HYPERBOLIC RANK-RIGIDITY AND FRAME FLOW

by

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For my parents

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CHAPTER I

Introduction

1.1 An overview of dynamics

In general terms, this dissertation contributes to the intersection of geometry and dynamics. It presents a result on geometric rigidity. Underlying the work here is an approach that emphasizes the interplay of dynamical techniques and the geometry of the spaces on which the dynamics take place. In this introductory chapter, I will begin by presenting some relevant background material, and attempt to motivate some of the main ideas behind the approaches taken in this work. I hope to give some idea of how various combinations of geometric and dynamical techniques can prove particularly fruitful. Introductory material more specific to the result here is presented in the final section of this chapter.

Classically, dynamics studies transformations of spaces, often called *phase spaces*. These transformations can be a single map from the space to itself, or a flow on the space. It is natural to think of these transformations as taking place in time: the iteration of a single transformation marks a discrete time sequence, a flow takes place in continuous time. Of particular interest in dynamics is long-term behavior associated to these transformations. How does the movement of a point under a flow or transformation – its orbit – behave in the long run?

How do average values of a function over segments of an orbit behave as that segment becomes longer? What measures or other objects are preserved by a given dynamical system?

Under a modern view, these transformations are examples of group actions and dynamics studies their behavior, particularly its long-term or asymptotic aspects. A group action is an assignment to each element of a group a transformation of the phase space in a way coherent with the algebraic structure of the group. More precisely, one has the following definition:

Definition 1.1.1 (Group action)**.** A left-action of a group G on a space X is map $G \times X \to X$ sending $(g, x) \mapsto g \cdot x$ satisfying

1. $g \cdot (h \cdot x) = (gh) \cdot x$ for any $g, h \in G$ and any $x \in X$
2. $e \cdot x = x$ for e the identity in G and any $x \in X$.

A right-action may be defined similarly. More concisely, the group action is a group homomorphism from G to $Bij(X)$, the group of bijections of X.

Of most interest are actions for which only e acts trivially on X, i.e. for which the kernel of the homomorphism into $Bij(X)$ is trivial; such actions are called *faithful*. In this framework, the discrete dynamics of an iterated transformation is a $\mathbb{Z}$-action and a flow is an $\mathbb{R}$-action. The acting group parametrizes 'time'; the extension to general acting groups allows, in some sense, a consideration of dynamical systems that evolve in 'generalized time.'

Consider two instructive examples:

Example 1.1.2 (Geodesic flow)**.** Let M be a smooth manifold, and take as phase space the unit tangent bundle T^1M. Let γ_v denote the geodesic with initial tangent vector v. The geodesic flow g_t takes a vector v to the vector tangent to γ_v a distance t along γ_v. To verify that the flow corresponds to an $\mathbb{R}$-action note that moving a tangent vector along a geodesic for distance t and then distance s is equivalent to moving it distance $t+s$. This dynamical system reflects the geometry of M very strongly; for example, periodic orbits correspond to closed geodesics. A dynamical system based on the geodesic flow will be of fundamental importance in chapter II.

Example 1.1.3. Let H be a group with subgroups Γ and G. G acts on the coset space H/Γ by left-multiplication: $g \cdot h\Gamma = gh\Gamma$. This example is particularly interesting when H is a Lie group and Γ is a discrete group. In this case H/Γ is a manifold. This dynamical system is strongly tied to the geometry of H/Γ as well as the underlying algebra of the groups H and G.

Dynamics only becomes interesting or useful if the group action preserves some structure on the phase space, for example a topology, a differentiable structure or a measure. For example, that the geodesic flow preserves the differentiable structure on the unit tangent bundle is crucial for the study of that system. This dissertation works with actions that preserve a measure. For example 1.1.2 this is the Liouville measure, the product measure of the Riemannian volumes on the manifold M and on the unit tangent spheres at each point; for example 1.1.3 if H has bi-invariant Haar measure the projection of this measure to H/Γ is invariant under the G-action.

Ergodic theory is the study of measure-preserving actions; its basic notion is that of an *ergodic measure*.

Definition 1.1.4 (Ergodic measure)**.** A G-invariant measure μ on X is called ergodic for a G-action if for any G-invariant, measurable set $A \subset X$, $\mu(A) = 0$ or $\mu(X \backslash A) = 0$.

This definition involves both the group action and the measure; one sometimes also says the action is ergodic for the measure. Ergodicity indicates that, up to zero measure sets, there is no way to cut up the phase space into smaller pieces preserved by the dynamics. The following theorem reinforces the utility of this idea and indicates why ergodic measures are the basic units of study in measure-preserving dynamics.

Theorem 1.1.5 (Ergodic decomposition)**.** *Any Borel probability measure μ on a space X invariant under a continuous group action can be written as a direct integral of ergodic measures. That is, there is a partition (modulo null sets) of X into invariant subsets X_α, with $\alpha \in A$, A a standard Borel space with measure ν, and an invariant ergodic measure μ_α on each X_α such that for any μ-measurable function f,*

$$\int_X f d\mu = \int_A \int_{X_\alpha} f d\mu_\alpha d\nu.$$

This theorem indicates that the basic building blocks of all invariant measures are the ergodic measures, and hence their study is fundamental to ergodic theory. The most fundamental tool of ergodic theory is the Birkhoff Ergodic Theorem:

Theorem 1.1.6 (Birkhoff Ergodic Theorem, see [23] Theorem 4.1.2)**.** *Let T be an invertible transformation on X preserving a probability measure μ and let f be in $L^1(X, \mu)$. Define the forward and backward time averages*

$$f^+(x) := lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x))$$

$$f^-(x) := lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^{-i}(x)).$$

Then these exist and agree for μ-almost every $x \in X$. Furthermore, if μ is ergodic, then

$$f^+(x) = f^-(x) = \int_X f d\mu$$

for μ-almost every $x \in X$.

The analogous version of this theorem for flows also holds. The intuitive content of this theorem is that, for ergodic measures, time averages equal the space average – the average of a function's value along the orbit of (almost) every point is equal to the integral of that function over the whole space. This tells us, in particular, that almost all orbits distribute themselves uniformly according to the ergodic measure. This connection between orbit behavior and ergodic measures is crucial for many rigidity results. A simple but useful result that follows easily from this theorem is the following lemma:

Lemma 1.1.7 (Dense orbits)**.** *Almost every point x in the support of an ergodic measure has a dense orbit in the support of that measure.*

A final fact from dynamics that should be introduced here is Poincaré recurrence.

Theorem 1.1.8 (Poincaré Recurrence, see [23] Theorem 4.1.19)**.** *Let T be a transformation of X preserving μ and let A be any measurable set. Then for any $N \in \mathbb{N}$*

$$\mu(\{x \in A : \{T^i(x)\}_{i \geq N} \subset X \backslash A\}) = 0.$$

Perhaps the simplest and most basic element of ergodic theory, this theorem states that orbits under measure-preserving transformations almost always return near themselves; this proves essential in controlling the behavior of these orbits and objects related to them. It is used throughout the arguments below.

1.2 Geometry and dynamics

In this thesis, dynamics is a tool for solving problems in geometry; its interest and usefulness come from the rich interplay between the dynamics and geometry of the objects at play. I discuss here a few aspects of their interaction, with particular attention to those that will play a role in the work presented later.

Geometry and dynamics are intertwined in the underlying phase space X. The geometric or topological properties of X affect what sort of group actions X admits, as well as the dynamical properties of these group actions. For example, any continuous, orientation-preserving transformation of the 2-sphere has a fixed point, whereas it is easy to find transformations of the 2-torus that do not; the differing topology of the phase spaces results in very distinct dynamical behavior.

An example of the role of geometry that is closely related to the work in chapter II is the ergodic theory of the geodesic flow. The following theorem asserts ergodicity of this flow for a compact manifold of negative curvature. It is due to Anosov for variable curvature; he builds on the argument of Eberhard Hopf for surfaces and spaces of constant constant curvature (see Brin's appendix to [3] for details).

Theorem 1.2.1. *The geodesic flow on a compact manifold of negative curvature is ergodic.*

The idea underlying the proof of this theorem (the Hopf argument) is the basic argument for ergodicity in smooth dynamical systems. It underlies, for example, the work of Brin on frame flow on which the results in chapter II rely.

To observe the importance of geometry for the dynamical result of this theorem, consider the case of a flat torus. If one fixes a vertical foliation of this torus by closed geodesics, then the angle between any geodesic and the leaves of this foliation is constant – the geodesic flow is clearly not ergodic. Questions relating the ergodic theory of the geodesic flow and the geometry of the underlying space abound. Though the geodesic flow is not ergodic on the flat torus, there are metrics on the torus with ergodic geodesic flow.

Another interesting line of questions relates to what dynamics can tell one about geometry. A conjugacy between the geodesic flows on M_1 and M_2 is an invertible map $\phi : T^1M_1 \to T^1M_2$ such that for all $v \in T^1M_1$ and all t, $\phi(g_t v) = g_t\phi(v)$. One can ask under what conditions the existence of such a conjugacy implies that the underlying manifolds M_1 and M_2 are isometric. This is true if the M_i are nonpositively curved compact surfaces ([25], [14]) or nonpositively curved locally symmetric spaces ([15]), but is open in many other cases.

Closely related to the dynamics of geodesic flows is the important idea of geometric rank. This will be the central notion in chapter II. Let M be a complete Riemannian manifold. For each vector v in T^1M, consider the space P_v of all parallel vector fields $w(t)$ orthogonal to γ_v such that the sectional curvature $K(w(t), \dot{\gamma}_v(t))$ is equal to zero for all t. We make the following definition:

Definition 1.2.2 (Geometric rank)**.** The minimum of $dim(P_v)$ over all $v \in T^1M$ is the geometric rank of M. If the geometric rank is at least one then we say M has *higher rank*.

Note that this definition is different from that usually offered for higher rank. In the usual definition, the manifold is said to have higher rank if along each geodesic the space of parallel Jacobi fields has dimension at least two. The two definitions are equivalent, though one must note that when counting dimensions of the relevant spaces, the definition here yields a dimension one smaller than the standard definition, as the velocity vector for the geodesic is not counted. The alternative definition is useful for its ability to generalize to other curvature settings as in the definition of hyperbolic rank below and in chapter II.

As indicated by the distinct behaviors of the geodesic flows for the flat torus and a hyperbolic surface, having higher rank has strong consequences for the behavior of the geodesic flow, particularly in non-positive curvature. These are explored in [4] and [5]. In work building on these results Ballmann [2] and Burns and Spatzier [12] proved that compact, non-positively curved manifolds with higher rank are locally symmetric. The result in chapter II is analogous for a related notion of rank, introduced by Hamenstädt in [20]. This notion is as follows:

Definition 1.2.3 (Higher hyperbolic rank)**.** Suppose for each geodesic γ in M there is a parallel vector field w_γ such that the sectional curvature $K(\dot{\gamma}, w_\gamma)$ is -1 always. Then M is said to have *higher hyperbolic rank.*

In her work, Hamenstädt proves a result analogous to that of Balmann and Burns-Spatzier, that a compact manifold with higher hyperbolic rank and sectional curvature is a locally symmetric space. The result in chapter II gives, under some conditions, a simpler proof of a certain case of her result; under other conditions the result is new.

