# Integer ratios of factorials, hypergeometric functions, and related step functions

by

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Theorem 1.1 appears in [8], and I would like to thank the anonymous referee of that paper for many helpful comments. Theorem 1.2 is joint work with Jason P. Bell, and appears in [5]. Also, the results of Theorem 1.2 were considerably strengthened with help from Kannan Soundararajan. Theorem 1.2 was also proved by Enrico Bombieri and Jean Bourgain [10] through different methods, and I would like to thank Bombieri for providing his notes. I also thank Alexander Borisov for providing a preprint of [12].

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# Background and results

## 1.1 Introduction

Fix two sets of natural numbers $\mathbf{a} = (a_1, a_2, \ldots, a_K)$ and $\mathbf{b} = (b_1, b_2, \ldots, b_L)$. We consider the *factorial ratio sequence*

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!}. \tag{1.1}$$

Our main question is: For which $\mathbf{a}$ and $\mathbf{b}$ is $u_n(\mathbf{a}, \mathbf{b})$ an integer for all $n \geq 0$? Nearly always we require that

$$\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l, \tag{1.2}$$

and we will call a sequence satisfying this condition *balanced*. This condition ensures that both $u_n$ and its inverse grow at most exponentially, instead of factorially.

The simplest examples of balanced integer factorial ratio sequences come from binomial coefficients: for any $a \geq b \geq 0$ we have the sequence

$$u_n(a; b, a-b) = \binom{an}{bn} = \frac{(an)!}{(bn)!((a-b)n)!}.$$

Another example, which was given by Catalan in 1874 [15], is

$$\frac{(2a)!(2b)!}{a!b!(a+b)!},$$

which is also an integer for all integers $a, b$. (We obtain an integer factorial ratio sequence by setting $a = a'n$ and $b = b'n$ for some integers $a'$ and $b'$.) If $u_n(\mathbf{a}, \mathbf{b})$ is balanced and is always an integer, then necessarily $L \geq K$, with $L = K$ only in the case that $u_n(\mathbf{a}, \mathbf{b}) = 1$ for all $n$, and we shall call the parameter $L - K$ the *height*. Similarly, we call $L+K$ the *length*. Thus both of the examples above give two parameter infinite families of integer factorial ratio sequences of height 1, one with length 3 and one with length 5. Surprisingly, there are few other examples of height 1, and they can be completely described. One of the main results of this thesis is that, other than one other two parameter family with length 5 there are just 52 balanced integer factorial ratio sequences of height 1, all of length 5, 7, or 9 (see Theorem 1.1). An example which does not belong to any of the infinite families is

$$u_n(30,1;15,10,6) = \frac{(30n)!n!}{(15n)!(10n)!(6n)!}. \tag{1.3}$$

One basic reason for interest in such sequences is their use in giving elementary estimates on the prime counting function $\pi(x) := \#\{p \le x \text{ such that } p \text{ is prime}\}$. Via a standard method (see [28, Section 5.1], for example), whenever $\sum a_k = \sum b_l$, one can use the integrality of such a sequence to prove the estimate originally due to Chebyshev [16] that

$$C_1(\mathbf{a},\mathbf{b})\frac{x}{\log x} \le \pi(x) \le C_2(\mathbf{a},\mathbf{b})\frac{x}{\log x}$$

for all large enough $x$, for some constants $C_1$ and $C_2$ depending on $\mathbf{a}$ and $\mathbf{b}$. The sequence

$$u_n = \binom{2n}{n}$$

is often used in textbooks to achieve these bounds with $C_1 = \log 2$ and $C_2 = 2\log 2$. The sequence (1.3), which is closely connected with Chebyshev's work, gives $C_1 \approx .9$ and $C_2 \approx 1.1$, which allowed Chebyshev to prove Bertrand's postulate that, for any positive integer $n$, there is always a prime number between $n$ and $2n$.

In studying factorial ratio sequences one is naturally led to consider the corresponding step function defined by

$$f(x;\mathbf{a},\mathbf{b}) = \sum_{k=1}^{K}\lfloor a_k x\rfloor - \sum_{l=1}^{L}\lfloor b_l x\rfloor \tag{1.4}$$

(where $\lfloor x\rfloor$ denotes the floor of $x$). With this definition, it is not hard to see the identity

$$u_n(\mathbf{a},\mathbf{b}) = \prod_{p \text{ prime}} p^{\sum_{\alpha=1}^{\infty} f\left(\frac{n}{p^\alpha}\right)}.$$

(Note that $f(x;\mathbf{a},\mathbf{b})$ is 0 in a neighborhood to the right of 0, so for any $n$ these infinite sums and this infinite product are actually finite.) In 1918 Landau [20] proved that $u_n(\mathbf{a}, \mathbf{b})$ is an integer for all $n$ if and only of $f(x; \mathbf{a}, \mathbf{b})$ is nonnegative for all $x > 0$ (see Theorem 2.1). Thus, classifying integer factorial ratios of the form (1.1) is the same as classifying step functions of the form (1.4).

Functions of the form (1.4) have a more direct connection to the distribution of prime numbers through the Beurling–Nyman criterion for the Riemann Hypothesis. Loosely speaking, the Beurling–Nyman criterion is the statement that the Riemann Hypothesis is true if and only if the constant function call be well-approximated by functions that look like (1.4). (This is discussed in more detaio in Section 1.4.)

With this in mind, V. I. Vasyunin [31] studied the classification of such functions which take only the values 0 and 1. Based on extensive computation, he gave a conjectural classification which our Theorem 1.1 proves: there are three infinite families and 52 more functions which do not fall into these families. In Chapter 2 we will show how Vasyunin's conjecture is equivalent to our Theorem 1.1.

One curious consequence of Theorem 1.1 is that the number of terms in such a function must be less than or equal to 9 if it is to take on only the values 0 and 1; that is, if

$$f(x) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{K+1} \lfloor b_l x \rfloor$$

with $\sum a_k = \sum b_j, a_k \neq b_l$ for all $k, l$ and $a_k, b_l \neq 0$, and $K > 4$, then $f(x) < 0$ for some $x$. It would be nice to have a more direct proof of this statement, but the only proof that will be presented here is one where we simply write down all of the possibilities and notice that none of them has more than nine terms.

Our second theorem is a generalization of this phenomenon, and answers a conjecture of A. Borisov related to cyclic quotient singularities [12, Conjecture 4]. We will prove that, subject to the obvious nondegeneracy conditions, if $L + K$ is large enough (as compared to $L - K$) then

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

cannot be always nonnegative. In terms of factorial ratio sequences, this says that a balanced sequence of large length must have sufficiently large height if it is to have a chance of being integral. In the specific case of height 1, the bounds that we achieve by this method will be quite far from what is true, however.

Finally, we may turn our attention to whether there exist classifications similar to that of Theorem 1.1 for integer factorial ratio sequences with fixed $D = L - K > 1$. Our Theorem 1.3 says that for fixed $K$ and $L$, if we require that $\sum a_k = \sum b_l$ then

$$u_n(\mathbf{a},\mathbf{b}) = \frac{(a_1 n)!(a_2 n)!\cdots(a_K n)!}{(b_1 n)!(b_2 n)!\cdots(b_L n)!}$$

is an integer for all $n$ if and only if $\mathbf{a}$ and $\mathbf{b}$ have nonnegative integer coordinates and lie in a set $W$ which is a finite union of subspaces of $\mathbb{R}^{K+L}$. Combining this with Theorem 1.2, we find that there exists a finite classification for any fixed height, of which Theorem 1.1 is just one example.

## 1.2 Statement of results

Our first theorem, which is proved in Chapter 3, describes completely the classification of integral factorial ratios of the form

$$u_n(\mathbf{a},\mathbf{b}) = \frac{(a_1 n)!(a_2 n)!\cdots(a_K n)!}{(b_1 n)!(b_2 n)!\cdots(b_{K+1} n)!}$$

such that $\sum a_k = \sum b_j$. As is explained in Chapter 2, this proves a conjecture of Vasyunin [31] concerning the classification of functions of the form

$$f(x;\mathbf{a},\mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

which take only the values 0 and 1. We will not use the connection with step functions to prove this theorem, however, but will instead use a connection with hypergeometric series.

The generating function

$$\mathbf{u}(z) = \sum_{n=0}^{\infty} u_n(\mathbf{a},\mathbf{b}) z^n$$

is a hypergeometric series, and when $u_n(\mathbf{a},\mathbf{b})$ is balanced, so that $\sum a_k = \sum b_l$, we show that it is in fact what is known as a $G$-function (see Section 2.3). Moreover, F. Rodriguez-Villegas [25] noticed that $\mathbf{u}(z)$ is algebraic if and only if $L - K = 1$, $\sum a_k = \sum b_l$, and $u_n(\mathbf{a},\mathbf{b})$ is an integer for all $n$. Combining this with work of Beukers and Heckman [6] which classifies all algebraic hypergeometric series of the form that we are interested in, we will be able to prove the following.

**Theorem 1.1.** *Let*

$$u_n(\mathbf{a},\mathbf{b}) = \frac{(a_1n)!(a_2n)!\cdots(a_Kn)!}{(b_1n)!(b_2n)!\cdots(b_{K+1}n)!}$$

*and suppose that $a_k \neq b_l$ for all $k, l$, that $\sum a_k = \sum b_l$, and that*

$$\gcd(a_1,\ldots,a_K,b_1,\ldots,b_{K+1}) = 1.$$

*Then $u_n(\mathbf{a},\mathbf{b})$ is an integer for all $n$ if and only if either*

1. *$u_n = u_n(\mathbf{a},\mathbf{b})$ takes one of the following forms:*

$$u_n = \frac{[(a+b)n]!}{(an)!(bn)!} \textit{ for } \gcd(a,b) = 1, \tag{1.5}$$

$$u_n = \frac{(2an)!(bn)!}{(an)!(2bn)![|a-b|\,n]!} \textit{ for } \gcd(a,b) = 1, \tag{1.6}$$

$$u_n = \frac{(2an)!(2bn)!}{(an)!(bn)![(a+b)n]!} \textit{ for } \gcd(a,b) = 1 \tag{1.7}$$

*or*

2. *$(\mathbf{a},\mathbf{b})$ is one of the 52 sporadic parameter sets listed in the second column of Table 3.2.*

As a consequence of this theorem, if

$$\frac{(a_1n)!(a_2n)!\cdots(a_Kn)!}{(b_1n)!(b_2n)!\cdots(b_{K+1}n)!}$$

is always an integer, then $K \leq 4$. Our second theorem generalizes thie observation to the case of $L-K > 1$. Instead of using this formulation, though, we will actually prove this theorem using the connection with the nonnegativity of $f(x;\mathbf{a},\mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$. More specifically, in Chapter 4 we will show that if $L+K$ is large compared to $L-K$, then

$$\int_0^1 f(x;\mathbf{a},\mathbf{b})^2 \mathrm{d}x$$

is large, from which it follows that $f(x;\mathbf{a},\mathbf{b})$ must take on some large values. From a relation between the maximum value of $f(x;\mathbf{a},\mathbf{b})$ and the minimum value of $f(x;\mathbf{a},\mathbf{b})$ in Chapter 2 we can conclude that $f(x;\mathbf{a},\mathbf{b})$ must also take negative values. To prove that the $L^2$ norm is large, we will use the fact that the Fourier coefficients of $f(x;\mathbf{a},\mathbf{b})$ have nice arithmetic properties, which allows us to understand the Dirichlet series whose coefficients are the Fourier coefficients of $f(x;\mathbf{a},\mathbf{b})$.

In the language of integral factorial ratios, this theorem takes the following form.

**Theorem 1.2.** *Fix $L - K = D$. Let*

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!}$$

*and suppose that $a_k \neq b_l$ for all $k, l$, that $\sum a_k = \sum b_l$, and that $u_n(\mathbf{a}, \mathbf{b})$ is an integer for all $n$. Then*

$$K + L \ll D^2 (\log D)^2$$

Our third theorem generalizes Theorem 1.1 in a different way. Theorem 1.1 can be thought of as saying that if $\sum a_k = \sum b_l$, then for fixed $K$,

$$\frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_{K+1} n)!}$$

is an integer if and only if **a** and **b** lie in one of a few specific subspaces of $\mathbb{R}^{2K+1}$. Although we will not write down such a classification for $L > K + 1$, we can show that this phenomenon does persist.

In this case our proof again will use the formulation of this theorem in terms of the nonnegativity of $f(x; \mathbf{a}, \mathbf{b})$, but the method is geometric and is a simple consequence of a theorem of Jim Lawrence [21] about closed subgroups of the torus. We also note that this theorem and proof are closely connected with a similar theorem by A. Borisov [11] about the classification of cyclic quotient singularities. Although we will not use the connection with quotient singularities to prove this theorem, it is possible we would never have been aware of it without knowing about Borisov's work.

The theorem, which we will prove in Chapter 5, is as follows.

**Theorem 1.3.** *Fix positive integers $K$ and $L$. The set $\mathcal{S}$ of $a_1, a_2, \ldots, a_K, b_1, b_2, \ldots b_L$ such that*

1. $\sum a_k = \sum b_l$
2. *$a_k$ and $b_l$ are nonnegative integers for all $k, l$*
3. $u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!} \in Z$ *for all $n$*

*can be written as*

$$\mathcal{S} = (\mathbb{Z}_{\geq 0})^{K+L} \bigcap \left( \bigcup_{n=1}^{N} V_n \right),$$

*where each $V_n$ is a subspace of $\mathbb{R}^{K+L}$.*

In Chapter 2 we will make more precise the connection between the factorial ratio sequence $u_n(\mathbf{a}, \mathbf{b})$ and the step function $f(x; \mathbf{a}, \mathbf{b})$ and we will restate these theorems in terms of the classification of nonnegative $f(x; \mathbf{a}, \mathbf{b})$. (To distinguish, the versions of these theorems in terms of step functions will be labeled Theorems 1.1*, 1.2* and 1.3*.) Additionally, in the next section we point out some applications to the classification of quotient singularities due to Alexander Borisov.

## 1.3 Applications to the classification of cyclic quotient singularities

By work of Alexander Borisov [12], our theorems have applications to the work of cyclic quotient singularities. We defer to [12] for details, but state here Borisov's theorem which makes the connection, and some consequences.

**Theorem 1.4** (A. Borisov, [12, Theorem 11])**.** *Suppose $u_1, u_2, \ldots, u_k$ and $v_1, v_2, \ldots, v_k$ are two finite sets of linear forms on $\mathbb{R}^d$ with coefficients in $\mathbb{N}$. Suppose further that $\sum_{k=1}^{K} u_i(X) = \sum_{l=1}^{L} v_l(X)$ for all $X = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Then the following two statements are equivalent.*

1. *For every $X = (x_1, \ldots, x_d) \in \mathbb{N}^d$,*

$$\frac{\prod_{k=1}^{K} u_k(X)!}{\prod_{l=1}^{L} v_l(X)!} \in \mathbb{N}.$$

2. *For any $n \in \mathbb{N}$ and all $X = (x_1, \ldots, x_d) \in \mathbb{Z}^d$ such that all $\left\{\frac{u_i(X)}{n}\right\}$ and $\left\{\frac{v_i(X)}{n}\right\}$ are nonzero, the following point in $\mathbb{T}^{n+k}$ defines a Gorenstein cyclic quotient with Shokurov minimal log-discrepancy at least $K$:*

$$\left(\left\{\frac{-u_1(X)}{n}\right\}, \left\{\frac{-u_2(X)}{n}\right\}, \cdots, \left\{\frac{-u_K(X)}{n}\right\}, \left\{\frac{v_1(X)}{n}\right\}, \left\{\frac{v_2(X)}{n}\right\}, \cdots, \left\{\frac{v_L(X)}{n}\right\}\right).$$

Through this connection Theorem 1.1 establishes a new proof of a classification of terminal cyclic quotient singularities in dimension 4 which was conjectured by Mori, Morrison, and Morrison [23] and first proved by Sankaran [27]. Additionally, Theorem 1.1 implies the following in higher dimensions.

**Proposition 1.5.** *Suppose $d \geq 5$ and we have a one-parameter family of Gorenstein cyclic quotient singularities of dimension $2d+1$ with Shukarov minimal log-discrepancy $d$. Then up to the permutation of the coordinates in the $T^{(2d+1)}$, the corresponding points lie in the subtorus $x_1 + x_2 = 1$.*

*Proof.* See [12, Conjecture 1]. □

Additionally, Borisov notes that Theorem 1.2* is equivalent to the following.

