pi}_\infty$ gives an arc in U_σ, which we denote by $\tilde{\pi}_\infty(\gamma)$. We have a commutative diagram

$$\begin{tikzcd}
\mathbb{C}[(\sigma')^\vee \cap M_\sigma] \arrow[r, "\gamma^\#"] & \mathbb{C}\,[[t]] \\
\mathbb{C}[\sigma^\vee \cap M] \arrow[u] \arrow[r, "\tilde{\pi}_\infty(\gamma)^\#"] & \mathbb{C}\,[[t]]. \arrow[u, "t \mapsto t^l"']
\end{tikzcd}$$

It follows that $\tilde{\pi}_\infty$ is $J_\infty(T)$-equivariant. Let w be an element of $\sigma \cap N$ and consider the notation of (3.8). With a slight abuse of notation, $(\tilde{\gamma}_w)^\#(\chi^{u_i/a_i|H|}) = t^{lw_i}$, and we compute

$$\tilde{\pi}_\infty(\tilde{\gamma}_w)^\#(\chi^{u_i}) = ((t^{1/l})^{lw_i})^{a_i|H|} = t^{a_i w_i |H|} = t^{\langle u_i, w\rangle}.$$

It follows from the definition of γ_w that $\tilde{\pi}_\infty(\tilde{\gamma}_w) = \gamma_w$. $\square$

Remark III.6. The morphism from a Deligne-Mumford stack to its coarse moduli space is proper [43]. Hence the fact that the map $\tilde{\pi}_\infty : |\mathcal{J}_\infty \mathcal{X}|' \to J_\infty(X)'$ is bijective follows from Proposition 3.37 of [67].

Remark III.7. We give a geometric description of the involution ι on $|\Sigma| \cap N$ from Section 2.1. Recall that the elements of $|\mathcal{IX}|$ are equivalence classes of pairs (x, α), where x is an object of $\mathcal{X}$ over $\mathbb{C}$ and α is an automorphism of x. There is a natural involution on $|\mathcal{IX}|$, taking a pair (x, α) to (x, α^{-1}). By Remark III.3, the connected components of $|\mathcal{IX}|$ are indexed by $\mathrm{Box}(\boldsymbol{\Sigma})$ and one verifies that I induces the involution ι on $\mathrm{Box}(\boldsymbol{\Sigma})$. More generally, I extends to a $J_\infty(T)$-equivariant involution $I : |\mathcal{J}_\infty \mathcal{X}|' \to |\mathcal{J}_\infty \mathcal{X}|'$, compatible with the projection $\pi : \mathcal{J}_\infty \mathcal{X} \to \mathcal{IX} = \mathcal{J}_0 \mathcal{X}$, satisfying $I(\tilde{\gamma}_v) = \tilde{\gamma}_{\iota(v)}$.

Remark III.8. For any non-negative integer n, we can consider $|\mathcal{J}_n \mathcal{X}|' = |\mathcal{J}_n \mathcal{X}| \smallsetminus \cup_{i=1}^r |\mathcal{J}_n \mathcal{D}_i|$ and an action of $J_n(T)$ on $|\mathcal{J}_n \mathcal{X}|'$. Recall that we have projection morphisms $\pi_n : \mathcal{J}_\infty \mathcal{X} \to \mathcal{J}_n \mathcal{X}$. For each non-zero cone τ of Σ, let

$$\overline{\mathrm{Box}}(\boldsymbol{n\tau}) = \{v \in N \mid v = \sum_{\rho_i \subseteq \tau} q_i b_i \text{ for some } 0 < q_i \leq n\}.$$

We set $\overline{\mathrm{Box}}(\boldsymbol{n\{0\}}) = \{0\}$ and $\overline{\mathrm{Box}}(\boldsymbol{n\Sigma}) = \cup_{\tau \in \Sigma} \overline{\mathrm{Box}}(\boldsymbol{n\tau})$. It can be shown that there is a decomposition of $|\mathcal{J}_n \mathcal{X}|'$ into $J_n(T)$-orbits

$$|\mathcal{J}_n \mathcal{X}|' = \coprod_{v \in \overline{\mathrm{Box}}(\boldsymbol{n\Sigma})} \pi_n(\tilde{\gamma}_v) \cdot J_n(T).$$

3.4 Contact Order along a Divisor

In this section, we compute the contact order of a twisted arc along a T-invariant divisor on $\mathcal{X}$. Recall that for each maximal cone σ in Σ, we have morphisms

$$\mathbb{A}^d \xrightarrow{p} [\mathbb{A}^d/N(\sigma)] \cong \mathcal{X}(\boldsymbol{\sigma}) \xrightarrow{q} \mathbb{A}^d/N(\sigma) \cong U_\sigma,$$

where $\mathbb{A}^d$ is the atlas of $\mathcal{X}(\boldsymbol{\sigma})$ via p and U_σ is the coarse moduli space of $\mathcal{X}(\boldsymbol{\sigma})$. For every ray ρ_i in Σ, there is a corresponding divisor $\mathcal{D}_i$ of $\mathcal{X}(\boldsymbol{\Sigma})$ (see Section 3.3). If $\rho_i \nsubseteq \sigma$ then $\mathcal{D}_i$ does not intersect $\mathcal{X}(\boldsymbol{\sigma})$. If $\rho_i \subseteq \sigma$, then $\mathcal{D}_i$ corresponds to the divisor $\{x_i = 0\}$ on $\mathbb{A}^d$, with an appropriate choice of coordinates. We say that a $\mathbb{Q}$-divisor $\mathcal{E}$ on $\mathcal{X}$ is *T-invariant* if $\mathcal{E} = \sum_i \beta_i \mathcal{D}_i$, for some $\beta_i \in \mathbb{Q}$. There is a natural isomorphism of Picard groups over $\mathbb{Q}$ (Example 6.7 [65])

$$q^* : \mathrm{Pic}_{\mathbb{Q}}(X(\Sigma)) \to \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}(\boldsymbol{\Sigma}))$$

$$D_i \mapsto q^* D_i = a_i \mathcal{D}_i.$$

The inverse map is induced by the pushforward functor

$$q_* : \mathrm{Pic}_{\mathbb{Q}}(\mathcal{X}(\boldsymbol{\Sigma})) \to \mathrm{Pic}_{\mathbb{Q}}(X(\Sigma)).$$

The canonical divisor on $\mathcal{X}$ is given by $K_{\mathcal{X}} = -\sum_i \mathcal{D}_i$, and so $q_* K_{\mathcal{X}} = -\sum_i a_i^{-1} D_i$. Recall that a T-invariant $\mathbb{Q}$-divisor $E = \sum_{\rho_i \in \Sigma} \alpha_i D_i$ on X corresponds to a real-valued piecewise $\mathbb{Q}$-linear function $\psi_E : |\Sigma| \to \mathbb{R}$ satisfying $\psi_E(v_i) = -\alpha_i$ [25]. Note that $\psi_{q_* K_{\mathcal{X}}}(b_i) = 1$ for $i = 1, \ldots, r$, and hence, with the notation of Section 3.1, $\psi = \psi_{q_* K_{\mathcal{X}}}$. Let $\mathcal{E} = \sum_i \beta_i \mathcal{D}_i$ be a T-invariant $\mathbb{Q}$-divisor on $\mathcal{X}$. Following Yasuda (Definition 4.17 [67]), we say that the pair $(\mathcal{X}, \mathcal{E})$ is *Kawamata log terminal* if $\beta_i < 1$ for $i = 1, \ldots, r$. Geometrically, this says that for each maximal cone σ, the representative of $\mathcal{E}$ in the atlas $\mathbb{A}^d$ of $\mathcal{X}(\boldsymbol{\sigma}) = [\mathbb{A}^d/N(\sigma)]$ is supported on the coordinate axes with all coefficients less than 1. Equivalently, the condition says that $\psi_{q_*\mathcal{E}}(b_i) > -1$ for $i = 1, \ldots, r$.

If $\mathcal{E} = \sum u_j \mathcal{E}_j$ is a $\mathbb{Q}$-divisor on $\mathcal{X}$, for some prime divisors $\mathcal{E}_j$ and rational numbers u_j, then Yasuda [67] defined a function

$$\operatorname{ord}\mathcal{E} : |\mathcal{J}_\infty \mathcal{X}| \setminus \cup_j |\mathcal{J}_\infty \mathcal{E}_j| \to \mathbb{Q},$$

$$\operatorname{ord}\mathcal{E} = \sum_j u_j \operatorname{ord}\mathcal{E}_j.$$

When $\mathcal{E}$ is a prime divisor, the function $\operatorname{ord}\mathcal{E}$ is defined as follows: if $\gamma : \mathcal{D}^l_{\infty,\mathbb{C}} \to \mathcal{X}$ is a representable morphism, then consider the composition of γ with the atlas $\operatorname{Spec}\mathbb{C}[[t]] \to \mathcal{D}^l_{\infty,\mathbb{C}}$, and choose a lifting to an arc $\bar{\gamma}$ of an atlas M of $\mathcal{X}$. If m is the contact order of $\bar{\gamma}$ along the representative of $\mathcal{E}$ in M, then $\operatorname{ord}\mathcal{E}(\gamma) = m/l$.

