Floer Homology and Symplectic Forms on $S^1 \times M^3$

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2009

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To my family

ACKNOWLEDGEMENTS

First, I would like to thank my thesis advisor Daniel M. Burns for his constant support and guidance. I learned a great deal from him about mathematics and the profession.

Theorems I.2 and I.4 are the result of joint work with my thesis co-advisor Clifford H. Taubes. I am indebted to him for sharing his time, his expertise and several of his ideas with me.

I am also grateful to the University of Michigan Rackham Graduate School for generously funding me during the last year of graduate school through the Rackham Pre-doctoral Fellowship and the Rackham Graduate Student Research Grant.

Last, but not the least, I dedicate this dissertation to my wonderful family whom I owe the most; my parents Nilüfer and Bora, and my brother Ögeday.

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CHAPTER I

Introduction

Since the early 1980's, starting with the work of S. K. Donaldson and E. Witten, gauge theory has become an important tool in understanding the geometry and topology of 3 and 4-dimensional manifolds. In [3], Donaldson showed that the moduli space of anti-self-dual connections on particular SU(2)-bundles over a smooth 4-dimensional manifold captures invariants of its smooth structure. Meanwhile, Witten used the language of quantum field theory to construct a refined version of Morse theory, known as *Morse–Witten theory* (see [30]). Witten's reinterpretation of Morse theory was used by A. Floer (see [4]) to study 3- and 4-dimensional manifolds. Floer used the anti-self-dual Yang–Mills equations to define a homology theory associated to closed, oriented 3-manifolds, the so-called *instanton Floer homology*. In certain cases, Floer's work has made it possible to calculate Donaldson invariants of 4-dimensional manifolds.

In 1994, Witten proposed an alternate point of view to Donaldson invariants of 4-dimensional manifolds after his joint work with N. Seiberg (see [18] and [31]). This new point of view involved counting solutions of a particular system of quasi-linear PDEs with abelian gauge group U(1), the so-called *Seiberg–Witten equations*. Compared to Donaldson theory, Seiberg-Witten theory proved much easier to work with. In fact, many problems which were considered out of reach in Donaldson theory at the time were handled relatively quickly using Seiberg–Witten theory.

The true strength of Seiberg–Witten theory was first revealed in the papers by Taubes. In his seminal work on the Seiberg–Witten invariants of closed symplectic 4-manifolds, Taubes showed that counting solutions of the Seiberg-Witten equations is the same as counting pseudo-holomorphic submanifolds representing a homology class fixed in advance (see [22]). A program initiated by Taubes aims at establishing a similar kind of correspondence between solutions of the Seiberg-Witten equations and embedded surfaces in the realm of a larger class of smooth 4-dimensional manifolds. As an interesting example, Taubes suggested studying 4-dimensional manifolds of the form $\mathrm{S}^1 \times \mathrm{M}$ where M is a closed, oriented 3-manifold. Here, the question of interest is when exactly such manifolds admit symplectic forms, and if they do, how could one identify the set of all symplectic forms up to equivalence. This dissertation grew out of an attempt to answer these questions.

It is known that if a closed, oriented 3-manifold, M, fibers over the circle, then $\mathrm{S}^1 \times \mathrm{M}$ admits a symplectic form. This was first observed by W. P. Thurston (see [27]). One way to see this is as follows: Suppose $f : \mathrm{M} \to \mathrm{S}^1$ is a smooth map with no critical values. Then, one can find a Riemannian metric on M so that $df := f^*\mathrm{dt}$ is a harmonic 1-form. Here, dt denotes the the volume form on S^1 with respect to the standard Riemannian metric. As a result, the 2-form $\omega_f := \mathrm{dt} \wedge df + *df$ is a symplectic form on $\mathrm{S}^1 \times \mathrm{M}$, which is self-dual and harmonic with respect to the product Riemanian metric. Conversely, the following was conjectured.

Conjecture I.1. *If* M *is a closed, oriented 3-manifold such that the 4-dimensional manifold* $\mathrm{S}^1 \times \mathrm{M}$ *admits a symplectic form, then* M *fibers over the circle.*

Conjecture I.1 is often attributed to Taubes in the literature.

Recently, S. Friedl and S. Vidussi have announced a complete proof of the above conjecture (see [7]). Their proof involves investigating certain topological invariants associated to finite index subgroups of the fundamental group of M. This dissertation presents a geometric approach to proving this conjecture.

The main ingredient in our approach is a Floer-type topological invariant for closed, oriented 3-manifolds, namely, *Seiberg–Witten Floer homology.* Seiberg-Witten Floer homology was constructed by P. Kronheimer and T. Mrowka in [9] using the Seiberg-Witten equations. Having fixed a *spin^c structure* $\mathfrak{s}$ on a closed, oriented 3-manifold M, Seiberg–Witten Floer homology associates to it three graded abelian groups, $\widehat{HM}(\mathrm{M}, \mathfrak{s})$, $\widetilde{HM}(\mathrm{M}, \mathfrak{s})$ and $\overline{HM}(\mathrm{M}, \mathfrak{s})$. Assuming that $\mathrm{S}^1 \times \mathrm{M}$ admits a symplectic form, our aim is to study these Floer homology groups so as to deduce conditions that are sufficient for M to fiber over the circle.

Before stating the first main result of this dissertation, we lay out the working assumptions. The set $\mathcal{S}(\mathrm{M})$ of spin^c structures (up to isomorphism) on M is a principal homogeneous space for $\mathrm{H}^2(\mathrm{M}; \mathbb{Z})$. When $\mathrm{S}^1 \times \mathrm{M}$ admits a symplectic form ω, $\mathcal{S}(\mathrm{M})$ is canonically identified with $\mathrm{H}^2(\mathrm{M}; \mathbb{Z})$. With this identification in mind, we may denote the spin^c structure corresponding to a given $e \in \mathrm{H}^2(\mathrm{M}; \mathbb{Z})$ by $\mathfrak{s}_e$. Now, let $i : \mathrm{M} \to \mathrm{S}^1 \times \mathrm{M}$ be the obvious inclusion map, and fix a spin^c structure $\mathfrak{s}_e$ such that $c_1(\mathfrak{s}_e) = \lambda[i^*\omega]$ in $\mathrm{H}^2(\mathrm{M}; \mathbb{R})$ for some $\lambda < 0$. From now on, we will refer to such spin^c structures as those that satisfy the *monotonicity condition.* Then,

Theorem I.2 (K.–Taubes)**.** *$\overline{HM}(\mathrm{M}, \mathfrak{s}_e)$ is trivial, $\widehat{HM}(\mathrm{M}, \mathfrak{s}_e) \cong \widetilde{HM}(\mathrm{M}, \mathfrak{s}_e)$, and the following hold true:*

- *If $e = 0$, then $\widetilde{HM}(\mathrm{M}, \mathfrak{s}_e) \cong \mathbb{Z}$.*
- *If $e \neq 0$ and $[\omega] \cdot e \leq 0$, then $\widetilde{HM}(\mathrm{M}, \mathfrak{s}_e)$ is trivial.*

Recently, Kronheimer and Mrowka introduced in [10] a topological invariant for null-homologous knots in a given closed, oriented 3-manifold using Seiberg–Witten Floer homology, so-called *monopole knot homology.* As an application, Kronheimer and Mrowka proved that monopole knot homology detects fibered knots in S^3, i.e. knots in S^3 whose complements fiber over the circle. Shortly afterwards, Ni proved in [15] the following analogue of this result for closed, oriented 3-manifolds using the ideas in [16].

Theorem I.3 (Ni)**.** *Let* M *be a closed, connected, oriented and irreducible 3-manifold and* $\Sigma \subset \mathrm{M}$ *be a non-separating, closed, oriented surface of genus* 2 *or more. Suppose that*

$$\widetilde{HM}(\mathrm{M}|[\Sigma]) := \bigoplus_{\mathfrak{s}\in\mathcal{S}(\mathrm{M})\,:\,\langle c_1(\mathfrak{s}),[\Sigma]\rangle = 2genus(\Sigma)-2} \widetilde{HM}(\mathrm{M},\mathfrak{s}) \cong \mathbb{Z}.$$

Then, M *fibers over the circle with* Σ *as a fiber.*

Now, with the help of Theorems I.2 and I.3 we can state the second main result of this dissertation as follows.

Theorem I.4 (K.–Taubes)**.** *Let* M *be a closed, connected, orientable, and irreducible 3-manifold with* $b_1(\mathrm{M}) = 1$*. Suppose that* $\mathrm{S}^1 \times \mathrm{M}$ *admits a symplectic form with non-torsion anticanonical class. Then,* M *fibers over the circle.*

A few remarks are in order regarding the hypotheses of Theorem I.4. First, if $\mathrm{S}^1 \times \mathrm{M}$ admits a symplectic form with torsion anticanonical class, then it follows from Theorem I.2, Theorem II.9 and Proposition 25.5.5 in [9] that M has vanishing Thurston (semi)-norm. As Friedl and Vidussi proved in [5], this implies that M fibers over the circle with torus fibers. Second, it follows from a theorem of J. D. McCarthy [13] with G. Perelman's proof of the Geometrization Conjecture that $\mathrm{S}^1 \times \mathrm{M}$ admits a symplectic form in the case when M is reducible if and only if $\mathrm{M} = \mathrm{S}^1 \times \mathrm{S}^2$.

The organization of this dissertation is as follows: Chapter II presents the required background material on Seiberg–Witten theory in dimension 3. There, we follow the book by Kronheimer and Mrowka [9], which presents an extensive treatment of the subject. Therefore, the reader is referred to their book for proofs of most of the results stated in that chapter. In Chapter III, we prove Theorems I.2 and I.4. The author would like to inform the reader of the fact that most of what is in that chapter has appeared in a joint paper with Taubes (see [11]). Finally, in Chapter IV, we conclude with some remarks as to how one could proceed in order to give a complete proof of Conjecture I.1 using our approach.

CHAPTER II

Background on Seiberg–Witten theory

In this chapter, we present a brief introduction to the theory of Seiberg–Witten invariants of 3-dimensional manifolds and Seiberg–Witten Floer homology as defined by Kronheimer and Mrowka in their book [9].

2.1 Spin^c structures and oriented two-plane fields

Spin^c structures constitute the geometric background on which Seiberg–Witten theory is built. Although the notion of a spin^c structure can be defined for manifolds of any dimension, we will focus on manifolds of dimension three.

Since $\pi_1(\mathrm{SO}(3)) \cong \mathbb{Z}_2$, there is a unique connected double cover of the Lie group $\mathrm{SO}(3)$, namely the group $\mathrm{Spin}(3) = \mathrm{SU}(2)$. The group $\mathrm{Spin}^c(3)$ is defined as the quotient of $\mathrm{U}(1) \times \mathrm{Spin}(3)$ by the diagonal action of $\mathbb{Z}_2$, thus the group $\mathrm{U}(2)$. Now, let M be a closed, oriented 3-manifold, and fix a Riemannian metric on M. Then, a spin^c structure $\mathfrak{s}$ on M is a principal $\mathrm{U}(2)$-bundle $\tilde{\mathrm{P}}$ such that $\tilde{\mathrm{P}} \times_\rho \mathrm{SO}(3) \cong \mathrm{P}_{\mathrm{SO}(3)}$, the principal $\mathrm{SO}(3)$-bundle associated to the tangent bundle of M. Here, ρ denotes the natural projection of $\mathrm{U}(2)$ onto $\mathrm{U}(2)/\mathrm{U}(1) = \mathrm{SO}(3)$.

A spin^c structure on M has an associated Hermitian $\mathbb{C}^2$-bundle $\mathbb{S}$, defined by the defining representation of $\mathrm{U}(2)$, and a Clifford multiplication $\mathfrak{cl} : \mathrm{T}^*\mathrm{M} \to \mathrm{End}_{\mathbb{C}}(\mathbb{S})$. The bundle $\mathbb{S}$ is called the *spinor bundle*. Its sections are called the *spinors*. The

Clifford multiplication is a bundle map that identifies $\mathrm{T}^*\mathrm{M}$ isometrically with the bundle $\mathfrak{su}(\mathbb{S})$ of traceless, skew-adjoint endomorphisms of $\mathbb{S}$ equipped with the inner product $\frac{1}{2}tr(\mathrm{A}^*\mathrm{B})$. It also respects the orientations, namely, $\mathfrak{cl}(e^1)\mathfrak{cl}(e^2)\mathfrak{cl}(e^3) = 1$ for an oriented orthonormal frame $\{e^1, e^2, e^3\}$. Moreover, having fixed an oriented orthonormal frame $\{e^1, e^2, e^3\}$ of the tangent bundle to M, there exists a splitting of the spinor bundle into a direct sum of two complex line bundles so that Clifford multiplication by each e^i has one of the following matrix representations:

$$\mathfrak{cl}(\mathrm{e}^1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad , \quad \mathfrak{cl}(\mathrm{e}^2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad , \quad \mathfrak{cl}(\mathrm{e}^3) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The Clifford multiplication can be extended to $\wedge\mathrm{T}^*\mathrm{M}$ by the rule

$$\mathfrak{cl}(\mathrm{a}\wedge\mathrm{b}) = \frac{1}{2}(\mathfrak{cl}(\mathrm{a})\mathfrak{cl}(\mathrm{b}) + (-1)^{deg(\mathrm{a})deg(\mathrm{b})}\mathfrak{cl}(\mathrm{b})\mathfrak{cl}(\mathrm{a})).$$

Therefore, by the orientation convention, $\mathfrak{cl}(*\mathrm{a}) = -\mathfrak{cl}(\mathrm{a})$.

There is also a map $det : \mathrm{U}(2) \to \mathrm{U}(1)$ defined by taking the determinant. This representation of U(2) yields a principal U(1)-bundle $\tilde{\mathrm{P}} \times_{det} \mathrm{U}(1)$. The complex line bundle associated to $\tilde{\mathrm{P}} \times_{det} \mathrm{U}(1)$ is called the determinant bundle of the $\mathrm{spin}^{\mathrm{c}}$ structure, which we denote by $det(\mathbb{S})$, because this line bundle is the second exterior power of the bundle $\mathbb{S}$. The first Chern class of $det(\mathbb{S})$ will be denoted by $c_1(\mathfrak{s})$.

Existence of $\mathrm{spin}^{\mathrm{c}}$ structures on M is due to the fact that the tangent bundle of an oriented 3-dimesional manifold is trivial. The set of $\mathrm{spin}^{\mathrm{c}}$ structures (up to isomorphism) on M form a principle bundle over a point for the additive group $\mathrm{H}^2(\mathrm{M};\mathbb{Z})$. To elaborate, a given cohomology class $e \in \mathrm{H}^2(\mathrm{M};\mathbb{Z})$ acts on a given $\mathrm{spin}^{\mathrm{c}}$ structure in such a way that the spinor bundle for the new $\mathrm{spin}^{\mathrm{c}}$ structure is obtained from that of the original one by tensoring with a complex line bundle E whose first Chern class is e. The Clifford multiplication changes by tensoring that of the original one with the identity endomorphism of E.

Next, we discuss an alternative and perhaps more topological way to describe the notion of a spin$^{\mathrm{c}}$ structure. In this regard, let $\mathbb{J}(\mathrm{M})$ denote the set of homotopy classes of oriented 2-plane fields on M. This set is non-empty because the tangent bundle of M is trivial. The following lemma provides a relationship between oriented 2-plane fields and spin$^{\mathrm{c}}$ structures on M.

Lemma II.1. [9, Lemma 28.1.1] *There exists a one-to-one correspondence between oriented 2-plane fields and isomorphism classes of pairs* $(\mathfrak{s}, \psi)$ *consisting of a spin*^{c} *structure* $\mathfrak{s}$ *and a unit-length spinor* ψ.

Proof. Given an oriented 2-plane field ξ, there exists a unique unit-length 1-form a on M such that ξ is the kernel of a and $*a$ restricts positively on ξ. Then, define a pair $(\mathfrak{s}, \psi)$ consisting of a spin$^{\mathrm{c}}$ structure $\mathfrak{s}$ and a unit-length spinor ψ as follows: Let $\mathbb{S} = \underline{\mathbb{C}} \oplus \xi$ and ψ denote the section $(1, 0)$ of $\mathbb{S}$. Then, define the Clifford multiplication $\mathfrak{cl} : \mathrm{T}^*\mathrm{M} \to \mathrm{End}_{\mathbb{C}}(\mathbb{S})$ by

$$\mathfrak{cl}(a) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \textit{and} \quad \mathfrak{cl}(b)(1,0) = (0, b^{\dagger})$$

for any 1-form b orthogonal to a. Here, $b^{\dagger}$ denotes the metric-dual of the 1-form b in ξ. This data is enough to define a spin$^{\mathrm{c}}$ structure $\mathfrak{s}$. Conversely, given a pair $(\mathfrak{s}, \psi)$ of a spin$^{\mathrm{c}}$ structure and a unit-length spinor, there exists a unique unit-length 1-form a on M such that the spinor bundle splits into eigenbundles of $\mathfrak{cl}(a)$ as $\mathbb{S} = \mathbb{C}\psi \oplus \psi^{\perp}$. Then, the 2-plane field defined by the kernel of a and oriented by $*a$ is isomorphic to the complex line bundle $\psi^{\perp}$. Note that the two constructions described above are inverses of each other. $\square$

Therefore, one can think of $\mathbb{J}(\mathrm{M})$ as the set of homotopy classes of pairs $(\mathfrak{s}, \psi)$ consisting of a spin$^{\mathrm{c}}$ structure $\mathfrak{s}$ and a unit-length spinor ψ. Note, in this regard, that by Lemma II.1 no two pairs $(\mathfrak{s}_0, \psi_0)$ and $(\mathfrak{s}_1, \psi_1)$ are homotopic unless $\mathfrak{s}_0$ and $\mathfrak{s}_1$ are isomorphic. The next lemma provides a classification of such pairs up to homotopy for a fixed isomorphism class of spin^c structures.

Lemma II.2. [9, Lemma 28.2.1] *Let $\mathfrak{s}_0$ be a spin^c structure on M. The pairs $(\mathfrak{s}, \psi)$ consisting of a spin^c structure $\mathfrak{s}$ isomorphic to $\mathfrak{s}_0$ and a unit-length spinor ψ are classified up to homotopy by the cokernel of the map*

$$\mathrm{H}_2(\mathrm{M}; \mathbb{Z}) \to \mathbb{Z} \quad : \quad [\sigma] \mapsto \langle c_1(\mathfrak{s}_0), [\sigma] \rangle.$$

Proof. Fix a unit-length section ψ_0 of the spinor bundle $\mathbb{S}_0$, and let ψ_1 be any other unit-length spinor. Then, the homotopy class of ψ_1 is determined by the Euler class of the pull-back of $\mathbb{S}_0$ onto $\mathrm{I} \times \mathrm{M}$ relative to the sections ψ_0 and ψ_1 at the boundary, namely, $\delta(\psi_0, \psi_1) = e(\mathrm{I} \times \mathbb{S}_0, \psi_0 \sqcup \psi_1)[\mathrm{I} \times \mathrm{M}, \partial \mathrm{I} \times \mathrm{M}]$. Since both ψ_0 and ψ_1 are of unit-length, we can write $\psi_1 = u\psi_0$ for some $u : \mathrm{M} \to \mathrm{S}^1$. On the other hand, by Lemma II.1, $\mathbb{S}_0 = \underline{\mathbb{C}} \oplus \mathrm{L}$ for some complex line bundle L. Now, the claim of the lemma follows from the fact that $\delta(\psi_0, \psi_1)$ is equal to the Euler number of the bundle over $\mathrm{S}^1 \times \mathrm{M}$ obtained from $\mathrm{I} \times \mathbb{S}_0$ by gluing the ends via the map $(1, u)$. Note that the latter is precisely $([u^{-1}\mathrm{d}u] \cup c_1(\mathfrak{s}_0))[\mathrm{M}]$. $\square$

Now, for a fixed spin^c structure $\mathfrak{s}$ on M, let $\mathbb{J}(\mathrm{M}, \mathfrak{s})$ denote the set of homotopy classes of oriented 2-plane fields that correspond to $\mathfrak{s}$. By Lemma II.2, $\mathbb{J}(\mathrm{M}, \mathfrak{s})$ is identified with $\mathbb{Z}/p$ where p is the greatest integer divisor of the class $c_1(\mathfrak{s})$ unless $c_1(\mathfrak{s})$ is torsion, in which case p is taken to be 0. Then, we can write $\mathbb{J}(\mathrm{M}) = \bigsqcup_{\mathfrak{s}} \mathbb{J}(\mathrm{M}, \mathfrak{s})$.

For the remainder of this chapter, $\mathfrak{s}$ will denote a spin^c structure on M with associated spinor bundle $\mathbb{S}$ and Clifford multiplication $\mathfrak{cl}$.

2.2 The Seiberg–Witten equations

In this section, we introduce the Seiberg–Witten equations and the numerical invariants associated to their solutions.

A unitary connection $\mathbb{A}$ on $det(\mathbb{S})$ together with the Levi-Civita connection on the orthonormal frame bundle of M determines a spin^c connection $\mathbf{A}$ on the spinor bundle $\mathbb{S}$, that is, a Hermitian connection on $\mathbb{S}$ that leaves $\mathfrak{cl}$ parallel. Then the Seiberg–Witten monopole equations are

$$\begin{aligned} *\mathrm{F}_{\mathbb{A}} &= \psi^\dagger \tau \psi \\ \mathcal{D}_{\mathbb{A}}\psi &= 0. \end{aligned} \tag{2.1}$$

Here, the notation is as follows: First, $\mathrm{F}_{\mathbb{A}} \in \Omega^2(\mathrm{M}, i\mathbb{R})$ denotes the curvature of the connection $\mathbb{A}$. Second, ψ is a section of the spinor bundle $\mathbb{S}$. Third, $\psi^\dagger\tau\psi$ denotes the section of $i\mathrm{T}^*\mathrm{M}$ which is the metric dual of the homomorphism $\psi^\dagger\mathfrak{cl}(\cdot)\psi : \mathrm{T}^*\mathrm{M} \to i\mathbb{R}$. Fourth, $\mathcal{D}_{\mathbb{A}}$ is the Dirac operator associated to $\mathbf{A}$, which is defined by

$$\Gamma(\mathbb{S}) \xrightarrow{\nabla_{\mathbf{A}}} \Gamma(\mathrm{T}^*\mathrm{M} \otimes \mathbb{S}) \xrightarrow{\mathfrak{cl}} \Gamma(\mathbb{S}).$$

The group of gauge transformations of a spin^c structure, the so-called *gauge group* $\mathcal{G} = \mathrm{C}^\infty(\mathrm{M}, \mathrm{S}^1)$, acts on the configuration space $\mathcal{C} = \mathrm{Conn}(det(\mathbb{S})) \times \mathrm{C}^\infty(\mathrm{M}; \mathbb{S})$ as

$$\begin{aligned} \mathcal{G} \times \mathcal{C} &\longrightarrow \mathcal{C} \\ (u, (\mathbb{A}, \psi)) &\longmapsto (\mathbb{A} - 2u^{-1}\mathrm{d}u, u\psi). \end{aligned}$$

The Seiberg–Witten equations are invariant under the action of the gauge group. Therefore, one can define the space of equivalence classes of solutions under the action of the gauge group. This is called the *moduli space* and we denote it by $\mathcal{M}$. The solutions of the Seiberg–Witten equations which are of the form $(\mathbb{A}, 0)$ are called reducible solutions because the stabilizer under the action of the gauge group is not trivial. Solutions with non-zero spinor component are called irreducible. We let $\mathcal{B} = \mathcal{C}/\mathcal{G}$. It is possible to prove that $\mathcal{M}$ is a sequentially compact subset of $\mathcal{B}$. The gauge group $\mathcal{G}$ acts freely on the space of irreducible solutions of the Seiberg–Witten equations. A suitable perturbation of the Seiberg–Witten equations guarantees that the quotient of this space by $\mathcal{G}$ is a finite set of points in $\mathcal{B}$.

To elaborate, let $\underline{\mathbb{R}}$ denote the trivial line bundle over M. Each $(\mathbb{A}, \psi) \in \mathcal{C}$ has an associated linear operator $\mathcal{L}_{(\mathbb{A},\psi)}$ that maps $\mathrm{C}^\infty(\mathrm{M}; i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}})$ onto itself. It is defined as

$$\mathcal{L}_{(\mathbb{A},\psi)}(\mathrm{b}, \phi, g) = \begin{pmatrix} *\mathrm{db} - \mathrm{d}g - (\psi^\dagger \tau \phi + \phi^\dagger \tau \psi) \\ \mathcal{D}_{\mathbb{A}}\phi + \frac{1}{2}\mathfrak{cl}(\mathrm{b})\psi + g\psi \\ -\mathrm{d}^*\mathrm{b} - \frac{1}{2}(\phi^\dagger\psi - \psi^\dagger\phi) \end{pmatrix}. \tag{2.2}$$

This operator extends to $\mathrm{L}^2(\mathrm{M}; i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}})$ as an unbounded, self-adjoint and Fredholm operator with dense domain ${\mathrm{L}^2}_1(\mathrm{M}; i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}})$. Furthermore, it has a discrete spectrum that is unbounded from above and below. The spectrum has no accumulation points, and each eigenvalue has finite multiplicity (see [9, Section 12]). An irreducible solution of the Seiberg–Witten equations is called non-degenerate if $\mathcal{L}$ has trivial kernel. A suitable perturbation of the Seiberg–Witten equations renders all irreducible solutions of the perturbed equations non-degenerate. In this case, irreducible solutions of the perturbed Seiberg–Witten equations define isolated points in $\mathcal{B}$. We shall denote the set of such points by $\mathcal{B}^*$.

The perturbations that are under consideration here result from the choice of a closed 2-form ρ on M. With such a 2-form chosen, the perturbed version of the

Seiberg–Witten equations read

$$
\begin{aligned}
*\mathrm{F}_{\mathbb{A}} &= \psi^{\dagger}\tau\psi - i * \rho \\
\mathcal{D}_{\mathbb{A}}\psi &= 0.
\end{aligned}
\tag{2.3}
$$

Now, suppose that $b_1(\mathrm{M}) > 0$. Then, having fixed a suitable perturbation ρ for the Seiberg–Witten equations, each point in $\mathcal{B}^*$ is assigned a canonically defined sign determined by a fixed orientation of $\mathrm{H}^1(\mathrm{M};\mathbb{R})$. The latter is a non-zero element $o \in \mathrm{H}^1(\mathrm{M};\mathbb{R})$. When $b_1(\mathrm{M}) = 1$, we further require that $([\rho] - 2\pi c_1(\mathfrak{s})) \cdot o > 0$. Then, a signed count of the points in $\mathcal{B}^*$ results in an integer which we denote by $SW(\mathrm{M}, \mathfrak{s})$. This is the so-called *Seiberg–Witten invariant* of M corresponding to the spin$^{\mathrm{c}}$ structure $\mathfrak{s}$. $SW(\mathrm{M}, \mathfrak{s})$ constitutes an invariant of the topology of M and the spin$^{\mathrm{c}}$ structure $\mathfrak{s}$.

