Galois deformation theory for norm fields and its arithmetic applications

by

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A dissertation submitted in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
(Mathematics)
in The University of Michigan
2009

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To my mentor in life, Mr. Daisaku Ikeda,
to a great teacher in mathematics, Brian Conrad,
to my mom and my sister,
and to whoever lent me support.

ACKNOWLEDGEMENTS

The author deeply thanks Brian Conrad for his guidance. The author especially appreciates his careful listening of my results and numerous helpful comments. The author would also like to thank Gebhard Böckle and Urs Hartl for various helpful comments. The author thanks Tong Liu for providing his idea to prove Corollary 12.2.6 when $p = 2$.

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CHAPTER I

Introduction

1.1 Motivation and overview

1.1.1 p-adic local Galois representations

Since the pioneering work of Wiles on the modularity of semi-stable elliptic curves over $\mathbb{Q}$, many classes of 2-dimensional (mod p or p-adic) global Galois representations are known to "come from" modular forms. One of the main difficulties of proving modularity lies in the study of local deformation problems with various p-adic Hodge theory conditions, for which one needs to understand Galois-stable $\mathbb{Z}_p$-lattices in (potentially) semi-stable p-adic representations and their reductions mod p^n. On the other hand, "*integral* p-adic Hodge theory" is much more delicate than "classical" p-adic Hodge theory, which makes it hard to study deformations satisfying various p-adic Hodge theory conditions.

This paper introduces a new technique of using the norm fields to study deformations and mod p reductions in p-adic Hodge theory, which is explained below. Let $\mathscr{K}/\mathbb{Q}_p$ be a finite extension. Choose a uniformizer $\pi \in \mathfrak{o}_{\mathscr{K}}$, and consider an infinite Kummer-type extension $\mathscr{K}_\infty := \mathscr{K}(\sqrt[p^\infty]{\pi})$. We put $\boldsymbol{\mathcal{G}}_{\mathscr{K}} := \mathrm{Gal}(\overline{\mathscr{K}}/\mathscr{K})$ and $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty} := \mathrm{Gal}(\overline{\mathscr{K}}/\mathscr{K}_\infty)$. Kisin [52] showed that the restriction to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ of a semi-stable $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representation with Hodge-Tate weights in $[0, h]$ is so-called a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-

representation "*of height* $\leqslant h$."[1] The precise definition will be given later in Definition 5.2.8. The point is that integral theory for $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representations "of height $\leqslant h$" is much simpler than integral p-adic Hodge theory, and that we lose no information by restricting crystalline $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$.[2] See §2.4 for a summary of Kisin [52].

In order to study (or even, to define) deformations "of height $\leqslant h$" one needs to define and study torsion representations "of height $\leqslant h$," which is carried out in §8–§9 of this paper. One of the main results of this paper is the existence of universal $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-deformation rings "of height $\leqslant h$" for any positive integer h:

Theorem (11.1.2). *Let $\mathbb{F}$ be a finite extension of $\mathbb{F}_p$ and let $\bar{\rho}_\infty$ be an $\mathbb{F}$-representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ of finite dimension. Then there exists a complete local noetherian $W(\mathbb{F})$-algebra $R^{\square,\leqslant h}_{\bar{\rho}_\infty}$ with residue field $\mathbb{F}$ and a framed deformation of $\bar{\rho}_\infty$ over $R^{\square,\leqslant h}_{\bar{\rho}_\infty}$ which is universal among all the framed deformations of $\bar{\rho}_\infty$ with "height $\leqslant h$." If $\mathrm{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}(\bar{\rho}_\infty) \cong \mathbb{F}$ then there exists a complete local noetherian $W(\mathbb{F})$-algebra $R^{\leqslant h}_{\bar{\rho}_\infty}$ with residue field $\mathbb{F}$ and a deformation of $\bar{\rho}_\infty$ over $R^{\leqslant h}_{\bar{\rho}_\infty}$ which is universal among all the deformations of $\bar{\rho}_\infty$ with "height $\leqslant h$."*

The existence of such $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-deformation rings is surprising because the usual 'unrestricted' $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-deformation functor has a infinite-dimensional tangent space (so 'unrestricted' $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-deformation rings do not exist in the category of complete local noetherian rings); see §11.7.1 for the proof of this claim. Note that $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ does not satisfy the cohomological finiteness condition that is usually used to prove the finite-dimensionality of the tangent space of interesting Galois deformation functors.

[1] Later in this paper, we use the terminology *$\mathcal{P}$-height* instead of *height* where $\mathcal{P}(u)$ is an Eisenstein polynomial over the maximal unramified subextension $\mathscr{K}_0$ of $\mathscr{K}$ such that $\mathcal{P}(\pi) = 0$. This is to avoid confusion with the analogous notion of height which uses the p-adic cyclotomic extension instead of an infinite Kummer-type extension.

[2] There is a semi-stable analogue of this statement. Roughly speaking, it says that by restricting the $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-action of a semi-stable representation to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$, we only lose the monodromy operator of the corresponding filtered (φ, N)-module.

Let $\bar\rho$ be a finite-dimensional $\mathbb{F}$-representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ such that $\bar\rho|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}} \cong \bar\rho_\infty$. Then "restricting the $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-action to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$" defines natural maps from $R^{\square,\leqslant h}_{\bar\rho_\infty}$ constructed in the above theorem into crystalline/semi-stable framed deformation rings[3] of $\bar\rho$ with Hodge-Tate weights in $[0,h]$. (If $\mathrm{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}(\bar\rho_\infty) \cong \mathbb{F}$ then we obtain the same result for deformation rings without framing.) By using these maps and analyzing the structure of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-deformation rings constructed above, we obtain the following results on crystalline/semi-stable deformation rings.

- The "ordinary" condition cuts out a union of connected components in (the $\mathbb{Q}_p$-fiber of) a crystalline or semi-stable (framed) deformation ring with Hodge-Tate weights in $[0,h]$ (where the crystalline and semi-stable deformation rings are as defined by Kisin [55] and Tong Liu [59]). This is done in Proposition 11.4.18.

- Assume $\dim_{\mathbb{F}} \bar\rho = 2$. Let $R^{\square,\mathbf{v}}_{\mathrm{fl}}$ be the quotient of the flat framed deformation ring with the property that the determinant of the action of the inertia group $I_{\mathscr{K}}$ is equal to the p-adic cyclotomic character[4]. Kisin gave a complete description of the connected components of $\mathrm{Spec}\, R^{\square,\mathbf{v}}_{\mathrm{fl}}[\frac{1}{p}]$, which is used as the main technical ingredient for the proof of his modularity lifting theorem [51, 53]. Assuming $p > 2$, the author gives a new proof of Kisin's description of the connected components of $\mathrm{Spec}\, R^{\square,\mathbf{v}}_{\mathrm{fl}}[\frac{1}{p}]$, which was crucially used in Kisin's modularity lifting theorem [51, 53]. The idea is to "resolve" $\mathrm{Spec}\, R^{\square,\mathbf{v}}_{\mathrm{fl}}$ using the Breuil-Kisin classification of finite flat group schemes. This paper presents another method to resolve $\mathrm{Spec}\, R^{\square,\mathbf{v}}_{\mathrm{fl}}$ using $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-deformation rings, so we eliminate the Bruil-Kisin classification from the proof of Kisin's modularity theorem. The virtue of this new method is that it works more uniformly in the case $p = 2$ (after

[3] A crystalline/semi-stable (framed) deformation ring "over $\mathbb{Q}_p$" was defined by Kisin [55], and later Tong Liu [59] defined it without inverting p We will use Tong Liu's definition, which recovers Kisin's ring after inverting p.

[4] This condition can be thought of as fixing a p-adic Hodge type.

minor modifications), while the Breuil-Kisin classification of finite flat group schemes is quite problematic when $p = 2$. Kisin needs a separate paper [53] to prove the classification of *connected* finite flat group schemes over a 2-adic base, which uses Zink's theory of windows and displays, and the full proof of Serre's conjecture by Khare-Wintenberger uses the modularity of 2-adic Barsotti-Tate liftings. See §11.6 for more details.

We digress to record the following result of separate interest, which is obtained as a byproduct of the study of torsion representations "of height $\leqslant h$." Observe that *a semi-simple mod p representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ can be uniquely recovered from its restriction to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$.* Indeed, since any semi-simple mod p representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ is tame, this assertion follows from the fact that the extension $\mathscr{K}_\infty/\mathscr{K}$ does not have any non-trivial tame subextension. By studying restrictions to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$, we thereby obtain an explicit description of mod p crystalline characters with Hodge-Tate weights in $[0, h]$ for any positive h. (See Proposition 9.4.8 for the case when the residue field of $\mathscr{K}$ is big enough. The author plans to generalize this results to accommodate "descent data for a tame extension" in a subsequent work.) Even the case $h = 1$ (i.e., finite flat mod p characters) is interesting. Savitt [70] obtained the same result for the case $p > 2$ and $h = 1$ via elaborate computations with Breuil modules, but the author's argument is much simpler and works in the case $p = 2$ as well (in addition to allowing any $h \geqslant 1$).

This result is a first step towards understanding the reduction mod p of crystalline $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations up to semisimplification, since any absolutely irreducible mod p representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ arises as an "unramified induction" of a character.

1.1.2 Equi-characteristic analogue

There exists an equi-characteristic "analogue" of Kisin's theory [52], which historically came first as initiated by Genestier-Lafforgue [35] and Hartl [39, 41] in an attempt to find an equi-characteristic analogue of Fontaine's theory of crystalline representations. To explain this we first introduce some notations. We fix a formal power series ring $\mathbb{F}_q[[\pi_0]]$, which will play the role of $\mathbb{Z}_p$ (and π_0 will play the role of p). We also fix a finite field k, a complete discrete valuation ring $\mathfrak{o}_K \cong k[[u]]$ with the fraction field $K \cong k(\!(u)\!)$ and a local map $\mathbb{F}_q[[\pi_0]] \to \mathfrak{o}_K$ which makes $\mathfrak{o}_K$ a finite $\mathbb{F}_q[[\pi_0]]$-module. In particular, this specifies an embedding $\mathbb{F}_q \hookrightarrow k$. Let $\boldsymbol{\mathcal{G}}_K$ denote the absolute Galois group for K. Genestier-Lafforgue and Hartl studied $\mathbb{F}_q[[\pi_0]]$-representations of $\boldsymbol{\mathcal{G}}_K$ which can be viewed as analogues of crystalline representations, and their theory bears an incredible resemblance with the class of p-adic $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representations "of finite height."

Before we discuss the work of Genestier-Lafforgue [35] and Hartl [39, 41], let us explain why their theory can be regarded as an equi-characteristic analogue of Fontaine's theory of crystalline representations. (The idea presented below is also found in Hartl's work [39, 41].) If one wants to find a class of $\mathbb{F}_q[[\pi_0]]$-representations of $\boldsymbol{\mathcal{G}}_K$ which can be viewed as an "analogue" of crystalline representations (or Barsotti-Tate representations), then the natural candidate is the π_0-adic Tate module of a "π_0-divisible group" G over $\mathfrak{o}_K$. But it turns out that in order to get a nice theory we need more assumptions on the π_0-divisible groups. We say that a π_0-divisible group G is *of "finite height"*[5] if the Verschiebung of G vanishes[6] and

[5]Hartl calls it a *divisible Anderson module* in [41, §3.1]. A π_0-divisible formal Lie group of height $\leqslant 1$ is also known as a Drinfeld formal $\mathbb{F}_q[[\pi_0]]$-module, and these have been widely studied since being introduced by Drinfeld in [25].

[6]The π_0-divisible group associated to a Drinfeld module or to any π_0-divisible formal Lie group has vanishing Verschiebung, so this is not a restrictive assumption. See [34, Ch.I, Prop 2.1.1] for the case of π_0-divisible formal Lie groups.

the induced $\mathbb{F}_q[[\pi_0]]$-action on the Lie algebra satisfies a certain natural assumption. We say a $\mathcal{G}_K$-representation over $\mathbb{F}_q[[\pi_0]]$ is *of finite height* if it is isomorphic to the π_0-adic Tate module of a π_0-divisible group of finite height. See [41, §3.1] or §7.3 of this paper.

An amusing fact is that whereas the p-adic Tate module of a Barsotti-Tate group always has its Hodge-Tate weights in $[0,1]$, the π_0-adic Tate module of a π_0-divisible group of finite height can have any non-negative "weights." To illustrate, consider the Lubin-Tate character $\chi_{\mathcal{LT}}$ of $\mathcal{G}_K$, which can be thought of as a representation of "weight 1." Then for any positive number h, the character $\chi_{\mathcal{LT}}^h$ comes from the π_0-adic Tate module of a certain 1-dimensional π_0-divisible formal Lie group over $\mathfrak{o}_K$ of "height h." It is reasonable to regard $\mathcal{G}_K$-representations of finite height as the equi-characteristic analogue of crystalline representations of non-negative Hodge-Tate weights.

The "Dieudonné-type classification" for finite flat group schemes with trivial Verschiebung [73, 3, Exp $\mathrm{VII_A}$, 7.4][7] induces a classification of π_0-divisible groups of finite height. This result was first announced by Hartl in [40], and is surely well-known to experts. Since the proof was not available to the author, we work out a proof in §7 of this paper.[8] The Frobenius modules which occur as the "Dieudonné module" of such π_0-divisible groups were studied by Genestier-Lafforgue [35] and Hartl [39, 41][9], and their theory exhibits many features that are remarkably similar to Kisin's theory [52] of Frobenius $\mathfrak{S}$-modules which classify $\mathcal{G}_{\mathscr{K}_\infty}$-representations "of finite height."

Although $\mathcal{G}_K$-representations of finite height have properties akin to those of

[7] For readers' convenience, we reproduce the proof in §7.2 of this paper.

[8] The classification of Drinfeld formal $\mathbb{F}_q[[\pi_0]]$-modules (i.e., π_0-divisible formal Lie groups of height $\leqslant 1$) is also proved in [34, §1].

[9] Such Frobenius modules are exactly the same as *effective local shtukas* in [35, 39, 41] (since $\mathfrak{o}_K$ is noetherian). See Proposition 7.1.9 of this paper.

$\mathcal{G}_{\mathscr{K}_\infty}$-representations of finite $\mathcal{P}$-height, it still makes sense to regard them as the equi-characteristic analogue of crystalline representations of the full Galois group $\mathcal{G}_{\mathscr{K}}$ (with non-negative Hodge-Tate weights) for the following reason. In a field of characteristic p, adjoining a pth root induces a purely inseparable extension and so does not change the absolute Galois group. Therefore the gap between $\mathcal{G}_K$ and the absolute Galois group of any infinite Kummer-type extension $K[\sqrt[q^\infty]{u}]$ collapses since $\mathrm{char}(K) = p > 0$.