**Proposition 1.6.** *Suppose $a \geq 0$ is any real number. Then for all large enough $d \in \mathbb{N}$, for all but finitely many $(x_1, x_2, \ldots, x_d) \in T^d$ that define a cyclic quotient singularity with Shokurov minimal log-discrepancy at least $d/2 - a$, for some pair of indices $1 \leq i < j \leq d$ we have $x_i + x_j = 1$.*

*Proof.* See [12, Conjecture 3]. □

### 1.4 The Beurling–Nyman criterion for the Riemann Hypothesis

The Riemann $\zeta$-function $\zeta(s)$, where $s = \sigma + it$, is classically defined in the half plane $\sigma > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

One way to obtain an analytic continuation of $\zeta(s)$ to the half plane $\sigma > 0$ is by writing this Dirichlet series as a Riemann–Stiltjes integral and integrating by parts. Doing so, we have

$$\begin{aligned} \zeta(s) &= \int_{1^-}^{\infty} y^{-s} \mathrm{d}\lfloor y \rfloor \\ &= s \int_{1}^{\infty} y^{-s-1} \lfloor y \rfloor \, \mathrm{d}y. \end{aligned}$$

Writing $\lfloor y \rfloor = y - \{y\}$, we obtain

$$\begin{aligned} \frac{\zeta(s)}{s} &= \int_1^{\infty} \frac{1}{y^s} \mathrm{d}y - \int_1^{\infty} \frac{\{y\}}{y^{s+1}} \mathrm{d}y \\ &= \frac{1}{s-1} - \int_1^{\infty} \frac{\{y\}}{y^{s+1}} \mathrm{d}y. \end{aligned}$$

Since the integral $\int_1^{\infty} \frac{\{y\}}{y^{s+1}} \mathrm{d}y$ converges absolutely for $\sigma > 0$, this expression gives an analytic continuation of the $\zeta$-function to this region, with the exception of a simple pole at $s = 1$.

Additionally, this expression indicates a relation between the functions $\{x\}$ and

$\zeta(s)$. In fact, if we instead start with

$$\alpha^s\zeta(s) = \int_{1^-}^{\infty} \left(\frac{\alpha}{y}\right)^s \mathrm{d}\lfloor y \rfloor = \int_{1^-}^{\infty} y^{-s}\mathrm{d}\lfloor \alpha y \rfloor,$$

which is valid for $0 \leq \alpha \leq 1$, and perform the same calculation, we find that

$$\frac{\alpha^s\zeta(s)}{s} = \frac{\alpha}{s-1} - \int_1^{\infty} \frac{\{\alpha y\}}{y^{s+1}}\mathrm{d}y.$$

Now suppose that $f(x)$ is a function of the form

$$f(x) = \sum_{n=1}^{N} c_n \lfloor \alpha_n x \rfloor \,,$$

where $\sum_{n=1}^{N} c_n\alpha_n = 0$. Using the relation $\lfloor x \rfloor = x - \{x\}$, we may rewrite this as

$$f(x) = -\sum_{n=1}^{N} c_n \{\alpha_n x\} \,,$$

and thus

$$\int_1^{\infty} \frac{f(y)}{y^{s+1}}\mathrm{d}y = \frac{\zeta(s)}{s}\sum_{n=1}^{N} c_n\alpha_n^s.$$

We will now see how the approximation of 1 by $f(x)$ leads to information about the zeros of $\zeta(s)$. Consider

$$\int_1^{\infty} \frac{(1-f(y))}{y}\frac{1}{y^s}\mathrm{d}y = \frac{1}{s}\left(1 - \zeta(s)\sum_{n=1}^{N} c_n\alpha_n^s\right). \tag{1.8}$$

By Holder's inequality, for $\sigma > (p-1)/p$ we have

$$\begin{aligned}\int_1^{\infty} \left|\frac{(1-f(y))}{y}\frac{1}{y^s}\right| \mathrm{d}y &\leq \left(\int_1^{\infty} \left|\frac{1-f(y)}{y}\right|^p \mathrm{d}y\right)^{1/p} \left(\int_1^{\infty} \frac{1}{y^{\sigma p/(p-1)}}\mathrm{d}y\right)^{(p-1)/p} \\ &= \left(\int_1^{\infty} \left|\frac{1-f(y)}{y}\right|^p \mathrm{d}y\right)^{1/p} \left(\frac{p-1}{\sigma p - p + 1}\right)^{(p-1)/p}.\end{aligned}$$

Let $\left(\int_1^{\infty} \left|\frac{1-f(y)}{y}\right|^p \mathrm{d}y\right)^{1/p} = \epsilon$. We then have

$$\left|1 - \zeta(s)\sum_{n=1}^{N} c_n\alpha_n^s\right|^{p/(p-1)} \leq \epsilon^{p/(p-1)} \left|s\right|^{p/(p-1)} \frac{p-1}{\sigma p - p + 1}.$$

Thus $\zeta(s) \neq 0$ whenever the right hand side is smaller than 1 and $\sigma > (p-1)/p$. For fixed $s$, the right hand side goes to 0 as $\epsilon \to 0$, so this proves half of the following theorem.

**Proposition 1.7** (Nyman, Beurling)**.** *$\zeta(s)$ has no zeros in the half plane $\sigma > \frac{p-1}{p}$ if and only if for any $\epsilon > 0$ there exists a function $f(x)$ of the form*

$$f(x) = \sum_{n=1}^{N} c_n \lfloor \alpha_n x \rfloor \tag{1.9}$$

*with $\sum c_n \alpha_n = 0$ and $0 \leq \alpha_n \leq 1$ for all $n$ such that*

$$\left( \int_1^\infty \left| \frac{1 - f(x)}{x} \right|^p \mathrm{d}x \right)^{1/p} < \epsilon.$$

*Remark* 1.8. This theorem appears for $p = 2$ in a different form in Nyman's thesis [24]. Beurling gives the generalization to $p > 2$ in [7]. Both Beurling and Nyman work instead with the function space $L^p((0,1), \mathrm{d}x)$, however. Here we are essentially considering $L^p((1,\infty), \frac{dx}{x})$, which corresponds to the change of variable $x \longleftrightarrow 1/x$. Also, Beurling and Nyman in fact show that $\zeta(s)$ being zero-free for $\sigma > \frac{p-1}{p}$ is equivalent to functions of the form (1.9) being dense in $L^p((1,\infty), \frac{dx}{x})$.

The argument above suggests a candidate for a function $f(x)$ that we might try to use to approximate 1. From equation (1.8) we see that we might want to try choosing $f(x) = \sum_{n=1}^{N} c_n \lfloor \alpha_n x \rfloor$ so that $\sum_{n=1}^{N} c_n \alpha^s \approx 1/\zeta(s)$. For $\sigma > 1$, (and for $\sigma > 1/2$ if we assume the Riemann Hypothesis), $1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n)/n^s$, so it seems we should choose $c_n = \mu(n)$ and $\alpha_n = 1/n$.

In fact, for any fixed $x$, we have

$$\sum_{n=1}^{\infty} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor = 1. \tag{1.10}$$

However, there is no $N$ for which the finite sum $\sum_{n=1}^{N} \frac{\mu(n)}{n}$ is 0, so we can not just take partial sums of this series. There are other candidates we may consider, though. For example, one possibility is to take a partial sum of the series $\sum_{n=1}^{\infty} \mu(n) \lfloor \frac{x}{n} \rfloor$ and add in a correction factor at the $n+1$-st place so that the condition $\sum c_n \alpha_n = 0$ is satisfied. The sequence of functions we get is

$$f_n(x) = \sum_{n=1}^{N} \mu(n) \left\lfloor \frac{x}{n} \right\rfloor - N \left( \sum_{n=1}^{N} \frac{\mu(n)}{n} \right) \left\lfloor \frac{x}{n} \right\rfloor.$$

This sequence converges to 1 pointwise, but not in $L^2((1,\infty), \frac{dx}{x})$ (see [2]). Nevertheless, as is perhaps suggested by (1.10), it is possible to restrict to $\alpha_n = \frac{1}{n}$ and get a theorem of the same type.

**Proposition 1.9** (Baez-Duárte [3]). *The Riemann Hypothesis is true if and only if for every $\epsilon > 0$ there is a function*

$$f(x) = \sum_{n=1}^{N} c_n \left\lfloor \frac{x}{n} \right\rfloor \tag{1.11}$$

*such that*

$$\int_1^\infty \left| \frac{1 - f(x)}{x} \right|^2 \mathrm{d}x < \epsilon.$$

Although Baez-Duarte had not yet proven this theorem, its conjectural existence was motivation for Vasyunin [31] to study functions of the form (1.11) taking only the values 0 and 1.

There is one more aspect of this approach that we should mention, though we will not use it. From the equation (1.8), we see that there is a relationship between approximating the constant function with linear combinations of floor functions and approximating the inverse of the zeta function. In fact, an "equivalent" formulation of the Beurling–Nyman criterion is the statement that the Riemann Hypothesis is true if and only if

$$\lim_{N\to\infty} \inf_{D_N} \int_{-\infty}^{\infty} \left( \frac{1 - \zeta(1/2+it) D_N(1/2+it)}{1/2+it} \right)^2 \mathrm{d}t = 0,$$

where $D_N$ ranges over all generalized Dirichlet polynomials of length $N$. (See [4].)

For more on the Beurling–Nyman criterion, one can also see the paper of Burnol [13], which proves Baez-Duarte's theorem from a more complex-analytic viewpoint, and the paper of Balazard and Saias [4], which proves the Beurling–Nyman criterion by using the theory of the Hardy space of the half plane $\sigma > 1/2$ and its relation to $L^2((0,1))$ through the Mellin transform.

# CHAPTER 2

# The connection between factorial ratios and step functions

## 2.1 Restatements of main theorems

The main object of this chapter is to prove the equivalence of Theorems 1.1, 1.2, 1.3 and the following restatements of those theorems in terms of step functions related to the Beurling-Nyman criterion.

Our first theorem was conjectured by Vasyunin [31] and is a complete description of functions of the form

$$f(x) = \sum_{k=1}^{N} \left\lfloor \frac{x}{m_k} \right\rfloor - \sum_{k=N+1}^{2N+1} \left\lfloor \frac{x}{m_k} \right\rfloor,$$

with $m_k > 0$ for all $k$ such that $f(x)$ takes only the values 0 and 1. Equivalently, through a change of variables, this is a description of functions of the form

$$f(x;\mathbf{a},\mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

which take only the values 0 and 1.

**Theorem 1.1*.** *Let*

$$f(x) = \sum_{k=1}^{N} \left\lfloor \frac{x}{m_k} \right\rfloor - \sum_{k=N+1}^{2N+1} \left\lfloor \frac{x}{m_k} \right\rfloor,$$

*with* $m_k > 0$ *for all* $k$, *and suppose that* $m_i \neq m_j$ *for all* $i \leq N, j \leq N+1$, *and that*

$$\gcd(m_1, m_2, \ldots, m_{2N+1}) = 1.$$

*Then* $f(x)$ *takes only the values* 0 *and* 1 *if and only if either*

1. *$f(x)$ takes one of the following forms:*

$$f(x) = \left\lfloor \frac{x}{ab} \right\rfloor - \left\lfloor \frac{x}{b(a+b)} \right\rfloor - \left\lfloor \frac{x}{a(a+b)} \right\rfloor \quad \textit{where } \gcd(a,b) = 1, \tag{2.1}$$

$$f(x) = \left\lfloor \frac{x}{b(a-b)} \right\rfloor + \left\lfloor \frac{x}{2a(a-b)} \right\rfloor - \left\lfloor \frac{x}{2b(a-b)} \right\rfloor - \left\lfloor \frac{x}{a(a-b)} \right\rfloor - \left\lfloor \frac{x}{2ab} \right\rfloor \tag{2.2}$$

*where*

$$\gcd(a,b) = \gcd(2, a-b) = 1 \textit{ and } a > b > 0,$$

$$f(x) = \left\lfloor \frac{x}{\frac{1}{2}b(a-b)} \right\rfloor + \left\lfloor \frac{x}{a(a-b)} \right\rfloor - \left\lfloor \frac{x}{b(a-b)} \right\rfloor - \left\lfloor \frac{x}{\frac{1}{2}a(a-b)} \right\rfloor - \left\lfloor \frac{x}{ab} \right\rfloor \tag{2.3}$$

*where*

$$\gcd(a,b) = \gcd(2,a) = \gcd(2,b) = 1 \textit{ and } a > b > 0,$$

$$f(x) = \left\lfloor \frac{x}{b(a+b)} \right\rfloor + \left\lfloor \frac{x}{a(a+b)} \right\rfloor - \left\lfloor \frac{x}{2b(a+b)} \right\rfloor - \left\lfloor \frac{x}{2a(a+b)} \right\rfloor - \left\lfloor \frac{x}{2ab} \right\rfloor \tag{2.4}$$

*where*

$$\gcd(a,b) = \gcd(2, a+b) = 1,$$

$$f(x) = \left\lfloor \frac{x}{\frac{1}{2}b(a+b)} \right\rfloor + \left\lfloor \frac{x}{\frac{1}{2}a(a+b)} \right\rfloor - \left\lfloor \frac{x}{b(a+b)} \right\rfloor - \left\lfloor \frac{x}{a(a+b)} \right\rfloor - \left\lfloor \frac{x}{ab} \right\rfloor \tag{2.5}$$

*where*

$$\gcd(a,b) = \gcd(2,a) = \gcd(2,b) = 1,$$

*or*

2. *$f(x)$ is one of the* 52 *sporadic step functions given by*

$$(m_1, m_2, \ldots, m_N) = \left( \frac{M}{a_1}, \frac{M}{a_2}, \ldots, \frac{M}{a_N} \right)$$

*and*

$$(m_{N+1}, m_{N+2}, \ldots, m_{2N+1}) = \left( \frac{M}{b_1}, \frac{M}{b_2}, \ldots, \frac{M}{b_{N+1}} \right)$$

*for some* $\mathbf{a}$ *(or permutation of* $\mathbf{a}$*) and* $\mathbf{b}$ *(or permutation of* $\mathbf{b}$*) listed in the second column of Table 3.2, where*

$$M = \operatorname{lcm}(a_1, a_2, \ldots, a_N, b_1, b_2, \ldots b_{N+1}).$$

As a consequence of Theorem 1.1*, if

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{K+1} \lfloor b_l x \rfloor$$

takes only the values 0 and 1, then $K \leq 4$; that is, the step function has at most 9 terms. In general, a function of the form

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

must take the values 0 and $L - K$. Our second theorem, which was conjectured by A. Borisov [12] generalizes this part of Theorem 1.1* by saying that if $f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$ has too many terms, then it must take values less than 0, and thus also values greater than $L - K$.

**Theorem 1.2*.** *Fix $L - K = D$. If*

$$f(x) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor \tag{2.6}$$

*where*

$$\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l$$

*and $a_k \neq b_l$, $a_k, b_l \in \{1, 2, 3, \ldots\}$ for all $k, l$, and if*

$$f(x) \geq 0$$

*for all $x$, then*

$$K + L \ll D^2 (\log D)^2.$$

Our third theorem generalizes a different aspect of Theorem 1.1*. When we fix $K$, Theorem 1.1* gives a finite list of subspaces of $\mathbb{R}^{2K+1}$ such that

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{K+1} \lfloor b_l x \rfloor$$

takes only the values 0 and 1 if and only if $(a_1, \ldots, a_K, b_1, \ldots, b_{K+1})$ lie in one of those subspaces. For other values of $K$ and $L$, we cannot at this time write down such a list of subspaces, but we can prove that in general such finite list of subspaces does exist.

**Theorem 1.3*.** *Fix positive integers $K$ and $L$, and an integer $A$. The set $\mathcal{S}$ of $a_1, a_2, \ldots, a_K, b_1, b_2, \ldots b_L$ such that*

1. $\sum a_k = \sum b_l$
2. *$a_k$ and $b_l$ are nonnegative integers for all $k, l$*
3. $A \leq \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor \leq L - K - A$

*can be written as*

$$\mathcal{S} = (\mathbb{Z}_{\geq 0})^{K+L} \bigcap \left( \bigcup_{n=1}^{N} V_n \right),$$

*where each $V_n$ is a subspace of $\mathbb{R}^{K+L}$.*

## 2.2 Equivalence of the restated theorems

To prove the equivalence of the starred and unstarred versions of our theorems, we begin with the following theorem of Landau.