2.3 Seiberg–Witten Floer homology

The Seiberg–Witten equations are the variational equations of a functional defined on the configuration space $\mathcal{C}$ by

$$\mathfrak{csd}(\mathbb{A}, \psi) = -\frac{1}{4}\int_{\mathrm{M}} (\mathbb{A} - \mathbb{A}_{\mathbb{S}}) \wedge (\mathrm{F}_{\mathbb{A}} + \mathrm{F}_{\mathbb{A}_{\mathbb{S}}}) + \int_{\mathrm{M}} \psi^{\dagger}\mathcal{D}_{\mathbb{A}}\psi.$$

Here, $\mathbb{A}_{\mathbb{S}}$ is any given connection fixed in advance on $det(\mathbb{S})$. This is the so-called *Chern-Simons-Dirac* functional. Seiberg–Witten Floer homology can be regarded as an infinite dimensional version of the Morse homology theory where $\mathcal{B}$ plays the role of the ambient manifold and the Chern-Simons-Dirac functional plays the role of the "Morse" function. As the critical points of the Chern-Simons-Dirac functional are solutions of the Seiberg–Witten equations, the latter are used, as in Morse theory, to label generators of the chain complex. The analogue of a non-degenerate critical point is a solution of the Seiberg–Witten equations whose version of $\mathcal{L}$ has trivial kernel if the solution is irreducible or the kernel is spanned by elements of the form $(0,0,i)$ and $(\mathrm{b},0,0)$ where b is a harmonic 1-form on M. Here, the point is that $\mathcal{L}$ is, formally, the Hessian of the Chern-Simons-Dirac functional.

Kronheimer and Mrowka describe in Chapter III of their book a *large*, separable Banach space $\mathcal{P}$ of *tame* perturbations to use with the Seiberg–Witten equations. The space $\mathcal{P}$ consists of formal gradients of smooth $\mathcal{G}$-invariant functions on $\mathcal{C}$ with certain additional properties. Moreover, Kronheimer and Mrowka prove that $\mathcal{P}$ contains a residual set of *admissible* perturbations. In particular, an admissible perturbation has the following property: The perturbed version of the Seiberg–Witten equations have only irreducible solutions unless $c_1(\mathfrak{s})$ is torsion, and all of the solutions are non-degenerate.

Having fixed an admissible perturbation to use with the Seiberg–Witten equations, Kronheimer and Mrowka define in Chapter VI of their book three chain complexes $(\check{C},\check{\partial})$, $(\hat{C},\hat{\partial})$, and $(\bar{C},\bar{\partial})$. Roughly speaking, the chain groups $\check{C}$, $\hat{C}$, and $\bar{C}$ are free abelian groups whose generators are obtained from gauge equivalence classes of irreducible and/or reducible solutions to the Seiberg–Witten equations via a blow-up of $\mathcal{B}$ along the set of gauge equivalence classes of reducible configurations. Blowing-up $\mathcal{B}$ results in one generator for each gauge equivalence class of an irreducible solution, and a countable set of generators for each gauge equivalence class of a reducible solution. We will explain what goes into the definitions of the three differentials $\check{\partial}$, $\hat{\partial}$, and $\bar{\partial}$ momentarily.

Remember that the differential in the Morse complex is defined via a signed count of the downward gradient flow lines of the Morse function. The analog in this context of a gradient flow line in finite dimensional Morse theory is a smooth map

$s \mapsto (\mathbb{A}(s), \psi(s))$ from $\mathbb{R}$ into $\mathcal{C}$ that obeys the rule

$$
\begin{aligned}
\frac{\partial}{\partial s}\mathbb{A} &= -*\mathrm{F}_{\mathbb{A}} + \psi^{\dagger}\tau\psi \\
\frac{\partial}{\partial s}\psi &= -\mathcal{D}_{\mathbb{A}}\psi.
\end{aligned}
\tag{2.4}
$$

This can also be written as $\frac{\partial}{\partial s}(\mathbb{A}, \psi) = -\nabla_{\mathrm{L}^2}\mathfrak{csd}|_{(\mathbb{A},\psi)}$ where ∇_{L^2} denotes the L^2-gradient of $\mathfrak{csd}$. An *instanton* is a solution of these equations on $\mathbb{R}\times\mathrm{M}$ that converges to a solution of the Seiberg–Witten equations on each end as $|s|$ tends to infinity. Note that the equations in (2.4) are also invariant under the action of $\mathcal{G}$.

The differentials $\check{\partial}$, $\hat{\partial}$, and $\bar{\partial}$ are defined using a suitably perturbed version of these instanton equations. As in finite dimensional Morse theory, a perturbation is in general necessary in order to have a well defined count of solutions. The perturbed equations can be viewed as defining the analog of what in finite dimensions would be the equations that define the flow lines of a pseudo-gradient vector field for the given function. In this regard, admissible perturbations guarantee that the resulting instanton equations can serve to define the differentials by means of a signed count of the gauge equivalence classes of solutions. Then, the chain complexes $(\check{C}, \check{\partial})$, $(\hat{C}, \hat{\partial})$, and $(\bar{C}, \bar{\partial})$ as defined by the perturbed version of the Seiberg–Witten equations yield the three versions of the Seiberg–Witten Floer homology denoted respectively by $\widecheck{HM}(\mathrm{M}, \mathfrak{s})$, $\widehat{HM}(\mathrm{M}, \mathfrak{s})$ and $\overline{HM}(\mathrm{M}, \mathfrak{s})$. These three groups constitute an invariant that depends only on the topology of M and the spin$^{\mathrm{c}}$ structure $\mathfrak{s}$.

Each of these Floer homology groups admits a $\mathbb{Z}/p$ grading, where p is the greatest integer divisor of $c_1(\mathfrak{s})$ unless $c_1(\mathfrak{s})$ is torsion, in which case p is taken to be 0. One way to see this is by associating to each generator of the three chain complexes $(\check{C}, \check{\partial})$, $(\hat{C}, \hat{\partial})$, and $(\bar{C}, \bar{\partial})$ a unit-length section of the spinor bundle, hence an element of $\mathbb{J}(\mathrm{M}, \mathfrak{s})$ by Lemma II.1 (see [9, Section 28]). Alternatively, as the Hessian in finite dimensional Morse theory can be used to define the grading of the Morse complex, it is also the case here that the operator $\mathcal{L}$ can be used to define gradings for the three Seiberg–Witten Floer homology chain complexes. In particular, $\mathcal{L}$ can be used to associate an integer degree to each non-degenerate solution of the Seiberg–Witten equations, in fact, to any given pair in $\mathcal{C}$ whose version of $\mathcal{L}$ has trivial kernel. It is enough to say here that this degree involves the notion of spectral flow for families of self adjoint operators such as $\mathcal{L}$. In general, only the $mod(p)$ reduction of this degree is gauge invariant. Therefore, it descends to a $\mathbb{Z}/p$ grading on the set of gauge equivalence classes of solutions to a suitably perturbed version of the Seiberg–Witten equations.

For a fixed spin^c structure $\mathfrak{s}$, the three Seiberg–Witten Floer homology groups $\widecheck{HM}(\mathrm{M}, \mathfrak{s})$, $\widehat{HM}(\mathrm{M}, \mathfrak{s})$ and $\overline{HM}(\mathrm{M}, \mathfrak{s})$ are finitely generated in each degree. However, if $c_1(\mathfrak{s})$ is torsion, then the degrees at which $\widecheck{HM}(\mathrm{M}, \mathfrak{s})$ is non-trivial are bounded from below, but unbounded from above. Similarly, the degrees at which $\widehat{HM}(\mathrm{M}, \mathfrak{s})$ is non-trivial are bounded from above, but unbounded from below. On the other hand, if $c_1(\mathfrak{s})$ is non-torsion, then both of these groups are finitely generated, and $\overline{HM}(\mathrm{M}, \mathfrak{s})$ is the trivial group. Furthermore, the two Seiberg–Witten Floer homology groups $\widecheck{HM}(\mathrm{M}, \mathfrak{s})$ and $\widehat{HM}(\mathrm{M}, \mathfrak{s})$ are isomorphic (see [9, Section 22]).

An important property of the Seiberg–Witten Floer homology is that it admits an involution. This involution is the result of a natural involution map defined on $\mathcal{S}(\mathrm{M})$, namely, *charge conjugation*. To elaborate, let $\mathfrak{s}^*$ denote the spin^c structure that is the complex conjugate of $\mathfrak{s}$. The former has the associated spinor bundle $\overline{\mathbb{S}}$ and Clifford multiplication the same as that associated to $\mathfrak{s}$ (as $\mathbb{R}$-linear maps). Then, Kronheimer and Mrowka prove that there are isomorphisms between each of the three versions of the Seiberg–Witten Floer homology for the conjugate spin^c

structures $\mathfrak{s}$ and $\mathfrak{s}^*$ (see [9, Propositions 25.5.5 and 25.5.7]).

The organization of the remainder of this section is as follows: First, we give an overview of the notion of *non-exact* perturbations for the Seiberg–Witten equations. These are the sort of perturbations used in this dissertation. Then, some relevant results on the vanishing/non-vanishing of the Seiberg–Witten Floer homology groups and their consequences are discussed.

2.3.1 Non-exact perturbations

As we mentioned in the beginning of this section, the definition of Seiberg–Witten Floer homology requires the choice of a perturbation from a residual subset of the Banach space $\mathcal{P}$. This Banach space consists of smooth $\mathcal{G}$-invariant functions on the configuration space $\mathcal{C}$, and the differential of such a function, $\mathfrak{g}$, at any given $(\mathbb{A}, \psi)$ defines a section $(\mathfrak{G}|_{(\mathbb{A},\psi)}, \mathfrak{H}_{(\mathbb{A},\psi)})$ of $i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S}$ by $\frac{d}{dt}\mathfrak{g}(\mathbb{A} + t\mathrm{b}, \psi + t\phi)|_{t=0} = \int_{\mathrm{M}}(\mathrm{b} \wedge *\mathfrak{G} - \frac{1}{2}(\phi^\dagger\mathfrak{H} + \mathfrak{H}^\dagger\phi))$. Then, the resulting perturbed version of the Seiberg–Witten equations read

$$\begin{aligned} *\mathrm{F}_{\mathbb{A}} &= \psi^\dagger\tau\psi + \mathfrak{G}|_{(\mathbb{A},\psi)} \\ \mathcal{D}_{\mathbb{A}}\psi &= \mathfrak{H}|_{(\mathbb{A},\psi)}. \end{aligned} \tag{2.5}$$

Note that since $\mathfrak{g}$ is $\mathcal{G}$-invariant, so are the sections $(\mathfrak{G}|_{(\mathbb{A},\psi)}, \mathfrak{H}_{(\mathbb{A},\psi)})$ and therefore the equations in (2.5).

The equations in (2.5) defines an operator on $\mathrm{C}^\infty(\mathrm{M}; i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}})$ by

$$\mathcal{L}^{\mathfrak{g}}_{(\mathbb{A},\psi)}(\mathrm{b}, \phi, g) = \begin{pmatrix} *\mathrm{db} - \mathrm{d}g - (\psi^\dagger\tau\phi + \phi^\dagger\tau\psi) - \mathfrak{D}\mathfrak{G}|_{(\mathbb{A},\psi)}(\mathrm{b}, \phi) \\ \mathcal{D}_{\mathbb{A}}\phi + \frac{1}{2}\mathfrak{cl}(\mathrm{b})\psi + g\psi - \mathfrak{D}\mathfrak{H}|_{(\mathbb{A},\psi)}(\mathrm{b}, \phi) \\ -\mathrm{d}^*\mathrm{b} - \frac{1}{2}(\phi^\dagger\psi - \psi^\dagger\phi) \end{pmatrix} \tag{2.6}$$

where $(\mathfrak{D}\mathfrak{G}|_{(\mathbb{A},\psi)}, \mathfrak{D}\mathfrak{H}|_{(\mathbb{A},\psi)})$ defines an operator on $\mathrm{C}^\infty(\mathrm{M}; i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S})$ by

$$(\mathfrak{D}\mathfrak{G}|_{(\mathbb{A},\psi)}(\mathrm{b}, \phi), \mathfrak{D}\mathfrak{H}|_{(\mathbb{A},\psi)}(\mathrm{b}, \phi)) = (\frac{d}{dt}\mathfrak{G}|_{(\mathbb{A}+t\mathrm{b},\psi+t\phi)}, \frac{d}{dt}\mathfrak{H}|_{(\mathbb{A}+t\mathrm{b},\psi+t\phi)})|_{t=0}.$$

The operator in (2.6) is symmetric and it extends to $L^2(M; iT^*M \oplus \mathbb{S} \oplus i\underline{\mathbb{R}})$ as an unbounded, self-adjoint operator with dense domain $L^2{}_1(M; iT^*M \oplus \mathbb{S} \oplus i\underline{\mathbb{R}})$.

The Banach space $\mathcal{P}$ contains a subspace, Ω, of smooth 1-forms σ for use in (2.5). In order to define this subspace, first denote by Ω_0 the space of finite linear combinations of eigenfunctions of the operator $*d$ on $C^\infty(M, T^*M)$. If $\sigma \in \Omega_0$, then the function $\mathfrak{g}_\sigma$, on $\mathcal{C}$ defined by $\mathfrak{g}_\sigma(\mathbb{A}, \psi) = i\int_M \sigma \wedge F_{\mathbb{A}}$ is a function in $\mathcal{P}$. Therefore, $\mathcal{P}$ contains the linear subspace $\{\mathfrak{g}_\sigma : \sigma \in \Omega_0\}$. Moreover, the induced norm on this linear space dominates all of the C^k-norms on $C^\infty(M; T^*M)$. Then, Ω is defined to be the completion of this linear subspace of $\mathcal{P}$ with respect to the induced norm. Because of the previously mentioned fact, each $\sigma \in \Omega$ is smooth. In fact, if M is assumed to have a real analytic structure, then each $\sigma \in \Omega$ is itself real analytic. With $\mathfrak{g} = \mathfrak{g}_\sigma$, the resulting version of the Seiberg–Witten equations read

$$
\begin{aligned}
*F_{\mathbb{A}} &= \psi^\dagger \tau \psi + i * d\sigma \\
\mathcal{D}_{\mathbb{A}}\psi &= 0,
\end{aligned} \tag{2.7}
$$

whereas the version of the operator $\mathcal{L}^{\mathfrak{g}}$ in this case is the same as in (2.2).

Note that the Chern-Simons-Dirac functional is not usually $\mathcal{G}$-invariant. To be more precise, for every $(\mathbb{A}, \psi) \in \mathcal{C}$ and $u \in \mathcal{G}$ we have

$$
\begin{aligned}
\mathfrak{csd}(\mathbb{A} - 2u^{-1}du, u\psi) - \mathfrak{csd}(\mathbb{A}, \psi) &= \int_M u^{-1}du \wedge F_{\mathbb{A}_{\mathbb{S}}} \\
&= (2\pi i[u] \cup -2\pi i c_1(\mathfrak{s})) \cap [M] \\
&= 4\pi^2([u] \cup c_1(\mathfrak{s})) \cap [M]
\end{aligned}
$$

where $[u] \in H^1(M; \mathbb{Z})$ corresponds to the base-free homotopy class of the map u. Denote by τ the integer $([u] \cup c_1(\mathfrak{s})) \cap [M]$. Then, the Chern-Simons-Dirac functional is said to have *period* $4\pi^2\tau$ for $\mathfrak{s}$. Note also that since the perturbations that are discussed above are all $\mathcal{G}$-invariant, the perturbed version of the Chern-Simon-Dirac functional, $\mathfrak{csd} + \mathfrak{g}$, has the same periods as the original one. We will refer to these perturbations as the *exact* perturbations.

If $c_1(\mathfrak{s})$ is non-torsion, the Chern-Simons-Dirac functional descends to a multi-valued function on $\mathcal{B}$, and the perturbations of the sort discussed above do not change this fact. Therefore, Kronheimer and Mrowka suggested in Chapter VIII of [9] to consider a larger class of perturbations, which would allow the periods of the Chern-Simons-Dirac functional to change, the so-called *non-exact* perturbations. Non-exact perturbations are smooth functions on $\mathcal{C}$ which are not $\mathcal{G}$-invariant but their differentials are. Such a function, $\mathfrak{f}$, is said to have *period class* $c \in \mathrm{H}^2(\mathrm{M};\mathbb{R})$ if for every $(\mathbb{A}, \psi) \in \mathcal{C}$ and $u \in \mathcal{G}$, $\mathfrak{f}(\mathbb{A} - 2u^{-1}\mathrm{d}u, u\psi) - \mathfrak{f}(\mathbb{A}, \psi) = ([u] \cup c) \cap [\mathrm{M}]$. An example of non-exact perturbations can be constructed as follows: Let ρ be a closed 2-form on M. Define a function $\mathfrak{f}_\rho$ on $\mathcal{C}$ by $\mathfrak{f}_\rho(\mathbb{A}, \psi) = -i\int_{\mathrm{M}}(\mathbb{A} - \mathbb{A}_{\mathbb{S}}) \wedge \rho$. The version of the Seiberg–Witten equations obtained from the perturbed Chern-Simons-Dirac functional $\mathfrak{csd} + \mathfrak{f}_\rho$ is exactly (2.3). The period class of this perturbation is $-4\pi[\rho]$, and the periods of the perturbed Chern-Simons-Dirac functional are calculated by $4\pi^2([u] \cup (c_1(\mathfrak{s}) + \frac{1}{4\pi^2}c)) \cap [\mathrm{M}]$.

Having fixed a spin$^{\mathrm{c}}$-structure, $\mathfrak{s}$, Kronheimer and Mrowka consider non-exact perturbations with period class c such that $4\pi^2 c_1(\mathfrak{s}) + c = \lambda 4\pi^2 c_1(\mathfrak{s})$ for some $\lambda \in \mathbb{R}$. Such non-exact perturbations are called *balanced*, *positively monotone* or *negatively monotone* respectively when $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$. Once the notion of an admissible perturbation is extended to the case of non-exact perturbations (as is explained in Chapter VIII of [9]), the results from the case of exact perturbations carry onto the balanced and monotone non-exact cases almost without any change, and there are canonical isomorphisms between the three Seiberg–Witten Floer homology groups and the Floer homology groups defined using admissible balanced or monotone non-exact perturbations (see [9, Theorems 31.1.1 and 31.1.2]).

2.3.2 Vanishing and non-vanishing theorems

We continue with a discussion of some well-known facts about Seiberg–Witten Floer homology that are relevant to the content of this dissertation. As the proofs of the following results involve tools that are beyond the scope of this dissertation, we omit the proofs and refer the reader to the appropriate references.

Remember that, for a fixed spin^c structure, the moduli space of solutions to a suitably perturbed version of the Seiberg–Witten equations is compact. The latter follows from the fact that there are uniform bounds on the L^2 norms of the curvature of the connection component and the covariant derivative of the spinor component of a solution. In fact, one can choose perturbations for each spin^c structure so as to guarantee existence of a uniform bound on the L^2 norm of the curvature which would work for every spin^c structure. The following proposition is a consequence of this last fact.

Proposition II.3. [9, Proposition 3.1.1] *The groups* $\widecheck{HM}(\mathrm{M},\mathfrak{s})$, $\widehat{HM}(\mathrm{M},\mathfrak{s})$ *and* $\overline{HM}(\mathrm{M},\mathfrak{s})$ *are non-trivial for only finitely many* spin^c *structures* $\mathfrak{s}$*.*

By taking a closer look at the uniform bound on the L^2 norm of the curvature and choosing a particular type of Riemannian metric on M, Kronheimer and Mrowka manage to obtain a sharper result on the set of spin^c structures for which the Seiberg–Witten Floer homology groups are non-trivial. We state this result in the following proposition. The reader is referred to [9] for the proof.

Proposition II.4. [9, Proposition 40.1.1] *Let* $\mathfrak{s}$ *be a spin^c structure with non-torsion first Chern class and* $\Sigma \subset \mathrm{M}$ *be a smoothly embedded, connected, oriented surface of* *non-zero genus. If*

$$|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| > 2genus(\Sigma) - 2 \tag{2.8}$$

then there is a Riemannian metric on M *for which the unperturbed Seiberg–Witten equations admit no solutions.*

Next, we state a non-vanishing result which proves the sharpness of the bound in (2.8). The former concerns the notion of a *taut foliation* from topology. Therefore, we start with a short discussion of this subject.

Definition II.5. A codimension-1 foliation $\mathfrak{F}$ on a 3-dimensional manifold N is a collection of pairwise disjoint, smoothly embedded, connected surfaces, called the *leaves*, that cover N in such a way that around each point $x \in$ N there exists a local coordinate chart which parametrizes each surface that intersects this chart by horizontal planes in $\mathbb{R}^3$.

It turns out that codimension-1 foliations exist in abundance on 3-dimensional manifolds. In fact, any 2-plane field on a 3-dimensional manifold is homotopic to one that is tangent to a foliation (see e.g. [26]). Somewhat harder to come by is a type of foliation which we shall define next.

Definition II.6. A codimension-1 foliation on a 3-dimensional manifold N is called taut if there exists an embedded closed curve in N which intersects each and every leaf transversally.

The question about the existence of taut foliations is addressed by D. Gabai in [8]. Before we state Gabai's theorem on the existence of taut foliations, a short digression follows.

It is known that every homology class in $\mathrm{H}_2(\mathrm{M};\mathbb{Z})$ can be represented by a closed, oriented surface smoothly embedded in M. Let $\mathbf{S} \in \mathrm{H}_2(\mathrm{M};\mathbb{Z})$ and $\Sigma = \bigsqcup_{i=1}^{n} \Sigma_i$ be a closed, oriented surface in M that represents the class $\mathbf{S}$. Next, we define a "norm" for Σ.

$$|\Sigma| := \sum_{i=1}^{n} max\{0, 2genus(\Sigma_i) - 2\}.$$

Note that for a given homology class $\mathbf{S} \in \mathrm{H}_2(\mathrm{M}; \mathbb{Z})$, a surface Σ which represents $\mathbf{S}$ can be chosen so as to satisfy the following three conditions.

1. Σ achieves the smallest possible norm among all representatives of $\mathbf{S}$.

2. No component of Σ is a sphere.

3. Each genus 1 component of Σ is homologically non-trivial.

Condition (2) can be guaranteed by attaching a 1-handle to any existing sphere component, whereas condition (3) is guaranteed by the condition (1). With the preceding understood, we will now state Gabai's theorem on the existence of taut foliations on irreducible 3-manifolds. Remember that a 3-dimensional manifold is irreducible if it does not contain any homotopically non-trivial sphere.

Theorem II.7 (Gabai)**.** *Let* M *be a closed, oriented, irreducible* 3*-manifold and* $\Sigma \subset \mathrm{M}$ *be a smoothly embedded, closed, oriented, surface representing a non-zero homology class in* $\mathrm{H}_2(\mathrm{M}; \mathbb{Z})$ *and satisfying the above three conditions. Then there exists an oriented, taut foliation on* M *where* Σ *is a union of closed leaves of the foliation. This foliation has smooth leaves and an associated* C^0 *tangent plane field. In fact, this foliation is smooth except possibly along genus* 1 *components of* Σ.

Now, suppose that M is a closed, oriented, irreducible 3-manifold that carries an oriented, taut foliation $\mathfrak{F}$ with smooth leaves and associated C^0 tangent plane field. Suppose further that the Euler class $e(\mathfrak{F})$ of the associated field of tangent planes is non-torsion. Let $\mathfrak{s}_{\mathfrak{F}}$ be the spin^c structure corresponding to the tangent plane field of $\mathfrak{F}$, which has $c_1(\mathfrak{s}_{\mathfrak{F}}) = e(\mathfrak{F})$. Then, the following proposition states the promised non-vanishing result.

Proposition II.8. [9, Corollary 41.4.2] $\widetilde{HM}(\mathrm{M}, \mathfrak{s}_{\mathfrak{F}})$ *has non-zero rank in the degree that corresponds to the tangent plane field of* $\mathfrak{F}$.

Note that if $\Sigma \subset \mathrm{M}$ is a smoothly embedded, closed, connected, oriented surface representing a non-torsion homology class in $\mathrm{H}_2(\mathrm{M};\mathbb{Z})$ and satisfying the above three conditions, then Gabai's theorem provides us with a taut foliation $\mathfrak{F}$ such that $\langle e(\mathfrak{F}), [\Sigma]\rangle = 2genus(\Sigma) - 2$. Furthermore, $\widetilde{HM}(\mathrm{M}, \mathfrak{s}_{\mathfrak{F}})$ has non-zero rank. This observation is central to the discussion in the next subsection.

2.3.3 Floer homology and the Thurston norm

Using Proposition II.4 and Proposition II.8, it is possible to identify the set of spin$^{\mathrm{c}}$ structures for which the Seiberg–Witten Floer homology has non-zero rank in a more quantitative fashion. In this regard, we will begin by defining a semi-norm on the homology $\mathrm{H}_2(\mathrm{M};\mathbb{R})$ of a closed, orientable 3-manifold M. This latter was introduced by W. P. Thurston in [28] and it measures the minimal complexity of an embedded surface.

Let $\mathbf{S} \in \mathrm{H}_2(\mathrm{M};\mathbb{Z})$, then the so-called *Thurston norm* of $\mathbf{S}$ is defined by

$$||\mathbf{S}||_{\mathrm{T}} := min\{|\Sigma| \ : \ \Sigma \subset \mathrm{M} \text{ represents } \mathbf{S}\}.$$

Thurston shows that $||\cdot||_{\mathrm{T}}$ defines a $\mathbb{Z}$-linear and subadditive function on $\mathrm{H}_2(\mathrm{M};\mathbb{Z})$ which extends to a semi-norm on $\mathrm{H}_2(\mathrm{M};\mathbb{R})$. The reason why $||\cdot||_{\mathrm{T}}$ might fail to define a norm on $\mathrm{H}_2(\mathrm{M};\mathbb{R})$ is because there might be non-torsion homology classes in $\mathrm{H}_2(\mathrm{M};\mathbb{Z})$ which are represented by unions of embedded tori in M.

With the above understood, there is a naturally defined dual norm on a subspace of $\mathrm{H}^2(\mathrm{M};\mathbb{R})$, which we denote by $||\cdot||^{\mathrm{T}}$. Let $\mathbf{H} \subset \mathrm{H}^2(\mathrm{M};\mathbb{R})$ denote the linear subspace consisting of the cohomology classes which annihilate any member of the span in $\mathrm{H}_2(\mathrm{M};\mathbb{R})$ of the homology classes with vanishing Thurston norm. Then, if $e \in \mathbf{H}$, we define its dual Thurston norm as follows:

$$||e||^{\mathrm{T}} := inf\{C \geq 0 \ : \ \langle e, \mathbf{S}\rangle \leq C||\mathbf{S}||_{\mathrm{T}} \text{ for each } \mathbf{S} \in \mathrm{H}_2(\mathrm{M};\mathbb{R})\}.$$

The unit ball of the dual Thurston norm is a convex polytope in $\mathbf{H}$, and if M is irreducible then the unit ball of the dual Thurston norm is the convex hull of the Euler classes $e(\mathfrak{F})$ as $\mathfrak{F}$ runs through all taut foliations on M. This remarkable result is due to Thurston and Gabai (see [28] and [8]). Having said that, it is not hard to prove the following theorem.