A key tool for the work of both Burns-Spatzier and Hamenstädt and an important object relating geometry and dynamics is the boundary at infinity of a nonpositively-curved manifold. It is defined as follows.

Definition 1.2.4 (Boundary at infinity)**.** Let M be a manifold with nonpositive sectional curvature and let $\gamma(t)$ and $\gamma'(t)$ for $t \in [0, \infty)$ be geodesic rays on the universal cover $\tilde{M}$. Declare γ to be equivalent to γ' if the distance between $\gamma(t)$ and $\gamma'(t)$ is bounded for all t. The boundary at infinity of $\tilde{M}$ is the set of equivalence classes for this relation and it denoted $\tilde{M}(\infty)$.

Burns and Spatzier study spherical building structures induced on $\tilde{M}(\infty)$ by higher-rank flat subspaces in $\tilde{M}$. Hamenstädt studies how the directions giving higher hyperbolic rank are recorded by a conformal structure on the boundary. In chapter II this space is used to construct certain paths whose segments are infinite geodesics and measures on the boundary are important in ensuring these geodesics have good dynamical properties.

A few remarks on homogeneous and symmetric spaces close this section.

Definition 1.2.5 (Homogeneous space)**.** Let a group H act on a space X. One calls X an *H-homogeneous space* if the action is transitive.

Of particular note is the case where H is a Lie group and X is a manifold. Fix some point o in X and let H_o be the set of transformations in H that fix o. Then it is easy to see that X is identified with the quotient space $H_o \backslash H$. Structures from the Lie group H are inherited by the homogeneous space. For example, the left-invariant metric and the left-Haar measure on H descend to X. If J is a closed subgroup of H then $J \backslash H$ is a homogeneous space and a manifold. If in addition Γ is a discrete subgroup of H such that Γ acts freely and properly discontinuously on $J \backslash H$ by right-multiplication, then $J \backslash H / \Gamma$ is a manifold which one says is modeled on the homogeneous space $J \backslash H$.

One particularly important class of homogeneous spaces is that of symmetric spaces. For a complete Riemannian manifold, define the *local geodesic symmetry* as follows. Fix a point $p \in M$ and let U be a neighborhood of p such that there is a unique shortest geodesic joining p to any point in U; write any such $x \in U$ as $\gamma(t)$ for such a geodesic. The local geodesic symmetry at p maps $x = \gamma(t)$ to $\gamma(-t)$.

Definition 1.2.6 (Symmetric Space)**.** M is a *locally symmetric space* if for each $p \in M$ there is a neighborhood of p on which the geodesic symmetry is an isometry. M is called a *(globally) symmetric space* if all local geodesic symmetries can be extended to global symmetries that are isometries.

There is an algebraic classification of symmetric spaces using Lie groups (see [22] for details), some points of which are recorded here:

Theorem 1.2.7. *Any symmetric space X is isomorphic to $K\backslash G$ for Lie groups K and G, where K is a compact subgroup. More precisely there is an involution σ of $\mathfrak{g}$ for which $\mathfrak{k}$ is the set of fixed points. Any locally symmetric space is of the form $K\backslash H/\Gamma$.*

There is a notion of rank for symmetric spaces coupled to the notion of geometric rank discussed above.

Definition 1.2.8 (Rank of a symmetric space)**.** The *rank* of a symmetric space X is the dimension of a maximal flat subspace in X.

This notion is a global version of the local geometric rank. Define the real rank of a Lie group as the dimension of a maximal, $\mathbb{R}$-split abelian subgroup of L. Then the rank of a symmetric space is the Lie group rank of the underlying Lie group L. Higher rank (rank two or greater) again implies many rigidity results about these spaces and groups.

1.3 Rank-rigidity and frame flow

Rank-rigidity theorems characterize locally symmetric spaces by the property of having higher geometric rank of some sort: manifolds with higher rank are shown, under some curvature conditions, to be locally symmetric spaces. The first notion of higher geometric rank is that of Definition 1.2.2 above. It is a local, geometric analogue of rank for symmetric spaces (see Definition 1.2.8). The more common equivalent definition of higher rank is the following:

Definition 1.3.1 (Euclidean rank)**.** Let M be a Riemannian manifold. The *rank* of a geodesic γ (or of a vector v tangent to γ) in M is the dimension of the space of parallel Jacobi fields along γ. The rank of M is the minimum of this number over all geodesics. M is said to have *higher Euclidean rank* if $rank(M) \geq 2$.

Euclidean rank is emphasized here in view of an extension to other notions of rank below; in most of the literature this is simply referred to as rank. Note that the dimension counted in this definition is always one higher than that in Definition 1.2.2, as it includes the tangent vector field for the geodesic itself.

The first rank-rigidity theorem was proved for Euclidean rank by Ballmann [2] and, using different methods, by Burns and Spatzier [12]. They proved that if an irreducible (the universal cover does not split isometrically as a product), compact, nonpositively curved manifold has higher Euclidean rank, then it is locally symmetric. Ballmann's proof works for finite volume as well and the most general version of this theorem is due to Eberlein and Heber, who prove it under only a dynamical condition on the isometry group of M's universal cover [19].

An extension of geometric rank is given by the following definition:

Definition 1.3.2. Let C be a real number and suppose that along every geodesic γ in M there exists a parallel, normal vector field w_γ such that the sectional curvature $K(\dot{\gamma}, w_\gamma)$ is always equal to C.

- If $C = 0$, M has *higher Euclidean rank.*
- If there is a constant a such that $C = -a^2$, M has *higher hyperbolic rank.*
- If there is a constant a such that $C = a^2$, M has *higher spherical rank.*

Hamenstädt was the first to work with a notion of rank other than Euclidean rank. She showed that a compact manifold with curvature bounded above by $-a^2$ which has higher hyperbolic rank is a locally symmetric space [20]. Shankar, Spatzier and Wilking extended rank-rigidity into positive curvature by defining spherical rank. They proved that a complete manifold with curvature bounded above by a^2 and with higher spherical rank is a compact, rank-one locally symmetric space [26].

These results settle many rank-rigidity questions, but leave questions about other curvature settings open (see [26] for an excellent overview). This thesis proves the following theorem, which can be applied to various settings in non-positive curvature.

Theorem 1. *Let M be a compact, (Euclidean) rank-1, nonpositively curved manifold. Suppose that along every geodesic in M there exists a parallel vector field making sectional curvature $-a^2$ with the geodesic direction, that is, M has higher hyperbolic rank. If M is odd-dimensional, or if M is even-dimensional and satisfies the sectional curvature pinching condition $-\Lambda^2 < K < -\lambda^2$ with $\lambda/\Lambda > .93$ then M has constant negative curvature equal to $-a^2$.*

Note that the spaces under consideration are non-positively curved, rank-1 spaces. The notion of rank involved in "rank-1" is Euclidean rank and the spaces here have higher hyperbolic rank but do not have higher Euclidean rank. Roughly speaking, a non-positively curved, rank-1 manifold behaves almost everywhere like a strictly negatively curved manifold.

The main tool involved in proving Theorem 1 is dynamical – the frame flow.

Definition 1.3.3 (Frame bundle)**.** For a Riemannian manifold M and integer $1 \leq k \leq n$ the *k-frame bundle* on M is the bundle with base space M and fiber over p the set of ordered, orthonormal k-frames of tangent vectors at p. One denotes this space by St_kM.

St_nM is a principal bundle over M, with structure group $SO(n)$. More often in this chapter one regards St_nM as a bundle over T^1M with a frame lying in the fiber over its first vector. Under this view, the bundle has structure group $SO(n-1)$.

Definition 1.3.4 (Frame flow)**.** The frame flow F_t is the flow on St_kM that takes a k-frame $\{v_1, \dots v_k\}$ to the k-frame $\{g_tv_1, \|_t\ v_2, \dots, \|_t\ v_k\}$, where $\|_t$ denotes parallel translation along the geodesic with initial tangent vector v_1.

Note that $St_1M = T^1M$ and the frame flow on this space is the geodesic flow. The frame flow is always a suspension of the geodesic flow as follows. Let $\pi : St_kM \to T^1M$ send any frame to its first vector. Then the following diagram commutes:

$$\begin{array}{ccc} St_kM & \xrightarrow{F_t} & St_kM \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ T^1M & \xrightarrow{g_t} & T^1M \end{array}$$

The proof of Theorem 1 simplifies considerably under the assumption that M has strictly negative curvature. In fact, for negative curvature the full frame flow is ergodic under the conditions of Theorem 1 in all dimensions but 7 and 8 ([9] for odd dimensions, [10] for even dimensions). Then the proof of Theorem 1 is immediate by considering a frame with dense orbit – such a frame will 'carry' the $-a^2$-curvature frame $\{\dot{\gamma}, w_\gamma\}$ arbitrarily close to any 2-frame, showing that the sectional curvature everywhere is $-a^2$. For dimensions 7 and 8 ergodicity of the full frame flow holds under very strong curvature pinching (see [11]) but under the curvature restrictions of Theorem 1 one only has ergodicity of the 2-frame flow. Note that ergodicity of this flow alone does not establish Theorem 1 since the set of 2-frames giving the distinguished sectional curvature $-a^2$ may, a priori, have zero measure. However, the ergodic theory of these types of flows, developed by Brin, proceeds via explicit geometric descriptions of the ergodic components and this allows Theorem 1 to be obtained from the 2-frame flow dynamics alone. The proof proceeding via 2-frame flow gives the result in the exceptional dimensions 7 and 8 in negative curvature. In addition, it suggests an adaptation to the rank-1 nonpositive curvature setting, where the ergodic theory of frame flows has not been developed. The simplifications possible in the strictly negative curvature setting will be noted throughout the chapter, but observe that although obtaining the result for nonpositively curved rank-1 spaces necessitates a more technical proof, the resulting theorem forms a better complement to the rank-rigidity theorem of Ballmann and Burns-Spatzier.

