Triangulation of locally semi-algebraic spaces

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2009

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ACKNOWLEDGEMENTS

I would like to thank Mircea Mustaţă and Bill Fulton for their tremendous assistance in the making of this thesis. I would also like to thank Andreas Blass for a helpful conversation about topoi and Clint McCrory for information on the history of triangulation.

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ABSTRACT

We give necessary and sufficient conditions for a locally semi-algebraic space to be homeomorphic to a simplicial complex. Our proof does not require the space to be embedded anywhere, and it requires neither compactness nor projectivity of the space. A corollary is that every real or complex algebraic variety is triangulable, a result which does not seem to be available in the literature when the variety is neither projective nor real and compact.

Chapter 1

Introduction

We will work exclusively over the real numbers. The goal of the present thesis is to prove the following theorem.

THEOREM. *Let X be a locally semi-algebraic space with its strong topology.*

(1) X is homeomorphic to a simplicial complex if and only if X is Hausdorff, paracompact, and locally compact.

(2) When X is homeomorphic to a simplicial complex, X admits a semi-algebraic triangulation which is unique up to semi-algebraic subdivision.

The new feature of this theorem is that X is not assumed to be embedded anywhere. The theorem is known when X is a compact semi-algebraic set [**27**]. If "simplicial complex" is replaced by "union of open simplices," then an analogous (non-embedded) theorem is known [**10**].

The theorem is proved in 3.30, 4.5, and 4.6. A corollary of this is that every real or complex algebraic variety and every locally compact constructible subset of a real or complex algebraic variety is triangulable. As far as we are aware, there is no proof in the literature of this fact when the variety is not either projective or real and compact. In the rest of the introduction, we will explain some of the prior work on triangulation and the definitions of the terms above, and we will give a sketch of the proofs of the results above.

The original motivation for triangulation came from Poincaré's introduction of homology theory in [**35**]. Poincaré's goal was to study period integrals on manifolds, and in his original work on homology, he depended on being able to move cycles into general position so that he could intersect them. However, his proofs had gaps. (Eventually, Hardt [**18**] proved that there is a homology theory for real analytic cycles that satisfies the Eilenberg-Steenrod axioms.) After these were pointed out to

Poincaré, he introduced in [**36**] the idea of *triangulation*. A triangulation is a homeomorphism from a simplicial complex to a topological space. If a space is triangulable, that is, if it admits a triangulation, then its homology is the simplicial homology of the simplicial complex, and for simplicial homology, Poincaré could give correct proofs of his results.

That left two problems. The first problem was to show the triangulability of a large class of interesting spaces. This was necessary to show that the simplicial techniques applied to interesting spaces. The second problem was to show that the triangulation was essentially unique. This would make it possible to define invariants of spaces in terms of invariants of triangulations.

In [**36**, §XI], Poincaré attempted to triangulate subsets of Euclidean space cut out by analytic equalities and inequalities. These are now called *semi-analytic sets*. Suppose that X is such a set. Poincaré's method was to choose a hyperplane, project X onto the hyperplane, triangulate the image Y by induction, and lift the triangulation back to X. There are several difficulties with this approach. First, it is not clear that the projection of a semi-analytic set is a semi-analytic set. Second, the projection needs to be chosen well. If the projection map has a positive dimensional fiber, then the triangulation of Y does not contain enough information to triangulate the positive dimensional fiber. Third, the triangulation of Y also needs to be chosen well. This is because the lifting is done by stratifying X so that each stratum projects isomorphically to Y. A simplex of Y that is contained in the image of a stratum can be lifted to that stratum. To ensure that this happens for every simplex and every stratum, the triangulation of Y needs to be aware of the structure of X.

Poincaré did not give all the details necessary to turn his ideas into a proof. Despite this, van der Waerden claimed in a 1930 paper on Schubert calculus [**47**] that Poincaré's procedure worked for compact subsets of Euclidean space cut out by polynomial equalities and inequalities. These are now called compact *semi-algebraic sets*. van der Waerden presented the procedure as an appendix barely longer than a page, and he asserted mostly without proof that the difficulties mentioned above can be solved. In order to show that this method applies to complex projective varieties, he noted that complex projective n-space can be embedded in real Euclidean $2n^2$-space as the set of all trace 1 complex Hermitian matrices:

$$[z_0 \colon \cdots : z_n] \mapsto \left(\frac{z_i \bar{z}_j}{\sum_k \|z_k\|^2} \right)_{1 \le i,j \le n}. \tag{0.1}$$

This embedding is originally due to Mannoury [**29**], whose goal was to attach geometric meaning to points with imaginary coordinates and points at infinity. It is real algebraic and its image is compact. Therefore it embeds complex projective varieties as compact real algebraic subsets of Euclidean space, where Poincaré's triangulation procedure applies.

The first rigorous proofs of triangulation were given for C^1-manifolds in the 1930s and 40s by Cairns [**7**] and Whitehead [**49**], who both used an approximation procedure. This left the triangulation of more singular spaces open.

An attempt to make Poincaré's methods rigorous was made in 1930 by Lefschetz [**25**] and in 1933 by Lefschetz and Whitehead [**26**]. To solve the third difficulty, they showed that given a set of real analytic subvarieties, there was around every point of the variety a triangulation in which each subvariety was a union of simplices. Then, rather than triangulating only Y at the inductive step, one simultaneously triangulates Y and the family of images of strata of X. The resulting simplices are each contained in the images of strata, so they can be lifted. Koopman and Brown [**24**] proved in 1932 that a real analytic variety is locally triangulable. To solve the second difficulty, they showed that good projections always existed by a Baire category argument.

Solving the first difficulty, that of showing that semi-algebraicity was preserved under projection, took the longest. The proofs mentioned above tried to prove this by showing that the equations for the variety could be written in a special form where projection was possible. No rigorous proof existed until Tarski [**45**] in 1948 and Seidenberg [**40**] in 1954 proved using quantifier elimination that the image of a semi-algebraic set under a polynomial map is semi-algebraic. This solved the first difficulty above in the semi-algebraic case. In 1964, Łojasiewicz [**27**] and Giesecke [**14**] both proved that the same is true for semi-analytic sets and used this to prove triangulability of semi-analytic sets. Łojasiewicz noted that his proof also gave triangulability of semi-algebraic sets. Hironaka [**20**] in 1975 and Hardt [**19**] in 1976 triangulated sub-analytic sets, which are a generalization of semi-analytic sets. Because of Grauert's theorem [**15**] on analytic embeddings of analytic manifolds, all of these imply simultaneous triangulation of a real analytic manifold and a locally finite family of semi-analytic subvarieties.

In 1975, Hironaka [**20**] adapted Łojasiewicz's work to the semi-algebraic setting, producing the best-known proof of triangulation of projective algebraic varieties. Replacing Mannoury's embedding of projective space with a semi-algebraic version of [**11**, Proposition 8.8] allows one to triangulate any complete real or complex variety, but this proof does not seem to have been published anywhere. For non-complete varieties, however, these techniques fail. They need to choose a finite stratification of X in order to ensure that the intersection of any family of strata is a semi-algebraic set, and therefore they produce only finite simplicial complexes. Finite simplicial complexes are compact, so the method cannot not directly apply to non-complete varieties.

The remedy to this is to glue local triangulations to give global triangulations. That this is possible is not obvious. Siebenmann [**42**, p. 138] found an example of a compact locally triangulable topological space which is not triangulable. Kirby and Siebenmann [**23**] proved in 1969 that there is an obstruction in $H^4(M, \mathbf{Z}/2\mathbf{Z})$ to a topological manifold being a piecewise linear manifold, that is, a manifold whose transition functions are piecewise-linear. In 1980, Galewski and Stern [**13**] showed that a topological manifold admits a triangulation if and only if a certain cohomology class related to the Kirby-Siebenmann obstruction vanishes. Freedman's E_8-manifold [**12**] is an example of a manifold where this class does not vanish; see [**1**]. Consequently, topological manifolds have non-trivial obstructions to triangulability.

Our theorem shows that there is no such obstruction in the semi-algebraic setting. (In particular, the E_8-manifold is not semi-algebraic.) The most natural objects to work with for this question are *locally semi-algebraic spaces*, which were introduced by Delfs and Knebusch [**10**]. Call a locally ringed space an *affine semi-algebraic space* if it is isomorphic to a semi-algebraic set with its sheaf of semi-algebraic morphisms to $\mathbf{R}$.[1] Then a locally semi-algebraic space is a locally ringed space locally isomorphic to an affine semi-algebraic space. These spaces turn up naturally. For example, the universal covering of an algebraic variety is a locally semi-algebraic space. The existence of triangulations of semi-algebraic sets implies that a locally compact locally semi-algebraic space is locally triangulable. We show that such spaces are globally triangulable and that the local compactness hypothesis cannot be dropped.

Delfs and Knebusch showed that with a more relaxed definition of a simplicial complex, the local compactness hypothesis can be weakened. Rather than considering complexes made from closed simplices, they considered complexes made from open simplices. Then it becomes possible to triangulate

[1]In fact, this is not strictly correct because real-valued semi-algebraic morphisms do not form a sheaf. However, there are standard ways around this problem. See section 2.

non-compact varieties by letting their missing points at infinity correspond to the missing faces of the open simplices. This is done in [**10**, Chapter 2, Theorem 4.4], which triangulates (in this sense) any Hausdorff paracompact locally semi-algebraic space.[2] For algebraic varieties, the same result follows quickly from Nagata's compactification theorem. One begins with a variety, chooses a compactification by Nagata's theorem, triangulates the compactification in such a way that the boundary corresponds to a union of simplices of the triangulation, and then removes those simplices. However, Nagata's theorem [**33, 34**] is a difficult result. (For proofs of Nagata's theorem using modern terminology, see Conrad [**8**] or Lütkebohmert [**28**]. As far as we are aware, this approach to triangulation has never been written down.)

If one is unwilling to relax the definition of a simplicial complex, then the local compactness hypothesis cannot be dropped. This is because a locally semi-algebraic space is locally metrizable, and it is well-known that a simplicial complex is locally metrizable if and only if it is locally compact. (For completeness, we have included this result in propositions 3.22 and 3.23.) There are other point-set topological obstructions to triangulability. Simplicial complexes are always Hausdorff, and locally finite simplicial complexes are paracompact. We summarize these restrictions in theorem 3.30.

In theorem 4.5 we show that being Hausdorff, paracompact, and locally compact are sufficient conditions for triangulability. Our proof works as follows. First, in proposition 4.4, we show that an affine semi-algebraic space can be triangulated. If it is compact, this is the usual triangulation theorem for semi-algebraic sets. Otherwise, we compactify the space and triangulate the compactification. To remove the boundary, we perform repeated barycentric subdivisions. These subdivide the simplicial complex infinitely often at the boundary, but only finitely many times at any other point. This produces a bijective continuous map from a simplicial complex to the space. After removing the boundary, the map is a homeomorphism exactly on the set of points where the space is locally compact.

Second, we glue local triangulations to produce a global triangulation. Theorem 2.61 shows that we can choose a cover of a paracompact locally compact locally semi-algebraic space X by compact affine subspaces. This cover has the additional property that each of the subspaces in the cover meets only finitely many other subspaces. To produce a global triangulation, we well-order the subspaces and proceed recursively. There are two interesting steps to this procedure. The first is to show that

[2]The theorem is stated for a regular paracompact space, but we can apply a standard result in point-set topology [**31**, Theorem 41.1] to see that a Hausdorff and paracompact space is regular and even normal.

given a triangulation of a subspace Y of X and an element K of the cover, there is a way of extending the triangulation to the union of Y and K. The second is to show that given an ascending chain of subspaces of X, each having a triangulation, there is a triangulation of the union of the subspaces.

The first step is done using theorem 4. This theorem takes a simplicial complex, a subcomplex, and a subdivision of the subcomplex, and it produces a subdivision of the larger complex with two properties. The first is that the subdivision of the larger complex contains an isomorphic copy of the subdivision of the subcomplex. The second is that the subdivision of the larger complex is the trivial subdivision on any simplex not meeting the subcomplex. It is the first property that allows us to extend the triangulation of Y to K. The triangulation of Y determines a triangulation of $Y \cap K$. We choose a triangulation of K which refines this triangulation of $Y \cap K$ using the triangulation theorem for affine semi-algebraic spaces. This determines a subdivision of the triangulation of $Y \cap K$, which we can extend to Y using theorem 3.36. Because this subdivision agrees with the triangulation of K on $Y \cap K$, we may take the union of the two triangulations to produce a triangulation of $Y \cup K$.

The second step, of forming an ascending union, is a consequence of the non-subdivision property of theorem 4 and of the fact that each K meets only finitely many other elements in the cover. Because of this finiteness, after finitely many steps, every element of the cover that meets K has been adjoined to Y. After this, Y does not change in a neighborhood of K. The non-subdivision property therefore implies that the triangulation of Y does not change in a neighborhood of K. Consequently the triangulation stabilizes near K. Taking the ascending union of the stable part of the triangulations is easy because it is just an ascending union of simplicial complexes, and this constructs a triangulation of X.

Next, we turn to uniqueness. Triangulations are obviously not unique, because whenever the space is positive dimensional, the triangulation can be subdivided to give a different triangulation. This is a trivial sort of non-uniqueness, and we wish to ignore it. To be precise, suppose that C and D are simplicial complexes and that $s: D \to C$ is an *affine subdivision* (or *subdivision* for short), meaning that s is a homeomorphism, maps each simplex of D into a simplex of C, and is an affine transformation on each simplex of D. If $\tau : C \to X$ is a triangulation, declare $\tau \circ s$ to be equivalent to τ. This generates an equivalence relation whose equivalence classes are called *piecewise-linear structures*, or, to distinguish them from a variant that will occur later, *affine piecewise-linear structures*. A space with a choice of PL structure is called a *PL space*.[3]

Any positive dimensional PL space admits many PL structures. For example, consider $[0,1]$, the convex hull of the points 0 and 1. Its standard PL structure is given by the identity map $x \mapsto x$. It has a second PL structure given by $x \mapsto x^2$ and a third given by $x \mapsto (e^x - 1)/(e-1)$. However, up to isotopy all three structures are identical.

The *Hauptvermutung* of Steinitz [**44**] and Tietze [**46**] predicted that this would always be the case: Every homeomorphism of PL spaces would be homotopic to a simplicial isomorphism. The first counterexample was constructed by Milnor [**30**]. Starting with the lens spaces $L(7,1)$ and $L(7,2)$, he constructed two homeomorphic finite simplicial complexes X_1 and X_2. A finite simplicial complex has an invariant called the torsion which is stable under refinement. If the homeomorphism $X_1 \to X_2$ was homotopic to a simplicial isomorphism, X_1 and X_2 would have the same torsion. Milnor showed that X_1 and X_2 had different torsion, and therefore no such homotopy exists.

On PL manifolds, isotopy classes of PL structures are well-understood by results of Kirby and Siebenmann. In the same paper [**23**] where they introduced the obstruction in H^4 mentioned above, they also proved that when the obstruction vanishes, the isotopy classes of PL structures which are PL manifolds are in one-to-one correspondence with $H^3(M, \mathbf{Z}/2\mathbf{Z})$. A particularly important situation is the isotopy classification of PL manifolds homeomorphic to the n-torus, $n \geq 5$, which was done independently by Hsiang and Shaneson [**21**], C. T. C. Wall [**48**], and A. Casson (unpublished). For these manifolds, $H^3(T^n, \mathbf{Z}/2\mathbf{Z})$ is isomorphic to the orbit space $(\bigwedge^{n-3} \mathbf{Z}^n \otimes \mathbf{Z}/2\mathbf{Z})/\mathrm{GL}_n \mathbf{Z}$. This is nontrivial, and hence there are many non-standard PL structures on tori of dimension at least 5. This is particularly relevant to the algebraic setting because complex abelian varieties are topological tori, and hence there are algebraic varieties with multiple PL structures, even up to isotopy.

However, it follows from the triangulation of semi-algebraic sets that T^n, and in fact any complete variety, has essentially only one triangulation in which all the simplices are semi-algebraic. Given any two triangulations, one can triangulate the variety and the family of all simplices occuring in either of the two triangulations. This produces a third triangulation which is finer than both of the

[3]This definition of a PL space is similar to the definition of a triangulation given in [**38**, p. 17], but it uses general homeomorphisms instead of PL homeomorphisms. Another approach to general piecewise-linear topology is that of Zeeman [**50**, Exposé 2], who introduced a notion he called a *polyspace*. Hudson [**22**, III.2] called the same object a PL space. All our PL spaces are polyspaces. By [**22**, p. 82], every Hausdorff second countable locally compact polyspace is one of our PL spaces.

original triangulations. The map from the third triangulation to either of the original triangulations satisfies all the conditions to be a subdivision except that it need only be semi-algebraic on simplices, not affine. We call such a map is a *semi-algebraic subdivision*, and we call the equivalence classes of triangulations with respect to semi-algebraic subdivision *semi-algebraic PL structures*. The triangulation theorem says that there is a unique semi-algebraic PL structure all of whose simplices are semi-algebraic.

It is natural to ask how semi-algebraic PL structures are related to PL structures. For example, are their isotopy classes the same? On compact semi-algebraic sets, a theorem of Shiota [**41**, Proposition 3.9 and Remark 3.10] implies that a semi-algebraic subdivision is semi-algebraically isotopic to an affine subdivision. Consequently, on a compact semi-algebraic set, isotopy classes of semi-algebraic PL structures are the same as isotopy classes of PL structures.

In particular, manifolds such as the torus admit exotic semi-algebraic PL structures. Each of these is a homeomorphism $h\colon C \to T^n$ from a finite simplicial complex such that the image of at least one simplex is not semi-algebraic. But because C is a finite simplicial complex, it is itself a semi-algebraic set, so h is a homeomorphism of semi-algebraic sets which is not isotopic to a semi-algebraic isomorphism. We conclude that the torus admits exotic semi-algebraic structures, in other words that there is a sheaf of local rings $\mathscr{O}$ on T^n such that $(T^n, \mathscr{O})$ is a locally semi-algebraic space but $\mathscr{O}$ is not isotopic to the usual structure sheaf of locally semi-algebraic functions on T^n. Furthermore, the isotopy classes of semi-algebraic structures are in one-to-one correspondence with the isotopy classes of PL structures. It seems reasonable to conjecture that the same is true on non-compact spaces.