Theorem II.9. [9, Theorem 41.5.2] *If* M *is a closed, orientable, irreducible* 3-*manifold, then the unit ball of the dual Thurston norm in* $\mathbf{H}$ *is the convex hull of the classes* $c_1(\mathfrak{s})$ *where* $\mathfrak{s}$ *runs through all spin*^{c} *structures for which* $\widetilde{HM}(\mathrm{M},\mathfrak{s})$ *has non-zero rank.*

Proof. First, if $\mathfrak{s}$ is a spin^c structure for which $\widetilde{HM}(\mathrm{M},\mathfrak{s})$ has non-zero rank, then by Proposition II.4 $|\langle c_1(\mathfrak{s}), [\Sigma]\rangle| \leq 2genus(\Sigma) - 2$ for any closed, connected, oriented surface Σ embedded in M. Therefore, $c_1(\mathfrak{s}) \in \mathbf{H}$ and $||c_1(\mathfrak{s})||^{\mathrm{T}} \leq 1$, i.e. $c_1(\mathfrak{s})$ is in the unit ball of the dual Thurston norm. Second, by Proposition II.8, the unit ball of the dual Thurston norm is inside the convex hull of the classes $c_1(\mathfrak{s})$ where $\mathfrak{s}$ runs through all spin^c structures for which $\widetilde{HM}(\mathrm{M},\mathfrak{s})$ has non-zero rank. This completes the proof of the theorem. $\square$

2.4 Sutured monopole homology

Seiberg–Witten Floer homology is defined only for closed manifolds. Recently, Kronheimer and Mrowka constructed a variant of Seiberg–Witten Floer homology for certain kinds of compact manifolds with boundary. The latter are the so-called *balanced sutured manifolds*. We will begin with the definitions of a balanced sutured manifold and *sutured monopole homology*, a Floer homology invariant for balanced sutured manifolds defined by Kronheimer and Mrowka in [10]. Then, we will see how sutured monopole homology can be used to detect closed, oriented 3-manifolds that fiber over the circle.

Definition II.10. A balanced sutured manifold is a pair (N, γ) that consists of a compact, oriented 3-manifold N with boundary $\partial\mathrm{N}$ and no closed components, and a collection γ of disjoint, closed, oriented curves in $\partial\mathrm{N}$, called the *sutures*, satisfying the following two conditions:

- Let $\mathrm{A}(\gamma)$ denote a collection of pairwise disjoint annuli around each component of γ and $\mathrm{R}(\gamma)$ denote the closure of $\partial\mathrm{N} \setminus \mathrm{A}(\gamma)$. Then, $\mathrm{R}(\gamma)$ does not contain any closed components.

- Orient the boundary of $\mathrm{A}(\gamma)$ in the same way as the curve γ, and orient $\mathrm{R}(\gamma)$ so that its oriented boundary coincides with the given orientation of the boundary of $\mathrm{A}(\gamma)$. As a result, $\mathrm{R}(\gamma)$ is divided into two regions $\mathrm{R}_+(\gamma)$ and $\mathrm{R}_-(\gamma)$ labeled according to whether the orientations determined by γ on either region agrees or disagrees with the boundary orientation. Then, $\chi(\mathrm{R}_+(\gamma)) = \chi(\mathrm{R}_-(\gamma))$.

Example II.11. The simplest example of a balanced sutured manifold is a *product sutured manifold*. A product sutured manifold is a pair $([-1, 1] \times \Sigma, \bar{\gamma})$ where Σ is a compact, oriented surface with non-empty boundary and no closed components, and $\bar{\gamma} = \{0\} \times \partial\Sigma$ oriented as the boundary of Σ. Then, $\mathrm{A}(\bar{\gamma}) = [-1, 1] \times \partial\Sigma$ and $\mathrm{R}_\pm(\bar{\gamma}) = \{\pm 1\} \times \Sigma$. Figure 2.1 shows the product sutured manifold where Σ is an annulus.

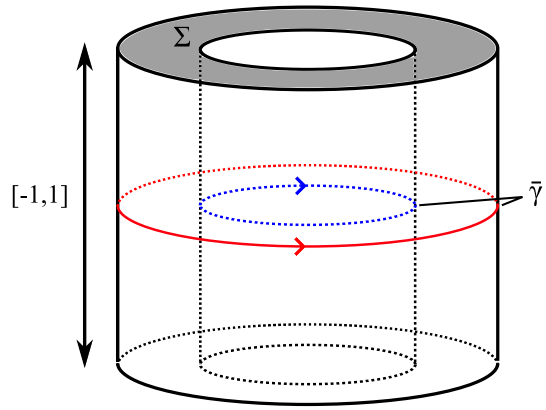

Figure 2.1: Product sutured manifold from an annulus.

Given a balanced sutured manifold (N, γ), Kronheimer and Mrowka construct a closed, oriented 3-manifold M as follows. Fix a compact, oriented surface Σ with as many boundary components as the number of components in γ, and construct the product sutured manifold $([-1,1] \times \Sigma, \bar{\gamma})$ as in the above example. Then, glue $\mathrm{A}(\bar{\gamma})$ onto $\mathrm{A}(\gamma)$ via an orientation-reversing map which maps $\mathrm{R}_{\pm}(\bar{\gamma})$ onto $\mathrm{R}_{\pm}(\gamma)$. The result is a 3-manifold with boundary which consists of two homeomorphic regions $\overline{\mathrm{R}}_{\pm} = \mathrm{R}_{\pm}(\bar{\gamma}) \cup \mathrm{R}_{\pm}(\gamma)$. Now, suppose that the genus of $\overline{\mathrm{R}}_{\pm}$ is at least 2. Then, glue $\overline{\mathrm{R}}_{+}$ and $\overline{\mathrm{R}}_{-}$ via a homeomorphism which respects their orientations. The resulting closed, oriented 3-manifold M contains a non-separating surface $\overline{\mathrm{R}}$ of genus 2 or more obtained via the identification of $\overline{\mathrm{R}}_{+}$ with $\overline{\mathrm{R}}_{-}$. With the preceding understood, we are ready to give the definition of sutured monopole homology.

Definition II.12 (Kronheimer and Mrowka)**.** The variant of the Seiberg–Witten Floer homology for the balanced sutured manifold (N, γ) is defined to be the finitely generated abelian group

$$SHM(\mathrm{N}, \gamma) := \widetilde{HM}(\mathrm{M}|[\overline{\mathrm{R}}]) = \bigoplus_{\mathfrak{s} \,:\, \langle c_1(\mathfrak{s}), [\overline{\mathrm{R}}]\rangle = 2genus(\overline{\mathrm{R}}) - 2} \widetilde{HM}(\mathrm{M}, \mathfrak{s}). \tag{2.9}$$

This group is independent of the choice of the surface Σ and the various gluing maps used to construct the closed, oriented 3-manifold M.

Now, a key step in proving Theorem I.3 requires using a theorem by Kronheimer and Mrowka which enables us to detect product sutured manifolds. First, we state this theorem without providing a proof. Then, we close this chapter by giving a sketch of the proof of Theorem I.3 following [16].

Theorem II.13. [10, Theorem 6.1] *Suppose that a balanced sutured manifold* (N, γ) *admits a taut foliation and* N *is a homology product, namely, the inclusions of* $\mathrm{R}_{\pm}(\gamma)$ *into* N *induce isomorphisms of the integer homology groups. Then,* (N, γ) *is a product sutured manifold if and only if* $SHM(\mathrm{N}, \gamma) \cong \mathbb{Z}$.

Remark II.14. The proof of Theorem II.13 is by contradiction, and the main step in the proof involves showing the existence of two taut foliations on N which extend to two taut foliations on M with different Euler classes and with associated plane fields tangent to $\overline{\mathrm{R}}$. This idea is originally due to P. Ghiggini which he used to prove a version of Theorem II.13 in the context of Heegaard-Floer homology.

Proof of Theorem I.3. Start by cutting M open along Σ so as to obtain a compact 3-manifold N' with boundary which consists of two copies of the surface Σ denoted by Σ_+ and Σ_- according to whether the boundary orientation agrees with the orientation of Σ or not. When $b_1(\mathrm{M}) > 1$, the assumption that $\widetilde{HM}(\mathrm{M}|[\Sigma]) \cong \mathbb{Z}$ implies that the Alexander polynomial of M is monic (see [14]), therefore N' is a homology product as is explained in [15, Section 3]. When $b_1(\mathrm{M}) = 1$, one can replace M by the double of N' along its boundary resulting in a manifold M' with first Betti number greater than 1. Now, there are two copies of the surface Σ inside M'. Cut M' open along one of these two copies of Σ and apply the previous argument. As a result, N' is again a homology product.

Next, Ni observes the following: One can assume, without loss of generality, that N' contains a submanifold of the form $G \times [-1,1]$ where G is a once-punctured torus and $G \times \{\pm 1\}$ are embedded inside $\Sigma_\pm$ in such a way that when we glue Σ_+ back onto Σ_-, $G \times \{1\}$ glues onto $G \times \{-1\}$ so as to yield a submanifold homeomorphic to $G \times S^1$ inside M. Now, denote by N the closure of $N' \setminus G \times [-1,1]$ and by γ the curve $\partial G \times \{0\}$ on the boundary of N. Then, (N, γ) is a balanced sutured manifold which is a homology product (see Figure 2.2). Moreover, there exists a taut foliation on N since there exists a taut foliation on N' that is tangent to the boundary of N' by Gabai's theorem. Thus, Theorem II.13 implies that (N, γ) is a product sutured manifold. In particular, N is a product manifold. Finally, glue (N, γ) and $(G \times [-1,1], \gamma)$ along $\gamma \times [-1,1]$ so as to retrieve N' and to see that N' is a product manifold. Therefore, M fibers over the circle with Σ as a fiber. $\square$

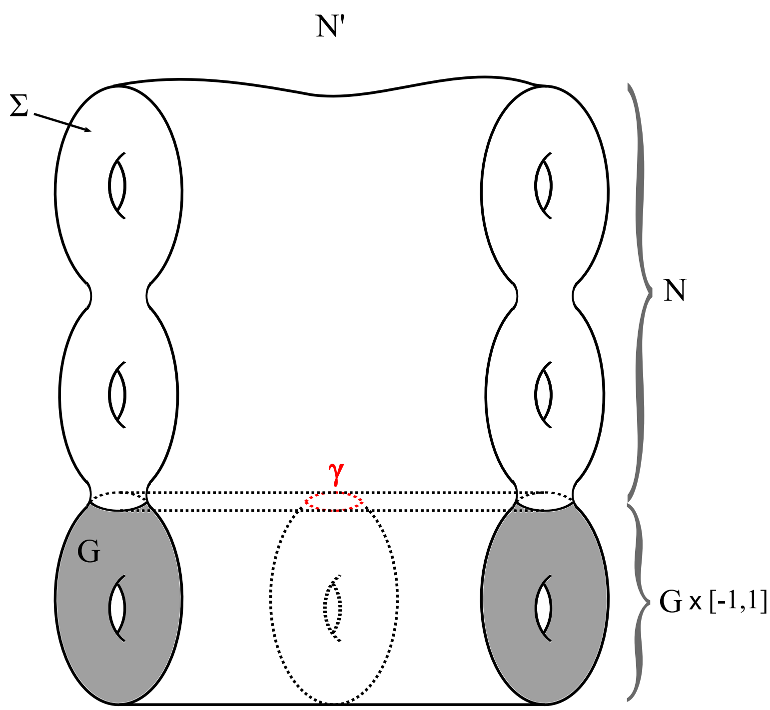

Figure 2.2: Proof of Theorem I.3.

CHAPTER III

Symplectic forms on $\mathbf{S^1 \times M^3}$

Let X denote a $2n$-dimensional manifold with a differentiable structure on it. A symplectic form on X is a closed 2-form ω such that $\omega^{\wedge n}$ is nowhere zero on X. The latter implies that X is an orientable manifold, and ω induces an orientation on X. A smooth, orientable $2n$-manifold X together with a symplectic form on it is called a symplectic manifold. An example of a symplectic manifold is $(\mathbb{R}^{2n}, \omega_0)$ where $\omega_0 = \sum_{i=1}^{n} \mathrm{d}x^i \wedge \mathrm{d}y^i$ and $(x^1, y^1, \cdots, x^n, y^n)$ are the coordinates on $\mathbb{R}^{2n}$. It is a theorem of G. Darboux [2] that any symplectic $2n$-manifold is locally diffeomorphic to $(\mathbb{R}^{2n}, \omega_0)$. In other words, dimension is the only local invariant of a symplectic manifold. Therefore, the question about the existence of symplectic forms on a given smooth $2n$-manifold concerns very much the topology of that manifold. Note in this regard that every closed, oriented Riemann surface admits a symplectic form, e.g. its area form. Therefore, the smallest dimension for which the existence question is non-trivial is 4.

Now, let M be a closed, connected, orientable 3-manifold and suppose that the 4-dimensional manifold $\mathrm{S}^1 \times \mathrm{M}$ admits a symplectic form. Let ω denote a symplectic form on $\mathrm{S}^1 \times \mathrm{M}$. Then, one can write ω as

$$\omega = \mathrm{dt} \wedge \nu + \mu \tag{3.1}$$

where dt is the volume form on S^1, ν is a section over $S^1 \times M$ of T^*M and μ is a section over $S^1 \times M$ of $\wedge^2 T^*M$. Let d denote the exterior derivative along M factor of $S^1 \times M$. Since ω is a closed 2-form, one has $\frac{\partial}{\partial t}\mu = d\nu$ and $d\mu = 0$. Thus, μ is a closed form on M at any given $t \in S^1$. Its cohomology class in $H^2(M;\mathbb{R})$ is denoted by $[\mu]$. As explained momentarily, the class $[\mu]$ is non-zero. To see why this is the case, first use the Künneth formula to write $H^2(S^1 \times M;\mathbb{R})$ as the direct sum $[dt] \cup H^1(M;\mathbb{R}) \oplus H^2(M;\mathbb{R})$ where $[dt]$ denotes the cohomology class of the 1-form dt. Let $[\omega]$ denote the cohomology class of the symplectic form ω. This class appears in the Künneth decomposition as $[dt] \cup [\bar{\nu}] + [\mu]$ where $\bar{\nu}$ is the push-forward from $S^1 \times M$ of the 2-form $dt \wedge \nu$. This understood, neither $[\bar{\nu}]$ nor $[\mu]$ are zero by virtue of the fact that $[\omega] \cup [\omega]$ is non-zero.

Our convention is to orient S^1 by dt, and $S^1 \times M$ by $\omega \wedge \omega$. Doing so finds that $\nu \wedge \mu$ is nowhere zero and so orients M at any given $t \in S^1$.

Here is what can be said about the topology of $S^1 \times M$. A smooth, orientable 4-manifold of the form $S^1 \times M$ has vanishing Euler characteristic and signature. Hence, $b_2^{\pm}(S^1 \times M) = b_1(M)$ and the intersection form of $S^1 \times M$ can be represented by a matrix of the form $b_1(M)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Moreover, $b_1(M) > 0$ since $S^1 \times M$ admits a symplectic form.

Now, fix a t-independent Riemannian metric on M, and let $*$ denote the Hodge star operator. At each $t \in S^1$, the 1-form $*\mu$ is a nowhere vanishing 1-form on M and so defines a homotopy class of oriented 2-plane fields by its kernel. This 2-plane field is denoted in what follows by K^{-1}. This bundle is oriented by μ and so has a corresponding Euler class which we write as $-c_1(K) \in H^2(M;\mathbb{Z})$. The latter is the so-called *anticanonical class* associated to the symplectic form ω on $S^1 \times M$.

Fix a spin^c structure $\mathfrak{s}$ on M with associated spinor bundle $\mathbb{S}$. At any $t \in S^1$, the eigenbundles for Clifford multiplication by $*\mu$ on $\mathbb{S}$ split $\mathbb{S}$ as a direct sum, $\mathbb{S} = E \oplus EK^{-1}$, where E is a complex line bundle over M. Here, our convention is to write the $+i|\mu|$ eigenbundle on the left. The *canonical spin^c* structure is that with $E = \underline{\mathbb{C}}$, the trivial complex line bundle. We use $det(\mathbb{S})$ to denote the complex line bundle $\wedge^2\mathbb{S} = E^2K^{-1}$ over M. Note that the assignment of $c_1(E) \in H^2(M;\mathbb{Z})$ to a given spin^c structure identifies the set of isomorphism classes of spin^c structures over M with $H^2(M;\mathbb{Z})$. This classification of the spin^c structures over M is independent of the choice of $t \in S^1$. For any given class $e \in H^2(M;\mathbb{Z})$, we use $\mathfrak{s}_e$ to denote the corresponding spin^c structure. Thus the spinor bundle $\mathbb{S}$ for $\mathfrak{s}_e$ splits as $E \oplus EK^{-1}$ with $c_1(E) = e$.

With the preceding understood, here is what can be said about the Seiberg–Witten invariants of M.

Theorem III.1 ([21]). *The Seiberg–Witten invariant of* M *for the canonical spin^c structure is non-zero, more precisely $SW(M, \mathfrak{s}_0) = \pm 1$. Moreover, suppose that $b_1(M) > 1$. Then, $SW(M, \mathfrak{s}_e) \neq 0$ only if $0 \leq e \cdot [\omega] \leq c_1(\mathfrak{s}_0) \cdot [\omega]$, and either equality holds if and only if $e = 0$ or $e = c_1(\mathfrak{s}_0)$, respectively.*

Theorem III.1 is the 3-dimensional version of Taubes' well-known result on the Seiberg–Witten invariants of symplectic 4-manifolds. Theorem III.1 and Theorem 1.1 in [14] were used by Vidussi in [29] to deduce the following facts about the Alexander polynomial Δ_M of M. Remember that Alexander polynomial of a closed, oriented 3-manifold M is an element of the group ring $\mathbb{Z}[H^2(M;\mathbb{Z})/Tor]$. Suppose without loss of generality that $[\omega] \in H^2(S^1 \times M;\mathbb{Z})$. Then, there exists a closed, connected, oriented, genus-minimizing surface $\Sigma \subset M$ such that $[\Sigma] \in H_2(M;\mathbb{Z})$ is primitive and some positive integer multiple of $[\Sigma]$ is the Poincaré dual of $[\bar{\nu}]$. Now, the group homomorphism $\Gamma : \mathrm{H}^2(\mathrm{M};\mathbb{Z})/Tor \to \mathbb{Z}$ defined by $\Gamma(c) = \langle c, [\Sigma]\rangle$ extends to a homomorphism $\widetilde{\Gamma} : \mathbb{Z}[\mathrm{H}^2(\mathrm{M};\mathbb{Z})/Tor] \to \mathbb{Z}[\mathrm{t}, \mathrm{t}^{-1}]$ of group rings. Then, $\widetilde{\Gamma}(\Delta_{\mathrm{M}})$ is monic, and its degree is equal to $2genus(\Sigma) - 2$, or $2genus(\Sigma)$ if $b_1(\mathrm{M}) = 1$. Unfortunately, the latter is not enough to determine whether M fibers over the circle or not. For example, the Alexander polynomial of the 3-dimensional manifold ${\mathrm{S}^3}_0(\mathrm{P})$ obtained by performing 0-framed surgery on the $(5, -3, 5)$ pretzel knot $\mathrm{P} \subset \mathrm{S}^3$ is $\Delta_{{\mathrm{S}^3}_0(\mathrm{P})} = 1 - 3\mathrm{t} + \mathrm{t}^2$, which is monic and has degree equal to twice the genus of P. However, ${\mathrm{S}^3}_0(\mathrm{P})$ does not fiber over the circle. Still, as Friedl and Vidussi showed in [6], the manifold $\mathrm{S}^1 \times {\mathrm{S}^3}_0(\mathrm{P})$ admits no symplectic forms.

3.1 A one-parameter family of equations

Our purpose in this section is to outline our proof of Theorem I.2. The proofs for most of the assertions made in this section are deferred to the subsequent sections of this dissertation.

Fix $t \in \mathrm{S}^1$, and let M_t denote the slice $\mathrm{M}_t = \{t\} \times \mathrm{M}$. A version of the Seiberg–Witten equations on M_t can be defined as follows: Let $\varpi_{\mathbb{S}}$ be the harmonic 2-form on M representing the class $2\pi c_1(det(\mathbb{S}))$. Fix a connection, $\mathbb{A}_{\mathbb{S}}$, on $det(\mathbb{S})$ with curvature 2-form $-i\varpi_{\mathbb{S}}$. Then, any given connection on $det(\mathbb{S})$ is of the form $\mathbb{A}_{\mathbb{S}} + 2\mathrm{a}$ for $\mathrm{a} \in \mathrm{C}^\infty(\mathrm{M}; i\mathrm{T}^*\mathrm{M})$. Now, fix $r \geq 1$ and $t \in \mathrm{S}^1$. We consider the equations

$$
\begin{aligned}
*\mathrm{da} &= r(\psi^\dagger\tau\psi - i * \mu) + \frac{i}{2} * \varpi_{\mathbb{S}} \\
\mathcal{D}_{\mathbb{A}}\psi &= 0,
\end{aligned}
\tag{3.2}
$$

where μ is the 2-form defined by the symplectic form. Suitably rescaling ψ, we see that these are a version of the equations in (2.3). These equations are the variational equations of a functional defined as

$$\mathfrak{a}(\mathbb{A}_{\mathbb{S}} + 2\mathrm{a}, \psi) = -\frac{1}{2}\int_{\mathrm{M}_t} \mathrm{a} \wedge (\mathrm{da} - i\varpi_{\mathbb{S}}) - ir\int_{\mathrm{M}_t} \mathrm{a} \wedge \mu + r\int_{\mathrm{M}_t} \psi^{\dagger}\mathcal{D}_{\mathbb{A}}\psi, \tag{3.3}$$

where $\mathrm{a} \in \mathrm{C}^{\infty}(\mathrm{M}; i\mathrm{T}^*\mathrm{M})$ and $\psi \in \mathrm{C}^{\infty}(\mathrm{M}; \mathbb{S})$.

For future purposes, we introduce a new functional on $\mathcal{C}$. Fix $r \geq 1$, $t \in \mathrm{S}^1$ and for $(\mathbb{A}, \psi) \in \mathcal{C}$ let

$$\mathcal{E}(\mathbb{A}, \psi) = i\int_{\mathrm{M}_t} \nu \wedge \mathrm{da}. \tag{3.4}$$

Our approach is to consider $\mathrm{S}^1 \times \mathrm{M}$ as a 1-parameter family of three-dimensional manifolds, each a copy of M and parametrized by $t \in \mathrm{S}^1$. We use the gauge equivalence classes of solutions of the equations in (3.2) on M_t (when non-degenerate) to define the generators of the Seiberg–Witten Floer homology. Here it is important to remark that the solutions of the equations in (3.2) can serve this purpose for any $r \geq 1$ because we assume that $c_1(det(\mathbb{S})) = \lambda[\mu]$ in $\mathrm{H}^2(\mathrm{M}; \mathbb{R})$ with $\lambda < 0$. For the same reason, (3.2) has no reducible solutions.

Here, we remark that what is written in (3.2) has *period class* $-4\pi[\mu]$. The assumption that $[\mu]$ is a negative multiple of $c_1(det(\mathbb{S}))$ guarantees that we are in the positively monotone case.

There is one more important point to make here: The only t-dependence in (3.2) is due to the appearance of the 2-form μ through the latter's t-dependence on $t \in \mathrm{S}^1$. to define generators of the corresponding Seiberg–Witten Floer homology. Note that the t-dependence is due entirely to the appearance of the 2-form μ and its dependence on t.

We suppose Theorem I.2 is false, and hence that there are at least two generators of the Seiberg–Witten Floer homology for each $t \in \mathrm{S}^1$ if $\mathrm{E} = \underline{\mathbb{C}}$ or that the Seiberg–Witten Floer homology is non-trivial for each $t \in \mathrm{S}^1$ if $\mathrm{E} \neq \underline{\mathbb{C}}$ with $c_1(\mathrm{E}) \cdot [\omega] < 0$.

Note in this regard that there is at least one generator for the $\mathrm{E} = \underline{\mathbb{C}}$ case because the fact that $\mathrm{S}^1 \times \mathrm{M}$ is symplectic implies, via Theorem III.1, that the Seiberg–Witten invariant for the canonical spin^c structure on $\mathrm{S}^1 \times \mathrm{M}$ is equal to 1. If there are at least two generators, then there are at least two solutions. Our plan is to use the large r behavior of at least one of these solutions to derive a contradiction from the assumed existence of two or more generators.

What follows describes what we would like to do. Given the existence of two or more non-zero Seiberg–Witten Floer homology classes, we would like to use a variant of the strategy from [24] and [25] to find, for large enough $r \geq 1$ and for each $t \in \mathrm{S}^1$, a set $\Theta_t \subset \mathrm{M}_t$ of the following sort: Θ_t is a finite set of pairs of the form (γ, m) with $\gamma \subset \mathrm{M}_t$ a closed integral curve of the vector field that generates the kernel of $\mu|_t$, and m is a positive integer. These are constrained so that no two pair have the same integral curve. In addition, with each γ oriented by $*\mu|_t$, the formal sum $\Sigma_{(\gamma,\mathrm{m})\in\Theta_t} \mathrm{m}\gamma$ represents the Poincaré dual to $c_1(\mathrm{E})$ in $\mathrm{H}_1(\mathrm{M}_t; \mathbb{Z})$. We would also like the graph $t \to \Theta_t$ to sweep out a smooth, oriented surface $\mathrm{S} \subset \mathrm{S}^1 \times \mathrm{M}$ whose fundamental class gives the Poincaré dual to $c_1(\mathrm{E})$ in $\mathrm{H}_2(\mathrm{S}^1 \times \mathrm{M}; \mathbb{Z})$. Note in this regard that such a surface is oriented by the vector field $\frac{\partial}{\partial t}$ and by the 1-form ν that appears when we write $\omega = \mathrm{dt} \wedge \nu + \mu$. In particular, $\omega|_{\mathrm{TS}}$ is positive and so the integral of ω over S is positive. On the other hand, the integral of ω over S must be non-positive if the cup product of $[\omega]$ with $c_1(\mathrm{E})$ is non-positive. This is the fundamental contradiction.

As it turns out, we cannot guarantee that Θ_t exists for all $t \in \mathrm{S}^1$, only for most t, where 'most' has a precise measure-theoretic definition. Even so, we have control over enough of S^1 to obtain a contradiction which is in the spirit of the one described from any violation to the assertion of Theorem I.2.