The analogy between Kisin's theory and its equi-characteristic analogue is further strengthened by the following theorem proved by the author, which is also a very useful tool in applying the theory of Genestier-Lafforgue and Hartl to Galois representations.

Theorem (5.2.3). *The π_0-adic Tate module functor from the category of π_0-divisible groups over $\mathfrak{o}_K$ of finite $\mathcal{P}$-height to the category of lattice $\mathbb{F}_q[[\pi_0]]$-representations of finite height is* fully faithful.

The statement of the above theorem is clearly reminiscent of Tate's theorem of the full faithfulness of the p-adic Tate module functor on Barsotti-Tate groups [75, §4.2]. For the proof, we use the "Dieudonné-type classification" to translate the theorem into a statement about Frobenius modules. The proof is completely analogous to that of [52, Proposition 2.1.12], except the following two modifications. First, we need to work with "isocrystals with weakly admissible Hodge-Pink structures"[10] instead of weakly admissible filtered isocrystals (or weakly admissible filtered (φ, N)-modules). Second, we need to eliminate the use of logarithmic connections over the open unit disk from the proof of [52, Proposition 2.1.12], which have no good equi-characteristic analogue.

[10] See Definition 2.3.1 and §2.3.7 for the definition.

Our modified argument works verbatim in the p-adic case and thus gives a variant of Kisin's proof of the p-adic version of Theorem 5.2.3; i.e., [52, Proposition 2.1.12]. In particular, we construct an analogue of weakly admissible Hodge-Pink structures in the $\mathbb{Z}_p$-coefficient case, and this is often useful. For example, one can give an explicit criterion, in terms of such "mixed characteristic" Hodge-Pink structures, to figure out whether an explicitly given $\mathfrak{M}[\frac{1}{p}] \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}]$ comes from a weakly admissible filtered isocrystal. See Remark 3.2.4 and Proposition 5.2.13 of this paper.

Thanks to the similarity between $\mathbb{Z}_p$-linear representations of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ "of finite height" and $\mathbb{F}_q[[\pi_0]]$-linear representations of $\boldsymbol{\mathcal{G}}_K$ of finite height, any discussion below for one adapts to the other. In particular, the same proof of Theorem 11.1.2 gives the existence of the universal deformation and framed deformation rings in the equi-characteristic setting, even though $\boldsymbol{\mathcal{G}}_K$ has infinite p-cohomological dimension in the equi-characteristic case.

1.2 Structure of the Paper

Since most of the results and proofs for p-adic $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representations "of finite height" and their equi-characteristic analogues are completely parallel, in §1.3 we give conventions to simultaneously discuss both cases simultaneously.

In §1–§7, we introduce various semilinear algebra objects which are used in the study of p-adic $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representations "of finite height" and their equi-characteristic analogues, and settle the relations between them (e.g. equivalences of categories). The following two results are the main theorems proved in §1–§7, which are crucially used in the study of deformations. First, we give another proof of the theorem of Genestier-Lafforgue [35, Théorème 3.3] which asserts the equivalence of categories between the category of local shtukas and the category of isocrystals with weakly admissible Hodge Pink structure. The argument presented in this paper is more akin to arguments of Kisin [52, §1.3] and also proves the analogous statement in the classical p-adic setting. (In the p-adic setting, "Kisin modules," or $(\varphi, \mathfrak{S})$-modules of finite height, play the same role as effective local shtukas. See Definitions 2.2.1 and 2.3.1 for the relevant definitions.) Second, we show the full faithfulness of natural functors from various categories of semi-linear algebra objects into the category of suitable Galois representations (Theorem 5.2.3). The p-adic case of this theorem was proved by Kisin [52, Proposition 2.1.12].

In §2, we define various semilinear algebra objects which are used to study p-adic $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representations "of finite height" and their equi-characteristic analogues. In §2.4, we outline the results of Kisin [52] in order to "preview" the discussions to follow.

In §3, we construct equivalences of categories between the category of isocrystals with "effective" Hodge-Pink structures and the category of certain vector bundles over the open unit disk with Frobenius structure. (We will define these objects in §2 for both the p-adic and equi-characteristic cases.) This chapter is "modeled" after [52, §1.2], except that we work with Hodge-Pink structures instead of filtered (φ, N)-modules.

In §4, we show the equivalence between the weak admissibility of an isocrystal with Hodge-Pink structure and the property that the corresponding vector bundle with Frobenius structure is pure of slope 0 in the sense of Kedlaya (in the p-adic setting) and Hartl (in the equi-characteristic setting). The key ingredient is the theory of slopes, which is due to Kedlaya in the p-adic case and due to Hartl in the equi-characteristic case. This chapter is "modeled" after [52, §1.3] except that we have to work solely with the Frobenius structure and eliminate the use of logarithmic connections on vector bundles over the open unit disk.

In §5.1, we review Fontaine's theory of étale φ-modules and develop its equi-characteristic analogue, which allows us to define natural functors from various categories of Frobenius modules we study into the category of suitable Galois representations. In §5.2, we finally prove the full faithfulness of these functors, using all the results in the previous chapters.

In §6, we prove the equi-characteristic analogue of Kedlaya's matrix factorization lemma [46, Prop 6.5] which was used in §4. This chapter could be replaced by the following single sentence: the same argument that proves the p-adic statement as appears in [46, §6] also proves the equi-characteristic analogue.

The main result of §7 is the equivalence of categories between the category "effective local shtukas"[11] and the category of π_0-divisible groups of finite height (Theorem 7.3.2). This result serves as an equi-characteristic analogue of the Breuil-Kisin classification of Barsotti-Tate groups [52, Theorem 2.2.7], which is also stated as Theorem 2.4.11(1) in this paper. This result was announced by Hartl [40], but since the proof was not available to the author, we work out the proof here.

The next two chapters §8–§9 develop the theory of torsion $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representations of finite height and its equi-characteristic analogue. In §8, we introduce torsion Frobenius modules which give rise to torsion Galois representations. In §9, we prove various results which play the same role in the study of deformations "of finite height" as Raynaud's theory [69] does in the study of flat deformations. As a byproduct, we obtain an explicit description of mod p crystalline $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-characters by studying mod p characters of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ with finite height. See Proposition 9.4.8 for the precise

[11]The definition we use (Definition 7.1.1) slightly differs from Hartl's, which is the reason why this term is in quotes: we modify the definition in order to be able to show the equivalence of categories with π_0-divisible groups. If either the base is locally noetherian or the image of π_0 is locally topologically nilpotent in the base, then our definition and Hartl's definition coincide (Proposition 7.1.9).

statement.

In the remaining chapters §10–§12, we apply all of the preceding results to study of deformations "of height $\leqslant h$." Since we work with the language of *deformation groupoids* instead of deformation functors, we provide a chapter (§10) to recall definitions and prove some basic properties that are needed. The discussion would be familiar to experts in stacks, except that we do not use a Grothendieck topology[12].

In §11.7, we show the existence of (framed) deformation rings for $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representations "of height $\leqslant h$" as well as for their equi-characteristic analogue (Theorem 11.1.2). In §11.1, we imitate the discussion in [51, (2.1)] to construct an analogue of Kisin's moduli space of finite flat group schemes over the (framed) deformation rings "of height $\leqslant h$." Here, we use the moduli of "$\mathfrak{S}$-lattices of height $\leqslant h$" in place of finite flat group schemes. In §11.2, we show that this auxiliary space we constructed over a deformation ring "of height $\leqslant h$" has generic fiber isomorphic to the generic fiber of the deformation ring (Proposition 11.2.6). This result crucially uses the full faithfulness of the natural functors from various categories of φ-modules into Galois representations (Theorem 5.2.3). Using this, we show that the generic fibers of deformation rings of "height $\leqslant h$" are formally smooth (Corollary 11.2.10). In §11.3, we define "types" on the generic fiber of a (framed) deformation ring "of height $\leqslant h$" and show that (under a suitable "separability" assumption which is automatic in the p-adic case) fixing a type cuts out a equi-dimensional union of connected components in the generic fiber. We also compute the dimension of the dimension in terms of a fixed "type." The discussion of this section is akin to [55, §3], except that we work with isocrystals with weakly admissible Hodge-Pink structure instead of weakly admissible filtered (φ, N)-modules.

[12] Or rather, one can view a category cofibered in groupoids as a stack by giving the "silly" Grothendieck topology on the base where only isomorphisms are coverings

The remaining sections are devoted to the study of connected components of the generic fibers of various (framed) deformation rings. In §11.4, we show that the "ordinary" condition cuts out a union of connected components in the generic fiber of a (framed) deformation ring "of height $\leqslant h$" for any positive h, and in the case of 2-dimensional representations we give a complete description of all connected components with a certain fixed "type." In the p-adic case we use the natural map into crystalline/semi-stable (framed) deformation rings to show that the "ordinary" condition cuts out a union of connected components in the $\mathbb{Q}_p$-fiber of crystalline/semi-stable (framed) deformation rings.

In §11.5, for 2-dimensional representations (under a suitable "separability" assumption which is automatic in the p-adic case) we determine the connected components of (framed) deformation rings "of height $\leqslant 1$" and of a certain fixed "type," using Deligne-Pappas local models for Hilbert-Blumenthal modular surfaces (and its equi-characteristic analogue). Since the "moduli of finite flat group schemes" and the "moduli of $\mathfrak{S}$-lattices of height $\leqslant 1$" are defined in a very similar manner, Kisin's argument [51, (2.4), (2.5)] applies with few modifications to show that if $p > 2$ then "restricting to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$" induces an isomorphism from the $\mathbb{Q}_p$-fiber of a framed $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-deformation ring "of height $\leqslant 1$" to the $\mathbb{Q}_p$-fiber of a framed flat deformation ring; we explain this in §11.6. The point is that this allows us to reduce the connected component analysis of flat deformation rings to that of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-deformation rings "of height $\leqslant 1$," which was carried out in §11.5. For the case $p = 2$, we prove a weaker statement which is good enough for the application to Kisin's modularity theorem for 2-adic potentially Barsotti-Tate representations [53]. The proof uses Breuil's theory of strongly divisible modules (§XII). We use strongly divisible lattices to produce some $\mathbb{Z}_p$-lattice crystalline representations with Hodge-Tate weights in $[0, 1]$ whose restriction to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ is naturally isomorphic to a specified one.

Acknowledgement.

The author deeply thanks Brian Conrad for his guidance. The author especially appreciates his careful listening of my results and numerous helpful comments. The author would also like to thank Gebhard Böckle and Urs Hartl for various helpful comments. The author thanks Tong Liu for providing his idea to prove Corollary 12.2.6 when $p = 2$.

1.3 Notations/Definitions

We define a *σ-ring* to be a pair (R, σ_R) where R is a ring and $\sigma_R : R \to R$ is a ring endomorphism. For example (R, id_R) is a σ-ring. We say (R, σ_R) is *σ-flat* if σ_R is flat. For two σ-rings (R, σ_R) and $(R', \sigma_{R'})$, we say that $(R', \sigma_{R'})$ *is defined over* (R, σ_R) if R' is an R-algebra and $\sigma_{R'}$ is σ_R-semilinear. In this paper, σ_R usually has an interpretation as a Frobenius endomorphism (or a partial Frobenius endomorphism) on R.

Let $\mathfrak{o}_0$ be either $\mathbb{Z}_p$ or $\mathbb{F}_q[[\pi_0]]$. We set $\pi_0 := p$ if $\mathfrak{o}_0 = \mathbb{Z}_p$. Let $F_0 := \mathfrak{o}_0[\frac{1}{\pi_0}]$ be the fraction field; i.e., $F_0 = \mathbb{Q}_p$ or $F_0 = \mathbb{F}_q(\!(\pi_0)\!)$. We view them as σ-rings by setting $\sigma := \mathrm{id}$. All the σ-rings (R, σ_R) that appears in this paper are defined over $(\mathfrak{o}_0, \mathrm{id})$. We let q denote the size of the residue field of $\mathfrak{o}_0$, so $q = p$ if $\mathfrak{o}_0 = \mathbb{Z}_p$.

Let K be a complete discretely valued field of characteristic p. Let $\mathfrak{o}_K$ be its valuation ring and let k be its residue field. We assume that k is perfect if $\mathfrak{o}_0 = \mathbb{Z}_p$, and that k has a finite p-basis and contains $\mathbb{F}_q$ if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$; i.e., k is a finite-dimensional k^p-vector space. In both cases, the qth power map on k (and hence, on K) is finite. We fix a uniformizer $u \in \mathfrak{o}_K$, so we often identify $\mathfrak{o}_K$ with $k[[u]]$. We fix a separable closure K^{sep} and set $\boldsymbol{\mathcal{G}}_K := \mathrm{Gal}(K^{\mathrm{sep}}/K)$. We would like to study a certain class of $\boldsymbol{\mathcal{G}}_K$-representations over $\mathfrak{o}_0$, F_0, or finite algebras thereof.

1.3.1 Motivating examples

We first describe some motivating examples of $\boldsymbol{\mathcal{G}}_K$-representations with p-adic and equi-characteristic coefficient. By letting $\mathfrak{o}_0$ denote either $\mathbb{Z}_p$ or $\mathbb{F}_q[[\pi_0]]$ and developing a consistent set of notations for each choice, we shall study p-adic and equi-characteristic $\boldsymbol{\mathcal{G}}_K$-representations simultaneously.

1.3.1.1 The case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$

Let us fix an injective local map $\mathfrak{o}_0 \hookrightarrow \mathfrak{o}_K$. We are interested in $\mathfrak{o}_0$-linear representations of $\boldsymbol{\mathcal{G}}_K$ which are obtained as the π_0-adic Tate modules of a certain class of π_0-divisible groups over $\mathfrak{o}_K$, namely "π_0-divisible groups of finite height" (Definition 7.3.1).

1.3.1.2 The case $\mathfrak{o}_0 = \mathbb{Z}_p$

Let $\mathscr{K}$ be a finite extension of $\mathbb{Q}_p$, and $\mathscr{K}_0$ the maximal unramified subfield of $\mathscr{K}$ (i.e., $\mathscr{K}_0 \cong W(k)[\frac{1}{p}]$ where k is the residue field of $\mathscr{K}$). Let us fix a uniformizer $\pi \in \mathfrak{o}_{\mathscr{K}}$ and an Eisenstein polynomial $\mathcal{P}(u) \in \mathfrak{o}_{\mathscr{K}_0}[u]$ such that $\mathcal{P}(\pi) = 0$. Pick $\pi^{(n)} \in \mathfrak{o}_{\overline{\mathscr{K}}}$ for $n \geq 0$ so that $\pi^{(0)} = \pi$ and $(\pi^{(n+1)})^p = \pi^{(n)}$. Set $\mathscr{K}_\infty := \bigcup_{n\geq 0} \mathscr{K}(\pi^{(n)})$ as subfields of a fixed algebraic closure $\overline{\mathscr{K}}$. The theory of norm fields provides a natural isomorphism $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty} \xrightarrow{\sim} \boldsymbol{\mathcal{G}}_K$ (call *norm-field* isomorphism) where $K \cong k(\!(u)\!)$. See §1.3.2 below for more discussions, and [78] for a complete exposition on norm fields.