The following lemma describes the function $\operatorname{ord}\mathcal{E}$ in the toric case. We will use the notation of Theorem III.5.

Lemma III.9. *If $\mathcal{E}$ is a T-invariant $\mathbb{Q}$-divisor on $\mathcal{X}$, then*

$$\operatorname{ord}\mathcal{E}(\tilde{\gamma}_w \cdot J_\infty(T)) = -\psi_{q_*\mathcal{E}}(w).$$

In particular,

$$\operatorname{ord} K_{\mathcal{X}}(\tilde{\gamma}_w \cdot J_\infty(T)) = -\psi(w).$$

Proof. It will be enough to prove the result for $\mathcal{E} = \mathcal{D}_i$ and twisted arcs of the form $\tilde{\gamma}_w$. Consider the representable morphism $\tilde{\gamma}_w : \mathcal{D}^l_{\infty,\mathbb{C}} \to \mathcal{X}(\boldsymbol{\sigma}) \subseteq \mathcal{X}$, for some lattice point w in a maximal cone σ. With the notation of (3.8), the composition of $\tilde{\gamma}_w$ with $\operatorname{Spec}\mathbb{C}[[t]] \to \mathcal{D}^l_{\infty,\mathbb{C}}$ lifts to an arc $(t^{lw_1}, \ldots, t^{lw_d})$ in $J_\infty(\mathbb{A}^d)$. If $\rho_i \nsubseteq \sigma$ then $\operatorname{ord}\mathcal{D}_i(\tilde{\gamma}_w) = 0 = -\psi_{a_i^{-1}D_i}(w)$. If $\rho_i \subseteq \sigma$ then the divisor $\mathcal{D}_i$ is represented by the divisor $\{x_i = 0\}$ in the atlas $\mathbb{A}^d$ and we conclude that $\operatorname{ord}\mathcal{D}_i(\tilde{\gamma}_w) = w_i = -\psi_{a_i^{-1}D_i}(w)$. The second statement follows since $\psi = \psi_{q_*K_{\mathcal{X}}}$. $\square$

3.5 Motivic Integration on Toric Stacks

We consider motivic integration on a Deligne-Mumford stack as developed by Yasuda in [67]. We will compute motivic integrals associated to T-invariant divisors on $\mathcal{X}$ and show that they correspond to weighted δ-power series of an associated polyhedral complex.

Recall that to any complex algebraic variety X of dimension r, we can associate its *Hodge polynomial* (see, for example, [53])

$$E_X(u,v) = \sum_{i,j=0}^{r} (-1)^{i+j} h_{i,j} u^i v^j \in \mathbb{Z}[u,v].$$

The Hodge polynomial is determined by the properties

1. $h_{i,j} = \dim H^j(X, \Omega^i_X)$ if X is smooth and projective,
2. $E_X(u,v) = E_U(u,v) + E_{X \smallsetminus U}(u,v)$ if U is an open subvariety of X,
3. $E_{X \times Y}(u,v) = E_X(u,v) E_Y(u,v)$.

The second property means we can consider the Hodge polynomial of a constructible subset of a complex variety. For example, $E_{\mathbb{A}^1}(u,v) = E_{\mathbb{P}^1}(u,v) - E_{\{\mathrm{pt}\}}(u,v) = uv$ and hence $E_{\mathbb{A}^n}(u,v) = (uv)^n$. Similarly, $E_{(\mathbb{C}^*)^n}(u,v) = (uv-1)^n$. More generally, we can compute the Hodge polynomial of any toric variety. Recall that if $\triangle$ is a fan in a lattice $N'_{\mathbb{R}}$, for some lattice N', then its associated h-vector $h_\triangle(t)$ is defined by

$$h_\triangle(t) = \sum_{\tau \in \triangle} t^{\dim \tau} (1-t)^{\operatorname{codim} \tau}.$$

Lemma III.10. *If $X = X(\triangle)$ is an r-dimensional toric variety associated to a fan $\triangle$, then*

$$E_X(u,v) = (uv)^r h_\triangle((uv)^{-1}).$$

Proof. We can write X as a disjoint union of torus orbits indexed by the cones of $\triangle$ (see, for example, [25]). The orbit corresponding to τ is isomorphic to $(\mathbb{C}^*)^{\operatorname{codim}\tau}$. We compute, using the example above,

$$E_X(u,v) = \sum_{\tau\in\triangle} E_{(\mathbb{C}^*)^{\operatorname{codim}\tau}}(u,v) = \sum_{\tau\in\triangle}(uv-1)^{\operatorname{codim}\tau} = (uv)^r h_{\triangle}((uv)^{-1}).$$

□

Remark III.11. If $\triangle$ is an r-dimensional, simplicial fan with convex support, then $X = X(\triangle)$ has no odd cohomology and the coefficient of $(uv)^i$ in $E_X(u,v)$ is equal to the dimension of the $2i^{\text{th}}$ cohomology group of X with compact support and rational coefficients. This follows from the above lemma, Lemma II.29 and Poincaré duality.

Recall that we have projection morphisms $\pi_n : |\mathcal{J}_\infty\mathcal{X}| \to |\mathcal{J}_n\mathcal{X}|$, for each non-negative integer n. A subset $A \subseteq |\mathcal{J}_\infty\mathcal{X}|$ is a *cylinder* if $A = \pi_n^{-1}\pi_n(A)$ and $\pi_n(A)$ is a constructible subset, for some non-negative integer n. In this case, the measure $\mu_{\mathcal{X}}(A)$ of A is defined to be

$$\mu_{\mathcal{X}}(A) = E_{\pi_n(A)}(u,v)(uv)^{-nd} \in \mathbb{Z}[u,u^{-1},v,v^{-1}].$$

By Lemma 3.18 of [67], the right hand side is independent of the choice of n. The collection of cylinders is closed under taking finite unions and finite intersections and $\mu_{\mathcal{X}}$ defines a finite measure on $|\mathcal{J}_\infty\mathcal{X}|$.

We will compute the measure of certain cylinders. Recall the decomposition of $|\mathcal{J}_\infty\mathcal{X}|'$ in Theorem III.5. Every w in $|\Sigma| \cap N$ can be uniquely written in the form $w = \{w\} + \tilde{w}$, where $\{w\}$ lies in $\operatorname{Box}(\boldsymbol{\tau})$ for some $\tau \subseteq \sigma(w)$ and $\tilde{w}$ lies in $N_{\sigma(w)}$. Recall that $\sigma(w)$ is the cone of Σ containing w in its relative interior and that $\psi : |\Sigma| \to \mathbb{R}$ is the piecewise $\mathbb{Q}$-linear function satisfying $\psi(b_i) = 1$ for $i = 1,\dots,r$.

Lemma III.12. *For any w in $|\Sigma| \cap N$, the orbit $\tilde{\gamma}_w \cdot J_\infty(T)$ is a cylinder in $|\mathcal{J}_\infty \mathcal{X}|$ with measure*

$$\mu_{\mathcal{X}}(\tilde{\gamma}_w \cdot J_\infty(T)) = (uv-1)^d (uv)^{-\psi(w)+\psi(\{w\})-\dim \sigma(\{w\})}.$$

Proof. Fix a d-dimensional cone σ containing w and consider the notation of (3.8). The twisted arc $\tilde{\gamma}_w$ lies in $J_\infty^{(g)} \mathbb{A}^d / N(\sigma)$ and $\pi_n(\tilde{\gamma}_w)$ lies in $J_{nl}^{(g)} \mathbb{A}^d / N(\sigma)$. If we consider the natural projection $\pi_n^{(g)} : J_\infty^{(g)} \mathbb{A}^d \to J_{nl}^{(g)} \mathbb{A}^d$, then $\pi_n^{(g)}((t^{lw_1}, \ldots, t^{lw_d}) \cdot J_\infty^{(l)}(T))$ is equal to

$$\{(\sum_{k=\lambda_1}^{n_1} \alpha_{1,l(k+q_1)} t^{l(k+q_1)}, \ldots, \sum_{k=\lambda_d}^{n_d} \alpha_{d,l(k+q_d)} t^{l(k+q_d)}) \mid \alpha_{i,w_i} \neq 0\},$$

for $n \geq \max\{\lambda_1 + 1, \ldots, \lambda_d + 1\}$. Here $n_i = n$ if $q_i = 0$ and $n_i = n - 1$ if $q_i \neq 0$. Note that $q_i \neq 0$ if and only if $\rho_i \subseteq \sigma(\{w\})$. Hence $(\pi_n^{(g)})^{-1} \pi_n^{(g)}((t^{lw_1}, \ldots, t^{lw_d}) \cdot J_\infty^{(l)}(T)) = (t^{lw_1}, \ldots, t^{lw_d}) \cdot J_\infty^{(l)}(T)$, and $\pi_n^{(g)}((t^{lw_1}, \ldots, t^{lw_d}) \cdot J_\infty^{(l)}(T))$ is isomorphic to

$$(\mathbb{C}^*)^d \times \mathbb{A}^{nd - \sum_{i=1}^d \lambda_i - \dim \sigma(\{w\})}. \tag{3.9}$$