Note that, unlike previous rank-rigidity results, Theorem 1 allows for situations where the distinguished curvature $-a^2$ is not extremal. However, in the cases where $-a^2$ is extremal the hypotheses of our theorem can be weakened, as demonstrated in section 2.4. The following two results are then easy corollaries of Theorem 1:

Corollary 1. *Let M be a compact, rank-1 manifold with sectional curvature $-1 \leq K \leq 0$. Suppose that along every geodesic in M there exists a Jacobi field making sectional curvature -1 with the geodesic direction. If M is odd-dimensional, or if M is even-dimensional and satisfies the sectional curvature pinching condition $-1 \leq K < -.93^2$ then M is hyperbolic.*

Corollary 2. (compare with Hamenstädt [20]) *Let M be a compact manifold with sectional curvature bounded above by -1. Suppose that along every geodesic in M there exists a Jacobi field making sectional curvature -1 with the geodesic direction. If M is odd-dimensional, or if M is even-dimensional and satisfies the sectional curvature pinching condition $-(1/.93)^2 < K \leq -1$ then M is hyperbolic.*

Corollary 1 is a new rank-rigidity result analogous to those described above. This is the first positive result for lower-rank, i.e. when the distinguished curvature value is the lower curvature bound; [26] provides a discussion of counterexamples to lower spherical and Euclidean rank-rigidity. Corollary 2 provides a shorter proof of Hamenstädt's result, under an added pinching constraint in even dimensions.

In [13], Connell showed that rank-rigidity results can be obtained using only a dynamical assumption on the geodesic flow, namely an assumption on the Lyapunov exponents at a full-measure set of unit tangent vectors. His paper deals with the upper-rank situations treated by Ballmann, Burns-Spatzier and

Hamenstädt. He proves that having the minimal Lyapunov exponent allowed by the curvature restrictions attained at a full-measure set of unit tangent vectors is sufficient to apply the results of Ballmann and Burns-Spatzier or Hamenstädt. In the lower-rank setting of this paper, this viewpoint translates into the following:

Theorem 2. *Let M be a compact, rank-1 manifold with sectional curvature $-a^2 \leq K \leq 0$, where $a > 0$. Endow T^1M with a fully supported ergodic measure; one can take the measure of maximal entropy or, if the curvature is known to be negative, the Liouville measure. Suppose that for a full-measure set of unit tangent vectors v on M the maximal Lyapunov exponent at v is a, the maximum allowed by the curvature restriction. If M is odd-dimensional, or if M is even-dimensional and satisfies the sectional curvature pinching condition $-a^2 \leq K < -\lambda^2$ with $\lambda/a > .93$ then M is of constant curvature $-a^2$.*

The adaptation of Connell's arguments for this setting is discussed in section 2.5.

The work of Brin and others on frame flow for negatively curved manifolds is the starting point for the arguments of this chapter; the results needed are summarized in section 2.1 (see also [8] for a survey of the area). Although none of his work is undertaken for rank-1 nonpositively curved manifolds, the ideas used in this paper to deal with that situation are clearly inspired by Brin's work. The proof will proceed as follows. One utilizes the transitivity group H_v, defined for any vector v in the unit tangent bundle of M, which acts on $v^\perp \subset T^1M$. Essentially, elements of H_v correspond to parallel translations around ideal polygons in M's universal cover (see Definition 2.1.2 below). In negative curvature, Brin shows that this group is the structure group for the ergodic components of the frame flow (see, e.g., [8] or [7]). For the rank-1 nonpositive curvature case the definition of this group must be adjusted (see Definition 2.2.4) and the proof uses only that it is the structure group for a subbundle of the frame bundle. The considerations for the rank-1 case are discussed in section 2.2. Section 2.3 shows that H_v preserves the parallel fields that make curvature $-a^2$ with the geodesic defined by v. Finally, application of the results on the 2-frame flow of Brin-Gromov (adapted to the rank-1 case) and Brin-Karcher imply that H_v acts transitively on $v^\perp$ and prove that the curvature of M is constant.

CHAPTER II

Frame Flow Dynamics and Hyperbolic Rank Rigidity

2.1 Notation and background

2.1.1 Notation

Begin by fixing the following notation:

- M: a compact, rank-1 Riemannian manifold with nonpositive sectional curvature, $\tilde{M}$ its universal cover, $\tilde{M}(\infty)$ the boundary at infinity.
- T^1M and $T^1\tilde{M}$: the unit tangent bundles of M and $\tilde{M}$, respectively.
- St_kM: the k-frame bundle of ordered, orthonormal k-frames on M.
- g_t: the geodesic flow on T^1M or $T^1\tilde{M}$.
- F_t: the frame flow on St_kM; when clear, k will not be referenced.
- W_g^s and W_g^u: the foliations of $T^1\tilde{M}$ given by inward and outward pointing normal vectors to horospheres.
- μ: the Bowen-Margulis measure of maximal entropy on T^1M.
- $\gamma_v(t)$: the geodesic in M or $\tilde{M}$ with velocity v at time 0.
- $w_v(t)$: a parallel normal vector field along $\gamma_v(t)$ making the distinguished curvature $-a^2$ with $\dot{\gamma}_v(t)$.

- $K(\cdot,\cdot)$: the sectional curvature operator.

Note that $\pi : St_kM \to T^1M$ mapping a frame to its first vector is a fiber bundle with structure group $SO(k-1)$ acting on the right; St_nM is a principal bundle. The measure μ is used in place of the standard Liouville measure as it has better known dynamical properties for rank-1, nonpositively curved spaces (see section 2.2). In the negative curvature setting Liouville measure can be used. Unless otherwise specified, μ and its product with the standard measure on the fibers of St_kM inherited from the Haar measure on $SO(n-1)$ will be the measures used in all that follows. In negative curvature, W_g^s and W_g^u are the stable and unstable foliations for the geodesic flow.

2.1.2 Background

In negative curvature, Brin develops the ergodic theory of frame flows as summarized below. Section 2.2 discusses how suitable portions of this setup can be generalized to the rank-1 setting.

First, the frame flow also gives rise to stable and unstable foliations W_F^s and W_F^u of St_kM as shown by Brin (see [8] Prop. 3.2). Brin notes that the existence of these foliations can be established in two ways, either by applying the work of Brin and Pesin on partially hyperbolic systems or by utilizing the exponential approach of asymptotic geodesics. In the second approach the leaves of the foliation are constructed explicitly – they sit above the stable/unstable leaves for the geodesic flow, and α and α' are in the same leaf if the distance between $F_t(\alpha)$ and $F_t(\alpha')$ goes to zero as $t \to \infty$ for the stable leaves, or $t \to -\infty$ for the unstable leaves. The following proposition makes possible this definition of $W_F^*(\alpha)$ by establishing that frames asymptotic to α exist and are unique. The proof follows the sketch given by Brin in [8].

Proposition 2.1.1. *Let v be a unit tangent vector and let α be a k-frame with first vector v. Let $v' \in W_g^s(v)$ (respectively $W_g^u(v)$) so that the distance between $g_t(v)$ and $g_t(v')$ goes to zero exponentially fast as $t \to \infty$ (resp. $t \to -\infty$). Then there exists a unique k-frame α' with first vector v' such that the distance between $F_t(\alpha)$ and $F_t(\alpha')$ goes to zero as $t \to \infty$ (resp. $t \to -\infty$).*

Note that in a compact, negatively curved manifold any two asymptotic vectors approach each other exponentially fast so this proposition allows one to define all leaves of the foliation. In rank-1 spaces this may no longer be the case; thus exponential approach has been added as a hypothesis to the proposition as it will be used for the rank-1 case later in the chapter.

Proof. Assume $v' \in W_g^s(v)$; the unstable case is analogous. Uniqueness of the limit is simple since it is clear that two different frames cannot both approach α. It remains only to show existence.

For t large enough, $g_t(v)$ and $g_t(v')$ are very close to each other, and then for every frame β with first vector $g_t(v)$ there exists a unique frame, call it $f(\beta)$, which minimizes the distance from β among frames with first vector $g_t(v')$. To approximate the unique frame α' one is looking for, consider the frames $\alpha'_t = F_{-t}(f(F_t(\alpha)))$. One wants to show that the α'_t have a limit as $t \to \infty$; this limit will clearly be α'.

Consider the sequence α'_n for $n \in \mathbb{N}$. Since the frame flow is smooth, by choosing large enough T the difference between the frame flow along the segments $[g_T(v), g_{T+1}(v)]$ and $[g_T(v'), g_{T+1}(v')]$ can be made arbitrarily small, and thus if the α'_n have a limit, it must be a limit for the α'_t. Again, since the frame flow is smooth and fibrewise isometric and since the distance between the geodesics decreases exponentially fast, the distance $d(\alpha'_n, \alpha'_{n+1})$ goes to zero exponentially fast as well. Note then that $d(\alpha'_n, \alpha'_m) \leq \sum_{i=n}^{m-1} d(\alpha'_i, \alpha'_{i+1}) \leq \sum_{i=n}^{\infty} d(\alpha'_i, \alpha'_{i+1})$. As the summands go to zero exponentially fast the last sum converges and given $\epsilon > 0$ one can pick n so large that this tail sum is less than ϵ. Then one sees that the α'_n form a Cauchy sequence so they have a limit as desired. $\square$

Let $p(v, v')$ be the map from the fiber of $St_k M$ over v to the fiber over v' that takes each α to $\alpha' = \pi^{-1}(v') \cap W^s_F(\alpha)$. Note that $p(v, v')$ corresponds to a unique isometry between $v^\perp$ and $v'^\perp$ and commutes with the right action of $SO(k-1)$. Most of this chapter will deal with the maps $p(v, v')$ acting on 2-frames. One can think of $p(v, v')(\alpha)$ as the result of parallel transporting α along $\gamma_v(t)$ out to the boundary at infinity of $\tilde{M}$ and then back to v' along $\gamma_{v'}(t)$. If v' and v belong to the same leaf of W^u_g there is similarly an isometry corresponding to parallel translation to the boundary at infinity along γ_{-v} and back along $\gamma_{-v'}$. This isometry will also be denoted by $p(v, v')$. In the spirit of Brin (see [8] Defn. 4.4) one defines the transitivity group at v as follows:

Definition 2.1.2. Given any sequence $s = \{v_0, v_1, \ldots, v_k\}$ with $v_0 = v, v_k = g_T(v)$ such that each pair $\{v_i, v_{i+1}\}$ lies on the same leaf of W^s_g or W^u_g, one has an isometry of $v^\perp$ given by

$$I(s) = F_{-T} \circ \prod_{i=0}^{k-1} p(v_i, v_{i+1}).$$

The closure of the group generated by all such isometries is denoted by H_v and is called the transitivity group.