We are interested in a certain class of p-adic representations of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$, which are called *semi-stable representations*. Kisin [52] observed that while the study of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable $\mathbb{Z}_p$-lattices in semi-stable representations is very subtle in general, their $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable $\mathbb{Z}_p$-lattices are much more accessible.

1.3.2 Norm-field isomorphism

In the case $\mathfrak{o}_0 = \mathbb{Z}_p$ we give a useful description of the norm-field isomorphism $\mathcal{G}_{\mathscr{K}_\infty} \xrightarrow{\sim} \mathcal{G}_K$, which will be used later in §9.4.

Consider the following ring $\mathfrak{R} := \varprojlim_{x^p \leftarrow x} \mathfrak{o}_{\overline{\mathscr{K}}}/(p)$ of characteristic $p > 0$. By [78, Théorème 4.1.2], $\mathfrak{R}$ is a complete valuation ring for the valuation $v_{\mathfrak{R}}$ defined as follows: for any $\underline{x} := \{x_n\}_{n \geqslant 0} \in \mathfrak{R}$, define $v_{\mathfrak{R}}(\underline{x}) := \mathrm{ord}_\pi \left(\lim_{n\to\infty} (\tilde{x}_n)^{p^n}\right)$ where $\tilde{x}_n \in \mathfrak{o}_{\mathbb{C}_{\mathscr{K}}}$ is *any* lift of $x_n \in \mathfrak{o}_{\overline{\mathscr{K}}}/(p) \xrightarrow{\sim} \mathfrak{o}_{\mathbb{C}_{\mathscr{K}}}/(p)$. (One can easily check that the sequence $\{(\tilde{x}_n)^{p^n}\}_n$ always converges in $\mathfrak{o}_{\mathbb{C}_{\mathscr{K}}}$, and its limit is independent of the choice of lifts $\tilde{x}_n$.) We have a natural surjection $\mathfrak{R} \twoheadrightarrow \bar{k}$ given by sending $\{x_n\}_{n\geqslant 0} \in \mathfrak{R}$ to $x_0 \bmod \mathfrak{m}_{\overline{\mathscr{K}}}$. This surjection has a natural section $\bar{k} \to \mathfrak{R}$ which sends $\alpha \in \bar{k}$ to $\{[\alpha^{p^{-n}}] \bmod p\}_{n\geqslant 0}$, where $[\alpha^{p^{-n}}] \in \mathfrak{o}_{\mathbb{C}_{\mathscr{K}}}$ denotes the Teichmüller lift of $\alpha^{p^{-n}}$. We view $\mathfrak{R}$ a $\bar{k}$-algebra via this map. Now, consider an element $\underline{\pi} := \{\pi^{(n)} \bmod p\}_{n\geq 0} \in \mathfrak{R}$, and clearly we have $v_{\mathfrak{R}}(\underline{\pi}) = 1$. So we obtain a continuous k-algebra embedding $\mathfrak{o}_K \cong k[[u]] \to \mathfrak{R}$ via $u \mapsto \underline{\pi}$, and we view $\mathfrak{R}$ as a complete ring extension of $\mathfrak{o}_K$ by this map. Note that $\mathcal{G}_{\mathscr{K}}$ continuously acts on $\mathfrak{R}$ via its natural action on each factor $\mathfrak{o}_{\overline{\mathscr{K}}}/(p)$, and the embedding $\mathfrak{o}_K \hookrightarrow \mathfrak{R}$ is stable under the $\mathcal{G}_{\mathscr{K}_\infty}$-action on the target.

By [78, Corollaires 3.2.3, 4.3.4], there exist a natural isomorphism $\mathcal{G}_{\mathscr{K}_\infty} \xrightarrow{\sim} \mathcal{G}_K$ (called *norm-field isomorphism*), and a natural $\mathfrak{o}_K$-isomorphism $\mathfrak{o}_{\mathbb{C}_K} \xrightarrow{\sim} \mathfrak{R}$ which "respects" the natural actions of $\mathcal{G}_{\mathscr{K}_\infty}$ on the source and $\mathcal{G}_K$ on the target (where $\mathcal{G}_{\mathscr{K}_\infty}$ and $\mathcal{G}_K$ are identified via the norm-field isomorphism).

1.3.3

We start with introducing some $\mathfrak{o}_0$-algebras over which various semilinear algebra objects shall be defined.

W or $\mathfrak{o}_{\mathscr{K}_0}$	if $\mathfrak{o}_0 = \mathbb{Z}_p$, then $W := W(k)$ is the ring of Witt vectors of k with the p-adic topology; if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then $W := \mathfrak{o}_0 \widehat{\otimes}_{\mathbb{F}_q} k \cong k[[\pi_0]]$ with the π_0-adic topology.
$\mathscr{K}_0 := W[\frac{1}{\pi_0}]$	the fraction field of W.
$\mathfrak{S} := W[[u]]$	with the natural $\mathfrak{o}_0$-algebra structure from the one on W.
$\mathfrak{o}_{\mathcal{E}}$	the π_0-adic completion of $\mathfrak{S}[\frac{1}{u}]$ (i.e., formal Laurent series $\sum a_n u^n$ with $a_n \in W$, $a_n \to 0$).
$\mathcal{E} := \mathfrak{o}_{\mathcal{E}}[\frac{1}{\pi_0}]$	the fraction field of $\mathfrak{o}_{\mathcal{E}}$.

Note that $\mathfrak{o}_{\mathcal{E}}$ is a complete discrete valuation ring with π_0 generating the maximal ideal and the residue field $K \cong k(\!(u)\!)$. Thus, $\mathfrak{o}_{\mathcal{E}}$ is a Cohen ring for K if $\mathfrak{o}_0 = \mathbb{Z}_p$. If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then under the identification $\mathfrak{o}_K \cong k[[u]]$ we have $\mathfrak{S} \cong \mathfrak{o}_k[[\pi_0]] \cong \mathfrak{o}_0 \widehat{\otimes}_{\mathbb{F}_q} \mathfrak{o}_K$. Similarly, we have $\mathfrak{o}_{\mathcal{E}} \cong K[[\pi_0]] \cong \mathfrak{o}_0 \widehat{\otimes}_{\mathbb{F}_q} K$. In particular, we are given inclusions $\mathfrak{o}_K \hookrightarrow \mathfrak{S}$ and $K \hookrightarrow \mathfrak{o}_{\mathcal{E}}$ in the equi-characteristic case.

We define a Frobenius endomorphism σ for each of above rings as follows. If $\mathfrak{o}_0 = \mathbb{Z}_p$, then let $\sigma_W : W \to W$ be the usual Witt vector Frobenius endomorphism. If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then define σ_W by $\sigma(\pi_0) = \pi_0$ and $\sigma(\alpha) = \alpha^q$ for all $\alpha \in k$. We extend it by continuity to $\mathfrak{S}$ by setting $\sigma_{\mathfrak{S}}(u) = u^q$, where $q = p$ if $\mathfrak{o}_0 = \mathbb{Z}_p$. This rule defines a unique endomorphism for each of rings defined above, which is *finite and flat.* (In the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, we need the assumption that k has a finite p-basis in order to show that σ is finite.) We always view above rings as σ-rings by this construction of σ. This σ lifts the usual qth power map modulo π_0 and fixes the image of $\mathfrak{o}_0$. In other words, all the above σ-rings are defined over $(\mathfrak{o}_0, \mathrm{id})$.

Now, we fix an element $\mathcal{P}(u) \in \mathfrak{S}$ which will play an important role throughout the paper, as follows.

The case $\mathfrak{o}_0 = \mathbb{Z}_p$. We view $W[u]$ as a subring of $\mathfrak{S}$. Let $\mathcal{P}(u) \in W[u]$ be an Eisenstein polynomial, and let e be the degree of $\mathcal{P}(u)$. We normalize $\mathcal{P}(u)$ so that $\mathcal{P}(0) = p = \pi_0$.[13] Note that $\mathcal{P}(u) \equiv p \bmod u$ and $\mathcal{P}(u) \equiv cu^e$ where $c \in W^\times$.

The case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. Fix a nonzero element $u_0 \in \mathfrak{m}_K$ (or equivalently, fix a continuous injective $\mathbb{F}_q$-map $\mathfrak{o}_0 \to \mathfrak{o}_K$ and let u_0 be the image of π_0). Put $\mathcal{P}(u) := \pi_0 - u_0 \in \mathfrak{S}$ and let $e := \mathrm{ord}_u(u_0)$.

Remark 1.3.4 (The case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$). We give another interpretation on the element $\mathcal{P}(u) := \pi_0 - u_0$. Within this remark, we give $\mathfrak{S}$ the $\mathfrak{m}_{\mathfrak{S}}$-adic topology, and we give the natural valuation topology to $\mathfrak{o}_0$ and $\mathfrak{o}_K$. Then we have an isomorphism $\mathfrak{S} \cong \mathfrak{o}_0 \widehat{\otimes}_{\mathbb{F}_q} \mathfrak{o}_K$ as a topological $\mathbb{F}_q$-algebra. Now, fix a "structure morphism" $\mathrm{Spf}\, \mathfrak{o}_K \to \mathrm{Spf}\, \mathfrak{o}_0$ as in §1.3.1.1, and let $\gamma : \mathrm{Spf}\, \mathfrak{o}_K \hookrightarrow \mathrm{Spf}\, \mathfrak{S} \cong \mathrm{Spf}\, \mathfrak{o}_0 \times_{\mathrm{Spec}\, \mathbb{F}_q} \mathrm{Spf}\, \mathfrak{o}_K$ be the graph morphism. Then, γ is a closed immersion defined by the (closed) ideal $\mathcal{P}(u) \cdot \mathfrak{S}$.

Since $\mathfrak{S}/(\mathcal{P}(u)) \cong \mathfrak{o}_K$ is a ring extension of W which induces the trivial extension on the residue field, we see that $\mathcal{P}(u)$ is a $\mathfrak{S}^\times$-multiple of some Eisenstein polynomial in u over W with degree e. This explains the notations.

Remark 1.3.5 (The case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$). As observed by G. W. Anderson [3] and Hartl [39], it is good to distinguish two roles of a uniformizer of $\mathfrak{o}_0$ by using different notations: a uniformizer π_0 of the "coefficient ring" $\mathfrak{o}_0$ of a $\boldsymbol{\mathcal{G}}_K$-representation (and hence, a uniformizer of W), and the image u_0 of π_0 in the "base ring" $\mathfrak{o}_K$. To illustrate, let us consider an $\mathfrak{o}_0$-linear representation coming from a "π_0-divisible group" over $\mathfrak{o}_K$. Then π_0 is an "operator" acting on the π_0-divisible group and u_0 is the function on the base scheme. They both act on the Lie algebra of the π_0-divisible group, but *a priori* they have nothing to do with each other. The situation is quite

[13] The definition of Tate objects $\mathfrak{S}(h)$ (Definition 2.2.6) depends on the choice of a specific polynomial $\mathcal{P}(u)$, not just on the ideal $\mathcal{P}(u) \cdot \mathfrak{S}$. Our normalization $\mathcal{P}(0) = p$ will be used later in §4.3.6 and §5.2.14.

different if $\mathfrak{o}_0 = \mathbb{Z}_p$. For a p-divisible group G over a p-adic ring $\mathfrak{o}_{\mathscr{K}}$, the action of $p \in \mathbb{Z}_p$ on G induces the multiplication by $p \in \mathfrak{o}_{\mathscr{K}}$ on the Lie algebra of G.

CHAPTER II

Frobenius modules and Hodge-Pink theory

2.1 Rigid-analytic objects

2.1.1 Rigid-analytic rings

We now introduce more notations from rigid-analytic geometry. We review some background in rigid-analytic geometry in Appendix §6.1, for the sake of completeness.

We normalize the absolute value $|\cdot|$ on $\mathscr{K}_0 = \operatorname{Frac} W$ and on any algebraic field extension of it so that $|\pi_0| = q^{-1}$. (Recall that q is the cardinality of the residue field of $\mathfrak{o}_0$.) Let $\mathbb{C}_{\mathscr{K}_0}$ be the completion of a fixed separable closure $\overline{\mathscr{K}}_0$. Let $I \subset [0,1)$ be a subinterval, and we always assume that all radii of disks and endpoints of I lie in $q^{\mathbb{Q}_{<0}}$, even if not stated.

- $\mathbf{\Delta}$ — the rigid-analytic open unit disk over $\mathscr{K}_0$ with u as a "coordinate." Concretely, its points x satisfy $|u(x)| < 1$.
- $\mathbf{\Delta}_I$ — the subdomain of $\mathbf{\Delta}$ whose points satisfy $|u(x)| \in I$, where $I \subset [0,1)$ is a subinterval (allowing $I = \{r\}$) whose endpoints lie in $q^{\mathbb{Q}_{<0}}$.
- $\mathcal{O}_{\mathbf{\Delta}}$ — the ring of rigid-analytic functions on $\mathbf{\Delta}$ (or the structure sheaf of $\mathbf{\Delta}$).
- $\mathcal{O}_{\mathbf{\Delta}_I}$ — the ring of rigid-analytic functions on $\mathbf{\Delta}_I$ (or the structure sheaf of $\mathbf{\Delta}_I$).

Concretely, an element of $\mathcal{O}_{\mathbf{\Delta}_I}$ is $f(u) = \sum_{n\in\mathbb{Z}} a_n u^n$ with $a_n \in \mathscr{K}_0$ such that $f(x)$ converges for any $x \in \mathbb{C}_{\mathscr{K}_0}$ with $|x| \in I$. We occasionally use the notation $\mathcal{O}_{\mathbf{\Delta}_I}$ to denote the structure sheaf on $\mathbf{\Delta}_I$ – for more detail, see §6.1. We point out that the construction of $\mathcal{O}_{\mathbf{\Delta}_I}$ relies on the fact that K is discretely valued, since we use a uniformizer u of K. In the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, one can take a different approach which allows K to be non-discretely valued (e.g. algebraically closed complete non-archimedean field); see §2.1.5 for more details.

Fix $r \in q^{\mathbb{Q}_{<0}}$, and put $\gamma := -\log_q r$. Let $f(u) = \sum_{i\in\mathbb{Z}} a_i u^i$ be a rigid-analytic function which converges in $\mathbf{\Delta}_{[r,r]}$; i.e., $f \in \mathcal{O}_{\mathbf{\Delta}_{[r,r]}}$. Note that $\mathcal{O}_{\mathbf{\Delta}_{[r,r]}}$ contains $\mathcal{O}_{\mathbf{\Delta}_I}$ if $r \in I$.