Note that (3.9) has a decomposition into a disjoint union of locally closed subspaces, each isomorphic to $(\mathbb{C}^*)^i$ for some i, which is preserved after quotienting by the induced action of the finite group $N(\sigma)$. The result follows by considering the corresponding spaces and maps after quotienting by $N(\sigma)$, and observing that this doesn't affect the Hodge polynomial of (3.9). $\square$

Let $F : |\mathcal{J}_\infty \mathcal{X}|' \to \mathbb{Q}$ be a $J_\infty(T)$-invariant function on the space of twisted arcs of $\mathcal{X}$ and let A be a $J_\infty(T)$-invariant subset of $|\mathcal{J}_\infty \mathcal{X}|$. By Theorem III.5, we have a decomposition

$$A \cap |\mathcal{J}_\infty \mathcal{X}|' = \coprod_{\substack{w \in |\Sigma| \cap N \\ \tilde{\gamma}_w \in A}} \tilde{\gamma}_w \cdot J_\infty(T).$$

We define

$$\int_A (uv)^F d\mu_{\mathcal{X}} := \sum_{\substack{w\in|\Sigma|\cap N \\ \tilde{\gamma}_w\in A}} \mu_{\mathcal{X}}(\tilde{\gamma}_w \cdot J_\infty(T))(uv)^{F(\tilde{\gamma}_w)},$$

when the right hand sum is a well-defined element in $\mathbb{Z}[(uv)^{1/N}]\,[[(uv)^{-1/N}]]$, for some positive integer N. With the definitions of Yasuda, this is the *motivic integral* of F over A.[2]

Recall the involution $\iota : \mathrm{Box}(\mathbf{\Sigma}) \to \mathrm{Box}(\mathbf{\Sigma})$ (Remark III.7). Following [67], we define the *shift function* $s_{\mathcal{X}} : |\mathcal{J}_\infty\mathcal{X}|' \to \mathbb{Q}$ by

$$s_{\mathcal{X}}(\tilde{\gamma}_w \cdot J_\infty(T)) = \dim \sigma(\{w\}) - \psi(\{w\}) = \psi(\iota(\{w\})).$$

The shift function factors as $|\mathcal{J}_\infty\mathcal{X}|' \xrightarrow{\pi} |\mathcal{J}_0\mathcal{X}| = |\mathcal{I}\mathcal{X}| \xrightarrow{\mathrm{sft}} \mathbb{Q}$, where the function sft is constant on the connected components of $|\mathcal{I}\mathcal{X}|$. Borisov, Chen and Smith [12] showed that the connected components of $|\mathcal{I}\mathcal{X}|$ are indexed by $\mathrm{Box}(\mathbf{\Sigma})$ (cf. Remark III.3). The value of sft on the component of $|\mathcal{I}\mathcal{X}|$ corresponding to g in $N(\sigma)$ is the age of g^{-1} (see (3.3)) and Remark III.7).

Consider a Kawamata log terminal pair $(\mathcal{X}, \mathcal{E})$, where $\mathcal{E}$ is a T-invariant $\mathbb{Q}$-divisor on $\mathcal{X}$ (see Section 3.4). Following Yasuda (Definition 4.1 [67]), we consider the invariant $\Gamma(\mathcal{X}, \mathcal{E})$ defined by

$$\Gamma(\mathcal{X}, \mathcal{E}) = \int_{|\mathcal{J}_\infty\mathcal{X}|} (uv)^{s_{\mathcal{X}} + \mathrm{ord}\,\mathcal{E}} d\mu_{\mathcal{X}}.$$

Yasuda showed that $\Gamma(\mathcal{X}, \mathcal{E})$ is a well-defined element in $\mathbb{Z}[(uv)^{1/N}]\,[[(uv)^{-1/N}]]$, for some positive integer N [67, Proposition 4.15]. Our goal is to compute such invariants and give them a combinatorial interpretation. We first recall some facts about weighted δ-power series from Chapter II. Consider the polyhedral complex $Q = \{v \in |\Sigma| \,|\, \psi(v) \leq 1\}$ and let $\lambda : |\Sigma| \to \mathbb{R}$ be a piecewise $\mathbb{Q}$-linear function

[2] To check the definitions agree, we can replace A by $A \cap |\mathcal{J}_\infty\mathcal{X}|'$, by Propositions 3.24 and 3.25 in [67]. Now apply the decomposition of Theorem III.5.

satisfying $\lambda(b_i) > -1$ for $i = 1, \ldots, r$. The *weighted δ-power series* $\delta^\lambda(t)$ is a power series in $\mathbb{Z}[[t^{1/N}]]$ and a rational function in $\mathbb{Q}((t^{1/N}))$, for some positive integer N, satisfying $\delta^\lambda(t) = t^d \delta^\lambda(t^{-1})$ if Σ is complete. We can write

$$\delta^\lambda(t) = \sum_{\tau \in \Sigma} h_\tau^\lambda(t) \sum_{v \in \mathrm{Box}(\boldsymbol{\tau})} t^{\psi(v)+\lambda(v)} \prod_{\rho_i \subseteq \tau} (t-1)/(t^{\lambda(b_i)+1} - 1), \tag{3.10}$$

where $h_\tau(t)$ is the weighted h-vector of τ, satisfying

$$t^{\operatorname{codim} \tau} h_\tau^\lambda(t^{-1}) = (t-1)^{\operatorname{codim} \tau} \sum_{\tau \subseteq \sigma} \prod_{\rho_i \subseteq \sigma \smallsetminus \tau} 1/(t^{\lambda(b_i)+1} - 1). \tag{3.11}$$

We also have the following expression

$$\delta^\lambda(t^{-1}) = \sum_{\tau \in \Sigma} h_\tau^\lambda(t^{-1}) \sum_{v \in \mathrm{Box}(\boldsymbol{\tau})} t^{-\psi(v)-\lambda(v)+\sum_{\rho_i \subseteq \tau} \lambda(b_i)} \prod_{\rho_i \subseteq \tau} (t-1)/(t^{\lambda(b_i)+1} - 1). \tag{3.12}$$

Theorem III.13. *If $(\mathcal{X}, \mathcal{E})$ is a Kawamata log terminal pair, where $\mathcal{E}$ is a T-invariant $\mathbb{Q}$-divisor on $\mathcal{X}$, then the corresponding piecewise $\mathbb{Q}$-linear function $\lambda = \psi_{q_* \mathcal{E}}$ satisfies $\lambda(b_i) > -1$ for $i = 1, \ldots, r$, and*

$$\Gamma(\mathcal{X}, \mathcal{E}) = (uv)^d \delta^\lambda((uv)^{-1}).$$

In particular, $\Gamma(\mathcal{X}, \mathcal{E})$ is a rational function in $\mathbb{Q}(t^{1/N})$, for some positive integer N. If Σ is a complete fan, then

$$\Gamma(\mathcal{X}, \mathcal{E})(u, v) = (uv)^d \Gamma(\mathcal{X}, \mathcal{E})(u^{-1}, v^{-1}) = \delta^\lambda(uv).$$

Moreover, every weighted δ-power series has the form

$$\delta^\lambda(uv) = (uv)^d \Gamma(\mathcal{X}, \mathcal{E})(u^{-1}, v^{-1}),$$

for some such pair $(\mathcal{X}, \mathcal{E})$.

Proof. We showed in Lemma III.9 that $\operatorname{ord} \mathcal{E}(\tilde{\gamma}_w \cdot J_\infty(T)) = -\lambda(w)$ and hence $(s_{\mathcal{X}} + \operatorname{ord} \mathcal{E})(\tilde{\gamma}_w \cdot J_\infty(T)) = \dim \sigma(\{w\}) - \psi(\{w\}) - \lambda(w)$. Using Lemma III.12, we compute

$$\Gamma(\mathcal{X}, \mathcal{E}) = (uv-1)^d \sum_{w \in |\Sigma| \cap N} (uv)^{-\psi(w)-\lambda(w)}.$$

For each w in $|\Sigma| \cap N$, we have a unique decomposition

$$w = \{w\} + w' + \sum_{\rho_i \subseteq \sigma(w) \smallsetminus \sigma(\{w\})} b_i,$$

where w' is a non-negative linear combination of $\{b_i \mid \rho_i \subseteq \sigma(w)\}$. Here $\sigma(w)$ is the cone containing w in its relative interior. Using this decomposition, we compute the following expression for $\Gamma(\mathcal{X}, \mathcal{E})$,

$$\sum_{\substack{\tau \in \Sigma \\ v \in \mathrm{Box}(\boldsymbol{\tau})}} (uv-1)^d (uv)^{-(\psi+\lambda)(v)} \sum_{\tau \subseteq \sigma} (uv)^{-\sum_{\rho_i \subseteq \sigma \smallsetminus \tau}(\lambda(b_i)+1)} \prod_{\rho_i \subseteq \sigma} 1/(1-(uv)^{-\lambda(b_i)-1}).$$

Rearranging and using (3.11) gives

$$\begin{aligned} \Gamma(\mathcal{X}, \mathcal{E}) = & \sum_{\tau \in \Sigma} (uv)^{\operatorname{codim} \tau} h_\tau^\lambda((uv)^{-1}) \times \\ & \sum_{v \in \mathrm{Box}(\boldsymbol{\tau})} (uv)^{\sum_{\rho_i \subseteq \tau} \lambda(b_i) + \dim \tau - \psi(v) - \lambda(v)} \prod_{\rho_i \subseteq \tau} (uv-1)/((uv)^{\lambda(b_i)+1}-1). \end{aligned}$$

Comparing with (3.12) gives $\Gamma(\mathcal{X}, \mathcal{E}) = (uv)^d \delta^\lambda((uv)^{-1})$. The second statement follows from Corollary II.12.