The idea of the transitivity group is that it is generated by isometries coming

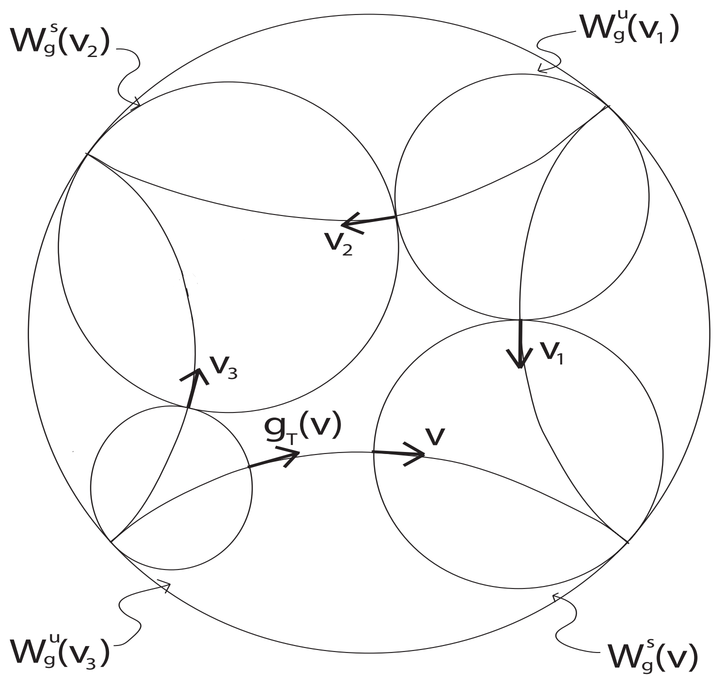

Figure 2.1: An element of the transitivity group

from parallel translation around ideal polygons in $\tilde{M}$ with an even number of sides, such as the one shown in figure 2.1.

Note that this definition differs slightly from that in Brin's work. Brin requires that $v_k = v$ and thus there is no F_{-T} term in his formula for $I(s)$. Brin proves that his group describes the ergodic components of the frame flow. He shows in [7] that the ergodic components are subbundles of St_kM with structure group a closed subgroup of $SO(n-1)$, now acting from the left (see also [8], section 5 for an overview). In addition, his proof demonstrates that the structure group for the ergodic component is the transitivity group (see [8] Remark 2 or [7] Proposition 2). Note that the action of H_v can be taken to be a left action as it commutes with the $SO(k-1)$ action of the structure group. This can be seen from noting that $p(v,v')(\alpha) \cdot g = p(v,v')(\alpha \cdot g)$ for any g in the structure group, and that these maps define the transitivity group. The proof that this group gives the ergodic components follows the Hopf argument for ergodicity, showing that the ergodic component is preserved under motion along stable and unstable leaves, and using the Birkhoff ergodic theorem to show that switching from stable to unstable also preserves the component.

The transitivity group as defined here is certainly at least as large as that defined by Brin. On the other hand, the added F_{-T} term preserves the ergodic components so this group still describes ergodic components and therefore is, in the end, the same as Brin's. The advantage to this formulation of the definition is that it allows all ideal polygons, not just those that are 'equilateral' in the sense that they can be traversed only by following leaves of the foliations. The explicit geometric description of the ergodic components given here is the central inspiration for the proof.

Two results on the ergodicity of the 2-frame flow are used in the proof.

Theorem 2.1.3. (Brin-Gromov [9] Proposition 4.3) *If M has negative sectional curvature and odd dimension then the 2-frame flow is ergodic.*

Theorem 2.1.4. (Brin-Karcher [10]) *If M has sectional curvature satisfying $-\Lambda^2 < K < -\lambda^2$ with $\lambda/\Lambda > .93$ then the 2-frame flow is ergodic.*

Theorem 2.1.4 is not directly stated as above in [10], rather it follows from remarks made in section 2 of that paper together with Proposition 2.9 and the extensive estimates carried out in the later sections.

2.1.3 A dynamical lemma

The following dynamical lemma is one of the main tools of the proof. It will be used in the proof of Lemma 2.2.3 and in the arguments of Section 2.3.

Lemma 2.1.5. *Suppose $\gamma(t)$ is a recurrent geodesic in M with a parallel normal field $P(t)$ along it such that $K(P(t), \dot\gamma(t)) \to C$ as $t \to \infty$. Then $K(P(t), \dot\gamma(t)) \equiv C$ for all t.*

Proof. Since $\gamma(t)$ is recurrent one can take an increasing sequence $\{t_k\}$ tending to infinity such that $\dot\gamma(t_k)$ approaches $\dot\gamma(0)$. As the parallel field $P(t)$ has constant norm and the set of vectors in $\dot\gamma_v(t)^\perp$ with this norm is compact, one can, by passing to a subsequence, assume that $P(t_k)$ has a limit $G(0)$. Extend $G(0)$ to a parallel vector field $G(t)$ along $\gamma(t)$.

By construction, $K(G(0), \dot\gamma(0)) = \lim_{k\to\infty} K(P(t_k), \dot\gamma(t_k)) = C$. In addition, for any real number T, the recurrence $\dot\gamma(t_k) \to \dot\gamma(0)$ implies recurrence $\dot\gamma(t_k + T) \to \dot\gamma(T)$. By continuity of the frame flow, $P(t_k + T) \to G(T)$ for the vector field G defined above. Thus $G(t)$ makes curvature C with $\dot\gamma(t)$ for any time t.

One can repeat the argument above, now letting $G(t)$ recur along the same sequence of times to produce $G_1(t)$, and likewise $G_i(t)$ recur to produce $G_{i+1}(t)$, forming a sequence of fields all making curvature identically C with the geodesic direction. Now, observe that $G(0) = P(0) \cdot g$ for some $g \in SO(n-1)$. Note here that g is not well defined by looking at P and G alone, but will be well defined if we consider n-frame orbits with second vector P recurring to n-frames with second vector $G(0)$; this is the g one utilizes. By construction and the fact that the $SO(n-1)$ action commutes with parallel translation, $G_i(0) = P(0) \cdot g^{i+1}$. $SO(n-1)$ is compact, so the $\{g^i\}$ have convergent subsequences. In addition, since the terms of this sequence are all iterates of a single element, one can choose a subsequence converging to the identity. Let g^{i_j+1} be such a sequence. Then $G_{i_j}(t)$ approach the original field $P(t)$ showing that P makes constant curvature

C with $\dot{\gamma}$ as well. $\square$

2.2 Extensions to rank-1 spaces

This section discusses some details of the extension to rank-1, nonpositively curved spaces, and notes how the necessary results on the dynamics of the frame flow can be appropriated to this situation.

2.2.1 The measure of maximal entropy

The measure of maximal entropy μ was developed for rank-1 spaces by Knieper in [24] and is constructed there as follows. Let $\{\nu_p\}_{p\in\tilde{M}}$ be the Patterson-Sullivan measures on $\tilde{M}(\infty)$. Fix any $p \in \tilde{M}$. Let $\mathcal{G}^E$ be the set of pairs (ξ, η) in $\tilde{M}(\infty)$ that can be connected by a geodesic. Then $d\bar{\mu}(\xi,\eta) = f(\xi,\eta)d\nu_p(\xi)d\nu_p(\eta)$ defines a measure on $\mathcal{G}^E$; f is a positive function which can be chosen to make the measure invariant under $\pi_1(M)$.

Let $P : T^1\tilde{M} \to \mathcal{G}^E$ be the projection $P(v) = (\gamma_v(-\infty), \gamma_v(\infty))$. One obtains a g_t and π_1 invariant measure $\tilde{\mu}$ on $T^1\tilde{M}$ by setting, for any Borel set A of $T^1\tilde{M}$,

$$\tilde{\mu}(A) = \int_{\mathcal{G}^E} vol(\pi(P^{-1}(\xi,\eta) \cap A))d\bar{\mu}(\xi,\eta),$$

where here $\pi : T^1\tilde{M} \to \tilde{M}$ is the standard projection and vol is the volume element on the submanifolds $P^{-1}(\xi,\eta)$. The projection of this measure to T^1M is μ, the measure of maximal entropy.

This chapter needs three key facts about this measure. First, μ is ergodic for the geodesic flow (see [24] Theorem 4.4). Second, μ has full support. This follows from the facts that μ is supported on the rank-1 vectors (see [24] again) and that the rank-1 vectors are dense in T^1M (see e.g. [1]). Third, μ is absolutely continuous for the foliations W_g^s and W_g^u. Absolute continuity of a measure for a foliation is a way of asking that a Fubini-like property hold for the foliation when integrating with respect to the measure (see Brin's appendix to [3]). In this situation, it is immediate from the definition of the measure.

2.2.2 The transitivity group

The next task is to extend the definition of the transitivity group to rank-1, nonpositively curved spaces. The central difficulty here is that the distance between asymptotic geodesics may approach a nonzero constant. Thus it is no longer clear whether foliations like W_F^s and W_F^u can be defined. This difficulty can be overcome by avoiding defining foliations for the frame flow, but still defining maps $p(v, v')$ used to produce a transitivity group. Section 2.3 will show that the transitivity group preserves the distinguished parallel fields. The technical points involved in the definitions of the p-maps and the transitivity group are necessary to make that proof work.

Note the following new terminology. As always, geodesic rays which remain within a bounded distance of one another are called asymptotic. To distinguish an important property, call rays where the distance is not only bounded but goes to zero *strictly asymptotic*.

Let v, v' be vectors in $T^1\tilde{M}$. The goal in this section is the definition of maps $p(v, v')$ mapping the k-frames in $St_k\tilde{M}$ with first vector v to k-frames with first vector v'. One suppresses reference to k in the notation as it will be clear from context what k is (usually 2). Begin by defining $\Omega' \subset \tilde{M}(\infty)$ as the set of all ξ satisfying

- ξ is the endpoint of a rank-1 geodesic which is recurrent (when projected to M)

- almost all (with respect to the Patterson-Sullivan measures) geodesics ending at ξ are recurrent (when projected to M).

Since recurrence and rank 1 are full-measure conditions for the ergodic measure μ and μ is absolutely continuous for g_t this is a full-measure set of $\tilde{M}(\infty)$ for any Patterson-Sullivan measure on $\tilde{M}(\infty)$. In particular, it is dense, as the Patterson-Sullivan measures have full support.

The first condition is placed on Ω' to allow the proof of the following lemma; the second will be needed for the work in section 2.3.

Lemma 2.2.1. *There exists a full-measure subset $\Omega \subset \Omega'$ such that if $\xi \in \Omega$ then any two geodesics $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $\tilde{M}$ with $\tilde{\gamma}_1(\infty) = \tilde{\gamma}_2(\infty) = \xi$ are exponentially strictly asymptotic.*

Proof. First one shows that the distance between $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ goes to zero (see also [24] Prop. 4.1). As $\xi \in \Omega'$, it is the end point of a rank-1, recurrent geodesic; call this geodesic $\tilde{\gamma}_v$. Suppose $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are not strictly asymptotic. Then $\tilde{\gamma}_v$ is not strictly asymptotic to one of these geodesics, without loss of generality, say $\tilde{\gamma}_1$. Since $\tilde{\gamma}_v$ is recurrent when projected to M there exists a sequence $\{\phi_i\}$ of isometries of $\tilde{M}$ and a sequence of real numbers $\{t_i\}$ tending to infinity such that $\phi_i(g_{t_i}(v)) \to v$ as $i \to \infty$. Consider the sequences of geodesics $\{\phi_i(\tilde{\gamma}_v)\}$ and $\{\phi_i(\tilde{\gamma}_1)\}$. By choice of the ϕ_i the first sequence converges to $\tilde{\gamma}_v$. Also, since $\tilde{\gamma}_1$ is asymptotic to $\tilde{\gamma}_v$, after perhaps passing to a subsequence, the second sequence converges to a geodesic, call it $\bar{\gamma}$. As $\tilde{\gamma}_v$ and $\tilde{\gamma}_1$ are not *strictly* asymptotic, $\bar{\gamma} \neq \tilde{\gamma}_v$, but since they are asymptotic, $\bar{\gamma}(-\infty) = \tilde{\gamma}_v(-\infty)$ and $\bar{\gamma}(\infty) = \tilde{\gamma}_v(\infty)$. Then the flat strip theorem (see [17]) implies that $\tilde{\gamma}_v$ and $\bar{\gamma}$ bound a totally geodesically embedded flat strip, contradicting the fact that $\tilde{\gamma}_v$ is rank 1.