$\|f\|_r$ The sup-norm on $\mathbf{\Delta}_{[r,r]}$.Concretely, $\|f\|_r := \max_i\{|a_i|\, r^i\}$.

$w_\gamma(f)$ The additive valuation: $w_\gamma(f) := -\log_q \|f\|_r = \min_i\{v(a_i) + \gamma\cdot i\}$.

We recall the following well-known properties of $\mathcal{O}_{\mathbf{\Delta}_I}$, which will be used later.

1. The ring $\mathcal{O}_{\mathbf{\Delta}_{[0,r]}}$ is complete with respect to $\|\cdot\|_r$, hence is a Banach $\mathscr{K}_0$-algebra. The ring $\mathcal{O}_{\mathbf{\Delta}_{[r,r']}}$ is complete with respect to a submultiplicative norm $\max\{\|\cdot\|_r, \|\cdot\|_{r'}\}$, hence is a Banach $\mathscr{K}_0$-algebra. If I is not closed then $\mathcal{O}_{\mathbf{\Delta}_I}$ is not a Banach algebra, but it is a Fréchet space for the (countable family of) norms $\|\cdot\|_r$ where $r \in I \cap q^{\mathbb{Q}_{<0}}$. Concretely, this means that any sequence $\{f_n\}$ in $\mathcal{O}_{\mathbf{\Delta}_I}$ converges if and only if $\{f_n\}$ is Cauchy with respect to the norm $\|\cdot\|_r$ for each $r \in I \cap q^{\mathbb{Q}_{<0}}$.

2. The ring $\mathcal{O}_{\mathbf{\Delta}_I}$ is a principal ideal domain if (and only if) I is a closed subinterval. In general, $\mathcal{O}_{\mathbf{\Delta}_I}$ does not even have to be noetherian. But since the base field $\mathscr{K}_0$ is discretely valued, the ring $\mathcal{O}_{\mathbf{\Delta}_I}$ is a Bézout domain for any I; i.e., any finitely generated ideal of $\mathcal{O}_{\mathbf{\Delta}}$ is principal. (This follows from the work of Lazard [57].) Finitely presented modules over a Bézout domain behave like finitely generated modules over a principal ideal domain. See §6.2.7 for an overview of where these properties come from, and [46, §2.4] or [48, §2.9] for more detail about the

Bézout properties.

For $f(u) := \sum_{n\in\mathbb{Z}} a_n u^n \in \mathcal{O}_{\mathbf{\Delta}_I}$ where $a_n \in \mathscr{K}_0$, one can check that $g(u) := \sum_{n\in\mathbb{Z}} \sigma_{\mathscr{K}_0}(a_n)u^{qn} \in \mathcal{O}_{\mathbf{\Delta}_{I^{1/q}}}$, where $I^{1/q} \subset [0,1)$ is the subinterval whose endpoints are qth root of the endpoints of I. So we obtain a $\sigma_{\mathscr{K}_0}$-semilinear ring morphism $\sigma : \mathcal{O}_{\mathbf{\Delta}_I} \to \mathcal{O}_{\mathbf{\Delta}_{I^{1/q}}}$ by setting $\sigma(f(u)) := g(u)$. Note that σ is *flat* because $\mathcal{O}_{\mathbf{\Delta}_I}$ is a Bézout domain and σ makes $\mathcal{O}_{\mathbf{\Delta}_{I^{1/q}}}$ into a torsion-free $\mathcal{O}_{\mathbf{\Delta}_I}$-module. Furthermore, one can check that σ is a finite map, granting that the qth power map on k is a finite map (which we assumed at the very beginning of §1.3). Since we have $I = I^{1/q}$ when $I = [0,1)$ or $I = (0,1)$ (and not otherwise), σ is an endomorphism of $\mathcal{O}_{\mathbf{\Delta}}$ and $\mathcal{O}_{\mathbf{\Delta}_{(0,1)}}$.

Since $\sigma : \mathcal{O}_{\mathbf{\Delta}_I} \to \mathcal{O}_{\mathbf{\Delta}_{I^{1/q}}}$ is *not* $\mathscr{K}_0$-linear but $\sigma_{\mathscr{K}_0}$-semilinear, it does not give rise to a morphism $\mathbf{\Delta}_{I^{1/q}} \to \mathbf{\Delta}_I$ in the sense of classical rigid-analytic geometry[1]. Instead, we should linearize σ to obtain $\mathbf{\Delta}_{I^{1/q}} \to \sigma^*_{\mathscr{K}_0}\mathbf{\Delta}_I$ the map induced on the rigid-analytic spaces, where $\sigma^*_{\mathscr{K}_0}\mathbf{\Delta}_I$ is the scalar extension of $\mathbf{\Delta}_I$ under $\sigma_{\mathscr{K}_0}$ in the sense of [8, §9.3.6]. The geometric map $\mathbf{\Delta} \to \sigma^*_{\mathscr{K}_0}\mathbf{\Delta}$ is not an endomorphism on $\mathbf{\Delta}$, whereas σ is an endomorphism of $\mathcal{O}_{\mathbf{\Delta}}$ (over $\sigma_{\mathscr{K}_0}$). This is not a serious problem but causes some annoying expository issues. We will avoid using rigid-analytic geometry when this issue comes up. Alternatively, one may handle this issue by identifying $\sigma^*_{\mathscr{K}_0}\mathbf{\Delta}$ with $\mathbf{\Delta}$; in other words, by identifying an $\mathcal{O}_{\sigma^*_{\mathscr{K}_0}\mathbf{\Delta}}$-module with a sheaf on $\mathbf{\Delta}$ where $\mathcal{O}_{\mathbf{\Delta}}$-multiplication has been twisted by $\sigma^{-1}_{\mathscr{K}_0}$ (for which we need to assume that k is perfect when $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$) – under this identification, $\sigma_{\mathbf{\Delta}}$ becomes an endomorphism of $\mathbf{\Delta}$ and induces the continuous $\mathscr{K}_0$-algebra map defined by $u \mapsto u^q$ on the global sections. We do not take this point of view.

Definition 2.1.2. The *Robba ring* $\mathcal{R}$ is the rising union of the rings of rigid-analytic

[1]This issue is resolved if we are willing to use Berkovich spaces, which has better functorial properties.

functions on some open annulus with outer radius 1. The *bounded Robba ring* $\mathcal{R}^{bd}$ is the rising union of the rings of rigid-analytic functions bounded near the outer radius. In other words,

$$\begin{aligned} \mathcal{R} &:= \varinjlim_{r\to 1^-} \Gamma(\mathbf{\Delta}_{[r,1)}, \mathcal{O}_{\mathbf{\Delta}}) \\ \mathcal{R}^{bd} &:= \varinjlim_{r\to 1^-} \Gamma(\mathbf{\Delta}_{[r,1)}, \mathcal{O}_{\mathbf{\Delta}})^{bd}, \end{aligned}$$

where $\Gamma(\mathbf{\Delta}_{[r,1)}, \mathcal{O}_{\mathbf{\Delta}})^{bd}$ denotes bounded rigid-analytic functions on $\mathbf{\Delta}_{[r,1)}$.

The Robba ring $\mathcal{R}$ is not noetherian, but is a Bézout domain (being a rising union of Bézout domains). The subring $\mathcal{R}^{bd}$ is a field with the following discrete valuation:

$$v_{\mathcal{R}^{bd}}(f) = \lim_{\gamma\to\infty} w_\gamma(f), \quad \text{for } f \in \mathcal{R}^{bd}, \tag{2.1.2.1}$$

where $w_\gamma(f) := -\log_q \|f\|_r$ is the additive valuation.

Let $\mathfrak{o}_{\mathcal{R}^{bd}}$ be the valuation ring. One can check that $\pi_0 \in \mathfrak{o}_{\mathcal{R}^{bd}}$ is a uniformizer, $k(\!(u)\!)$ is the residue field, and $\mathfrak{o}_{\widehat{\mathcal{R}^{bd}}} \cong \mathfrak{o}_{\mathcal{E}}$ where the completion on the left-hand side is with respect to the π_0-adic topology. We also remark that $\sigma_{\mathbf{\Delta}} : \mathcal{O}_{\mathbf{\Delta}_I} \to \mathcal{O}_{\mathbf{\Delta}_{I^{1/q}}}$ induces "Frobenius" endomorphisms of $\mathcal{R}$, $\mathcal{R}^{bd}$, and $\mathfrak{o}_{\mathcal{R}^{bd}}$.

It is immediate that:

$$\mathfrak{S}[1/\pi_0] = \mathcal{O}_{\mathbf{\Delta}} \cap \mathcal{R}^{bd}. \tag{2.1.2.2}$$

In particular, $\mathfrak{S} = \mathcal{O}_{\mathbf{\Delta}} \cap \mathfrak{o}_{\mathcal{R}^{bd}}$.

2.1.3

Let $\mathcal{P}(u) \in \mathfrak{S}(= W[[u]])$, as defined in §1.3.3. Recall that $\mathcal{P}(u)$ is a $\mathfrak{S}^\times$-multiple of an Eisenstein polynomial in $W[u]$ (and in fact, is an Eisenstein polynomial if $\mathfrak{o}_0 = \mathbb{Z}_p$). Therefore $\sigma^n(\mathcal{P}(u))$ is also a $\mathfrak{S}^\times$-multiple of an Eisenstein polynomial in $W[u]$, and in particular generates a maximal ideal in $\mathfrak{S}[\frac{1}{\pi_0}]$.

Denote by $x_n \in \mathbf{\Delta}$ the unique point where $\sigma^n(\mathcal{P}(u))$ vanishes. Note that if the residue field $\mathscr{K}_0(x_0)$ at x_0 is separable over $\mathscr{K}_0$, then the residue field $\mathscr{K}_0(x_n)$ is separable for all $n \geq 0$. Now we define a convergent infinite product

$$\lambda := \prod_{n\geq 0} \sigma^n\left(\frac{\mathcal{P}(u)}{\mathcal{P}(0)}\right), \tag{2.1.3.1}$$

which is a rigid-analytic function on $\mathbf{\Delta}$ and has simple zeroes exactly at $\{x_n\}_{n\geq 0}$ and no other zeroes. From the construction, we have

$$\sigma(\lambda) = \frac{\mathcal{P}(0)}{\mathcal{P}(u)} \cdot \lambda \tag{2.1.3.2}$$

In particular, $\mathcal{O}_{\mathbf{\Delta}}[1/\lambda]$ is stable under σ inside $\mathrm{Frac}(\mathcal{O}_{\mathbf{\Delta}})$.

Let $\mathcal{O}_{\mathbf{\Delta},x_n}$ be the ring of germs of rigid-analytic functions at $x_n \in \mathbf{\Delta}$, which is known to be a discrete valuation ring [8, §7.3.2]. Since $\widehat{\mathcal{O}_{\mathbf{\Delta},x_n}}$ is faithfully flat over $\mathcal{O}_{\mathbf{\Delta},x_n}$, we may study analytic local properties of a coherent sheaf at $x_n \in \mathbf{\Delta}$ via completed stalks at x_n. In fact, $\widehat{\mathcal{O}_{\mathbf{\Delta},x_n}}$ can be thought of as the $\sigma^n(\mathcal{P}(u))$-adic completion of $\mathcal{O}_{\mathbf{\Delta}}$, or equivalently, the $\sigma^n(\mathcal{P}(u))$-adic completion of $\mathfrak{S}[\frac{1}{\pi_0}]$; for the proof, we take the global sections of the short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{O}_{\mathbf{\Delta}} \xrightarrow{\sigma^n\mathcal{P}(u)^i} \mathcal{O}_{\mathbf{\Delta}} \longrightarrow \mathcal{O}_{\mathbf{\Delta}}/(\sigma^n\mathcal{P}(u))^i \longrightarrow 0$$

and use that the global sections functor $\Gamma(\mathbf{\Delta},\cdot)$ is exact on coherent sheaves. As a consequence of this argument, the residue field $\mathscr{K}_0(x_n)$ at $x_n \in \mathbf{\Delta}$ is isomorphic to $\mathcal{O}_{\mathbf{\Delta}}/(\sigma^n\mathcal{P}(u)) \xleftarrow{\sim} \mathfrak{S}[\frac{1}{\pi_0}]/(\sigma^n\mathcal{P}(u))$. We often write $\mathscr{K} := \mathscr{K}_0(x_0) \cong \mathfrak{S}[\frac{1}{\pi_0}]/\mathcal{P}(u)$.

We have a canonical $\mathscr{K}_0$-algebra isomorphism $\widehat{\mathcal{O}_{\mathbf{\Delta},x_n}} \cong \mathscr{K}_0(x_n)[[\sigma^n(\mathcal{P}(u))]]$ lifting the residue field identification, when $\mathscr{K}/\mathscr{K}_0$ is separable. But if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ then such an isomorphism can fail to exist, so in general we avoid using this isomorphism.

For $n, m \geq 0$, the Frobenius endomorphism $\sigma^n : \mathcal{O}_{\mathbf{\Delta}} \to \mathcal{O}_{\mathbf{\Delta}}$ induces, on completed local rings, local injections

$$\sigma^n : \widehat{\mathcal{O}_{\mathbf{\Delta},x_m}} \hookrightarrow \widehat{\mathcal{O}_{\mathbf{\Delta},x_{n+m}}}, \tag{2.1.3.3}$$

which are σ^n-semilinear inclusions of $\mathcal{O}_{\mathbf{\Delta}}$-algebras carrying the uniformizer $\sigma^m(\mathcal{P}(u))$ to the uniformizer $\sigma^{n+m}(\mathcal{P}(u))$. By linearizing it over $\mathcal{O}_{\mathbf{\Delta}}$, we obtain the following isomorphism:

$$\gamma_{n,m} : \mathcal{O}_{\mathbf{\Delta}} \otimes_{\sigma^n, \mathcal{O}_{\mathbf{\Delta}}} \mathcal{O}_{\widehat{\mathbf{\Delta}}, x_m} \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathbf{\Delta}}, x_{n+m}} . \tag{2.1.3.4}$$

That this natural map is isomorphism uses that $\sigma : \mathcal{O}_{\mathbf{\Delta}} \to \mathcal{O}_{\mathbf{\Delta}}$ is finite and flat. Recall that in the case when $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, the finiteness of σ follows from the assumption that k has a finite p-basis.