If $\lambda : |\Sigma| \to \mathbb{R}$ is a piecewise $\mathbb{Q}$-linear function satisfying $\lambda(b_i) > -1$ for $i = 1, \ldots, r$, then we may consider the corresponding T-invariant $\mathbb{Q}$-divisor E on X. The pair $(\mathcal{X}, q^*E)$ is a Kawamata log terminal pair and, by the above argument, $\delta^\lambda(uv) = (uv)^d \Gamma(\mathcal{X}, q^*E)(u^{-1}, v^{-1})$. Hence every weighted δ-power series corresponds to a motivic integral on $\mathcal{X}$. $\square$

Remark III.14. By Theorem III.13 and (3.12), we obtain a formula for the invariant $\Gamma(\mathcal{X}, \mathcal{E})$. This formula is a stacky analogue of Batyrev's formula for the motivic integral of a simple normal crossing divisor on a smooth variety (see [64], [5]).

Remark III.15. When $\mathcal{E} = 0$, $\Gamma(\mathcal{X}, 0)$ is a polynomial in uv of degree d and the coefficient of $(uv)^j$ is equal to the dimension of the $2j^{\text{th}}$ orbifold cohomology group of $\mathcal{X}$ with compact support [66]. When Σ is complete, the symmetry $\Gamma(\mathcal{X},0)(u,v) = (uv)^d\Gamma(\mathcal{X},0)(u^{-1},v^{-1})$ is a consequence of Poincaré duality for orbifold cohomology [14].

3.6 The Transformation Rule

A morphism $f : \mathcal{Y} \to \mathcal{X}$ of smooth Deligne-Mumford stacks is *birational* if there exist open, dense substacks $\mathcal{Y}_0$ of $\mathcal{Y}$ and $\mathcal{X}_0$ of $\mathcal{X}$ such that f induces an isomorphism $\mathcal{Y}_0 \cong \mathcal{X}_0$ (Definition 3.36 [67]). If f is proper and birational then Yasuda [67] proved a transformation rule relating motivic integrals on $\mathcal{X}$ to motivic integrals on $\mathcal{Y}$, generalising a classic result of Kontsevich for smooth varieties (see, for example, [64]). The goal of this section is to interpret the transformation rule in our context in order to give a geometric proof of Proposition II.13.

If $\boldsymbol{\triangle} = (N, \triangle, \{\bar{b}_i\})$ and $\boldsymbol{\Sigma} = (N, \Sigma, \{b_i\})$ are stacky fans, we say that $\boldsymbol{\triangle}$ *refines* $\boldsymbol{\Sigma}$ if

1. The fan $\triangle$ refines Σ in $N_{\mathbb{R}}$,
2. For any ray $\bar{\rho}_i$ in $\triangle$, $\bar{b}_i$ is an integer combination of the lattice points $\{b_j \mid \rho_j \subseteq \sigma\}$, where σ is any cone of Σ containing $\bar{\rho}_i$.

If $\boldsymbol{\triangle}$ *refines* $\boldsymbol{\Sigma}$, we have an induced morphism of toric stacks $f : \mathcal{X}(\boldsymbol{\triangle}) \to \mathcal{X}(\boldsymbol{\Sigma})$ (Remark 4.5 [12]) such that f restricts to the identity map on the torus and hence is birational. Note that the corresponding map of coarse moduli spaces is proper [25]. Since the morphism from a Deligne-Mumford stack to its coarse moduli space is proper [40], it follows that f is proper [43]. We will give a local description of f. Let τ be a maximal cone in $\triangle$, contained in a maximal cone σ of Σ. By Property (2), we have an inclusion of lattices $N_\tau \hookrightarrow N_\sigma$, inducing a homomorphism of groups $j : N(\tau) \to N(\sigma)$. Let τ' be the cone generated by $\{\bar{b}_i \mid \bar{\rho}_i \subseteq \tau\}$ in N_τ and let σ' be the cone generated by $\{b_i \mid \rho_i \subseteq \sigma\}$ in N_σ. The inclusion $\tau' \cap N_\tau \hookrightarrow \sigma' \cap N_\sigma$ induces a j-equivariant map $\phi : \operatorname{Spec} \mathbb{C}[\check{\tau}' \cap M_\tau] \to \operatorname{Spec} \mathbb{C}[(\sigma')^\vee \cap M_\sigma]$. Taking stacky quotients of both sides yields the restriction of f to $\mathcal{X}(\boldsymbol{\tau})$, $f : \mathcal{X}(\boldsymbol{\tau}) = [\operatorname{Spec} \mathbb{C}[\check{\tau}' \cap M_\tau]/N(\tau)] \to \mathcal{X}(\boldsymbol{\sigma}) = [\operatorname{Spec} \mathbb{C}[(\sigma')^\vee \cap M_\sigma]/N(\sigma)]$.

Given a morphism $g : \mathcal{Y} \to \mathcal{Z}$ of Deligne-Mumford stacks, Yasuda described a natural morphism $g_\infty : |\mathcal{J}_\infty \mathcal{Y}| \to |\mathcal{J}_\infty \mathcal{Z}|$ (Proposition 2.14 [67]). We outline his construction. Given a representable morphism $\gamma : \mathcal{D}^l_{\infty,\mathbb{C}} \to \mathcal{Y}$, we can consider the composition $\gamma' : \mathcal{D}^l_{\infty,\mathbb{C}} \to \mathcal{Y} \to \mathcal{X}$. If l' is a positive integer dividing l, then we have a group homomorphism $\mu_l \to \mu_{l'}$, $\zeta_l \mapsto (\zeta_l)^{l/l'} = \zeta_{l'}$, and an equivariant map $\operatorname{Spec} \mathbb{C}\,[[t]] \to \operatorname{Spec} \mathbb{C}[[t]]$, $t \mapsto t^{l/l'}$. Taking stacky quotients of both sides gives a morphism $\mathcal{D}^l_{\infty,\mathbb{C}} \to \mathcal{D}^{l'}_{\infty,\mathbb{C}}$. By Lemma 2.15 of [67], there exists a unique positive integer l' dividing l so that γ' factors (up to 2-isomorphism) as $\mathcal{D}^l_{\infty,\mathbb{C}} \to \mathcal{D}^{l'}_{\infty,\mathbb{C}} \xrightarrow{\psi} \mathcal{X}$, where ψ is representable, and then $g_\infty(\gamma) = \psi$.

Let Y be the coarse moduli space of $\mathcal{Y}$ and let Z be the coarse moduli space of $\mathcal{Z}$. Then $g : \mathcal{Y} \to \mathcal{Z}$ induces a morphism $g' : Y \to Z$. Note that, by considering coarse moduli spaces, the morphism $\mathcal{D}^l_{\infty,\mathbb{C}} \to \mathcal{D}^{l'}_{\infty,\mathbb{C}}$ yields the identity morphism on $\operatorname{Spec} \mathbb{C}\,[[t]]$. It follows that we have a commutative diagram

$$\begin{tikzcd} {|\mathcal{J}_\infty \mathcal{Y}|} \arrow[r, "g_\infty"] \arrow[d, "\tilde{\pi}_\infty"] & {|\mathcal{J}_\infty \mathcal{Z}|} \arrow[d, "\tilde{\pi}_\infty"] \\ J_\infty(Y) \arrow[r, "g'_\infty"] & J_\infty(Z). \end{tikzcd} \tag{3.13}$$

Consider the notation of Theorem III.5.