Now one shows that this convergence is exponential for almost all ξ. Taking Ω to be the intersection with Ω' of the full-measure set of $\tilde{M}(\infty)$ for which the following part of the proof works, one obtains the lemma.

Let $\tilde{\gamma}$ be a rank-1 geodesic in $\tilde{M}$ such that, when projected to γ on M, returns to any neighborhood of $\dot{\gamma}(0)$ a positive fraction of the time. Using the Birkhoff ergodic theorem, it is easy to see that this is a full-measure set of geodesics, and the forward endpoints of these geodesics will be the full-measure set in $\tilde{M}(\infty)$ one uses to obtain Ω. Take any $\tilde{\gamma}'$ forward asymptotic to $\tilde{\gamma}$. The argument below will show that these two geodesics approach each other exponentially fast, which is enough to show that any two geodesics with endpoint $\tilde{\gamma}(\infty)$ approach each other exponentially fast.

By the work above, $\tilde{\gamma}'$ is strictly asymptotic to $\tilde{\gamma}$; hence, after a finite time, $\tilde{\gamma}'$ can be assumed to be very close to $\tilde{\gamma}$, so one can assume that $\dot{\tilde{\gamma}}'(0) \in \bar{B}_\epsilon(\dot{\tilde{\gamma}}(0))$, the closed ϵ-ball around $\dot{\tilde{\gamma}}(0)$ for some positive ϵ. (Here, parametrize geodesics so that $\dot{\tilde{\gamma}}'(0)$ lies on the stable manifold for $\dot{\tilde{\gamma}}(0)$.) For any such $\tilde{\gamma}'$, there exists a time $T_{\tilde{\gamma}'}$ such that

$$d(\tilde{\gamma}(T_{\tilde{\gamma}'}), \tilde{\gamma}'(T_{\tilde{\gamma}'})) < \frac{1}{2} d(\tilde{\gamma}(0), \tilde{\gamma}'(0)).$$

By continuity of the geodesic flow and compactness of the closed ϵ-ball, there exists a single time T such that this equation holds for all $\tilde{\gamma}'$.

One now notes that the geodesic flow must decrease distances similarly nearby $\dot{\tilde{\gamma}}(0)$. In particular, it follows from the continuity of the geodesic flow and the stable foliation that there exists a $\delta > 0$ (depending on $\dot{\tilde{\gamma}}(0)$ and chosen smaller than ϵ) such that for all $v \in T^1\tilde{M}$ δ-close to $\dot{\tilde{\gamma}}(0)$ the geodesic flow for time T decreases distance between $\tilde{\gamma}_v$ and asymptotic geodesics $\epsilon/2$-close to it by a factor of at least one half as well. Since, by assumption, $\tilde{\gamma}$ returns to such a δ-ball about its starting point a positive fraction of the time, and since on a non-positively curved manifold the distances between asymptotic geodesics never increase, it is clear that $\tilde{\gamma}$ and $\tilde{\gamma}'$ approach each other at an exponential rate, as desired. $\square$

Let $v' \in W^s_g(v)$ (the case $v' \in W^u_g(v)$ proceeds in a similar manner). One makes definitions of $p(v, v')$ in two cases.

Case I: $\gamma_v(\infty) \in \Omega$. Lemma 2.2.1 provides exponential convergence of the geodesics in question. Then Proposition 2.1.1 allows one to define $p(v, v')$ mapping frames over v to frames over v' as in the negative curvature case.

Case II: $\gamma_v(\infty) \notin \Omega$. Define a family of maps $\{p_{\{\xi_n\}}(v, v')\}$ in the following manner. Let $\gamma_v(\infty) = \xi$. Consider all sequences of points $\{\xi_n\}$ in Ω that approach ξ in the sphere topology on $\tilde{M}(\infty)$. As noted, Ω is dense in $\tilde{M}(\infty)$, so one can find such sequences approaching any ξ. Let c_n and c_n' be the geodesics connecting the footpoints of v and v' to ξ_n such that $c_n(0)$ is the footpoint of v and $\dot{c}_n'(0) \in W^s_g(\dot{c}_n(0))$ (see figure 2.2). The maps $p(\dot{c}_n(0), \dot{c}_n'(0))$ are defined under Case I. As n tends to infinity, $\dot{c}_n(0) \to v$ and $\dot{c}_n'(0) \to v'$ so limit points of the maps $\{p(\dot{c}_n(0), \dot{c}_n'(0))\}$ will give maps from frames over v to frames over v'. Let us restrict the allowed sequences $\{\xi_n\}$ to only those for which $\{p(\dot{c}_n(0), \dot{c}_n'(0))\}$ has a unique limit; call that limit $p_{\{\xi_n\}}(v, v')$. These will be the allowed maps for the second case of the definition.

As before, the transitivity group will be defined as a composition of the p- and

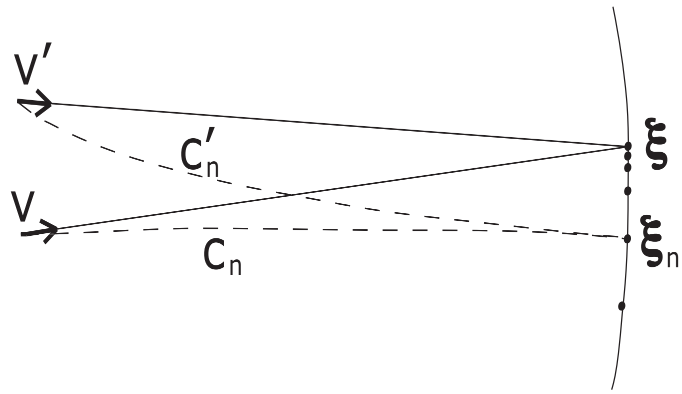

Figure 2.2: Adjusting the definition of the transitivity group

$p_{\{\xi_n\}}$-maps corresponding to translations around ideal polygons. Again, nonpositive curvature necessitates some technical considerations. First, as noted in [18] 1.11, not all pairs of points on $\tilde{M}(\infty)$ can be connected by geodesics. However, as noted in the following lemma, a given point can be connected to almost all other points on $\tilde{M}(\infty)$.

Lemma 2.2.2. *Let ξ be an element of $\tilde{M}(\infty)$ and let A_ξ be the subset of $\tilde{M}(\infty)$ consisting of points that can be connected to ξ by geodesics. Then A_ξ contains an open, dense set.*

Proof. This lemma simply works out some consequences of [1]. Let $A'_\xi \subset A_\xi$ be the set of all points at infinity which can be connected to ξ by a geodesic that does not bound a flat half-plane. Ballmann's Theorem 2.2 (iii) tells one that A'_ξ contains all endpoints of periodic geodesics that do not bound a flat half-plane. Together with his Theorem 2.13, this implies that the set A'_ξ is dense. In addition, Ballmann's Lemma 2.1 implies that A'_ξ is open, proving the Lemma. $\square$

The following technical criterion is also necessary for the new definition of the transitivity group:

Composition Criterion. Let v be in $T^1\tilde{M}$ and let $\xi = \gamma_v(\infty), \eta = \gamma_v(-\infty) \notin \Omega$ be the endpoints at infinity of γ_v. Let $\xi_n \to \xi$ and $\eta_n \to \eta$ be sequences in Ω. One says the pair $(\{\xi_n\}, \{\eta_n\})$ satisfies the *composition criterion for* v if $\eta_n \in A_{\xi_n}$ for all n and $\gamma_n \to \gamma_v$ as $n \to \infty$, where γ_n is the geodesic connecting ξ_n and η_n.

This criterion will be required of pairs $(\{\xi_n\}, \{\eta_n\})$ if one is to compose the maps $p_{\{\xi_n\}}$ and $p_{\{\eta_n\}}$ in forming elements of the transitivity group. It will be important in the proof of Proposition 2.3.2. One must first, however, establish that, given $\{\xi_n\}$ and v, there exist sequences $\{\eta_n\}$ that satisfy the composition criterion for v with $\{\xi_n\}$. Without this fact the definition of the transitivity group could be vacuous.

Lemma 2.2.3. *Given $v \in T^1\tilde{M}$ with $\gamma_v(\infty) = \xi, \gamma_v(-\infty) = \eta$ and a sequence $\xi_n \to \xi$ in Ω, there exists a sequence $\eta_n \to \eta$ in Ω such that $\{\xi_n\}$ and $\{\eta_n\}$ satisfy the composition criterion for v.*

Proof. For any $\zeta \in \tilde{M}(\infty)$ let $pr_\zeta : \tilde{M} \to \tilde{M}(\infty)$ be the projection defined by setting $pr_\zeta(y)$ equal to $\gamma(\infty)$ where γ is the geodesic with $\gamma(-\infty) = \zeta$ and $\gamma(0) = y$. Let x be the footpoint of v and $B_R(x)$ be the ball of radius R around x in $\tilde{M}$.

Lemma 3.5 in [24] provides that given x, there exists an $R > 0$ such that $pr_{\xi_n}(B_R(x))$ contains an open set U in $\tilde{M}(\infty)$. Examining Knieper's proof one sees that R can be taken to be any number greater than the distance from x to some rank-1 geodesic c which has an endpoint at ξ_n. Consider the geodesic c_n joining x and ξ_n. Since $\xi_n \in \Omega$, it is the endpoint of a rank-1, recurrent geodesic, call it γ. Lemma 2.2.1 then implies that the geodesic c_n is strictly asymptotic to γ and then Lemma 2.1.5 can be applied with $C = 0$ to show that c_n must be of rank 1 as γ is. Thus the rank-1 geodesic c needed by Knieper can be taken to be c_n. It is distance 0 from x, therefore R can be taken to be arbitrarily small; in particular take $R_n = 1/2^n$ for ξ_n.

For each R_n, the open set U_n provided by Knieper contains elements of the set $\Omega \cap A_{\xi_n}$ since A_{ξ_n} is open and dense and Ω has full measure. Pick $\eta_n \in U_n \cap \Omega \cap A_{\xi_n}$ to form the sequence $\{\eta_n\}$. Then the geodesics γ_n connecting ξ_n and η_n enter $B_{1/2^n}(x)$ for all n, and as $\xi_n \to \xi$ and $\eta_n \to \eta$ one must have $\gamma_n \to \gamma_v$. Thus $(\{\xi_n\}, \{\eta_n\})$ satisfies the composition criterion for v as desired. $\square$

The transitivity group is defined via the following two definitions. One starts by defining its action on the frames above one particular vector v.