**Theorem 2.1** (Landau [20]). *Let $a_{k,s}, b_{l,s} \in \mathbb{Z}_{\geq 0}, 1 \leq k \leq K, 1 \leq l \leq L, 1 \leq s \leq r$ and let*

$$A_k(x_1, x_2, \ldots, x_r) = \sum_{s=1}^{r} a_{k,s} x_s$$

*and*

$$B_l(x_1, x_2, \ldots, x_r) = \sum_{s=1}^{r} b_{l,s} x_s.$$

*(That is, $A_k$ and $B_l$ are linear forms in $r$ variables with nonnegative integral coefficients.) Then the factorial ratio*

$$\frac{\prod_{k=1}^{K} A_k(x_1, x_2, \ldots, x_r)!}{\prod_{l=1}^{L} B_l(x_1, x_2, \ldots, x_r)!}$$

*is an integer for all $(x_1, \ldots, x_r) \in \mathbb{Z}_{\geq 0}^r$ if and only if the step function*

$$F(y_1, \ldots, y_r) = \sum_{k=1}^{K} \lfloor A_k(y_1, \ldots, y_r) \rfloor - \sum_{l=1}^{L} \lfloor B_l(y_1, \ldots, y_r) \rfloor$$

*is nonnegative for all $(y_1, \ldots, y_r) \in [0, 1]^r$.*

*Proof.* See [20]. □

The special case of this that we will use is the following.

**Lemma 2.2.** *Let*

$$u_n = u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!}.$$

*Then $u_n$ is an integer for all $n$ if and only if the function*

$$f(x) = f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

*is nonnegative for all $x$ between 0 and 1.*

*Proof.* Take $A_k(x) = a_k x$ and $B_l(x) = b_l x$ in Theorem 2.1. □

It also turns out that if $f(x; \mathbf{a}, \mathbf{b})$ is ever negative, then every prime that is large enough occurs as a factor in the denominator of $u_n(\mathbf{a}, \mathbf{b})$ for some $n$.

**Lemma 2.3.** *Let*

$$u_n = u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!}.$$

*If $u_n$ is not an integer for some $n$, then there exists some integer $P$ such that for each prime $p > P$ there exists some $n$ such that $v_p(u_n) < 0$ (where $v_p(u_n)$ is the $p$-adic valuation of $u_n$).*

*Proof.* Consider the $p$-adic valuation of $n!$. We have

$$v_p(n!) = \sum_{\alpha=1}^{\infty} \left\lfloor \frac{n}{p^\alpha} \right\rfloor .$$

Thus we have

$$v_p(u_n) = \sum_{\alpha=1}^{\infty} f\left(\frac{n}{p^\alpha}\right),$$

where $f(x) = f(x; \mathbf{a}, \mathbf{b})$.

Assuming that $u_n$ is not always an integer, we know from Lemma 2.2 that $f(x)$ is negative for some $x$. Since $f$ is a step function, it follows that there is some interval, say $[\beta, \beta + \epsilon]$ such that $f(x) < 0$ for all $x \in [\beta, \beta + \epsilon]$. Additionally, we know that there is some $\delta > 0$ such that $f(x) = 0$ for all $x \in [0, \delta]$. If we could find some $n$ and $p$ such that $n/p \in [\beta, \beta + \epsilon]$ and $n/p^2 \in [0, \delta]$, then we would have $f(n/p) < 0$ and $f(n/p^\alpha) = 0$ for all $\alpha > 1$, and so we would clearly have $\nu_p(u_n) < 0$.

Now, such an $n$ and $p$ need to simultaneously satisfy the two inequalities

$$p\beta \le n \le p(\beta + \epsilon)$$

and

$$0 \le n \le p^2\delta.$$

For $p$ large enough, say $p > P_1$, we have $p^2\delta > p(\beta+\epsilon)$, so it is sufficient for $n$ and $p$ to satisfy the first of the inequalities. Moreover, for any $p$ large enough, say $p > P_2$, we have $p\epsilon > 1$, so that there will in fact be an integer $n$ between $p\beta$ and $p(\beta+\epsilon)$. So in fact, for any $p > P = \max(P_1, P_2)$ we have that there exists an $n$ such that $\nu_p(u_n) < 0$. □

Along with Lemma 2.2, the following lemma, which is a simple generalization of [31, Proposition 3] will yield the full equivalence of Theorems 1.1, 1.2, and 1.3 and Theorems 1.1*, 1.2*, and 1.3*.

**Lemma 2.4.** *Suppose that $f(x)$ is a function of the form*

$$f(x) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

*with $a_k, b_l$ positive integers, and that $f(x)$ is bounded for all $x \in \mathbb{R}$. Then $\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l$ and, for any $n$, there exists some $x$ such that $f(x) = -n$ if and only if there exists some $x'$ such that $f(x') = L - K + n$. In particular, $f(x)$ is nonnegative if and only if the maximum value of $f$ is $L - K$.*

*Proof.* The first assertion is clear, as $f(n) = n\left(\sum a_k - \sum b_j\right)$ for $n \in \mathbb{Z}$, so if $\sum a_k \neq \sum b_l$, then $f(x)$ is unbounded. Now we know that $f(x)$ is periodic with period 1. Now, for any $z$ that is not an integer we have $\lfloor z \rfloor + \lfloor -z \rfloor = -1$, so for any $z$ for which none of $a_i z$, $b_j z$ is an integer, we have

$$f(z) + f(-z) = L - K,$$

from which the assertion follows. □

The following lemma describes explicitly the equivalence between the main theorems stated in the introduction and the theorems stated in this section.

**Lemma 2.5.** *Let* $\mathbf{a} = (a_1, a_2, \ldots, a_K), \mathbf{b} = (b_1, b_2, \ldots, b_L)$*, and put*

$$M = \operatorname{lcm}(a_1, a_2, \ldots, a_K, b_1, b_2, \ldots, b_L).$$

*Set*

$$(m_1, m_2, \ldots, m_{K+L}) = \left(\frac{M}{a_1}, \frac{M}{a_2}, \ldots, \frac{M}{a_K}, \frac{M}{b_1}, \frac{M}{b_2}, \ldots, \frac{M}{b_L}\right).$$

*Then the following are equivalent:*

1. $$f(x) = \sum_{i=1}^{K} \left\lfloor \frac{x}{m_i} \right\rfloor - \sum_{i=K+1}^{K+L} \left\lfloor \frac{x}{m_i} \right\rfloor$$

   *takes on values only in the range* $0, 1, \ldots, L - K$.

2. $\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l$ *and*

   $$u_n = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!}$$

   *is an integer for all* $n \in \mathbb{N}$.

*Proof.* $f(x)$ differs from $f(x; \mathbf{a}, \mathbf{b})$ only by a change of variables, so Lemma 2.2 tells us that $u_n \in \mathbb{Z}$ for all $n \geq 0$ if and only if $f(x) > 0$ for all $x \in [0, 1]$. Additionally, the boundedness of $f(x)$ is equivalent to the statement that $\sum a_k = \sum b_l$, and Lemma 2.4 tells us that $f(x)$ is bounded and nonnegative if and only if it maximum value is $L - K$.

□

The equivalences of Theorems 1.2 and 1.2* and of Theorems 1.3 and 1.3* are immediate from the above lemma. In Theorems 1.1 and 1.1*, the only complication that remains is that of classifying solutions with greatest common divisor 1.

*Proof of Theorem 1.1* (using Theorem 1.1).* The only complication that remains is that of classifying solutions with greatest common divisor 1. Consider the map $\phi : \mathbb{N}^K \times \mathbb{N}^L \to \mathbb{N}^K \times \mathbb{N}^L$ given by

$$\phi(a_1, a_2, \ldots, a_K, b_1, \ldots, b_L) = \left(\frac{M}{a_1}, \frac{M}{a_2}, \ldots, \frac{M}{a_K}, \frac{M}{b_1}, \ldots, \frac{M}{b_L}\right),$$

where

$$M = \operatorname{lcm}(a_1, a_2, \ldots, a_K, b_1, \ldots, b_L).$$

The image of $\phi$ is all $(K+L)$-tuples with greatest common divisor 1 and $\phi$ is bijective on this subset. Thus $\phi$, in combination with Lemma 2.5, gives a bijection between integral factorial ratios with greatest common divisor 1 and nonnegative step functions whose terms have greatest common divisor 1.

When we apply this map to the three families of factorial ratios listed in Theorem 1.1 we get the five families of step functions listed in Theorem 1.1* and the 52 sporadic step functions are given by the 52 sporadic integer factorial ratios. □

### 2.3 $G$-functions and the balancing condition

In Chapter 3, we will attach a generating function to a factorial ratio sequence by defining

$$\mathbf{u}(z) = \sum_{n=0}^{\infty} u_n(\mathbf{a}, \mathbf{b}) z^n.$$

One main reason that it is useful to look at one this function is that $\mathbf{u}(z)$ is algebraic if and only if $L = K + 1$, $\sum a_k = \sum b_l$, and $u_n(\mathbf{a}, \mathbf{b})$ is an integer for all $n$. One example of this is

$$\sum_{n=0}^{\infty} \binom{2n}{n} z^n = \frac{1}{\sqrt{1-4z}}.$$

In this section we point out that the balancing condition that $\sum a_k = \sum b_l$ corresponded exactly to the condition that $\mathbf{u}(z)$ is what is known as a $G$-function. A $G$-function is simply an analytic function that is given by a power series whose coefficients satisfy some growth and divisibility conditions. The precise definition is:

**Definition 2.6.** *An analytic function $f(z)$ given by a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

*with a positive radius of convergence is called a $G$-*function *if all of the following hold:*

1. *$a_i \in \overline{\mathbb{Q}}$ for all $i$.*

2. *$f(z)$ satisfies a linear differential equation with coefficients in $\mathbb{Q}(z)$.*

3. *There exists a $C$ such that $|a_n| \leq C^n$ for all sufficiently large $n$.*

4. *There exists a $C$ such that $\operatorname{lcm}(\operatorname{den}(a_1), \operatorname{den}(a_2), \ldots, \operatorname{den}(a_n)) \leq C^n$ for all sufficiently large $n$.*

$G$-functions grew out of Siegel's work in transcendence theory in 1929 [29]. Siegel was successful in proving results for $E$-functions (which have an identical definition except that $f(z) = \sum_{n=0}^{\infty} a_n \frac{z^n}{n!}$) and he gave some indication of what might be possible for $G$-functions. Later some irrationality results were obtained by Galočkin in 1974 [19] and Bombieri in 1981 [9]. For more on $G$-functions, also see the books by Dwork, Gerotto, and Sullivan [17] and André [1].

We now show that for the factorial ratios we consider the balancing condition that $\sum a_k = \sum b_l$, is exactly the condition which ensures $\mathbf{u}(z)$ is a $G$-function.

**Theorem 2.7.** *For positive integers $a_1, a_2, \ldots, a_K, b_1, b_2, \ldots, b_L$, let*

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!}.$$

*The following are equivalent.*

*(i) The function $\mathbf{u}(z) = \sum_{n=0}^{\infty} u_n(\mathbf{a}, \mathbf{b}) z^n$ is a $G$-function.*

*(ii) $\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l$.*

*(iii) $f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$ is bounded.*

*Proof.* The equivalence of (ii) and (iii) is part of Lemma 2.4.

For implication (i) $\implies$ (ii) we note that from Stirling's approximation it follows easily that if $\sum a_k \neq \sum b_l$, then either $u_n(\mathbf{a}, \mathbf{b})$ or $\frac{1}{u_n(\mathbf{a}, \mathbf{b})}$ grows faster than exponentially, so that conditions (3) and (4) in Definition 2.6 cannot be simlutaneously satisfied.

It remains to show that ((ii) and (iii)) $\implies$ (i), so we assume that $\sum a_k = \sum b_l$. Condition (1) of Definition 2.6 is satisfied because all the coefficients are in fact rational. Condition (2) follows from the fact that $\mathbf{u}(z)$ is a hypergeometric function, which will be proved as Lemma 3.3, and which implies that $\mathbf{u}(z)$ satisfies an associated hypergeometric differential equation. Condition (3) follows easily from Stirling's approximation, but we will also prove it below since it easily follows from the proof that condition (4) is satisfied.

The important point in proving that condition (4) is satisfied is that the associated step function $f(x) = f(x; \mathbf{a}, \mathbf{b})$ is bounded, so let us assume that $|f(x)| \leq B$. For any $p$, the number of times that $p$ divides $u_n$ is given by

$$v_p(u_n) = \sum_{\alpha=1}^{\infty} f(n/p^{\alpha}).$$

We also know that $f(x)$ is identically 0 on some half-open interval to the right of 0, so let us assume that $f(x) = 0$ for all $x \in [0, 1/A)$, for some integer $A$. Then in fact we may write

$$v_p(u_n) = \sum_{\alpha=1}^{\lfloor \log(An)/\log p \rfloor} f(n/p^\alpha).$$

Thus, for all $p$,

$$v_p(u_n) \leq B \frac{\log(An)}{\log p}, \tag{2.7}$$

and for $p > An$, $v_p(u_n) = 0$. Moreover, since the right hand side of (2.7) is increasing in $n$, we have that

$$\max_{m \leq n}(v_p(u_m)) \leq B \frac{\log(An)}{\log p}.$$

It follows that both $u_n$ and $\mathrm{lcm}(\mathrm{den}(u_1), \mathrm{den}(u_2), \ldots, \mathrm{den}(u_n))$ are bounded by $h(n)$, where

$$h(n) = \prod_{p \leq An} p^{B \log(An)/\log p} = \left( \prod_{p \leq An} p \right)^B (An)^{\pi(An)}$$

We will show shortly that $h(n)$ is grows at most exponentially, from which it follows that both conditions (3) and (4) are satisfied.

From the Prime Number Theorem (or from easier Chebyshev-type estimates), we know that for some $\epsilon$ and for all sufficiently large $x$, we have

$$\vartheta(x) := \sum_{p \leq x} \log p \leq x(1+\epsilon)$$

and

$$\pi(x) \leq \frac{x}{\log x}(1+\epsilon).$$

Thus

$$\prod_{p \leq An} p^B = e^{B\vartheta(An)} \leq e^{ABn(1+\epsilon)}$$

and

$$(An)^{\pi(An)} \leq (An)^{\frac{An}{\log An}(1+\epsilon)} = e^{An(1+\epsilon)}.$$

From these estimates it follows that for all sufficiently large $n$,

$$h(n) \leq C^n,$$

where $C = e^{(AB+A)(1+\epsilon)}$. □

# CHAPTER 3

## Factorial ratios of height 1

### 3.1 Introduction

In this chapter we prove Theorem 1.1, which gives a complete classification of balanced integer factorial ratio sequences of height one.

**Theorem 1.1.** *Let*

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_{K+1} n)!}$$

*and suppose that* $a_k \neq b_l$ *for all* $k, l$*, that* $\sum a_k = \sum b_l$*, and that*

$$\gcd(a_1, \ldots, a_K, b_1, \ldots, b_{K+1}) = 1.$$

*Then* $u_n(\mathbf{a}, \mathbf{b})$ *is an integer for all* $n$ *if and only if either*

1. $u_n = u_n(\mathbf{a}, \mathbf{b})$ *takes one of the following forms:*

$$u_n = \frac{[(a+b)n]!}{(an)!(bn)!} \text{ for } \gcd(a, b) = 1, \tag{3.1}$$

$$u_n = \frac{(2an)!(bn)!}{(an)!(2bn)![(a-b)n]!} \text{ for } \gcd(a, b) = 1 \text{ and } a > b, \tag{3.2}$$

$$u_n = \frac{(2an)!(2bn)!}{(an)!(bn)![(a+b)n]!} \text{ for } \gcd(a, b) = 1 \tag{3.3}$$

*or*

2. $(\mathbf{a}, \mathbf{b})$ *is one of the* 52 *sporadic parameter sets listed in the second column of Table 3.2.*

### 3.1.1 Notation

As before, throughout this chapter $\mathbf{a}$ and $\mathbf{b}$ denote ordered tuples of positive integers

$$\mathbf{a} = (a_1, a_2, \ldots, a_K)$$

and

$$\mathbf{b} = (b_1, b_2, \ldots b_L),$$

and $u_n(\mathbf{a}, \mathbf{b})$ denotes the factorial ratio

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!}.$$

In this chapter we are primarily concerned with the case when $L = K + 1$, but we will only specify this condition later.

We also note some other notation we will use in this chapter. The Pochhammer symbol $(\alpha)_n$ denotes the rising factorial

$$(\alpha)_n := (\alpha)(\alpha+1)(\alpha+2)\cdots(\alpha+n-1).$$

For $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_m)$, ${}_nF_m(\alpha; \beta; z)$ is the hypergeometric function

$${}_nF_m(\alpha; \beta; z) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k \cdots (\alpha_n)_k}{(\beta_1)_k(\beta_2)_k \cdots (\beta_m)_k} \frac{z^k}{k!}.$$

Also, $e(x) := \exp(2\pi i x) := e^{2\pi i x}$ and $\zeta_n = e(1/n)$ denotes the primitive $n$th root of unity with smallest positive argument.