Lemma III.16. *If $\triangle = (N, \triangle, \{\bar{b}_i\})$ is a stacky fan refining $\boldsymbol{\Sigma} = (N, \Sigma, \{b_i\})$, then the birational morphism $f : \mathcal{X}(\triangle) \to \mathcal{X}(\boldsymbol{\Sigma})$ induces a $J_\infty(T)$-equivariant map*

$$f_\infty : |\mathcal{J}_\infty \mathcal{X}(\triangle)|' \to |\mathcal{J}_\infty \mathcal{X}(\boldsymbol{\Sigma})|'$$

$$f_\infty(\tilde{\gamma}_v) = \tilde{\gamma}_v.$$

Proof. By considering coarse moduli spaces, $f : \mathcal{X}(\triangle) \to \mathcal{X}(\mathbf{\Sigma})$ gives rise to the toric morphism $f' : X(\triangle) \to X(\Sigma)$, with induced map of arc spaces $f'_\infty : J_\infty(X(\triangle))' \to J_\infty(X(\Sigma))'$. Consider the commutative diagram (3.13). By Theorem III.5, the maps $\tilde{\pi}_\infty$ are $J_\infty(T)$-equivariant bijections satisfying $\tilde{\pi}_\infty(\tilde{\gamma}_v) = \gamma_v$. Hence we only need to show that f'_∞ is $J_\infty(T)$-equivariant and satisfies $f'_\infty(\gamma_v) = \gamma_v$. This fact is well-known but we recall a proof for the convenience of the reader. Suppose v lies in a cone τ of $\triangle$ and let σ be a cone in Σ containing τ. The arc γ_v corresponds to the ring homomorphism $\mathbb{C}[\check{\tau} \cap M] \to \mathbb{C}\,[[t]]$, $\chi^u \mapsto t^{\langle u,v\rangle}$ and hence $f'_\infty(\gamma_v)$ corresponds to the ring homomorphism $\mathbb{C}[\sigma^\vee \cap M] \hookrightarrow \mathbb{C}[\check{\tau} \cap M] \to \mathbb{C}[[t]]$, $\chi^u \mapsto \chi^u \mapsto t^{\langle u,v\rangle}$. We see that f'_∞ is $J_\infty(T)$-equivariant and $f'_\infty(\gamma_v) = \gamma_v$. $\square$

We now state Yasuda's transformation rule in the case of toric stacks.

Theorem III.17 (The Transformation Rule, [67])**.** *Let $\mathbf{\triangle} = (N, \triangle, \{\bar{b}_i\})$ be a stacky fan refining $\mathbf{\Sigma} = (N, \Sigma, \{b_i\})$, with corresponding birational morphism $f : \mathcal{X}(\mathbf{\triangle}) \to \mathcal{X}(\mathbf{\Sigma})$. If $F : |\mathcal{J}_\infty \mathcal{X}(\mathbf{\Sigma})|' \to \mathbb{Q}$ is a $J_\infty(T)$-invariant function and A is a $J_\infty(T)$-invariant subset of $|\mathcal{J}_\infty \mathcal{X}(\mathbf{\Sigma})|'$, then*

$$\int_A (uv)^{s_{\mathcal{X}(\Sigma)} + F} d\mu_{\mathcal{X}} = \int_{f_\infty^{-1}(A)} (uv)^{s_{\mathcal{X}(\triangle)} + F \circ f_\infty - \operatorname{ord} K_{\mathcal{X}(\triangle)/\mathcal{X}(\Sigma)}} d\mu_{\mathcal{X}'},$$

where $K_{\mathcal{X}(\triangle)/\mathcal{X}(\Sigma)} = K_{\mathcal{X}(\triangle)} - f^* K_{\mathcal{X}(\Sigma)}$.

We deduce the following geometric proof of Proposition II.13.

Proof. Let $\tilde{\Sigma}$ be a common refinement of Σ and $\triangle$. We can choose lattice points $\{\tilde{b}_i\}$ on the rays of $\tilde{\Sigma}$ so that $\tilde{\mathbf{\Sigma}} = (N, \tilde{\Sigma}, \{\tilde{b}'_i\})$ is a stacky fan refining $\mathbf{\Sigma}$ and $\mathbf{\triangle}$. Hence we can reduce to the case when $\mathbf{\triangle}$ refines $\mathbf{\Sigma}$. In this case, consider the corresponding birational morphism $f : \mathcal{X}(\mathbf{\triangle}) \to \mathcal{X}(\mathbf{\Sigma})$. By Lemma III.9, there is a Kawamata log terminal pair $(\mathcal{X}(\mathbf{\Sigma}), \mathcal{E})$ such that $\operatorname{ord} \mathcal{E}(\tilde{\gamma}_w \cdot J_\infty(T)) = -\lambda(w)$. It follows from Lemma

III.16 that $(\operatorname{ord}\mathcal{E} \circ f_\infty)(\tilde{\gamma}_w \cdot J_\infty(T)) = -\lambda(w)$ and Lemma III.9 and Lemma III.16 imply that $\operatorname{ord} K_{\mathcal{X}(\triangle)/\mathcal{X}(\Sigma)}(\tilde{\gamma}_w \cdot J_\infty(T)) = \psi_\Sigma(w) - \psi_\triangle(w)$. The result now follows from Theorem III.13 and Theorem III.17, with $F = \operatorname{ord}\mathcal{E}$ and $A = |\mathcal{J}_\infty\mathcal{X}(\mathbf{\Sigma})|$. □

3.7 Remarks

More generally, we can replace N by a finitely generated abelian group of rank d. All the results go through with minor modifications. We mention the local construction of the toric stack [12]. Let $\bar{N}$ be the lattice given by the image of N in $N_\mathbb{R}$ and for each v in N, let $\bar{v}$ denote the image of v in $\bar{N}$. Let Σ be a complete, simplicial, rational fan in $N_\mathbb{R}$ and let $\bar{v}_1, \ldots, \bar{v}_r$ be the primitive integer generators of Σ in $\bar{N}$. Fix elements $b_1, \ldots, b_r$ in N such that $\bar{b}_i = a_i\bar{v}_i$, for some positive integer a_i. The data $\mathbf{\Sigma} = (N, \Sigma, \{b_i\})$ is a *stacky fan*. For each maximal cone σ of Σ, let N_σ denote the subgroup of N generated by $\{b_i \,|\, \rho_i \subseteq \sigma\}$. We obtain a homomorphism of finite groups $N(\sigma) = N/N_\sigma \to \bar{N}(\sigma) = \bar{N}/\bar{N}_\sigma$. Composing with the injection $\bar{N}(\sigma) \to (\mathbb{C}^*)^d$ from Section 3.1 gives a homomorphism $N(\sigma) \to \bar{N}(\sigma) \to (\mathbb{C}^*)^d$, and $\mathcal{X}(\boldsymbol{\sigma}) = [\mathbb{A}^d/N(\sigma)]$.

Using this more general setup, one can apply Theorem III.5 to the T-invariant closed substacks of $\mathcal{X}$ to give a decomposition of $|\mathcal{J}_\infty\mathcal{X}|$ into $J_\infty(T)$-orbits.

CHAPTER IV

Inequalities and Ehrhart δ-Vectors

4.1 Inequalities and Ehrhart δ-Vectors

We will continue with the definitions and notation from previous chapters. In this chapter, P will denote a d-dimensional lattice polytope (not necessarily containing an interior lattice point) in a lattice N. Recall that the degree s of P equals the degree of the δ-polynomial $\delta_P(t)$ and the codegree of P equals $l = d + 1 - s$. Our main object of study will be the polynomial

$$\bar{\delta}_P(t) = (1 + t + \cdots + t^{l-1})\delta_P(t).$$

Since $\delta_P(t)$ has degree s and non-negative integer coefficients, it follows that $\bar{\delta}_P(t)$ has degree d and non-negative integer coefficients. In fact, we will show that $\bar{\delta}_P(t)$ has positive integer coefficients (Theorem IV.14). Observe that we can recover $\delta_P(t)$ from $\bar{\delta}_P(t)$ if we know the codegree l of P. If we write

$$\bar{\delta}_P(t) = \bar{\delta}_0 + \bar{\delta}_1 t + \cdots + \bar{\delta}_d t^d,$$

then

$$\bar{\delta}_i = \delta_i + \delta_{i-1} + \cdots + \delta_{i-l+1}, \tag{4.1}$$

for $i = 0, \ldots, d$. Note that $\bar{\delta}_0 = 1$ and $\bar{\delta}_d = \delta_s$.

Example IV.1. Let N be a lattice with basis $e_1, \ldots, e_d$ and let P be the *standard simplex* with vertices $0, e_1, \ldots, e_d$. It can be shown that $\delta_P(t) = 1$ and hence $\bar{\delta}_P(t) = 1 + t + \cdots + t^d$. On the other hand, if Q is the *standard reflexive simplex* with vertices $e_1, \ldots, e_d$ and $-e_1 - \cdots - e_d$, then $\bar{\delta}_Q(t) = \delta_Q(t) = 1 + t + \cdots + t^d$. We conclude that $\bar{\delta}_P(t)$ does not determine $\delta_P(t)$.

Remark IV.2. We can interpret $\bar{\delta}_P(t)$ as the Ehrhart δ-vector of a $(d+l)$-dimensional polytope. More specifically, let Q be the standard reflexive simplex of dimension $l-1$ in a lattice M as above. Henk and Tagami [29] defined $P \otimes Q$ to be the convex hull in $(N \times M \times \mathbb{Z})_{\mathbb{R}}$ of $P \times \{0\} \times \{0\}$ and $\{0\} \times Q \times \{1\}$. By Lemma 1.3 in [29], $P \otimes Q$ is a $(d+l)$-dimensional lattice polytope with Ehrhart δ-vector $\delta_{P \otimes Q}(t) = \delta_P(t)\delta_Q(t) = \delta_P(t)(1 + t + \cdots + t^{l-1}) = \bar{\delta}_P(t)$

Our main objects of study will be the polynomials $a(t)$ and $b(t)$ in the following elementary lemma.