Given what has been said so far, we have the desired sets $\Theta_t \subset \mathrm{M}_t$ for points t

in the complement of a closed set with non-empty interior in S^1. On the face of it, this is far from what we need, which is a surface $S \subset S^1 \times M$ that is swept out by such points. As we show below, we can make due with what we have. In particular, we first change our point of view and interpret integration of ω over a surface in $S^1 \times M$ as integration over $S^1 \times M$ of the product of ω and a closed 2-form Φ that represents the Poincaré dual of the surface. We then construct a 2-form Φ on $S^1 \times M$ that is localized near the surface swept out by Θ_t on most of $S^1 \times M$. This partial localization is enough to prove that $\int_{S^1 \times M} \omega \wedge \Phi > 0$ when this integral should be zero or negative. The existence of such a form gives the fundamental contradiction that proves Theorem I.2.

3.2 Properties of solutions

In this section, we discuss certain analytic properties of solutions to the equations in (3.2) and their geometric significance. Our goal is to understand under what conditions would solutions to the equations in (3.2) yield the sets Θ_t. We start by deriving some fundamental estimates on the norms of solutions and their derivatives.

3.2.1 Basic analytic estimates

Many of the following arguments in this section exploit two fundamental a priori bounds for solutions of the large r versions of (3.2). To start with, write a section ψ of $\mathbb{S} = E \oplus EK^{-1}$ as $\psi = (\alpha, \beta)$ where α is a section of E and β is a section of EK^{-1}. Given a spin^c connection $\mathbf{A}$ on $\mathbb{S}$, denote by ∇_E the covariant derivative operator on sections of E defined by $\nabla_E \alpha = \frac{1}{2}(1 - \frac{i}{|\mu|} * \mu) \nabla_{\mathbf{A}}(\alpha, 0)$. Similarly, denote by $\nabla_{EK^{-1}}$ the covariant derivative operator on sections of EK^{-1} defined by $\nabla_{EK^{-1}} \beta = \frac{1}{2}(1 + \frac{i}{|\mu|} * \mu) \nabla_{\mathbf{A}}(0, \beta)$. Then, the next lemma supplies the fundamental estimates on the norms of α and β.

Lemma III.2. *Fix a bound on the* C^3*-norm of* μ. *Then, there are constants* $c, c' > 0$ *with the following significance: Suppose that* $(\mathbb{A}, \psi = (\alpha, \beta))$ *is a solution of a given* $t \in \mathrm{S}^1$ *and* $r \geq 1$ *version of the equations in (3.2). Then,*

- $|\alpha| \leq |\mu|^{1/2} + c\, r^{-1}$
- $|\beta|^2 \leq c'\, r^{-1}(|\mu| - |\alpha|^2) + c\, r^{-2}$.

Proof. This lemma is the same as Lemma 2.2 in [24] except for the inevitable appearance of $|\mu|$. We will give the proof in this new context.

Since $\mathcal{D}_{\mathbb{A}}\psi = 0$, one has ${\mathcal{D}_{\mathbb{A}}}^2\psi = 0$ as well. Then, the Weitzenböck formula for ${\mathcal{D}_{\mathbb{A}}}^2$ yields

$$(3.5) \qquad {\mathcal{D}_{\mathbb{A}}}^2\psi = \nabla^\dagger\nabla\psi + \frac{1}{4}\mathcal{R}\,\psi - \frac{1}{2}\mathfrak{cl}(*\mathrm{F}_{\mathbb{A}})\psi = 0$$

where $\mathcal{R}$ denotes the scalar curvature of the Riemannian metric. Contract this equation with ψ to see that

$$(3.6) \qquad \frac{1}{2}\mathrm{d}^*\mathrm{d}|\psi|^2 + |\nabla\psi|^2 + \frac{r}{2}|\psi|^2(|\psi|^2 - |\mu| - \frac{c_0}{r}) \leq 0.$$

where $c_0 > 0$ is a constant depending only on the supremum of $|\varpi_{\mathbb{S}}|$ and the infimum of the scalar curvature.

Now, introduce $\psi = |\mu|^{1/2}\ \psi'$, therefore $\alpha = |\mu|^{1/2}\ \alpha'$ and $\beta = |\mu|^{1/2}\ \beta'$. Then, one can rewrite (3.6) as follows:

$$(3.7) \qquad \begin{aligned} &\frac{|\mu|}{2}\mathrm{d}^*\mathrm{d}|\psi'|^2 - <\mathrm{d}|\mu|, \mathrm{d}|\psi'|^2> + \frac{1}{2}|\psi'|^2\mathrm{d}^*\mathrm{d}|\mu| \\ &+ \frac{r}{2}|\mu||\psi'|^2(|\mu||\psi'|^2 - |\mu| - \frac{c_0}{r}) \leq 0 \end{aligned}$$

Manipulating (3.7), one obtains

$$(3.8) \qquad \frac{1}{2}\mathrm{d}^*\mathrm{d}|\psi'|^2 - \frac{1}{|\mu|} <\mathrm{d}|\mu|, \mathrm{d}|\psi'|^2> + \frac{r}{2}|\mu||\psi'|^2(|\psi'|^2 - 1 - \frac{c_1}{r}) \leq 0$$

where $c_1 > 0$ is a constant depending on c_0. An application of the maximum principle to (3.8) yields

$$|\psi'|^2 \leq 1 + \frac{c_1}{r} \tag{3.9}$$

from which the first bullet of Lemma III.2 follows immediately.

As for the claimed estimate on the norm of β, start by contracting (3.5) first with $(\alpha, 0)$ and then with $(0, \beta)$ to get

$$\begin{aligned}
&\frac{1}{2}\mathrm{d}^*\mathrm{d}|\alpha|^2 + |\nabla_{\mathrm{E}}\alpha|^2 + \frac{r}{2}|\alpha|^2(|\alpha|^2 + |\beta|^2 - |\mu|) + \kappa_1|\alpha|^2 + \kappa_2(\alpha, \beta)\\
&+\kappa_3(\alpha, \nabla_{\mathrm{E}}\alpha) + \kappa_4(\alpha, \nabla_{\mathrm{EK}^{-1}}\beta) = 0\\
&\frac{1}{2}\mathrm{d}^*\mathrm{d}|\beta|^2 + |\nabla_{\mathrm{EK}^{-1}}\beta|^2 + \frac{r}{2}|\beta|^2(|\alpha|^2 + |\beta|^2 + |\mu|) + \kappa_1'(\beta, \alpha) + \kappa_2'|\beta|^2\\
&+\kappa_3'(\beta, \nabla_{\mathrm{E}}\alpha) + \kappa_4'(\beta, \nabla_{\mathrm{EK}^{-1}}\beta) = 0
\end{aligned} \tag{3.10}$$

where κ_i's and κ_i''s depend only on the Riemannian metric. Then, the equations in (3.10) yield the following equations in terms of α' and β':

$$\begin{aligned}
&\frac{1}{2}\mathrm{d}^*\mathrm{d}|\alpha'|^2 + |\nabla_{\mathrm{E}}\alpha'|^2 + \frac{r}{2}|\mu||\alpha'|^2(|\alpha'|^2 + |\beta'|^2 - 1) + \lambda_1|\alpha'|^2\\
&+\lambda_2(\alpha', \beta') + \lambda_3(\alpha', \nabla_{\mathrm{E}}\alpha') + \lambda_4(\alpha', \nabla_{\mathrm{EK}^{-1}}\beta') = 0\\
&\frac{1}{2}\mathrm{d}^*\mathrm{d}|\beta'|^2 + |\nabla_{\mathrm{EK}^{-1}}\beta'|^2 + \frac{r}{2}|\mu||\beta'|^2(|\alpha'|^2 + |\beta'|^2 + 1) + \lambda_1'(\beta', \alpha')\\
&+\lambda_2'|\beta'|^2 + \lambda_3'(\beta', \nabla_{\mathrm{E}}\alpha') + \lambda_4'(\beta', \nabla_{\mathrm{EK}^{-1}}\beta') = 0
\end{aligned} \tag{3.11}$$

where λ_i's and λ_i''s depend only on the Riemannian metric.

Now, introduce $\mathrm{w} = 1 - |\alpha'|^2$. Then, the top equation in (3.11) can be rewritten as

$$\begin{aligned}
&-\frac{1}{2}\mathrm{d}^*\mathrm{d}\mathrm{w} + |\nabla_{\mathrm{E}}\alpha'|^2 - \frac{r}{2}|\mu||\alpha'|^2\mathrm{w} + \frac{r}{2}|\mu||\alpha'|^2|\beta'|^2 +\\
&\lambda_1|\alpha'|^2 + \lambda_2(\alpha', \beta') + \lambda_3(\alpha', \nabla_{\mathrm{E}}\alpha') + \lambda_4(\alpha', \nabla_{\mathrm{EK}^{-1}}\beta') = 0.
\end{aligned} \tag{3.12}$$

Using the estimate in (3.9), manipulating the lower order terms and maximizing positive valued functions that do not depend on the value of r or the particular solution (α, β), the bottom equation in (3.11) and the equation (3.12) yield the following inequalities:

$$\begin{aligned}
&-\frac{1}{2}\mathrm{d}^*\mathrm{dw} + \zeta_0|\nabla_{\mathrm{E}}\alpha'|^2 - \frac{r}{2}|\mu||\alpha'|^2\mathrm{w} \leq \zeta_1 + \zeta_2|\nabla_{\mathrm{EK}^{-1}}\beta'|^2 \\
&\frac{1}{2}\mathrm{d}^*\mathrm{d}|\beta'|^2 + \eta_0|\nabla_{\mathrm{EK}^{-1}}\beta'|^2 + \frac{r}{2}\eta_1|\mu||\beta'|^2 + \frac{r}{2}|\mu||\alpha'|^2|\beta'|^2 \leq \frac{\eta_2}{r} + \frac{\eta_3}{r}|\nabla_{\mathrm{E}}\alpha'|^2
\end{aligned}$$

(3.13)

where ζ_i's and η_i's are positive constants depending only on the Riemannian metric and the constant c_0.

Multiplying the top inequality in (3.13) by $\frac{\mathrm{k}}{r}$ where k is a positive constant large enough to satisfy

- $\mathrm{k}\zeta_0 \geq \eta_3$ and
- $\eta_0 \geq \mathrm{k}\zeta_2$,

and adding the resulting inequality to the bottom inequality in (3.13), we deduce that there are positive constants c_2 and c_3 that depend only on the Riemannian metric and the constant c_0 such that

$$\mathrm{d}^*\mathrm{d}(|\beta'|^2 - \frac{c_2}{r}\mathrm{w} - \frac{c_3}{r^2}) + r|\mu||\alpha'|^2(|\beta'|^2 - \frac{c_2}{r}\mathrm{w} - \frac{c_3}{r^2}) \leq 0. \tag{3.14}$$

Then, an application of the maximum principle to (3.14) yields

$$|\beta'|^2 \leq \frac{c_2}{r}(1 - |\alpha'|^2) + \frac{c_3}{r^2}$$

which, eventually, gives rise to the second bullet of Lemma III.2 after multiplying both sides of the inequality by $|\mu|$. □

Given Lemma III.2, the next lemma finds a priori bounds on the derivatives of α and β.

Lemma III.3. *Fix a bound on the* C^3*-norm of* μ*. Given* $r \geq 1$ *and* $t \in S^1$*, let* $(\mathbb{A}, \psi = (\alpha, \beta))$ *denote a solution of the* t *and* r *version of the equations in (3.2). Then, for each integer* $n \geq 1$ *there exists a constant* $c_n \geq 1$*, which is independent of the value of* $t \in S^1$*, the value of* $r \geq 1$ *and the solution* $(\mathbb{A}, \psi = (\alpha, \beta))$*, with the following significance:*

- $|\nabla_E^n \alpha| \leq c_n r^{n/2}$
- $|\nabla_{EK^{-1}}^n \beta| \leq c_n r^{(n-1)/2}$.

The following is also true: Fix $\epsilon > 0$*. There exists* $\delta > 0$ *and* $\kappa > 1$ *such that if* $r > \kappa$ *and if* $|\alpha| \geq |\mu|^{1/2} - \delta$ *in any given ball of radius* $2\kappa r^{-1/2}$ *in* M_t*, then* $|\nabla_E^n \alpha| \leq \epsilon c_n r^{n/2}$ *for* $n \geq 1$ *and* $|\nabla_{EK^{-1}}^n \beta| \leq \epsilon c_n r^{(n-1)/2}$ *for all* $n \geq 0$ *in the concentric ball with radius* $\kappa r^{-1/2}$.

Proof. The proof is essentially identical to that of Lemma 2.3 in [24]. This is to say that the proof is local in nature: Fix a Gaussian coordinate chart centered at any given point in M so as to view the equations in (3.2) as equations on a small ball in $\mathbb{R}^3$. Then rescale coordinates by writing $x = r^{-1/2} y$ so that the resulting equations are on a ball of radius $\mathcal{O}(r^{1/2})$ in $\mathbb{R}^3$. The r-dependence of these rescaled equations is such that standard elliptic regularity techniques provide uniform bounds on the rescaled versions of β and the derivatives of the rescaled α and β in the unit radius ball about the origin. Rescaling back to the original coordinates will give what is claimed by the lemma. □

One of the key implications of Lemma III.2 is a priori bounds on the values of $\mathcal{E}$. First, note that since $\nu \wedge \mu > 0$ at each $t \in S^1$, it follows that

$$\nu = *\frac{q}{|\mu|}\mu + \upsilon \tag{3.15}$$

where $\mathrm{q} =< \nu, *\mu > |\mu|^{-1}$ is a positive valued function on M_t at each $t \in \mathrm{S}^1$, and $\upsilon \wedge \mu = 0$. The following lemma states the a priori lower bound on $\mathcal{E}$ that follows from this last observation.

Lemma III.4. *There exists a constant $\kappa > 1$ with the following significance: Suppose that $r \geq \kappa$, $t \in \mathrm{S}^1$, and $(\mathbb{A}, \psi)$ is a solution of the corresponding version of the equations in (3.2). Then, $\mathcal{E}(\mathbb{A}, \psi) \geq -\kappa$.*

Proof. Fix $r \geq 1$ and $t \in \mathrm{S}^1$. Let $(\mathbb{A}, \psi)$ be a solution of the t and r version of the equations in (3.2). Write $\mathbb{A} = \mathbb{A}_{\mathbb{S}} + 2\mathrm{a}$ and $\psi = (\alpha, \beta)$. Then, by (3.15) we can write

$$\mathcal{E}(\mathbb{A}, \psi) = i \int_{\mathrm{M}} \nu \wedge \mathrm{da} = r \int_{\mathrm{M}} \mathrm{q}(|\mu| - |\alpha|^2) + i \int_{\mathrm{M}} \upsilon \wedge \mathrm{da}. \tag{3.16}$$

Now, it follows from (3.2) and Lemma III.2 that

$$\mathcal{E}(\mathbb{A}, \psi) \geq \frac{1}{2} r \int_{\mathrm{M}} \mathrm{q}(|\mu| - |\alpha|^2) - c_4 \geq -c_5 \tag{3.17}$$

where $c_4, c_5 > 0$ are constants depending only on the Riemannian metric. □

The next lemma concerns an estimate for the connection itself.

Lemma III.5. *Fix $t \in \mathrm{S}^1$ and $r \geq 1$. Suppose that $(\mathbb{A} = \mathbb{A}_{\mathbb{S}} + 2\mathrm{a}, \psi = (\alpha, \beta))$ is a solution of the corresponding version of (3.2). Then, there exists a smooth map $u : \mathrm{M} \to \mathrm{S}^1$ and a constant $c > 0$ such that $\hat{\mathrm{a}} = \mathrm{a} - u^{-1}\mathrm{d}u$ obeys $|\hat{\mathrm{a}}| \leq c(r^{2/3}|\mathcal{E}(\mathrm{A}, \psi)|^{1/3} + 1)$.*

Proof. Remember that any given connection on $det(\mathbb{S})$ is of the form $\mathbb{A} = \mathbb{A}_{\mathbb{S}} + 2\mathrm{a}$ where a is an imaginary valued 1-form on M. Then, for each such a there is a smooth map $u : \mathrm{M} \to \mathrm{S}^1$ such that $\mathrm{a} - u^{-1}\mathrm{d}u$ is co-closed and the norm of its L^2-orthogonal projection onto the space of harmonic 1-forms has an upper bound c_0 depending only on the Riemannian metric. To see this, remember that each cohomology class in $\mathrm{H}^1(\mathrm{M};\mathbb{Z})$ has a unique harmonic representative of the form $iu^{-1}\mathrm{d}u$ where $u : \mathrm{M} \to \mathrm{S}^1$ is smooth. Therefore, there is a smooth map $u_1 : \mathrm{M} \to \mathrm{S}^1$ such that the L^2-orthogonal projection of $\mathrm{a} - {u_1}^{-1}\mathrm{d}u_1$ onto the space of harmonic 1-forms has a bound on its norm depending only on the Riemannian metric. Next, note that by Hodge theory $\mathrm{d}^*(\mathrm{a} - {u_1}^{-1}\mathrm{d}u_1) = \mathrm{d}^*\mathrm{d}h$ where h is a smooth, imaginary valued function uniquely determined up to a constant. Let $u_2 = e^h$ so that ${u_2}^{-1}\mathrm{d}u_2 = \mathrm{d}h$. As a result, $\mathrm{d}^*(\mathrm{a} - {u_1}^{-1}\mathrm{d}u_1 - {u_2}^{-1}\mathrm{d}u_2) = 0$, and the L^2-orthogonal projection of $\hat{\mathrm{a}} = \mathrm{a} - {u_1}^{-1}\mathrm{d}u_1 - {u_2}^{-1}\mathrm{d}u_2$ onto the space of harmonic 1-forms is the same as that of $\mathrm{a} - {u_1}^{-1}\mathrm{d}u_1$.

Now, using the Green's function for the operator $*\mathrm{d}$ acting on the space of co-closed 1-forms on M, we obtain

$$|\hat{\mathrm{a}}|(x) \leq c_0 + \int_{\mathrm{M}} \frac{1}{\mathfrak{d}(x,\cdot)^2} |*\mathrm{d}\hat{\mathrm{a}}|, \tag{3.18}$$

where $\mathfrak{d}(x,\cdot)$ denotes the distance from a fixed point $x \in \mathrm{M}$. Let $diam(\mathrm{M}) \geq \mathfrak{r}_0 > 0$. Then, break the integral on the right-hand side of (3.18) into two parts

$$\int_{\mathfrak{d}(x,\cdot)>\mathfrak{r}_0} \frac{1}{\mathfrak{d}(x,\cdot)^2} |*\mathrm{d}\hat{\mathrm{a}}| + \int_{\mathfrak{d}(x,\cdot)\leq\mathfrak{r}_0} \frac{1}{\mathfrak{d}(x,\cdot)^2} |*\mathrm{d}\hat{\mathrm{a}}|.$$

Now, the portion of the integral over the region where $\mathfrak{d}(x,\cdot) > \mathfrak{r}_0$ yields the following inequality after an appeal to Lemma III.2:

$$\int_{\mathfrak{d}(x,\cdot)>\mathfrak{r}_0} \frac{1}{\mathfrak{d}(x,\cdot)^2} |*\mathrm{d}\hat{\mathrm{a}}| \leq r \int_{\mathfrak{d}(x,\cdot)>\mathfrak{r}_0} \frac{1}{\mathfrak{d}(x,\cdot)^2} ||\upmu| - |\upalpha|^2| + c_1, \tag{3.19}$$

where $c_1 > 0$ is a constant depending only on the Riemannian metric. Then, using (3.16), it is easy to see that the right-hand side of (3.19) is no greater than

$$c_2{\mathfrak{r}_0}^{-2}\mathcal{E}(\mathrm{A},\uppsi) + c_3 \tag{3.20}$$

where $c_2, c_3 > 0$ are constants depending only on the Riemannian metric.

As for the portion of the integral over the region where $\mathfrak{d}(x,\cdot) \leq \mathfrak{r}_0$, using Lemma III.2 and the inequality $||\mu| - |\alpha|^2| \leq |\mu| + \frac{2c}{r}$, which follows from (3.16), we obtain

$$\int_{\mathfrak{d}(x,\cdot)\leq\mathfrak{r}_0} \frac{1}{\mathfrak{d}(x,\cdot)^2}|*\mathrm{d}\hat{\mathrm{a}}| \leq c_4 r\mathfrak{r}_0 + c_5, \tag{3.21}$$

where $c_4, c_5 > 0$ are constants depending only on the Riemannian metric.

Finally, the desired estimate follows from (3.20) and (3.21) once we set $\mathfrak{r}_0 = r^{-1/3}|\mathcal{E}(\mathrm{A},\psi)|^{1/3}$. □

3.2.2 Existence and uniqueness

Here, we address the question of existence of the sets Θ_t. What follows is the key to this question.

Proposition III.6. *Fix a bound on the* C^3*-norm of* μ*, and fix constants* $\mathcal{K} > 1$ *and* $\delta > 0$*. There exists* $\kappa > 1$ *with the following significance: Suppose that* $r \geq \kappa$*,* $t \in \mathrm{S}^1$*, and* $(\mathbb{A} = \mathrm{A}_0 + 2\mathrm{A}, \psi = (\alpha, \beta))$ *is a solution of the equations in (3.2) with* $\mathcal{E}(\mathbb{A},\psi) \leq \mathcal{K}$ *and with* $\sup_{\mathrm{M}}(|\mu| - |\psi|^2) > \delta$*. Then,*

- *There exists a finite set* Θ_t *whose typical element is a pair* (γ, m) *with* $\gamma \subset \mathrm{M}_t$ *a closed integral curve tangent to the kernel of* μ*, and with* m *a positive integer. Distinct pairs in* Θ_t *have distinct curves, and* $\Sigma_{(\gamma,\mathrm{m})\in\Theta_t}\mathrm{m}\gamma$ *generates the Poincaré dual to* $c_1(\mathrm{E})$ *in* $\mathrm{H}_1(\mathrm{M}_t;\mathbb{Z})$*.*

- *Each point where* $|\alpha|^2 < |\mu| - \delta$ *has distance* $\kappa r^{-1/2}$ *or less from a curve in* Θ_t*, and also from some point in* $\alpha^{-1}(0)$*.*

- *Fix* $(\gamma, \mathrm{m}) \in \Theta_t$*. Let* $\mathrm{D} \subset \mathbb{C}$ *denote the closed unit disk centered at the origin and* $\varphi : \mathrm{D} \to \mathrm{M}_t$ *denote a smooth embedding such that all the points in* $\varphi(\partial\mathrm{D})$ *have distance* $\kappa r^{-1/2}$ *or more from any loop in* Θ_t*. Assume in addition that* $\varphi(\mathrm{D})$ *has intersection 1 with* γ*. Fix a trivialization of the bundle* $\varphi^*\mathrm{E}$ *over*

D *so as to view* $\varphi^*\alpha$ *as a smooth map from* D *into* $\mathbb{C}$*. The resulting map is non-zero on* $\partial\mathrm{D}$ *and has degree* m *as a map from* $\partial\mathrm{D}$ *into* $\mathbb{C}\setminus\{0\}$*.*

Proof. Given Lemmas III.2, III.3 and III.5, the proof of this proposition is identical but for minor changes to the proof of Theorem 2.1 given in Section 6 of [24]. The proof of the second bullet is just as in Lemma 6.5 in [24]. □

Proposition III.6 raises the following, perhaps obvious, question:

How do we find solutions with $\mathcal{E}$ *bounded at large* r*?*

To say something about this absolutely crucial question, remark that Proposition III.6 here has an almost exact analog that played a central role in [24] and [25]. These papers use the analog of (3.2) with $*\mu$ replaced by a contact 1-form to prove the existence of Reeb vector fields. The contact 1-form version of $\mathcal{E}$ replaces the form ν with the contact 1-form also. The existence of an r-independent bound on the contact 1-form version of $\mathcal{E}$ played a key role in the arguments given in [24] and [25]. The existence of the desired bound on the contact 1-form version of $\mathcal{E}$ exploits the r**-dependence** of the functional $\mathfrak{a}$.

We obtain the desired r-independent bound on our version of $\mathcal{E}$ for most $t \in \mathrm{S}^1$ by exploiting the t**-dependence** of $\mathfrak{a}$. To say more about this, it proves useful now to introduce a spectral flow function, $\mathcal{F}$, for certain configurations in $\mathcal{C}$. There are three parts to its definition. Here is the first part: Fix a section ψ_{E} of $\mathbb{S}$ so that the $(\mathbb{A}_{\mathbb{S}}, \psi_{\mathrm{E}})$ version of the operator $\mathcal{L}$ as defined in Section 2.2 is non-degenerate. Use $\mathcal{L}_{\mathrm{E}}$ to denote the latter operator. The second part introduces the version of $\mathcal{L}$ that is relevant to (3.2); it is obtained from the original by taking into account the rescaling of ψ. In particular, it is defined by

$$
\mathcal{L}_{(\mathbb{A},\psi)}(\mathrm{b},\phi,g)=\begin{pmatrix} *\mathrm{db}-\mathrm{d}g-2^{-1/2}r^{1/2}(\psi^{\dagger}\tau\phi+\phi^{\dagger}\tau\psi) \\ \mathcal{D}_{\mathbb{A}}\phi+2^{1/2}r^{1/2}(\mathfrak{cl}(\mathrm{b})\psi+g\psi) \\ -\mathrm{d}^{*}\mathrm{b}-2^{-1/2}r^{1/2}(\phi^{\dagger}\psi-\psi^{\dagger}\phi) \end{pmatrix} \tag{3.22}
$$

for each $(\mathrm{b},\phi,g)\in \mathrm{C}^{\infty}(\mathrm{M}; i\mathrm{T}^{*}\mathrm{M}\oplus\mathbb{S}\oplus i\underline{\mathbb{R}})$. Thus, $\mathcal{L}_{\mathrm{E}}$ is the $r=1$ version of (3.22) as defined using $(\mathbb{A}_{\mathbb{S}},\psi_{\mathrm{E}})$. To start the third part of the definition, suppose that $(\mathbb{A},\psi)\in\mathcal{C}$ is non-degenerate in the sense that the operator $\mathcal{L}_{(\mathbb{A},\psi)}$ as depicted in (3.22) has trivial kernel. As explained in [24] and [25], there is a well defined spectral flow from the operator $\mathcal{L}_{\mathrm{E}}$ to $\mathcal{L}_{(\mathbb{A},\psi)}$ (see, also [23]). This integer is the value of $\mathcal{F}$ at $(\mathbb{A},\psi)$. Note that $\mathcal{F}(\cdot)$ is defined on the complement of a codimension-1 subvariety in $\mathcal{C}$. As such, it is piecewise constant. In general, only the $mod(p)$ reduction of $\mathcal{F}$ is gauge invariant where p is the greatest divisor of the class $c_1(det(\mathbb{S}))$.