We also obtain $\sigma^n_{\mathscr{K}_0}$*-semilinear* inclusions $\sigma^n : \mathscr{K}_0(x_m) \hookrightarrow \mathscr{K}_0(x_{n+m})$ by reducing the map (2.1.3.3) modulo maximal ideals. When $\mathscr{K}/\mathscr{K}_0$ is separable then via the canonical isomorphism $\mathcal{O}_{\widehat{\mathbf{\Delta}}, x_m} \cong \mathscr{K}_0(x_m)[[\sigma^m \mathcal{P}(u)]]$ for each m we can view the map (2.1.3.3) as $\sigma^n : \mathscr{K}_0(x_m)[[\sigma^m(\mathcal{P}(u))]] \to \mathscr{K}_0(x_{n+m})[[\sigma^{n+m}\mathcal{P}(u)]]$ which restricts to the natural map $\sigma^n : \mathscr{K}_0(x_m) \hookrightarrow \mathscr{K}_0(x_{n+m})$ on coefficients and $\sigma^m(\mathcal{P}(u)) \mapsto \sigma^{n+m}(\mathcal{P}(u))$. We do not us this later, since it is not available when $\mathscr{K}/\mathscr{K}_0$ is not separable.

Remark 2.1.4 (The case $\mathfrak{o}_0 = \mathbb{Z}_p$). Using the notations from §1.3.1.2, if $n > 0$ then $\mathscr{K}_0(x_n)$ and $\mathscr{K}_0(\pi^{(n)})$ do not *not* have to be isomorphic extensions of $\mathscr{K}_0$. The former is generated over $\mathscr{K}_0$ bya root of the irreducible polynomial $\sigma^n(\mathcal{P}(u))$, while the latter is generated over $\mathscr{K}_0$ by a root of $\mathcal{P}(u^{p^n})$. We have $\sigma^n_{\mathscr{K}_0}\mathcal{P}(u^{p^n}) = \sigma^n \mathcal{P}(u)$, where $\sigma_{\mathscr{K}_0}$ acts on the coefficients.

2.1.5 "Conversion" from Hartl's Dictionary

We momentarily assume that $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. Then we may consider the rigid-analytic open unit disk over K and use π_0 as its "coordinate."[2] This open unit disk will be denoted, in this paper, by $\mathbf{\Delta}_K$, to emphasize that the disk is defined over

[2]This has no "geometric" analogue for $\mathfrak{o}_0 = \mathbb{Z}_p$, but $\mathcal{O}^{\text{int}}_{\mathbf{\Delta}_{(0,1)}}$ can be thought of as an analogue of $\mathcal{O}_{\mathbf{\Delta}_K}$.

K. For a subinterval $J \subset [0,1)$ with endpoints in $q^{\mathbb{Q}_{\leqslant 0}} \cup \{0\}$, we let $\mathbf{\Delta}_{K,J}$ denote the subdomain of $\mathbf{\Delta}_K$ whose points x satisfy $\pi_0(x) \in J$. For $f(\pi_0) := \sum_{n\in\mathbb{Z}} \alpha_n \pi_0^n \in \mathcal{O}_{\mathbf{\Delta}_{K,J}}$ where $\alpha_n \in K$, one can check that $g(\pi_0) := \sum_{n\in\mathbb{Z}} (\alpha_n)^q \pi_0^n \in \mathcal{O}_{\mathbf{\Delta}_{K,J^{1/q}}}$, where $J^{1/q} \subset [0,1)$ is the subinterval whose endpoints are qth root of the endpoints of J. So we obtain a σ_K-semilinear ring morphism $\sigma : \mathcal{O}_{\mathbf{\Delta}_{K,J}} \to \mathcal{O}_{\mathbf{\Delta}_{K,J^{1/q}}}$ by setting $\sigma(f(\pi_0)) := g(\pi_0)$. Since we assumed that K has a finite p-basis, the qth power map $\sigma_K : K \to K$ is finite and flat so σ is finite and flat. In [39, 41], Hartl works with $\mathbf{\Delta}_K$ instead of $\mathbf{\Delta}$.

Put $I := [q^{-s}, q^{-r}]$ and $J := [q^{-1/r}, q^{-1/s}]$ for some positive rational numbers r, s. The K-algebra $\mathcal{O}_{\mathbf{\Delta}_{K,J}}$ naturally sits inside of the K-vector space $K[[\pi_0, \frac{1}{\pi_0}]]$, which naturally sits in the k-vector space $k[[\pi_0, \frac{1}{\pi_0}, u, \frac{1}{u}]]$ of 2-variable infinite-tailed formal Laurent series over k. On the other hand, the $\mathscr{K}_0$-algebra $\mathcal{O}_{\mathbf{\Delta}_I}$ sits inside of the $\mathscr{K}_0$-vector space $\mathscr{K}_0[[u, \frac{1}{u}]]$, which naturally sits in the k-vector space $k[[\pi_0, \frac{1}{\pi_0}, u, \frac{1}{u}]]$. One can see that $\mathcal{O}_{\mathbf{\Delta}_I}$ and $\mathcal{O}_{\mathbf{\Delta}_{K,J}}$ define the same subspace of $k[[\pi_0, \frac{1}{\pi_0}, u, \frac{1}{u}]]$, and has the same multiplication law. (Indeed, one can characterize the $\mathscr{K}_0$-subspace $\mathcal{O}_{\mathbf{\Delta}_I} \subset \mathscr{K}_0[[u, \frac{1}{u}]]$ via some "growth condition" of the coefficients as worked out in §6.1.2, and one has a similar description of the K-subspace $\mathcal{O}_{\mathbf{\Delta}_{K,J}} \subset K[[\pi_0, \frac{1}{\pi_0}]]$. Then, one directly checks that they define the same k-subspace of $k[[\pi_0, \frac{1}{\pi_0}, u, \frac{1}{u}]]$.) From this, one can also see that the functions bounded near the boundary of $\mathbf{\Delta}$ correspond functions which have an isolated pole at the origin of $\mathbf{\Delta}_K$, and vice versa. In particular, one can recover $\mathcal{O}_{\mathbf{\Delta}}$, the Robba ring, and the bounded Robba ring using $\mathbf{\Delta}_K$.

$$\begin{aligned}
\mathcal{O}_{\mathbf{\Delta}} &= \{\sum_{i\in\mathbb{Z}} a_i\pi_0^i \in \Gamma(\dot{\mathbf{\Delta}}_K, \mathcal{O}_{\mathbf{\Delta}_K}) |\, a_i \in \mathfrak{o}_K,\ \forall i\} \\
\mathcal{R} &= \varinjlim_{r\to 0^+} \Gamma(\mathbf{\Delta}_{K,(0,r]}, \mathcal{O}_{\mathbf{\Delta}_K}) \\
\mathcal{R}^{bd} &= \varinjlim_{r\to 0^+} \Gamma(\mathbf{\Delta}_{K,[0,r]}, \mathcal{O}_{\mathbf{\Delta}_K})[\frac{1}{\pi_0}],
\end{aligned}$$

where $\dot{\mathbf{\Delta}}_K$ is the punctured open unit disk over K with coordinate π_0.

The advantage of using Hartl's $\mathbf{\Delta}_K$ over using $\mathbf{\Delta}$ is that K does not have to be discretely valued.[3] (The right sides of above equations make sense even if K is not discretely valued.) One can even replace K by any affinoid K-algebra and develop the theory for "families", which makes the argument in [39, §3] work. It is very useful to allow K to be an algebraically closed ground field. For example, using $\mathbf{\Delta}_{\mathbb{C}_K}$ we can give natural definitions for the following analytic rings, which are also defined in §6.1.10:

$$\mathcal{R}^{\mathrm{alg}} = \varinjlim_{r\to 0^+} \Gamma(\mathbf{\Delta}_{\mathbb{C}_K,(0,r)}, \mathcal{O}_{\mathbf{\Delta}_{\mathbb{C}_K}}) \tag{2.1.5.1}$$

$$\mathcal{R}^{\mathrm{alg},bd} = \varinjlim_{r\to 0^+} \Gamma(\mathbf{\Delta}_{\mathbb{C}_K,[0,r)}, \mathcal{O}_{\mathbf{\Delta}_{\mathbb{C}_K}})[\frac{1}{\pi_0}] \tag{2.1.5.2}$$

These rings play a crucial role in "Dieudonné-Manin type" classification (Theorem 4.1.2).

2.2 φ-modules of finite $\mathcal{P}$-height

Let (R, σ) be a σ-ring, and we always assume σ-flatness unless stated otherwise. For any R-module M, we write $\sigma^*M := R\otimes_{\sigma,R} M$. A finitely presented R-module M equipped with an R-linear map $\varphi : \sigma^*M \to M$ is called a *(φ, R)-module*, or simply a *φ-module* if there is no risk of confusions. A morphism $(M, \varphi_M) \to (N, \varphi_N)$ of

[3] In fact, Hartl proves Theorem 4.3.4 of this paper allowing more general K than discretely valued ones, with the statement modified if K is not discretely valued.

φ-modules is an R-linear map $f: M \to N$ such that $f \circ \varphi_M = \varphi_N \circ \sigma^* f$. For two φ-modules (M, φ_M) and (N, φ_N), the tensor product $M \otimes_R N$ is again a φ-module via $\varphi_M \otimes \varphi_N$.

From now on, assume further that (R, σ) is defined over $(\mathfrak{S}, \sigma)$, so $\mathcal{P}(u)$ is viewed as an element of R. We further assume that π_0 and $\mathcal{P}(u)$ are *not* zero-divisors in R. The main examples of such R are $\mathfrak{S}$, $\mathfrak{o}_{\mathcal{E}}$, $\mathcal{E}$, $\mathcal{O}_{\boldsymbol{\Delta}}$, $\mathcal{R}$, and $\mathcal{R}^{bd}$.

Definition 2.2.1 (φ-module of finite $\mathcal{P}$-height)**.** We call a (φ, R)-module (M, φ) is *of finite $\mathcal{P}$-height* if M is a locally free[4] R-module and $\operatorname{coker} \varphi$ is killed by some power of $\mathcal{P}(u)$. We say that (M, φ) is *of $\mathcal{P}$-height* $\leqslant h$ if $\mathcal{P}(u)^h \cdot \operatorname{coker} \varphi = 0$. We let $\underline{\mathrm{Mod}}_R(\varphi)$ denote the category of φ-modules over R of finite $\mathcal{P}$-height, and let $\underline{\mathrm{Mod}}_R(\varphi)^{\leqslant h}$ denote the full subcategory of $\underline{\mathrm{Mod}}_R(\varphi)$ whose objects are of $\mathcal{P}$-height $\leqslant h$.

If $\mathcal{P}(u) \in R^\times$ (for example, if $R = \mathfrak{o}_{\mathcal{E}}, \mathcal{R}, \mathcal{R}^{bd}$), then a φ-module (M, φ_M) is of finite $\mathcal{P}$-height if and only if φ_M is an isomorphism. Hence we make the following definition.

Definition 2.2.2. An φ-module (M, φ) over R is *étale* if φ is bijective. The category of étale φ-modules over R is denoted by $\underline{\mathrm{Mod}}_R^{\text{ét}}(\varphi)$ taking morphisms to be those of φ-modules. We denote by $\underline{\mathrm{Mod}}_R^{\text{ét,free}}(\varphi)$ the full subcategory of étale φ-modules whose underlying R-modules are free. We denote by $\underline{\mathrm{Mod}}_R^{\text{ét,tor}}(\varphi)$ the full subcategory of étale φ-modules whose underlying R-modules are annihilated by some power of π_0.

Since torsion étale φ-modules play important roles in proofs (even though statements may only concerns finite free étale φ-modules), we do not require R-freeness in the definition of étale φ-modules.

[4]A locally free module is always assumed to be of constant rank.

2.2.3 Injectivity of φ

The following lemma can be useful to prove the injectivity of φ in many cases.

Lemma 2.2.3.1. *Let (R, σ) be a σ-ring over $(\mathfrak{S}, \sigma_{\mathfrak{S}})$ and let (M, φ_M) be a φ-module over R. Suppose that there exists an R-algebra R' (not necessarily a σ-ring) such that the natural maps $M \to R' \otimes_R M$ and $\mathrm{id}_{R'} \otimes \varphi_M : R' \otimes_R (\sigma^* M) \to R' \otimes_R M$ are injective. Then, the map φ_M is injective.*

Proof. It follows from chasing the diagram below.

$$\begin{array}{ccc} \sigma^* M & \hookrightarrow & R' \otimes_R (\sigma^* M) \\ {\scriptstyle \varphi_M} \downarrow & & \downarrow {\scriptstyle \mathrm{id}_{R'} \otimes \varphi_M} \\ M & \hookrightarrow & R' \otimes_R M \end{array}$$

□

Corollary 2.2.3.2. *Assume that $\mathcal{P}(u) \in R$ is not a zero-divisor (as assumed at the beginning of the chapter). For any $M \in \underline{\mathrm{Mod}}_R(\varphi)$,[5] the map $\varphi_{\mathfrak{M}} : \sigma^* \mathfrak{M} \to \mathfrak{M}$ is injective.*

Proof. Since $\mathcal{P}(u) \in R$ is not a zero-divisor, the free R-module M has no non-trivial $\mathcal{P}(u)$-torsion. So we obtain the corollary by applying the above lemma to $R' = R[\frac{1}{\mathcal{P}(u)}]$. □

2.2.4 Formal Properties

Here we record some immediate properties, which mostly follow from σ-flatness of R.

(1) For a short exact sequence $0 \to M' \to M \to M'' \to 0$ of φ-modules, if two of them are étale (respectively, of finite $\mathcal{P}$-height and all three terms are free), then

[5] We write $M \in \underline{\mathrm{Mod}}^{\text{ét}}_R(\varphi)$ to mean $M \in \mathrm{Ob}(\underline{\mathrm{Mod}}^{\text{ét}}_R(\varphi))$. We keep this convention throughout the paper.

so is the third. If M is of $\mathcal{P}$-height $\leqslant h$, then M' and M'' are also of $\mathcal{P}$-height $\leqslant h$. To verify the claims on $\mathcal{P}$-height, we use the injectivity of φ (Corollary 2.2.3.2).

(2) (*scalar extension*)[6] Let $(R,\sigma) \xrightarrow{f} (R',\sigma')$ be a morphism of σ-flat rings where σ' lies over σ; i.e., $\sigma' \circ f = f \circ \sigma$. Let M be a (φ, R)-module. Then, the "scalar extension" $R' \otimes_R M$ is naturally a (φ, R')-module via $R' \otimes \varphi$: this makes sense as a Frobenius structure, using

$$R' \otimes_{\sigma', R'} (R' \otimes_{f,R} M) \cong R' \otimes_{f,R} (R \otimes_{\sigma,R} M) \xrightarrow{\mathrm{id} \otimes \varphi} R' \otimes_{f,R} M$$

Moreover, if M is of finite $\mathcal{P}$-height (respectively, étale), so is $R' \otimes_{f,R} M$.