Lemma IV.3. *The polynomial $\bar{\delta}_P(t)$ has a unique decomposition*

$$\bar{\delta}_P(t) = a(t) + t^l b(t), \tag{4.2}$$

where $a(t)$ and $b(t)$ are polynomials with integer coefficients satisfying $a(t) = t^d a(t^{-1})$ and $b(t) = t^{d-l} b(t^{-1})$.

Proof. Let a_i and b_i denote the coefficients of t^i in $a(t)$ and $b(t)$ respectively, and set

$$a_{i+1} = \delta_0 + \cdots + \delta_{i+1} - \delta_d - \cdots - \delta_{d-i}, \tag{4.3}$$

$$b_i = -\delta_0 - \cdots - \delta_i + \delta_s + \cdots + \delta_{s-i}. \tag{4.4}$$

We compute, using (4.1) and since $s + l = d + 1$,

$$\begin{aligned}
a_i + b_{i-l} &= \delta_0 + \cdots + \delta_i - \delta_d - \cdots - \delta_{d-i+1} - \delta_0 - \cdots - \delta_{i-l} + \delta_s + \cdots + \delta_{s-i+l} \\
&= \delta_{i-l+1} + \cdots + \delta_i = \bar{\delta}_i, \\
a_i - a_{d-i} &= \delta_0 + \cdots + \delta_i - \delta_d - \cdots - \delta_{d-i+1} - \delta_0 - \cdots - \delta_{d-i} + \delta_d + \cdots + \delta_{i+1} \\
&= 0, \\
b_i - b_{d-l-i} &= -\delta_0 - \cdots - \delta_i + \delta_s + \cdots + \delta_{s-i} + \delta_0 + \cdots + \delta_{s-i-1} - \delta_s - \cdots - \delta_{i+1} \\
&= 0,
\end{aligned}$$

for $i = 0, \ldots, d$. Hence we obtain our desired decomposition and one easily verifies the uniqueness assertion. □

Example IV.4. If N is a lattice with basis $e_1, \ldots, e_5$, then let P be the 5-dimensional lattice polytope with vertices 0, e_1, e_1+e_2, e_2+2e_3, $3e_4+e_5$ and e_5. Henk and Tagami showed that $\delta_P(t) = (1+t^2)(1+2t) = 1 + 2t + t^2 + 2t^3$ (Example 1.1 in [29]). It follows that $s = l = 3$ and $\bar{\delta}_P(t) = 1 + 3t + 4t^2 + 5t^3 + 3t^4 + 2t^5$. We calculate that $a(t) = 1 + 3t + 4t^2 + 4t^3 + 3t^4 + t^5$ and $b(t) = 1 + 0t + t^2$.

We may view our proposed inequalities on the coefficients of the Ehrhart δ-vector as conditions on the coefficients of $\bar{\delta}_P(t)$, $a(t)$ and $b(t)$.

Lemma IV.5. *With the notation above,*

- *Inequality* (1.2) *holds if and only if the coefficients of* $a(t)$ *are non-negative.*
- *Inequality* (1.3) *holds if and only if the coefficients of* $b(t)$ *are non-negative.*
- *Inequality* (1.6) *holds if and only if* $a_1 \leq a_i$ *for* $i = 2, \ldots, d-1$.
- *Inequality* (1.7) *holds if and only if the coefficients of* $b(t)$ *are non-negative.*
- *Inequality* (1.8) *holds if and only if* $\bar{\delta}_1 \leq \bar{\delta}_i$ *for* $i = 2, \ldots, d-1$.

- *Inequality* (1.9) *holds if and only if the coefficients of* $a(t)$ *are positive.*

Proof. The result follows by substituting (4.1),(4.3) and (4.4) into the right hand sides of the above statements. ☐

Remark IV.6. The coefficients of $a(t)$ are *unimodal* if $a_0 = 1 \leq a_1 \leq \cdots \leq a_{\lfloor d/2 \rfloor}$. It follows from (4.3) that $a_{i+1} - a_i = \delta_{i+1} - \delta_{d-i}$ for all i. Hence the coefficients of $a(t)$ are unimodal if and only if $\delta_{i+1} \geq \delta_{d-i}$ for $i = 0, \ldots, \lfloor d/2 \rfloor - 1$. In Remark IV.17, we show that the coefficients of $a(t)$ are unimodal for $d \leq 5$. As explained in the next remark, in higher dimensions, this might not hold.

Remark IV.7 (cf. Remark II.25). A lattice polytope P is *reflexive* if the origin is the unique lattice point in its relative interior and each facet F of P has the form $F = \{v \in P \mid \langle u, v \rangle = -1\}$, for some $u \in \mathrm{Hom}(N, \mathbb{Z})$. Equivalently, P is reflexive if it contains the origin in its relative interior and, for every positive integer m, every non-zero lattice point in mP lies on $\partial(nP)$ for a unique positive integer $n \leq m$. It is a result of Hibi [33] that $\delta_P(t) = t^d \delta_P(t^{-1})$ if and only if P is a translate of a reflexive polytope (cf. Corollary IV.18). We see from Lemma IV.3 that $\delta_P(t) = t^d \delta_P(t^{-1})$ if and only if $\delta_P(t) = \bar{\delta}_P(t) = a(t)$. Payne and Mustaţă gave examples of reflexive polytopes where the coefficients of $\delta_P(t) = a(t)$ are not unimodal [49]. Further examples are given by Payne for all $d \geq 6$ [51].

Remark IV.8. It follows from (4.4) that $b_{i+1} - b_i = \delta_{s-(i+1)} - \delta_{i+1}$ for all i. Hence the coefficients of $b(t)$ are unimodal if and only if $\delta_i \leq \delta_{s-i}$ for $i = 1, \ldots, \lfloor (s-1)/2 \rfloor$. We see from Example IV.4 that the coefficients of $b(t)$ are not necessarily unimodal.

Our next goal is to express $\bar{\delta}_P(t)$ as a sum of shifted h-vectors, using a result of Payne (Theorem 1.2 [51]). We first fix a lattice triangulation $\mathcal{T}$ of ∂P and recall what it means for $\mathcal{T}$ to be regular. Translate P by an element of $N_{\mathbb{Q}}$ so that the origin lies in its interior and let Σ denote the fan over the faces of $\mathcal{T}$. Then $\mathcal{T}$ is *regular* if Σ can be realised as the fan over the faces of a rational polytope. Equivalently, $\mathcal{T}$ is regular if the toric variety $X(\Sigma)$ is projective. We may always choose $\mathcal{T}$ to be a regular triangulation (see, for example, [3]). We regard the empty face as a face of $\mathcal{T}$ of dimension -1. For each face F of $\mathcal{T}$, consider the h-vector of F,

$$h_F(t) = \sum_{F \subseteq G} t^{\dim G - \dim F}(1-t)^{d-1-\dim G}.$$

We recall a slight extension of Lemma II.29 to this setting.

Lemma IV.9. *Let $\mathcal{T}$ be a regular lattice triangulation of ∂P. If F is a face of $\mathcal{T}$, then the h-vector of F is a polynomial of degree $d-1-\dim F$ with symmetric, unimodal integer coefficients. That is, if h_i denotes the coefficient of t^i in $h_F(t)$, then $h_i = h_{d-1-\dim F - i}$ for all i and $1 = h_0 \leq h_1 \leq \cdots \leq h_{\lfloor (d-1-\dim F)/2 \rfloor}$. The coefficients satisfy the Upper Bound Theorem,*

$$h_i \leq \binom{h_1 + i - 1}{i},$$

for $i = 1, \ldots, d-1-\dim F$.

Proof. As above, translate P by an element of $N_{\mathbb{Q}}$ so that the origin lies in its interior and let Σ denote the fan over the faces of $\mathcal{T}$. For each face F of $\mathcal{T}$, let $\operatorname{span} F$ denote the smallest linear subspace of $N_{\mathbb{R}}$ containing F and let Σ_F be the complete fan in $N_{\mathbb{R}}/\operatorname{span} F$ whose cones are the projections of the cones in Σ containing F. We can interpret h_i as the dimension of the $2i^{\text{th}}$ cohomology group of the projective toric variety $X(\Sigma_F)$. The symmetry of the h_i follows from Poincaré Duality on $X(\Sigma_F)$, while unimodality follows from the Hard Lefschetz Theorem. The cohomology ring $H^*(X(\Sigma_F), \mathbb{Q})$ is isomorphic to the quotient of a polynomial ring in h_1 variables of degree 2 and hence h_i is bounded by the number of monomials of degree i in h_1 variables of degree 1. $\square$