The function $\mathfrak{a}$ is not invariant under the action of $\mathcal{G}$ on $\mathcal{C}$; and, as just noted, neither is $\mathcal{F}$ when $c_1(det(\mathbb{S}))$ is non-torsion. However, our assumption that $c_1(det(\mathbb{S}))=\lambda[\mu]$ in $\mathrm{H}^2(\mathrm{M};\mathbb{R})$ implies the following: There exists a constant $\mathfrak{C}$ independent of $r\geq 1$ and $t\in \mathrm{S}^1$ such that

$$
\mathfrak{a}^{\mathcal{F}}=\mathfrak{a}+r\mathfrak{C}\mathcal{F}
$$

is invariant under the action of $\mathcal{G}$. We will say more about the role of $\mathfrak{a}^{\mathcal{F}}$ with regard to the question we addressed above in the next section.

The next proposition says something about when we can guarantee Proposition III.6's condition on $|\psi|$:

Proposition III.7. *Fix a bound on the* C^3*-norm of* μ*. Then, there exists* $\kappa>1$ *such that if* $r\geq\kappa$*, then the following are true:*

- *Suppose that* $\mathbb{S}=\underline{\mathbb{C}}\oplus\mathrm{K}^{-1}$*. Then, for any* $t\in \mathrm{S}^1$*, there exists a unique gauge equivalence class of solutions* $(\mathbb{A}_{\underline{\mathbb{C}}},\psi_{\underline{\mathbb{C}}})$ *of the* t *and* r *version of the equations* *in (3.2) with* $|\psi_{\underline{\mathbb{C}}}| \geq |\mu|^{1/2} - \kappa^{-1}$. *Moreover, these solutions are non-degenerate with* $|\psi_{\underline{\mathbb{C}}}| \geq |\mu|^{1/2} - \kappa r^{-1/2}$ *and* $\mathcal{E}(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}}) \leq \kappa$.

- *Suppose that* $\mathbb{S} = \mathrm{E} \oplus \mathrm{EK}^{-1}$ *with* $c_1(\mathrm{E}) \neq 0$. *If* $(\mathbb{A}, \psi)$ *is a solution of any given* $t \in \mathrm{S}^1$ *version of the equations in (3.2), then there exists points in* M *where* $|\psi| \leq \kappa r^{-1/2}$.

Proof. In the case when $c_1(\mathrm{E}) \neq 0$, the claim about $|\psi|$ follows from Lemma III.2 given that α is a section of E. This understood, we now assume that $\mathrm{E} = \underline{\mathbb{C}}$. To start, let $1_{\underline{\mathbb{C}}}$ denote a unit length trivializing section of the $\underline{\mathbb{C}}$ summand. There exists a unique connection A_0 on K^{-1} such that the section $\psi_0 = (1_{\underline{\mathbb{C}}}, 0)$ of $\mathbb{S}_0 = \underline{\mathbb{C}} \oplus \mathrm{K}^{-1}$ obeys $\mathcal{D}_{\mathrm{A}_0}\psi_0 = 0$. Now, we look for a solution of the equations in (3.2) of the form

$$(\mathbb{A}, \psi) = (\mathrm{A}_0 + 2(2r)^{1/2}\mathrm{b}, |\mu|^{1/2}\psi_0 + \phi)$$

with $(\mathrm{b}, \phi) \in \mathrm{C}^\infty(\mathrm{M}; i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S})$. Then, $(\mathbb{A}, \psi)$ will solve the equations in (3.2) if $\mathfrak{b} = (\mathrm{b}, \phi, g) \in \mathrm{C}^\infty(\mathrm{M}; i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}})$ solves the following system of equations:

$$\begin{gathered} *\mathrm{db} - \mathrm{d}g - 2^{-1/2}r^{1/2}[|\mu|^{1/2}({\psi_0}^\dagger\tau\phi + \phi^\dagger\tau\psi_0) + \phi^\dagger\tau\phi] = -2^{-3/2}r^{-1/2} * \mathrm{F}_{\mathrm{A}_0} \\ \mathcal{D}_{\mathrm{A}_0}\phi + 2^{1/2}r^{1/2}[|\mu|^{1/2}(\mathfrak{cl}(\mathrm{b})\psi_0 + g\psi_0) + (\mathfrak{cl}(\mathrm{b})\phi + g\phi)] = -\mathfrak{cl}(\mathrm{d}|\mu|^{1/2})\psi_0 \\ -\mathrm{d}^*\mathrm{b} - 2^{-1/2}|\mu|^{1/2}r^{1/2}(\phi^\dagger\psi_0 - {\psi_0}^\dagger\phi) = 0. \end{gathered}$$

(3.23)

For notational convenience, we denote by $\mathcal{L}_0$ the operator $\mathcal{L}_{(\mathrm{A}_0, |\mu|^{1/2}\psi_0)}$ as defined in (3.22). Then, the equations in (3.23) can be rewritten as

$$\mathcal{L}_0(\mathrm{b}, \phi, g) + r^{1/2}\begin{pmatrix} -2^{-1/2}\phi^\dagger\tau\phi \\ 2^{1/2}(\mathfrak{cl}(\mathrm{b})\phi + g\phi) \\ 0 \end{pmatrix} = \begin{pmatrix} -2^{-3/2}r^{-1/2} * \mathrm{F}_{\mathrm{A}_0} \\ -\mathfrak{cl}(\mathrm{d}|\mu|^{1/2})\psi_0 \\ 0 \end{pmatrix}.$$

(3.24)

Now, for $\mathfrak{b} = (\mathrm{b}, \phi, g)$ and $\mathfrak{b}' = (\mathrm{b}', \phi', g')$ in $\mathrm{C}^\infty(\mathrm{M}; i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}})$, let $(\mathfrak{b}, \mathfrak{b}') \mapsto \mathfrak{b} * \mathfrak{b}'$ be the bilinear map defined by

$$\mathfrak{b} * \mathfrak{b}' = \frac{1}{2} \begin{pmatrix} -2^{-1/2}(\phi^\dagger \tau \phi' + \phi'^\dagger \tau \phi) \\ 2^{1/2}(\mathfrak{cl}(\mathrm{b})\phi' + g\phi' + \mathfrak{cl}(\mathrm{b}')\phi + g'\phi) \\ 0 \end{pmatrix}, \tag{3.25}$$

and let $\mathfrak{u}$ denote the section defined by $(-2^{-3/2} r^{-1/2} * \mathrm{F}_{\mathrm{A}_0}, -\mathfrak{cl}(\mathrm{d}|\mu|^{1/2})\psi_0, 0)$ of $i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}}$. Then, (3.24) has the schematic form

$$\mathcal{L}_0 \mathfrak{b} + r^{1/2} \mathfrak{b} * \mathfrak{b} = \mathfrak{u}. \tag{3.26}$$

Our plan is to use the contraction mapping theorem to solve (3.26) in a manner much like what is done in the proof of Proposition 2.8 of [25]. To set the stage for this, we first introduce the Hilbert space $\mathbb{H}$ as the completion of $\mathrm{C}^\infty(\mathrm{M}; i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}})$ with respect to the norm whose square is:

$$||\xi||_{\mathbb{H}}{}^2 = \int_\mathrm{M} |\nabla_0 \xi|^2 + \frac{1}{4} r \int_\mathrm{M} |\xi|^2, \tag{3.27}$$

where ∇_0 denotes the covariant derivative on sections of $i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}}$ that acts as the Levi-Civita covariant derivative on sections of $i\mathrm{T}^*\mathrm{M}$, the covariant derivative defined by A_0 on sections of $\mathbb{S}$, and that defined by the exterior derivative on sections of $i\underline{\mathbb{R}}$.

Lemma III.8. *There exists $\kappa \geq 1$ such that*

- $||\xi||_6 \leq \kappa ||\xi||_{\mathbb{H}}$ *and* $||\xi||_4 \leq \kappa r^{-1/8} ||\xi||_{\mathbb{H}}$ *for all* $\xi \in \mathbb{H}$.
- *If* $r \geq \kappa$, *then* $\kappa^{-1} ||\xi||_{\mathbb{H}} \leq ||\mathcal{L}_0 \xi||_2 \leq \kappa ||\xi||_{\mathbb{H}}$ *for all* $\xi \in \mathbb{H}$.

Proof. The first bullet follows using a standard Sobolev inequality with the fact that $|\mathrm{d}|\xi|| \leq |\nabla_0 \xi|$. The right hand inequality in the second bullet follows by simply from the appearance of only first derivatives in $\mathcal{L}_0$. To obtain the left hand inequality of the second bullet, use the Bochner-type formula for the operator ${\mathcal{L}_0}^2$ (see (5.21) in [25]). To elaborate, let f be any given function on M. Write a section ξ of $i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}}$ as (b, ϕ, g). Then, ${\mathcal{L}_{(\mathrm{A}_0, \mathrm{f}\psi_0)}}^2(\mathrm{b}, \phi, g)$ has respective $i\mathrm{T}^*\mathrm{M}$, $\mathbb{S}$ and $i\underline{\mathbb{R}}$ components

$$\begin{aligned}
&\nabla^\dagger\nabla \mathrm{b} + 2r\mathrm{f}^2\mathrm{b} + r^{1/2}\mathbb{V}_1(\xi)\\
&{\nabla_{\mathrm{A}_0}}^\dagger\nabla_{\mathrm{A}_0}\phi + 2r\mathrm{f}^2\phi + r^{1/2}\mathbb{V}_2(\xi)\\
&\mathrm{d}^*\mathrm{d}g + 2r\mathrm{f}^2 g + r^{1/2}\mathbb{V}_3(\xi),
\end{aligned} \tag{3.28}$$

where $\mathbb{V}_\mathrm{i}$ are zero'th order endomorphisms with absolute value bounded by an r-independent constant. In the case at hand, $\mathrm{f} = |\mu|^{1/2}$ is strictly bounded away from zero. This last point understood, then the left hand inequality in the second bullet of the lemma follows by first taking the L^2 inner product of ${\mathcal{L}_0}^2\xi$ with ξ and then integrating by parts to rewrite the resulting integral. $\square$

It follows from Lemma III.8 that the operator $\mathcal{L}_0$ is invertible when r is large. This understood, write $\mathfrak{y} = {\mathcal{L}_0}^{-1}\mathfrak{u}$,

Lemma III.9. *There exists $\kappa \geq 1$ for use in Lemma III.8 such that when $r \geq \kappa$, then the corresponding $\mathfrak{y} = {\mathcal{L}_0}^{-1}\mathfrak{u}$ obeys $|\mathfrak{y}| \leq c_0 r^{-1/2}$.*

Proof. Let Δ denote the operator that is obtained from what is written in the $\mathrm{f} = |\mu|^{1/2}$ version of (3.28) by setting $\mathbb{V}_\mathrm{i}$ all equal to zero. The latter has Green's function G, a positive, symmetric function on $\mathrm{M} \times \mathrm{M}$ with pole along the diagonal. Moreover, there exists an r-independent constant $c > 1$ such that if $\mathrm{x}, \mathrm{y} \in \mathrm{M}$, then

$$\begin{aligned}
&\mathrm{G}(\mathrm{x}, \mathrm{y}) \leq \frac{c}{dist(\mathrm{x}, \mathrm{y})} e^{-\sqrt{r}\frac{dist(\mathrm{x}, \mathrm{y})}{c}},\\
&|\mathrm{dG}|(\mathrm{x}, \mathrm{y}) \leq c\left(\frac{1}{dist(\mathrm{x}, \mathrm{y})^2} + \frac{\sqrt{r}}{dist(\mathrm{x}, \mathrm{y})}\right) e^{-\sqrt{r}\frac{dist(\mathrm{x}, \mathrm{y})}{c}}.
\end{aligned} \tag{3.29}$$

Both of these bounds follow by using the maximum principle with a standard parametrix for G near the diagonal in $\mathrm{M} \times \mathrm{M}$.

Now write (3.28) as $\Delta\xi + r^{1/2}\mathbb{V}\xi$, and then use G, the fact that ${\mathcal{L}_0}^2\mathfrak{y} = \mathcal{L}_0\mathfrak{u}$, and the uniform bounds on the terms $\mathbb{V}_{\mathrm{i}}$ to see that

$$|\mathfrak{y}|(\mathrm{x}) \leq c' \int_{\mathrm{M}} \mathrm{G}(\mathrm{x}, \cdot)(1 + r^{1/2}(1 + |\mathfrak{y}|)),$$

where c' is independent of r. This last equation together with (3.29) yields

$$|\mathfrak{y}|(\mathrm{x}) \leq c'' r^{-1/2}(1 + \sup_{\mathrm{M}}|\mathfrak{y}|),$$

where c'' is also independent of r. The lemma follows from this bound. □

With $\mathfrak{y}$ in hand, it follows that $\xi \in \mathbb{H}$ is a solution of the equations in (3.26) if $\tilde{\xi} = \xi - \mathfrak{y}$ is a solution of the equation $\mathcal{L}_0\tilde{\xi} + r^{1/2}(\tilde{\xi} * \tilde{\xi} + 2\mathfrak{y} * \tilde{\xi}) = -r^{1/2}\mathfrak{y} * \mathfrak{y}$. To find a solution $\tilde{\xi}$ of the latter equation, introduce the map $\mathbb{T} : \mathbb{H} \to \mathbb{H}$ defined by

$$\mathbb{T} : \tilde{\xi} \mapsto -r^{1/2}{\mathcal{L}_0}^{-1}(\mathfrak{y} * \mathfrak{y} + \tilde{\xi} * \tilde{\xi} + 2\mathfrak{y} * \tilde{\xi}). \tag{3.30}$$

Note in this regard that Sobolev inequalities in Lemma III.8 guarantee that $\mathbb{T}$ does indeed define a smooth map from $\mathbb{H}$ onto itself when r is larger than some fixed constant. Our goal now is to show that the map $\mathbb{T}$ has a unique fixed point with small norm. Given $\mathrm{R} \geq 1$, we let $\mathrm{B}_{\mathrm{R}} \in \mathbb{H}$ denote the ball of radius $r^{-1/2}\mathrm{R}$ centered at the origin. We next invoke

Lemma III.10. *There exists* $\kappa > 1$, *and given* $\mathrm{R} \geq \kappa$, *there exists* κ_{R} *such that if* $r \geq \kappa_{\mathrm{R}}$, *then* $\mathbb{T}$ *maps* B_{R} *onto itself as a contraction mapping.*

Proof. Let $\mathrm{R} > 1$ be such that $||\mathfrak{y}||_\infty \leq \frac{1}{2^{10}} r^{-1/2}\mathrm{R}^{1/2}$. We first show that if r is large, then $\mathbb{T}$ maps B_{R} into itself. Indeed, this follows from Lemma III.8 using the following chain of inequalities:

$$
\begin{aligned}
||\mathbb{T}(\tilde{\xi})||_{\mathbb{H}} &\leq || - r^{1/2}\mathfrak{y} * \mathfrak{y} - r^{1/2}(\tilde{\xi} * \tilde{\xi} + 2\mathfrak{y} * \tilde{\xi})||_2 \\
&\leq r^{1/2}||\mathfrak{y} * \mathfrak{y}||_2 + r^{1/2}||\tilde{\xi} * \tilde{\xi} + 2\mathfrak{y} * \tilde{\xi}||_2 \\
&\leq \frac{1}{4}r^{-1/2}\mathrm{R} + r^{1/2}(||\tilde{\xi} * \tilde{\xi}||_2 + 2||\mathfrak{y} * \tilde{\xi}||_2) \\
&\leq \frac{1}{4}r^{-1/2}\mathrm{R} + r^{1/2}(||\tilde{\xi}||_4{}^2 + 2||\mathfrak{y}||_4||\tilde{\xi}||_4) \\
&\leq \frac{1}{4}r^{-1/2}\mathrm{R} + r^{1/2}(\kappa r^{-1/4}||\tilde{\xi}||_{\mathbb{H}}{}^2 + r^{-1/2}\mathrm{R}^{1/2}\kappa r^{-1/8}||\tilde{\xi}||_{\mathbb{H}}) \\
&\leq \frac{1}{4}r^{-1/2}\mathrm{R} + r^{1/2}(\kappa r^{-1/4}r^{-1}\mathrm{R}^2 + r^{-1/2}\mathrm{R}^{1/2}\kappa r^{-1/8}r^{-1/2}\mathrm{R}) \\
&\leq r^{-1/2}\mathrm{R}(\frac{1}{4} + 2\kappa\mathrm{R}r^{-1/8}). \qquad (3.31)
\end{aligned}
$$

Next, using similar arguments, we show that $\mathbb{T}|_{\mathrm{B_R}}$ is a contraction mapping. In this regard, let $\tilde{\xi}_1, \tilde{\xi}_2 \in \mathrm{B_R}$, then

$$
\begin{aligned}
||\mathbb{T}(\tilde{\xi}_1) - \mathbb{T}(\tilde{\xi}_2)||_{\mathbb{H}} &\leq || - r^{1/2}(\tilde{\xi}_1 * \tilde{\xi}_1 + 2\mathfrak{y} * \tilde{\xi}_1) + r^{1/2}(\tilde{\xi}_2 * \tilde{\xi}_2 + 2\mathfrak{y} * \tilde{\xi}_2)||_2 \\
&\leq r^{1/2}(||(\tilde{\xi}_1 * \tilde{\xi}_1 - \tilde{\xi}_2 * \tilde{\xi}_2)||_2 + 2||\mathfrak{y} * \tilde{\xi}_1 - \mathfrak{y} * \tilde{\xi}_2||_2) \\
&\leq r^{1/2}(||(\tilde{\xi}_1 + \tilde{\xi}_2) * (\tilde{\xi}_1 - \tilde{\xi}_2)||_2 + ||\mathfrak{y} * (\tilde{\xi}_1 - \tilde{\xi}_2)||_2) \\
&\leq r^{1/2}(||\tilde{\xi}_1 + \tilde{\xi}_2||_4||\tilde{\xi}_1 - \tilde{\xi}_2||_4 + 2||\mathfrak{y}||_4||\tilde{\xi}_1 - \tilde{\xi}_2||_4) \\
&\leq r^{1/2}(||\tilde{\xi}_1||_4 + ||\tilde{\xi}_2||_4 + 2||\mathfrak{y}||_4)||\tilde{\xi}_1 - \tilde{\xi}_2||_4 \\
&\leq r^{1/2}(2\kappa r^{-1/8}r^{-1/2}\mathrm{R} + r^{-1/2}\mathrm{R}^{1/2})\kappa r^{-1/8}||\tilde{\xi}_1 - \tilde{\xi}_2||_{\mathbb{H}} \\
&\leq 3\kappa^2\mathrm{R}r^{-1/8}||\tilde{\xi}_1 - \tilde{\xi}_2||_{\mathbb{H}}. \qquad (3.32)
\end{aligned}
$$

Therefore, by the contraction mapping theorem, there exists a unique fixed point of the map $\mathbb{T}$ in the ball $\mathrm{B_R}$. Moreover, by standard elliptic regularity arguments, it follows that the fixed point is smooth, therefore it lies in $\mathrm{C}^\infty(\mathrm{M}; i\mathrm{T}^*\mathrm{M} \oplus \mathbb{S} \oplus i\underline{\mathbb{R}})$. $\square$

We next find an r-independent constant κ and prove that the norm of $\psi = |\mu|^{1/2}\psi_0 + \phi$ is bounded from below by $|\mu|^{1/2} - \kappa r^{-1/2}$. To this end, note that $\tilde{\xi}$

obeys the equation

$$\Delta\tilde{\xi} + r^{1/2}\mathbb{V}\tilde{\xi} = -r^{1/2}\mathcal{L}_0(\mathfrak{y} * \mathfrak{y} + \tilde{\xi} * \tilde{\xi} + 2\mathfrak{y} * \tilde{\xi}). \tag{3.33}$$

What with (3.29) and the bound $|\mathfrak{y}| \le 2r^{-1/2}\mathrm{R}$ this last equation implies is

$$|\tilde{\xi}|(\mathrm{x}) \le c_0 r^{-1/2} + c_0 r^{1/2}\int_{\mathrm{M}} \Big(\frac{1}{dist(\mathrm{x},\cdot)^2} + \frac{\sqrt{r}}{dist(\mathrm{x},\cdot)}\Big)e^{-\sqrt{r}\frac{dist(\mathrm{x},\cdot)}{c}}(|\tilde{\xi}|^2 + r^{-1/2}|\tilde{\xi}|)] \tag{3.34}$$

where c_0 is independent of x and r. Bound the term $r^{-1/2}|\tilde{\xi}|$ in the integral by $|\tilde{\xi}|^2 + r^{-1}$. The contribution to the right hand side of (3.34) of the resulting term with r^{-1} factor is bounded by $c_1 r^{-1/2}$ where c_1 is independent of r. To say something about the term with $|\tilde{\xi}|^2$, note that the function $\frac{1}{dist(\mathrm{x},\cdot)}|\tilde{\xi}|$ is square integrable with L^2-norm bounded by an x-independent multiple of the ${\mathrm{L}^2}_1$-norm of $|\tilde{\xi}|$; and thus by $c_2||\tilde{\xi}||_{\mathbb{H}}$ with c_2 independent of r and $\tilde{\xi}$. This understood, the term in the integral with $|\tilde{\xi}|^2$ contributes at most $c_3(r^{1/2}||\tilde{\xi}||_{\mathbb{H}}{}^2 + r||\tilde{\xi}||_2||\tilde{\xi}||_{\mathbb{H}})$ with c_3 independent of r and $\tilde{\xi}$. The latter is bounded by an r-independent multiple of $r^{-1/2}$. Thus, we see that $|\tilde{\xi}| \le c_4 r^{-1/2}$ which proves our claim that $|\psi| \ge |\mu|^{1/2} - \kappa r^{-1/2}$.

We now turn to the claim about uniqueness. To this end, let $\delta \in (0, \frac{\inf_{\mathrm{M}}|\mu|}{2})$ and let $(\mathbb{A}, \psi)$ be a solution of some $t \in \mathrm{S}^1$ and $r \ge 1$ version of the equations in (3.2) with the property that $|\psi| \ge |\mu|^{1/2} - \delta$ at each point in M. Granted such is the case, it follows from Lemma III.2 that $|\alpha| \ge |\mu|^{1/2} - \delta - \kappa r^{-1/2}$ at each point in M, with C_0 independent of r. We now make use of Lemma III.3 to see the following: Given $\epsilon > 0$, there exists $\delta_\epsilon > 0$ such that if $\delta < \delta_\epsilon$, then

$$\begin{aligned}
&|\mu|^{1/2} - \epsilon \le |\alpha| \le |\mu|^{1/2} + \epsilon \text{ and } |\beta| \le \epsilon r^{-1/2},\\
&|\nabla_{\mathrm{E}}\alpha| \le \epsilon r^{1/2} \text{ and } |\nabla_{\mathrm{EK}^{-1}}\beta| \le \epsilon,\\
&|\nabla^2_{\mathrm{E}}\alpha| \le \epsilon r \text{ and } |\nabla^2_{\mathrm{EK}^{-1}}\beta| \le \epsilon r^{1/2}.
\end{aligned} \tag{3.35}$$

Since α is nowhere zero for sufficiently large $r > 1$, one has $u = \bar{\alpha}/|\alpha| \in \mathcal{G}$. Now, change $(\mathbb{A}, \psi)$ to a new gauge by u, and denote the resulting pair of gauge and spinor fields again by $(\mathbb{A}, \psi)$. Since $u\alpha = |\alpha| 1_{\underline{\mathbb{C}}}$, one has $\mathbb{A} = \mathrm{A}_0 + 2i\mathrm{a}$ where

$$\mathrm{a} = -\frac{i}{2}(\alpha^{-1}\nabla_{\mathrm{E}}\alpha - \bar{\alpha}^{-1}\nabla_{\mathrm{E}}\bar{\alpha}). \tag{3.36}$$

Then, (3.35) and (3.36) imply

$$r^{-1/2}|\mathrm{a}| + r^{-1}|\nabla \mathrm{a}| \leq c_0 \epsilon. \tag{3.37}$$

We now change $(\mathbb{A}, \psi)$ to yet another gauge so as to write the resulting pair of connection and spinor as $(\mathrm{A}_0 + 2(2r)^{1/2}\mathrm{b}, |\mu|^{1/2}\psi_0 + \phi)$ where $(\mathrm{b}, \phi, 0)$ obey (3.23). This gauge transformation is written $e^{i\mathrm{x}}$ where $\mathrm{x} : \mathrm{M} \to \mathbb{R}$. Thus, the pair (b, ϕ) is

$$\begin{aligned} \mathrm{b} &= i(2r)^{-1/2}(\mathrm{a} - \mathrm{dx}) \\ \phi &= e^{i\mathrm{x}}\psi - |\mu|^{1/2}\psi_0. \end{aligned} \tag{3.38}$$

Equation (3.23) is obeyed if and only if x obeys the equation

$$\mathrm{d}^*\mathrm{dx} + 2|\mu|^{1/2} r |\alpha| \sin \mathrm{x} = \mathrm{d}^*\mathrm{b}. \tag{3.39}$$

We can now proceed along the lines of what is done in [25] to solve an analogous equation, namely (2.16) in [25]. In particular, the arguments in [25] can be used with only small modifications to find an r-independent constant κ such that if the constant ϵ in (3.35) is bounded by κ^{-1} and $r \geq \kappa$, then (3.39) has a unique solution, x, with

$$|\mathrm{x}| + r^{1/2}|\mathrm{dx}| \leq \kappa\epsilon. \tag{3.40}$$

Granted this, it follows that $\mathfrak{b} = (\mathrm{b}, \phi, 0)$ with (b, ϕ) as in (3.38) obeys (3.26) and that

$$|\mathfrak{b}| \leq c\epsilon \tag{3.41}$$

with $c > 0$ a constant that is independent of ϵ and r. Then, $\mathfrak{h} = \mathfrak{b} - \mathfrak{y}$ obeys $\mathcal{L}_0\mathfrak{h} = r^{1/2}(\mathfrak{y} * \mathfrak{y} + \mathfrak{h} * \mathfrak{h} + 2\mathfrak{y} * \mathfrak{h})$ and $||\mathfrak{h}||_\infty \leq c_0\epsilon$ where c_0 is independent of $(\mathbb{A}, \psi)$ and r. This understood, it follows from Lemma III.8 that

$$||\mathfrak{h}||_{\mathbb{H}} \leq \frac{1}{4}\mathrm{R}_{\mathfrak{y}} r^{-1/2} + c_1 r^{1/2}||\mathfrak{h}||_\infty||\mathfrak{h}||_2 \leq \frac{1}{4}\mathrm{R}_{\mathfrak{y}} r^{-1/2} + c_2 r^{1/2}\epsilon||\mathfrak{h}||_2, \tag{3.42}$$

where $\mathrm{R}_{\mathfrak{y}}$ is an r independent constant such that $||\mathfrak{y}||_\infty \leq \frac{1}{2^{10}} r^{-1/2}\mathrm{R}_{\mathfrak{y}}$ and $c_1, c_2 > 0$ are constants which are both independent of $(\mathbb{A}, \psi)$ and r. This last inequality implies that $||\mathfrak{h}||_{\mathbb{H}} < \mathrm{R}_{\mathfrak{y}} r^{-1/2}$ when $\epsilon < c_4$ with c_4 an r and $(\mathbb{A}, \psi)$ independent constant. This understood, it follows from Lemma III.10 that $(\mathbb{A}, \psi)$ is gauge equivalent to the solution of (3.2) that was constructed from Lemma III.10's fixed point of the map $\mathbb{T}$ when r is larger than some fixed constant. This then proves the uniqueness assertion made by Proposition III.7.