It is useful to attach certain polynomials to $u_n(\mathbf{a}, \mathbf{b})$ as follows.

**Definition 3.1.** *Given positive integers $a_1, \ldots, a_K$ and $b_1, \ldots, b_L$ with $\sum a_k = \sum b_l$, define $P(x) = P(\mathbf{a}, \mathbf{b}; x) \in \mathbb{Z}[x]$ and $Q(x) = Q(\mathbf{a}, \mathbf{b}; x) \in \mathbb{Z}[x]$ to be relatively prime polynomials such that*

$$\frac{P(x)}{Q(x)} = \frac{(x^{a_1}-1)(x^{a_2}-1)\cdots(x^{a_K}-1)}{(x^{b_1}-1)(x^{b_2}-1)\cdots(x^{b_L}-1)}.$$

*Then for some $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_d$ and $\beta_1 \leq \beta_2 \leq \ldots \beta_d$, with $0 < \alpha_i, \beta_j \leq 1$, $P$ and $Q$ factor in $\mathbb{C}[x]$ as*

$$P(x) = (x - e(\alpha_1)) \cdots (x - e(\alpha_d))$$

*and*

$$Q(x) = (x - e(\beta_1)) \cdots (x - e(\beta_d)).$$

*where* $e(x) = e(2\pi i x)$.

*Set* $\alpha(\mathbf{a}, \mathbf{b}) = \{\alpha_1, \ldots, \alpha_d\}$ *and* $\beta(\mathbf{a}, \mathbf{b}) = \{\beta_1, \ldots, \beta_d\}$.

We will occasionally make use of the notion of the interlacing of two sets, so we state the following formally as a definition.

**Definition 3.2** (Interlacing)**.** *We say that two finite sets of real numbers $A$ and $B$ interlace if the function*

$$f(x) = \#\left((-\infty, x) \cap A\right) - \#\left((-\infty, x) \cap B\right)$$

*either takes only the values* $0$ *and* $1$*, or takes only the values* $-1$ *and* $0$*. In other words, there is an element of $A$ in between any two elements of $B$, and an element of $B$ in between any two elements of $A$.*

*We say that two sets $A$ and $B$ of complex numbers on the unit circle* interlace on the unit circle *if their arguments interlace on the real line, where we take the argument of a complex number to be in* $[0, 2\pi)$*.*

## 3.2 Connection between factorial ratios and hypergeometric series

Rodriguez-Villagas [25] observed a connection between hypergeometric series and factorial ratio sequences. The purpose of this section is to formulate this connection explicitly in order to use it for our classification.

We begin with a lemma to show that the generating function for $u_n(\mathbf{a}, \mathbf{b})$ is in fact a hypergeometric series.

**Lemma 3.3.** *Given positive integers $a_1, \ldots, a_L$, and $b_1, \ldots, b_L$ with $\sum a_k = \sum b_l$, let*

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!}$$

*and*

$$\mathbf{u}(\mathbf{a}, \mathbf{b}; z) = \sum_{n=0}^{\infty} u_n(\mathbf{a}, \mathbf{b}) z^n.$$

*Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d) = \alpha(\mathbf{a}, \mathbf{b})$ and $\beta = (\beta_1, \beta_2, \ldots, \beta_d) = \beta(\mathbf{a}, \mathbf{b})$, as in Definition 3.1, and let*

$$C = \frac{a_1^{a_1} \cdots a_K^{a_K}}{b_1^{b_1} \cdots b_L^{b_L}}.$$

*If $L > K$, then $\mathbf{u}(\mathbf{a}, \mathbf{b}; z)$ is the hypergeometric series*

$$\mathbf{u}(\mathbf{a}, \mathbf{b}; z) = {}_dF_{d-1}\left( \begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_d \\ \beta_1, \beta_2, \ldots, \beta_{d-1} \end{matrix} ; Cz \right).$$

*Otherwise, if $L \leq K$, then $\mathbf{u}(\mathbf{a}, \mathbf{b}; z)$ is the hypergeometric series*

$$\mathbf{u}(\mathbf{a}, \mathbf{b}; z) = {}_{d+1}F_d\left( \begin{matrix} \alpha_1, \alpha_2, \ldots, \alpha_d, 1 \\ \beta_1, \beta_2, \ldots, \beta_d \end{matrix} ; Cz \right).$$

*Proof.* Examine the ratio between two consecutive terms

$$\begin{aligned} A(n+1) &= \frac{u_{n+1}(\mathbf{a}, \mathbf{b})}{u_n(\mathbf{a}, \mathbf{b})} \\ &= \frac{(a_1(n+1))!(a_2(n+1))! \cdots (a_K(n+1))!}{(b_1(n+1))!(b_2(n+1))! \cdots (b_L(n+1))!} \times \left[ \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!} \right]^{-1}. \end{aligned}$$

After cancellation, this can be written as

$$A(n+1) = \frac{(a_1 n + 1)(a_1 n + 2) \cdots (a_1 n + a_1)(a_2 n + 1) \cdots (a_K n + a_K)}{(b_1 n + 1)(b_1 n + 2) \cdots (b_1 n + b_1)(b_2 n + 1) \cdots (b_L n + b_L)}.$$

Now if we factor out the coefficients of $n$ in each term we get

$$A(n+1) = C \frac{\left(n + \frac{1}{a_1}\right)\left(n + \frac{2}{a_1}\right) \cdots \left(n + \frac{a_1}{a_1}\right)\left(n + \frac{1}{a_2}\right) \cdots \left(n + \frac{a_K}{a_K}\right)}{\left(n + \frac{1}{b_1}\right)\left(n + \frac{2}{b_1}\right) \cdots \left(n + \frac{b_1}{b_1}\right)\left(n + \frac{1}{b_2}\right) \cdots \left(n + \frac{b_L}{b_L}\right)},$$

where

$$C = \frac{a_1^{a_1} \cdots a_K^{a_K}}{b_1^{b_1} \cdots b_L^{b_L}}.$$

If we remove the common factors in the fraction then for exactly the same $\alpha$ and $\beta$ as in Definition 3.1 we have

$$A(n+1) = C \frac{(n + \alpha_1) \cdots (n + \alpha_d)}{(n + \beta_1) \cdots (n + \beta_d)}.$$

Now, $u_0(\mathbf{a}, \mathbf{b}) = 1$, so we have in general

$$u_n(\mathbf{a}, \mathbf{b}) = \prod_{k=1}^{n} A(k) = C^n \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_d)_n}.$$

Now, if $L > K$, then $\beta_d = 1$, so

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{C^n}{n!} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_{d-1})_n}$$

and

$$\mathbf{u}(\mathbf{a}, \mathbf{b}; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_d)_n}{(\beta_1)_n \cdots (\beta_{d-1})_n} \frac{(Cz)^n}{n!} = {}_dF_{d-1}\left( \begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_d \\ \beta_1, \beta_2, \dots, \beta_{d-1} \end{matrix} ; Cz \right).$$

If, on the other hand, $L \leq K$, then $\beta_d \neq 1$, so we instead write

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{C^n}{n!} \frac{(\alpha_1)_n \cdots (\alpha_d)_n (1)_n}{(\beta_1)_n \cdots (\beta_d)_n},$$

and we find that

$$\mathbf{u}(\mathbf{a}, \mathbf{b}; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_d)_n (1)_n}{(\beta_1)_n \cdots (\beta_d)_n} \frac{(Cz)^n}{n!} = {}_{d+1}F_d\left( \begin{matrix} \alpha_1, \alpha_2, \dots, \alpha_d, 1 \\ \beta_1, \beta_2, \dots, \beta_d \end{matrix} ; Cz \right).$$

□

**Example 3.4.** *Let* $\mathbf{a} = (30, 1)$ *and* $\mathbf{b} = (15, 10, 6)$ *and*

$$u_n = u_n(\mathbf{a}, \mathbf{b}) = \frac{(30n)!n!}{(15n)!(10n)!(6n)!}.$$

*Consider the ratio* $\frac{u_{n+1}}{u_n}$. *This is*

$$\frac{(30n+1)(30n+2)\dots(30n+30)(n+1)}{(15n+1)\dots(15n+15)(10n+1)\dots(10n+10)(6n+1)\dots(6n+6)}.$$

*Factoring out the coefficients of* $n$ *in each term in the products, we get*

$$\frac{30^{30}(n+\frac{1}{30})(n+\frac{2}{30})\dots(n+\frac{30}{30})(n+1)}{15^{15}10^{10}6^6(n+\frac{1}{15})\dots(n+\frac{15}{15})(n+\frac{1}{10})\dots(n+\frac{10}{10})(n+\frac{1}{6})\dots(n+\frac{6}{6})}.$$

*Now there is a lot of clear cancellation in the fraction, and we see that this is*

$$\frac{30^{30}(n+\frac{1}{30})(n+\frac{7}{30})(n+\frac{11}{30})(n+\frac{13}{30})(n+\frac{17}{30})(n+\frac{19}{30})(n+\frac{23}{30})(n+\frac{29}{30})}{15^{15}10^{10}6^6(n+\frac{1}{5})(n+\frac{1}{3})(n+\frac{2}{5})(n+\frac{1}{2})(n+\frac{3}{5})(n+\frac{2}{3})(n+\frac{4}{5})(n+1)},$$

*which tells us that*

$$\sum_{n\geq 1} u_n z^n = {}_8F_7\left(\tfrac{1}{30},\tfrac{7}{30},\tfrac{11}{30},\tfrac{13}{30},\tfrac{17}{30},\tfrac{19}{30},\tfrac{23}{30},\tfrac{29}{30};\tfrac{1}{5},\tfrac{1}{3},\tfrac{2}{5},\tfrac{1}{2},\tfrac{3}{5},\tfrac{2}{3},\tfrac{4}{5};Cz\right),$$

*where*

$$C = \frac{30^{30}}{15^{15}10^{10}6^{6}}.$$

We will need to know that the hypergeometric series attached to a factorial ratio is essentially unique. We prove this in the next two lemmas.

**Lemma 3.5.** *Suppose that* $a_1 \geq a_2 \geq \ldots a_K > 0$*,* $b_1 \geq b_2 \geq \ldots, b_L > 0$ *and that*

$$u_n(\mathbf{a},\mathbf{b}) = \frac{(a_1n)!(a_2n)!\cdots(a_Kn)!}{(b_1n)!(b_2n)!\cdots(b_Ln)!} = 1$$

*for all* $n \geq 1$*. Then* $K = L$ *and* $\mathbf{a} = \mathbf{b}$*.*

*Proof.* The cases $K \geq L$ and $K \leq L$ are symmetric, so we may as well assume that $K \leq L$. We will prove the case $K = 1$ and then proceed by induction on $K$.

If $K = 1$ and $a_1 < b_1$, then it is clear that $u_n \to 0$ as $n \to \infty$. On the other hand, if $a_1 > b_1$, then by Dirichlet's Theorem on primes in arithmetic progressions, there exists some $m > 1$ such that $a_1m - 1 = p$ is prime. Then $p$ divides the numerator of $u_m$ but not the denominator, so $u_m \neq 1$. Now, if $a_1 = b_1$, then it is clear that $L = K = 1$.

The case for general $K$ proceeds similarly. We need only show that $a_1 = b_1$, and we are finished by induction. Again, if $a_1 > b_1$, then there is some $m > 1$ such that $a_1m - 1 = p$ is prime, and $p$ divides the numerator of $u_m$ but not the denominator. If, on the other hand, $b_1 > a_1$, we just reverse the argument and find an $m$ and $p$ such that $p$ divides the denominator of $u_m$ but not the numerator. By induction on $K$, we prove the lemma. □

**Lemma 3.6.** *The map*

$$(\mathbf{a},\mathbf{b}) \to u(\mathbf{a},\mathbf{b};z)$$

*is one-to-one on the set of pairs* $(\mathbf{a},\mathbf{b})$ *such that* $a_k \neq b_l$ *for all* $k, l$ *and* $a_1 \geq a_2 \cdots \geq a_K$*,* $b_1 \geq b_2 \geq \ldots \geq b_L$*.*

*Proof.* For some $(\mathbf{a},\mathbf{b})$ and $(\mathbf{a}',\mathbf{b}')$, we have

$$u(\mathbf{a},\mathbf{b};z) = u(\mathbf{a}',\mathbf{b}';z)$$

if and only if

$$u_n(\mathbf{a}, \mathbf{b}) = u_n(\mathbf{a}', \mathbf{b}')$$

for all $n$. In this case, we can rewrite this as

$$u_n(\mathbf{a}, \mathbf{b})(u_n(\mathbf{a}', \mathbf{b}'))^{-1} = u_n(\mathbf{a} \cup \mathbf{b}', \mathbf{b} \cup \mathbf{a}') = 1.$$

Now it follows from Lemma 3.5 that $\mathbf{a} \cup \mathbf{b}'$ is a permutation of $\mathbf{b} \cup \mathbf{a}'$. Thus $\mathbf{a}'$ is a permutation of $\mathbf{a}$ and $\mathbf{b}'$ is a permutation of $\mathbf{b}$. □

*Remark* 3.7. It is also possible to state Lemma 3.6 in an algorithmic manner. Roughly speaking, given parameters $\alpha = (\alpha_1, \ldots, \alpha_d)$ and $\beta = (\beta_1, \ldots, \beta_{d-1})$ that come from a factorial ratio, we can form the polynomials $P(x)$ and $Q(x)$ from Definition 3.1. It is then possible to add extra factors to $P(x)$ and $Q(x)$ together to obtain

$$\frac{P(x)}{Q(x)} = \frac{(x^{a_1} - 1)(x^{a_2} - 1) \cdots (x^{a_K} - 1)}{(x^{b_1} - 1)(x^{b_2} - 1) \cdots (x^{b_L} - 1)}$$

and to recover $\mathbf{a}$ and $\mathbf{b}$. In this manner, if we did not already know about the 52 sporadic integer factorial ratio sequences from Vasyunin's work, we could recover them from the work of Beukers and Heckman [6] described in Section 3.3.1.

The main interest in looking at hypergeometric series attached to factorial ratio sequences comes from the following observation of Rodriguez-Villegas [25].

**Theorem 3.8** (Rodriguez-Villegas [25])**.** *Let*

$$u_n(\mathbf{a}, \mathbf{b}) = \frac{(a_1 n)!(a_2 n)! \cdots (a_K n)!}{(b_1 n)!(b_2 n)! \cdots (b_L n)!}$$

*with* $\sum_{K=1}^{K} a_k = \sum_{l=1}^{L} b_l$ *and let*

$$\mathbf{u}(\mathbf{a}, \mathbf{b}; z) = \sum_{n=0}^{\infty} u_n(\mathbf{a}, \mathbf{b}) z^n.$$

*Then* $\mathbf{u}(\mathbf{a}, \mathbf{b}; z)$ *is algebraic over* $\mathbb{Q}(z)$ *if and only if* $L - K = 1$ *and* $u_n(\mathbf{a}, \mathbf{b}) \in \mathbb{Z}$ *for all* $n \geq 0$.

*Proof of Theorem* 3.8*, part 1.* We begin by proving that if the generating function is algebraic, then $u_n$ is in fact integral. In particular, it follows from Lemmas 2.2 and 2.4 that this will imply that we must have $L - K \geq 1$, which is, in fact, all that we need from this part of the proof.

A theorem of Eisenstein (see [18]) asserts that if $\mathbf{u}(\mathbf{a};\mathbf{b};z)$ is algebraic, then there exists an $N$ such that $u_n(\mathbf{a},\mathbf{b}) \cdot N^n$ in an integer for all $n$. But Lemma 2.3 implies that the set of primes occurring in the denominator of some $u_n(\mathbf{a},\mathbf{b})$ is either empty or infinite. So, if such an $N$ exists, then we are able to take $N = 1$, which implies that $u_n(\mathbf{a},\mathbf{b})$ is an integer for all $n$. □

The remainder of this proof relies on Landau's theorem and the following lemma of Beukers and Heckman [6].

**Lemma 3.9** (Beukers–Heckman [6]). *Let $\alpha_1, \alpha_2, \ldots \alpha_n$ and $\beta_1, \beta_2, \ldots \beta_{n-1}$ be rational numbers with common denominator $M$. The hypergeometric function*

$$_nF_{n-1}(\alpha_1, \alpha_2, \ldots \alpha_n; \beta_1, \beta_2, \ldots \beta_{n-1}; z)$$

*is algebraic if and only if for all $k$ relatively prime to $M$ the sequences*

$$e(k\alpha_1), \ldots, e(k\alpha_n)$$

*and*

$$e(k\beta_1), \ldots, e(k\beta_{n-1}), 1$$

*interlace on the unit circle.*

*Proof.* This follows from [6, Theorem 4.8] and the fact that this function is algebraic if and only if its monodromy group is finite. □

For the case of the hypergeometric functions that are generating series for $u_n(\mathbf{a},\mathbf{b})$ we can make this lemma slightly stronger.