Recall that l is the smallest positive integer such that lP contains a lattice point in its relative interior and fix a lattice point $\bar{v}$ in $lP \smallsetminus \partial(lP)$. Let $N' = N \times \mathbb{Z}$ and let $u : N' \to \mathbb{Z}$ denote the projection onto the second factor. We write σ for the cone over $P \times \{1\}$ in $N'_{\mathbb{R}}$ and ρ for the ray through $(\bar{v}, l)$. For each face F of $\mathcal{T}$, let σ_F denote the cone over F and let σ'_F denote the cone generated by σ_F and ρ. The empty face corresponds to the origin and ρ respectively. The set of cones σ_F and $\sigma_{F'}$, for various F, forms a simplicial fan $\triangle$ refining σ. Recall from Chapter II that, for each non-zero cone τ in $\triangle$, with primitive integer generators $v_1, \ldots, v_r$, we consider the open parallelopiped

$$\mathrm{Box}(\tau) = \{a_1 v_1 + \cdots + a_r v_r \in N' \mid 0 < a_i < 1\},$$

and observe that we have an involution

$$\iota : \mathrm{Box}(\tau) \to \mathrm{Box}(\tau)$$

$$\iota(a_1 v_1 + \cdots + a_r v_r) = (1 - a_1)v_1 + \cdots + (1 - a_r)v_r.$$

We also set $\mathrm{Box}(\{0\}) = \{0\}$ and $\iota(0) = 0$, and observe that $\mathrm{Box}(\rho) = \emptyset$. For each face F of $\mathcal{T}$, we define

$$B_F(t) = \sum_{v \in \mathrm{Box}(\sigma_F)} t^{u(v)}$$

$$B'_F(t) = \sum_{v \in \mathrm{Box}(\sigma'_F)} t^{u(v)}.$$

If $\mathrm{Box}(\sigma_F) = \emptyset$ or $\mathrm{Box}(\sigma'_F) = \emptyset$ then we define $B_F(t) = 0$ or $B'_F(t) = 0$ respectively. For example, when F is the empty face, $B_F(t) = 1$ and $B'_F(t) = 0$. We will need the following lemma.

Lemma IV.10. *For each face F of $\mathcal{T}$, $B_F(t) = t^{\dim F + 1} B_F(t^{-1})$ and $B'_F(t) = t^{\dim F + l + 1} B'_F(t^{-1})$.*

Proof. Using the involution ι above,

$$t^{\dim F+1}B_F(t^{-1}) = \sum_{v\in\mathrm{Box}(\sigma_F)} t^{\dim F+1-u(v)} = \sum_{v\in\mathrm{Box}(\sigma_F)} t^{u(\iota(v))} = B_F(t).$$

Similarly,

$$t^{\dim F+l+1}B'_F(t^{-1}) = \sum_{v\in\mathrm{Box}(\sigma'_F)} t^{\dim F+1+l-u(v)} = \sum_{v\in\mathrm{Box}(\sigma'_F)} t^{u(\iota(v))} = B'_F(t).$$

□

Consider any element v in $\sigma \cap N'$ and let G be the smallest face of $\mathcal{T}$ such that v lies in σ'_G. Set $r = \dim G + 1$ and let $v_1, \ldots, v_r$ denote the vertices of G. Then v can be uniquely written in the form

$$v = \{v\} + \sum_{(v_i,1)\notin\tau} (v_i, 1) + w, \tag{4.5}$$

where $\{v\}$ lies in $\mathrm{Box}(\tau)$ for some subcone τ of σ'_G, and w is a non-negative integer sum of $(v_1, 1), \ldots, (v_r, 1)$ and $(\bar{v}, l)$. If we write $w = \sum_{i=1}^{r} a_i(v_i, 1) + a_{r+1}(\bar{v}, l)$, for some non-negative integers $a_1, \ldots, a_{r+1}$, then

$$u(v) = u(\{v\}) + \dim G - \dim F + \sum_{i=1}^{r} a_i + a_{r+1}l. \tag{4.6}$$

Conversely, given $\tilde{v}$ in $\mathrm{Box}(\tau)$, for some $\tau \subseteq \sigma'_G$, and w a non-negative integer sum of $(v_1, 1), \ldots, (v_r, 1)$ and $(\bar{v}, l)$, then $v = \tilde{v} + \sum_{(v_i,1)\notin\tau}(v_i, 1) + w$ lies in $\sigma \cap N'$ and G is the smallest face of $\mathcal{T}$ such that v lies in σ'_G.

Remark IV.11. With the above notation, observe that $\tau = \sigma_F$ for some $F \subseteq G$ if and only if v lies on a translate of $\partial\sigma$ by a non-negative multiple of $(\bar{v}, l)$. Note that if $\tau = \sigma'_F$ for some (necessarily non-empty) $F \subseteq G$, then $\{v + nv_1\} = \{v\}$ and $u(v + nv_1) = u(v) + n$, for any non-negative integer n. We conclude that $B'_F(t) = 0$ for all faces F of $\mathcal{T}$ if and only if every element v in $\sigma \cap (N \times l\mathbb{Z})$ can be written as the sum of an element of $\partial(mlP)\times\{ml\}$ and $m'(\bar{v},l)$, for some non-negative integers m and m'.

The generating series of $f_P(m)$ can be written as $\sum_{v\in\sigma\cap N'} t^{u(v)}$. Payne described this sum by considering the contributions of all v in $\sigma\cap N'$ with a fixed $\{v\}\in \mathrm{Box}(\tau)$. We have the following application of Theorem 1.2 in [51]. We recall the proof in this situation for the convenience of the reader.

Lemma IV.12. *With the notation above,*

$$\bar{\delta}_P(t) = \sum_{F\in\mathcal{T}} (B_F(t) + B'_F(t))h_F(t).$$

Proof. Using (4.5) and (4.6), we compute

$$\begin{aligned}
\bar{\delta}_P(t) &= (1+t+\cdots+t^{l-1})\delta_P(t) = (1-t^l)(1-t)^d \sum_{v\in\sigma\cap N'} t^{u(v)} \\
&= (1-t)^d \sum_{F\in\mathcal{T}} (B_F(t)+B'_F(t)) \sum_{F\subseteq G} t^{\dim G-\dim F}/(1-t)^{\dim G+1} \\
&= \sum_{F\in\mathcal{T}} (B_F(t)+B'_F(t))h_F(t).
\end{aligned}$$

$\square$

Remark IV.13. We can write $\bar{\delta}_P(t) = (1-t)^{d+1}\sum_{v\in\sigma\cap N'}(1+t+\cdots+t^{l-1})t^{u(v)}$. Ehrhart Reciprocity states that, for any positive integer m, $f_P(-m)$ is $(-1)^d$ times the number of lattice points in the relative interior of mP (see, for example, [32]). Hence $f_P(-1) = \cdots = f_P(1-l) = 0$ and the generating series of the polynomial $f_P(m) + f_P(m-1) + \cdots + f_P(m-l+1)$ has the form $\bar{\delta}_P(t)/(1-t)^{d+1}$.

We will now prove our first main result of this chapter. When $s = d$, $\bar{\delta}_P(t) = \delta_P(t)$ and the theorem below is due to Betke and McMullen (Theorem 5 [10]), while a geometric description of the decomposition was given in Remark II.18.

Theorem IV.14. *The polynomial $\bar{\delta}_P(t)$ has a unique decomposition*

$$\bar{\delta}_P(t) = a(t) + t^l b(t),$$

where $a(t)$ and $b(t)$ are polynomials with integer coefficients satisfying $a(t) = t^d a(t^{-1})$ and $b(t) = t^{d-l} b(t^{-1})$. Moreover, the coefficients of $b(t)$ are non-negative and, if a_i denotes the coefficient of t^i in $a(t)$, then

$$1 = a_0 \leq a_1 \leq a_i, \tag{4.7}$$

for $i = 2, \ldots, d-1$.

Proof. Let $\mathcal{T}$ be a regular lattice triangulation of ∂P. We may assume that $\mathcal{T}$ contains every lattice point of ∂P as a vertex (see, for example, [3]). By Lemma IV.12, if we set

$$a(t) = \sum_{F \in \mathcal{T}} B_F(t) h_F(t) \tag{4.8}$$

$$b(t) = t^{-l} \sum_{F \in \mathcal{T}} B'_F(t) h_F(t), \tag{4.9}$$

then $\bar{\delta}_P(t) = a(t) + t^l b(t)$. Since mP contains no lattice points in its relative interior for $m = 1, \ldots, l-1$, if v lies in $\mathrm{Box}(\sigma'_F)$ for some face F of $\mathcal{T}$, then $u(v) \geq l$. We conclude that $b(t)$ is a polynomial. By Lemma IV.9, the coefficients of $b(t)$ are non-negative integers. Since every lattice point of ∂P is a vertex of $\mathcal{T}$, if v lies in $\mathrm{Box}(\sigma_F)$ for some non-empty face F of $\mathcal{T}$, then $u(v) \geq 2$. If we write $a(t) = h_\emptyset(t) + t^2 \sum_{F \in \mathcal{T}, F \neq \emptyset} t^{-2} B_F(t) h_F(t)$ then Lemma IV.9 implies that $1 = a_0 \leq a_1 \leq a_i$ for $i = 2, \ldots, d-1$. By Lemma IV.3, we are left with verifying that $a(t) = t^d a(t^{-1})$ and $b(t) = t^{d-l} b(t^{-1})$. Using Lemmas IV.9 and IV.10, we compute

$$t^d a(t^{-1}) = t^d \sum_{F \in \mathcal{T}} B_F(t^{-1}) h_F(t^{-1}) = \sum_{F \in \mathcal{T}} B_F(t) t^{d - \dim F - 1} h_F(t^{-1}) = a(t),$$

$$t^{d-l}b(t^{-1}) = t^{d-l}t^{l}\sum_{F\in\mathcal{T}} B'_F(t^{-1})h_F(t^{-1}) = t^{-l}\sum_{F\in\mathcal{T}} B'_F(t)t^{d-\dim F-1}h_F(t^{-1}) = b(t).$$

□

Remark IV.15. It follows from the above theorem that expressions (4.8) and (4.9) are independent of the choice of lattice triangulation $\mathcal{T}$ and the choice of $\bar{v}$ in $lP \smallsetminus \partial(lP)$.