CONJECTURE. *Let X be a Hausdorff paracompact locally compact topological space. Then the isotopy classes of semi-algebraic structures on X are in one-to-one correspondence with the isotopy classes of PL structures on X.*

The conjecture would follow if Shiota's theorem were true on a locally finite simplicial complex, not just a finite simplicial complex. Our theorem implies that when X is locally semi-algebraic, it admits at least one isotopy class of PL structures.

It would be interesting to classify the isotopy classes of semi-algebraic structures on a locally semi-algebraic space.

The thesis is structured as follows. In section 2, we describe the foundations of semi-algebraic geometry, that is, we describe semi-algebraic sets, affine semi-algebraic spaces, locally semi-algebraic spaces, and semi-algebraic spaces. Section 3 is about piecewise linear topology. In particular, subsection 2 proves the results in point-set topology that we use in 3 to give restrictions on the kinds of locally semi-algebraic spaces that are triangulable. Subsection 4 gives the main technical result we will need about simplicial complexes, a theorem that allows us to extend a subdivision. Section 4 proves the main triangulation theorem.

Chapter 2

Locally semi-algebraic and semi-algebraic spaces

Over an ordered field such as the real numbers, it is natural to allow sets defined by polynomial inequalities as well as those defined by polynomial equalities. These are called *semi-algebraic sets*, and they are the subject of section 1. A basic reference for them is [**4**].

Semi-algebraic sets can be globalized in two ways. The first way is through *semi-algebraic spaces* [**9**]. These are spaces which are covered by finitely many semi-algebraic sets. We recall their theory in section 2. The second way is through *locally semi-algebraic spaces* [**10**]. These are spaces which are covered by semi-algebraic spaces, and we recall their theory in section 3. In section 4, we recall the theory of dimension for locally semi-algebraic spaces. Finally, in section 5, we give a precise description of the semi-algebraic structures discussed in the introduction.

1. Semi-algebraic sets

DEFINITION 2.1. A *semi-algebraic set* is a subset of $\mathbf{R}^n$ which is a finite union of sets cut out by polynomial equalities and inequalities. That is, it is a set $X \subseteq \mathbf{R}^n$ such that there are natural numbers r and $s_1, \ldots, s_r$ and polynomials f_{ij} and g_{ij}, where $1 \le i \le r$ and $1 \le j \le s_i$, such that:

$$X = \bigcup_{i=1}^{r} \bigcap_{j=1}^{s_i} (\{x \mid f_{ij}(x) = 0\} \cup \{x \mid g_{ij}(x) > 0\}). \tag{1.1}$$

Semi-algebraic sets were first introduced under that name by Łojasiewicz [**27**]. They had been studied earlier by Brakhage [**6**] and had appeared implicitly in the work of Tarski [**45**].

Semi-algebraic sets constitute the smallest Boolean algebra of subsets of $\mathbf{R}^n$ containing sets defined by polynomial inequalities $\{f > 0\}$. That is, they are the smallest family containing sets defined by polynomial inequalities and closed under finite union, finite intersection, and complementation. It is clear from the definition that semi-algebraic sets are closed under finite unions and finite intersections, and to see that they are closed under complementation, notice that $\mathbf{R}^n \setminus \{f > 0\} = \{-f \geq 0\} = \{-f > 0\} \cup \{f = 0\}$ and $\mathbf{R}^n \setminus \{f = 0\} = \{f > 0\} \cup \{-f > 0\}$. For the converse, notice that $\{f = 0\} = \mathbf{R}^n \setminus (\{f > 0\} \cup \{-f > 0\})$. It is also clear that if $X \subseteq \mathbf{R}^n$ and $Y \subseteq \mathbf{R}^m$ are semi-algebraic sets, then $X \times Y \subseteq \mathbf{R}^{n+m}$ is a semi-algebraic set because it is cut out by the equations and inequalities in the first n variables that define X and the equations and inequalities in the next m variables that define Y.

We give a semi-algebraic set the *Euclidean* or *strong topology*, that is, the subspace topology with respect to the metric topology on $\mathbf{R}^n$. In subsection 2 we will discuss another topology, the semi-algebraic topology, but topological properties such as openness, closedness, and compactness will refer to the Euclidean topology unless we explicitly say otherwise.

LEMMA 2.2. *A semi-algebraic set has a basis consisting of open semi-algebraic subsets.*

PROOF. If $x \in U \subseteq X \subseteq \mathbf{R}^n$, where X is a semi-algebraic set and U is open in X, then there is an open ball B in $\mathbf{R}^n$ such that $B \cap X \subseteq U$. $B \cap X$ is an open semi-algebraic set. □

DEFINITION 2.3. Let $X \subseteq \mathbf{R}^n$ and $Y \subseteq \mathbf{R}^m$ be semi-algebraic sets. A *semi-algebraic morphism* $f : X \to Y$ is a continuous function such that the graph of f is a semi-algebraic subset of $\mathbf{R}^n \times \mathbf{R}^m$.

A semi-algebraic morphism in the above sense is a continuous semi-algebraic map in the sense of [**4**, Definition 2.2.5]. It will follow from corollary 2.8 that semi-algebraic sets (considered with their embeddings) and semi-algebraic morphisms form a category.

Any rational function is a semi-algebraic morphism wherever it is defined, and in particular any regular morphism of affine algebraic varieties is a semi-algebraic morphism. However, many more functions are semi-algebraic than are algebraic. For example, the graph of $f(x) = \sqrt{x}$ is the semi-algebraic set $\{(x, y) \mid y^2 = x, y \geq 0\}$, so f is a semi-algebraic morphism.

The most fundamental theorem on semi-algebraic sets is that they are closed under polynomial maps:

THEOREM 2.4 (Tarski [**45**], Seidenberg [**40**]). *If $f : \mathbf{R}^n \to \mathbf{R}^m$ is a polynomial map and $X \subseteq \mathbf{R}^n$ is a semi-algebraic set, then $f(X)$ is semi-algebraic.*

The Tarski-Seidenberg theorem is effective, meaning that the defining equations for $f(X)$ can be computed from those for X. See [**3**, Algorithm 14.20] for a primitive recursive algorithm.

LEMMA 2.5 ([**9**, p. 181]). *Let X be a semi-algebraic set. Then the diagonal $\Delta\colon X \to X \times X$ is a semi-algebraic morphism.*

PROOF. The graph of Δ is the intersection of $X \times X \times X$ with the semi-algebraic set $\{(x,y,z) \mid x = y = z\}$. $\square$

LEMMA 2.6 ([**9**, Theorem 6.12]). *Let $X \subseteq \mathbf{R}^n$, $X' \subseteq \mathbf{R}^{n'}$, $Y \subseteq \mathbf{R}^m$, and $Y' \subseteq \mathbf{R}^{m'}$ be semi-algebraic sets, and let $f : X \to Y$ and $f' : X' \to Y'$ be semi-algebraic morphisms. Then $f \times f' : X \times X' \to Y \times Y'$ is a semi-algebraic morphism.*

PROOF. The graph of $f \times f'$ is $\{(x,x',y,y') \mid y = f(x),\ y' = f'(x')\}$, which is the intersection of $\{(x,x',y,y') \mid y = f(x)\} \cong \Gamma_f \times \mathbf{R}^{n'} \times \mathbf{R}^{m'}$ and $\{(x,x',y,y') \mid y' = f(x')\} \cong \Gamma_{f'} \times \mathbf{R}^n \times \mathbf{R}^m$. $\square$

LEMMA 2.7 ([**4**, Proposition 2.2.7]). *Let $X \subseteq \mathbf{R}^n$ and $Y \subseteq \mathbf{R}^m$ be semi-algebraic sets and $f : X \to Y$ be a semi-algebraic morphism. Then $f(X)$ is a semi-algebraic set.*

PROOF. Γ_f, the graph of f, is a semi-algebraic set by assumption, so we apply theorem 2.4 to the second projection $\Gamma_f \subseteq \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^m$. $\square$

COROLLARY 2.8 ([**4**, Proposition 2.2.6]). *If $X \subseteq \mathbf{R}^n$, $Y \subseteq \mathbf{R}^m$ and $Z \subseteq \mathbf{R}^\ell$ are semi-algebraic sets, and $f : X \to Y$ and $g : Y \to Z$ are semi-algebraic morphisms, then $g \circ f$ is semi-algebraic.*

PROOF. Write Γ_f and $\Gamma_{g\circ f}$ for the graphs of f and $g \circ f$, respectively. $\Gamma_{g\circ f} = \{(x,z) \mid z = (g \circ f)(x)\} = (1_X \times g)(\Gamma_f)$, and by lemma 2.7 this is semi-algebraic. $\square$

LEMMA 2.9 ([**4**, Proposition 2.2.7]). *The preimage of a semi-algebraic set under a semi-algebraic morphism is a semi-algebraic set.*

PROOF. Let $X \subseteq \mathbf{R}^n$, $Y \subseteq \mathbf{R}^m$, and $Z \subseteq Y$ be semi-algebraic sets, and suppose that $f : X \to Y$ is a semi-algebraic morphism. Write $f^{-1}(Z) = \pi_X(\Gamma_f \cap (\mathbf{R}^n \times Z))$, where Γ_f is the graph of f and $\pi_X : \mathbf{R}^n \times \mathbf{R}^m \to \mathbf{R}^n$ is the first projection. Apply theorem 2.4 to deduce that $f^{-1}(Z)$ is a semi-algebraic set. $\square$

The converse of this lemma is false: Semi-algebraicity of a morphism is much stronger than requiring that the preimage of a semi-algebraic set is semi-algebraic. Notice that the exponential function $\exp\colon \mathbf{R} \to \mathbf{R}$ is not semi-algebraic, but it is monotone, so the preimage of a semi-algebraic set under the exponential map is a semi-algebraic set.

LEMMA 2.10 ([**4**, Proposition 2.2.6]). *If X is a semi-algebraic set, then the semi-algebraic morphisms $X \to \mathbf{R}$ form a ring under pointwise addition and multiplication. An element of this ring is a unit if and only if it is non-vanishing.*

PROOF. Let $f: X \to \mathbf{R}$. The map $x \mapsto -x$ is a semi-algebraic morphism because its graph is $\{(x,y) \mid x + y = 0\}$, so by corollary 2.8, $-f$ is a semi-algebraic morphism. The map $x \mapsto 1/x$ is a semi-algebraic morphism because its graph is $\{(x,y) \mid xy - 1 = 0\}$, so by corollary 2.8, $1/f$ is a semi-algebraic morphism if it is defined. If f vanishes at some point, then, because multiplication is defined pointwise, f cannot have an inverse.

If $g\colon X \to \mathbf{R}$, then we apply lemmas 2.5 and 2.6 and corollary 2.8 to see that $(f,g)\colon X \to \mathbf{R}^2$ is a semi-algebraic morphism because it is the composite of the diagonal map $X \to X \times X$ and the product $f \times g\colon X \times X \to \mathbf{R} \times \mathbf{R}$. The addition and multiplication maps $\mathbf{R}^2 \to \mathbf{R}$ are semi-algebraic morphisms because their graphs $\{(x,y,z) \mid x + y = z\}$ and $\{(x,y,z) \mid xy = z\}$ are cut out by polynomial equations. By lemma 2.8, $f + g$ and $f \cdot g$ are semi-algebraic morphisms.

The axioms for addition and multiplication are true because they are true pointwise. □

LEMMA 2.11 ([**9**, Theorem 7.10]). *Let $S \subseteq \mathbf{R}^k$, $X \subseteq \mathbf{R}^n$, and $Y \subseteq \mathbf{R}^m$ be semi-algebraic sets, and let $f\colon X \to S$ and $g\colon Y \to S$ be semi-algebraic morphisms. Then the fibered product of sets $X \times_{f,S,g} Y \subseteq \mathbf{R}^n \times \mathbf{R}^m$ is a semi-algebraic set and is a fibered product in the category of semi-algebraic sets and semi-algebraic morphisms.*

PROOF. The set-theoretic fibered product $X \times_S Y$ is $\{(y,z) \in X \times Y \mid f(y) - g(z) = 0\}$. The function $f(y) - g(z)\colon X \times Y \to \mathbf{R}^k$ is semi-algebraic by lemma 2.10, and the preimage of 0 under this map equals $X \times_S Y$. The preimage of 0 is semi-algebraic by lemma 2.9, so $X \times_S Y$ is semi-algebraic.

To see that $X \times_S Y$ is a fibered product, suppose that $W \subseteq \mathbf{R}^n$ is a semi-algebraic set and that $s\colon W \to X$ and $t\colon W \to Y$ are two semi-algebraic morphisms such that $fs = gt$. The morphism $(s,t)\colon W \to$

$\mathbf{R}^n \times \mathbf{R}^m$ is the composite of the diagonal morphism $W \to W \times W$ and the product $s \times t$, so it is a semi-algebraic morphism. $\square$

PROPOSITION 2.12 ([**4**, Proposition 2.2.2]). *Let X be a semi-algebraic subset of $\mathbf{R}^n$. Then the closure, interior, and boundary of X are semi-algebraic sets.*

REMARK 2.13. The closure and the interior are not constructed by relaxing strict inequalities to weak ones and vice versa. [**4**, pp. 27–28] gives the following example: The closure of $\{y^2 < x^2(x-1)\} \subseteq \mathbf{R}^2$ is not $\{y^2 \le x^2(x-1)\}$ because the latter set has an extra point at the origin. The proof that closures and interiors are semi-algebraic instead relies on the fact that the square of the Euclidean distance function is semi-algebraic.

PROOF. The closure of X can be written as

$$\overline{X} = \{x \mid \forall t \in \mathbf{R}\ \exists y \in X \text{ such that } (\|y-x\|^2 < t^2 \text{ or } t = 0)\}$$

If we write π for the projection $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n \times \mathbf{R}$ that sends $(x, y, t) \mapsto (x, t)$ and ρ for the projection $\mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ that sends $(x, t) \mapsto x$, then $\overline{X}$ is equal to

$$\mathbf{R}^n \setminus \rho\left((\mathbf{R}^n \times \mathbf{R}) \setminus \pi\left(\{(x, y, t) \mid y \in X \text{ and } (\|y-x\|^2 < t^2 \text{ or } t = 0)\}\right)\right).$$

π and ρ are polynomial, so this expression and theorem 2.4 shows that $\overline{X}$ is semi-algebraic. By taking complements, we deduce that the interior and boundary of X are also semi-algebraic sets. $\square$

PROPOSITION 2.14 ([**9**, §7, Example 2]). *Let X and Y be semi-algebraic sets, and let $\{U_i\}$ be a finite cover of X by open semi-algebraic subsets. Suppose that $f_i \colon U_i \to Y$ is a semi-algebraic morphism for each i and that $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ for all i and j. Then there is a unique semi-algebraic morphism $f \colon X \to Y$ such that $f|_{U_i} = f_i$ for all i.*

PROOF. There is a unique continuous function f which restricts to f_i for all i. The graph of f is the union of the graphs of $\{f_i\}$. Since there are only finitely many f_i and since the graph of each f_i is a semi-algebraic set, the graph of f is also a semi-algebraic set. Therefore f is a semi-algebraic morphism. $\square$

2. Semi-algebraic spaces

Semi-algebraic spaces were introduced in [**9**] to provide a notion of semi-algebraicity that is independent of an embedding. Like schemes or complex analytic spaces, semi-algebraic spaces come with a structure sheaf. However, semi-algebraic morphisms from a semi-algebraic set to $\mathbf{R}$ do not form a sheaf with respect to the Euclidean topology. Proposition 2.14 guarantees that they form a sheaf with respect to finite covers, but they do not form a sheaf with respect to infinite covers.

EXAMPLE 2.15. [**4**, Remark 7.3.3] Define $f\colon \mathbf{R} \to \mathbf{R}$ by $f(x) = |x - \frac{1}{2} - \lfloor x \rfloor|$. The graph of f is a zigzag, and $f^{-1}(0)$ is the set $\mathbf{Z} + \frac{1}{2}$. By [**4**, Theorem 2.4.5], $f^{-1}(0)$ is not a semi-algebraic subset of $\mathbf{R}$. Therefore f does not satisfy the conclusion of lemma 2.9, so it is not a semi-algebraic morphism. But the restriction of f to any bounded interval is a semi-algebraic morphism because its graph is a finite union of line segments.

This cannot be fixed by keeping the same underlying topological space and using a different topology. The problem is that only finite covers should be allowed. There are two methods for repairing this defect. The first is to introduce a Grothendieck pretopology in which the covering families are finite. This is the method pursued by Delfs and Knebusch in [**9**]. As they mention in [**10**, Appendix A], there is a second approach which uses real spectra, an ordered analog of prime spectra. This introduces new generic points into the space, making it compact. This solves the problem posed by example 2.15 by ensuring that all open covers are, in effect, finite. We will not discuss the latter approach further, except to say that it has been thoroughly worked out by N. Schwartz [**39**] and that his real closed spaces are more general objects than Delfs and Knebusch's locally semi-algebraic spaces.

Because the theory of locally semi-algebraic spaces has already been fully worked out, we provide proofs only when Delfs and Knebusch do not provide the exact statement we need. Nevertheless we do not claim any of these statements as new, since they are all obvious in light of the work already done in [**9**], and were surely known to Delfs and Knebusch. For statements that appear in [**9**], we will provide precise restatements.

Rather than working directly with sites and Grothendieck topologies, Delfs and Knebusch encode an equivalent set of data in a new object they call a restricted topological space.