**Lemma 3.10.** *For $\mathbf{a}$ and $\mathbf{b}$ positive integer vectors, the function*

$$\mathbf{u}(\mathbf{a};\mathbf{b};z) = \sum_{n=0}^{\infty} u_n(\mathbf{a},\mathbf{b}) z^n$$

*is an algebraic function if and only if $\alpha = \alpha(\mathbf{a},\mathbf{b})$ and $\beta = \beta(\mathbf{a},\mathbf{b})$ interlace on $[0,1]$.*

*Proof.* In our case the $\alpha_i$ and $\beta_j$ are rational numbers in $(0,1]$. Suppose that they have common denominator $M$. Recall that the numbers $e(\alpha_i)$ are roots of the polynomial $P(\mathbf{a},\mathbf{b};x)$, and that $P$ is the product of cyclotomic polynomials, say $P = \Phi_{m_1}\Phi_{m_2}\cdots\Phi_{m_l}$. Then for any $(k,M) = 1$ we also have $(k,m_i) = 1$ for all $m_i$. So the map $\alpha \to \alpha^k$ simply permutes the roots of any $\Phi_{m_i}$. In particular, it permutes the roots of $P$, and hence permutes the numbers $e(\alpha_i)$.

The exact same argument applies for $\beta$. Thus we have that $\alpha$ and $\beta$ interlace on $[0,1]$ if and only if $e(\alpha_i)^k$ and $e(\beta_j)^k$ interlace on the unit circle for all $k$ with $(k, M) = 1$.

In particular, $\mathbf{u}(\mathbf{a};\mathbf{b};z)$ is algebraic if and only if $\alpha$ and $\beta$ interlace on $[0,1]$. □

Combining these lemmas finishes the proof of 3.8.

*Proof of Theorem* 3.8*, part 2.* Suppose that $\mathbf{u}(\mathbf{a},\mathbf{b};z)$ is algebraic. Then we know that $L - K \geq 1$, from the first part of the proof. Note that the number of copies of the number 1 in the set $\beta$ is $L - K$. However, if $\alpha$ and $\beta$ are to interlace, no values can be repeated, so we must have $L - K = 1$.

Now, if $L - K = 1$, then from Lemma 2.5 we know that $u_n(\mathbf{a},\mathbf{b})$ is integral if and only if $\alpha$ and $\beta$ interlace. From Lemma 3.10 we know that this is equivalent to $\mathbf{u}(\mathbf{a},\mathbf{b};z)$ being algebraic. □

## 3.3 A Classification of Integral Factorial Ratios

### 3.3.1 Monodromy for Hypergeometric Functions ${}_nF_{n-1}$

This section is an application of the work of Beukers and Heckman [6], so we begin by restating a few necessary theorems and definitions.

**Definition 3.11** (Hypergeometric Groups)**.** *Let* $w_1, \ldots, w_n$ *and* $z_1, \ldots z_n$ *be complex numbers with* $w_i \neq z_j$ *for all* $i$ *and* $j$*. The hypergeometric group* $H(\mathbf{w},\mathbf{z})$ *with numerator parameters* $w_1, \ldots, w_n$ *and denominator parameters* $z_1, \ldots, z_n$ *is a subgroup of* $GL_n(\mathbb{C})$ *generated by elements*

$$h_0, h_1, \text{ and, } h_\infty$$

*such that*

$$h_0 h_1 h_\infty = 1$$

*and*

$$\det(t - h_\infty) = \prod_{j=1}^{n} (t - w_j)$$

$$\det(t - h_0^{-1}) = \prod_{j=1}^{n} (t - z_j)$$

*and such that* $h_1 - 1$ *has rank* 1*.*

Hypergeometric groups are precisely those groups which occur as monodromy groups for hypergeometric functions. Specifically, we have the following.

**Proposition 3.12.** *The monodromy group for the hypergeometric function*

$$ {}_nF_{n-1}(\alpha_1, \dots, \alpha_n; \beta_1, \dots, \beta_{n-1}; z), $$

*with* $\alpha_i \beta_j \in \mathbb{C}$, *is a hypergeometric group with numerator parameters*

$$ e(\alpha_1), e(\alpha_2), \dots, e(\alpha_n) $$

*and denominator parameters*

$$ e(\beta_1), e(\beta_2), \dots, e(\beta_{n-1}), 1, $$

*where* $e(z) = e^{2\pi i z}$.

*Proof.* This is [6, Proposition 3.2]. □

In categorizing hypergeometric groups it is useful to consider the following special subgroup.

**Definition 3.13.** *The subgroup* $H_r(\mathbf{w}, \mathbf{z})$ *of* $H(\mathbf{w}, \mathbf{z})$ *generated by* $h_\infty^k h_1 h_\infty^{-k}$ *for* $k \in \mathbb{Z}$ *is called the reflection subgroup of* $H(\mathbf{w}, \mathbf{z})$.

Also, the classification of hypergeometric groups splits into a primitive case and an imprimitive case, as in the following definition.

**Definition 3.14.** *Let* $G \subset GL(V)$ *be a subgroup acting irreducibly on a complex vector space* $V$. $G$ *is called* imprimitive *if there exists a direct sum decomposition* $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$ *with* $n > 1$ *and* $\dim V_i > 0$ *for all* $i$ *such that the action of* $G$ *on* $V$ *permutes the subspaces* $V_i$. *Otherwise,* $G$ *is called* primitive.

The existence of the following two theorems explains some of the usefulness of considering the reflection subgroup of a hypergeometric group and in considering when a hypergeometric group is primitive.

**Theorem 3.15.** *The reflection subgroup* $H_r(\mathbf{w}, \mathbf{z})$ *of* $H(\mathbf{w}, \mathbf{z})$ *acts reducibly on* $\mathbb{C}^n$ *if and only if there exists a root of unity* $\zeta \neq 1$ *such that multiplication by* $\zeta$ *permutes both the elements of* $\mathbf{w}$ *and the elements of* $\mathbf{z}$. *Moreover, if* $H_r(\mathbf{w}, \mathbf{z})$ *is reducible, then* $H(\mathbf{w}, \mathbf{z})$ *is imprimitive.*

*Proof.* This is [6, Theorem 5.3]. □

**Theorem 3.16.** *Suppose that $H_r(\mathbf{w}, \mathbf{z})$ is irreducible. Then $H$ is imprimitive if and only if there exist $p, q \in N$ with $p + q = n$ and $(p, q) = 1$, and $A, B, C \in \mathbb{C}^*$ such that $A^n = B^p C^q$ and such that*

$$\{w_1, \ldots, w_n\} = \{A, A\zeta_n, A\zeta_n^2, \ldots, A\zeta_n^{n-1}\}$$

*and*

$$\{z_1, \ldots, z_n\} = \{B, B\zeta_p, B\zeta_p^2, \ldots, B\zeta_p^{p-1}, C, C\zeta_q, C\zeta_q^2, \ldots, C\zeta_q^{q-1}\}$$

*where $\zeta_n = e(1/n)$, or with the same sets of equalities with $\mathbf{w}$ and $\mathbf{z}$ reversed.*

*Proof.* This is [6, Theorem 5.8]. □

As defined, hypergeometric groups are subgroups of $GL_n(\mathbb{C})$. The following proposition tells us when a hypergeometric group is defined over $GL_n(R)$ for $R \subset \mathbb{C}$.

**Proposition 3.17.** *Suppose $w_1, \ldots, w_n, z_1, \ldots, z_n \in \mathbb{C}^*$ with $w_i \neq z_j$ for all $i, j$. Let $A_1, \ldots A_n, B_1, \ldots, B_n$ be defined by*

$$\prod_{j=1}^{n}(t - w_j) = t^n + A_1 t^{n-1} + \cdots + A_n,$$

*and*

$$\prod_{j=1}^{n}(t - z_j) = t^n + B_1 t^{n-1} + \cdots + B_n.$$

*Then relative to a suitable basis, the hypergeometric group $H(\mathbf{w}, \mathbf{z})$ is defined over the ring $\mathbb{Z}[A_1, \ldots, A_n, B_1, \ldots, B_n, A_n^{-1}, B_n^{-1}]$.*

*Proof.* This is [6, Corollary 3.6], and follows directly from a theorem of Levelt [22, Theorem 1.1]. □

We need to state one more definition, and then we will be ready to state the main classification theorem of Beukers and Heckman that we are interested in.

**Definition 3.18.** *A scalar shift of the hypergeometric group $H(\mathbf{w}, \mathbf{z})$ is a hypergeometric group $H(d\mathbf{w}, d\mathbf{z}) = H(dw_1, dw_2, \ldots, dw_n; dz_1, dz_2, \ldots, dz_n)$ for some $d \in C^*$.*

Our main interest in the work of Beukers and Heckman comes from the following Theorem.

**Theorem 3.19.** *Let $n \geq 3$ and let $H(\mathbf{w}, \mathbf{z}) \subset GL_n(\mathbb{C})$ be a primitive hypergeometric group whose parameters are roots of unity and generate the field $\mathbb{Q}(\zeta_h)$. Then $H(\mathbf{w}, \mathbf{z})$ is finite if and only if, up to a scalar shift, the parameters have the form $w_1^k, w_2^k, \ldots, w_n^k; z_1^k, z_2^k, \ldots, z_n^k$, where $\gcd(h, k) = 1$ and the exponents of either $w_1, \ldots, w_n; z_1, \ldots, z_n$ or $z_1, \ldots, z_n; w_1, \ldots w_n$ are listed in [6, Table 8.3].*

*Proof.* This is [6, Theorem 7.1]. □

### 3.3.2 The Classification

From now on we set $L = K + 1$. We are interested in ratios where the parameters have greatest common divisor 1. The following lemma shows that this condition translates nicely into the reflection group of the monodromy group being irreducible.

**Lemma 3.20.** *Let $u_n = u_n(\mathbf{a}, \mathbf{b})$ and $\mathbf{u} = \mathbf{u}(\mathbf{a}, \mathbf{b}; z)$. Let $H(\mathbf{u})$ be the hypergeometric group associated to $\mathbf{u}$ and let $H_r(\mathbf{u})$ be the reflection subgroup of $H(\mathbf{u})$. Suppose that $u_n$ is an integer for all $n$. Then $H_r(\mathbf{u})$ acts reducibly on $\mathbb{C}$ if and only if*

$$\gcd(a_1, a_2, \ldots, a_K, b_1, b_2, \ldots, b_{K+1}) > 1.$$

*Moreover, if $H_r(\mathbf{u})$ is reducible, then $H(\mathbf{u})$ is imprimitive.*

*Proof.* Let $P = P(\mathbf{a}, \mathbf{b}; x)$ and $Q = Q(\mathbf{a}, \mathbf{b}; x)$. Then we have

$$\frac{P}{Q} = \frac{(x - e(\alpha_1)) \cdots (x - e(\alpha_d))}{(x - e(\beta_1)) \cdots (x - e(\beta_d))} = \frac{(x^{a_1} - 1) \cdots (x^{a_K} - 1)}{(x^{b_1} - 1) \cdots (x^{b_{K+1}} - 1)},$$

and $H(\mathbf{u})$ is a hypergeometric group with numerator parameters

$$e(\alpha_1), e(\alpha_2), \ldots, e(\alpha_d)$$

and denominator parameters

$$e(\beta_1), e(\beta_2), \ldots, e(\beta_d).$$

From Theorem 3.15 we know that $H_r(\mathbf{u})$ acts reducibly on $\mathbb{C}$ if and only if there exists some $\gamma \not\equiv 0 \pmod 1$ such that

$$\{e(\alpha_1), e(\alpha_2), \ldots, e(\alpha_d)\} = \{e(\alpha_1 + \gamma), e(\alpha_2 + \gamma), \ldots, e(\alpha_d + \gamma)\}$$

and

$$\{e(\beta_1), e(\beta_2), \ldots, e(\beta_d)\} = \{e(\beta_1 + \gamma), e(\beta_2 + \gamma), \ldots, e(\beta_d + \gamma)\}.$$

In this case, $e(\gamma)$ is necessarily a primitive $M$th root of unity for some $M$, and by raising it to an appropriate power we may assume that $\gamma$ is a primitive $p$th root of unity for some prime $p$. We will show that if multiplication by $e(\gamma)$ gives the desired permutations, then $p$ divides all of the $a_k$ and all of the $b_j$.

Recall that $P$ and $Q$ are products of cyclotomic polynomials, and let $Z_M$ denote the set of primitive $M$-th roots of unity. Notice that if $(M, p) = 1$, then multiplication by $e(\gamma)$ maps all primitive $M$-th roots of unity to primitive $Mp$-th roots of unity, while some primitive $Mp$-th roots of unity are mapped to others, and some are mapped to primitive $p$-th roots of unity. Thus, multiplication by $e(\gamma)$ gives a permutation of $Z_M \cup Z_{Mp}$. On the other hand, multiplication by $e(\gamma)$ simply permutes each of the sets $Z_{Mp^e}$ for $e > 1$.

Thus, if multiplication by $e(\gamma)$ separately permutes the $\alpha_i$ and the $\beta_j$, then whenever $P$ or $Q$ has a factor $\Phi_M$ with $(M, p) = 1$, it must also have a factor $\Phi_{Mp}$. Suppose then that there were some $a_l$ or $b_k$ coprime to $p$, and assume $M$ be the largest such. Then $(x^M - 1)$ would have $\Phi_M$ as a factor and any other term $(x^{a_k} - 1)$ or $(x^{b_j} - 1)$ which had $\Phi_M$ as a factor would necessarily have $\Phi_{Mp}$ as a factor as well. Ultimately, one of the polynomials $P$ or $Q$ would not have the terms $\Phi_M$ and $\Phi_{pM}$ properly paired as factors. Thus, if $p$ does not divide $\gcd(a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l)$, then multiplication by a primitive $p$th root of unity does not separately permute $\alpha(\mathbf{a}, \mathbf{b})$ and $\beta(\mathbf{a}, \mathbf{b})$.

On the other hand, if $p$ does divide $\gcd(a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_l)$, then multiplication by $e(1/p)$ permutes the roots of each of the factors $(x^{a_k} - 1)$ and $(x^{b_l} - 1)$, and hence it separately permutes the roots of $P$ and the roots of $Q$.

□

Vasyunin noticed that the step functions corresponding to

$$u_n = \frac{(2an)!(bn)!}{(an)!(2bn)![(a-b)n]!} \text{ with } b < a \tag{3.4}$$

and

$$u_n = \frac{(2an)!(2bn)!}{(an)!(bn)![(a+b)n]!} \tag{3.5}$$

are nonnegative. Thus for both of these families, $u_n$ is an integer for all $n$. It turns out that when $a$ and $b$ are not both odd these two infinite families give exactly those with factorial ratios with gcd 1 for which the hypergeometric group is imprimitive.

On the other hand, when $a$ and $b$ are both odd, these come from scalar shifts of the hypergeometric groups associated to binomial coefficients.

**Lemma 3.21.** *Let $u_n = u_n(\mathbf{a}, \mathbf{b})$ and $\mathbf{u} = \mathbf{u}(\mathbf{a}, \mathbf{b}; z)$. Let $H(\mathbf{u})$ be the hypergeometric group associated to $\mathbf{u}$ and let $H_r(\mathbf{u})$ be the reflection subgroup of $H(\mathbf{u})$. Suppose that $u_n$ is an integer for all $n$ and that $H_r(\mathbf{u})$ is irreducible; i.e.* $\gcd(a_1, \ldots, a_K, b_1, \ldots, b_L = 1$. *Then $H(\mathbf{u})$ is imprimitive if and only if $u_n$ is of the form* (3.4) *or* (3.5) *with $a$ and $b$ not both odd.*

*Proof.* Again let $P = P(\mathbf{a}, \mathbf{b}; x)$ and $Q = Q(\mathbf{a}, \mathbf{b}; x)$. Suppose that $u_n$ is the of form (3.4). Then we have

$$\frac{P}{Q} = \frac{(x^{2a}-1)(x^b-1)}{(x^{2b}-1)(x^a-1)(x^{a-b}-1)} = \frac{(x^a+1)}{(x^b+1)(x^{a-b}-1)},$$

which is in lowest terms if $a$ and $b$ are not both odd. Then the numerator parameters for $H(\mathbf{u})$ are

$$A, A\zeta_a^1, A\zeta_a^2, \ldots, A\zeta_a^{a-1}$$

and the denominator parameters are

$$B, B\zeta_b, B\zeta_b^2, \ldots, B\zeta_b^{b-1}, C\zeta_{a-b}, C\zeta_{a-b}^2, \ldots, C\zeta_{a-b}^{a-b-1}, 1$$

where $A = \zeta_{2a}$, $B = \zeta_{2b}$, and $C = 1$ satisfy $A^a = B^b C^{a-b} = -1$, so these parameters satisfy the condition of Theorem 3.16, and $H(\mathbf{u})$ is imprimitive.