(3) The condition of being of finite $\mathcal{P}$-height (respectively, étale) is stable under $\otimes$-product. The rank-1 free module R together with the linearization of $\varphi := \mathrm{id}_R \otimes \sigma$ defines the "neutral object" among φ-modules in the sense that it is the "left and right identity" under $\otimes$-product. (Under the identification $R \otimes_{R,\sigma} R \cong R$ by $\sum a_i \otimes b_i \mapsto \sum a_i \sigma(b_i)$, the map $\varphi = \mathrm{id}_R \otimes \sigma$ induces $\mathrm{id}_R : R \to R$.) We often let R denote this neutral object.

Etale φ-modules enjoy further nice properties.

(4) Internal Hom is defined in $\underline{\mathrm{Mod}}_R^{\text{ét}}(\varphi)$: since $\mathrm{Hom}_R(\sigma^* M, \sigma^* M') \cong \sigma^* \mathrm{Hom}_R(M, M')$ for finitely presented R-modules M and M', we define $(\mathrm{Hom}_R(M, M'), f \mapsto \varphi_{M'} \circ f \circ \varphi_M^{-1}) \in \underline{\mathrm{Mod}}_R^{\text{ét}}(\varphi)$, where $f \in \sigma^* \mathrm{Hom}_R(M, M') \cong \mathrm{Hom}_R(\sigma^* M, \sigma^* M')$.

(5) On finite free objects $M \in \underline{\mathrm{Mod}}_R^{\text{ét,free}}(\varphi)$, one can define the duality functor $M^* := \mathrm{Hom}_R(M, R)$ by taking the internal hom into the "neutral object" $(R, \mathrm{id}_R \otimes \sigma)$.

[6] We do not have to require σ-flatness for these claims, except for the étaleness assertion.

(6) Duality for torsion étale φ-modules is not a good concept in general. But if R is a discrete valuation ring, we may show that $M^* := \mathrm{Hom}_R(M, \mathrm{Frac}(R)/R)$ with a natural φ_{M^*} is a good duality functor. More specifically, we essentially interpret this as in (4) except $\mathrm{Frac}(R)/R$ is not an object of $\underline{\mathrm{Mod}}_R^{\text{ét,tor}}(\varphi)$. Nonetheless, all R-linear morphism from M into $\mathrm{Frac}(R)/R$ factors through some finite submodule $\mathfrak{m}_R^{-N}R/R \subset \mathrm{Frac}(R)/R$ for $N \gg 0$ since M is of finite length, so there is no problem.

Remark 2.2.5. To give a natural φ-module structure on $\mathrm{Hom}_R(M, M')$ in (4), we need to *invert* φ_M. If we try to carry out the same construction for non-étale φ-module M of finite $\mathcal{P}$-height, then the φ-structure on the internal Hom $\mathrm{Hom}_R(M, M')$ will pick up a "pole" at the ideal $\mathcal{P}(u)R$. (At the beginning of the chapter, we assume that $\mathcal{P}(u)$ is not a zero divisor in R.)

Next, we define *Tate objects* and *Tate twist*.

Definition 2.2.6. For $n \in \mathbb{Z}_{\geq 0}$, the *Tate object* $R(n)$ is the φ-module

$$R(n) := \big(R,\, \mathcal{P}(u)^n \cdot (\mathrm{id}_R \otimes \sigma)\big).$$

For $(M, \varphi_M) \in \underline{\mathrm{Mod}}_R(\varphi)$, the *Tate twist* $M(n)$ is the tensor product $M(n) := M \otimes_R R(n) \cong (M,\, \mathcal{P}(u)^n \cdot \varphi_M)$.

It is clear that $R(n) \cong R(1)^{\otimes n}$. For $n > 0$, we write $M(-n) := (M,\, \mathcal{P}(u)^{-n} \cdot \varphi_M)$ if $\mathcal{P}(u)^{-n} \cdot \varphi_M$ is well defined, which is always the case if $\mathcal{P}(u) \in R^\times$. It follows that $\big(M(n)\big)(n') \cong M(n+n')$ whenever both sides are well-defined.

Note that the definition of $R(n)$ depends upon the specific element $\mathcal{P}(u) \in R$, not just upon the ideal $\mathcal{P}(u) \cdot R$. In the case $\mathfrak{o}_0 = \mathbb{Z}_p$, our normalization $\mathcal{P}(0) = p$ will play a role in §4.3.6 and §5.2.14.

2.2.7 Isogeny

Recall that $\underline{\mathrm{Mod}}_R(\varphi)$ denotes the category of (φ, R)-modules of $\mathcal{P}$-height $\leqslant h$ (Definition 2.2.1). A morphism $f : M \to M'$ in $\underline{\mathrm{Mod}}_R(\varphi)$ is called an *isogeny* if f is injective and $\operatorname{coker} f$ is killed by some power of π_0, say by π_0^N. Then, there exists a unique $g : M' \to M$ such that $f \circ g = \pi_0^N$ and $g \circ f = \pi_0^N$, by the following commutative diagram

$$\begin{array}{ccccc} M & \xrightarrow{f} & M' & \twoheadrightarrow & \operatorname{coker} f \\ \downarrow{\scriptstyle \pi_0^N} & \swarrow{\scriptstyle \exists ! g} & \downarrow{\scriptstyle \pi_0^N} & & \downarrow{\scriptstyle 0} \\ M & \xrightarrow[f]{} & M' & \twoheadrightarrow & \operatorname{coker} f. \end{array}$$

Here the uniqueness of g follows from our assumption that π_0 is not a zero divisor in R. Hence we can define the isogeny category $\underline{\mathrm{Mod}}_R(\varphi)[\frac{1}{\pi_0}]$ by formally inverting π_0 on morphisms.

The natural functor $\underline{\mathrm{Mod}}_R(\varphi)[\frac{1}{\pi_0}] \to \underline{\mathrm{Mod}}_{R[\frac{1}{\pi_0}]}(\varphi)$ which sends M to $M[\frac{1}{\pi_0}]$ is fully faithful. Using this, we identify the isogeny class containing M with $M[\frac{1}{\pi_0}]$. This functor does not have to be essentially surjective unless $R = R[\frac{1}{\pi_0}]$. For example, if $R = \mathfrak{S}$ or $\mathfrak{o}_{\mathcal{E}}$ then the functor is not essentially surjective.

2.2.8 Vector Bundle on $\boldsymbol{\Delta}$ with Frobenius Structure

We will see later (in §6.1.5) that one can view $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$ as the category of vector bundles on $\boldsymbol{\Delta}$ equipped with a certain nice Frobenius structure in the following sense. For $\mathcal{M} \in \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$, let $\widetilde{\mathcal{M}}$ and $\widetilde{(\sigma^*\mathcal{M})}$ be the vector bundles over $\boldsymbol{\Delta}$ with global sections $\mathcal{M}$ and $\sigma^*\mathcal{M}$, respectively. Then $\varphi : \sigma^*\mathcal{M} \to \mathcal{M}$ corresponds to a map $\widetilde{\varphi} : \widetilde{\sigma^*\mathcal{M}} \to \widetilde{\mathcal{M}}$ of coherent $\mathcal{O}_{\boldsymbol{\Delta}}$-modules, and this is an isomorphism outside $x_0 \in \boldsymbol{\Delta}$ (which is the point cut out by $\mathcal{P}(u) = 0$).

By the discussion of §2.2.4, the scalar extension $\mathfrak{M} \to \mathcal{O}_{\boldsymbol{\Delta}} \otimes_{\mathfrak{S}} \mathfrak{M}$ defines a functor $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi) \to \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$ that factors through the isogeny category of the source category, so we obtain a functor

$$\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi). \tag{2.2.8.1}$$

We will see, after some nontrivial work, that the essential image of this functor is precisely the objects pure of slope 0 in the sense of Kedlaya (for the case $\mathfrak{o}_0 = \mathbb{Z}_p$) and Hartl (for the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$). This is proved in Proposition 4.3.3 of this paper.

2.2.9 Hodge-Pink type

We now work with the case $R = \mathfrak{S}[\frac{1}{\pi_0}]$ or $\mathcal{O}_{\mathbf{\Delta}}$. Since $\mathcal{O}_{\mathbf{\Delta}}$ is a Bézout domain and $\mathfrak{S}[\frac{1}{\pi_0}]$ is a principal ideal domain, we have a structure theorem for finitely presented R-modules. Furthermore, the natural inclusion $\mathfrak{S}[\frac{1}{\pi_0}] \hookrightarrow \mathcal{O}_{\mathbf{\Delta}}$ induces an isomorphism between $\mathcal{P}(u)$-adic completions; in particular, we have an isomorphism $\mathfrak{S}[\frac{1}{\pi_0}]/\mathcal{P}(u)^w \xrightarrow{\sim} \mathcal{O}_{\mathbf{\Delta}}/\mathcal{P}(u)^w$ for any $w > 0$.

A *Hodge-Pink type* $\mathbf{v}$ is a collection of integers m_w for each non-negative integer w, such that only finitely many m_w are nonzero. We call $n := \sum_w m_w$ the rank of $\mathbf{v}$. If $m_w = 0$ for all $w \notin [0, h]$, we say v is *of $\mathcal{P}$-height* $\leqslant h$, and we then define a quotient $\bar{\Lambda}^{\mathbf{v}}$ of $(R/\mathcal{P}(u)^h)^{\oplus n}$ as follows.

$$\bar{\Lambda}^{\mathbf{v}} \cong \bigoplus_{w>0} \left(\frac{\mathfrak{S}[\frac{1}{\pi_0}]}{(\mathcal{P}(u)^w)} \right)^{m_w} \cong \bigoplus_{0 \le w \le h} \left(\frac{\mathcal{O}_{\mathbf{\Delta}}}{(\mathcal{P}(u)^w)} \right)^{m_w}. \tag{2.2.9.1}$$

Although the term for $w = 0$ does not influence $\bar{\Lambda}^{\mathbf{v}}$, m_0 may be positive and in (2.2.9.1) we are viewing $\bar{\Lambda}^{\mathbf{v}}$ as a quotient of $(R/\mathcal{P}(u)^h)^{\oplus n}$. Any $R/\mathcal{P}(u)^h$-module which can be generated by n elements is isomorphic to $\bar{\Lambda}^{\mathbf{v}}$ for a unique $\mathbf{v}$ with rank n and $\mathcal{P}$-height $\leqslant h$.

Let $\mathcal{M}$ be a (φ, R)-module of $\mathcal{P}$-height $\leqslant h$. Assume furthermore that $\operatorname{rank} \mathcal{M} = n$. Then the cokernel of $\varphi_{\mathcal{M}}$, being annihilated by $\mathcal{P}(u)^h$, is isomorphic to $\bar{\Lambda}^{\mathbf{v}}$ for a unique Hodge-Pink type $\mathbf{v}$ of rank n (and necessarily of $\mathcal{P}$-height $\leqslant h$). We say $\mathcal{M}$ is *of Hodge-Pink type* $\mathbf{v}$ if $\operatorname{rank}\mathcal{M} = \operatorname{rank}\mathbf{v}$ and $\operatorname{coker}\varphi_{\mathcal{M}} \cong \bar{\Lambda}^{\mathbf{v}}$ as R-modules. We say $w \in \mathbb{Z}_{\geq 0}$ is a *Hodge-Pink weight* of $\mathcal{M}$ if $m_w \neq 0$, and we call m_w the *multiplicity* of w for $\mathcal{M}$.

The following equivalent formulation can be useful. Keeping the notations as above, $\mathcal{M}$ is of Hodge-Pink type $\mathbf{v}$ if and only if there exists a choice of R-basis for $\mathcal{M}$ which induces the following commutative diagram:

(2.2.9.2)
$$\begin{array}{ccc} (R/\mathcal{P}(u)^h)^{\oplus n} & \xrightarrow{\sim} & \mathcal{M}/\mathcal{P}(u)^h\cdot\mathcal{M} \\ \downarrow & & \downarrow \\ \bar{\Lambda}^{\mathbf{v}} & \xrightarrow{\sim} & \operatorname{coker}\varphi_{\mathcal{M}} \end{array}$$

For $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, the cokernel of $\varphi_{\mathfrak{M}}$ can be a non-trivial extension among $\mathfrak{S}/\mathcal{P}(u)^w$s, so inverting π_0 is crucial to obtain the simple form as above. The point is that $\mathfrak{S}[\frac{1}{\pi_0}]$ is a principal ideal domain while $\mathfrak{S}$ is not.

Remark 2.2.10. In due course, we discuss the relationship between the notion of Hodge-Pink type/weights and the notion of Hodge-type/Hodge-Tate weights for crystalline $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations in the case $\mathfrak{o}_0 = \mathbb{Z}_p$.

2.2.11 Generalized φ-module of finite $\mathcal{P}$-height

As previously, assume that R be a $\mathfrak{S}$-algebra with no non-zero $\mathcal{P}(u)$-torsion (i.e., we have $R \subset R[\frac{1}{\mathcal{P}(u)}]$). This condition is satisfied if R is a domain and $\mathcal{P}(u) \neq 0$ in R. Then we can make the following generalization of $\underline{\mathrm{Mod}}_R(\varphi)$ by allowing φ to have a "pole" at $\mathcal{P}(u)\cdot R$. Consider a finitely generated locally free R-module M, equipped with a $R[\frac{1}{\mathcal{P}(u)}]$-linear map $\varphi : (\sigma^* M)[\frac{1}{\mathcal{P}(u)}] \xrightarrow{\sim} M[\frac{1}{\mathcal{P}(u)}]$. We call such a pair (M, φ) a *generalized* (φ, R)*-module of finite* $\mathcal{P}$*-height* or a *generalized* (φ, R)*-module* if $\mathcal{P}$ is understood. If $\mathcal{P}(u) \in R^\times$, then they are just étale φ-modules. In general, the category of generalized φ-modules of finite $\mathcal{P}$-height contains $\underline{\mathrm{Mod}}_R(\varphi)$ as the full subcategory of objects (M, φ) such that φ restricts to a map $\sigma^* M \to M$, and is thereby equivalent to $\underline{\mathrm{Mod}}_R(\varphi)$ if $\mathcal{P}(u) \in R^\times$. For any $N \gg 0$ (depending on M), the map $\mathcal{P}(u)^N \cdot \varphi$ restricts to $\sigma^* M \to M$, so $(M, \mathcal{P}(u)^N \cdot \varphi) \in \underline{\mathrm{Mod}}_R(\varphi)$.

We can extend all the natural operations on $\underline{\mathrm{Mod}}_R(\varphi)$ as in §2.2.4 to generalized φ-modules. For example, we can define duals and internal homs for generalized φ-modules of finite $\mathcal{P}$-height, as suggested earlier at Remark 2.2.5. In particular, we can define the Tate objects $R(n)$ for all $n \in \mathbb{Z}$, so $R(-n) = R(n)^*$. For any generalized φ-module (M, φ), the Tate twist $M(n) := M \otimes_R R(n)$ for $n \gg 0$ becomes an actual φ-module.