We introduce $(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$ to denote the solution that is obtained from Lemma III.10's fixed point. This solution is of the form $(\mathrm{A}_0 + 2(2r)^{1/2}\mathrm{b}, |\mu|^{1/2}\psi_0 + \phi)$. Our final task is to prove that the $(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$ version of the operator in (3.22) has trivial kernel. To see that such is the case, remember that (b, ϕ) has norm bounded by $c_0 r^{-1/2}$ with c_0 independent of r. This being the case, the operator in question differs from the operator $\mathcal{L}_0$ by a zero'th order term with bound independent of r. As a consequence, there is a constant $c > 0$ which is independent of r and such that

$$||\mathcal{L}_{(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})}\xi||_2 \geq c||\xi||_{\mathbb{H}} \tag{3.43}$$

for all $\xi \in \mathbb{H}$ when r is large. This understood, the fact that $(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$ is non-degenerate when r is large follows from Lemma III.8. This finishes the proof of Proposition III.7. $\square$

3.3 Dependence of solutions on the parameter

In this section, we investigate the behavior of solutions to the equations in (3.2) as $t \in \mathrm{S}^1$ varies. Our purpose is to find solutions of the equations in (3.2) for large $r > 1$ so as to guarantee existence of the sets Θ_t for every t outside a set of small measure.

3.3.1 Bifurcation analysis

In [25], Taubes proves the existence of a residual subset of Ω such that for each perturbation from this residual set there exists a locally finite set of $r > 1$ values in the complement of which all solutions of the corresponding version of the equations under consideration are non-degenerate. Moreover, for such values of r, the values of the perturbed version of the Chern-Simons-Dirac functional on pairs of configurations which are not gauge equivalent are different. These results can be carried over to the solutions of the equations in (3.2) for fixed values of $t \in \mathrm{S}^1$. Here, we shall prove a similar result for $t \in \mathrm{S}^1$ values with $r \geq 1$ fixed.

Proposition III.11. *Fix $r \geq 1$ and $\delta > 0$. Then there exist a t-independent 1-form $\sigma \in \Omega$ with $\mathcal{P}$ norm bounded by δ such that the following is true: Replace μ by $\mu + \mathrm{d}\sigma$.*

- *The resulting 2-form $\omega = \mathrm{d}t \wedge \nu + \mu$ is symplectic.*

- *There exists finite sets $\mathfrak{T}_r$ and $\mathfrak{T}_r'$ in S^1 such that if $t \in \mathrm{S}^1 \setminus \mathfrak{T}_r$, then $\mathfrak{a}^{\mathcal{F}}$ distinguishes distinct gauge equivalence classes of solutions of the t and r version of the equations in (3.2). On the other hand, if $t \in \mathrm{S}^1 \setminus \mathfrak{T}_r'$ all solutions of the t and r version of the equations in (3.2) are non-degenerate.*

- *There exists a countable set $\mathfrak{S}_r \in \mathrm{S}^1$ that contains $\mathfrak{T}_r \cup \mathfrak{T}_r'$ with accumulation points on the latter such that if $t \in \mathrm{S}^1 \setminus \mathfrak{S}_r$, then the gauge equivalence classes* *of solutions of the equations in (3.2) can be used to label the generators of the Seiberg–Witten Floer complex. In this regard, the degree of any generator can be taken to be mod(p) reduction of the negative of the spectral flow function $\mathcal{F}$.*

Proof. The claim in the first bullet of the proposition is obvious. As for the second and third bullets, the proof of these two follow from the arguments similar to those used in Sections 2a and 2b of [25]. There are three parts to the proof.

Part 1: We shall start by changing the symplectic structure in its isotopy class. Let $\Delta \subset \Omega$ denote an open ball of small radius consisting of 1-forms σ on M such that $dt \wedge \nu + (\mu + d\sigma)$ is a symplectic form on $S^1 \times M$. Being an open subset of Ω, Δ is a smooth Banach manifold. Note also that perturbing the symplectic form via forms in Δ does not change the canonical spin^c structure. Then, with $dt \wedge \nu + (\mu + d\sigma)$ as the new symplectic form, the equations in (3.2) read

$$\begin{aligned} *da &= r(\psi^\dagger \tau \psi - i * (\mu + d\sigma)) \\ \mathcal{D}_A \psi &= 0. \end{aligned} \tag{3.44}$$

Let $\mathcal{H}_2$ and $\mathcal{H}_3$ denote respectively the Banach spaces $L^2{}_2(M; iT^*M \oplus \mathbb{S})$ and $L^2{}_3(M; iT^*M \oplus \mathbb{S})$. Given $t \in S^1$ and $r \geq 1$, let $\mathcal{Y}$ denote the set of triples $(\sigma, (a, \psi))$ in $\Delta \times \mathcal{H}_3$ that solves the corresponding version of the equations in (3.44). The set $\mathcal{Y}$ is the zero locus of some smooth section of a smooth vector bundle over $\Delta \times \mathcal{H}_3$ whose fiber over any $(\sigma, (a, \psi))$ is the subspace in $\mathcal{H}_2$ of pairs (q, ξ) satisfying

$$-d^*q - 2^{-1/2} r^{1/2} (\xi^\dagger \psi - \psi^\dagger \xi) = 0. \tag{3.45}$$

The aforementioned section of this smooth vector bundle is defined by

$$(\sigma, (a, \psi)) \mapsto (*da - r(\psi^\dagger \tau \psi - i * (\mu + d\sigma)), 2r^{1/2} \mathcal{D}_A \psi). \tag{3.46}$$

In what follows, we shall denote this section by $\mathcal{S}$. We will next show that $\mathcal{Y}$ has the structure of a smooth Banach manifold.

The image of a vector (η, b, ϕ) under the differential of the section $\mathcal{S}$ at any $(\sigma, (\mathrm{a}, \psi))$ has respective $i\mathrm{T}^*\mathrm{M}$ and $\mathbb{S}$ components

$$
\begin{aligned}
&* \mathrm{db} - 2^{-1/2} r^{1/2}((\psi^\dagger \tau \phi + \phi^\dagger \tau \psi) - i * \mathrm{d}\eta) \\
&\mathcal{D}_\mathrm{A} \phi + 2^{1/2} r^{1/2} \mathfrak{cl}(\mathrm{b}) \psi.
\end{aligned}
\tag{3.47}
$$

First of all, observe that the image of vectors of the form $(0, \mathrm{b}, \phi)$ under the differential of the section $\mathcal{S}$ at an arbitrary $(\sigma, (\mathrm{a}, \psi))$ has respective $i\mathrm{T}^*\mathrm{M}$ and $\mathbb{S}$ components which are the $g = 0$ versions of the first two components in the image of the operator $\mathcal{L}$ as it is defined in (3.22). Therefore, the differential of $\mathcal{S}$ has finite dimensional co-kernel at any $(\sigma, (\mathrm{a}, \psi))$. Let (q, ξ) be a vector in this co-kernel. Then, (q, ξ) is L^2-orthogonal to any vector in the image of the differential of $\mathcal{S}$. In particular, it is L^2-orthogonal to the image of any vector of the form $(0, \mathrm{b}, \phi)$ under the differential of $\mathcal{S}$, and since $\mathcal{L}$ is a self-adjoint operator, (q, ξ) obeys the coupled equations

$$
\begin{aligned}
* \mathrm{dq} - 2^{-1/2} r^{1/2}(\psi^\dagger \tau \xi + \xi^\dagger \tau \psi) &= 0 \\
\mathcal{D}_\mathrm{A} \xi + 2^{1/2} r^{1/2} \mathfrak{cl}(\mathrm{q}) \psi &= 0 \\
-\mathrm{d}^* \mathrm{q} - 2^{-1/2} r^{1/2}(\xi^\dagger \psi - \psi^\dagger \xi) &= 0.
\end{aligned}
\tag{3.48}
$$

Since (q, ξ) is L^2-orthogonal to the image of any vector of the form $(\eta, 0, 0)$ under the differential of $\mathcal{S}$ as well, one has $-i \int_\mathrm{M} \mathrm{d}\eta \wedge *\mathrm{q} = 0$ for any η. The latter implies $\mathrm{dq} = 0$. Hence, one could write $\mathrm{q} = \mathrm{h} + \mathrm{df}$ where h is an imaginary valued harmonic 1-form and f is an imaginary valued function on M. Then, the middle equation in (3.48) requires that $\xi = -2^{1/2} r^{1/2} \mathrm{f} \psi + \zeta$ where $\mathcal{D}_\mathrm{A} \zeta = -2^{1/2} r^{1/2} \mathfrak{cl}(\mathrm{p}) \psi$. But now, the top equation in (3.48) implies that $\zeta = \kappa\psi$ for some imaginary valued function κ defined on the set where $\psi \neq 0$. Therefore, $\mathrm{h} = 2^{-1/2}r^{-1/2}\mathrm{d}\kappa$ on this set, and it follows by the unique continuation property of the Dirac operator that $\mathrm{h} = 0$ and κ extends to a constant function on M. To be more explicit, the unique continuation property and the ellipticity of the Dirac operator requires that the zero set of ψ is neither an open subset of M nor it disconnects some open ball in M. Therefore, any loop representing a homology class in M can be homotoped so as to avoid the zero set of ψ. Since h is exact outside the zero set of ψ, its integral on any generator of the first homology of M yields zero. Next, an application of the maximum principle to the bottom equation in (3.48) shows that f is a constant function. Therefore, $\mathrm{q} = 0$ which in turn requires that $\xi = 0$.

The set $\mathcal{Y}$ has the structure of a smooth Banach manifold as a result of the above discussion. Now, consider the quotient of $\mathcal{Y}$ by the action of the gauge group. This is also a smooth Banach manifold which we will denote by $\mathcal{Y}/\mathcal{G}$. Furthermore, the projection $\pi : \mathcal{Y}/\mathcal{G} \to \Delta$ is a Fredholm map of index zero. This is because the restriction of the section $\mathcal{S}$ onto $\mathcal{Y} \cap \{\sigma\} \times \mathcal{H}_3$ for any $(\sigma, (\mathrm{a}, \psi)) \in \mathcal{Y}$ is Fredholm of index zero. Therefore, by the Sard-Smale theorem [19] there exists a residual set of regular values of the map π in Δ. Note that σ is a regular value of π if and only if all solutions of the corresponding version of equations in (3.44) are non-degenerate.

Now, fix $r \geq 1$ and consider the smooth vector bundle with base space $\mathrm{S}^1 \times \Delta \times \mathcal{H}_3$ and with fibers over any $(t, \sigma, (\mathrm{a}, \psi))$ being the subspace in $\mathcal{H}_2$ of pairs (q, ξ) satisfying (3.45). Let $\mathcal{X}$ denote the zero set of the smooth section of this bundle defined by (3.46). Denote this section by $\mathcal{S}$ as well. The image of $(\mathrm{s}, \eta, \mathrm{b}, \phi)$ under the differential of this section at an arbitrary $(t, \sigma, (\mathrm{a}, \psi))$ has respective $i\mathrm{T}^*\mathrm{M}$ and

$\mathbb{S}$ components

$$
\begin{aligned}
&*\mathrm{db} - 2^{-1/2} r^{1/2}((\psi^{\dagger}\tau\phi + \phi^{\dagger}\tau\psi) - i * (\dot{\mu} + \mathrm{d}\dot{\sigma})\mathrm{s} - i * \mathrm{d}\eta) \\
&\mathcal{D}_{\mathrm{A}}\phi + 2^{1/2} r^{1/2}\mathfrak{cl}(\mathrm{b})\psi.
\end{aligned} \tag{3.49}
$$

An argument very much the same as the one provided above can be used to prove that the differential of $\mathcal{S}$ is surjective at any $(t, \sigma, (\mathrm{a}, \psi))$. Therefore, $\mathcal{X}$ has the structure of a smooth Banach manifold. Its quotient $\mathcal{X}/\mathcal{G}$ by the action of the gauge group is also a smooth Banach manifold, and the projection $\pi_1 : \mathcal{X}/\mathcal{G} \to \Delta$ is a Fredholm map of index 1. Then, by the Sard-Smale theorem there exists a residual set of regular values of the map π_1 in Δ. If σ is a regular value, then its pre-image under the map π_1 is a smooth 1-dimensional manifold consisting of solutions of the equations in (3.44). Now, let σ be a regular value of the map π_1 and $\pi_1^{\sigma} : \pi_1^{-1}(\sigma) \to \mathrm{S}^1$ denote the projection map. Then, by Sard's theorem, critical values of the map π_1^{σ} form a compact set of measure zero in S^1, and $t \in \mathrm{S}^1$ is a regular value if and only if all solutions of the t, r and σ version of the equations in (3.44) are non-degenerate. In fact, critical values of the map π_1^{σ} form a finite subset of S^1. To see this, suppose that $(t, \sigma, (\mathrm{a}, \psi))$ is a critical point of π_1^{σ}. As is explained in Section 7 of [24], $\pi_1^{-1}(\sigma)$ can be endowed with the structure of a real analytic set near $(t, \sigma, (\mathrm{a}, \psi))$. Since π_1^{σ} is a proper map, this implies that the set of critical values of π_1^{σ} is locally finite. We denote this set by $\mathfrak{T}_r{}'$.

Part 2: Denote by $\mathcal{W}$ the subset of $\mathcal{X} \times \mathcal{X}$ whose elements are of the form $((t, \sigma, \mathfrak{c}_1), (t, \sigma, \mathfrak{c}_2))$ such that both $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are non-degenerate solutions of the t, r and σ version of the equations in (3.44), and $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are not gauge equivalent. Then, $\mathcal{W}$ has the structure of a smooth Banach manifold which one can show as follows: The subset $\mathcal{V}$ of $\mathcal{X}$ consisting of elements of the form $(t, \sigma, \mathfrak{c})$ where $\mathfrak{c}$ is a non-degenerate solution of the t, r and σ version of the equations in (3.44) is open. Therefore, $\mathcal{V}\times\mathcal{V}$ is open in $\mathcal{X}\times\mathcal{X}$ as well. Consider the projection $\mathcal{V}\times\mathcal{V} \to (\mathrm{S}^1 \times \Delta)^{\times 2}$ under which $((t_1, \sigma_1, \mathfrak{c}_1), (t_2, \sigma_2, \mathfrak{c}_2))$ is mapped to $((t_1, \sigma_1), (t_2, \sigma_2))$. Then, the pre-image of the diagonal under this projection map is a smooth Banach manifold since for any $((t, \sigma, \mathfrak{c}_1), (t, \sigma, \mathfrak{c}_2))$ in the pre-image of the diagonal, both $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are non-degenerate solutions of the corresponding versions of the equations in (3.44). To elaborate, note that the tangent space of $\mathcal{V}$ at an arbitrary $(t, \sigma, \mathfrak{c})$ is the set of vectors of the form $(\mathrm{s}, \eta, \mathrm{b}, \phi)$ that are in the kernel of the differential of the section $\mathcal{S}$ as described in (3.49), and when $\mathfrak{c}$ is a non-degenerate solution of the t, r and σ version of the equations in (3.44), all possible values of s and η appear among these vectors. The differential of the projection map at an arbitrary $((t, \sigma, \mathfrak{c}_1), (t, \sigma, \mathfrak{c}_2))$ maps a tangent vector $((\mathrm{s}_1, \eta_1, \mathrm{b}_1, \phi_1), (\mathrm{s}_2, \eta_2, \mathrm{b}_2, \phi_2))$ to $((\mathrm{s}_1, \eta_1), (\mathrm{s}_2, \eta_2))$. Therefore, it is surjective at any point in the pre-image of the diagonal. Then, $\mathcal{W}$ has the structure of a smooth Banach manifold as well because of being an open subset of $\mathcal{V} \times \mathcal{V}$.

Now consider the functional

$$\mathfrak{w} : \mathcal{W} \to \mathbb{R}$$

defined by $\mathfrak{w}((t, \sigma, \mathfrak{c}_1), (t, \sigma, \mathfrak{c}_2)) = \mathfrak{a}^{\mathcal{F}}(\mathfrak{c}_2) - \mathfrak{a}^{\mathcal{F}}(\mathfrak{c}_1)$. We will show that the functional $\mathfrak{w}$ has no critical values.

Let $(t, \sigma, \mathfrak{c} = (\mathbb{A}_{\mathbb{S}} + 2\mathrm{a}, \psi)) \in \mathcal{W}$. Then, the image of a tangent vector $(\mathrm{s}, \eta, \mathrm{b}, \phi)$ under the differential of $\mathfrak{a}$ is given by

$$-ir(\frac{\partial \mathfrak{a}}{\partial t}(\mathfrak{c}))\mathrm{s} - ir \int_{\mathrm{M}} \eta \wedge \mathrm{da} + r \int_{\mathrm{M}} \psi^{\dagger} \mathfrak{cl}(\mathrm{b})\psi \tag{3.50}$$

Therefore, if $((t, \sigma, \mathfrak{c}_1), (t, \sigma, \mathfrak{c}_2)) \in \mathcal{W}$ where $\mathfrak{c}_{\mathrm{i}} = (\mathbb{A}_{\mathrm{i}} = \mathbb{A}_{\mathbb{S}} + 2\mathrm{a}_{\mathrm{i}}, \psi_{\mathrm{i}})$ for $\mathrm{i} = 1, 2$, then the differential of $\mathfrak{w}$ is identically zero at $((t, \sigma, \mathfrak{c}_1), (t, \sigma, \mathfrak{c}_2))$, only if

$$\begin{aligned} \mathrm{da}_1 &= \mathrm{da}_2, \\ \psi_1{}^{\dagger}\mathfrak{cl}(\cdot)\psi_1 &= \psi_2{}^{\dagger}\mathfrak{cl}(\cdot)\psi_2. \end{aligned} \tag{3.51}$$

The first equation in (3.51) implies that $\mathrm{a}_2 = \mathrm{a}_1 - i\gamma$ for some closed 1-form γ. On the other hand, the second equation in (3.51) requires that $\psi_2 = u\psi_1$ for a smooth S^1-valued function defined on the set where $\psi_1 \neq 0$. Since both $\mathcal{D}_{\mathbb{A}_1}\psi_1 = 0$ and $\mathcal{D}_{\mathbb{A}_2}\psi_2 = 0$, $\gamma = u^{-1}\mathrm{d}u$ on this set. The unique continuation property of the Dirac operator requires that u extends to a smooth S^1-valued function on M, and hence $\mathfrak{c}_1$ and $\mathfrak{c}_2$ become gauge equivalent which contradicts with the working assumptions. Therefore, the differential of the functional $\mathfrak{w}$ is surjective at $((t, \sigma, \mathfrak{c}_1), (t, \sigma, \mathfrak{c}_2))$ as is claimed. In particular, 0 is a regular value of the functional $\mathfrak{w}$, and $\mathfrak{w}^{-1}(0)$ is a co-dimension 1 sub-manifold of $\mathcal{W}$.

Now, consider the quotient $\mathfrak{w}^{-1}(0)/\mathcal{G}$ of $\mathfrak{w}^{-1}(0)$ by the action of $\mathcal{G} \times \mathcal{G}$. The projection $\pi_2 : \mathfrak{w}^{-1}(0)/\mathcal{G} \to \Delta$ is a Fredholm map of index zero. Hence, by the Sard-Smale theorem, there exists a residual set of regular values of this map, and if σ is a regular value, then the pre-image of σ in $\mathfrak{w}^{-1}(0)/\mathcal{G}$ is a zero-dimensional manifold thus a locally finite set of points. In particular, the projection of this zero-dimensional manifold onto S^1 is a finite set. Denote this set by $\mathfrak{T}_r$.

Part 3: With what is said in Parts 1 and 2 understood, the claim in the third bullet of the proposition follows from the arguments that are almost exactly the same as those used in the proof of Proposition 2.3 in [25]. Therefore, we shall not repeat those arguments here. □

In order to clarify the claim in the third bullet of Proposition III.11, we shall give a pictorial explanation of what should be expected as the values of t varies in S^1. Having fixed $r \geq 1$, Proposition III.11 lets us find a canonical basis for the Seiberg–Witten Floer complex as defined by the solutions of the equations in (3.2) at each $t \in \mathrm{S}^1 \setminus \mathfrak{S}_r$. This basis consists of gauge equivalence classes of solutions $\{[\mathfrak{c}_i]\}_{i=1}^{n_{r,t}}$ to the perturbed version of the equations in (3.2) ordered in such a way that $\mathfrak{a}^{\mathcal{F}}(\mathfrak{c}_i) > \mathfrak{a}^{\mathcal{F}}(\mathfrak{c}_{i+1})$ for $i = 1, \ldots, n_{r,t} - 1$. As t varies in one of the connected components of $\mathrm{S}^1 \setminus \mathfrak{S}_r$ this basis varies smoothly without any change. On the other hand, three different things could happen as t crosses a point in $\mathfrak{S}_r$.

- **Handle slide:** As t crosses a point in $\mathfrak{T}_r$, pairs of generators from this basis could change order.

- **Pair annihilation/creation:** As t crosses a point in $\mathfrak{T}_r{}'$, pairs of generators from this basis could cancel each other or a new pair of generators could be born.

- As t crosses a point in $\mathfrak{S}_r \setminus \mathfrak{T}_r \cup \mathfrak{T}_r{}'$, a change of the bases could occur which would be represented by an upper triangular matrix with all diagonal entries equal to 1.

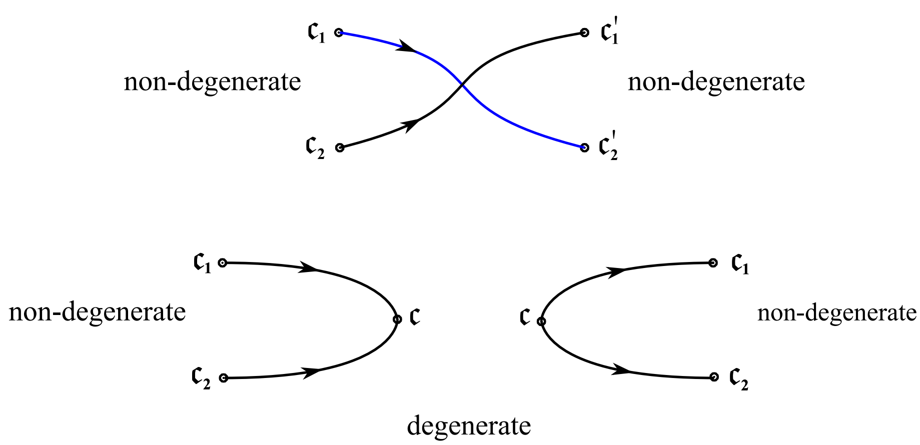

Figure 3.1: Bifurcation diagrams.

Figure 3.1 explains schematically first two of the three things that could occur as t crosses a point in $\mathfrak{S}_r$. The top diagram in Figure 3.1 refers to a handle slide which corresponds to crossing a point in $\mathfrak{T}_r$, and the two diagrams in the bottom of Figure 3.1 refer respectively to a pair annihilation and a pair creation as t crosses a point in ${\mathfrak{T}_r}'$.

3.3.2 Min-Max generators

Suppose now that $t \in \mathrm{S}^1 \setminus \mathfrak{S}_r$ and that θ is a non-zero Seiberg–Witten Floer homology class. Let $\{\mathfrak{c}_i\}_{i=1}^{n_{r,t}}$ be the canonical basis for the Seiberg–Witten Floer complex as found by Proposition III.11 and $\mathfrak{n} = \Sigma \mathrm{z_i}[\mathfrak{c}_\mathrm{i}]$ denote a cycle that represents θ . Here $\mathrm{z_i} \in \mathbb{Z}$ and $[\mathfrak{c}_\mathrm{i}] \in \mathcal{B}$ is a gauge equivalence class of solutions of the t and r version of the equations in (3.2). Let $\mathfrak{a}^{\mathcal{F}}[\mathfrak{n}; t]$ denote the maximum value of $\mathfrak{a}^{\mathcal{F}}$ on the set of generators $\{[\mathfrak{c}_\mathrm{i}]\}$ with $\mathrm{z_i} \neq 0$. Set ${\mathfrak{a}^{\mathcal{F}}}_\theta$ to denote the minimal value in the resulting set $\{\mathfrak{a}^{\mathcal{F}}[\mathfrak{n}; t]\}$ when $\mathfrak{n}$ runs through all possible representatives of the class θ. The value ${\mathfrak{a}^{\mathcal{F}}}_\theta$ is attained by a unique generator $[\mathrm{c}_\Theta]$ from the canonical basis and this generator is called the *min-max generator*. The min-max generators change in a smooth fashion as t varies in $\mathrm{S}^1 \setminus \mathfrak{T}_r$. As is stated in the next proposition, this fact could be used to construct a continuous, piecewise differentiable function on S^1.

Proposition III.12. *The various $t \in \mathrm{S}^1 \setminus \mathfrak{S}_r$ versions of the Seiberg–Witten Floer homology groups can be identified in a degree preserving manner so that if θ is any given non-zero class, then the function ${\mathfrak{a}^{\mathcal{F}}}_\theta(\cdot)$ on $\mathrm{S}^1 \setminus \mathfrak{S}_r$ extends to the whole of S^1 as a continuous, Lipschitz function that is smooth on the complement of $\mathfrak{T}_r$. Moreover, if $\mathrm{I} \subset \mathrm{S}^1 \setminus \mathfrak{T}_r$ is a component, then there exists $\mathrm{I}' \subset \mathrm{S}^1$ containing the closure of I and a smooth map $\mathfrak{c}_{\theta,\mathrm{I}} : \mathrm{I}' \to \mathcal{C}$ that solves the corresponding version of the equations in (3.2) at each $t \in \mathrm{I}'$ and is such that ${\mathfrak{a}^{\mathcal{F}}}_\theta(t) = \mathfrak{a}^{\mathcal{F}}(\mathfrak{c}_{\theta,\mathrm{I}}(t))$ at each $t \in \mathrm{I}'$.*

Proof. The proof is, but for notational changes and two additional remarks, identical to that of Proposition 2.5 in [25]. To set the stage for the first remark, fix a base point $0 \in \mathrm{S}^1 \setminus \mathfrak{S}_r$. The identifications of the Seiberg–Witten Floer homology groups given by adapting what is done in [25] may result in the following situation: As t increases from 0, these identifications results at $t = 2\pi$ in an automorphism, U, on the $t = 0$ version of the Seiberg–Witten Floer homology. This automorphism need not obey $\mathfrak{a}^{\mathcal{F}}{}_{\mathrm{U}\theta} = \mathfrak{a}^{\mathcal{F}}{}_{\theta}$. If not, then it follows using Proposition III.11 that the identifications made at $t < 2\pi$ to define U can be changed if necessary as t crosses points in $\mathfrak{T}_r$ so that the new version of U does obey $\mathfrak{a}^{\mathcal{F}}{}_{\mathrm{U}\theta} = \mathfrak{a}^{\mathcal{F}}{}_{\theta}$. The second remark concerns the fact that any given $\mathfrak{c}_{\theta,\mathrm{I}}$ is unique up to gauge equivalence. This follows from Proposition III.11's assertion that the function $\mathfrak{a}^{\mathcal{F}}$ distinguishes the Seiberg–Witten solutions when $t \in \mathrm{S}^1 \setminus \mathfrak{T}_r$. □

When $\mathrm{E} = \underline{\mathbb{C}}$, we need to augment what is said in Proposition III.12 with the following:

Proposition III.13. *Suppose that* $\mathrm{E} = \underline{\mathbb{C}}$ *and that there are at least two non-zero Seibeg-Witten Floer homology classes. Then, the identifications made by Proposition III.12 between the various* $t \in \mathrm{S}^1$ *versions of the Seiberg–Witten Floer homology groups can be assumed to have the following property. There is a non-zero class* θ *such that none of Proposition III.12's maps* $\mathfrak{c}_{\theta,\mathrm{I}}$ *send the corresponding interval* I' *to a solution in the gauge equivalence class of Proposition III.7's solution* $(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$.