DEFINITION 2.16 ([**9**, §7, Definitions 1 and 2]). A *restricted topological space* is a pair $(M, \mathfrak{S}^\circ(M))$, where $\mathfrak{S}^\circ(M)$ is a family of subsets of M called the *open subsets*, such that the following axioms hold:

(1) The empty set and M are open.

(2) If U and V are open, then $U \cup V$ and $U \cap V$ are open.

A *continuous function* from one restricted topological space to another is a function such that the preimage of an open set is open.

The point of this definition is that only finite unions of open sets are assumed to be open. An infinite union of open sets may no longer be open.

The usual notion of a sheaf carries over easily to restricted topological spaces. That is, a sheaf is a presheaf which satisfies the normalization axiom that there are no sections over the empty set and the gluing axiom that compatible sections on a collection of open sets induce a unique section on the union of the open sets. However, the gluing axiom is now restricted to those covers $\{U_i\}_{i\in I}$ such that $\bigcup_{i\in I} U_i$ is open. Continuous functions induce pullback and pushforward functors on presheaves and sheaves in the same way as for classical topological spaces. These have the formulas

$$f_*\mathscr{F}(V) = \mathscr{F}(f^{-1}(V)),$$

$$f^{-1}\mathscr{G}(U) = \varinjlim_{V \supseteq f(U)} \mathscr{G}(V),$$

where $f : X \to Y$ is a continuous function, $U \subseteq X$ and $V \subseteq Y$ are open, and $\mathscr{F}$ and $\mathscr{G}$ are presheaves on X and Y, respectively. f_* sends sheaves to sheaves. f^{-1} does not, just as in the topological situation, so if $\mathscr{G}$ is a sheaf, $f^{-1}\mathscr{G}$ is the sheaf associated to the presheaf defined above.

As Delfs and Knebusch comment, the notion of a restricted topological space can also be seen as a very special example of the abstract machinery of sites and topoi. We call this site the *site associated to the restricted topological space*. The objects of the site are the open sets, and the morphisms are the inclusions. The covering families of an open set U are finite families of open sets $\{U_i\}$ such that $\bigcup U_i = U$. A sheaf on a restricted topological space is the same as a sheaf on this site. A continuous function from one restricted topological space to another determines a functor between the associated sites by sending an open set to its inverse image. We call this functor the *functor associated to the continuous function*.

Recall that a *topos* is a category of sheaves on a site [**2**, IV 1.1]. Therefore the topos of the site associated to a restricted topological space is just the category of sheaves on that space. A *(geometric) morphism of topoi* $f : E \to F$ is a triple (f^*, f_*, ϕ), where $f^* : F \to E$ preserves finite limits, $f_* : E \to F$ is right adjoint to f^*, and ϕ is the natural isomorphism of bifunctors defining the adjunction between f^* and f_* [**2**, IV 3.1]. A precise statement of the fact that a continuous function between restricted topological spaces induces pushforward and pullback morphisms is the fact that a continuous function induces a morphism of topoi between the topoi of the sites associated to the restricted topological spaces. Above we did not check that f_* and f^{-1} were functors or were adjoint. This can be done easily by invoking the abstract machinery of sites and topoi. Since Delfs and Knebusch do not check this detail, we include it.

To do this, we show that a continuous function induces a morphism of sites. To be precise, if $\mathscr{C}$ and $\mathscr{C}'$ are sites, then a functor $f : \mathscr{C} \to \mathscr{C}'$ is *continuous* if for every sheaf F on $\mathscr{C}'$, $F \circ f$ is a sheaf on $\mathscr{C}$ [**2**, III 1.1]. A continuous functor f extends to a morphism of categories of sheaves $f^s : \widetilde{\mathscr{C}} \to \widetilde{\mathscr{C}'}$ [**2**, III 1.2]. f is a morphism of sites $\mathscr{C}' \to \mathscr{C}$ (*not* $\mathscr{C} \to \mathscr{C}'$) if it is continuous and f^s is left exact, and in this case, f^s is the pullback functor for a morphism of topoi $\widetilde{\mathscr{C}} \to \widetilde{\mathscr{C}'}$ [**2**, IV 4.91].

LEMMA 2.17. *The functor associated to a continuous function is a morphism of sites.*

PROOF. Let $f : M \to N$ be a continuous function between two restricted topological spaces. By [**2**, IV 4.9.2], it suffices to check that the functor f^{-1} associated to the continuous function is a continuous functor, that the site associated to a restricted topological space admits all finite limits, and that f^{-1} preserves finite limits. The site associated to a restricted topological space has a terminal object, namely the whole space, and admits all fibered products, because fibered products are intersections. Therefore it admits finite limits [**5**, Proposition 2.8.2]. f^{-1} preserves products and fibered products because $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$, so f^{-1} preserves finite limits [**2**, I 2.4.2].

To see that f^{-1} is continuous, suppose that we have a sheaf F on M. To see that the presheaf $F \circ f^{-1}$ is a sheaf, suppose that V is an open set of N and that $V_1, \ldots, V_k$ is a covering of V. To say that $F \circ f^{-1}$ is a sheaf is to say that the diagram

$$(F \circ f^{-1})(V) \to \coprod_i (F \circ f^{-1})(V_i) \rightrightarrows \coprod_{i,j} (F \circ f^{-1})(V_i \cap V_j)$$

is an equalizer. But the diagram is equal to

$$F(f^{-1}(V)) \to \coprod_i F(f^{-1}(V_i)) \rightrightarrows \coprod_{i,j} F(f^{-1}(V_i) \cap f^{-1}(V_j))),$$

so it suffices to note that $f^{-1}(V_1), \ldots, f^{-1}(V_k)$ is a covering of $f^{-1}(V)$. $\square$

DEFINITION 2.18. [**9**, §7, Definition 2] A *ringed space* over $\mathbf{R}$, also called an $\mathbf{R}$*-ringed space*, is a pair $(M, \mathcal{O}_M)$ where M is a restricted topological space and $\mathcal{O}_M$ is a sheaf of $\mathbf{R}$-algebras. A *morphism* of $\mathbf{R}$-ringed spaces $(\phi, \theta)\colon (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is a continuous function $\phi\colon M \to N$ together with an $\mathbf{R}$-algebra morphism $\theta\colon \mathcal{O}_N \to \phi_* \mathcal{O}_M$.

EXAMPLE 2.19. [**9**, §7, Examples 1 and 2] Let M be a semi-algebraic set. M has a restricted topology whose open sets are the open semi-algebraic subsets of M. With this restricted topology, M has a sheaf of rings $\mathcal{O}_M$ which on an open semi-algebraic subset U is the $\mathbf{R}$-algebra of $\mathbf{R}$-valued semi-algebraic morphisms. This makes $(M, \mathcal{O}_M)$ a ringed space over $\mathbf{R}$. We call this ringed space the *affine semi-algebraic space associated to* M.

DEFINITION 2.20. An *affine semi-algebraic space* is an $\mathbf{R}$-ringed space $(M, \mathcal{O}_M)$ which is isomorphic to the affine semi-algebraic space associated to a semi-algebraic set, that is, to one of the ringed spaces arising by the construction of example 2.19. A *semi-algebraic space* is a ringed space $(M, \mathcal{O}_M)$ which admits a finite covering $\{M_1, \ldots, M_k\}$ by open sets such that the $\mathbf{R}$-ringed spaces $(M_i, \mathcal{O}_M|_{M_i})$ are affine semi-algebraic spaces. A *morphism* between semi-algebraic spaces is a morphism of $\mathbf{R}$-ringed spaces.

THEOREM 2.21 ([**9**, Proposition 7.1 and Theorem 7.2]). *A semi-algebraic morphism of semi-algebraic sets determines a morphism of the associated affine semi-algebraic spaces. Furthermore, this determines a full and faithful functor from the category of semi-algebraic sets to the category of semi-algebraic spaces.* $\square$

DEFINITION 2.22 ([**9**, p. 185]). The *strong topology* on a restricted topological space is the topology (in the usual sense) generated by the open sets of the restricted topological space.

That is, the open sets of the strong topology are the unions of the open sets of the restricted topology. Following [**9**, p. 185], we will adhere to the convention that topological properties such as being open, closed, compact, and so on always refer to the strong topology. The other topology on a semi-algebraic space will be called the *restricted topology*. When we wish to refer to an open or closed set in the restricted topology, we will always refer to it as a semi-algebraic open set or a semi-algebraic closed set. (This will be consistent with the definition we will make below of a semi-algebraic subset. It will also be true that semi-algebraic open sets and semi-algebraic closed sets are semi-algebraic spaces.) A function between two semi-algebraic spaces will be called *continuous* if it is continuous in the strong topology and *strictly continuous* if it is continuous in the restricted topology.

DEFINITION 2.23 ([**9**, p. 185]). A subset of a semi-algebraic space M is called *semi-algebraic* if it is a member of the smallest family of subsets of M which contains the semi-algebraic open sets and is closed with respect to taking finite unions and complements in M.

PROPOSITION 2.24 ([**9**, Proposition 7.4]). *A subset A of M is a semi-algebraic subset if and only if there exists an open semi-algebraic subset U of M, natural numbers r and $s_1, \dots, s_r$, and $\mathbf{R}$-valued semi-algebraic morphisms f_{ij} and g_{ij} on U, where $1 \le i \le r$ and $1 \le j \le s_i$, such that equation* (1.1) *is satisfied.*

THEOREM 2.25 ([**9**, Theorem 7.6]). *Let $f : M \to N$ be a function between two semi-algebraic spaces. The following are equivalent:*

(1) f is continuous, and its graph Γ_f is a semi-algebraic subset of $M \times N$.
(2) Pullback by f induces a morphism of sheaves $f^\# : \mathscr{O}_N \to f_* \mathscr{O}_M$, and the pair $(f, f^\#)$ is a morphism of semi-algebraic spaces.

Moreover, if either of the above conditions hold, then for any semi-algebraic subset A of M, $f(A)$ is semi-algebraic in N, and for any semi-algebraic subset B of N, $f^{-1}(B)$ is semi-algebraic in M. □

THEOREM 2.26 ([**9**, Theorem 7.7]). *Let A be a semi-algebraic subset of a semi-algebraic space M. Then the closure, interior, and boundary of A in the strong topology of M are semi-algebraic subsets.*

PROOF. Delfs and Knebusch do not prove that the boundary of A is semi-algebraic, but to show this it suffices to remark that $\partial A = \overline{A} \setminus A^\circ$. □

DEFINITION 2.27. [**9**, p. 186] Let A be a non-empty semi-algebraic subset of a semi-algebraic space M. Give A the restricted topology whose open subsets are semi-algebraic in M and relatively open in A with respect to the strong topology on M. For each such subset V, $\mathcal{O}_A(V)$ is the set of all functions $f : V \to \mathbf{R}$ which are continuous with respect to the strong topologies on M and $\mathbf{R}$ and whose graph is semi-algebraic in $M \times \mathbf{R}$. Then $(A, \mathcal{O}_A)$ is a semi-algebraic space. The inclusion of A into M induces a morphism of semi-algebraic spaces, and the resulting morphism $(A, \mathcal{O}_A) \to (M, \mathcal{O}_M)$ is a monomorphism. The semi-algebraic space $(A, \mathcal{O}_A)$ together with the morphism to $(M, \mathcal{O}_M)$ is called a *semi-algebraic subspace* of M.

REMARK 2.28. Delfs and Knebusch refer to all the facts asserted in the above paragraph as "clear".

PROPOSITION 2.29. [**9**, Proposition 7.9] *Let $f : M \to N$ be a semi-algebraic morphism. If B is a semi-algebraic subspace of N with $f(M) \subseteq B$, then the map $g : M \to B$ obtained by restricting the range of f is semi-algebraic. In particular, since $f(M)$ is a semi-algebraic subspace of N, f has a canonical factorization as $i \circ \widetilde{f}$, where $\widetilde{f}$ is the restriction of the range of f to $f(M)$ and i is the inclusion of $f(M)$ into N.*

THEOREM 2.30 ([**9**, Theorem 7.10]). *The category of semi-algebraic spaces and semi-algebraic morphisms admits fibered products. Moreover, the underlying topological space of a fibered product is the fibered product of the underlying topological spaces of its factors.*

PROPOSITION 2.31. [**9**, Proposition 7.11] *Let A_1 and A_2 be semi-algebraic subspaces of semi-algebraic spaces M_1 and M_2, respectively. Then $A_1 \times A_2$ is a semi-algebraic subset of $M_1 \times M_2$. Furthermore, $A_1 \times A_2$ considered as a semi-algebraic subspace of $M_1 \times M_2$ is isomorphic to $A_1 \times A_2$ considered as the product of the semi-algebraic subspaces A_1 and A_2.*

3. Locally semi-algebraic spaces

Delfs and Knebusch generalized the notion of a semi-algebraic set even further in [**10**] to give the category of locally semi-algebraic spaces. This category includes, for example, all universal covers of semi-algebraic spaces. Again their definition relies on a special type of site, but this site is not a restricted topological space.

DEFINITION 2.32 ([**10**, I, §1, Definitions 1 and 2]). A *generalized topological space* is a triple $(M, \tau^\circ(M), \mathrm{Cov}_M)$, where M is a set, $\tau^\circ(M)$ is a set of subsets of M called the *open subsets* of M, and Cov_M is a set of subsets of $\tau^\circ(M)$ called the *admissible coverings*, such that the following axioms hold:

(1) The empty set and M are open.

(2) If U and V are open, then $U \cup V$ and $U \cap V$ are open.

(3) Every family $\{U_i\}_{i\in I}$ with I finite is an admissible covering.

(4) If $\{U_i\}_{i\in I}$ is an admissible covering, then $\bigcup_{i\in I} U_i$ is open. For any open subset U, the set of all admissible coverings $\{U_i\}_{i\in I}$ such that $U = \bigcup_{i\in I} U_i$ is called the *admissible coverings of U* and is denoted $\mathrm{Cov}_M(U)$.

(5) If $\{U_i\}_{i\in I}$ is an admissible covering of U, and if $V \subseteq U$ is an open set, then $\{U_i \cap V\}_{i\in I}$ is an admissible covering of V.

(6) If $\{U_i\}_{i\in I}$ is an admissible covering of U, and if for every $i \in I$, $\{V_{ij}\}_{j\in J_i}$ is an admissible covering of U_i, then $\{V_{ij}\}_{i\in I, j\in J_i}$ is an admissible covering of U.

(7) If $\{U_i\}_{i\in I}$ is a collection of open sets whose union U is an open set, and if $\{V_j\}_{j\in J}$ is an admissible covering of U and a *refinement* of $\{U_i\}$, meaning that for every V_j there is a U_i such that $V_j \subseteq U_i$, then $\{U_i\}_{i\in I}$ is an admissible covering of U.

(8) If U is an open subset of M, $\{U_i\}_{i\in I}$ is an admissible covering of U, and V is a subset of U with $V \cap U_i$ open for all $i \in I$, then V is open.

A *continuous function* between two generalized topological spaces M and N is a function $f : M \to N$ such that for every open subset V of N, $f^{-1}(V)$ is an open subset of M, and for every admissible covering $\{V_i\}_{i\in I}$ of V, the family $\{f^{-1}(V_i)\}_{i\in I}$ is an admissible covering of $f^{-1}(V)$.

A restricted topological space determines a generalized topological space. The open sets are the same for both, and the covering families for the generalized topological space are the families of open sets $\{U_i\}_{i\in I}$ whose union is open.

Delfs and Knebusch refer to their definition as "ad hoc." The triple $(M, \tau^\circ(M), \mathrm{Cov}_M)$ is abbreviated by M. Again, there is a notion of a sheaf on a generalized topological space. It is essentially the same as the notion of a sheaf on a topological space, but now the gluing axiom is required for all covering families. Again, continuous functions induce pushforwards and pullbacks of sheaves satisfying the usual formulas.

Delfs and Knebusch remark that a generalized topological space is a type of site. We call this site the *site associated to the generalized topological space*. The objects of this site are the open subsets of M, and the morphisms of this site are the inclusions. The covering families are the admissible coverings. A sheaf on a generalized topological space is a sheaf on this site. A continuous function from one generalized topological space to another determines a functor between the associated sites by sending an open set to its inverse image. We call this functor the *functor associated to the continuous function*.

As in the case of restricted topological space, Delfs and Knebusch omit the verification that a continuous function between two generalized topological spaces induces pullback and pushforward functions, that is, that it induces a morphism of topoi. We include this detail below. To check it, we show that a continuous function induces a morphism of sites as in the case of restricted topological spaces. This proof is very similar to the case of restricted topological spaces.

LEMMA 2.33. *The functor associated to a continuous function is a morphism of sites.*

PROOF. Let $f: M \to N$ be a continuous function between two generalized topological spaces. Again we apply [**2**, IV 4.9.2], so it suffices to check that the functor f^{-1} associated to the continuous function is a continuous functor, that the site associated to a generalized topological space admits all finite limits, and that f^{-1} preserves finite limits. The only condition whose proof is not identical to the proof presented in lemma 2.17 is the proof that f^{-1} is continuous.

To see that f^{-1} is continuous, suppose that we have a sheaf F on M. To see that the presheaf $F \circ f^{-1}$ is a sheaf, suppose that V is an open set of N and that $\{V_i\}_{i \in I}$ is a covering of V. To say that $F \circ f^{-1}$ is a sheaf is to say that the diagram

$$(F \circ f^{-1})(V) \to \coprod_i (F \circ f^{-1})(V_i) \rightrightarrows \coprod_{i,j} (F \circ f^{-1})(V_i \cap V_j)$$

is an equalizer. But the diagram is equal to

$$F(f^{-1}(V)) \to \coprod_i F(f^{-1}(V_i)) \rightrightarrows \coprod_{i,j} F(f^{-1}(V_i) \cap f^{-1}(V_j))),$$

so it suffices to note that applying f^{-1} to an admissible covering of V produces an admissible covering of $f^{-1}(V)$ by assumption. $\square$

Definition 2.34 ([**10**, I, §1, Definition 2]). A *ringed space* over $\mathbf{R}$, also called an $\mathbf{R}$-*ringed space*, is a pair $(M,\mathcal{O}_M)$ where M is a generalized topological space and $\mathcal{O}_M$ is a sheaf of commutative $\mathbf{R}$-algebras. A *morphism* of $\mathbf{R}$-ringed spaces $(\phi,\theta)\colon (M,\mathcal{O}_M)\to(N,\mathcal{O}_N)$ is a continuous function $\phi\colon M\to N$ together with an $\mathbf{R}$-algebra morphism $\theta\colon \mathcal{O}_N\to\phi_*\mathcal{O}_M$.