Most of our results on $\underline{\mathrm{Mod}}_R(\varphi)$ can extend to generalized φ-modules by Tate twist, and some results and definitions can be stated more neatly using generalized φ-modules. But we do not crucially use this notion. For $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, the definition of generalized φ-module over $\mathfrak{S}$ is exactly that of a *local shtuka* over $\mathfrak{o}_K$. (See Definition 7.1.1.)

2.3 Hodge-Pink structure

In this section, we define the objects (so called, isocrystals with Hodge-Pink structure) which are the equi-characteristic replacement for "filtered isocrystals". In §3.2 we will see how these objects arise from $\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ and $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}]$. This section is written based on [39, §2.2].

We call an étale φ-module over $\mathscr{K}_0$ an *isocrystal*, or more precisely a *isocrystal* over k. Recall that $x_0 \in \mathbf{\Delta}$ is the point cut out by $\mathcal{P}(u) = 0$, and we denoted by $\mathscr{K}$ the residue field at $x_0 \in \mathbf{\Delta}$. In §2.1.3 we have seen that there is a canonical isomorphism $\widehat{\mathcal{O}_{\mathbf{\Delta}, x_0}} \cong \mathscr{K}[[\mathcal{P}(u)]]$ as $\mathscr{K}_0$-algebras when $\mathscr{K}/\mathscr{K}_0$ is separable, and in general $\mathcal{P}(u)$ is a uniformizer of $\widehat{\mathcal{O}_{\mathbf{\Delta}, x_0}}$.

Definition 2.3.1. For a finite-dimensional $\mathscr{K}_0$-vector space D, we put[7] $\widehat{\mathcal{D}}_{x_0} := \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \otimes_{\mathscr{K}_0} D$. A *Hodge-Pink structure*[8] on D is a $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}$-lattice Λ inside $\widehat{\mathcal{D}}_{x_o}[\frac{1}{\mathcal{P}(u)}] \cong \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}[\frac{1}{\mathcal{P}(u)}] \otimes_{\mathscr{K}_0} D$. A Hodge-Pink structure Λ on D is *effective* if Λ contains the standard lattice $\widehat{\mathcal{D}}_{x_0}$. An effective Hodge-Pink structure Λ is *of $\mathcal{P}$-height $\leqslant h$* if Λ is contained in $\mathcal{P}(u)^{-h} \cdot \widehat{\mathcal{D}}_{x_0}$.

Let Λ and Λ' be Hodge-Pink structures on D and D', respectively. We say that a $\mathscr{K}_0$-linear map $f : D \to D'$ *respects Hodge-Pink structures* if $\mathrm{id} \otimes f : \widehat{\mathcal{D}}_{x_0}[\frac{1}{\mathcal{P}(u)}] \to \widehat{\mathcal{D}}'_{x_0}[\frac{1}{\mathcal{P}(u)}]$ takes Λ into Λ', where $\widehat{\mathcal{D}}'_{x_0} := \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \otimes_{\mathscr{K}_0} D'$.

An *isocrystal with Hodge-Pink structure* (respectively, *with effective Hodge-Pink structure*) is a tuple (D, φ, Λ), where (D, φ) is an isocrystal and Λ is a Hodge-Pink structure (respectively, an effective Hodge-Pink structure) on the underlying $\mathscr{K}_0$-vector space D. We denote by $\mathcal{HP}_K(\varphi)$ the category of isocrystals with Hodge-Pink structure, where a *morphism* is a $\mathscr{K}_0$-linear map on the underlying vector spaces which is φ-compatible and respects Hodge-Pink structures. We denote by $\mathcal{HP}_K^{\geqslant 0}(\varphi)$ (respectively, $\mathcal{HP}_K^{[0,h]}(\varphi)$) the full subcategory of isocrystals with effective Hodge-Pink structure (respectively, with Hodge-Pink structures of $\mathcal{P}$-height $\leqslant h$).

Remark. Originally, Hodge-Pink structures were defined by Pink [67] in the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, as a “correct” analogue of Hodge structures in function field arithmetic.

2.3.2

Let $D := (D, \varphi, \Lambda)$ and $D' := (D', \varphi', \Lambda')$ be objects in $\mathcal{HP}_K(\varphi)$. The category $\mathcal{HP}_K(\varphi)$ is equipped with the $\otimes$-product

$$(D,\, \varphi,\, \Lambda) \otimes (D',\, \varphi',\, \Lambda') := (D \otimes_{\mathscr{K}_0} D', \varphi \otimes \varphi', \Lambda \otimes_{\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}} \Lambda')$$

[7] Later, we will put $\mathcal{D} := \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} D$, so $\widehat{\mathcal{D}}_{x_0}$ is the completed stalk of $\mathcal{D}$ at $x_0 \in \mathbf{\Delta}$.

[8] The usual definition of a Hodge-Pink structure is a $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}$-lattice Λ inside $(\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0})[\frac{1}{\mathcal{P}(u)}] \otimes \sigma^* D$. But via the isomorphism $\varphi : \sigma^* D \to D$ one can pass between this definition and the usual one - including all the statements involving “Hodge-Pink structure.”

and the internal hom via the identification $\mathrm{Hom}_{\mathscr{K}_0}(\sigma^*D, \sigma^*D') \cong \sigma^*\,\mathrm{Hom}_{\mathscr{K}_0}(D, D')$:

$$\mathrm{Hom}((D,\Lambda),(D',\Lambda')) := (\mathrm{Hom}_{\mathscr{K}_0}(D,D'), f \mapsto \varphi' \circ f \circ \varphi^{-1}, \mathrm{Hom}_{\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}}(\Lambda,\Lambda')),$$

which satisfy all the expected properties. One can check that $\mathbf{1} := (\mathscr{K}_0,\ \mathrm{id}\otimes\sigma,\ \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0})$ is the "neutral object" in $\mathcal{HP}_K(\varphi)$ and the contravariant functor $(D,\Lambda) \mapsto (D^*,\Lambda^*) = \mathrm{Hom}((D,\Lambda),\mathbf{1})$ defines a duality. The category $\mathcal{HP}_K^{\geqslant 0}(\varphi)$ is stable under $\otimes$-product, but not under internal hom or duality.

For any integer n, we define the *Tate object* $\mathbf{1}(n)$ to be:

(2.3.2.1) $$\mathbf{1}(n) := (\mathscr{K}_0\mathbf{e},\ \varphi(\sigma^*\mathbf{e}) = \pi_0^n\mathbf{e},\ \mathcal{P}(u)^{-n}\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}).$$

For any $(D,\Lambda) \in \mathcal{HP}_K(\varphi)$, we define the *$n$-fold Tate twist* to be $(D,\Lambda)\otimes\mathbf{1}(n) \cong (D,\ \mathcal{P}(u)^{-n}\cdot\Lambda)$. Clearly for any Hodge-Pink structure $(D,\Lambda) \in \mathcal{HP}_K(\varphi)$, the Hodge-Pink structure $(D,\Lambda)\otimes\mathbf{1}(n) = (D,\ \mathcal{P}(u)^{-n}\cdot\Lambda)$ is effective for $n \gg 0$.

A *subobject* $(D',\Lambda') \subset (D,\Lambda)$ in $\mathcal{HP}_K(\varphi)$ simply means that the natural inclusion is a morphism of $\mathcal{HP}_K(\varphi)$; i.e., $D' \subset D$ is φ-stable and $\Lambda' \subset \Lambda \cap \left(\widehat{\mathcal{D}}'_{x_0}[\frac{1}{\mathcal{P}(u)}]\right)$. We say that a subobject $(D',\Lambda') \subset (D,\Lambda)$ is *saturated*[9] if $\Lambda' = \Lambda \cap \left(\widehat{\mathcal{D}}'_{x_0}[\frac{1}{\mathcal{P}(u)}]\right)$ holds, where the intersection is taken inside $\widehat{\mathcal{D}}_{x_0}[\frac{1}{\mathcal{P}(u)}]$. Similarly, a *quotient* (D'',Λ'') of (D,Λ) means that D'' is a quotient of D as a $\mathscr{K}_0$-vector space and that Λ'' coincides with the image of Λ under the map $\widehat{\mathcal{D}}_{x_0}[\frac{1}{\mathcal{P}(u)}] \twoheadrightarrow \widehat{\mathcal{D}}''_{x_0}[\frac{1}{\mathcal{P}(u)}]$ induced by the natural projection. For any saturated subobjects $(D',\Lambda') \subset (D,\Lambda)$, we can form the quotient $(D/D',\Lambda/\Lambda')$, and the kernel (D',Λ') of the natural projection $(D,\Lambda) \twoheadrightarrow (D'',\Lambda'')$ onto a quotient is a saturated subobject such that the natural projection induces an isomorphism $(D/D',\Lambda/\Lambda') \xrightarrow{\sim} (D'',\Lambda'')$.

A *short exact sequence* in $\mathcal{HP}_K(\varphi)$ ($\mathcal{HP}_K^{\geqslant 0}(\varphi)$, respectively) is defined as a short exact sequence of underlying $\mathscr{K}_0$-vector spaces which induce a short exact sequence

[9] In [39, §2.2], saturated subobjects are called "strict subobjects." We chose to call them "saturated" because a subobject $(D',\Lambda') \subset (D,\Lambda)$ is saturated if and only if $\Lambda' \subset \Lambda$ is saturated.

on the Hodge-Pink structures (i.e., on $\widehat{\mathcal{O}}_{\Delta,x_0}$-lattices Λ's). The left/right flanking term is a saturated submodule/quotient of the middle term, and conversely, any saturated submodule or quotient can be placed in a short exact sequence in an evident manner.

2.3.3 Hodge-Pink type and Hodge-Pink structures

Let $\mathbf{v}$ be a Hodge-Pink type; i.e., a collection of non-negative integers m_w for each integer w, such that only finitely many m_w are nonzero. In §2.2.9 we only considered m_w when w is non-negative. Now we are allowing "negative weights."

Now we associate such a $\mathbf{v}$ to a Hodge-Pink structure Λ on a $\mathscr{K}_0$-vector space D. First, we define a decreasing filtration on $\widehat{\mathcal{D}}_{x_0}$ from the Hodge-Pink structure as follows:

$$\text{(2.3.3.1)} \qquad \mathrm{Fil}^w_\Lambda\left(\widehat{\mathcal{D}}_{x_0}\right) := (\widehat{\mathcal{D}}_{x_0}) \cap (\mathcal{P}(u)^w \cdot \Lambda) \quad \text{for } w \in \mathbb{Z}$$

where the intersections are taken inside $\widehat{\mathcal{D}}_{x_0}[\frac{1}{\mathcal{P}(u)}]$. In turn, we obtain a separated and exhaustive filtration $\mathrm{Fil}^\bullet D_{\mathscr{K}}$ on $D_{\mathscr{K}} := \mathscr{K} \otimes_{\mathscr{K}_0} D$ by taking the image of this filtration $\mathrm{Fil}^\bullet_\Lambda\left(\widehat{\mathcal{D}}_{x_0}\right)$ under the natural projection map $\widehat{\mathcal{D}}_{x_0} \twoheadrightarrow \widehat{\mathcal{D}}_{x_0}/\mathcal{P}(u)\widehat{\mathcal{D}}_{x_0} \cong D_{\mathscr{K}}$. Note that $\mathrm{gr}^w D_{\mathscr{K}} := \frac{\mathrm{Fil}^w D_{\mathscr{K}}}{\mathrm{Fil}^{w+1} D_{\mathscr{K}}} = 0$ for $w \ll 0$ and for $w \gg 0$.

Definition 2.3.3.2. Let $\mathbf{v} := \{m_w := \dim_{\mathscr{K}}(\mathrm{gr}^w(D_{\mathscr{K}}))\}$. We say (D, Λ) is of *Hodge-Pink type* $\mathbf{v}$. The *Hodge-Pink weights* for (D, Λ) are $w \in \mathbb{Z}$ such that $m_w \neq 0$, and we call m_w the *multiplicity* of w. The *Hodge-Pink type* for an isocrystal (D, φ, Λ) with Hodge-Pink structure means the Hodge-Pink type for (D, Λ).

If (D, Λ) is of Hodge-Pink type $\mathbf{v} = \{m_w\}_{w\in\mathbb{Z}}$, then $\sum_w m_w$ equals $\dim_{\mathscr{K}}(\mathscr{K} \otimes_{\mathscr{K}_0} D = \dim_{\mathscr{K}_0} D$. Clearly a Hodge-Pink structure (D, Λ) is effective (respectively, of $\mathcal{P}$-height $\leqslant h$) if and only if $m_w = 0$ for all $w < 0$ (respectively, $m_w = 0$ for all $w \notin [0, h]$).

The following proposition shows the behavior of Hodge-Pink types/weights under the natural operations, such as duality, tensor product, and internal Hom.

Proposition 2.3.4. *Consider $(D,\Lambda) \in \mathcal{HP}_K(\varphi)$ which has Hodge-Pink weights $\{w_1,\cdots,w_s\}$ and each weight w_j has multiplicity m_j.*

1. *The dual (D^*,Λ^*) has Hodge-Pink weights exactly $\{-w_1,\cdots,-w_s\}$ and each weight $-w_j$ has multiplicity m_j; i.e., the duality inverts the signs of the Hodge-Pink weights.*

2. *Assume $(D',\Lambda') \in \mathcal{HP}_K(\varphi)$ has Hodge-Pink weights $\{w'_1,\cdots,w'_{s'}\}$ and each weight $w'_{i'}$ has multiplicity $m'_{i'}$. Then the tensor product $(D\otimes D',\Lambda\otimes\Lambda')$ induces the tensor product filtration on $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \otimes_{\mathscr{K}_0} (D\otimes D')$. In particular the Hodge-Pink weights for the tensor product are $\{w_i+w'_{i'}\}_{i=1,\cdots,s,\, i'=1,\cdots,s'}$ and each weight w has multiplicity $\sum_{j,j'} m_j + m'_{j'}$ where the summation is over (j,j') such that $w = w_j + w'_{j'}$.*

3. *For the Tate twist $(D,\Lambda)\otimes\mathbf{1}(n) \cong (D,\mathcal{P}(u)^{-n}\Lambda)$, we have that $\mathrm{Fil}^i_{\mathcal{P}(u)^{-n}\Lambda} = \mathrm{Fil}^{i-n}_{\Lambda}$; i.e., the Tate twist shifts the filtration. In particular, the Hodge-Pink weights for the twist $(D,\mathcal{P}(u)^{-n}\Lambda)$ are exactly w_j+n with multiplicity m_j.*

Using (1) and (2), we can obtain the filtration, Hodge-Pink weights, and multiplicities for the internal hom, which is left to the reader.