Proof. At any given $t \in \mathrm{S}^1 \setminus \mathfrak{T}_r$, there is a class θ with $\mathfrak{c}_\theta$ not gauge equivalent to $(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$. To see this, first assume the contrary. Then, it should be the case that any two non-zero homology classes have the same degree, otherwise we would be able to find another homology class whose associated min-max generator is not gauge equivalent to $(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$. Now, suppose that θ_1 and θ_2 are two non-zero homology classes. Then, by assumption, both $\mathfrak{c}_{\theta_1}$ and $\mathfrak{c}_{\theta_2}$ are gauge equivalent to $(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$. Therefore, there exist two relatively prime integers z_1 and z_2 such that $z_2\theta_1 = z_1\theta_2$. As a result, the homology is generated by a single class which contradicts with our working assumptions. This understood, Proposition III.12's isomorphisms can be changed as t crosses a point in $\mathfrak{T}_r$ while increasing from $t = 0$ to insure that no version of $\mathfrak{c}_{\theta,\mathrm{I}}$ gives the same gauge equivalence class as $(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$. $\square$

Let I denote a component of $\mathrm{S}^1\backslash\mathfrak{T}_r$. The assignment of $t \in \mathrm{I}$ to $\mathcal{E}(\mathfrak{c}_{\theta,\mathrm{I}}(t))$ associates to θ a smooth function on I. View this function on I as the restriction from $\mathrm{S}^1 \setminus \mathfrak{T}_r$ of a function, $\mathcal{E}_\theta$. Note that the latter need not extend to S^1 as a continuous function.

With the function $\mathfrak{a}^{\mathcal{F}}{}_\theta$ understood, we come to the heart of the matter, which is the formula for the derivative for this function on any given interval $\mathrm{I} \subset \mathrm{S}^1 \setminus \mathfrak{T}_r$: Let $\mathfrak{c}_{\theta,\mathrm{I}}$ be as described in Proposition III.12. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}^{\mathcal{F}}(\mathfrak{c}_{\theta,\mathrm{I}}(t)) = -ir\int_{\mathrm{M}_t} \nu \wedge \mathrm{da} = -r\mathcal{E}_\theta. \tag{3.52}$$

To explain, keep in mind that $\mathfrak{c}_{\mathrm{I}}$ is a critical point of $\mathfrak{a}^{\mathcal{F}}$ and so the chain rule for the derivative of $\mathfrak{a}^{\mathcal{F}}(\mathfrak{c}_{\theta,\mathrm{I}}(\cdot))$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{a}^{\mathcal{F}}(\mathfrak{c}_{\theta,\mathrm{I}}(t)) = -ir\int_{\mathrm{M}_t} \mathrm{a} \wedge \frac{\partial}{\partial t}\mu; \tag{3.53}$$

and this is the same as (3.52) because ω is a closed form. Indeed, write $\omega = \mathrm{d}t\wedge\nu+\mu$ to see that the equation $\mathrm{d}\omega = 0$ requires $\frac{\partial}{\partial t}\mu = \mathrm{d}\nu$. This understood, an integration by parts equates (3.53) to (3.52).

We get bounds on $\mathcal{E}_\theta$ after integrating (3.52) around S^1. Given that $\mathfrak{a}^{\mathcal{F}}{}_\theta$ is a continuous function, integration of the left-hand side over S^1 gives zero. Thus, we conclude that

$$\int_{\mathrm{S}^1} \mathcal{E}_\theta = 0. \tag{3.54}$$

This formula tells us that $\mathcal{E}_\theta$ is bounded at some points in S^1. Granted the lower bound on $\mathcal{E}$ provided by Lemma III.4, the next result follows as a corollary:

Lemma III.14. *There exists a constant $\kappa > 1$ with the following significance: Fix $r \geq \kappa$ so as to define the set $\mathfrak{S}_r \subset \mathrm{S}^1$. Let θ denote a non-zero Seiberg–Witten Floer homology class. Let n denote a positive integer.Then, the measure of the set in $\mathrm{S}^1 \setminus \mathfrak{S}_r$ where $\mathcal{E}_\theta \geq 2^n$ is less than $\kappa 2^{-n}$.*

Proof. Given the lower bound provided by Lemma III.4, the claim of the lemma follows easily from (3.54). $\square$

3.4 Proofs of Theorem I.2 and Theorem I.4

We now fix r very large so as to define the set $\mathfrak{T}_r = \{t_\mathrm{i}\}_{\mathrm{i}=1,..,\mathrm{N}_r}$. We set $t_{\mathrm{N}_r+1} = t_1$ and take the index i to increase in accordance with the orientation of S^1. For each i, we use Propositions III.12 and III.13 to provide $\mathfrak{c}_{\theta,[t_\mathrm{i},t_{\mathrm{i}+1}]}$ which we write as $(\mathbb{A}_{\mathrm{i},\mathrm{i}+1}, \psi_{\mathrm{i},\mathrm{i}+1})$. We view the connection $\mathbb{A}_{\mathrm{i},\mathrm{i}+1}$ as defining a connection on the line bundle $det(\mathbb{S})$ over $\mathrm{I}' \times \mathrm{M}$ where $\mathrm{I}' \in \mathrm{S}^1$ is some open neighborhood of $[t_\mathrm{i}, t_{\mathrm{i}+1}]$. We also view the $t \in [t_\mathrm{i}, t_{\mathrm{i}+1}]$ versions of Proposition III.7's connection $\mathbb{A}_{\underline{\mathbb{C}}}$ as a connection on the bundle K^{-1} over $[t_\mathrm{i}, t_{\mathrm{i}+1}] \times \mathrm{M}$. Note in this regard that K^{-1} is the determinant line bundle for the canonical spin$^\mathrm{c}$ structure with spinor bundle $\mathbb{S}_0 = \underline{\mathbb{C}} \oplus \mathrm{K}^{-1}$.

With r large and $\delta > 0$ very small, we define Φ on $[t_\mathrm{i} + \delta, t_{\mathrm{i}+1} - \delta] \times \mathrm{M}$ to be $\frac{i}{2\pi}(\mathrm{F}_{\mathbb{A}_{\mathrm{i},\mathrm{i}+1}} - \mathrm{F}_{\mathbb{A}_{\underline{\mathbb{C}}}})$. This done, we have yet the task of describing Φ on the part of $\mathrm{S}^1 \times \mathrm{M}$ where $t \in [t_\mathrm{i} - \delta, t_\mathrm{i} + \delta]$. We do this as follows: If $\delta > 0$ is sufficiently small, then Proposition III.6 asserts that $\mathfrak{c}_{\theta,[t_\mathrm{i},t_{\mathrm{i}+1}]}$ is defined on the interval $[t_\mathrm{i} - \delta, t_{\mathrm{i}+1} + \delta]$, and likewise $\mathfrak{c}_{\theta,[t_{\mathrm{i}-1},t_\mathrm{i}]}$ is defined on the interval $[t_{\mathrm{i}-1} - \delta, t_\mathrm{i} + \delta]$. This understood, we find a suitable gauge transformations so as to write $\mathbb{A}_{\mathrm{i}-1,\mathrm{i}} = \mathbb{A}_\mathbb{S} + 2\mathrm{a}_{\mathrm{i}-1,\mathrm{i}}$ and $\mathbb{A}_{\mathrm{i},\mathrm{i}+1} = \mathbb{A}_\mathbb{S} + 2\mathrm{a}_{\mathrm{i},\mathrm{i}+1}$ on $[t_\mathrm{i} - \delta, t_\mathrm{i} + \delta] \times \mathrm{M}$. In particular, these gauge transformations should be chosen so that the spectral flow between the respective $(\mathbb{A}_{\mathrm{i}-1,\mathrm{i}}, \psi_{\mathrm{i}-1,\mathrm{i}})$ and $(\mathbb{A}_{\mathrm{i},\mathrm{i}+1}, \psi_{\mathrm{i},\mathrm{i}+1})$ versions of (3.22) is zero. We then interpolate between $\mathrm{a}_{\mathrm{i}-1,\mathrm{i}}$ and $\mathrm{a}_{\mathrm{i},\mathrm{i}+1}$ on $[t_{\mathrm{i}} - \delta, t_{\mathrm{i}} + \delta] \times \mathrm{M}$ using a smooth bump function, v so as to define a connection $\mathbb{A}_i = \mathbb{A}_{\mathbb{S}} + 2(1-\mathrm{v})\mathrm{a}_{\mathrm{i}-1,\mathrm{i}} + 2\mathrm{v}\mathrm{a}_{\mathrm{i},\mathrm{i}+1}$ on $det(\mathbb{S})$ over $[t_{\mathrm{i}} - \delta, t_{\mathrm{i}} + \delta] \times \mathrm{M}$. The "Poincaré dual" of this gluing process and what we hope to get from it is illustrated in Figure 3.2. With this connection in hand, we define Φ to be $\frac{i}{2\pi}(\mathrm{F}_{\mathbb{A}_{\mathrm{i}}} - \mathrm{F}_{\mathbb{A}_{\underline{\mathbb{C}}}})$ on $[t_{\mathrm{i}} - \delta, t_{\mathrm{i}} + \delta] \times \mathrm{M}$.

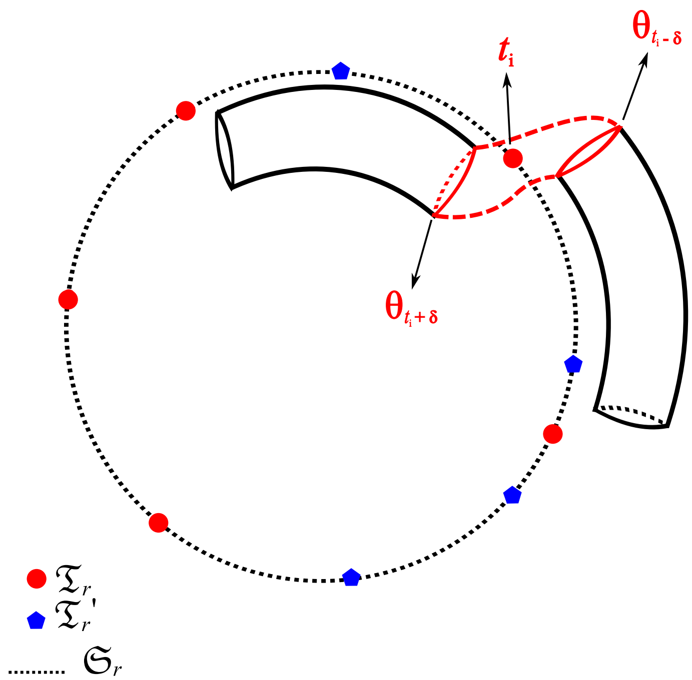

Figure 3.2: Gluing one-parameter family of solutions.

The continuity of the function $t \to \mathfrak{a}^{\mathcal{F}}{}_{\theta}(t)$ is then used to prove the following:

Proposition III.15. *Fix a bound on the* C^3*-norm of* μ*. There exists* $\kappa > 1$ *such that if* $r \geq \kappa$ *and if* $\delta > 0$ *is sufficiently small, then*

- Φ *is twice the first Chern class of a bundle of the form* $\mathrm{E} \otimes \mathrm{L}$ *where* $c_1(\mathrm{L})$ *has zero cup product with* $[\omega]$*.*

- $\int_{\mathrm{S}^1 \times \mathrm{M}} \omega \wedge \Phi > 0.$

What is claimed by Proposition III.15 is not possible given that the first Chern class of E is assumed to have non-positive cup product with the class defined by ω. Thus there can be no counter example to the claim made by Theorem I.2. We prove Proposition III.15 in this section and thus complete the proof of Theorem I.2. The proof that follows has nine parts.

Part 1: Here we say more about the solution of each $t \in S^1$ version of the equations in (3.2) provided by Proposition III.7. We denote this solution as $(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$ and write it at times as $(\mathbb{A}_{\underline{\mathbb{C}}} = \mathbb{A}_{\mathbb{S}_0} + 2A_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}} = (\alpha_{\underline{\mathbb{C}}}, \beta_{\underline{\mathbb{C}}}))$ where $\mathbb{A}_{\mathbb{S}_0}$ is a t-independent connection on the line bundle $K^{-1} = det(\mathbb{S}_0)$ with harmonic curvature form, and where $A_{\underline{\mathbb{C}}}$ is a connection on the trivial bundle $\underline{\mathbb{C}}$. Since each $t \in S^1$ version of these solutions is non-degenerate, the family parametrized by $t \in S^1$ can be changed by t-dependent gauge transformations to define a smooth map from the universal cover, $\mathbb{R}$, of S^1 into $\mathcal{C}$. Moreover, because $\alpha_{\underline{\mathbb{C}}}$ is nowhere zero, a further gauge transformation can be applied if necessary to obtain a 2π-periodic map from $\mathbb{R}$ into $\mathcal{C}$ and thus a map from S^1 into $\mathcal{C}$. This understood, we can view $A_{\underline{\mathbb{C}}}$ as a connection on the trivial bundle over $S^1 \times M$. We write its curvature form as

$$F_{A_{\underline{\mathbb{C}}}} = F_{A_{\underline{\mathbb{C}}}|_t} + dt \wedge \dot{A}_{\underline{\mathbb{C}}}. \tag{3.55}$$

where $F_{A_{\underline{\mathbb{C}}}|_t}$ denotes the component long M_t. Note that the integral of $\frac{i}{2\pi}\omega \wedge dt \wedge \dot{A}_{\underline{\mathbb{C}}}$ over $S^1 \times M$ is zero since $(\mathbb{A}_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$ is a 1-parameter family of solutions of the equations in (3.2). To see this, use an integration by parts, the fact that $d\nu = \dot{\mu}$ and the equation in (3.52) to get

$$\begin{aligned}
\frac{i}{2\pi}\int_{S^1\times M}\omega\wedge dt\wedge\dot{A}_{\underline{\mathbb{C}}} &= \int_{S^1}\Big(\int_M \dot{A}_{\underline{\mathbb{C}}}\wedge\mu\Big)dt \\
&= -\frac{i}{2\pi}\int_{S^1}\Big(\int_M \nu\wedge dA_{\underline{\mathbb{C}}}\Big)dt \\
&= \frac{2\pi}{r}\int_{S^1}\frac{d}{dt}\mathfrak{a}^{\mathcal{F}}(\mathbb{A}_{\underline{\mathbb{C}}},\psi_{\underline{\mathbb{C}}})dt = 0. \qquad (3.56)
\end{aligned}$$

Therefore,

$$\frac{i}{2\pi}\int_{S^1\times M}\omega\wedge F_{A_{\underline{\mathbb{C}}}} = \frac{i}{2\pi}\int_{S^1\times M}\omega\wedge F_{A_{\underline{\mathbb{C}}}|_t}. \qquad (3.57)$$

We also note that the left hand side in (3.57) is equal to zero since $A_{\underline{\mathbb{C}}}$ is a connection on the trivial bundle.

Part 2: Fix $r \geq 1$ large in order to define $\mathfrak{T}_r$ as in Proposition III.11. Let $\mathfrak{T}_r = \{t_i\}_{i=1,..,N-r}$. Given $\delta > 0$ very small we shall use I_i to denote the interval $[t_i - \delta, t_i + \delta]$ and we shall use $J_{i,i+1}$ to denote the interval $[t_i + \delta, t_{i+1} - \delta]$. We write the connection $\mathbb{A}_{i,i+1}$ as $\mathbb{A}_{i,i+1} = \mathbb{A}_{\mathbb{S}_0} + 2A_{i,i+1}$ where $A_{i,i+1}$ is viewed as a connection on the bundle E over $(I_i \cup J_{i,i+1} \cup I_{i+1}) \times M$. The curvature of $A_{i,i+1}$ over $J_{i,i+1} \times M$ is given by

$$F_{A_{i,i+1}} = F_{A_{i,i+1}|_t} + dt\wedge\dot{A}_{i,i+1}. \qquad (3.58)$$

We now write the integral of $\frac{i}{2\pi}\omega\wedge(F_{A_{i,i+1}} - F_{A_{\underline{\mathbb{C}}}|_t})$ over $J_{i,i+1}\times M$ as

$$\frac{i}{2\pi}\int_{J_{i,i+1}\times M} dt\wedge\nu\wedge(F_{A_{i,i+1}|_t} - F_{A_{\underline{\mathbb{C}}}|_t}) + \frac{i}{2\pi}\int_{J_{i,i+1}\times M}\mu\wedge dt\wedge\dot{A}_{i,i+1}. \qquad (3.59)$$

We will first examine the left most integral in (3.59) and then the right most integral. Moreover, in order to consider the left most integral, we fix an integer n to define $J_{i,i+1;n}$ to be the set of $t \in J_{i,i+1}$ where $\mathcal{E}_\theta(t) < 2^n$. We then consider separately the contribution to the left most integral from $(\mathrm{J}_{\mathrm{i,i+1}} \setminus \mathrm{J}_{\mathrm{i,i+1};n}) \times \mathrm{M}$ and from $\mathrm{J}_{\mathrm{i,i+1};n} \times \mathrm{M}$.

Part 3: Little can be said about the contribution from $(\mathrm{J}_{\mathrm{i,i+1}} \setminus \mathrm{J}_{\mathrm{i,i+1};n}) \times \mathrm{M}$ to the left most integral in (3.59) except what is implied by Lemma III.2. In particular, it follows from the latter using (3.15) that if $t \in \mathrm{J}_{\mathrm{i,i+1}} \setminus \mathrm{J}_{\mathrm{i,i+1};n}$, then

$$\frac{i}{2\pi}\int_{\mathrm{M}_t} \nu \wedge (\mathrm{F}_{\mathrm{A}_{\mathrm{i,i+1}}|_t} - \mathrm{F}_{\mathrm{A}_{\mathbb{C}}|_t}) \geq c_0{}^{-1}\mathcal{E}_\theta(t) - c_0 \tag{3.60}$$

where $c_0 > 0$ is independent of n, the index i, t, and also r. Note in particular that (3.60) is positive if $2^n > c_0{}^2$.

As we show momentarily, there is a positive lower bound for the contribution to the left most integral in (3.59) from $\mathrm{J}_{\mathrm{i,i+1};n} \times \mathrm{M}$. To this end, we exhibit constants $c_* > 0$ and $r_n > 1$ with the former independent of n, both independent of r and the index i; and such that

$$\frac{i}{2\pi}\int_{\mathrm{M}_t} \nu \wedge (\mathrm{F}_{\mathrm{A}_{\mathrm{i,i+1}}|_t} - \mathrm{F}_{\mathrm{A}_{\mathbb{C}}|_t}) \geq c_* \tag{3.61}$$

at each fixed $t \in \mathrm{J}_{\mathrm{i,i+1};n}$ when $r \geq r_n$. What follows is an outline of how this is done. We first appeal to Proposition III.6 to find r_n such that if $r > r_n$, then each point of $\alpha_{\mathrm{i,i+1}}{}^{-1}(0)$ has distance $c_0 r^{-1/2}$ or less from a curve of the vector field that generates the kernel of μ. We then split the integral in (3.61) so as to write it as a sum of two integrals, one whose integration domain consists of points with distance $\mathcal{O}(r^{-1/2})$ or less from the loops in M_t, and the other whose integration domain is complementary part in M_t. We show that the contribution to the former is bounded away from zero by some constant $\mathfrak{L} > 0$ which is essentially the length of the shortest closed integral curve of this same vector field. We then show that the contribution from the rest of M_t is much smaller than this when r is large.

Part 4: Fix $t \in \mathrm{J}_{\mathrm{i,i+1};n}$. Given $\epsilon > 0$, Proposition III.6 finds a constant $r_{n,\epsilon}$, and if $r > r_{n,\epsilon}$, a collection Θ_t of pairs (γ, m) with various properties of which the most salient for the present purposes are that γ is a closed integral curve of the vector field that generates the kernel of $\mu|_t$ such that $||\alpha_{\mathrm{i,i+1}}| - |\mu|^{1/2}| < \epsilon$ at points with distance $c_\epsilon r^{-1/2}$ from any loop in Θ_t. Here, $c_\epsilon \geq 1$ depends on ϵ but not on r, t, or the index i. This understood, fix some very small ϵ and let $\mathrm{M}_{t,\epsilon} \subset \mathrm{M}_t$ denote the set of points with distance $2^7 c_\epsilon r^{-1/2}$ or greater from all loops in Θ_t.

To consider the contribution to (3.61) from $\mathrm{M}_t \setminus \mathrm{M}_{t,\epsilon}$, we write the 1-form ν as in (3.15). Then, by Lemma III.2, it follows that

$$\frac{i}{2\pi}\int_{\mathrm{M}_t\setminus\mathrm{M}_{t,\epsilon}} |\upsilon \wedge (\mathrm{F}_{\mathrm{A}_{\mathrm{i,i+1}}|_t} - \mathrm{F}_{\mathrm{A}_{\underline{\mathbb{C}}}|_t})| \leq c_\epsilon r^{-1/2}\mathfrak{L}_t, \tag{3.62}$$

where $\mathfrak{L}_t = \Sigma_{(\gamma,\mathrm{m})}\mathrm{m} \cdot \mathrm{length}(\gamma)$.

To see about the rest of the $\mathrm{M}_t \setminus \mathrm{M}_{t,\epsilon}$ contribution, note that Lemma 6.1 in [24] has a verbatim analogue in the present context. In particular, the latter implies that

$$\frac{i}{2\pi} * (*\mu \wedge \mathrm{F}_{\mathrm{A}_{\mathrm{i,i+1}}|_t}) \geq \frac{1}{8\pi} r|\mu|(|\mu| - |\alpha_{\mathrm{i,i+1}}|^2) \tag{3.63}$$

at all points in $\mathrm{M}_t\setminus\mathrm{M}_{t,\epsilon}$ if r is large. It follows from this, the third item in Proposition III.6 and (3.62) that

$$\frac{i}{2\pi}\int_{\mathrm{M}_t\setminus\mathrm{M}_{t,\epsilon}} \nu \wedge (\mathrm{F}_{\mathrm{A}_{\mathrm{i,i+1}}|_t} - \mathrm{F}_{\mathrm{A}_{\underline{\mathbb{C}}}|_t}) \geq c_0\mathfrak{L}_t, \tag{3.64}$$

when r is larger than some constant that depends only on ϵ and n. Here, $c_0 > 0$ is independent of r, t, n, ϵ and the index i.

Part 5: Turn now to the contribution to (3.61) from $\mathrm{M}_{t,\epsilon}$. By Lemma III.3, no generality is lost by taking $r_{n,\epsilon}$ so that

$$||\mu|^{1/2} - |\alpha_{i,i+1}|| < \epsilon \text{ and } |\nabla_{A_{i,i+1}}{}^k \alpha_{i,i+1}| \leq \epsilon r^{k/2} \text{ for } k = 1, 2;$$

$$|\nabla_{A_{i,i+1}}{}^k \beta_{i,i+1}| \leq \epsilon r^{(k-1)/2} \text{ for } k = 0, 1, 2 \tag{3.65}$$

at all points in M_t with distance $c_\epsilon r^{-1/2}$ or more from any loop in Θ_t. Let M′ denote the latter set. Note in this regard that $M_{t,\epsilon}$ is the set of points with distance $2^7 c_\epsilon r^{-1/2}$ or more from any loop in Θ_t, so $M_{t,\epsilon} \subset M'$. Meanwhile, we can also assume that (3.65) holds at all points in M_t when $(A_{i,i+1}, (\alpha_{i,i+1,\beta_{i,i+1}}))$ is replaced by $(A_{\underline{\mathbb{C}}}, (\alpha_{\underline{\mathbb{C}}}, \beta_{\underline{\mathbb{C}}}))$. Granted these last observations, we change the gauge for $(A_{i,i+1}, \psi_{i,i+1})$ on M′ so that $\alpha_{i,i+1} = h\alpha_{\underline{\mathbb{C}}}$ where h is a real and positive valued function. Having done so, we write $A_{i,i+1}$ on M′ as $A_{i,i+1} = A_{\underline{\mathbb{C}}} + (2r)^{1/2} b$ with b a smooth imaginary valued 1-form. This understood, then the contribution to (3.61) from $M_{t,\epsilon}$ is no greater than

$$c_1 \int_{M_{t,\epsilon}} |db| \tag{3.66}$$

where c_1 depends only on ω. Our task now is to show that (3.66) is small if r is sufficiently large.

To start this task, we note that with our choice of gauge, it follows from (3.65) and its $(A_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})$ analogue that

$$|\alpha_{i,i+1} - \alpha_{\underline{\mathbb{C}}}| + |b| \leq c_0 \epsilon \tag{3.67}$$

on M′. Here, c_0 is independent of ϵ and r.

Introduce $M'' \subset M'$ to denote the set of points with distance $2^6 c_\epsilon r^{-1/2}$ or more from any loop in Θ_t. We now see how to find a function $x : M \to \mathbb{R}$ with the following properties: First, $\mathfrak{b} = (b - i(2r)^{-1/2} dx, e^{ix}\psi - \psi_{\underline{\mathbb{C}}}, 0)$ obeys the equation

$$\mathcal{L}_{(A_{\underline{\mathbb{C}}}, \psi_{\underline{\mathbb{C}}})} \mathfrak{b} + r^{1/2} \mathfrak{b} * \mathfrak{b} = 0 \tag{3.68}$$

on M″. Second, $|\mathfrak{b}| \le \mathrm{z}\epsilon$ where $\mathrm{z} > 0$ is independent of r and ϵ.