Remark 2.35. Delfs and Knebusch refer to ringed spaces on restricted topological spaces and ringed spaces on generalized topological spaces as just "ringed spaces" with no qualifier. In practice this does not lead to confusion.

Remark 2.36. Strangely, Delfs and Knebusch here impose the restriction that the sheaf $\mathcal{O}_M$ is commutative, whereas they do not impose this restriction for a ringed space on a restricted topological space. [**16**, 4.1.1] does not assume that the structure sheaf of a ringed space is commutative.

Example 2.37 ([**10**, p. 4]). Let M be a semi-algebraic space. The restricted topological space underlying M determines the structure of a generalized topological space on M as follows. The open semi-algebraic subsets of M are the open subsets of the generalized topological space. A family $\{U_i\}_{i\in I}$ is an admissible covering of U if $U=\bigcup_{i\in I}U_i$ and if there is a finite subset $i_1,\dots,i_k$ of I such that $U=U_{i_1}\cup\dots\cup U_{i_k}$. Delfs and Knebusch assert that it is clear that this determines a generalized topological space and that the sheaves on this generalized topological space are the same as the sheaves on the restricted topological space underlying M. In particular, $\mathcal{O}_M$ is a sheaf on the generalized topological space, so $(M,\mathcal{O}_M)$ is an $\mathbf{R}$-ringed space called the *locally semi-algebraic space associated to the semi-algebraic space M*.

Remark 2.38 ([**10**, p. 4]). Given an $\mathbf{R}$-ringed space $(M,\mathcal{O}_M)$ and an open subset U of M, we define the a ringed space $(U,\mathcal{O}_M|_U)$ as follows. The open subsets of U are the open subsets of M which are contained in U. The covering families of U are the covering families of M each of whose members are contained in U. $\mathcal{O}_M|_U$ is the restriction of $\mathcal{O}_M$ to U. $(U,\mathcal{O}_M|_U)$ is said to arise by *restriction*. These subspaces are called *open subspaces* of $(M,\mathcal{O}_M)$. An open subspace is an *open semi-algebraic subset* if it is the locally semi-algebraic space associated to a semi-algebraic space as in example 2.37.

Remark 2.39. It seems more natural to say that an open semi-algebraic subset is a locally ringed space isomorphic to the locally semi-algebraic space associated to a semi-algebraic space, but this is not the definition used by Delfs and Knebusch.

DEFINITION 2.40 ([**10**, I, §1, Definitions 3 and 5]). A *locally semi-algebraic space* is a ringed space $(M, \mathcal{O}_M)$ over $\mathbf{R}$ which possesses an admissible covering $\{M_\alpha\}_{\alpha \in A}$ such that all M_α are open semi-algebraic subsets of M. A *locally semi-algebraic morphism* is a morphism of $\mathbf{R}$-ringed spaces.

DEFINITION 2.41 ([**10**, I, §1, Definition 4]). A family $\{X_\lambda\}_{\lambda \in \Lambda}$ of subsets of a locally semi-algebraic space M is *locally finite* if any open semi-algebraic subset W of M meets only finitely many X_λ.

PROPOSITION 2.42 ([**10**, I, Theorem 1.2]). *Let $(f, \phi) \colon (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ be a locally semi-algebraic morphism of locally semi-algebraic spaces. Then ϕ is pullback by f. That is, if V is an open semi-algebraic subset of N, then for every $h \in \mathcal{O}_N(V)$, $\phi(V) \colon \mathcal{O}_M(f^{-1}(V)) \to \mathcal{O}_N(V)$ sends $h \colon V \to \mathbf{R}$ to $h \circ f \colon f^{-1}(V) \to \mathbf{R}$.*

PROPOSITION 2.43 ([**10**, I, §1, Proposition 1.3]). *Let $f \colon M \to N$ be a function, and let $\{M_i\}_{i \in I}$ and $\{N_j\}_{j \in J}$ be admissible coverings of M and N by open semi-algebraic subsets. Assume that for every i, there exists a j such that $f(M_i) \subseteq N_j$. Then f is locally semi-algebraic if and only if the restriction $f|_{M_i} \colon M_i \to N_j$ is a semi-algebraic morphism for every i.*

An immediate corollary of this is that the semi-algebraic morphisms between two semi-algebraic spaces are locally semi-algebraic morphisms, and that a locally semi-algebraic morphism between two semi-algebraic spaces is a semi-algebraic morphism. Consequently example 2.37 determines a full and faithful embedding of the category of semi-algebraic spaces into the category of locally semi-algebraic spaces.

LEMMA 2.44 ([**10**, I, Lemma 2.2]). *Let M be a set, and let $\{M_i\}_{i \in I}$ be a family of subsets of M whose union is M. Suppose that each M_i has a structure sheaf $\mathcal{O}_{M_i}$ such that $(M_i, \mathcal{O}_{M_i})$ is a locally semi-algebraic space. Assume that I is a partially ordered set and that if $i < j$, then M_i is an open subspace of M_j. Then the inductive limit $\varinjlim_i (M_i, \mathcal{O}_{M_i})$ exists in the category of $\mathbf{R}$-ringed spaces, has $\{M_i\}_{i \in I}$ as an admissible cover, and each $(M_i, \mathcal{O}_{M_i})$ is an open subspace of $(M, \mathcal{O}_M)$. Furthermore, the underlying generalized topological space of this inductive limit is M.*

DEFINITION 2.45 ([**10**, I, §3, Definition 1]). A subset X of a locally semi-algebraic space M is *locally semi-algebraic* if, for every open semi-algebraic subset W of M, the set $X \cap W$ is semi-algebraic in W.

REMARK 2.46 ([**10**, p. 27]). Every open subset (of the generalized topological space underlying a locally semi-algebraic space) is a locally semi-algebraic set. Furthermore, locally semi-algebraic sets are stable under complementation, and the union and intersection of any locally finite family of locally semi-algebraic sets is locally semi-algebraic.

DEFINITION 2.47 ([**10**, p. 28]). Let X be a locally semi-algebraic subset of a locally semi-algebraic space M. Choose an admissible covering $\{M_i\}_{i\in I}$ of M all of whose members are open semi-algebraic subsets and which is stable under finite unions. (Any admissible covering can be enlarged so that it satisfies this condition.) For every $i \in I$, the intersection $M_i \cap X$ is a semi-algebraic subspace of the semi-algebraic space M_i, and it has a structure sheaf $\mathcal{O}_{M_i\cap X}$. Let $(X,\mathcal{O}_X)$ be the inductive limit of the directed system of semi-algebraic spaces $\{(M_i \cap X,\mathcal{O}_{M_i\cap X})\}$. Then $(X,\mathcal{O}_X)$ is a *subspace* of M.

REMARK 2.48 ([**10**, p. 28]). Delfs and Knebusch note that a subspace of a locally semi-algebraic space is a locally semi-algebraic space, and that an open subspace is a subspace. Furthermore, the inclusion of X into M, together with the restriction map on sheaves, is a locally semi-algebraic morphism.

PROPOSITION 2.49 ([**10**, I, Proposition 3.2]). *Let $f : L \to M$ be a locally semi-algebraic map between two locally semi-algebraic spaces, and assume that $f(L) \subseteq X$, where X is a locally semi-algebraic subset of M. Then f has a canonical factorization as $i \circ \widetilde{f}$, where $\widetilde{f}$ is f with its range restricted to X and i is the inclusion of X into M.*

A consequence of this proposition is that the space $(X,\mathcal{O}_X)$ does not depend on the choice of admissible covering made in definition 2.47 [**10**, p. 28]. Another consequence is that if X is a locally semi-algebraic subset of M and Y is any subset of X, then Y is locally semi-algebraic in X if and only if it is locally semi-algebraic in M. Furthermore, if Y is locally semi-algebraic, then the locally semi-algebraic space structure on Y determined by considering Y as a subspace of X is the same as the locally semi-algebraic space structure on Y determined by considering Y as a subspace of M [**10**, I, Proposition 3.4].

LEMMA 2.50 ([**10**, I, Example 2.5]). *Let M and N be locally semi-algebraic spaces. Then M and N admit a direct product $M \times N$. Furthermore, the underlying point set of $M \times N$ is the product of the* *underlying point sets of M and N, and if $\{M_i\}_{i\in I}$ and $\{N_j\}_{j\in J}$ are admissible coverings of M and N, then $\{M_i \times N_j\}_{i\in I, j\in J}$ is an admissible covering of $M \times N$.*

THEOREM 2.51 ([**10**, I, Propositions 3.5 and 3.6]). *Let M, N, and S be locally semi-algebraic spaces, and let $\phi\colon M \to S$ and $\psi\colon N \to S$ be locally semi-algebraic morphisms. Then there is a fibered product $M \times_S N$ whose underlying point set is the fibered product of the underlying point sets of M, N, and S. Furthermore, $M \times_S N$ is a subspace of $M \times N$.*

DEFINITION 2.52 ([**10**, I, §3, Definition 2 and I, §5, Definition 1]). A subset X of a locally semi-algebraic space M is *semi-algebraic* if it is locally semi-algebraic and if the subspace $(X, \mathscr{O}_X)$ is a semi-algebraic space. A locally semi-algebraic morphism $f: M \to N$ of locally semi-algebraic spaces is *semi-algebraic* if the preimage of a semi-algebraic subset of N is a semi-algebraic subset of M.

Delfs and Knebusch say that it is "clear" that an open semi-algebraic subset is also a semi-algebraic subset in this sense [**10**, p. 30].

DEFINITION 2.53 ([**10**, p. 31]). The *strong topology* on a generalized topological space is the topology (in the usual sense) generated by the open sets of the generalized topological space.

Again, just like for a restricted topological space, the open sets of the strong topology are the arbitrary unions of the open sets of the generalized topological space. The strong topology on a product is the direct product of the strong topologies, and the strong topology on a subspace is the subspace topology for the strong topology on the larger space [[**10**, p. 31]].

Following [**10**, p. 32], when we refer to topological properties such as openness or closedness, we will mean that these are true in the strong topology. The open sets of the generalized topological space underlying a locally semi-algebraic space will be called the *open locally semi-algebraic sets*. A function between two semi-algebraic spaces is *continuous* if it is continuous for the strong topologies on those spaces and *strictly continuous* if it is a continuous map of generalized topological spaces.

PROPOSITION 2.54 ([**10**, I, Proposition 3.14]). *The closure, interior, complement, and boundary of a locally semi-algebraic subset are locally semi-algebraic subsets.*

PROOF. Delfs and Knebusch do not prove that the boundary of A is semi-algebraic, but to show this it suffices to remark that $\partial A = \overline{A} \setminus A^\circ$. □

PROPOSITION 2.55 ([**10**, I, Proposition 3.1]). *Let $f : M \to N$ be a locally semi-algebraic morphism of locally semi-algebraic spaces, and let $Y \subseteq N$ be a locally semi-algebraic set. Then $f^{-1}(Y)$ is a locally semi-algebraic subset of M.*

It is not true that the image of a locally semi-algebraic morphism is a locally semi-algebraic set. See [[**10**, I, Counterexample 6.3 (a)]].

DEFINITION 2.56 ([**10**, I, §5, Defintion 2 and I, §7, Definition 1]). Let $f : M \to N$ be a locally semi-algebraic morphism of locally semi-algebraic spaces. f is *proper* if, for every locally semi-algebraic morphism $g : N' \to N$, the locally semi-algebraic morphism $f' : M \times_N N' \to N'$ obtained by base change of f with respect to g maps every closed locally semi-algebraic subset of $M \times_N N'$ to a closed locally semi-algebraic subset of N'. M is *complete* if the map from M to the one-point space is proper. M is *locally complete* if every point has a neighborhood which is a complete semi-algebraic space.

LEMMA 2.57 ([**9**, pp. 198–199]). *A semi-algebraic space is complete if and only if it is compact.*

LEMMA 2.58 ([**10**, I, Lemma 7.5]). *A locally complete semi-algebraic space M can be embedded as a closed subspace of a Euclidean space.*

Note that the lemma does not claim that every locally semi-algebraic space is a closed subspace of a Euclidean space.

LEMMA 2.59. *X is locally complete if and only if X is locally compact.*

PROOF. If X is locally complete, then every point has a complete semi-algebraic neighborhood K. K is a closed subspace of a Euclidean space by lemma 2.58, so it is locally compact in its strong topology.

Conversely, assume that X is locally compact in its strong topology. Every point x of X admits an open affine neighborhood, and this neighborhood can be embedded as a semi-algebraic set Z in Euclidean space. Because X is locally compact, there is a compact neighborhood K of x contained in Z. K contains an open set around x, so there is a ball B in the ambient Euclidean space that contains x and satisfies $B \cap Z \subseteq K$. Choose a ball B' containing x whose closure is contained in B, and let $L = \overline{B'} \cap Z$. L is a closed subspace of K and therefore it is compact. L is the intersection of two semi-algebraic sets, and hence it is a complete semi-algebraic neighborhood of x. $\square$

LEMMA 2.60. *Let X be a locally semi-algebraic space. Then X is locally metrizable.*

PROOF. Around every point of X, there is an open affine semi-algebraic subspace U. U can be embedded as a semi-algebraic set in Euclidean space. The strong topology on U is the subspace topology for the strong topology on Euclidean space, and the strong topology on Euclidean space is the metric topology. $\square$

THEOREM 2.61. *Let X be a locally compact locally semi-algebraic space. The following are equivalent.*

(1) X is paracompact.

(2) Every open cover $\{U_\alpha\}$ of X admits a refinement $\{K_\beta\}$ by compact affine semi-algebraic subspaces satisfying the following two conditions.

(a) The interiors of the members of $\{K_\beta\}$ cover X.

(b) For each K_{β_0}, there are only finitely many β such that $K_\beta \cap K_{\beta_0}$ is non-empty.

(3) Every open cover $\{U_\alpha\}$ of X admits a refinement $\{U_\beta\}$ by open affine semi-algebraic subspaces such that each U_{β_0} meets only finitely many other U_β.

[**10**, I, Corollary 4.19] proves a related statement: If M is a Hausdorff connected paracompact locally semi-algebraic space, then M has a locally finite covering by open semi-algebraic sets $\{M_n\}_{n\in\mathbf{N}}$ such that $M_n \cap M_m = \emptyset$ when $|n-m| \geq 2$.

PROOF. It is clear that the second statement implies the third by taking interiors and that the third statement implies the first. We will show that the first statement implies the second.

Let $\{U_\alpha\}$ be an open cover of X. Each point x of X has a compact affine semi-algebraic neighborhood A_x. By the paracompactness of X, there is a locally finite refinement $\{W_\gamma\}$ of $\{U_\alpha \cap A_x\}$. Each W_γ is an open subset of a compact set, but it need not be semi-algebraic and the collection of all $\overline{W_\gamma}$ need not be locally finite. To repair this, we will replace each W_γ by a sequence of semi-algebraic sets.

The first step in constructing this sequence is to fill out W_γ with an ascending chain of semi-algebraic sets. To construct this chain, fix an embedding $W_\gamma \subseteq A_x \subseteq \mathbf{R}^n$. Because A_x is compact, ∂W_γ is compact. (By ∂W_γ we mean $\partial_{A_x} W_\gamma$, not $\partial_{\mathbf{R}^{n+1}} W_\gamma$.) The elements of the chain will be semi-algebraic sets $Z_1 \subseteq Z_2 \subseteq Z_3 \subseteq \cdots$, each open in A_x, such that Z_k contains all points whose distance to ∂W_γ is at least 2^{-k} and does not contain any point whose distance to ∂W_γ is less than 2^{-k-1}.

To be precise, let $W_{\gamma,k}$ be the set of all points in $\mathbf{R}^n$ whose distance to ∂W_γ is less than or equal to 2^{-k-1}. Choose a cover of $W_{\gamma,k}$ by open balls in $\mathbf{R}^n$ of radius 2^{-k} with center on ∂W_γ. Because $W_{\gamma,k}$ is compact, this cover may be taken to be finite. Let N_k be the union of these balls. N_k is semi-algebraic. Note that $\overline{N_{k+1}} \subseteq W_{\gamma_k} \subseteq N_k$ because the balls were chosen to be centered on ∂W_γ. The sets $Z_k = (\mathbf{R}^n \setminus \overline{N_k}) \cap W_\gamma$ are open and $\overline{Z_k} \subseteq Z_{k+1}$. The Z_k fill out W_γ, but they are not obviously semi-algebraic.

To see that they are semi-algebraic, recall that W_γ is an open subset of A_x, so it is the intersection of A_x with an open set T in $\mathbf{R}^n$. Let $\overline{Z_k}$ be the closure of Z_k in W_γ. Because N_{k+1} is an open neighborhood of ∂W_γ, this closure is the same as the closure of Z_k in A_x, and hence it is compact. Therefore it is covered by finitely many balls, say $B_1,\dots,B_\ell$. Then $Z_k = (B_1 \cup \dots \cup B_\ell) \cap A_x \cap (\mathbf{R}^n \setminus \overline{N_k})$ because $B_i \cap A_x \subseteq W_\gamma$ for all B_i. Consequently Z_k is semi-algebraic.