Remark IV.16 (cf. Remark II.15). Let K be the *pyramid* over ∂P. That is, K is the truncation of $\partial\sigma$ at level 1 and can be written as

$$K := \{(x,\lambda) \in (N\times\mathbb{Z})_{\mathbb{R}} \mid x \in \partial(\lambda P), 0 < \lambda \leq 1\} \cup \{0\}.$$

We may view K as a polyhedral complex and consider its Ehrhart polynomial $f_K(m)$ and associated Ehrhart δ-polynomial $\delta_K(t)$ ([35, p1], [32, Chapter XI]). By the proof of Theorem IV.14,

$$a(t)/(1-t)^{d+1} = \sum_{F\in\mathcal{T}} \sum_{\substack{v\in\sigma\cap N' \\ \{v\}\in\mathrm{Box}(\sigma_F)}} (1+t+\cdots+t^{l-1})t^{u(v)}.$$

It follows from Remark IV.11 that $a(t)/(1-t)^{d+1}$ is the generating series of $f_K(m)$ and hence that $a(t) = \delta_K(t)$. With the terminology of [35], K is *star-shaped* with respect to the origin and the fact that $1 = a_0 \leq a_1 \leq a_i$, for $i = 2,\ldots,d-1$, is a consequence of Hibi's results in [35].

Remark IV.17. By Theorem IV.14, $a_0 \leq a_1 \leq a_2$ and hence the coefficients of $a(t)$ are unimodal for $d \leq 5$ (cf. Remark IV.6).

As a corollary, we obtain a combinatorial proof of a result of Stanley [55]. Recall, from Remark IV.7, that a lattice polytope P is reflexive if and only if it contains the origin in its relative interior and, for every positive integer m, every non-zero lattice point in mP lies on $\partial(nP)$ for a unique positive integer $n \leq m$.

Corollary IV.18. *If P is a lattice polytope of degree s and codegree l, then $\delta_P(t) = t^s\delta_P(t^{-1})$ if and only if lP is a translate of a reflexive polytope.*

Proof. Since $t^d\bar{\delta}_P(t^{-1}) = t^s\delta_P(t^{-1})(1 + t + \cdots + t^{l-1})$, we see that $\delta_P(t) = t^s\delta_P(t^{-1})$ if and only if $t^d\bar{\delta}_P(t^{-1}) = \bar{\delta}_P(t)$. By Lemma IV.3, we need to show that $b(t) = 0$ if and only if lP is a translate of a reflexive polytope. By Remark IV.11, $b(t) = 0$ if and only if every element v in $\sigma \cap (N \times l\mathbb{Z})$ can be written as the sum of an element of $\partial(mlP) \times \{ml\}$ and $m'(\bar{v}, l)$, for some non-negative integers m and m'. That is, $b(t) = 0$ if and only if $lP - \bar{v}$ is a reflexive polytope. □

We now prove our second main result.

Theorem IV.19. *Let P be a d-dimensional lattice polytope of degree s and codegree l. The Ehrhart δ-vector $(\delta_0, \ldots, \delta_d)$ of P satisfies the following inequalities.*

$$\delta_1 \geq \delta_d,$$

$$\delta_2 + \cdots + \delta_{i+1} \geq \delta_{d-1} + \cdots + \delta_{d-i} \textit{ for } i = 0, \ldots, \lfloor d/2 \rfloor - 1,$$

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i} \textit{ for } i = 0, \ldots, d,$$

$$\delta_{2-l} + \cdots + \delta_0 + \delta_1 \leq \delta_i + \delta_{i-1} + \cdots + \delta_{i-l+1} \textit{ for } i = 2, \ldots, d-1.$$

Proof. We observed in the introduction that $\delta_1 \geq \delta_d$. By Lemma IV.5, the second inequality is equivalent to $a_1 \leq a_i$, for $i = 2, \ldots, d-1$, and the third inequality is equivalent to $b_i \geq 0$ for all i. Hence these inequalities follow from Theorem IV.14. When $l \geq 2$, the conditions above imply that $\bar{\delta}_1 \leq \bar{\delta}_i$ for $i = 2, \ldots, d-1$. By Lemma IV.5, this proves the final inequality when $l > 1$. When $l = 1$, the last inequality is Hibi's result (1.4). □

A lattice triangulation $\mathcal{T}$ of ∂P is *unimodular* if for every non-empty face F of $\mathcal{T}$, the cone over $F \times \{1\}$ in $N'_{\mathbb{R}}$ is non-singular. Equivalenty, $\mathcal{T}$ is unimodular if and only if $\mathrm{Box}(\sigma_F) = \emptyset$, for every non-empty face F of $\mathcal{T}$.

Theorem IV.20. *Let P be a d-dimensional lattice polytope. If ∂P admits a regular unimodular lattice triangulation, then*

$$\delta_{i+1} \geq \delta_{d-i}$$

$$\delta_0 + \cdots + \delta_{i+1} \leq \delta_d + \cdots + \delta_{d-i} + \binom{\delta_1 - \delta_d + i + 1}{i+1},$$

for $i = 0, \ldots, \lfloor d/2 \rfloor - 1$.

Proof. From the above discussion, if ∂P admits a regular unimodular triangulation then $B_F(t) = 0$ for every non-empty face F of $\mathcal{T}$. By (4.8), $a(t) = h_F(t)$, where F is the empty face of $\mathcal{T}$. By Lemma IV.3, $1 = a_0 \leq a_1 \leq \cdots \leq a_{\lfloor d/2 \rfloor}$ and $a_i \leq \binom{a_1+i-1}{i}$ for $i = 1, \ldots, d$. The result now follows from the expression $a_{i+1} = \delta_0 + \cdots + \delta_{i+1} - \delta_d - \cdots - \delta_{d-i}$ (see (4.3)). □

Remark IV.21. When P is the regular simplex of dimension d, $\delta_P(t) = 1$ and both inequalities in Theorem IV.20 are equalities.

Remark IV.22. Recall that if P is a reflexive polytope then the coefficients of the Ehrhart δ-vector are symmetric (Remark IV.7). In this case, Theorem IV.20 implies that if ∂P admits a regular, unimodular lattice triangulation then the coefficients of the δ-vector are symmetric and unimodal. Note that if P is reflexive then P admits a regular unimodular lattice triangulation if and only if ∂P admits a regular unimodular lattice triangulation. Hence this result is a consequence of the theorem of Athanasiadis stated in the introduction (Theorem 1.3 [3]). This special case was first proved by Hibi in [31].

Remark IV.23. Recall that $\delta_1 = |P \cap N| - (d+1)$ and $\delta_d = |(P \smallsetminus \partial P) \cap N|$. Hence $\delta_1 = \delta_d$ if and only if $|\partial P \cap N| = d + 1$. If $|\partial P \cap N| = d+1$ and ∂P admits a regular, unimodular lattice triangulation then, by Theorems IV.19 and

IV.20, $\delta_0 + \cdots + \delta_{i+1} = \delta_d + \cdots + \delta_{d-i} + 1$, for $i = 0, \ldots, \lfloor d/2 \rfloor - 1$. This implies that $\delta_{i+1} = \delta_{d-i}$ for $i = 0, \ldots, \lfloor d/2 \rfloor - 1$. If, in addition, P is reflexive then the coefficients of the δ-vector are symmetric and hence $\delta_P(t) = 1 + t + \cdots + t^d$.

Remark IV.24. Let P be a lattice polytope of dimension d in N and let Q be the convex hull of $P \times \{1\}$ and the origin in $(N \times \mathbb{Z})_{\mathbb{R}}$. That is, Q is the pyramid over P. If ∂Q admits a regular unimodular lattice triangulation $\mathcal{T}$ then $\mathcal{T}$ restricts to give regular unimodular lattice triangulations of P and ∂P.

Remark IV.25. Hibi gave an example of a 4-dimensional reflexive lattice polytope whose boundary does not admit a regular unimodular lattice triangulation (Example 36.4 [32]). By Remark IV.24, there are examples of d-dimensional lattice polytopes P such that ∂P does not admit a regular unimodular lattice triangulation for $d \geq 4$. On the other hand, if P is a lattice polytope of dimension $d \leq 3$, then ∂P always admits a regular, unimodular lattice triangulation. In fact, any regular triangulation of ∂P containing every lattice point as a vertex is necessarily unimodular. This follows from the fact that if Q is a lattice polytope of dimension $d' \leq 2$ and $|Q \cap N| = d' + 1$, then Q is isomorphic to the regular d'-simplex.

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BIBLIOGRAPHY

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