Similarly, if $u_n$ is of the form (3.5) then we have

$$\frac{P}{Q} = \frac{(x^{2a}-1)(x^{2b}-1)}{(x^a-1)(x^b-1)(x^{a+b}-1)} = \frac{(x^a+1)(x^b+1)}{x^{a+b}-1},$$

again in lowest terms if $a$ and $b$ are not both odd, and so $H(\mathbf{u})$ is a hypergeometric group with numerator parameters

$$A, A\zeta_a^1, A\zeta_a^2, \ldots, A\zeta_a^{a-1}, B, B\zeta_b, B\zeta_b^2, \ldots, B\zeta_b^{b-1}$$

and denominator parameters

$$C\zeta_{a+b}, C\zeta_{a+b}^2, \ldots, C\zeta_{a+b}^{a+b}$$

where $A = \zeta_{2a}$, $B = \zeta_{2b}$, and $C = 1$. $A$ and $B$ satisfy $A^a = B^b = -1$, so $A^a B^b = 1 = C^{a+b}$, so $H(\mathbf{u})$ again satisfies the conditions of the theorem.

To see the converse, suppose that the numerator parameters of $H(u)$ are of the form

$$A, A\zeta_a, A\zeta_a^2, \ldots, A\zeta_a^{a-1}.$$

These parameters must have the property that, if they contain one primitive $M$th root of unity for some $M$, then they contain all of them. Thus, by symmetry considerations, we find that the only possibilities are that $A = 1$ or $A = \zeta_{2a}$. However, we cannot have $A = 1$, as one of the denominator parameters for a hypergeometric group coming from an integral factorial ratio sequence will be 1, and the numerator and denominator parameters must be distinct. Thus, $A = \zeta_{2a}$. Similarly, for the denominator parameters we find that either $B$ or $C$ is 1, and so without loss of generality, we assume that $C = 1$ and find that $B = \zeta_{2b}$ is the only possibility. Indeed, whenever $(a, b) = 1$ and $a$ and $b$ are not both odd, this does work and gives $u_n$ of the form (3.4).

A similar argument works for the second case. □

We now examine the case where both $a$ and $b$ are odd.

**Lemma 3.22.** *Let* $u_n = u_n(\mathbf{a}, \mathbf{b})$ *and* $\mathbf{u} = \mathbf{u}(\mathbf{a}, \mathbf{b}; z)$. *Let* $H(\mathbf{u})$ *be the hypergeometric group associated to* $\mathbf{u}$.

*(i) If* $u_n$ *is of the form* (3.4) *with* $a$ *and* $b$ *both odd, then* $H(\mathbf{u})$ *is a scalar shift by* $-1 = e(1/2)$ *of* $H(\mathbf{u}')$, *where*

$$u_n' = \binom{an}{bn}.$$

*(ii) If* $u_n$ *is of the form* (3.5) *with* $a$ *and* $b$ *both odd, then* $H(\mathbf{u})$ *is obtained by taking a scalar shift of* $H(\mathbf{u}')$ *and reversing the numerator and denominator parameters, where*

$$u_n' = \binom{(a+b)n}{an}.$$

*Proof.* (i) Suppose that $u_n$ is of the form (3.4). Then, as in the previous lemma, we have for $P(x) = P(\mathbf{a}, \mathbf{b}; x)$ and $Q(x) = Q(\mathbf{a}, \mathbf{b}; x)$,

$$\frac{P(x)}{Q(x)} = \frac{(x^a + 1)}{(x^b + 1)(x^{a-b} - 1)}.$$

This is not in lowest terms, but it is in a convenient form for computing the scalar shift of $H(\mathbf{u})$. The scalar shift corresponds to multiplying each root of $P$ and $Q$ by

$-1$, in which case we obtain polynomials $P^*$ and $Q^*$ with

$$\frac{P^*(x)}{Q^*(x)} = \frac{(x^a - 1)}{(x^b - 1)(x^{a-b} - 1)},$$

which very clearly come from $u'_n = \binom{an}{bn}$.

(ii) If $u_n$ is of the form (3.5), we proceed similarly, except that this time we find that

$$\frac{P^*(x)}{Q^*(x)} = \frac{(x^a - 1)(x^b - 1)}{(x^{a+b} - 1)},$$

which we can see comes from $u'_n = \binom{(a+b)n}{an}$, with the numerator and denominator parameters reversed. □

We are now ready to prove Theorem 1.1

*Proof of Theorem* 1.1. Lemma 3.21 classifies all of those integral factorial ratios whose associated hypergeometric group is imprimitive, so it remains to classify those with $\gcd(a_1, \ldots, a_K, b_1, \ldots, b_L)$, whose associated hypergeometric groups are primitive. Beukers and Heckman have categorized all finite primitive hypergeometric groups, so we can examine [6, Table 8.3] to find all integral factorial ratios with associated primitive hypergeometric groups.

Specifically, if $H$ is a primitive hypergeometric group that comes from an integral factorial ratio sequence, then $H$ is either one of the entries in [6, Table 8.3] or a scalar shift of one of the entries, possibly with the numerator and denominator parameters reversed. Moreover, if $H$ has numerator parameters $\alpha_1, \ldots, \alpha_d$ and denominator parameters $\beta_1, \ldots, \beta_d$, then the polynomials $P(x) = \prod(x - e(\alpha_i))$ and $Q(x) = \prod(x - e(\beta_j))$ must have coefficients in $\mathbb{Z}$. Aside from Line 1, there are 26 entries in the table with this property. Moreover, Beukers and Heckman list the entries in the table so that if the polynomials $P$ and $Q$ formed from the numerator and denominator parameters of an entry have coefficients in a field $K$, then the polynomials formed from a scalar shift of that entry will never have coefficients in a proper subfield of $K$, so we only need to consider these 26 entries, along with Line 1, which we will mention momentarily.

As the denominator parameters for a hypergeometric group coming from a factorial ratio must contain a 1, there are only a finite number of scalar shifts of each entry that we need to consider. It is easy to enter the data from [6, Table 8.3] into a computer program and to check each of the scalar shifts of each of these 26 entries. It turns out that there are 52 hypergeometric group parameter sets coming from integral factorial ratios, and these are all given by these 26 entries and scalar shifts by $-1$ of these entries, possibly with numerator and denominator parameters reversed.

It is easily seen that Line 1 of [6, Table 8.3] corresponds to the infinite family of binomial coefficient sequences

$$\frac{[(a+b)n]!}{(an)!(bn)!} \text{ with } \gcd(a,b)=1,$$

and, as we have seen in Lemma 3.22, scalar shifts of this family by $-1$ yield factorial ratios of the forms (3.4) and (3.5) already considered. We need only to rule out other scalar shifts for this family.

The scalar shifts of this family that we need to consider are shifts by roots of unity of the form $e(-n/a)$, $e(-n/b)$ and $e\left(\frac{-n}{a+b}\right)$. The cases of multiplication by $e(-n/a)$ and $e(-n/b)$ are symmetric, so let us consider multiplication by $\zeta = e(-n/a)$. $\zeta$ is a primitive $Mth$ root of unity for some $M$ dividing $a$, so we may as well assume that $\zeta = e(1/M)$. But then $\zeta \cdot e(1/b)$ is a primitive $bM$-th root of unity, as $(M,b)=1$. If $M \neq 2$ then some of the primitive $bM$-th roots of unity are mapped to themselves by multiplication by $e(1/M)$, so not all primitive $bM$-th roots of unity can be obtained in this way, so the polynomials $P$ and $Q$ obtained from this shift cannot have integer coefficients unless $M=2$, which is the case of a scalar shift by $-1$.

The case for multiplication by a root of the form $e\left(\frac{-n}{a+b}\right)$ is completely similar, since $a$, $b$, and $a+b$ are relatively prime in pairs.

Lemma 3.6 assures us that this must be all integer factorial ratio sequences in which the parameters have greatest common divisor 1, giving us the "only if" part of the theorem. □

### 3.4 A Listing of all Integral Factorial Ratios with $L-K=1$

The following tables contain a listing of all integer factorial ratios with gcd 1, organized as follows. The second column lists the parameters for $u_n(\mathbf{a},\mathbf{b})$. The parameter $d$ of the third column is the dimension of the monodromy group of $\mathbf{u}(\mathbf{a},\mathbf{b};z)$, and the fourth column lists the specific parameters for ${}_dF_{d-1}$. With the exception of lines 2 and 3 of Table 3.1 all the entries have primitive monodromy groups, and so they have a corresponding entry in [6, Table 8.3].

It should be noted that lines 1, 2, and 3 of Table 3.1 only correspond to solutions with gcd 1 when $\gcd(a,b)=1$.

Table 3.1: The three infinite families of integral factorial ratio sequences

| Line # | $u_n(\mathbf{a},\mathbf{b})$ | $d$ | ${}_dF_{d-1}$ parameters | [6] Line # |
|---|---|---|---|---|
| $1_{a,b}$ | [a+b]<br>[a,b] | $a+b-1$ | $[\frac{1}{a+b},\frac{2}{a+b},\ldots,\frac{a+b-1}{a+b}]$<br>$[\frac{1}{a},\frac{2}{a},\ldots,\frac{a-1}{a},\frac{1}{b},\frac{2}{b},\ldots,\frac{b-1}{b}]$ | 1 |
| $2_{a,b}$ | [2a,b]<br>[a,2b,a−b] | $a$ | $[\frac{1}{2a},\frac{3}{2a},\ldots\frac{2a-1}{2a}]$<br>$[\frac{1}{2b},\frac{3}{2b},\ldots\frac{2b-1}{2b},\frac{1}{a-b},\frac{2}{a-b},\ldots,\frac{a-b-1}{a-b}]$ | None |
| $3_{a,b}$ | [2a,2b]<br>[a,b,a+b] | $a+b$ | $[\frac{1}{2a},\frac{3}{2a},\ldots\frac{2a-1}{2a},\frac{1}{2b},\frac{3}{2b},\ldots\frac{2b-1}{2b}]$<br>$[\frac{1}{a+b},\frac{2}{a+b},\ldots,\frac{a+b-1}{a+b}]$ | None |

Table 3.2: The 52 sporadic integral factorial ratio sequences

| Line # | $u_n(\mathbf{a},\mathbf{b})$ | $d$ | ${}_dF_{d-1}$ parameters | [6] Line # |
|---|---|---|---|---|
| 1 | [12,1]<br>[6,4,3] | 4 | $\left[\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}\right]$<br>$\left[\frac{1}{3},\frac{1}{2},\frac{2}{3}\right]$ | 37 |
| 2 | [12,3,2]<br>[6,6,4,1] | 4 | $\left[\frac{1}{12},\frac{5}{12},\frac{7}{12},\frac{11}{12}\right]$<br>$\left[\frac{1}{6},\frac{1}{2},\frac{5}{6}\right]$ | 37 |
| 3 | [12,1]<br>[8,3,2] | 6 | $\left[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{7}{12},\frac{5}{6},\frac{11}{12}\right]$<br>$\left[\frac{1}{8},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{7}{8}\right]$ | 45 |
| 4 | [12,3]<br>[8,6,1] | 6 | $\left[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{7}{12},\frac{2}{3},\frac{11}{12}\right]$<br>$\left[\frac{1}{8},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{7}{8}\right]$ | 45 |
| 5 | [12,3]<br>[6,5,4] | 6 | $\left[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{7}{12},\frac{2}{3},\frac{11}{12}\right]$<br>$\left[\frac{1}{5},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{4}{5}\right]$ | 46 |
| 6 | [12,5]<br>[10,4,3] | 6 | $\left[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{7}{12},\frac{5}{6},\frac{11}{12}\right]$<br>$\left[\frac{1}{10},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{9}{10}\right]$ | 46 |
| 7 | [18,1]<br>[9,6,4] | 6 | $\left[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{11}{18},\frac{13}{18},\frac{17}{18}\right]$<br>$\left[\frac{1}{4},\frac{1}{3},\frac{1}{2},\frac{2}{3},\frac{3}{4}\right]$ | 47 |
| 8 | [9,2]<br>[6,4,1] | 6 | $\left[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}\right]$<br>$\left[\frac{1}{6},\frac{1}{4},\frac{1}{2},\frac{3}{4},\frac{5}{6}\right]$ | 47 |
| 9 | [9,4]<br>[8,3,2] | 6 | $\left[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}\right]$<br>$\left[\frac{1}{8},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{7}{8}\right]$ | 48 |
| 10 | [18,4,3]<br>[9,8,6,2] | 6 | $\left[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{11}{18},\frac{13}{18},\frac{17}{18}\right]$<br>$\left[\frac{1}{8},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{7}{8}\right]$ | 48 |
| 11 | [9,1]<br>[5,3,2] | 6 | $\left[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}\right]$<br>$\left[\frac{1}{5},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{4}{5}\right]$ | 49 |
| 12 | [18,5,3]<br>[10,9,6,1] | 6 | $\left[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{11}{18},\frac{13}{18},\frac{17}{18}\right]$<br>$\left[\frac{1}{10},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{9}{10}\right]$ | 49 |

| Line # | $u_n(\mathbf{a},\mathbf{b})$ | $d$ | ${}_dF_{d-1}$ parameters | [6] Line # |
|---|---|---|---|---|
| 13 | [18,4]<br>[12,9,1] | 7 | $\left[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}\right]$<br>$\left[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{7}{12},\frac{2}{3},\frac{11}{12}\right]$ | 58 |
| 14 | [12,2]<br>[9,4,1] | 7 | $\left[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{5}{6},\frac{11}{12}\right]$<br>$\left[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}\right]$ | 58 |
| 15 | [18,2]<br>[9,6,5] | 7 | $\left[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}\right]$<br>$\left[\frac{1}{5},\frac{1}{3},\frac{2}{5},\frac{3}{5},\frac{2}{3},\frac{4}{5}\right]$ | 59 |
| 16 | [10,6]<br>[9,5,2] | 7 | $\left[\frac{1}{10},\frac{1}{6},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{5}{6},\frac{9}{10}\right]$<br>$\left[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}\right]$ | 59 |
| 17 | [14,3]<br>[9,7,1] | 7 | $\left[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}\right]$<br>$\left[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{5}{9},\frac{7}{9},\frac{8}{9}\right]$ | 60 |
| 18 | [18,3,2]<br>[9,7,6,1] | 7 | $\left[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}\right]$<br>$\left[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7}\right]$ | 60 |
| 19 | [12,2]<br>[7,4,3] | 7 | $\left[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{5}{6},\frac{11}{12}\right]$<br>$\left[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7}\right]$ | 61 |
| 20 | [14,6,4]<br>[12,7,3,2] | 7 | $\left[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}\right]$<br>$\left[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{7}{12},\frac{2}{3},\frac{11}{12}\right]$ | 61 |
| 21 | [14,1]<br>[7,5,3] | 7 | $\left[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}\right]$<br>$\left[\frac{1}{5},\frac{1}{3},\frac{2}{5},\frac{3}{5},\frac{2}{3},\frac{4}{5}\right]$ | 62 |
| 22 | [10,6,1]<br>[7,5,3,2] | 7 | $\left[\frac{1}{10},\frac{1}{6},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{5}{6},\frac{9}{10}\right]$<br>$\left[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{4}{7},\frac{5}{7},\frac{6}{7}\right]$ | 62 |
| 23 | [15,1]<br>[9,5,2] | 8 | $\left[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}\right]$<br>$\left[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{1}{2},\frac{5}{9},\frac{7}{9},\frac{8}{9}\right]$ | 63 |
| 24 | [30,9,5]<br>[18,15,10,1] | 8 | $\left[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\right]$<br>$\left[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}\right]$ | 63 |
| 25 | [15,4]<br>[12,5,2] | 8 | $\left[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}\right]$<br>$\left[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{5}{6},\frac{11}{12}\right]$ | 64 |
| 26 | [30,5,4]<br>[15,12,10,2] | 8 | $\left[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\right]$<br>$\left[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{2}{3},\frac{11}{12}\right]$ | 64 |
| 27 | [15,4]<br>[8,6,5] | 8 | $\left[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}\right]$<br>$\left[\frac{1}{8},\frac{1}{6},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{5}{6},\frac{7}{8}\right]$ | 65 |
| 28 | [30,5,4]<br>[15,10,8,6] | 8 | $\left[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\right]$<br>$\left[\frac{1}{8},\frac{1}{3},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{2}{3},\frac{7}{8}\right]$ | 65 |
| 29 | [15,2]<br>[10,4,3] | 8 | $\left[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}\right]$<br>$\left[\frac{1}{10},\frac{1}{4},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{3}{4},\frac{9}{10}\right]$ | 66 |
| 30 | [30,3,2]<br>[15,10,6,4] | 8 | $\left[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\right]$<br>$\left[\frac{1}{5},\frac{1}{4},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{3}{4},\frac{4}{5}\right]$ | 66 |
| 31 | [30,1]<br>[15,10,6] | 8 | $\left[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\right]$<br>$\left[\frac{1}{5},\frac{1}{3},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{2}{3},\frac{4}{5}\right]$ | 67 |