$\{Z_1, Z_2, Z_3, \dots\}$ is an open cover of W_γ by semi-algebraic sets. We set $L_k = \overline{Z_k} \setminus Z_{k-2}$. Each L_k is compact, and because $L_k \cap L_\ell = \emptyset$ unless ℓ is $k-1$, k, or $k+1$, $\{L_k\}$ is locally finite. We claim that $\bigcup_k L_k^\circ = W_\gamma$, where L_k° denotes the interior of L_k as a subspace of W_γ. This follows immediately from the claim that $L_k^\circ \supseteq Z_k \setminus \overline{Z_{k-2}}$. This is a consequence of the fact that N_k and N_{k-2} are finite unions of balls: One has $L_k = (\mathbf{R}^n \setminus N_k) \cap \overline{N_{k-2}} \cap W_\gamma$, so

$$L_k^\circ = (\mathbf{R}^n \setminus N_k)^\circ \cap (\overline{N_{k-2}})^\circ \cap W_\gamma = (\mathbf{R}^n \setminus \overline{N_k}) \cap N_{k-2} \cap W_\gamma = Z_k \setminus \overline{Z_{k-2}}.$$

Therefore $\{L_k\}$ is a cover of W_γ with the desired properties. For notational convenience in the following, we rename these sets $L_{\gamma,k}$.

To conclude, we let $\{K_\beta\}$ be the collection of all the compact sets $L_{\gamma,k}$ as γ and k vary. To see that each K_{β_0} meets only finitely many other K_β, notice that around each point x of X, there is an open neighborhood V_x which meets only finitely many W_γ. For each such W_γ, we may shrink V_x so that it meets only finitely many members of $\{L_{\gamma,k}\}$. Therefore we may choose V_x so that it meets only finitely many members of $\{K_\beta\}$. The collection $\{V_x\}_{x \in K_{\beta_0}}$ is an open cover of K_{β_0}, so it admits a finite subcover. Each of the finitely many elements in this subcover meets only finitely many members of $\{K_\beta\}$, so we conclude that K_{β_0} meets only finitely many members of $\{K_\beta\}$. The remaining conditions on $\{K_\beta\}$ are clear. $\square$

Proposition 2.62. *Let k be $\mathbf{R}$ or $\mathbf{C}$. There is a functor S from locally finite type k-schemes to locally semi-algebraic spaces with the following properties.*

(1) *S sends the variety $\{f_1 = \cdots = f_k = 0\} \subseteq \mathbf{A}^n$ to the locally semi-algebraic space corresponding to the semi-algebraic set cut out by the same equations.*
(2) *S sends a scheme to a locally compact locally semi-algebraic space.*
(3) *S sends quasi-compact schemes to paracompact semi-algebraic spaces and quasi-compact morphisms to semi-algebraic morphisms.*
(4) *S sends a separated scheme to a Hausdorff locally semi-algebraic space.*

Furthermore, the first of the above properties characterizes S uniquely.

[**9**, §7, Example 2] states the weaker fact that the real points of an algebraic variety determine a semi-algebraic space.

Proof. For a finite type k-scheme X and an open affine subscheme U of X, fix an embedding $U \subseteq \mathbf{A}^N$, where N depends on U. U is cut out by polynomial equations, and these polynomial equations also cut out the semi-algebraic subset $U(k)$ of $\mathbf{A}^N(k)$. ($\mathbf{A}^N(k)$ is $\mathbf{R}^N$ if $k = \mathbf{R}$ and $\mathbf{R}^{2N}$ if $k = \mathbf{C}$.) $U(k)$ has a corresponding affine semi-algebraic space, and hence a corresponding locally semi-algebraic space. We can glue these by proposition 2.44 to give a locally semi-algebraic space $S(X)$. The underlying point set of $S(X)$ is the gluing of the $U(k)$ and hence is equal to $X(k)$.

Suppose that $f: X \to Y$. If U and V are open affine subschemes of X and Y, respectively, and $f(U) \subseteq V$, then the graph of $f|_U$ is a subvariety of $U \times V$ and hence, after taking k-points, determines a morphism of semi-algebraic sets from U to V. This determines a morphism of affine semi-algebraic spaces $S(f|_U): S(U) \to S(V)$. Furthermore, if $U_1 \subseteq U$, then $S(f|_{U_1}): S(U_1) \to S(V)$ is the restriction of $S(f|_U)$. Therefore the morphisms $S(f|_U)$ can be glued to produce the desired $S(f)$. $S(f)$ is functorial in f because taking k-points is a functor.

$S(X)$ has a canonical representation as the inductive limit of all open affine subspaces, and consequently the value of the functor S is determined by its value on affine semi-algebraic spaces. By the assumption that S sends a variety cut out by equations into the semi-algebraic set cut out by the same equations, it follows that any two choices of $S(X)$ are isomorphic, and this gives the uniqueness of S on objects.

For an affine scheme U, $S(U)$ is locally compact because U can be written as a closed subspace of affine space, which is locally compact. Therefore $S(X)$ is locally compact for any X.

If X is quasi-compact, then X can be covered by only finitely many open affine subschemes, so $S(X)$ admits a finite cover by affine semi-algebraic spaces and is therefore a semi-algebraic space. Furthermore, each open affine subscheme determines a paracompact semi-algebraic space because it is a closed subvariety of an affine space and closed subspaces of paracompact spaces are paracompact. Therefore $S(X)$ is covered by finitely many paracompact open subspaces and hence is paracompact.

If f is quasi-compact and V is an open affine subscheme of Y, then $f^{-1}(V)$ is quasi-compact, so it is covered by finitely many open affines U_i. $S(f)\colon S(U_i) \to S(V)$ is a morphism of affine semi-algebraic spaces, so it is semi-algebraic. Since semi-algebraicity is local on the base and local on the source with respect to finite covers, we deduce that $S(f)$ is semi-algebraic.

If X is separated, then the diagonal morphism $X(k) \to X(k) \times X(k)$ is a closed immersion, and this implies that $S(X)$ is Hausdorff. □

4. Dimension of locally semi-algebraic spaces

DEFINITION 2.63 ([**4**, Proposition 2.8.2]). If $X \subseteq \mathbf{R}^n$ is a semi-algebraic set, then $\dim X$ is the dimension of the Zariski closure of X. If X is an affine semi-algebraic space and $i\colon X \hookrightarrow \mathbf{R}^n$ is any semi-algebraic embedding, then $\dim X$ is $\dim i(X)$.

THEOREM 2.64 ([**9**, p. 189]). *The dimension of an affine semi-algebraic space is well-defined. That is, if X is an affine semi-algebraic space and $i\colon X \hookrightarrow \mathbf{R}^n$ and $j\colon X \hookrightarrow \mathbf{R}^m$ are two semi-algebraic embeddings, then the dimensions of $i(X)$ and $j(X)$ are the same.*

PROOF. This follows from [**9**, Theorem 8.1]. □

DEFINITION 2.65 ([**10**, I, §3, Definition 4]). Let X be a locally semi-algebraic space. $\dim X$ is the supremum of $\dim U$ as U varies over the open affine semi-algebraic subsets of X.

PROPOSITION 2.66 ([**10**, I, Proposition 3.21 (d)]). *Let Z be a non-empty finite dimensional locally semi-algebraic subspace of a locally semi-algebraic space X. Then* $\dim(\overline{Z} \setminus Z) < \dim Z$.

5. Semi-algebraic structures on topological space

DEFINITION 2.67. Let X be a topological space. A *semi-algebraic structure* on X is a locally semi-algebraic space $(Y, \mathscr{O}_Y)$ and a homeomorphism θ from X to Y considered with its strong topology. A *morphism* of semi-algebraic structures $(Y_1, \mathscr{O}_{Y_1}, \theta_1) \to (Y_2, \mathscr{O}_{Y_2}, \theta_2)$ is a morphism of locally semi-algebraic spaces $f : Y_1 \to Y_2$ such that $\theta_2 = f \circ \theta_1$.

DEFINITION 2.68. Let X be a topological space, and let $\mathbf{Y}_1 = (Y_1, \mathscr{O}_{Y_1}, \theta_1)$ and $\mathbf{Y}_2 = (Y_2, \mathscr{O}_{Y_2}, \theta_2)$ be semi-algebraic structures on X. $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are *homotopic* if there exists a homotopy $H : X \times [0,1] \to X$ such that $H_0 = 1_X$, $\theta_2 \circ H_1 = \theta_1$, and $\theta_2 \circ H_1 \circ \theta_1^{-1}$ is an isomorphism of locally semi-algebraic spaces. They are *isotopic* if H is an isotopy, that is, if each H_t is a homeomorphism.

Chapter 3

Piecewise linear topology

In this section, we recall the foundations of piecewise linear (PL) topology. We will need very few facts from PL topology, mostly basic definitions. All of this material is standard and can be found in sources such as [**22**], [**32**], [**37**], [**43**], and [**50**].

Section 1 recalls the basic definitions of simplicial topology, and section 2 recalls some basic facts about the topology of simplicial complexes. In section 4, we prove a refined version of a well-known theorem that a subdivision of a subcomplex can be extended to a subdivision of the entire complex. In section 5, we prove that differences of simplicial complexes are almost simplicial complexes.

1. Simplicial complexes

DEFINITION 3.1 ([**17**, 1.1, 2.3, 4.1] and [**32**, §1]). Let $p_1,\dots,p_k$ be a finite set of points in $\mathbf{R}^N$. $\{p_1,\dots,p_k\}$ is called *affinely independent* if, for each j, the set $\{p_i - p_j \mid i \neq j\}$ is linearly independent. The *convex hull* of $p_1,\dots,p_k$ is

$$\mathrm{Conv}(p_1,\dots,p_k) = \{\alpha_1 p_1 + \cdots + \alpha_k p_k \mid \alpha_1 + \cdots + \alpha_k = 1,\ \alpha_1,\dots,\alpha_k \geq 0\}. \tag{1.1}$$

The convex hull of $n+1$ affinely independent points is called a *(closed) n-simplex*. n is the *dimension* of the simplex. The numbers $\alpha_1,\dots,\alpha_k$ are are *barycentric coordinates* on the simplex.

According to this definition, the empty set is a simplex of dimension -1.

DEFINITION 3.2 ([**38**, p. 1]). Let S and T be two simplices in $\mathbf{R}^N$. The *join* of S and T, denoted $S \cdot T$, is the convex hull of S and T.

If $S = \mathrm{Conv}(s_1,\dots,s_{k+1})$, $T = \mathrm{Conv}(t_1,\dots,t_{\ell+1})$, and if $\{s_1,\dots,s_{k+1},t_1,\dots,t_{\ell+1}\}$ is an affinely independent set, then $S \cdot T$ is a $k+\ell+1$-simplex.

DEFINITION 3.3 ([**17**, 4.1]). A *face* of a simplex $S = \mathrm{Conv}(s_1,\ldots,s_{k+1})$ is the simplex formed as the convex hull of a subset of $\{s_1,\ldots,s_{k+1}\}$. The *dimension* of a face is its dimension as a simplex. A zero-dimensional face is called a *vertex*.

The empty set is a face of every simplex. Every n-simplex has a unique n-dimensional face, namely the simplex itself.

DEFINITION 3.4 ([**32**, §1]). The *interior* of S is the set S° of all points not contained in a proper face of S. A set which is the interior of a simplex is an *open simplex*.

Note that the interior of a simplex is not necessarily the same as its interior in the sense of point-set topology.

A simplex is a semi-algebraic set, so as above there is a notion of a semi-algebraic morphism on a simplex. Semi-algebraic morphisms are not very natural for convex geometry, where the right notion is an affine transformation. If $p \in \mathbf{R}^N$, we denote the translation $x \mapsto x+p$ by $t_p : \mathbf{R}^N \to \mathbf{R}^N$.

DEFINITION 3.5. An *affine transformation* $f : \mathbf{R}^n \to \mathbf{R}^m$ is a composite $t_p \circ T$, where $T : \mathbf{R}^n \to \mathbf{R}^m$ is a linear transformation and $t_p : \mathbf{R}^m \to \mathbf{R}^m$ is a translation. An *affine isomorphism* is an affine transformation which admits an inverse which is also an affine transformation.

LEMMA 3.6. *The composite of any two affine transformations is an affine transformation.*

PROOF. $(t_q \circ U) \circ (t_p \circ T) = t_{q+U(p)} \circ U \circ T$. □

LEMMA 3.7. *An affine transformation is an affine isomorphism if and only if it is bijective.*

PROOF. Let $f : \mathbf{R}^n \to \mathbf{R}^m$ be the affine transformation. $f = t_p \circ g$ for some translation t_p and some linear transformation g. t_p is always both an isomorphism and bijective, so f is an isomorphism or is bijective if and only if $g = t_p^{-1} \circ f$ is. Finally, g is an isomorphism if and only if it is bijective because it is a linear transformation. □

LEMMA 3.8. *Let $p_1,\ldots,p_{n+1}$ be a set of affinely independent points in $\mathbf{R}^n$, and let $q_1,\ldots,q_{n+1}$ be any set of points in $\mathbf{R}^m$. Then there is a unique affine transformation $f : \mathbf{R}^n \to \mathbf{R}^m$ that sends p_i to q_i for all i.*

PROOF. The set $p_1 - p_{n+1}, \dots, p_n - p_{n+1}$ is linearly independent, so there is a unique linear transformation $g\colon \mathbf{R}^n \to \mathbf{R}^m$ that sends $p_i - p_{n+1} \mapsto q_i - q_{n+1}$ for all $1 \le i \le n$. We define $f(p) = g(p) + q_{n+1} - g(p_{n+1})$. Then

$$f(p_i) = g(p_i - p_{n+1}) + g(p_{n+1}) + q_{n+1} - g(p_{n+1}) = q_i - q_{n+1} + q_{n+1} = q_i$$

for $1 \le i \le n$, and

$$f(p_{n+1}) = g(p_{n+1}) + q_{n+1} - g(p_{n+1}) = q_{n+1}.$$

If f' is a second affine transformation sending $p_i \mapsto q_i$, then $f'(p_i) - f'(p_{n+1}) = q_i - q_{n+1}$, so, by the uniqueness of g, f' is also a translate of g. It follows that $f' = f$. $\square$

DEFINITION 3.9. If $S \subseteq \mathbf{R}^n$ is a simplex, an *affine morphism* $S \to \mathbf{R}^m$ is the restriction of an affine transformation $\mathbf{R}^n \to \mathbf{R}^m$. If $T \subseteq \mathbf{R}^m$ is a simplex, then an *affine morphism* $S \to T$ is an affine transformation $S \to \mathbf{R}^m$ whose image is in T. The affine morphism is *simplicial* if the image of every face of S is a face of T.

By lemma 3.8, the affine morphisms on S are independent of the ambient space containing S. They depend only on the images of the vertices of S. Notice that a morphism is simplicial if and only if the image of each vertex is a vertex.

DEFINITION 3.10 ([**32**, §2]). Let J be a set, and let $E^J \subseteq \mathbf{R}^J$ be the set of all J-tuples having only finitely many non-zero entries. That is, E^J has a basis whose members are the J-tuples having a 1 in a single position and zeroes in all other positions. Give E^J the topology induced by the $\|\cdot\|_\infty$ norm. A *simplicial complex* C in E^J is a set of simplices such that every face of a simplex of C is in C and the intersection of any two simplices of C is a face of each of them. A face is *maximal* if it is not contained in any strictly larger face. The *dimension* of C is the supremum of the dimensions of its simplices. The *k-skeleton* of C is the subcomplex C_k of all simplices of dimension at most k. C is *finite* if its set of faces is finite. If $x \in C$, then C is *locally finite at* x if x is contained in at most finitely faces. C is *locally finite* if it is locally finite at every point. A *subcomplex* of C is a simplicial complex D such that D is a subset of C.

We will usually write C for both the simplicial complex C and the union of the simplices of C.

DEFINITION 3.11. An *affine morphism* (resp. *locally semi-algebraic morphism*) from a simplicial complex C to a simplicial complex D is a continuous function $\phi: C \to D$ whose restriction to every simplex of C is an affine morphism (resp. locally semi-algebraic morphism). An affine or semi-algebraic morphism is *simplicial* if the image of every simplex of C is simplex of D, that is, if its restriction to each simplex of C is simplicial.

All our simplicial complexes will be finite dimensional. We will assume this without further comment.

The empty set is a face of every simplicial complex. If S is a simplex in $\mathbf{R}^N$, then S is a simplicial complex with simplex set equal to the set of all faces of S. Any union of simplices in $\mathbf{R}^N$ which meet along faces determines a simplicial complex.

DEFINITION 3.12. Let C be a simplicial complex. The *star* of a subset X of C is the subcomplex $\operatorname{Star} X$ generated by the simplices containing X. Equivalently, it is the subcomplex of simplices contained in a simplex meeting X. The *open star* of X is the topological space $\operatorname{Star}^\circ X$ formed by the union of all open simplices whose closures meet X. The *link* of a subset X is the subcomplex $\operatorname{Link} X$ of $\operatorname{Star} X$ whose simplices do not meet X. The *boundary* of a simplex S is the subcomplex ∂S of all proper faces of S. (Note that this is not the same as the boundary in the sense of homological algebra because the faces of ∂S are not oriented.)

Before continuing, we prove some elementary lemmas.

LEMMA 3.13. *Let C be a simplicial complex. Let $\mathscr{S}_C$ be the subcategory of the category of simplicial complexes and simplicial morphisms whose objects are the simplices of C and whose morphisms are the inclusions among the simplices of C. Then C is a colimit of $\mathscr{S}_C$.*

PROOF. Let D be a simplicial complex, and suppose that we have a morphism $C \to D$. By restriction we get a morphism $S \to D$ for every simplex S of C, and these morphisms commute with inclusions. Conversely, given morphisms $S \to D$ which commute with inclusions, we define $f: C \to D$ to equal $S \to D$ on the simplex S. This is well-defined by the commutativity assumption, and it is affine because it is affine on each simplex by assumption. Clearly these two processes are inverse, so if we can check that f is continuous, then $C = \varinjlim \mathscr{S}_C$.