Definition 2.2.4. Let $v \in T^1\tilde{M}$ be such that $\gamma_v(\infty)$ and $\gamma_v(-\infty)$ are in Ω. Consider any sequence $s = \{v_0, v_1, \ldots, v_k\}$ with $v_0 = v, v_k = g_T(v)$ for some real T, such that each pair $\{v_i, v_{i+1}\}$ lies on the same leaf of W_g^s or W_g^u. Furthermore, take for each pair $\{v_i, v_{i+1}\}$ with v_i falling under Case II a choice of a sequence $\{\xi_n^i\} \subset \Omega$ as described above. One requires that $(\{\xi_n^i\}, \{\xi_n^{i+1}\})$ satisfies the composition criterion for v_{i+1}. Then one has an isomorphism of $v^\perp$ given by

$$I(s) = F_{-T} \circ \prod_{i=0}^{k-1} p_-(v_i, v_{i+1}).$$

Here $p_-(v_i, v_{i+1}) = p(v_i, v_{i+1})$ when v_i falls under Case I and $p_-(v_i, v_{i+1}) = p_{\{\xi_n^i\}}(v_i, v_{i+1})$ when v_i falls under Case II. The closure of the group generated by all such isometries is denoted by H_v.

One now extends the action of this group to any $w \in T^1\tilde{M}$ by connecting v to w by a segment of an ideal polygon. To do so one simply needs a point ξ in $A_{\gamma_v(\infty)} \cap A_{\gamma_w(\infty)}$. By Lemma 2.2.2 this set is open and dense, so in fact one can choose $\xi \in \Omega \cap A_{\gamma_v(\infty)} \cap A_{\gamma_w(\infty)}$. Let g be the isometry from $v^\perp$ to $w^\perp$ given by frame flow along the segment connecting v and w via ξ. More specifically, let v_1 lie on the geodesic connecting $\gamma_v(\infty)$ and ξ such that $v_1 \in W_g^s(v)$, let v_2 lie on the geodesic connecting ξ and $\gamma_w(\infty)$ such that $v_2 \in W_g^u(v_1)$, and let $T \in \mathbb{R}$ be such that $g_T(w) \in W_g^s(v_2)$. Then let

$$g = F_{-T} \circ p_-(v_2, g_T(w)) \circ p_-(v_1, v_2) \circ p(v, v_1). \tag{2.1}$$

In the negative-curvature case, it is clear that $H_w = gH_vg^{-1}$. Thus one completes the definition of the transitivity group by making the following definition:

Definition 2.2.5. Let $H_w := gH_vg^{-1}$.

Note that the choices of v and ξ only affect the group H_w up to multiplication by an element of H_v, so the specific choices are not relevant. In addition, attempting to define elements of H_w for vectors w for which neither endpoint is in Ω by ideal polygons based at γ_w is problematic as, due to the composition criterion, the composition of such elements may not be in the group. Hence one defines such H_w via H_v where no such issues arise. The result is a well-defined action of an abstract group H isomorphic to H_v on the frame bundle, which in the negatively curved case essentially reduces to Brin's definition. Again, as the $p_-(v, v')$-maps constructed here are invariant under elements of the structure group $SO(k-1)$, the action of H commutes with the action of $SO(k-1)$ and thus takes the form of a left action. Finally, the work above takes place in $T^i\tilde{M}$. Since everything in the definition is invariant under action of isometries of $\tilde{M}$, one has a well-defined transitivity group for any $v \in T^1M$.

2.2.3 The subbundle given by H

This section constructs a subbundle of St_kM for any $k \leq n$ with an action of H on it.

Definition 2.2.6. Given a k-frame α based above a vector $v \in \Omega$ let $Q(\alpha) \subseteq St_kM$ be the smallest set containing α and closed under all $h \in H_v$, F_t for all t and all isometries g as in equation 2.1.

Proposition 2.2.7. *$Q(\alpha)$ is a subbundle of St_kM.*

Proof. Since for any $w \in T^1M$ one has an isometry g as in Definition 2.2.5, we see that $\pi(Q(\alpha)) = T^1M$.

Let $\bar{\alpha}$ be an extension of the k-frame α to an n-frame with first k vectors given by α. One first shows that $Q(\bar{\alpha})$ is a subbundle. By construction, $Q(\bar{\alpha})$ admits an action of H, an abstract group isomorphic to H_v. It is clear that $Q(\bar{\alpha}) \cap \pi^{-1}(w)$ is the H_w orbit of $g(\bar{\alpha})$ for any $w \in T^1M$, where g is as in Equation 2.1. Furthermore, H acts freely on St_nM so all orbit types of this action are the same. Thus Theorem 5.8 from [6] applies and $\pi : Q(\bar{\alpha}) \to T^1M$ is a fiber bundle with structure group H as desired.

For $k < n$, embed $SO(n-k)$ into $SO(n-1)$, the structure group for St_nM so that it acts on the last $n-k$ vectors in a given frame. The map $\bar{\pi} : St_nM/SO(n-k) = St_kM \to T^1M$ is the subbundle of k-frames. To produce $Q(\alpha)$ one would like to apply the same process to $Q(\bar{\alpha})$ but must proceed carefully. Let

$$K_{\bar{\alpha}} = \{\kappa \in SO(n-k) | \bar{\alpha} \cdot \kappa = h(\kappa) \cdot \bar{\alpha} \text{ for some } h(\kappa) \in H_v\}$$

where $SO(n-k)$ acts on $\bar{\alpha}$ via the same embedding. This is the stabilizer of the first k vectors in $\bar{\alpha}$ (that is, α) in the subgroup of the structure group that preserves the H_v-orbit of $\bar{\alpha}$. Examine this stabilizer for any other frame $\bar{\alpha}'$ in $Q(\bar{\alpha})$. Any such $\bar{\alpha}'$ takes the form $h' \cdot g \cdot \bar{\alpha}$ for some $h' \in H_w$ and g as in equation 2.1. Then compute

$$\begin{aligned} K_{\bar{\alpha}'} = & \;\{\kappa \in SO(n-k) | \bar{\alpha}' \cdot \kappa = h(\kappa) \cdot \bar{\alpha}' \text{ for some } h(\kappa) \in H_w\} \\ = & \;\{\kappa \in SO(n-k) | h' \cdot g \cdot \bar{\alpha} \cdot \kappa = h(\kappa) h' \cdot g \cdot \bar{\alpha} \text{ for some } h(\kappa) \in H_w\}. \end{aligned}$$

But $h' \cdot g \cdot \bar{\alpha} \cdot \kappa = h(\kappa)h' \cdot g \cdot \bar{\alpha}$ if and only if $\bar{\alpha} \cdot \kappa = g^{-1} \cdot (h')^{-1} h(\kappa) h' \cdot g \cdot \bar{\alpha}$, and $g^{-1} \cdot (h')^{-1} h(\kappa) h' \cdot g$ is an element of H_v so one sees that $K_{\bar{\alpha}'} = K_{\bar{\alpha}}$ for all $\bar{\alpha}' \in Q(\bar{\alpha})$. Refer to this group simply as K, and note that $K \hookrightarrow H$ by $\kappa \mapsto h(\kappa)^{-1}$. Thus, one obtains $\pi : Q(\alpha) \to T^1M$ as $\bar{\pi} : Q(\bar{\alpha})/K \to T^1M$. The fibers of this map are of the form H/K everywhere so again apply [6] to see that one has a fibration, as desired. $\square$

Remark 2.2.8. The argument here proves that $Q(\alpha)$ is a topological sub-fiber bundle – nothing has been claimed about smoothness. C^1-smoothness of $Q(\alpha)$ in the negative curvature case is proven by Brin and is key to his proof that $Q(\alpha)$ is the ergodic component containing α; here, however, one needs only the topological result to appropriate the needed results from Brin-Gromov.

Proposition 2.2.9. *The transitivity group H acts transitively on the fiber of 2-frames over any $v \in T^1M$.*

Proof. First, note that when n is even one is restricted to strict negative curvature and this result is Theorem 2.1.4 due to Brin and Karcher. When n is odd the result is Theorem 2.1.3 due to Brin and Gromov and found in section 4 of [9]. They discuss the proof only in the strict negative-curvature case, but it works perfectly well in nonpositive curvature. It is included here for completeness.

The work in Proposition 2.2.7 produced a subfibration $\pi : Q(\alpha) \to T^1M$ with fiber H/K of the fibration $\pi : St_kM \to T^1M$ with fiber $SO(n-1)/SO(n-k)$. Restrict attention now to 2-frames, and specifically to $St_2M|_p$, the restriction of the 2-frame bundle to those frames based at a point p of M. This provides bundles

$$\begin{tikzcd}
H/K \arrow[r] & Q(\alpha)|_p/K \arrow[dd, "\pi"] & & S^{n-2} \arrow[r] & St_2M|_p \arrow[dd, "\pi"] \\
& & {} \arrow[r, hook, "i"] & {} & \\
& S^{n-1} & & & S^{n-1}
\end{tikzcd}$$

where $S^{n-2} = SO(n-1)/SO(n-2)$ and S^{n-1} is the unit tangent sphere above p. Take $b_0 \in S^{n-1}$ and $x_0 \in \pi^{-1}(b_0) \subset H/K$. These fibrations, together with the inclusion map i, give the following commutative diagram, which connects the homotopy long exact sequences for the fibrations by the induced inclusion map i_* (see [21] Theorem 4.41):

$$\begin{tikzcd}
\pi_{n-1}(Q(\alpha)|_p/K, x_0) \arrow[r, "\pi_*"] \arrow[d, hook, "i_*"] & \pi_{n-1}(S^{n-1}, b_0) \arrow[r, "\bar{\partial}"] \arrow[d, "\cong"] & \pi_{n-2}(H/K, x_0) \arrow[d, hook, "i_*"] \\
\pi_{n-1}(St_2M|_p, x_0) \arrow[r, "\pi_*"] & \pi_{n-1}(S^{n-1}, b_0) \arrow[r, "\partial"] & \pi_{n-2}(S^{n-2}, x_0)
\end{tikzcd}$$

Note that $\partial = i_* \circ \bar{\partial}$. Now suppose H does not act transitively on the fiber of two frames over some $v \in T^1M$. Then $H/K \subsetneq S^{n-2}$ so $i_* = 0$ on $\pi_{n-2}(H/K, x_0)$ and thus $\partial = 0$ on $\pi_{n-1}(S^{n-1}, b_0)$. This implies that the map π admits a section, thus giving a nonvanishing vector field on S^{n-1}. If n is odd this is a contradiction. $\square$

2.3 The transitivity group and distinguished vector fields

As noted above, the transitivity group is crucial to this chapter's argument. This section will show that certain distinguished vector fields $w_v(t)$ along $\gamma_v(t)$ are preserved under the action of the transitivity group and use this result to prove Theorem 1. Throughout it utilizes the dynamical lemma, Lemma 2.1.5, with $C = -a^2$. Consider, for example, the ideal rectangle defined by v, v_1, v_2 and v_3 as pictured in Figure 2.1. If γ_{v_1} and γ_{v_3} are positively recurrent and If γ_v and γ_{v_2} are negatively recurrent, Lemma 2.1.5 implies that the element of H corresponding to this ideal polygon preserves the distinguished fields. The following arguments show how this idea can be worked out for *all* ideal polygons, first in the negative-curvature case and then in the general case.