| Line # | $u_n(\mathbf{a},\mathbf{b})$ | $d$ | ${}_dF_{d-1}$ parameters | [6] Line # |
|---|---|---|---|---|
| 32 | [15,2]<br>[10,6,1] | 8 | $\left[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}\right]$<br>$\left[\frac{1}{10},\frac{1}{6},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{5}{6},\frac{9}{10}\right]$ | 67 |
| 33 | [15,7]<br>[14,5,3] | 8 | $\left[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}\right]$<br>$\left[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}\right]$ | 68 |
| 34 | [30,5,3]<br>[15,10,7,6] | 8 | $\left[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\right]$<br>$\left[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{1}{2},\frac{4}{7},\frac{5}{7},\frac{6}{7}\right]$ | 68 |
| 35 | [30,5,3]<br>[15,12,10,1] | 8 | $\left[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\right]$<br>$\left[\frac{1}{12},\frac{1}{4},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{3}{4},\frac{11}{12}\right]$ | 69 |
| 36 | [15,6,1]<br>[12,5,3,2] | 8 | $\left[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}\right]$<br>$\left[\frac{1}{12},\frac{1}{4},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{3}{4},\frac{11}{12}\right]$ | 69 |
| 37 | [15,1]<br>[8,5,3] | 8 | $\left[\frac{1}{15},\frac{2}{15},\frac{4}{15},\frac{7}{15},\frac{8}{15},\frac{11}{15},\frac{13}{15},\frac{14}{15}\right]$<br>$\left[\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4},\frac{7}{8}\right]$ | 70 |
| 38 | [30,5,3,2]<br>[15,10,8,6,1] | 8 | $\left[\frac{1}{30},\frac{7}{30},\frac{11}{30},\frac{13}{30},\frac{17}{30},\frac{19}{30},\frac{23}{30},\frac{29}{30}\right]$<br>$\left[\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{3}{4},\frac{7}{8}\right]$ | 70 |
| 39 | [20,3]<br>[12,10,1] | 8 | $\left[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}\right]$<br>$\left[\frac{1}{12},\frac{1}{6},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{5}{6},\frac{11}{12}\right]$ | 71 |
| 40 | [20,6,1]<br>[12,10,3,2] | 8 | $\left[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}\right]$<br>$\left[\frac{1}{12},\frac{1}{3},\frac{5}{12},\frac{1}{2},\frac{7}{12},\frac{2}{3},\frac{11}{12}\right]$ | 71 |
| 41 | [20,1]<br>[10,8,3] | 8 | $\left[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}\right]$<br>$\left[\frac{1}{8},\frac{1}{3},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{2}{3},\frac{7}{8}\right]$ | 72 |
| 42 | [20,3,2]<br>[10,8,6,1] | 8 | $\left[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}\right]$<br>$\left[\frac{1}{8},\frac{1}{6},\frac{3}{8},\frac{1}{2},\frac{5}{8},\frac{5}{6},\frac{7}{8}\right]$ | 72 |
| 43 | [20,1]<br>[10,7,4] | 8 | $\left[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}\right]$<br>$\left[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{1}{2},\frac{4}{7},\frac{5}{7},\frac{6}{7}\right]$ | 73 |
| 44 | [20,7,2]<br>[14,10,4,1] | 8 | $\left[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}\right]$<br>$\left[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}\right]$ | 73 |
| 45 | [20,3]<br>[10,9,4] | 8 | $\left[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}\right]$<br>$\left[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{1}{2},\frac{5}{9},\frac{7}{9},\frac{8}{9}\right]$ | 74 |
| 46 | [20,9,6]<br>[18,10,4,3] | 8 | $\left[\frac{1}{20},\frac{3}{20},\frac{7}{20},\frac{9}{20},\frac{11}{20},\frac{13}{20},\frac{17}{20},\frac{19}{20}\right]$<br>$\left[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}\right]$ | 74 |
| 47 | [24,1]<br>[12,8,5] | 8 | $\left[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}\right]$<br>$\left[\frac{1}{5},\frac{1}{4},\frac{2}{5},\frac{1}{2},\frac{3}{5},\frac{3}{4},\frac{4}{5}\right]$ | 75 |
| 48 | [24,5,2]<br>[12,10,8,1] | 8 | $\left[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}\right]$<br>$\left[\frac{1}{10},\frac{1}{4},\frac{3}{10},\frac{1}{2},\frac{7}{10},\frac{3}{4},\frac{9}{10}\right]$ | 75 |
| 49 | [24,4,1]<br>[12,8,7,2] | 8 | $\left[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}\right]$<br>$\left[\frac{1}{7},\frac{2}{7},\frac{3}{7},\frac{1}{2},\frac{4}{7},\frac{5}{7},\frac{6}{7}\right]$ | 76 |
| 50 | [24,7,4]<br>[14,12,8,1] | 8 | $\left[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}\right]$<br>$\left[\frac{1}{14},\frac{3}{14},\frac{5}{14},\frac{1}{2},\frac{9}{14},\frac{11}{14},\frac{13}{14}\right]$ | 76 |

| Line # | $u_n(\mathbf{a},\mathbf{b})$ | $d$ | ${}_dF_{d-1}$ parameters | [6] Line # |
|---|---|---|---|---|
| 51 | [24,4,3]<br>[12,9,8,2] | 8 | $\left[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}\right]$<br>$\left[\frac{1}{9},\frac{2}{9},\frac{4}{9},\frac{1}{2},\frac{5}{9},\frac{7}{9},\frac{8}{9}\right]$ | 77 |
| 52 | [24,9,6,4]<br>[18,12,8,3,2] | 8 | $\left[\frac{1}{24},\frac{5}{24},\frac{7}{24},\frac{11}{24},\frac{13}{24},\frac{17}{24},\frac{19}{24},\frac{23}{24}\right]$<br>$\left[\frac{1}{18},\frac{5}{18},\frac{7}{18},\frac{1}{2},\frac{11}{18},\frac{13}{18},\frac{17}{18}\right]$ | 77 |

# CHAPTER 4

# Proof of Theorem 1.2

## 4.1 Introduction

In this chapter, which is joint work with Jason Bell [5], we give a proof of Theorem 1.2*, which we recall gives a relation between the height and the length of a balanced nonnegative step function of the form

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor .$$

From Lemma 2.5, this also proves Theorem 1.2. We recall that the theorem we wish to prove is

**Theorem 1.2*.** *Fix $L - K = D$. If*

$$f(x) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

*where*

$$\sum_{k=1}^{K} a_k = \sum_{l=1}^{L} b_l$$

*and $a_k \neq b_l$, $a_k, b_l \in \{1, 2, 3, \ldots\}$ for all $k, l$, and if*

$$f(x) \geq 0$$

*for all $x$, then*

$$K + L \ll D^2 (\log D)^2.$$

From Lemma 2.4 we know that if $\sum a_k = \sum b_l$, then $f(x; \mathbf{a}, \mathbf{b})$ is nonnegative if and only if its maximum value is exactly $L - K$. Therefore in order to prove that such a function cannot be nonnegative if $L+K$ is large (in terms of $L-K$) we will show that $f(x)$ must take on large values if $L+K$ is large. We do this by giving the following explicit lower bound on the $L^2$ norm of $f$.

**Theorem 4.1.** *Let*

$$f(x)=\sum_{k=1}^{K}\lfloor a_k x\rfloor-\sum_{l=1}^{L}\lfloor b_l x\rfloor$$

*where*

$$\sum_{k=1}^{K}a_k=\sum_{l=1}^{L}b_l$$

*and* $a_k\neq b_l$, $a_k,b_l\in\{1,2,3,\ldots\}$ *for all* $k,l$. *Then for all* $M>285$

$$\int_0^1 f(x)^2\mathrm{d}x\geq\frac{(K-L)^2}{4}+\frac{(K+L)}{2\pi^2}\left[\frac{e^{-\gamma}}{(\log M)}\left(1-\frac{1}{2(\log M)^2}\right)\right]^2 - \frac{(K+L)^2}{2\pi^2}\left[\frac{2}{(M-1)^{1/2}}\left(\frac{e^{-\gamma}}{(\log M)}\left(1-\frac{1}{2(\log M)^2}\right)\right)^{-1}+\frac{1}{M-1}\right], \quad (4.1)$$

*where* $\gamma:=\lim_{N\to\infty}\left(\sum_{n=1}^{N}\frac{1}{n}-\log n\right)\approx 0.577215664901533$ *is Euler's constant.*

*Remark* 4.2. In this theorem $M$ is a parameter that we will be free to choose to optimize this lower bound. We obtain easily stated asymptotic results below by choosing $M=(K+L)^4$ and give examples of optimized bounds that can be obtained in Section 4.3. (The condition that $M>285$ in the above statement comes from our use of Rosser and Schoenfeld's [26] explicit formulation of Merten's Theorem (see Lemma 4.8).)

Granting this theorem for a moment, we may now easily prove Theorem 1.2*.

*Proof of Theorem 1.2*.* Take $M$ to be $(K+L)^4$. Then for fixed $K-L$, equation (4.1) gives

$$\max_{x\in\mathbb{R}}|f(x)|^2\geq\int_0^1 f(x)^2\mathrm{d}x\gg\frac{(K+L)}{(\log(K+L))^2}.$$

Now let $N=K+L$ and $A=\max_{x\in\mathbb{R}}|f(x)|$. Then we have

$$\frac{N}{(\log N)^2}\ll A^2.$$

Additionally, since this implies that $A\gg N^{1/3}$, we may write

$$N\ll N\frac{(\log A)^2}{(\log N)^2}\ll A^2(\log A)^2. \quad (4.2)$$

Now take $D = L - K$ fixed. Lemma 2.4 says that if $f(x) \geq 0$ for all $x$, then the maximum value of $f(x)$ is $D$. Thus $A = D$ in this case, and Equation (4.2) gives the desired bound. □

### 4.1.1 Notation

$\lfloor x \rfloor$ denotes the *floor* of $x$, which is the largest integer less than or equal to $x$. $\{x\} := x - \lfloor x \rfloor$ is the *fractional part* of $x$. Also $e(x) := \exp(2\pi i x) := e^{2\pi i x}$, and $\gamma$ denotes Euler's constant

$$\gamma := \lim_{N\to\infty} \left( \sum_{n=1}^{N} \frac{1}{n} - \log n \right) \approx 0.577215664901533.$$

## 4.2 An explicit lower bound for the $L^2$ norm

To prove Theorem 4.1, we will use Parseval's identity, so we start by computing the Fourier coefficients of $f(x; \mathbf{a}, \mathbf{b})$. Note that if $f$ is balanced, so that $\sum a_k = \sum b_k$, then using the relation $x = \lfloor x \rfloor + \{x\}$ we can rewrite $f(x; \mathbf{a}, \mathbf{b})$ as $f(x; \mathbf{a}, \mathbf{b}) = \sum_{l=1}^{L} \{b_l x\} - \sum_{k=1}^{K} \{a_k x\}$.

**Lemma 4.3.** *Suppose that* $f(x) = \sum_{l=1}^{L} \{b_l x\} - \sum_{k=1}^{K} \{a_k x\}$. *Then the Fourier expansion of* $f$ *is*

$$f(x) = \frac{L-K}{2} + \frac{1}{2\pi i} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{1}{n} \left[ \sum_{a_k | n} a_k - \sum_{b_l | n} b_l \right] e(nx).$$

*That is,* $\hat{f}(0) = (L-K)/2$ *and for* $n \neq 0$

$$\hat{f}(n) = \frac{1}{2\pi i n} \left[ \sum_{a_k | n} a_k - \sum_{b_l | n} b_l \right].$$

*Proof.* The Fourier expansion for the fractional part of $x$ is

$$\{x\} = \frac{1}{2} - \frac{1}{2\pi i} \sum_{\substack{n\in\mathbb{Z} \\ n\neq 0}} \frac{e(nx)}{n}. \tag{4.3}$$

Using this we can write

$$\sum_{l=1}^{L}\{b_l x\} - \sum_{k=1}^{K}\{a_k x\} = \frac{L-K}{2} - \frac{1}{2\pi i}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{1}{n}\left[\sum_{l=1}^{L} e(nb_l x) - \sum_{k=1}^{K} e(na_k x)\right].$$

Changing the order of summation we get

$$\sum_{k=1}^{K}\lfloor a_k x\rfloor - \sum_{l=1}^{L}\lfloor b_l x\rfloor = \frac{L-K}{2} + \frac{1}{2\pi i}\left[\sum_{k=1}^{K}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{e(na_k x)}{n} - \sum_{l=1}^{L}\sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}}\frac{e(nb_l x)}{n}\right].$$

Now extracting the coefficient of $e(mx)$ in the sum we see that for $m \neq 0$ we have

$$\hat{f}(m) = \frac{1}{2\pi i}\left[\sum_{\substack{n,a_k\\ na_k=m}}\frac{1}{n} - \sum_{\substack{n,b_l\\ nb_l=m}}\frac{1}{n}\right].$$

Now the result follows on replacing the $n$ in the sum with $n = m/a_k$. □

*Remark* 4.4. From now on we set $D = L - K$. It is convenient to subtract off the first Fourier coefficient of $f(x)$ and to consider

$$\int_0^1 \left|f(x) - \frac{D}{2}\right|^2 \mathrm{d}x.$$

On considering the Fourier expansion, it is easy to see that

$$\int_0^1 |f(x)|^2 \,\mathrm{d}x = \int_0^1 \left|f(x) - \frac{D}{2}\right|^2 \mathrm{d}x + \frac{D^2}{4}, \tag{4.4}$$

so it is simple to transfer a lower bound on $\int_0^1 \left|f(x) - \frac{D}{2}\right|^2 \mathrm{d}x$ to a lower bound on $\int_0^1 |f(x)|^2 \,\mathrm{d}x$. Notice also that $\left|\hat{f}(n)\right| = \left|\hat{f}(-n)\right|$, so it follows from Parseval's theorem that

$$\int_0^1 |f(x) - D/2|^2 \,\mathrm{d}x = 2\sum_{n=1}^{\infty}\left|\hat{f}(n)\right|^2. \tag{4.5}$$

*Remark* 4.5. We now notice a Möbius inversion-type formula for the Fourier coefficients of $f(x) = f(x;\mathbf{a},\mathbf{b})$. Let

$$g(n) = g(n;\mathbf{a},\mathbf{b}) = \#\{a_k : a_k = n\} - \#\{b_l : b_l = n\}. \tag{4.6}$$

Then from Lemma 4.3 we see that for $n \geq 1$ we have

$$\hat{f}(n) = \frac{1}{2\pi i} \sum_{d|n} \frac{dg(d)}{n}.$$

Or, forming the Dirichlet series

$$G(s) = D(g, s) = \sum_{n=1}^{\infty} \frac{g(n)}{n^s}$$

(note that $G$ is actually given by a finite sum) and

$$F(s) = D(\hat{f}, s) = \sum_{n=1}^{\infty} \frac{\hat{f}(n)}{n^s},$$

we have the relation

$$G(s)\zeta(s+1) = 2\pi i F(s), \tag{4.7}$$

where $\zeta(s) = D(1, s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann $\zeta$-function.

To estimate the mean square of $f$ we use the following theorem of Carlson [14] to relate $\sum_{n=1}^{\infty} \left|\hat{f}(n)\right|^2$ to a mean value of $|G(it)|^2$.