To see that f is continuous, note that C is contained in some E^J, and E^J has the colimit topology with respect to its finite-dimensional subspaces. Consequently, to check that a function is continuous, it suffices to check that it is continuous on each finite dimensional subspace. The intersection of C with a finite dimensional subspace is a finite subcomplex, so checking that the function to D is continuous amounts to checking that a function on a finite complex is continuous if and only if it is continuous on each simplex. Therefore we may assume that C is a finite complex. Let $F \subseteq D$ be closed. Simplices are compact Hausdorff spaces, so $f^{-1}(F)$ is a finite union of compact Hausdorff spaces because it meets each simplex in a closed subset. In particular, it is compact, and hence closed in C. Therefore f is continuous. $\square$

COROLLARY 3.14. *Let C be a simplicial complex. A subset of C is open (resp. closed) if and only if its intersection with each simplex S of C is open (resp. closed) in S.*

PROOF. By lemma 3.13, C has the colimit topology with respect to its simplices, and by definition this gives the conclusion of the lemma. $\square$

This implies that subcomplexes are closed, and therefore stars are closed and determine canonical closed neighborhoods. Open stars are open because they are complements of subcomplexes, so they will determine canonical open neighborhoods. In a locally finite complex, stars will determine canonical compact neighborhoods by proposition 3.22.

LEMMA 3.15. *Let $f : C \to D$ be a bijective simplicial locally semi-algebraic morphism. Then f is a locally semi-algebraic isomorphism.*

PROOF. Let S be a simplex of D. $f^{-1}(S)$ is also a simplex because f is bijective. The graph of the restriction of f to $f^{-1}(S)$ is a semi-algebraic subset of $f^{-1}(S) \times S$. Consequently the graph of the restriction of f^{-1} to S is a semi-algebraic subset of $S \times f^{-1}(S)$ because it is cut out by the same equations and inequalities as those defining the graph of f but with the order of the variables changed. Therefore f^{-1} is a locally semi-algebraic morphism. f^{-1} is inverse to f by definition, so f is a locally semi-algebraic isomorphism. $\square$

LEMMA 3.16. *If $f : A \to B$ and $g : A \to C$ are simplicial affine injections, then there is a pushout $B \amalg_A C$.*

PROOF. Let $M = K \amalg_J L$ be the pushout of the vertex sets of B and C along A determined by f and g. In E^M there is a simplicial complex X whose vertices are the basis vectors of E^M and whose faces correspond to the faces of K and of L. That is, basis vectors $e_1, \ldots, e_n$ span a face if and only if they correspond to vertices of K spanning a face of K or vertices of L spanning a face of L. It is clear that X is a simplicial complex. Furthermore, it is a pushout of B and C along A. The natural morphisms from B and C to X send a vertex to the basis vector corresponding to that vertex. Given simplicial morphisms from B and C to a simplicial complex D which determine the same simplicial morphism on A, then there is a corresponding simplicial morphism from X to D which sends a basis vector of E^M corresponding to a vertex of B to the image of that vertex of B and similarly for basis vectors corresponding to vertices of C. It is clear that a morphism from X to D induces morphisms from B and C to D which agree on A, so X is a pushout. □

LEMMA 3.17. *Direct limits of simplicial affine injections exist. That is, if I is a well-ordered index set such that for each $i \in I$ there is a simplicial complex A_i and for each $i \leq j$ there is a simplicial affine injection $A_i \to A_j$, then $\varinjlim_{i \in I} A_i$ exists.*

PROOF. For each i, let V_i be the set of vertices of A_i. The simplicial affine injections $A_i \to A_j$ determine injections $V_i \to V_j$ for all i and j. Let V be the inductive limit of these injections of sets. For each $v \in V$, denote the corresponding basis vector of E^V by e_v. We construct a simplicial complex X in E^V by letting $e_{v_1}, \ldots, e_{v_k}$ span a face if and only if $v_1, \ldots, v_k$ are the image of a collection of vertices in some V_{i_0} which span a face in A_{i_0}. It is clear that X is a simplicial complex. Furthermore, it is the direct limit of the A_i. Given compatible simplicial morphisms from the A_i to a simplicial complex B, there is a simplicial morphism from X to B determined by sending a vertex of X to the image of a vertex of some A_i which corresponds to that vertex of X. The images of the other points are determined by linear interpolation. Each simplex, being contained in a finite dimensional subspace of E^V, has its usual topology, so this map is continuous on each simplex and hence is continuous. Conversely, it is clear that a morphism from X to B induces compatible morphisms from each A_i to B, so X is the direct limit. □

2. Topology of simplicial complexes

Recall that a space X is *normal* if points in X are closed and if for every two closed sets A and B of X, there exist disjoint open sets U and V such that U is a neighborhood of A and V is a neighborhood of B.

LEMMA 3.18. *Let C be a simplicial complex. Then C is normal. In particular, C is Hausdorff.*

PROOF. Every simplex is closed, and points of simplices are closed, so points of C are closed.

We prove an equivalent formulation of normality: If A is a closed set and U is an open neighborhood of A, then there exists an open neighborhood V of A such that $\overline{V} \subseteq U$. We do this by induction on dimension. We will construct for each dimension d an open neighborhood V_d of $A \cap C_d$ in C_d such that $\overline{V_d} \subseteq U \cap C_d$.

If $d = 0$, then $A \cap C_d$ is a set of points, and we let $V_d = A \cap C_d$. Assume inductively that we have constructed V_{d-1}, and let S be a simplex of dimension d. S is a simplex, so it has a metric. For each point p of $A \cap S^\circ$, choose $\epsilon_p > 0$ such that $B(p, 2\epsilon_p)$, the ball of radius $2\epsilon_p$ around p, is contained in $S^\circ \cap U$. For each point p of $V_{d-1} \cap S$, choose $\epsilon_p > 0$ such that $B(p, \epsilon_p) \cap C_{d-1} \subseteq V_{d-1}$. Let $V_d \cap S = \bigcup_{p \in (V_{d-1} \cap S) \cup (A \cap S^\circ)} B(p, \epsilon_p)$. $V_d \cap S$ is open in S. Its closure is contained in $U \cap S$, and its intersection with ∂S is $V_{d-1} \cap \partial S$. We let V_d be the union of all $V_d \cap S$. V_d has the required properties because it has these properties in each simplex.

Finally, we let $V = \bigcup V_d$. □

PROPOSITION 3.19. *Let C be a simplicial complex and x a point of C. The following are equivalent:*

(1) C is locally finite at x.

(2) x admits an open neighborhood which meets only finitely many simplices.

(3) The star of x contains at most finitely many simplices.

The following are also equivalent:

(1) C is locally finite.

(2) Every point of C admits an open neighborhood which meets only finitely many simplices.

(3) The star of every point of C contains at most finitely many simplices.

(4) Every vertex of C is contained in at most finitely many simplices.

(5) Every vertex of C admits an open neighborhood which meets only finitely many simplices.

(6) The star of every vertex of C contains at most finitely many simplices.

PROOF. The star of x is the set of all simplices contained in a simplex containing x. If C is locally finite at x, then x is contained in at most finitely many simplices, and because each of these simplices has finitely many faces, the star of x is a finite complex. The open star of x is an open neighborhood of x contained in the star of x, so if the star of x contains at most finitely many simplices, then x admits an open neighborhood meeting only finitely many simplices. Finally, if x admits an open neighborhood meeting only finitely many simplices, then in particular at most finitely many simplices meet x, so C is locally finite at x.

The open stars of vertices cover C. This, together with the previous paragraph, implies the remaining equivalences. □

COROLLARY 3.20. *The set of all points at which C is locally finite is open.* □

Next, we will describe the relationship between various kinds of finiteness and compactness.

PROPOSITION 3.21 ([**32**, Chapter 1, 2.5]). *Let C be a simplicial complex, and let $F \subset C$ be closed. Then F is compact if and only if F is contained in a finite subcomplex of C.*

PROOF. Every simplex S is a compact Hausdorff space, so every $F \cap S$ is compact. If F meets only finitely many simplices of C, then F is a finite union of compact sets and hence is compact.

Conversely, assume that F is compact and not contained in a finite subcomplex of C. For each simplex S of C, choose a point $p_S \in F \cap S^\circ$ if the intersection is non-empty. Let $G = \{p_S \mid F \cap S^\circ \neq \emptyset\}$. Every subset of G is closed because its intersection with every simplex is finite. Consequently G is discrete. But G is an infinite subset of the compact set F, so this is a contradiction. □

PROPOSITION 3.22 ([**32**, Chapter 1, 2.5 and 2.6]). *Let C be a simplicial complex.*

(1) C is finite if and only if C is compact.

(2) C is locally finite at x if and only if C is locally compact at x.

(3) C is locally finite if and only if C is locally compact.

PROOF. For the first statement, apply proposition 3.21 with $F = C$.

For the second statement, if C is locally finite at x, then the star of x is compact by proposition 3.21 and contains an open neighborhood of x, the open star of x. Therefore C is locally compact at x. Conversely, assume that C is not locally finite at x. If C is locally compact, then there is a compact subset K of C containing an open neighborhood U of x. Because C is not locally finite at v, proposition 3.19 implies that U meets infinitely many simplices of C, and hence K does also. But by compactness of K and lemma 3.18, K is closed, so by proposition 3.21 with $F = K$, K meets only finitely many simplices. This is a contradiction.

The third statement follows immediately from the second. □

We will need to know the relationship between metric topologies on a simplicial complex and the usual topology. This is useful because locally semi-algebraic spaces are locally metrizable, so the proposition implies that a triangulable locally semi-algebraic space is locally compact.

PROPOSITION 3.23. [**32**, Chapter 1, exercise 2.7] *Let C be a simplicial complex, and let d be a metric on C. Suppose that for every simplex S of C, the metric topology on S induced by d is the same as the subspace topology on S.*

(1) If (C,d) denotes C with the metric topology, then the map $i: C \to (C,d)$ which is the identity on points is continuous.

(2) i is a homeomorphism on the open subset U of C consisting of all points at which C is locally compact.

(3) i is not a homeomorphism on any open subset of C not contained in U.

In particular, if C is metrizable, then C is locally finite.

PROOF. Let $B(x,\epsilon)$ be an open ball with respect to d. For every simplex S, $B(x,\epsilon) \cap S$ is open in S by assumption, and therefore $B(x,\epsilon)$ is open in C. Consequently i is continuous.

Suppose that $F \subseteq U$ is a closed set. For each simplex S, $S \cap F$ is closed in the metric topology. $F = \bigcup_S S \cap F$, and since $F \subseteq U$, this union is locally finite. Therefore F is closed in the metric topology, and consequently i is a homeomorphism on U.

Let V be an open subset of C not contained in U, and let $x \in V \setminus U$. Let $\mathscr{M}$ be the set of all minimal faces of C meeting x, that is, the set of all faces of C which properly contain x and each of whose subfaces do not properly contain x. Every face of C containing x contains a face of $\mathscr{M}$. If

$\mathscr{M}$ were finite, then there would be at most finitely many vertices in the star of x, so C would be locally finite at x. This contradicts our choice of x, so instead we may choose countably many distinct simplices $S_1, S_2, \ldots$ in $\mathscr{M}$. Let $W_i = C \setminus \{y \in S_i \mid d(x,y) \geq 1/i\}$. W_i is an open neighborhood of x, and it contains no ball around x of radius greater than $1/i$. $W = V \cap \bigcap_i W_i$ contains no ball around x of radius greater than $1/i$ for every i, so W contains no ball and hence is not metrically open. But W is open in the simplicial complex topology. To see this, let S be a simplex. S meets at most finitely many members of $\mathscr{M}$, so $W \cap S$ is a finite intersection of open sets and is therefore open. Consequently W is open in C. Therefore i is not open on V, so it is not a homeomorphism.

The last claim follows from proposition 3.22. □

Using these facts, we can prove that a simplicial complex is a locally semi-algebraic space if and only if it is locally finite.

DEFINITION 3.24. Let $S \subseteq \mathbf{R}^n$ be a simplex. The *natural semi-algebraic structure* on S is the semi-algebraic structure determined by S as a semi-algebraic set, that is, it is the semi-algebraic structure determined by the affine semi-algebraic space associated to S.

Note that the semi-algebraic structure on S is independent of n and the embedding of S.

PROPOSITION 3.25. *Let C be a simplicial complex. Then the following are equivalent.*

(1) C is locally finite.
(2) C admits a semi-algebraic structure, called the natural semi-algebraic structure *on C, which on each simplex of C restricts to the natural semi-algebraic structure on S, and for which the collection of all open stars of vertices is an admissible covering.*

PROOF. If C is not locally finite at x, then by propositions 3.22 and 3.23, it is not locally metrizable at x. Lemma 2.60 therefore implies that in any neighborhood of x, C is not a locally semi-algebraic space.

Conversely, assume that C is locally finite and that v is a vertex of C. $\operatorname{Star} v$ is a finite simplicial complex, so it, and hence $\operatorname{Star}^\circ v$, can be embedded in Euclidean space as a semi-algebraic set. This determines a semi-algebraic structure on $\operatorname{Star}^\circ v$ which on each simplex restricts to the natural semi-algebraic structure. Now we apply lemma 2.44 to the collection of all open stars and their overlaps in C. □

Now we have two notions of a locally semi-algebraic morphism on a locally finite simplicial complex, one coming from the semi-algebraic structure and the other coming from definition 3.11. A locally semi-algebraic morphism in the simplicial complex sense determines a locally semi-algebraic morphism in the locally semi-algebraic space sense because the morphism restricts to a locally semi-algebraic morphism on each open star, locally semi-algebraic morphisms can be glued with respect to admissible covers, and the morphism of sheaves can be given by pullback. Conversely, a locally semi-algebraic morphism in the locally semi-algebraic space sense determines a locally semi-algebraic morphism in the simplicial complex sense because its restriction to the open star of a simplex is a semi-algebraic morphism. These two determinations are inverse because the morphism of sheaves is always given by pullback. Therefore the two notions agree.

PROPOSITION 3.26. *Let C be a locally finite simplicial complex. Then C is paracompact.*

PROOF. Let the collection of open stars of vertices of C be denoted $\{S_v\}_{v\in V}$. Each S_v is a paracompact space, so C has a locally finite covering by paracompact spaces. If $\{U_\alpha\}_{\alpha\in A}$ is an open cover of C, then $\{U_\alpha \cap S_v\}_{\alpha\in A}$ is an open cover of S_v for each v. Therefore it admits a locally finite refinement $\{V_{v\beta}\}_{\beta\in B_v}$. The collection $\{V_{v\beta}\}_{v\in V,\beta\in B_v}$ is then a locally finite refinement of $\{U_\alpha\}$. Consequently C is paracompact. □

3. Subdivisions and piecewise linear structures

DEFINITION 3.27. Let $s\colon D \to C$ be a continuous function between two simplicial complexes. Assume that s is a homeomorphism and that for every simplex T of D, there is a simplex S of C such that $s(T) \subseteq S$. If s is affine on every simplex of C, then s is an *affine subdivision* or a *subdivision* for short. If s is a semi-algebraic morphism of semi-algebraic sets on every simplex of C, then s is a *semi-algebraic subdivision*.

All affine subdivisions are also semi-algebraic subdivisions. An affine subdivision is the same as a homeomorphism which is also an affine morphism. Note that "subdivision" without any qualifier will always mean affine subdivision.

DEFINITION 3.28. Let X be a topological space. A *triangulation* of X is a homeomorphism $t\colon C \to X$, where C is a simplicial complex. If D is another simplicial complex and $r\colon D \to C$ is an affine subdivision, then we say that tr is an *affine subdivision* of t. Affine subdivision generates an equivalence relation whose equivalence classes are called *affine PL structures* on X. A topological space with a choice of affine PL structure is an *affine PL space*. If X and Y are affine PL spaces, then a function $f: X \to Y$ is *affine PL* if it is continuous and if for some representatives $t: C \to X$ and $u: D \to Y$ of the affine PL structures on X and Y, there is an affine map $\phi: C \to D$ such that $f = u \circ \phi \circ t^{-1}$. The function is *affine simplicial* if C, D, and ϕ can be chosen so that ϕ is simplicial.

The definition of a semi-algebraic PL structure is parallel to the definition of an affine PL structure.

DEFINITION 3.29. Let $t: C \to X$ be a triangulation. If D is another simplicial complex and $r: D \to C$ is a semi-algebraic subdivision, then we say that tr is a *semi-algebraic subdivision* of t. Semi-algebraic subdivision generates an equivalence relation whose equivalence classes are called *semi-algebraic PL structures* on X. A topological space with a choice of semi-algebraic PL structure is a *semi-algebraic PL space*. If X and Y are semi-algebraic PL spaces, then a function $f: X \to Y$ is *semi-algebraic PL* if it is continuous and if for some representatives $t: C \to X$ and $u: D \to Y$ of the semi-algebraic PL structures on X and Y, there is a locally semi-algebraic morphism $\phi: C \to D$ such that $f = u \circ \phi \circ t^{-1}$. The function is *semi-algebraic simplicial* if C, D, and ϕ can be chosen so that ϕ is simplicial.

THEOREM 3.30. *Let X be a locally semi-algebraic space that admits a triangulation. Then X is Hausdorff, paracompact, and locally compact.*

PROOF. Let $t: C \to X$ be the triangulation. By lemma 3.18, all simplicial complexes are Hausdorff, so if X admits a triangulation then it is also Hausdorff. The pullback of the structure sheaf of X to C makes C a locally semi-algebraic space, so by proposition 3.25, C is locally finite. Therefore propositions 3.22 and 3.26 imply that C, and hence X, is locally compact and paracompact. □

Finally, when we are working with locally semi-algebraic spaces, it will be useful to consider only subdivisions in which all the simplices are semi-algebraic subspaces. By the previous theorem, we may always assume that C is a locally finite simplicial complex, and so proposition 3.25 implies that C is naturally a locally semi-algebraic space.