2.3.1 The negative curvature case

In the negative-curvature case the argument of this section is considerably simpler. Consider the situation depicted in Figure 2.1. Lemma 2.1.5 shows that, when γ_{v_1} is recurrent in forward time, the map $p(v, v_1)$ preserves the distinguished vector fields in the sense that it sends a vector from one such field, $w_v(0)$, to a vector from another such field along γ_{v_1}. Thus, if in Figure 2.1 γ_{v_1} and γ_{v_3} are recurrent in positive time and γ_v and γ_{v_2} are recurrent in negative time, then the element of H_v given by parallel translation around this ideal rectangle will map $w_v(0)$ to another element of $v^\perp$ which is in a parallel field along γ_v making curvature $-a^2$. If these sort of recurrence properties held for all 'equilateral' ideal polygons based at v one would have that the transitivity group preserves the distinguished vector fields. One cannot assure that these recurrence properties are always present, but ergodicity of the geodesic flow on M indicates that they will be present almost all the time. Furthermore, the fact that elements of the transitivity group are defined using the continuous foliations provided by Brin can be used to argue that elements of the transitivity group depend continuously on the choice of ideal polygon. Thus the transitivity group will preserve the distinguished vector fields.

2.3.2 The general case

For the general case one needs arguments to deal with the problem of pairs of geodesics that are asymptotic but not strictly asymptotic, and the fact that one no longer knows one has a continuous foliation. By assumption, the frame flow preserves the distinguished vector fields. The only question in terms of how they behave under the action of elements from the transitivity group is how they behave when they are transferred across corners of the ideal polygons. As Lemma 2.1.5 shows, when the geodesics involved are strictly asymptotic and the second geodesic is recurrent, the fields are transferred as desired. Thus, there are two problems to deal with: when the second geodesic is not recurrent, and when the geodesics are not strictly asymptotic. The new definition of the transitivity group provides a way to deal with both of these issues.

First, note that under Case II of Definition 2.2.5, one defines the maps $p_{\{\xi_n\}}(v, v')$ as limits of the maps from the first case of the definition. Thus, to show that a distinguished field w_γ is preserved by some $p_{\{\xi_n\}}(v, v')$ one needs to realize w_γ as a limit of distinguished fields along the geodesics c_n used to

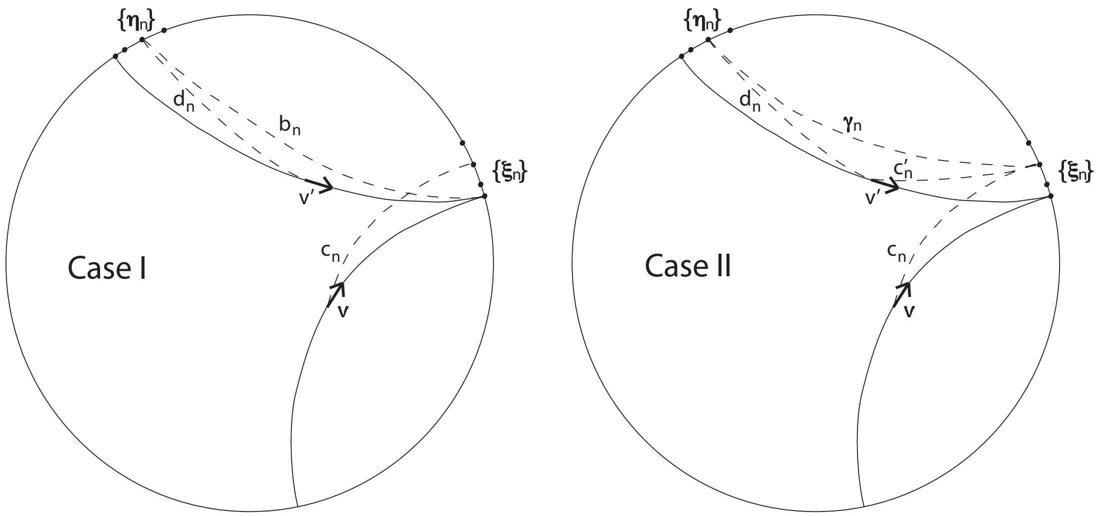

Figure 2.3: Geodesic configurations for Prop. 2.3.1 and 2.3.2

define the map $p_{\{\xi_n\}}(v, v')$ (see Figure 2.2). In view of this fact one works with distinguished fields that can be realized as limits, and ensures that this property of arising as a limit is also preserved by the $p_-(v, v')$-maps. In particular, one will consider distinguished fields w_v that arise as limits of distinguished fields w_{c_n} along geodesics c_n as depicted in Figure 2.3 and show that such a field is transferred by a map $p_-(v, v')$ to a field $w_{v'}$ arising as a limit of fields w_{d_n} along geodesics d_n which connect $\gamma_{v'}(0)$ to $\{\eta_n\} \to \gamma_{v'}(-\infty)$. If the next corner to be traversed falls under Case II, the sequence $\{\eta_n\}$ is determined by the map $p_{\{\eta_n\}}(v', v'')$; otherwise one is free to take any sequence. The arguments are slightly different in the two cases and so are addressed separately:

Case I:

Proposition 2.3.1. *Suppose $v \in T^1M$ falls under Case I, and that w_v is the limit of w_{c_n}. Then for any $v' \in W^s_g(v)$ (respectively $W^u_g(v)$), $w_{v'} := p(v, v')(w_v)$ is a distinguished field along $\gamma_{v'}$ arising as a limit of distinguished fields w_{d_n}.*

Proof. Assume $v' \in W^s_g(v)$; the proof for the unstable case is essentially the same. Under Case I, $\gamma_{v'}$ is limit of recurrent geodesics $\gamma_{v'_i}$ for $v'_i \in W^s_g(v)$. The maps $p(v, v'_i)$ preserve distinguished fields as demonstrated in Lemma 2.1.5 and $p(v, v'_i) \to p(v, v')$, so $w_{v'}$ will be a distinguished field as well.

Now one needs to demonstrate $w_{\gamma_{v'}}$ as a limit in the proper way. Construct geodesics d_n connecting $v'(0)$ and η_n and b_n connecting $\gamma_v(\infty)$ and η_n as in Figure 2.3. The field $w_{b_n} := p(v, \dot{b}_n(0))(w_v)$ will be a distinguished field by the argument of the previous paragraph. Likewise, the fields $w_{d_n} := p(\dot{b}(0)_n, \dot{d}_n(0))(w_{b_n})$ will be distinguished fields as $\eta_n \in \Omega$ falls under Case I. Since the $w_{b_n} \to w_{v'}$ it is clear that the $w_{d_n} \to w_{v'}$ and the proof is done. □

Case II:

Proposition 2.3.2. *Suppose $v \in T^1M$ falls under Case II, and that w_v is the limit of w_{c_n}. Then for any $v' \in W^s_g(v)$ (respectively $W^u_g(v)$), $w_{v'} := p_{\{\xi_n\}}(v, v')(w_v)$ is a distinguished field along $\gamma_{v'}$ arising as a limit of distinguished fields w_{d_n}.*

Proof. Again, assume $v' \in W^s_g(v)$. Since the maps $p(\dot{c}_n(0), \dot{c}'_n(0))$ are under Case I, they preserve distinguished fields. Thus, $w_{v'}$, which is defined as the limit of $p(\dot{c}_n(0), \dot{c}'_n(0))(w_{c_n})$, will be a distinguished field.

Immediately, one has that $w_{v'}$ arises as a limit of distinguished fields along the geodesics c'_n. As in Figure 2.3, let γ_n be the geodesic connecting ξ_n and η_n and let d_n be the geodesic connecting the footpoint of v' and η_n. If $\gamma_{v'}(-\infty)$ is not in Ω the map $p_{\{\eta_n\}}(v', v'')$ supplies the sequence $\{\eta_n\}$. In this case one requires that $(\{\xi_n\}, \{\eta_n\})$ satisfies the composition criterion for v', so c'_n, d_n and γ_n all approach each other (and $\gamma_{v'}$) as $n \to \infty$. If $\gamma_{v'}(-\infty)$ is in Ω it is easy to see that these geodesics still all converge as otherwise one would find a flat strip along $\gamma_{v'}$. Using p-maps under Case I, the fields $w_{c'_n}$ can be transfered to distinguished fields w_{γ_n} along γ_n and subsequently to distinguished fields w_{d_n} along d_n. It is then clear that $w_{c'_n}, w_{\gamma_n}$ and w_{d_n} all limit on $w_{v'}$; specifically, $w_{d_n} \to w_{v'}$ shows that $w_{v'}$ arises as a limit in the desired manner. □

This work proves the following proposition:

Proposition 2.3.3. *The transitivity group preserves distinguished vector fields that arise as limits of distinguished fields in the correct manner.*

2.3.3 Proof of the main theorem

One can now apply the results of Brin-Karcher from and of Brin-Gromov as adapted to the rank-1 situation in section 2.2 and prove Theorem 1 easily.

Theorem 1. *Let M be a compact, rank-1, nonpositively curved manifold. Suppose that along every geodesic in M there exists a parallel vector field making sectional curvature $-a^2$ with the geodesic direction. If M is odd-dimensional, or if M is even-dimensional and satisfies the sectional curvature pinching condition $-\Lambda^2 < K < -\lambda^2$ with $\lambda/\Lambda > .93$ then M has constant negative curvature equal to $-a^2$.*

Proof. Proposition 2.3.3 showed that the sectional curvature $-a^2$ fields that arise in the desired way as limits are preserved by the transitivity group. In the setting of the theorem, the adapted results of Brin-Gromov and Brin-Karcher show that the transitivity group acts transitively on $v^\perp \subset T^1M$. In particular, by considering the orbit of a distinguished field that arises correctly as a limit one sees that $K(\cdot, v)$ is identically $-a^2$, and the theorem is proved. □

2.4 Parallel fields and Jacobi fields

In [26] a distinction is made between 'weak' and 'strong' rank. The existence of *parallel* fields making extremal curvature is called strong rank; the existence only of *Jacobi* fields making extremal curvature is called weak rank. A parallel field scaled by a solution to the real variable version of the Jacobi equation (where the standard derivative replaces the covariant derivative) produces a Jacobi field. Thus, a proof under the less stringent hypothesis of weak rank implies a proof for strong rank. Hamenstädt's is the sole result prior to the result of this chapter. She states her main theorem for parallel fields only, but she shows in Lemma 2.1 that in negative curvature a Jacobi field making maximal curvature is a parallel field scaled by a function [20]. Essentially, she shows that Jacobi fields making maximal curvature grow at precisely the rate one finds for the constant curvature case. Connell accomplishes the same in [13] Lemma 2.3. This, together with some of the arguments below, shows that these Jacobi fields are in fact parallel fields scaled by an appropriate function. Therefore, Corollary 2 is a weak-rank result, needing only the Jacobi field hypothesis.