**Proposition 4.6.** *Let $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ in some half-plane. Then if $f(s)$ analytically continues to the half plane $\sigma \geq \alpha$ and satisfies $|f(\sigma + it)| \ll |t|^A$ on all vertical lines $\sigma \geq \alpha$, and if*

$$\frac{1}{2T} \int_{-T}^{T} |f(\alpha + it)|^2 \, \mathrm{d}t$$

*is bounded as $T \to \infty$, then*

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + it)|^2 \, \mathrm{d}t = \sum_{n=1}^{\infty} \frac{|a_n|^2}{n^{2\sigma}}.$$

*Proof.* See [30, Section 9.51]. □

It follows immediately from this proposition and the relation (4.7) that for all $\sigma > 0$ we have

$$\sum_{n=1}^{\infty} \frac{\left|\hat{f}(n)\right|}{n^{2\sigma}} = \lim_{T \to \infty} \int_{-T}^{T} |G(\sigma + it)\zeta(1 + \sigma + it)|^2 \, \mathrm{d}t. \tag{4.8}$$

Eventually we will let $\sigma$ tend to zero, but first we estimate the right hand side of

(4.8) using a truncated Euler product for the $\zeta$-function. Define the main term

$$\zeta_M(s) = \prod_{p \le M} \left(1 - \frac{1}{p^s}\right)^{-1},$$

and let $\zeta_R(s)$ be defined by $\zeta(s) = \zeta_M(s) + \zeta_R(s)$. We will use the following lemmas which give bounds on the size of $\zeta_M$.

**Lemma 4.7.** *If* $\mathrm{Re}(s) \ge 1$ *then*

$$\frac{1}{\zeta_M(1)} \le |\zeta_M(s)| \le \zeta_M(1). \tag{4.9}$$

*Proof.* This is just a restatement of the fact that for all $s$ with real part greater than 1, and all integers $n > 0$, we have

$$\left(1 - \frac{1}{n}\right) \le \left|1 - \frac{1}{n^s}\right|^{-1} \le \left(1 - \frac{1}{n}\right)^{-1}.$$

□

**Lemma 4.8** (Effective Mertens' bound)**.** *For all* $M > 285$,

$$\zeta_M(1) \le \frac{\log M}{e^{-\gamma}} \left(1 - \frac{1}{2(\log M)^2}\right)^{-1} \tag{4.10}$$

*Proof.* See Rosser and Schoenfeld [26, Theorem 7]. □

*Proof of Theorem 4.1.* We now write

$$\frac{1}{2T}\int_{-T}^{T} |G(\sigma+it)\zeta(1+\sigma+it)|^2 \,\mathrm{d}t = \frac{1}{2T}\int_{-T}^{T} |G(\sigma+it)\zeta_M(1+\sigma+it)|^2 \,\mathrm{d}t + E(M,T),$$

where

$$E(M,T) = \frac{1}{2T}\int_{-T}^{T} |G(\sigma+it)|\,|\zeta_M(1+\sigma+it) + \zeta_R(1+\sigma+it)|^2 \,\mathrm{d}t - \frac{1}{2T}\int_{-T}^{T} |G(\sigma+it)\zeta_M(1+\sigma+it)|^2 \,\mathrm{d}t.$$

Applying the triangle inequality in the form $||A| - |B|| \le |A - B|$ to the above, we obtain

$$|E(M,T)| \le \frac{1}{2T}\int_{-T}^{T} |G(\sigma+it)|^2 \, |\zeta_R(1+\sigma+it)|^2 \,\mathrm{d}t$$
$$+ \frac{1}{2T}\int_{-T}^{T} 2\left|G(\sigma+it)^2\zeta_M(1+\sigma+it)\zeta_R(1+\sigma+it)\right| \mathrm{d}t.$$

On inserting the bound for $\zeta_M$ from Lemma 4.7, the bound $|G(\sigma+it)| \le K+L$, and applying Cauchy's inequality to the integral $\int_{-T}^{T} |\zeta_R(1+\sigma+it)| \,\mathrm{d}t$, we have

$$|E(M,T)| \le 2(K+L)^2\zeta_M(1)\left[\frac{1}{2T}\int_{-T}^{T} |\zeta_R(1+\sigma+it)|^2 \,\mathrm{d}t\right]^{1/2}$$
$$+ (K+L)^2\frac{1}{2T}\int_{-T}^{T} |\zeta_R(1+\sigma+it)|^2 \,\mathrm{d}t.$$

Define $E(M) := \lim_{T\to\infty} E(M,T)$. Then

$$\begin{aligned} |E(M)| &\le 2(K+L)^2\zeta_M(1)\left[\sum_{n\ge M}\frac{1}{n^{2+2\sigma}}\right]^{1/2} + (K+L)^2\sum_{n\ge M}\frac{1}{n^{2+2\sigma}} \\ &\le 2(K+L)^2\zeta_M(1)\left[\sum_{n\ge M}\frac{1}{n^{2}}\right]^{1/2} + (K+L)^2\sum_{n\ge M}\frac{1}{n^{2}}. \end{aligned}$$

Inserting the bound

$$\sum_{n\ge M}\frac{1}{n^2} \le \frac{1}{M-1},$$

we get

$$|E(M)| \le \frac{2(K+L)^2\zeta_M(1)}{(M-1)^{1/2}} + \frac{(K+L)^2}{M-1}.$$

Now,

$$\frac{1}{2T}\int_{-T}^{T} |F(\sigma+it)|^2 \,\mathrm{d}t = \frac{1}{(2\pi)^2}\frac{1}{2T}\int_{-T}^{T} |G(\sigma+it)\zeta_M(1+\sigma+it)|^2 \,\mathrm{d}t + \frac{E(M,T)}{4\pi^2},$$

so inserting the bound from our Lemma 4.7 and taking the limit as $T\to\infty$ we get

$$\begin{aligned} \sum_{n=1}^{\infty}\left|\hat{f}(n)\right|^2 &\ge \frac{1}{\zeta_M(1)^2}\frac{1}{(2\pi)^2}\sum_{n=1}^{\infty}|g(n)|^2 + \frac{E(M)}{4\pi^2} \\ &\ge \frac{1}{\zeta_M(1)^2}\frac{1}{(2\pi)^2}\sum_{n=1}^{\infty}|g(n)|^2 - \frac{|E(M)|}{4\pi^2}. \end{aligned}$$

Putting in the bounds for $E(M)$, we get

$$\sum_{n=1}^{\infty}\left|\hat{f}(n)\right|^2 \geq \frac{1}{\zeta_M(1)^2}\frac{1}{(2\pi)^2}(K+L) - \frac{(K+L)^2}{4\pi^2}\left[\frac{2\zeta_M(1)}{(M-1)^{1/2}} + \frac{1}{M-1}\right].$$

And finally, putting in the bounds on $\zeta_M(1)$, we get

$$\sum_{n=1}^{\infty}\left|\hat{f}(n)\right|^2 \geq \frac{(K+L)}{4\pi^2}\left[\frac{e^{-\gamma}}{(\log M)}\left(1-\frac{1}{2(\log M)^2}\right)\right]^2 - \frac{(K+L)^2}{4\pi^2}\left[\frac{2}{(M-1)^{1/2}}\left(\frac{e^{-\gamma}}{(\log M)}\left(1-\frac{1}{2(\log M)^2}\right)\right)^{-1} + \frac{1}{M-1}\right]. \quad (4.11)$$

The theorem now follows immediately from (4.11) and fact that

$$\int_0^1 f(x)^2 \mathrm{d}x = 2\sum_{n=1}^{\infty}\left|\hat{f}(n)\right|^2 + \frac{(K-L)^2}{4}.$$

□

### 4.3 Explicit bounds and remarks

Let $\mathbf{B}(D)$ denote the maximal number of terms that a step function of the form

$$f(x;\mathbf{a},\mathbf{b}) = \sum_{k=1}^{K}\lfloor a_k x\rfloor - \sum_{l=1}^{L}\lfloor b_l x\rfloor$$

may have if $D = L - K$, $a_i, b_j \in \mathbb{N}$, $\sum a_i = \sum b_j$ and $a_i \neq b_j$ for all $i, j$, and $f(x) \geq 0$ for all $x$. With this notation, Theorem 1.2* says that $\mathbf{B}(D)$ is finite, and that, moreover $\mathbf{B}(D) \ll D^2 \log D$. Using equation 4.11 we may compute some explicit upper bounds on $\mathbf{B}(D)$.

For example, when $D = 1$ such a function $f(x)$ that only takes on the values 0 and 1 must have $\sum_{n=1}^{\infty}\left|\hat{f}(n)\right|^2 = \frac{1}{4}$. If $K + L = 112371$ and $M = 112371^{4.96215}$, the right side of 4.11 is $\approx 0.250000802$. Additionally, the right side of (4.11) is an increasing function of $K + L$ when $M$ is set to be a fixed power of $K + L$, so it follows that $\mathbf{B}(1) < 112371$.

Similarly we obtain $\mathbf{B}(2) < 502827$ by taking $L + K = 502827$ and $M = 502827^{4.6602}$, in which case the right side of (4.11) is $\approx 1.00000138$, but it must be the case that $\sum_{n=1}^{\infty}\left|\hat{f}(b)\right|^2 \leq 1$ if $f(x)$ takes values in precisely in the range $\{0, 1, 2\}$.

These bounds are far from best possible. From Theorem 1.1* we know that $\mathbf{B}(1) = 9$, and computations suggest that it might be the case that $\mathbf{B}(2) = 18$. Additionally, for the case of $D = 3$, a direct computer search has checked that there are no functions with 29 terms that take values only in the range $\{0, 1, 2, 3\}$ when $\sum a_i = \sum b_j \leq 60$. (However, the search space would have to be greatly enlarged for this to be truly convincing evidence that $\mathbf{B}(3) = 27$.)

# CHAPTER 5

## Proof of Theorem 1.3

In this section we prove a generalization of Theorem 1.1* which says that for any fixed $D = L - K$, there exists a classification of nonnegative step functions

$$f(x; \mathbf{a}, \mathbf{b}) = \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor$$

such that $\sum a_k = \sum b_l$ similar to the classification given for $D = 1$. Specifically, if we fix $L$ and $K$, then the function $f(x; \mathbf{a}, \mathbf{b})$ is nonnegative for all $x$ if and only if $\mathbf{a}$ and $\mathbf{b}$ lie in a subset of $\mathbb{R}^{K+L}$ which can be written as the union of a finite number of subspaces of $\mathbb{R}^{K+L}$.

The theorem we will prove is

**Theorem 1.3*.** *Fix positive integers $K$ and $L$, and an integer $A \leq 0$. The set $\mathcal{S}$ of $a_1, a_2, \ldots, a_K, b_1, b_2, \ldots b_L$ such that*

1. $\sum a_k = \sum b_l$
2. *$a_k$ and $b_l$ are nonnegative integers for all $k, l$*
3. $A \leq \sum_{k=1}^{K} \lfloor a_k x \rfloor - \sum_{l=1}^{L} \lfloor b_l x \rfloor \leq L - K - A$

*can be written as*

$$\mathcal{S} = (\mathbb{Z}_{\geq 0})^{K+L} \bigcap \left( \bigcup_{n=1}^{N} V_n \right),$$

*where each $V_n$ is a subspace of $\mathbb{R}^{K+L}$.*

We will prove this theorem as an application of a theorem of J. Lawrence [21] about closed subgroups of the torus. Lawrence's theorem is a theorem about the set of subgroups which miss a given *full* subset of either $\mathbb{R}^n$ or $\mathbb{T}^n$. We will not need the full strength of Lawrence's result here, though, because open subsets of the torus are a particular example of full subsets. The result that we will use is

**Theorem 5.1** ([21, Corollary 1.A]). *If $U$ is an open subset of $\mathbb{T}^n$, then there are only finitely many maximal open subgroups $H$ of $\mathbb{T}^n$ such that $H \cap U = \emptyset$.*

*Proof of Theorem 1.3*.* Let $F(x_1, \ldots x_K, y_1, \ldots, y_L) = -\sum_{k=1}^{K}\{x_k\} + \sum_{l=1}^{L}\{y_l\}$, and let $V$ denote the subspace of $\mathbb{R}^{K+L}$ defined by the equation $\sum_{k=1}^{K} x_k = \sum_{l=1}^{L} y_l$. We may consider $F$ as a function from $\mathbb{R}^{K+L}$ to $\mathbb{R}$, since $F$ has period 1 in each of its variables, we may consider it as a function from $\mathbb{T}^{K+L} = \mathbb{R}^{K+L}/\mathbb{Z}^{K+L}$ to $\mathbb{R}$. Note that, when restricted to the subspace $V$, $F(x_1, \ldots x_K, y_1, \ldots, y_L) = \sum_{k=1}^{K}\lfloor x_k\rfloor - \sum_{l=1}^{L}\lfloor y_l\rfloor$.

For each subspace $G$ of $V$ let $S_G$ denote the set

$$S_G := \{(x_1, \ldots, x_K, y_1, \ldots, y_L) \in G \text{ such that } F(x_1, \ldots, x_K, y_1, \ldots y_L) < A\}.$$

Note that as $F$ is not continuous, $S_G$ is not, in general, open. We will consider in general, however whether or not $S_G$ contains an open set. In fact, we will show that the set of subspaces $G$ of $V$ such that $S_G$ does not contain a nonempty open subset (with respect to the subspace topology induced on $G$) has finitely many maximal elements.

Granting this for a moment, let $V_1, V_2, \ldots, V_N$ denote these subspaces. The conditions that $f(x; \mathbf{a}, \mathbf{b}) \geq A$ for all $x$ and $f(x; \mathbf{a}, \mathbf{b}) \leq L - K - A$ for all $x$ are equivalent, so we focus on the first. Suppose now that $f(x; \mathbf{a}, \mathbf{b}) \geq A$ for all $x$, and for some $\mathbf{a}, \mathbf{b}$ satisfying the desired properties. Then $F(a_1x, a_2x, \ldots, a_Kx, b_1x, \ldots, b_Lx) \geq A$ for all $x$, so we have a one dimensional subspace of $\mathbb{R}^{K+L}$ on which $F$ is larger than $A$, which must be contained within one of the $V_n$. In particular, $(a_1, a_2, \ldots, a_K, b_1, \ldots, b_L)$ is contained in one of the $V_n$.

On the other hand, suppose that some set of positive integers

$$(a_1, a_2, \ldots, a_K, b_1, b_2, \ldots, b_L) \in V_n, \text{ for some } n.$$

We must show that $f(x) = f(x; \mathbf{a}, \mathbf{b}) \geq A$ for all $x$. We argue by contradiction, constructing an open set in $V_n$ where $F(x_1, \ldots, x_K, y_1, \ldots, y_L) < A$. Suppose that $f(c) < A$. As all $a_1, \ldots, a_K, b_1, \ldots, b_L$ are positive, $f(x)$ is right continuous and piecewise constant, so we may assume that $f(x) < A$ for all $x \in [c, d)$. Furthermore, we may choose $d$ small enough so that none of $a_kx$ or $b_lx$ is an integer for $x \in (c, d)$. Let $x_0 = (c + d)/2$. Then $F(x_1, \ldots, x_K, y_1, \ldots, y_L)$ is continuous at $P = (a_1x_0, a_2x_0, \ldots, a_Kx_0, b_1x_0, \ldots, b_Lx_0)$, and it is strictly less than $A$ at this point. Thus it is strictly less than $A$ in an open neighborhood $U$ of $P$. In particular, $F < A$ in $U \cap V_n$, which is open with respect to the subspace topology on $V_n$, which is a contradiction.

We now construct these maximal subgroups inductively using Lawrence's theorem. If $S_V$ does not contain a nonempty open subset, then we are done.

Otherwise, consider $\tilde{V} \subset \mathbb{T}^{K+L} = \mathbb{R}^{K+L}/\mathbb{Z}^{K+L}$, the image of $V$ in the quotient. Since $V$ has a basis of rational vectors, $\tilde{V}$ is a proper closed subgroup of $\mathbb{T}^{K+L}$. $F$ is periodic modulo 1 in each of its variables, so it is well defined on $\mathbb{T}^{K+L}$, so we may consider $S_V$ as a subset of $\tilde{V}$. By assumption, it contains a nonempty open subset $U$.

$\tilde{V}$ itself is a $K + L - 1$ dimensional torus, so by Theorem 5.1 the set of closed subgroups $G$ of $V$ which miss $U$ has finitely many maximal elements $G_1, \ldots, G_m$. Each of these is a subgroup of dimension strictly less than the dimension of $\tilde{V}$, and, in turn, for each of these subgroups $S_G$ either does or does not contain a nonempty open set. If $S_G$ does not contain an open set, it is one of the maximal subgroups that we are looking for. Otherwise, we proceed recursively, applying this same argument to each $G$ such that $S_G$ contains an open set, and ignoring discrete subgroups, Since the dimension is reduced at each step, this process eventually terminates. Non-discrete subgroups of $\mathbb{T}^{K+L}$ are in one to one correspondence with subspaces of $\mathbb{R}^{K+L}$, so by lifting these subgroups to $\mathbb{R}^{K+L}$ we obtain the subgroups we are looking for. □

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