DEFINITION 3.31. Let X be a locally semi-algebraic space, and let $t\colon C \to X$ be a triangulation. Give C its natural semi-algebraic structure. If t is a locally semi-algebraic morphism (so that the pair $(t, t^{\#})$, where $t^{\#}$ is pullback by t, is a morphism of locally semi-algebraic spaces), then t is a *semi-algebraic triangulation* of X.

4. Extension of subdivisions

In theorem 4, we will prove that, given a subdivision of a subcomplex, we can find a subdivision of the whole complex which, on the subcomplex, is the given subdivision. Furthermore, this can be done in a way which does not change the simplices of the larger complex except where they meet the subcomplex. We are aware of two similar results in the literature. Zeeman [**50**, Exposé 1, Lemma 3, (ii)] states a version of this theorem without claiming functoriality or non-subdivision of simplices not meeting the subcomplex. Delfs and Knebusch [**10**, Chapter II, Lemma 4.3] prove a special case of the theorem for triangulations of certain locally semi-algebraic spaces. They prove non-subdivision of simplices, but not functoriality. The primary difference between theorem 4 and these results is that in our approach, the new simplicial complex is assembled in a canonical way using functoriality, whereas in the approaches of Zeeman and of Delfs and Knebusch, the new simplicial complex is constructed inductively and in a non-canonical way.

Before proving the theorem, we first give a characterization of subdivisions, and we use this to describe a particular way of subdividing a simplex. Note that the preimage of a subcomplex under an affine morphism is a subcomplex of the domain.

PROPOSITION 3.32. *Let $s\colon D \to C$ be a bijective semi-algebraic morphism between two simplicial complexes. Then s is a semi-algebraic subdivision if and only if for every simplex S of C, the subcomplex $s^{-1}(S) \subseteq D$ is finite. In particular, if s is a semi-algebraic subdivision, then C is finite (resp. locally finite) if and only if D is also finite (resp. locally finite).*

PROOF. Suppose that s is a semi-algebraic subdivision. Then $s^{-1}(S)$ is a subcomplex of D homeomorphic to S. S is compact, so $s^{-1}(S)$ is also compact. Therefore by proposition 3.22 it is a finite complex.

Conversely, if s is bijective and semi-algebraic, it suffices to show that s is homeomorphism. For this, it suffices to show that each map $s^{-1}(S) \to S$ is closed. If $F \subseteq s^{-1}(S)$ is closed and $D \subseteq s^{-1}(S)$ is a simplex, then $s(D\cap F)$ is closed because $D\cap F$ is compact. Since $s^{-1}(S)$ is finite, this implies that $s(F)$ is closed. □

DEFINITION 3.33. Let C be a simplicial complex in E^J. The *abstract cone* on C, denoted $\operatorname{Cone} C$, is the simplicial complex in $E^{J\cup\{*\}}$ whose simplices are the simplices of C, the simplex $\{e_*\}$ whose only point is the basis vector corresponding to element $*$ of $J\cup\{*\}$, and the joins $S\cdot e_*$, where S is a simplex of C. The vertex e_* is called the *apex* of the cone.

LEMMA 3.34. *Let C be a simplex in $\mathbf{R}^n$, and let $s\colon D\to\partial C$ be a semi-algebraic subdivision. If $p\in C^\circ$, then there is a semi-algebraic subdivision $\hat{s}\colon \operatorname{Cone} D\to C$ that sends the apex to p and restricts to s on D.*

PROOF. We construct $\hat{s}$ as follows. On D, $\hat{s}=s$. We set $\hat{s}(e_*)=p$. Finally, $\hat{s}$ linearly interpolates the remaining points. That is, it sends $\alpha x+(1-\alpha)e_*$ to $\alpha s(x)+(1-\alpha)p$. We will show that $\hat{s}$ is a subdivision using proposition 3.32.

Our first step in showing that $\hat{s}$ is a subdivision is to show that its image is C. We may assume that $n=\dim C$. $s(D)\subseteq C$ and $p\in C^\circ$, so the convexity of C implies that $\hat{s}(\operatorname{Cone} D)\subseteq C$. For the reverse inclusion, first note that C° is open and convex, so it is a connected component of $\mathbf{R}^n\setminus\partial C$. If $q\in C\setminus\{p\}$, then the ray starting at p and passing through q meets both C° and $\mathbf{R}^n\setminus C$ by boundedness of C, so the ray must pass through ∂C at some point r. Because the ray is one-dimensional and $p\in C^\circ$, r is unique. There is a unique minimal simplex S of ∂C containing r, and $q\in S\cdot p$. Therefore $q\in\hat{s}(\operatorname{Cone} D)$, so $\hat{s}(\operatorname{Cone} D)=C$.

Next we will show that $\hat{s}$ is injective. Suppose that there are two points of $\operatorname{Cone} D$ which map to the same point of C. Write these points as $\alpha x+(1-\alpha)e_*$ and $\beta y+(1-\beta)e_*$. Construct the rays starting at p and passing through the images of these two points. These rays meet ∂C at the same point r, so r is the image under $\hat{s}$ of both x and of y. Since s is injective, we find that $x=y$. Since $p\in C^\circ$ is not contained in $s(D)$, p is affinely independent of the image of the simplex of D containing x, and consequently $\alpha=\beta$. Therefore $\hat{s}$ is injective, hence bijective.

It is clear that $\hat{s}$ is semi-algebraic from the formula given above. Finally, note that ∂C is a finite complex, so D is a finite complex by proposition 3.32, and hence $\operatorname{Cone} D$ is a finite complex. Therefore another application of proposition 3.32 implies that $\hat{s}$ is a semi-algebraic subdivision. □

The previous construction will be most important when p is the barycenter (that is, the center of gravity) of the simplex.

DEFINITION 3.35. Let $S = \mathrm{Conv}(v_1, \ldots, v_r)$ be a simplex in $\mathbf{R}^N$. The *barycenter* of S is the point $b = \frac{1}{r}\sum_{i=1}^{r} v_i$.

The barycenter of S is always in S. If $f\colon S \to \mathbf{R}^n$ is an affine injection, then f sends the barycenter of S to the barycenter of $f(S)$. This implies that the process of finding the barycenter is functorial with respect to affine injections.

THEOREM 3.36. *Let C and E be simplicial complexes, and let $j\colon E \to C$ be an injective simplicial morphism. Assume that we have a semi-algebraic subdivision $s\colon F \to E$. Then there is a simplicial complex D, an injective simplicial morphism $i\colon F \to D$, and a semi-algebraic subdivision $t\colon D \to C$ such that the diagram*

$$\begin{tikzcd} F \arrow[r, dotted, "i"] \arrow[d, "s"'] & D \arrow[d, dotted, "t"] \\ E \arrow[r, "j"'] & C \end{tikzcd} \tag{4.1}$$

commutes. Also, if S is a simplex of C which does not meet $j(E)$, then $t^{-1}(S)$ is a simplex of D. (That is, t does not subdivide simplices which do not meet $j(E)$.)

Furthermore, injective simplicial affine morphisms of the initial data induce compatible morphisms of the subdivision D. That is, suppose that C' and E' are simplicial complexes, $j'\colon E' \to C'$ is an injective simplicial morphism, $s'\colon F' \to E'$ is a semi-algebraic subdivision, and D', $i'\colon F' \to D'$ and $t'\colon D' \to C'$ are the simplicial complex, simplicial morphism, and semi-algebraic subdivision constructed by the first part of the theorem for C', E', F', j', and s'. If $f\colon C' \to C$, $g\colon E' \to E$, and $h\colon F' \to F$ are injective simplicial affine morphisms such that $fj' = jg$, $sh = gs'$, and $E' = f^{-1}(E)$, then there exists an injective simplicial morphism $k\colon D' \to D$ such that $ki' = ih$ and $tk = ft'$. In other words, if the *diagram*

(4.2)
$$
\begin{tikzcd}
F \arrow[rrr, "i"] \arrow[ddd, "s"'] & & & D \arrow[ddd, "t"] \\
& F' \arrow[ul, "h"'] \arrow[r, "i'"] \arrow[d, "s'"'] & D' \arrow[ur, dotted, "k"'] \arrow[d, "t'"] & \\
& E' \arrow[dl, "g"] \arrow[r, "j'"'] \arrow[r, phantom, "\square"', shift right=4ex] & C' \arrow[dr, "f"'] & \\
E \arrow[rrr, "j"'] & & & C
\end{tikzcd}
$$

commutes and the bottom trapezoid is cartesian, then k exists and makes the diagram commute.

Finally, the morphism k is functorial.

PROOF. We work by induction on the dimension of C. We break down the proof in each dimension into two steps. First, we prove the existence of the diagram (4.1) and functoriality for all simplices. Second, we use functoriality to extend this to simplicial complexes.

Suppose that C is a simplex of dimension d. If j is an isomorphism, then we take $D = F$, $i = 1_F$, and $t = js$. If $E = \emptyset$, then we take $D = C$, $i = \emptyset$, and $t = 1_C$. If $d = 0$, these are the only cases, so this completes the $d = 0$ case.

Otherwise, $d > 0$ and E is contained in ∂C. Because ∂C is $d-1$-dimensional, we may apply the $d-1$-dimensional case of the theorem to the diagram

(4.3)
$$
\begin{tikzcd}
F \arrow[r, dotted, "\partial i"] \arrow[d, "s"'] & \partial D \arrow[d, dotted, "\partial t"] \\
E \arrow[r, "j"'] & \partial C
\end{tikzcd}
$$

The $d-1$-dimensional case produces a complex and two morphisms that fit into the upper right-hand corner of the diagram (4.3). Call the complex ∂D and the two morphisms $\partial i\colon F \to \partial D$ and $\partial t\colon \partial D \to \partial C$. By induction, ∂i is an injective simplicial morphism, ∂t is a semi-algebraic subdivision, ∂i and ∂t make (4.3) commute, and ∂D does not subdivide simplices which do not meet E.

Let b denote the barycenter of C. Using lemma 3.34 and the semi-algebraic subdivision ∂t of ∂C, we construct a semi-algebraic subdivision $t\colon D \to C$. D is $\operatorname{Cone}\partial D$, t restricts to ∂t on ∂D, and t sends the apex of D to b. We let i be the composite of ∂i and the inclusion of ∂D into D. This gives the diagram (4.1), and D, i, and t have the required properties.

Next we show that there is a functorial and simplicial morphism of the complexes D whenever there is a simplicial morphism of the initial data. Assume that C' is a simplicial complex of dimension $e \le d$, and suppose that we have morphisms $f: C' \to C$, $g: E' \to E$, and $h: F' \to F$ as in the statement. Because t and t' are homeomorphisms, we may take $k = t^{-1} f t'$. The formula immediately implies that k is injective and functorial. Furthermore,

$$ki' = t^{-1} f t' i' = t^{-1} f j' s' = t^{-1} j g s' = t^{-1} j s h = t^{-1} t i h = ih.$$

It remains to check that k is simplicial. Because t, f, and t' are simplicial morphisms, it suffices to check that the image of a simplex of D' in C is the image of a simplex of D. It further suffices to consider only the case where C' is a simplex. We do this by induction on e. If $e = 0$, then C' is a point, so k is trivially simplicial. We now assume inductively that k is simplicial whenever C' has dimension strictly less than e, and we consider all the possible cases for D and D'.

The first case is when $E = \emptyset$. Then $E' = \emptyset$, so t and t' are isomorphisms. Hence k is simplicial.

The second case is when j is an isomorphism. Then $D = F$, $i = 1_F$, and $t = js$, so $k = s^{-1} j^{-1} f t'$. j', being the pullback of an isomorphism, is also an isomorphism, and since t' and s' are semi-algebraic subdivisions, we deduce that i' is a bijection. By lemma 3.15, we find that i' is an isomorphism. Consequently, we have

$$k = s^{-1} g (j')^{-1} t' = h (s')^{-1} (j')^{-1} t' = h (i')^{-1},$$

so k is simplicial. This and the previous case together complete the $d = 0$ case.

The remaining case is when $d > 0$ and E is a non-empty proper subcomplex of C, so $D = \operatorname{Cone}\partial D$. Again we consider cases. If $E' = \emptyset$, then $f(C') \cap j(E) = \emptyset$, so t does not subdivide $f(C')$. Therefore $(t')^{-1}|_{f(C')}$ is a simplicial isomorphism, and hence k is simplicial. If j' is an isomorphism, then i' is an isomorphism, so $k = ih(i')^{-1}$ is simplicial. Finally, suppose $D' = \operatorname{Cone}\partial D'$. Because f is an affine injection, it preserves barycenters, that is, the image of the barycenter b' of C' is the barycenter b of C. Restricting k to $\partial D'$ gives a simplicial morphism by induction. Consequently, if S' is a simplex of $\partial D'$ and $k(S') = S$ for some simplex S of ∂D, then $(ft')(S' \cdot b') = t(S \cdot b)$, and hence $k(S' \cdot b') = S \cdot b$. Therefore k is simplicial.

This completes all of the cases in all dimensions. Induction now implies that the statement is true for any morphism from a simplicial complex of dimension at most d to a simplex of dimension at most d. This concludes the case of simplices.

Now assume that C is a simplicial complex. Consider the collection of all simplices in C and all inclusions among these simplices. By lemma 3.13, C is the colimit of this diagram of objects and morphisms. For each simplex S, we let $E_S = j^{-1}(S)$ and $F_S = s^{-1}(E_S)$. Just as with C, E is the colimit of the diagram of all E_S and their inclusions, and F is the colimit of the diagram of all F_S and their inclusions. We apply the above construction to each $F_S \to E_S \to S$ and to each inclusion of diagrams coming from an inclusion of simplices of C. This produces simplicial complexes D_S, maps $i_S \colon F_S \to D_S$ and $t_S \colon D_S \to S$, and morphisms of the corresponding diagrams. Taking the colimit of these gives a complex D and morphisms i and t which give the desired diagram (4.1). The condition on not subdividing simplices of C which do not meet $j(E)$ is true because it is true for each simplex S. Finally, the existence of k is clear from the universal property of the colimit. Because it satisfies the formula $k = t^{-1}ft'$ on each simplex, it satisfies this formula on the entire complex. Functoriality is clear from the formula, and this completes the proof. □

5. Differences of simplicial complexes

We will show in this section that if $C_1 \supseteq C_2 \supseteq \cdots \supseteq C_k$ is a decreasing chain of simplicial complexes, then there is a simplicial complex whose underlying point set is $A = C_1 \setminus (C_2 \setminus \cdots (C_{k-1} \setminus C_k) \cdots)$. There is a natural morphism from this simplicial complex to A which is a homeomorphism on the set of points where A is locally compact and not a homeomorphism elsewhere.

The first step in doing this is to describe a special kind of subdivision.

PROPOSITION 3.37. *There is a unique functor, denoted* Bary *and called* relative barycentric subdivision*, and a unique natural transformation s having the following properties.*

(1) Bary *is a functor from the category $\mathscr{C}$ whose objects are pairs (C, D), where C is a simplicial complex and D is a subcomplex of C, and whose morphisms are affine morphisms preserving the subcomplex to the category of simplicial complexes and affine morphisms*

(2) *If U is the forgetful functor from $\mathscr{C}$ to the category of simplicial complexes that sends (C,D) to C, then s is a natural transformation from* Bary *to U,*

(3) *For every pair (C,D), $s\colon \mathrm{Bary}_D C \to C$ is the identity map on underlying topological spaces and a subdivision,*

(4) $\mathrm{Bary}_C C = C$,

(5) *If C is zero-dimensional, then* $\mathrm{Bary}_D C = C$,

(6) *If C is zero-dimensional and $f : (C,D) \to (C',D')$ is a morphism, then* $\mathrm{Bary} f : C \to \mathrm{Bary}_{D'} C'$ *equals f,*

(7) *If S is an n-dimensional simplex with barycenter b and $T \subseteq \partial S$, then* $\mathrm{Bary}_T S = \mathrm{Cone}_b(\mathrm{Bary}_T \partial S)$.

PROOF. Bary and s are defined by induction on dimension. For a zero-dimensional simplicial complex or a morphism whose source is a zero-dimensional complex, Bary and s are determined by the statement of the proposition. If Bary and s have been defined and are unique up to dimension $n-1$, then the statement of the proposition uniquely determines the value of Bary on a simplex of dimension n. The functoriality assumption then determines Bary uniquely on a complex of dimension n by lemma 3.13. Because an affine morphism of an n-simplex is determined by its values on its vertices, an affine morphism from a simplicial complex is determined by its values on its 0-skeleton, so Bary is determined on an affine morphism from an n-dimensional complex by the assumptions of the proposition. s is defined for an n-complex by the assumption that it is the identity on underlying topological spaces. □

LEMMA 3.38. *Let $C = C_1 \supseteq C_2 \supseteq \cdots \supseteq C_k$ be a decreasing chain of simplicial complexes. Let $A = C_1 \setminus (C_2 \setminus \cdots (C_{k-1} \setminus C_k) \cdots)$ and $B = \partial_{\partial_C A}(A \cap \partial_C A)$. Assume that C is locally finite. Then A is locally compact at every point of $A \setminus B$ and is not locally compact at any point of B.*

Rather than assuming that A has the specific form given above, it is equivalent to assume that A is a union of open simplices. However, in our applications, we will use only the specific form given above.

PROOF. C is locally finite, hence by proposition 3.22 it is locally compact. Therefore $\overline{A}$ is also locally compact. To show that $A \setminus B$ is locally compact, therefore, it suffices to show that its complement in $\overline{A}$, which is $B \cup (\overline{A} \setminus A)$, is closed. B itself is closed, since it is a boundary, so we need only show that if S is a simplex whose interior is contained in $\overline{A} \setminus A$, then S is contained in $B \cup (\overline{A} \setminus A)$. Clearly such an S is contained in $\partial_C A$, so every point of S not in $\overline{A} \setminus A$ must be in $\partial_{\partial_C A}(\overline{A} \setminus A) = B$. Therefore $S \subseteq B \cup (\overline{A} \setminus A)$, so $A \setminus B$ is locally compact.