To explain our final destination, fix a smooth, non-increasing function $\chi : [0,\infty) \to [0,1]$ with value 0 on $[0, \frac{3}{4}]$ and with value 1 on $[1,\infty)$. Set $\chi_\epsilon{}'$ to denote the function on M given by

$$(3.69) \qquad \chi_\epsilon{}' = \chi(dist(\cdot, \cup_{(\gamma,\mathrm{m})\in\Theta_t}\gamma)/2^7 c_\epsilon r^{-1/2}).$$

Let $\mathfrak{b}' = \chi_\epsilon{}'\mathfrak{b}$. This function has compact support in M″ and it obeys the equation

$$(3.70) \qquad \mathcal{L}_{(\mathrm{A}_{\underline{\mathbb{C}}},\psi_{\underline{\mathbb{C}}})}\mathfrak{b}' + r^{1/2}\mathfrak{b} * \mathfrak{b}' = \mathfrak{h},$$

where $|\mathfrak{h}| \le c_0\mathrm{z}|\mathrm{d}\chi_\epsilon{}'|\epsilon$ where c_0 is independent of r, t, ϵ and the index i. Note in particular that the L^2-norm of $\mathfrak{h}$ is bounded by $c_1\mathrm{z}\mathfrak{L}_t\epsilon$ where c_1 is also independent of the same parameters. This understood, it follows from (3.43) that

$$(3.71) \qquad ||\mathfrak{b}'||_{\mathbb{H}} \le c_2\mathrm{z}\epsilon r^{1/2}||\mathfrak{b}'||_2 + c_1\mathrm{z}\epsilon\mathfrak{L}_t.$$

Equation (3.71) gives the bound $||\mathfrak{b}'||_{\mathbb{H}} \le 2c_1\mathrm{z}\epsilon\mathfrak{L}_t$ when $\epsilon < \frac{1}{4}(c_2\mathrm{z})^{-1}$. As a final consequence, (3.66) is seen to be no greater than $c_3\mathrm{z}\epsilon\mathfrak{L}_t$ with c_3 again independent of r, t, ϵ and the index i.

To find the desired function x, introduce again the function χ, and define $\chi_\epsilon : \mathrm{M} \to [0,1]$ by replacing $2^7 c_\epsilon r^{-1/2}$ in (3.69) by $2^6 c_\epsilon r^{-1/2}$. Equation (3.70) is then satisfied on M″ if x obeys the equation

$$(3.72) \qquad \mathrm{d}^*\mathrm{dx} + 2|\mu|^{1/2} r|\alpha_{\mathrm{i,i+1}}|\sin\mathrm{x} = \chi_\epsilon \mathrm{d}^*\mathrm{b}.$$

This equation has the same form as that in (3.28). In particular, the arguments in [25] that find a solution of the equation (2.16) in [25] can be applied only with minor modifications to find a solution, x, of the equation in (3.72) that obeys the bounds in (3.40). This being the case, the resulting $\mathfrak{b} = (\mathrm{b} - i(2r)^{-1/2}\mathrm{dx}, e^{i\mathrm{x}}\psi - \psi_{\underline{\mathbb{C}}}, 0)$ is such that $|\mathfrak{b}| \leq z\epsilon$.

Part 6: It follows from what is said in Parts 4 and 5 that there exists $c_* > 0$ and $r_n \geq 1$ such that if $r \geq r_n$, then (3.61) holds. Moreover, c_* is independent of n because it is larger than some fixed fraction of the shortest closed integral curve of any given $t \in S^1$ version of the kernel of μ. With (3.60), this implies that the left most integral in (3.59) obeys

$$\frac{i}{2\pi}\int_{J_{i,i+1}\times M} dt \wedge \nu \wedge (F_{A_{i,i+1}|_t} - F_{A_{\underline{\mathbb{C}}}|_t}) \geq c_{**}\text{length}(J_{i,i+1}), \tag{3.73}$$

where c_{**} is also independent of n and r which are both very large.

To say something about the right most integral in (3.59), we write $A_{i,i+1} = A_E + a_{i,i+1}$ where A_E is the t-independent connection on E with harmonic curvature form chosen so that $\mathbb{A}_{\mathbb{S}} = \mathbb{A}_{\mathbb{S}_0} + 2A_E$. We then use the fact that the equations in (3.2) are the variational equations of the functional $\mathfrak{a}$ as in (3.3) to write

$$\frac{i}{2\pi}\int_M \mu \wedge \dot{a}_{i,i+1} = -\frac{1}{4\pi r}\int_M a_{i,i+1} \wedge da_{i,i+1}. \tag{3.74}$$

Here, we use the fact that $\mathcal{D}_{A_{i,i+1}}\psi_{i,i+1} = 0$ to dispense with the derivative of the right most integral in (3.3) with respect to t. Granted (3.74), we identify the right most integral in (3.59) with

$$\frac{1}{4\pi r}[-\int_M (a_{i,i+1} \wedge (da_{i,i+1} - i\varpi_{\mathbb{S}}))|_{t_{i+1}-\delta} + \int_M (a_{i,i+1} \wedge (da_{i,i+1} - i\varpi_{\mathbb{S}}))|_{t_i+\delta}]. \tag{3.75}$$

Equations (3.73) and (3.75) summarize what we say for now about (3.59).

Part 7: Recall that $\mathrm{I_i} = [t_\mathrm{i} - \delta, t_\mathrm{i} + \delta]$. We now review how we define the connection $\mathrm{A_i}$ on E over $\mathrm{I_i} \times \mathrm{M}$. This is done using a 'bump' function, $\mathrm{v} : \mathrm{I_i} \to [0,1]$. This function is non-decreasing, it is equal to 0 near $t_\mathrm{i} - \delta$ and equal to 1 near $t_\mathrm{i} + \delta$. Meanwhile, we chose gauges for $\mathrm{A_{i-1,i}}$ and $\mathrm{A_{i,i+1}}$ so that there is no spectral flow between the respective $(\mathbb{A}_{\mathrm{i-1,i}}, \psi_{\mathrm{i-1,i}})$ and $(\mathbb{A}_{\mathrm{i,i+1}}, \psi_{\mathrm{i,i+1}})$ versions of (3.22). Having done so, we write $\mathrm{A_{i-1,i}} = \mathrm{A_E} + \mathrm{a_{i-1,i}}$ and $\mathrm{A_{i,i+1}} = \mathrm{A_E} + \mathrm{a_{i,i+1}}$. We then defined $\mathbb{A}_\mathrm{i} = \mathbb{A}_\mathbb{S} + 2(1-\mathrm{v})\mathrm{a_{i-1,i}} + 2\mathrm{v}\mathrm{a_{i,i+1}}$ and we used the latter to define Φ on $\mathrm{I_i} \times \mathrm{M}$ by $\frac{i}{2\pi}(\mathrm{F}_{\mathbb{A}_\mathrm{i}} - \mathrm{F}_{\mathbb{A}_{\underline{\mathbb{C}}}})$.

In order to say something about

$$\int_{\mathrm{I_i}\times\mathrm{M}} \omega \wedge \frac{i}{2\pi}(\mathrm{F}_{\mathbb{A}_\mathrm{i}} - \mathrm{F}_{\mathbb{A}_{\underline{\mathbb{C}}}}) \tag{3.76}$$

we write $\mathrm{F}_{\mathrm{A_i}} - \mathrm{F}_{\mathrm{A}_{\underline{\mathbb{C}}}|_t}$ as

$$\begin{aligned} &\mathrm{v}\,(\mathrm{F}_{\mathrm{A_{i,i+1}}|_t} - \mathrm{F}_{\mathrm{A}_{\underline{\mathbb{C}}}|_t}) + (1-\mathrm{v})(\mathrm{F}_{\mathrm{A_{i-1,i}}|_t} - \mathrm{F}_{\mathrm{A}_{\underline{\mathbb{C}}}|_t}) \\ &+\mathrm{dt} \wedge \frac{\partial}{\partial t}(\mathrm{v}\mathrm{a_{i,i+1}}) + \mathrm{dt} \wedge \frac{\partial}{\partial t}((1-\mathrm{v})\mathrm{a_{i-1,i}}). \end{aligned} \tag{3.77}$$

As we saw in Parts 4 and 5 above, the two left most terms in (3.77) give positive contribution to the integral in (3.76). The contribution of the two right most terms are

$$\frac{i}{2\pi}\int_{\mathrm{I_i}\times\mathrm{M}} (\mathrm{dt} \wedge \mu \wedge \frac{\partial}{\partial t}(\mathrm{v}\mathrm{a_{i,i+1}})) + \frac{i}{2\pi}\int_{\mathrm{I_i}\times\mathrm{M}} (\mathrm{dt} \wedge \mu \wedge \frac{\partial}{\partial t}((1-\mathrm{v})\mathrm{a_{i-1,i}})). \tag{3.78}$$

We analyze (3.78) using an integration by parts to write it as the sum of

$$-\frac{i}{2\pi}\int_{\mathrm{I_i}\times\mathrm{M}} (\mathrm{dt} \wedge \mathrm{d}\nu \wedge \mathrm{v}\mathrm{a_{i,i+1}} + (1-\mathrm{v})\mathrm{a_{i-1,i}}), \tag{3.79}$$

and

$$\frac{i}{2\pi}\int_{\mathrm{M}} (\mu \wedge \mathrm{a_{i,i+1}})|_{t_\mathrm{i}+\delta} - \frac{i}{2\pi}\int_{\mathrm{M}} (\mu \wedge \mathrm{a_{i-1,i}})|_{t_\mathrm{i}-\delta}. \tag{3.80}$$

Our only remark about the term in (3.79) is that it is bounded below by $-\mathcal{K}\delta$, where $\mathcal{K}$ is a constant that is independent of δ. This is all we need to know. Meanwhile, we use (3.3) to write (3.80) as the sum of the two terms:

$$-\frac{1}{2\pi r}(\mathfrak{a}(\mathfrak{c}_{\theta,[t_{\mathrm{i}},t_{\mathrm{i}+1}]})|_{t_{\mathrm{i}}+\delta} - \mathfrak{a}(\mathfrak{c}_{\theta,[t_{\mathrm{i}-1},t_{\mathrm{i}}]})|_{t_{\mathrm{i}}-\delta}) \tag{3.81}$$

and

$$\frac{1}{4\pi r}[\int_{\mathrm{M}}(\mathrm{a}_{\mathrm{i}-1,\mathrm{i}} \wedge (d\mathrm{a}_{\mathrm{i}-1,\mathrm{i}} - i\varpi_{\mathbb{S}}))|_{t_{\mathrm{i}}-\delta} - \int_{\mathrm{M}}(\mathrm{a}_{\mathrm{i},\mathrm{i}+1} \wedge (d\mathrm{a}_{\mathrm{i},\mathrm{i}+1} - i\varpi_{\mathbb{S}}))|_{t_{\mathrm{i}}+\delta}]. \tag{3.82}$$

To say something about (3.81), recall that we choose gauges when defining $\mathrm{a}_{\mathrm{i}-1,\mathrm{i}}$ and $\mathrm{a}_{\mathrm{i},\mathrm{i}+1}$ on $\mathrm{I}_{\mathrm{i}} \times \mathrm{M}$ so that the spectral flow $\mathcal{F}$ take the same value on $(\mathbb{A}_{\mathrm{i}-1,\mathrm{i}}, \psi_{\mathrm{i}-1,\mathrm{i}})$ and $(\mathbb{A}_{\mathrm{i},\mathrm{i}+1}, \psi_{\mathrm{i},\mathrm{i}+1})$. As a consequence,

$$-\frac{1}{2\pi r}(\mathfrak{a}(\mathfrak{c}_{\theta,[t_{\mathrm{i}},t_{\mathrm{i}+1}]})|_{t_{\mathrm{i}}+\delta} - \mathfrak{a}(\mathfrak{c}_{\theta,[t_{\mathrm{i}-1},t_{\mathrm{i}}]})|_{t_{\mathrm{i}}-\delta}) = -\frac{1}{2\pi r}(\mathfrak{a}^{\mathcal{F}}{}_{\theta}(t_{\mathrm{i}+\delta}) - \mathfrak{a}^{\mathcal{F}}{}_{\theta}(t_{\mathrm{i}-\delta})). \tag{3.83}$$

Because the function $\mathfrak{a}^{\mathcal{F}}{}_{\theta}$ is continuous and piecewise differentiable, what appears on the right hand side of (3.83) is bounded below by $-\mathcal{K}\delta$, with $\mathcal{K}$ again a constant that is independent of δ.

We comment on (3.82) in Part 8.

Part 8: The terms in (3.82) are fully gauge invariant. This understood, we observe that the term with integral of $\mathrm{a}_{\mathrm{i},\mathrm{i}+1} \wedge d\mathrm{a}_{\mathrm{i},\mathrm{i}+1}$ is identical but for its sign to the right most term in (3.75). As the signs are, in fact, opposite, these two terms cancel. Meanwhile, the term with $\mathrm{a}_{\mathrm{i}-1,\mathrm{i}} \wedge d\mathrm{a}_{\mathrm{i}-1,\mathrm{i}}$ is identical but for the opposite sign, to the left most term in the version of (3.75) over the interval $\mathrm{J}_{\mathrm{i}-1,\mathrm{i};\delta}$. Thus, it cancels the latter term. This understood, the sum of the various $\{\mathrm{J}_{\mathrm{i},\mathrm{i}+1}\}_{\mathrm{i}=1,..,\mathrm{N}_r}$ version of (3.75) is exactly minus the sum of the various $\{\mathrm{I}_{\mathrm{i}}\}_{\mathrm{i}=1,..,\mathrm{N}_r}$ versions of (3.82). Thus, they cancel when we sum up the various contributions to $\int_{S^1\times M} \omega\wedge\Phi$. This we now do. In particular, we find from (3.71) and from what is said above and in Part 7 that

$$\int_{S^1\times M} \omega\wedge\Phi \geq 4\pi c_{**} - N_r\mathcal{K}\delta \tag{3.84}$$

where $\mathcal{K}$ is a constant that is independent of δ. Thus, if we take $\delta > 0$ sufficiently small, we see that

$$\int_{S^1\times M} \omega\wedge\Phi > 0. \tag{3.85}$$

Part 9: With (3.85) understood, our proof of Proposition III.15 is complete with a suitable idenfication of the class defined by Φ in $H^2(M;\mathbb{Z})$. To this end, remark that it follows from our definition of each $\mathbb{A}_{i,i+1}$ and each $\mathbb{A}_i$, that Φ can be written as $\frac{i}{2\pi}(F_{\mathbb{A}}-F_{\mathbb{A}_{\underline{\mathbb{C}}}})$ where $\mathbb{A}$ can be written as $\mathbb{A}_{\mathbb{S}_0}+2A$ where A is a connection on a line bundle E' over $S^1\times M$ whose first Chern class restricts to each M_t as that of E. Indeed, $\mathbb{A}$ is defined first on each of $\{J_{i,i+1}\times M\}_{i=1,..,N_r}$ as $\{\mathbb{A}_{i,i+1}=\mathbb{A}_{\mathbb{S}_0}+2A_{i,i+1}\}_{i=1,..,N_r}$, and then on each of $\{I_i\times M\}_{i=1,..,N_r}$ as $\{\mathbb{A}_i=\mathbb{A}_{\mathbb{S}_0}+2A_E+2(1-v)a_{i-1,i}+2va_{i,i+1}\}_{i=1,..,N_r}$. These various connections were then glued on the overlaps using maps from M to S^1.

We write E' as $E\otimes L$. Let $0\in S^1$ denote any chosen point. Given what was just said, L over $[0,2\pi)\times M$ is isomorphic to the trivial bundle. As such, it is obtained from the trivial bundle over $[0,2\pi]\times M$ by identifying the fiber over $\{2\pi\}\times M$ with that over $\{0\}\times M$ using a map $u: M\to U(1)$. To say more about L, we define for each $t\in S^1$, a section $\psi|_t$ of $\mathbb{S}$ as follows: For any given index $i\in\{1,..,N_r\}$, define $\psi|_t=\psi_{i,i+1}$ on $J_{i,i+1}\times M$. We then define ψ at $t\in I_i$ to be $v\psi_{i,i+1}+(1-v)\psi_{i-1,i}$ using the same gauge choices that are used above to define $\mathbb{A}_i$. This done, the pair $(\mathbb{A}=\mathbb{A}_{\mathbb{S}_0}+2A,\psi)$ defines a pair of connection over $S^1\times M$ for the line bundle $det(\mathbb{S})\otimes L^2$ and section of the spinor bundle $\mathbb{S}\otimes L$. We now trivialize L over $[0,2\pi)\times M$ so as to view the restrictions to any given M_t of $(\mathbb{A},\psi)$ as defining a smooth map from $[0, 2\pi)$ into $\mathcal{C}$. There is then the corresponding 1-parameter family of operators whose $t \in [0, 2\pi)$ member is the $(\mathbb{A}, \psi)|_t$ version of (3.22). This family has zero spectral flow. Indeed, this is the case because $\mathbb{A}$ was defined over $\mathrm{I_i}$ by interpolating between $\mathbb{A}_{\mathrm{i-1,i}}$ and $\mathbb{A}_{\mathrm{i,i+1}}$ in gauges where there is zero spectral flow between the respective $(\mathrm{A_{i-1,i}}, \psi_{\mathrm{i-1,i}})$ and $(\mathrm{A_{i-1,i}}, \psi_{\mathrm{i-1,i}})$ versions of (3.22).

Because $(\mathbb{A}, \psi)|_{2\pi} = (\mathbb{A}|_0 - 2u^{-1}\mathrm{d}u, u\psi|_0)$ and there is no spectral flow between the respective $(\mathbb{A}, \psi)|_0$ and $(\mathbb{A}, \psi)|_{2\pi}$ versions of (3.22), it follows from [1] that the cup product of $c_1(\mathrm{L})$ with $c_1(det(\mathbb{S}))$ is zero. Keeping this last point in mind, and given that L restricts as the trivial bundle to each M_t, we use the Künneth formula to see that the cup product of $c_1(\mathrm{L})$ with the class defined by ω is the same as that between $c_1(\mathrm{L})$ and the class defined by $\mu|_0$. By assumption, the latter class is proportional to $c_1(det(\mathbb{S}))$ in $\mathrm{H}^2(\mathrm{M}; \mathbb{R})$. Therefore, $c_1(\mathrm{L})$ has zero cup product with $[\omega]$. $\square$

We end this chapter with the proof of Theorem I.4, which is a special case of Conjecture I.1.

Proof of Theorem I.4. Note that if $-c_1(\mathrm{K})$ is not torsion in $\mathrm{H}^2(\mathrm{M}; \mathbb{Z})$, then $c_1(\mathrm{K}) = \lambda[\mu]$ in $\mathrm{H}^2(\mathrm{M}; \mathbb{R})$ with $\lambda > 0$. This is because the cup product pairing between $c_1(\mathrm{K})$ and $[\omega]$ has the same sign as λ. If $\lambda < 0$, then it follows from [12] or [17] that $\mathrm{M} = \mathrm{S}^1 \times \mathrm{S}^2$.

Now, let $\mathbf{S}$ denote the generator of $\mathrm{H}_2(\mathrm{M}; \mathbb{Z})$ with the property that $\langle c_1(\mathrm{K}), \mathbf{S} \rangle > 0$. Note that such a class exists by virtue of the fact noted above that $c_1(\mathrm{K}) = \lambda[\mu]$ in $\mathrm{H}^2(\mathrm{M}; \mathbb{R})$ with $\lambda > 0$. Let Σ denote a closed, connected, oriented and genus-minimizing representative for the class $\mathbf{S}$. Note that $||\mathbf{S}||_{\mathrm{T}} = 2genus(\Sigma) - 2$, which one can easily show using the fact that $b_1(\mathrm{M}) = 1$. Then it is a consequence of Proposition II.4 that $2genus(\Sigma) - 2 \geq \langle c_1(\mathrm{K}), \mathbf{S} \rangle$. This is to say that $c_1(\mathrm{K})$ lies in the unit ball of the dual Thurston norm on $\mathbf{H}$. In fact, $c_1(\mathrm{K})$ is an extremal point in this ball, which is to say that $\langle c_1(\mathrm{K}), \mathbf{S}\rangle = 2genus(\Sigma) - 2$. Here is why: Theorem I.2 in the present context says that

$$\begin{aligned} \bigoplus_{e \in \mathrm{H}^2(\mathrm{M};\mathbb{Z})\,:\,\langle e, \mathbf{S}\rangle < 0} \widetilde{HM}(\mathrm{M}, \mathfrak{s}_e) &\cong \{0\}, \\ \bigoplus_{e \in \mathrm{H}^2(\mathrm{M};\mathbb{Z})\,:\,\langle e, \mathbf{S}\rangle = 0} \widetilde{HM}(\mathrm{M}, \mathfrak{s}_e) &\cong \mathbb{Z}. \end{aligned}$$

Meanwhile, Proposition 25.5.5 in [9] asserts isomorphisms between the Seiberg–Witten Floer homology groups for the spin^c structure $\mathfrak{s}_e$ and those for the spin^c structure $\mathfrak{s}_{c_1(\mathrm{K})-e}$. Thus, Theorem I.2 also finds that

$$\begin{aligned} \bigoplus_{e \in \mathrm{H}^2(\mathrm{M};\mathbb{Z})\,:\,\langle e, \mathbf{S}\rangle > \langle c_1(\mathrm{K}), \mathbf{S}\rangle} \widetilde{HM}(\mathrm{M}, \mathfrak{s}_e) &\cong \{0\}, \\ \bigoplus_{e \in \mathrm{H}^2(\mathrm{M};\mathbb{Z})\,:\,\langle e, \mathbf{S}\rangle = \langle c_1(\mathrm{K}), \mathbf{S}\rangle} \widetilde{HM}(\mathrm{M}, \mathfrak{s}_e) &\cong \mathbb{Z}. \end{aligned} \tag{3.86}$$

These last results together with Theorem II.9 imply that $c_1(\mathrm{K})$ is an extremal point of the unit ball as defined by the dual of the Thurston norm, that is to say $\langle c_1(\mathrm{K}), \mathbf{S}\rangle = 2genus(\Sigma) - 2$. The unit ball of the dual Thurston norm is shown in Figure 3.3.

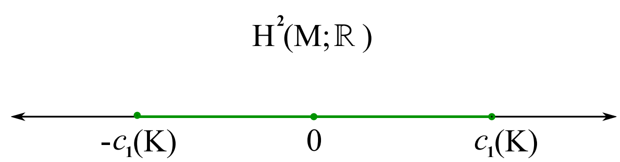

Figure 3.3: The unit ball of the dual Thurston norm.

Finally, given (3.86), the assertion made by Theorem I.4 follows directly from Theorem I.3. □

CHAPTER IV

Conclusion and Remarks

As we mentioned in the Introduction, Friedl and Vidussi have recently announced a complete proof of Conjecture I.1. Given a closed, oriented, irreducible 3-manifold, M, Firedl and Vidussi find in [6] conditions on the *twisted Alexander polynomials* of M that are necessary for $\mathrm{S}^1 \times \mathrm{M}$ to admit a symplectic form. In [7], using Stallings' criterion (see [20]), they show that these conditions are sufficient to deduce that M fibers over the circle. In this dissertation, we present an alternative way of proving Conjecture I.1 using Seiberg–Witten Floer homology. However, the monotonicity condition imposed in the statement of Theorem I.2 restricts *a priori* our ability to extend the statement of Theorem I.4 to manifolds with first Betti number 2 or more. Yet, if a closed, oriented 3-manifold M fibers over the circle and $f : \mathrm{M} \to \mathrm{S}^1$ is a smooth fiber bundle map, then for any class $\Xi \in \mathrm{H}^2(\mathrm{M}; \mathbb{R})$ with $[df] \cdot \Xi > 0$ there exists a symplectic form ω on $\mathrm{S}^1 \times \mathrm{M}$ such that $[\omega] = [\mathrm{dt}] \cup [df] + \Xi$ (see [7]). Moreover, the anticanonical class for the symplectic form ω_f satisfies the monotonicity condition. This motivates the following question.

Question IV.1. *Is it possible to prove that when* $\mathrm{S}^1 \times \mathrm{M}$ *admits a symplectic form with non-torsion anticanonical class, it also admits a symplectic form whose anticanonical class satisfies the monotonicity condition?*

At the moment, the author does not know how to answer this question. In fact, it seems that answering this question is as hard as proving Conjecture I.1. Still, if the answer to Question IV.1 were affirmative, then we would be able to use Theorem I.2 to extend our proof of Conjecture I.1 to manifolds with first Betti number 2 or more as follows: Let $\omega = dt \wedge \nu + \mu$ be a symplectic form on $S^1 \times M$ with non-torsion anticanonical class which satisfies the monotonicity condition. Decompose $[\omega]$ into its Künneth components as $[\omega] = [dt] \cup [\bar{\nu}] + [\mu]$. Since the set of cohomology classes represented by a symplectic form constitute an open cone in $H^2(S^1 \times M; \mathbb{R})$, we could wiggle $[\bar{\nu}]$ to make sure that $e \cdot [\omega] = 0$ for any $e \in H^2(M;\mathbb{Z})$ for which $\widetilde{HM}(M; \mathfrak{s}_e)$ is non-trivial. We could also guarantee that the resulting class $[\bar{\nu}]$ lies in $H^1(M;\mathbb{Q})$. Note that none of these changes to the symplectic form affect the canonical spin^c structure. Therefore, we could proceed as in the proof of Theorem I.4 in order to complete our proof.

Even if we were able to answer Question IV.1, the case when the symplectic form on $S^1 \times M$ has torsion anticanonical class still needs special treatment. In this case, we could appeal to the fact that torsion anticanonical class implies the vanishing of the Thurston norm. Then, Friedl and Vidussi prove in a rather short way that M fibers over the circle with torus fibers. Alternatively, we could try to prove an analogue of Theorem I.2 for Seiberg–Witten Floer homology with *twisted coefficients* where the twisting is defined using the class $[\mu]$ (see [9]). In this case, we would also need an analogue of Theorem I.3 for torus bundles over the circle in order to complete the proof of Conjecture I.1. Such an analogue of Theorem I.3 has already been proven by Kronheimer and Mrowka in [9].

It was suggested to the author by Kronheimer that one could extend our proof of Conjecture I.1 to closed, oriented 3-manifolds with first Betti number 2 or more in the following way: Suppose ω is a symplectic form on $S^1 \times M$ that represents a cohomology class in $H^2(S^1 \times M; \mathbb{Z})$. Once again, write $[\omega] = [dt] \cup [\bar{\nu}] + [\mu]$, and consider a closed, connected, oriented and genus-minimizing surface $\Sigma \subset M$ such that $[\Sigma] \in H_2(M; \mathbb{Z})$ is primitive and a positive integer multiple of $[\Sigma]$ is the Poincaré dual of the class $[\bar{\nu}]$. Then, cut M open along Σ and reglue with a diffeomorphism of Σ so that the first Betti number of the resulting manifold, M', is equal to 1. This is possible because the manifold that we obtain by cutting M open along Σ is a homology product. Now, the following question remains to be answered.

Question IV.2. *Does* $S^1 \times M'$ *admit a symplectic form?*

If one can give an affirmative answer to this question, then Theorem I.4 implies that both M' and M fiber over the circle with Σ as a fiber. The author hopes to answer this question in the near future.

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