This section shows that Jacobi fields making *minimal* curvature with the geodesic direction are also scaled parallel fields. This will justify the phrasing of Corollary 1 as a weak-rank result. In this section, $\langle \cdot, \cdot \rangle$ will denote the Riemannian inner product and $R(\cdot, \cdot)\cdot$ the curvature tensor.

First, note that one need only consider non-vanishing Jacobi fields; hence it will be enough to prove that stable and unstable Jacobi fields are scaled parallel fields. Stable Jacobi fields are those which have norm approaching zero as $t \to \infty$; unstable Jacobi fields have the same property in the negative time direction.

Suppose $J(t)$ is a stable Jacobi field along the geodesic $\gamma(t)$ making curvature $-a^2$ with the geodesic (take $a > 0$ now), where $-a^2$ is the curvature minimum for the manifold (the modifications of what follows for unstable Jacobi fields are straightforward). The Rauch Comparison Theorem (see [16] Chapt 10, Theorem 2.3) can be used to show that

$$|J(t)| \geq |J(0)|e^{-at}. \tag{2.2}$$

One would like to show that equality is achieved in (2.2). Write $J(t) = j(t)U(t)$ where $j(t) = |J(t)|$ and $U(t)$ is a unit vector field. Then the Jacobi equation for J reads:

$$j''U + 2j'U' + jU'' + jR(\dot{\gamma}, U)\dot{\gamma} = 0 \tag{2.3}$$

where j' denotes the standard derivative and U' denotes covariant derivative. Taking the inner product of (2.3) with U and noting that $\langle U'', U\rangle = -\langle U', U'\rangle$ one obtains

$$j'' - j(\langle U', U'\rangle + a^2) = 0. \tag{2.4}$$

One now knows the following about the magnitude of J: $j \geq 0$ by definition, $\lim_{t\to\infty} j(t) = 0$ since J is a stable Jacobi field, and $j'' \geq a^2 j$ by (2.4). These allow the following conclusion; its proof was shown to the author by Jeffrey Rauch:

Lemma 2.4.1. *Let j be a non-negative, real valued function satifsying $j'' \geq a^2 j$ and* $\lim_{t\to\infty} j(t) = 0$. *Then $j(t) \leq j(0)e^{-at}$.*

Proof. One has that $a^2 j - j'' \leq 0$. On the interval $0 \leq t \leq R$ for $R \gg 1$ define g_R by $g_R(0) = j(0)$, $g_R(R) = j(R)$ and $a^2 g_R - g_R'' = 0$. Note that as $R \to \infty$, $g_R \to j(0)e^{-at}$. The claim is that $j \leq g_R$; the Lemma follows in the limit.

This claim is essentially the maximum principle. First, $j \leq g_R$ holds at 0 and R. Now suppose $j - g_R$ has a positive maximum at $c \in (0, R)$. Then $(j'' - g_R'')(c) \leq 0$. However, one knows $a^2(j - g_R) - (j'' - g_R'') \leq 0$, so a positive value of $j - g_R$ at c together with a nonpositive value of $j'' - g_R''$ yields a contradiction. Therefore $j \leq g_R$ holds on all of $[0, R]$ as desired. $\square$

This Lemma, together with equation (2.2), provides that $|J(t)| = |J(0)|e^{-at}$. Examining equation (2.4) one sees that having the growth rate e^{-at}, as in the constant curvature $-a^2$ case, implies that $U' = 0$, that is, J is a scaled parallel field, as desired.

2.5 The dynamical perspective

This section discusses how the results of Connell in [13] can be adapted to prove Theorem 2 as a simple consequence of Corollary 1. The necessary changes are for the most part cosmetic; the discussion here is included for completeness, but the author does not claim to have added anything of substance to Connell's work. The notation below that has not already been assigned follows Connell's for ease of reference.

Recall that Lyapunov exponents are a tool for measuring long-term asympotic growth rates in dynamical systems (see Katok and Mendoza's Supplement to [23], section S.2, for an exposition). In the setting of the geodesic flow they can be defined as follows. Let $v \in T^1M$ and $u \in v^\perp$. Let $J_u(t)$ be the unstable Jacobi field along γ_v with initial condition $J_u(0) = u$. Then, the *positive Lyapunov* *exponent at v in the u-direction* is

$$\lambda_v^+(u) = \limsup_{t\to\infty} \frac{1}{t} log|J_u(t)|.$$

Define

$$\lambda_v^+ = \max_{u\in v^\perp} \lambda_v^+(u).$$

This is the maximal Lyapunov exponent at v; the curvature bound $-a^2 \leq K$ (again, take $a > 0$) implies that $\lambda_v^+ \leq a$. Let

$$\Omega = \{v \in T^1M : \lambda_v^+ = a\}.$$

One can rephrase Theorem 2 more succinctly.

Theorem 2. *Let M be a compact manifold with sectional curvature $-a^2 \leq K \leq 0$. Suppose that Ω has full measure with respect to a geodesic flow-invariant measure μ with full support. If M is odd dimensional, or if M is even dimensional and satisfies the sectional curvature pinching condition $-a^2 \leq K < -\lambda^2$ with $\lambda/a > .93$ then M is of constant curvature $-a^2$.*

Connell shows in the upper rank case that the dynamical assumption implies the geometric one, that is, that the manifold in fact has higher rank, allowing the application of an appropriate rank-rigidity theorem. He first shows ([13], Proposition 2.4) that along a closed geodesic $\lambda_v^+ = a$ implies the existence of an unstable Jacobi field making curvature $-a^2$ with the geodesic direction. Essentially, if the Jacobi field giving rise to the Lyapunov exponent does not have this curvature, it will continually see non-extremal curvature a positive fraction of the time as it moves around the closed geodesic. This contradicts the supposed value of the Lyapunov exponent. The lower curvature bound version of the argument is exactly the same as that presented by Connell, with the proper inequalities reversed; also note that the work in section 2.4 of this paper gives the results analogous to Connell's Lemma 2.3 necessary for the argument.

It is clear that if a dense set of geodesics have the distinguished Jacobi fields, then all geodesics will. Since the velocity vectors for closed geodesics are dense in T^1M, Connell finishes his proof in section 3 of [13] by showing that these vectors are all in Ω and using the argument of the previous paragraph. Adapted to the setting of Theorem 2 the argument runs as follows. If $w \in T^1M$ is tangent to a closed geodesic and $\lambda_w^+ < a$ the previous paragraph implies that any unstable Jacobi field along γ_w must make curvature strictly greater than $-a^2$ a positive fraction of the time. By continuity, this will also be true of any unstable Jacobi field along a geodesic γ_v in a sufficiently small neighborhood of γ_w (in the Sasaki metric on T^1M). The ergodic theorem implies that for a full-measure set of $v \in T^1M$, γ_v will spend a positive fraction of its life in this small neighborhood of the periodic geodesic γ_w; the positivity follows from the fact that μ has full support. The intersection of this full-measure set with the full-measure set Ω thus contains vectors v which have $\lambda_v^+ = a$ but spend a positive fraction of their life so close to γ_w that no Jacobi fields along them can make the minimal curvature $-a^2$ with the geodesic direction during this fraction of the time. In fact, since γ_w is compact, so is the closure of this small neighborhood and therefore the curvature between these Jacobi fields and the geodesics, when in this neighborhood, can be bounded away from $-a^2$, i.e. $K(J_u, \dot{\gamma}_v) > c > -a^2$ for a fixed c. Having this curvature bound a positive fraction of the time contradicts $\lambda_v^+ = a$; therefore all closed geodesics must lie in Ω and the argument is complete.

Again, this version of the argument, relevant for the lower curvature bound situation, is the same as that presented by Connell with the proper inequalities reversed. Thus, the dynamical assumption implies the geometric assumption of Corollary 1 and Theorem 2 follows. Note that for these arguments the extremality of the distinguished curvature is essential and one does not obtain a result that parallels Theorem 1 in allowing non-extremal distinguished curvature.

CHAPTER III

Conclusion

I conclude this thesis with a few remarks on possible extensions of this work and some open questions. It is useful to include here a summary of rank-rigidity results taken from [26]. The table refers to results on compact manifolds M; weak rank refers to the condition of having a Jacobi field along every geodesic making extremal curvature with the geodesic direction, strong rank refers to the same condition for parallel fields. The implication arrows in the table indicate when a positive result for the weaker, Jacobi field condition implies a positive result for the stronger, parallel field condition (as in the $K \leq -1$ case) or when a counterexample for the stronger condition furnishes a counterexample to the weaker condition (as in the $K \geq 0$ case).

Compact manifolds		
Curvature bound	weak rank	strong rank
$K \leq 0$	?	M is locally symmetric, or some finite cover is isometrically a product; [2], [12]
$K \geq 0$	$\Longleftarrow$	There are simply connected, irreducible M not homeomorphic to a symmetric space; [27]
$K \leq -1$	M is locally symmetric; [20], Corollary 2 (partial result)	$\Longrightarrow$
$K \geq -1$	M is hyperbolic under curvature pinching; Corollary 1 (partial result)	$\Longrightarrow$
$K \leq 1$	Non-symmetric examples exist; [26]	M is symmetric; [26]
$K \geq 1$	Non-symmetric examples exist; [26]	?

The central direction for extending this work is in removing the curvature pinching condition. Since parallel translation preserves the complex structure on a Kähler manifold the 2-frame flow will not be ergodic. These known counterexamples to ergodic frame flow are excluded by requiring $-1 < K < -1/4$, leading Brin to conjecture that strict 1/4-pinching implies that the frame flow is ergodic ([8], Conjecture 2.6). A positive answer to this conjecture, or any extended results for ergodicity of the 2-frame flow in negative curvature would extend the results on rank-rigidity presented here correspondingly, using the same proof as presented above.

Note, however, that in even dimension a result directly parallel to the odd-dimensional result cannot be hoped for. There are negatively curved manifolds with higher hyperbolic rank which are not hyperbolic; complex hyperbolic manifolds are examples. One still hopes that hyperbolic rank-rigidity (in the sense that higher rank implies the space is locally symmetric) could be true in even dimensions under only the condition $0 < K \leq -1$, and the result here as well as the extensive analogous results for the other rank-rigidity theorems seem to make such a theorem more likely. However, such a result would call for a significantly different method of proof from that presented here.

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