Next, suppose that $x \in B$. We will show that A is not locally compact at x. We claim that there is a simplex S of C such that $x \in S$, $S^\circ \subseteq A^\circ$, and $S \nsubseteq A$. Suppose otherwise. Then if S is a simplex such that $x \in S$ and $S^\circ \subseteq A^\circ$, we must have $S \subseteq A$. A is a union of open simplices, so this implies that $A \cap \mathrm{Star}_C\, x$ is closed in $\mathrm{Star}_C\, x$. But then we would have $B = \partial\partial A = \emptyset$, contradicting the existence of x. Therefore S exists.

Every open neighborhood U of x in S meets S° because $x \in S$. Because B is a boundary, U also meets points of $\partial_C A$ not in $A \cap \partial_C A$. $A \cap \partial_C A$ is not equal to $\partial_C A$, because that would imply that A is closed and hence that B is empty, so U it meets points of C not in A. In particular, there is a sequence $\{y_i\}$ in $U \cap S^\circ$ which converges to a point of $S \setminus A$. If $A \cap S$ were locally compact, then x would admit a compact neighborhood, but then we could choose a $\{y_i\}$ in this compact neighborhood whose only limit point would be in $S \setminus A$, not in the compact neighborhood. This is impossible, so $A \cap S$ is not locally compact. □

THEOREM 3.39. *Let $C = C_1 \supseteq C_2 \supseteq \cdots \supseteq C_k$ be a decreasing chain of simplicial complexes. Let $A = C_1 \setminus (C_2 \setminus \cdots (C_{k-1} \setminus C_k) \cdots)$ and $B = \partial_{\partial_C A}(A \cap \partial_C A)$. Then there exists a simplicial complex D and an affine bijection $i : D \to A$ such that*

(1) i restricts to a homeomorphism $i^{-1}(A \setminus B) \to A \setminus B$.

(2) i is not a local homeomorphism at any point of B.

PROOF. We perform repeated relative barycentric subdivisions. Let $D_0 = C_1$. Assuming that we have defined D_n, let E_n be the set of all simplices of D_n which are in A. Set $D_{n+1} = \mathrm{Bary}_{E_n} D_n$. We see from this construction that every simplex of D_n is either a simplex in the nth iterated barycentric subdivision of C or is contained in A.

Say that a subset S of A is a face if there exists an N such that for all $n \geq N$, S is a simplex of D_n. It is clear that a face of a simplex is a face of A and that the intersection of two faces of A is a face of A. Therefore these faces determine a simplicial complex D and a bijection $i : D \to A$. To see that i is continuous, choose an open set $V \subseteq C$. The intersection of V with a simplex of D is an open subset of the simplex, and hence $i^{-1}(V)$ is open. i is an affine morphism on each simplex of D because it is the identity on underlying point sets.

Choose a point $x \in A \setminus B$. We will show that i is a local homeomorphism at x. To do this, choose a simplex S containing x, and choose a metric on S. Let ϵ be less than half the distance from x to $S \cap B$. By [**32**, Chapter 2, Theorem 15.4], there is an n such that every simplex in the nth iterated barycentric subdivision of S has diameter less than ϵ. Consequently every simplex S' of $D_n \cap S$ containing x is contained in A, because either it is contained in some E_i with $i < n$ or it is a simplex of the nth iterated barycentric subdivision. Therefore each such simplex is not further subdivided in $D_{n+1}, D_{n+2}, \ldots$, that is, the star of x in $D_n \cap S$ is fixed. Consequently i determines a homeomorphism from the preimage of this star to the star. The star contains the open star, which is an open neighborhood of x, and consequently i is a local homeomorphism at x.

Let $x \in B$. By the argument of lemma 3.38, there is a simplex S such that $x \in S$, $S^\circ \subseteq A^\circ$, and $S \nsubseteq A$. If i were a local homeomorphism at x, then $i|_{i^{-1}(S)}$ would be a local homeomorphism. Because S is a simplex, it is metrizable, hence $A \cap S$ is also metrizable. Proposition 3.23 would imply that $A \cap S$ is locally compact. But lemma 3.38 to $S \cap C_1 \supseteq S \cap C_2 \supseteq \cdots$ implies that $A \cap S$ is not locally compact at x, so i is not a local homeomorphism at x. $\square$

Chapter 4

Triangulation

1. Triangulation of affine semi-algebraic spaces

The best-known statement of triangulation constructs embedded triangulations of finite collections of bounded semi-algebraic subsets of Euclidean space.

THEOREM 4.1 ([**27**], [**20**]). *Let $\{X_\alpha\}_{\alpha\in A}$ be a finite collection of bounded semi-algebraic subsets of $\mathbf{R}^n$. Then there exists a locally finite simplicial complex C and a homeomorphism $t\colon C \to \mathbf{R}^n$ such that:*

(1) The image of every open simplex of C is a semi-algebraic set and a real analytic manifold.

(2) Every X_α is the union of images of open geometric simplices of C.

(3) There is a compact subset K of $\mathbf{R}^n$ such that for every simplex S of C, if $t(S)$ is not contained in K, then $t(S)$ is the convex hull of its vertices. $\square$

Note that for every simplex S of C, $t(S)$ is also a semi-algebraic set because it is the closure of $t(S^\circ)$.

An immediate consequence of this theorem is that if the union X of $\{X_\alpha\}$ is closed, then there is a simplicial complex which is homeomorphic to X.

COROLLARY 4.2 ([**4**, Theorem 9.2.1]). *Let $\{X_\alpha\}_{\alpha\in A}$ be a finite collection of bounded semi-algebraic subsets of $\mathbf{R}^n$, and assume that $X = \bigcup_{\alpha\in A} X_\alpha$ is closed. Then there exists a finite simplicial complex C and a homeomorphism $t\colon C \to X$ such that:*

(1) The image of every open simplex of C is a semi-algebraic set and a real analytic manifold.

(2) Every X_α is the union of images of open geometric simplices of C.

PROOF. Apply theorem 4.1 to the collection $\{X_\alpha\}_{\alpha\in A}$ to produce a locally finite simplicial complex $\widetilde{C}$ and a homeomorphism $\widetilde{t}$ as in the theorem. X is closed, so the collection C of all faces which map into X is a subcomplex of $\widetilde{C}$. X is the union of images of open geometric simplices, so $\widetilde{t}(C) = X$. X is compact, so by lemma 3.22, C is finite. The remaining conditions of the corollary follow immediately from the theorem. $\square$

We will want to reduce statements about arbitrary subspaces to statements about collections of closed subspaces. This is done by the following lemma.

LEMMA 4.3. *Let X be a locally semi-algebraic space, and let $Y \subseteq X$ be a locally semi-algebraic subspace. Then there exists closed subspaces $Z_0 \supset Z_1 \supset \cdots \supset Z_m$ such that $\dim Z_i < \dim Z_{i+1}$ and $Y = Z_0 \setminus (Z_1 \setminus \cdots (Z_{m-1} \setminus Z_m) \cdots)$.*

PROOF. Let $Y_0 = Y$. Assuming that we have defined Y_i, set $Y_{i+1} = \overline{Y_i} \setminus Y_i$. Since $Y_{i+1} \subseteq \partial Y_i$, we have $\dim Y_{i+1} < \dim Y_i$, so eventually the sequence terminates. We have

$$Y = \overline{Y_0} \setminus Y_1 = \overline{Y_0} \setminus (\overline{Y_1} \setminus Y_2) = \overline{Y_0} \setminus (\overline{Y_1} \setminus (\overline{Y_2} \setminus Y_3)) = \cdots$$

Therefore we set $Z_i = \overline{Y_i}$. $\square$

PROPOSITION 4.4. *Let X be an affine semi-algebraic space, and let $Y_1, \ldots, Y_k$ be a finite collection of affine semi-algebraic subspaces. Then there exists a simplicial complex D and a bijective continuous map $t\colon D \to X$ such that:*

(1) The image of every open simplex of D is an affine semi-algebraic space and a real analytic manifold.

(2) Each of $Y_1, \ldots, Y_k$ is a union of images of open geometric simplices of D.

(3) t is a homeomorphism on the maximal locally compact subspace of X, and is not a local homeomorphism at any point outside that subspace.

PROOF. Embed X as a semi-algebraic set in $\mathbf{R}^n$. Stereographic projection makes $\mathbf{R}^n$ a semi-algebraic subset of $S^n \subseteq \mathbf{R}^{n+1}$, so X embeds in $\mathbf{R}^{n+1}$ as a bounded semi-algebraic set. Apply lemma 4.3 to X to write it as an iterated difference of the closed and bounded subspaces $Z_1, \ldots, Z_\ell$. Apply corollary 4.2 to $Y_1, \ldots, Y_k, Z_1, \ldots, Z_\ell$ to produce a triangulation $s\colon C \to Z_1 \cup \cdots \cup Z_\ell$ in which each Z_i corresponds to a subcomplex C_i of C and in which each Y_i is a union of open geometric simplices of C. Then apply theorem 3.39 to $C_1 \supseteq \cdots \supseteq C_k$ to produce a simplicial complex D and a bijective affine morphism $i\colon D \to C_1$. Let $t = s \circ i$. Because the image of each open simplex in C_1 under s is a semi-algebraic set and a real analytic manifold, the same is true of the image under t of each open simplex of D. Furthermore, C is finite, hence compact, so lemma 3.38 implies that t is a homeomorphism on the points of C where X is locally compact and not a local homeomorphism at any other point. □

2. Triangulation of locally semi-algebraic spaces

THEOREM 4.5. *Let X be a locally semi-algebraic space such that X is Hausdorff, paracompact, and locally compact, and let $\{Y_\alpha\}_{\alpha\in A}$ be a locally finite family of locally semi-algebraic subspaces of X. Then there exists a simplicial complex C and a triangulation $t\colon C \to X$ such that:*

(1) The image of every open simplex of C is a real analytic manifold and an affine semi-algebraic subspace of X,

(2) Each Y_α is the union of images of open simplices of C.

PROOF. Around every point of X, there is an open neighborhood meeting only finitely many members of $\{Y_\alpha\}$. Within that open neighborhood, there is a smaller open neighborhood whose intersection with each of those members of $\{Y_\alpha\}$ is an affine semi-algebraic space. These neighborhoods determine a cover of X. We refine this cover by lemma 2.61 to produce a cover $\{K_\beta\}_{\beta\in B}$ of X. Every member of $\{K_\beta\}$ is a compact affine semi-algebraic subspace and meets only finitely many other members of $\{K_\beta\}$.

Choose a well ordering $\le$ of B, and set $L_\beta = \bigcup_{\beta'\le\beta} K_{\beta'}$. We will use transfinite recursion to construct, for each $\beta \in B$, a triangulation of L_β. Call this triangulation $t_\beta\colon C_\beta \to L_\beta$. t_β will have the following properties:

(1) t_β is semi-algebraic.

(2) The image of every open simplex of C_β is a real analytic manifold and an affine semi-algebraic subspace of X,

(3) For all $\alpha \in A$, $Y_\alpha \cap L_\beta$ is the union of images of open simplices of C_β,

(4) For all $\gamma \in B$, $K_\gamma \cap L_\beta$ is the union of images of open simplices of C_β,

(5) If $\beta' \le \beta$, then $t_{\beta'}^{-1} \circ t_\beta\colon C_\beta \to t_{\beta'}^{-1}(t_\beta(C_\beta))$ is a semi-algebraic subdivision,

(6) If $\beta' \le \beta$, then, when $t_{\beta'}^{-1} \circ t_\beta$ is restricted to the subcomplex of all simplices of C_β whose images in L_β do not meet any K_γ for $\beta' \le \gamma \le \beta$, $t_{\beta'}^{-1} \circ t_\beta$ becomes an isomorphism of simplicial complexes.

To begin the recursion, suppose that $0 \in B$ is the minimal element. Then $L_0 = K_0$ is a compact affine semi-algebraic space, so we would like to apply proposition 4.4 to K_0 and to the family of subspaces $\{Y_\alpha \cap K_0\} \cup \{K_\gamma \cap K_0\}$, where α and γ are arbitrary. This family is finite by our choice of cover. Therefore the proposition applies, and it produces a triangulation $t_0 : C_0 \to K_0$ satisfying all the properties listed above.

Next, suppose that we have constructed the triangulation up to some ordinal β. Let β^+ be the successor of β. We will construct t_{β^+} by gluing t_β to a triangulation of K_{β^+}. K_{β^+} is a compact affine semi-algebraic space, so we apply proposition 4.4 to K_{β^+} and the family of subspaces $\{Y_\alpha \cap K_{\beta^+}\} \cup \{K_\gamma \cap K_{\beta^+}\} \cup \{t_\beta(S) \cap K_{\beta^+}\}$, where α and γ are arbitrary and where S ranges over the simplices of C_β. Again, the proposition applies because our choice of cover implies that the family of subspaces is finite. Call the resulting triangulation $s_{\beta^+} : D_{\beta^+} \to K_{\beta^+}$.

To construct t_{β^+}, we glue t_β and s_{β^+}. We do this in two steps. The first step is to refine C_β so that it agrees with D_{β^+} on their overlap $K_{\beta^+} \cap L_\beta$. The second step is to construct the pushout of D_{β^+} and the refinement of C_β. For the first step, apply theorem 3.36. For the complex C of the theorem we choose C_β. For E we choose the subcomplex $t_\beta^{-1}(K_{\beta^+} \cap L_\beta)$ of C_β, and for j we choose the canonical inclusion of this subcomplex into C_β. E is a subcomplex by our recursive hypotheses on t_β and L_β. For the refinement F of E we choose the subcomplex $t_{\beta^+}^{-1}(K_{\beta^+} \cap L_\beta)$ of D_{β^+}, and for s we choose $t_\beta^{-1} \circ t_{\beta^+}$. t_β and t_{β^+} are semi-algebraic, and D_{β^+} was chosen so that $t_\beta^{-1} \circ t_{\beta^+}$ is a subdivision. Therefore theorem 3.36 applies. It produces a semi-algebraic subdivision $C'_\beta \to C_\beta$ which is the trivial subdivision on every simplex of C_β not meeting K_{β^+} and which is identical to D_{β^+} on $K_{\beta^+} \cap L_\beta$.

Now we apply lemma 3.16. We construct the pushout of C'_β and D_{β^+} along the subcomplex corresponding to $K_{\beta^+} \cap L_\beta$ and call it C_{β^+}. Because it is a pushout, it comes with a semi-algebraic isomorphism t_{β^+} to L_{β^+}. The image of every open simplex of C_{β^+} is the image of a simplex of C'_β or D_{β^+} and hence is a real analytic manifold and an affine semi-algebraic space. Each $Y_\alpha \cap C_{\beta^+}$ and each $K_\gamma \cap C_{\beta^+}$ is the union of open simplices because each $Y_\alpha \cap C'_\beta$, $Y_\alpha \cap D_{\beta^+}$, $K_\gamma \cap C'_\beta$, and $K_\gamma \cap D_{\beta^+}$ is. If $\beta' \leq \beta$, then $t_{\beta'}^{-1} \circ t_\beta$ is a semi-algebraic subdivision because C'_β is a semi-algebraic subdivision of C_β. Finally, because our application of theorem 3.36 did not subdivide any simplex not meeting K_{β^+}, we find that t_{β^+} and C_{β^+} satisfy the non-subdivision condition, and hence all the recursive hypotheses, given above. This completes the case of a successor ordinal.

It remains to consider the case of a limit ordinal β. For all $\beta' < \beta$, let $E_{\beta'}$ denote the subcomplex of $C_{\beta'}$ formed by all simplices which do not meet any K_γ for $\beta' \le \gamma \le \beta$. By the recursive hypotheses, we have simplicial affine injections $E_{\beta'} \to E_{\beta''}$ whenever $\beta' \le \beta'' < \beta$. By lemma 3.17, $E = \varinjlim_{\beta'<\beta} E_{\beta'}$ exists. By the universal property, it has a semi-algebraic morphism $u : E \to L_\beta$ which restricts on each $E_{\beta'}$ to $t_{\beta'}$. Therefore it is straightforward to verify all of the recursive hypotheses given above for E. To see that every point of $\bigcup_{\beta'<\beta} K_{\beta'}$ is in the image of u, notice that every point lies in some K_{β_0} and that K_{β_0} meets only finitely many $K_{\beta'}$. Therefore there is a $\beta_1 \ge \beta_0$ such that K_{β_0} meets no $K_{\beta'}$ with $\beta_1 \le \beta' < \beta$. In particular, by the non-subdivision condition, the given point lies in E_{β_1}. Consequently u and E determine a triangulation of $\bigcup_{\beta'<\beta} K_{\beta'}$. However, K_β need not be empty, so E might not triangulate L_β. To repair this, we repeat the argument for the successor ordinal case, replacing C_β by E and t_β by u. This produces t_β and C_β. (Another solution to this deficiency is to replace $\{K_\beta\}$ by a different collection in which $K_\beta = \emptyset$ whenever β is a limit ordinal.)

The output of the transfinite recursion is the desired triangulation of X, and it satisfies the conclusion of the theorem because of the recursive hypotheses. □

COROLLARY 4.6. *Let X be a locally semi-algebraic space such that X is Hausdorff, paracompact, and locally compact. Then X admits at most one semi-algebraic PL structure.*

PROOF. Any two triangulations $t_1 : C_1 \to X$ and $t_2 : C_2 \to X$ determine two locally finite families of semi-algebraic subspaces of X. Applying theorem 4.5 to these two families produces a common semi-algebraic subdivision of the two triangulations, and hence they are equivalent. □

COROLLARY 4.7. *Let X be a separated finite type k-scheme, where k is $\mathbf{R}$ or $\mathbf{C}$. Then $X(k)$ admits one and only one semi-algebraic PL structure.*

PROOF. By proposition 2.62, $X(k)$ is Hausdorff, paracompact, and locally compact. Therefore by theorem 4.5 and corollary 4.6, $X(k)$ admits a unique semi-algebraic PL structure. □

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