The following easy lemma shows how to recover the Hodge-Pink structure Λ from the filtration $\mathrm{Fil}^\bullet_\Lambda$ defined by Λ.

Lemma 2.3.5. *Let $(D,\Lambda) \in \mathcal{HP}_K(\varphi)$, and let $\mathrm{Fil}^w_\Lambda(\widehat{\mathcal{D}}_{x_0})$ be the filtration on $\widehat{\mathcal{D}}_{x_0}$ associated to the Hodge-Pink structure Λ. Then,*

$$\Lambda = \sum_{w\in\mathbb{Z}} \left(\mathcal{P}(u)^{-w}\cdot \mathrm{Fil}^w_\Lambda(\widehat{\mathcal{D}}_{x_0})\right) = \mathrm{Fil}^0\left(\widehat{\mathcal{D}}_{x_0}[1/\mathcal{P}(u)]\right),$$

where the last term is the 0*th filtration for the tensor product filtration on* $\widehat{\mathcal{D}}_{x_0}[\frac{1}{\mathcal{P}(u)}] \cong \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}[\frac{1}{\mathcal{P}(u)}] \otimes_{\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}} \widehat{\mathcal{D}}_{x_0}$*, where we put the* $\mathcal{P}(u)$*-adic filtration on* $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}[\frac{1}{\mathcal{P}(u)}]$.

Let $\mathbf{v} := \{m_w\}_{w\in\mathbb{Z}}$ be a Hodge-Pink type, and assume that $m_w = 0$ for all $w < 0$. In §2.2.9, we associated to $\mathbf{v}$ a $\mathfrak{S}[\frac{1}{\pi_0}]$-module $\bar{\Lambda}^{\mathbf{v}}$ killed by some power of $\mathcal{P}(u)$ which recovers all m_w except m_0. The following corollary shows how $\bar{\Lambda}^{\mathbf{v}}$ is related to any Hodge-Pink structure Λ on D of Hodge-Pink type $\mathbf{v}$.

Corollary 2.3.6. *Consider an effective Hodge-Pink structure* $(D,\Lambda) \in \mathcal{HP}_K^{\geqslant 0}(\varphi)$ *that is of Hodge-Pink type* $\mathbf{v} := \{m_w\}$*, so* $m_w = 0$ *for all* $w < 0$*. Then we have*

$$\Lambda/\widehat{\mathcal{D}}_{x_0} \;\cong\; \bar{\Lambda}^{\mathbf{v}} \;=\; \bigoplus_{m_w\neq 0} \left(\frac{\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}}{(\mathcal{P}(u)^w)}\right)^{m_w} \tag{2.3.6.1}$$

In §2.2.9, we also defined the notion of Hodge-Pink type on $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}]$. Later in §3.2.6, we will define a functor $\underline{\mathbb{H}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \mathcal{HP}_K^{\geqslant 0}(\varphi)$ which preserves Hodge-Pink types, so the notion of Hodge-Pink type for these two categories is compatible.

2.3.7 Weak admissibility

Let $(D,\varphi,\Lambda) \in \mathcal{HP}_K(\varphi)$ be of rank 1; i.e., D is a 1-dimensional vector space over $\mathscr{K}_0$. Necessarily, $\Lambda = \mathcal{P}(u)^{-h} \cdot \widehat{\mathcal{D}}_{x_0} \subset \widehat{\mathcal{D}}_{x_0}[\frac{1}{\mathcal{P}(u)}]$ for a unique $h \in \mathbb{Z}$. We define the *Hodge number* for (D,φ,Λ) to be $t_H(D,\Lambda) := h$. We often write $t_H(D)$ if Λ is understood. For any $\mathscr{K}_0$-basis $\mathbf{e} \in D$, there is a nonzero element $\alpha_{\mathbf{e}} \in \mathscr{K}_0^{\times}$ such that $\varphi(\sigma^*\mathbf{e}) = \alpha_{\mathbf{e}} \cdot \mathbf{e}$. Note that $\mathrm{ord}_{\pi_0}(\alpha_{\mathbf{e}})$ is *independent* of the choice of basis though $\alpha_{\mathbf{e}}$ is not. We define the *Newton number* for the isocrystal (D,φ) to be $t_N(D) := \mathrm{ord}_{\pi_0}(\alpha_{\mathbf{e}})$.

Since the category $\mathcal{HP}_K(\varphi)$ has an obvious notion of exterior products (using $\otimes$-products and quotients), we define Hodge and Newton numbers for any $(D,\varphi,\Lambda) \in \mathcal{HP}_K(\varphi)$ as follows: $t_H(D) := t_H(\det D)$ and $t_N(D) := t_N(\det D)$. Now, we can define "weak admissibility" for Hodge-Pink structures.

Definition 2.3.7.1. An object $(D, \Lambda) \in \mathcal{HP}_K(\varphi)$ is called *weakly admissible* if the following properties hold:

1. $t_H(D) = t_N(D)$.

2. For any subobject $(D', \Lambda') \subset (D, \Lambda)$, we have $t_H(D', \Lambda') \le t_N(D')$.

The full subcategory of isocrystals D with a weakly admissible Hodge-Pink structure Λ will be denoted by $\mathcal{HP}_K^{wa}(\varphi)$. We similarly define $\mathcal{HP}_K^{wa,\geqslant 0}(\varphi)$ and $\mathcal{HP}_K^{wa,[0,h]}(\varphi)$ as full subcategories in $\mathcal{HP}_K^{\geqslant 0}(\varphi)$ and $\mathcal{HP}_K^{[0,h]}(\varphi)$ consisting of weakly admissible objects.

Lemma 2.3.7.2. *Condition (2) in Definition 2.3.7.1 is equivalent to:*

(2)′ For any saturated *subobject* $(D', \Lambda') \subset (D, \Lambda)$*, we have* $t_H(D', \Lambda') \le t_N(D')$

In particular, an isocrystal (D, Λ) *of rank* 1 *is weakly admissible if and only if* $t_H(D) = t_N(D)$.

Proof. It is enough to show $(2)'$ implies (2). By passing to the determinant of (D', Λ'), we may assume that D' has rank 1. Let $(D', \Lambda'_{\mathrm{sat}})$ be the "saturation" of (D', Λ'); i.e., $\Lambda'_{\mathrm{sat}} := \Lambda \cap \left(\widehat{\mathcal{D}}'_{x_0}[\frac{1}{\mathcal{P}(u)}]\right)$. The saturation Λ'_{sat} necessarily contains Λ' by the definition of subobject, so we have $t_H(D', \Lambda'_{\mathrm{sat}}) \ge t_H(D', \Lambda')$. But the Newton numbers of both subobjects are the same because they only depend on the underlying isocrystals, not on the Hodge-Pink structure. Therefore, the inequality $t_H(D', \Lambda') \le t_N(D')$ follows if it holds for the saturation $(D', \Lambda'_{\mathrm{sat}})$. □

Proposition 2.3.8. *The full subcategory* $\mathcal{HP}_K^{wa}(\varphi)$ *of* $\mathcal{HP}_K(\varphi)$ *is closed under the formation of tensor, symmetric and exterior products, internal homs and duality, extensions and direct sums. A direct sum* $(D, \Lambda) \oplus (D', \Lambda')$ *is weakly admissible if and only if both factors are weakly admissible. Moreover,* $\mathcal{HP}_K^{wa}(\varphi)$ *is an abelian category.*

A direct proof of this proposition is presented in [67, §4,§5]. (Note that Pink uses the terminology "semistability" to mean our weak admissibility.) The direct proof is rather tedious but elementary except the assertion about tensor products which can be proved by adapting Totaro's argument for weakly admissible filtered φ-modules [76]. It is also possible to deduce these using the rigid-analytic interpretation of weak admissibility (Theorem 4.3.4) and the theory of slopes.

Since Tate objects $\mathbf{1}(n)$ are weakly admissible for any $n \in \mathbb{Z}$, an isocrystal with weakly admissible Hodge-Pink structure (D, Λ) is weakly admissible if and only if its Tate twist $(D, \Lambda)(n)$ is weakly admissible for some n, by the previous proposition. One can also directly see this since the Tate twist $(D, \Lambda)(n)$ increases t_N and t_H by n for all subobjects and quotient objects. We also note that if the residue field k is *algebraically closed*, then any rank-1 isocrystal with weakly admissible Hodge-Pink structure is isomorphic to $\mathbf{1}(n)$ for some $n \in \mathbb{Z}$. As mentioned in Remark 4.1.3, this is a direct consequence of the Dieudonné-Manin classification (Theorem 4.1.2).

2.4 Filtered isocrystals, crystalline $\mathcal{G}_{\mathscr{K}}$-representations, and resumé of [52]

We assume that $\mathfrak{o}_0 = \mathbb{Z}_p$ throughout the section and follow the notations from §1.3.1.2. We fix the uniformizer $\pi_0 = p$ of $\mathbb{Z}_p$. The main purpose of this section is to explain the relationship between crystalline $\mathcal{G}_{\mathscr{K}}$-representations and the semilinear algebra objects introduced so far, which will motivate the later discussions. Most of the results in this section are proved in [52]. We assume some basic knowledge of crystalline and semi-stable representations (and p-adic Hodge theory), for which we refer to [32, 33].

2.4.1 Filtered isocrystals

A *filtered isocrystal* is $(D, \varphi, \mathrm{Fil}^\bullet D_{\mathscr{K}})$, where (D, φ) is an étale φ-module[10] which is finite-dimensional over $\mathscr{K}_0$ (i.e., an isocrystal) and $\mathrm{Fil}^\bullet D_{\mathscr{K}}$ is a decreasing separated and exhaustive filtration on $D_{\mathscr{K}} := \mathscr{K} \otimes_{\mathscr{K}_0} D$ by $\mathscr{K}$-linear subspaces. We also define a *filtered* (φ, N)*-module* to be $(D, \varphi, N, \mathrm{Fil}^\bullet D_{\mathscr{K}})$ where $(D, \varphi, \mathrm{Fil}^\bullet D_{\mathscr{K}})$ is a filtered isocrystal and $N : D \to D$ is a (necessarily nilpotent) $\mathscr{K}_0$-linear endomorphism such that $N\varphi = p\varphi N$. We call N a *monodromy operator*. We view a filtered isocrystal as a filtered (φ, N)-module by setting $N = 0$. We let $\mathcal{MF}_{\mathscr{K}}(\varphi)$ denote the category of filtered isocrystals, and $\mathcal{MF}_{\mathscr{K}}(\varphi, N)$ the category of filtered (φ, N)-module with the obvious notions of morphisms. We have natural definitions of subobjects and quotients; direct sums; tensor products; internal homs; and duality. We leave the exact formulation to readers, or refer to [33].

Recall that a "Hodge-Pink type" in the sense of §2.3.3 is a collection $\mathbf{v}$ of non-negative integers m_w for each integer $w \in \mathbb{Z}$ such that only finitely many m_w are nonzero. We say $\mathbf{v} := \{m_w := \dim_{\mathscr{K}}(\mathrm{gr}^w D_{\mathscr{K}})\}$ is the *p-adic Hodge type* for $(D, \varphi, N, \mathrm{Fil}^\bullet D_{\mathscr{K}})$, or *Hodge type* for $(D, \varphi, N, \mathrm{Fil}^\bullet D_{\mathscr{K}})$ in short. Note that the numerical datum $\mathbf{v}$ determines the decreasing separated and exhaustive filtration $\mathrm{Fil}^\bullet D_{\mathscr{K}}$ of $D_{\mathscr{K}}$ by its $\mathscr{K}$-subspaces, uniquely up to $\mathscr{K}$-automorphism of $D_{\mathscr{K}}$. We call w for which $m_w \neq 0$ a *Hodge-Tate weight* for $(D, \varphi, N, \mathrm{Fil}^\bullet D_{\mathscr{K}})$, and m_w the *multiplicity* of w. Note that the definitions of Hodge type, Hodge-Tate weights, and their multiplicities have nothing to do with φ and N but only use $\mathrm{Fil}^\bullet D_{\mathscr{K}}$. We let $\mathcal{MF}^{\geqslant 0}_{\mathscr{K}}(\varphi)$ (respectively, $\mathcal{MF}^{[0,h]}_{\mathscr{K}}(\varphi)$) denote the full subcategory of filtered isocrystals such that all the Hodge-Tate weights are non-negative (respectively, are in $[0, h]$). We make similar definitions for $\mathcal{MF}^{\geqslant 0}_{\mathscr{K}}(\varphi, N)$ and $\mathcal{MF}^{[0,h]}_{\mathscr{K}}(\varphi, N)$.

[10] Following the usual convention, φ is a σ-semilinear endomorphism throughout this section.

We now define the Hodge and the Newton numbers for $D := (D, \varphi, N, \mathrm{Fil}^\bullet D_{\mathscr{K}})$. We first assume that D is 1-dimensional. Then we define the *Hodge number* $t_H(D) = t_H(D, \mathrm{Fil}^\bullet D_{\mathscr{K}})$ to be the unique Hodge-Tate weight. To define the Newton number t_N, choose a basis $D \cong \mathscr{K}_0\mathbf{e}$ so $\varphi(\mathbf{e}) = \alpha_{\mathbf{e}}\mathbf{e}$ for some $\alpha_{\mathbf{e}} \in \mathscr{K}_0^\times$. The *Newton number* $t_N(D) = t_N(D, \varphi)$ is $\mathrm{ord}_p(\alpha_{\mathbf{e}})$. If D is of arbitrary dimension, we define $t_H(D) := t_H(\det D)$ and $t_N(D) := t_N(\det D)$. Note that the Hodge number only uses the filtration, while the Newton number only uses the Frobenius structure.

A filtered (φ, N)-module $(D, \varphi, N, \mathrm{Fil}^\bullet D_{\mathscr{K}})$ is called *weakly admissible* if $t_H(D) = t_N(D)$ and the inequality $t_H(D') \leq t_N(D')$ holds for any φ-stable subspace $D' \subset D$ where $D'_{\mathscr{K}}$ is given the subspace filtration. We let $\mathcal{MF}^{wa}_{\mathscr{K}}(\varphi)$ (respectively, $\mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi)$, $\mathcal{MF}^{wa,[0,h]}_{\mathscr{K}}(\varphi)$) denote the full subcategory of weakly admissible filtered isocrystals (respectively, weakly admissible filtered isocrystals with the conditions on Hodge-Tate weights). We similarly define $\mathcal{MF}^{wa}_{\mathscr{K}}(\varphi, N)$, $\mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi, N)$, and $\mathcal{MF}^{wa,[0,h]}_{\mathscr{K}}(\varphi, N)$, where now $D' \subset D$ ranges over $\mathscr{K}_0$-subspace stable under φ and N.

From Fontaine's "period ring formalism," we obtain a contravariant functor $\underline{D}^*_{\mathrm{cris}}$: $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}}) \to \mathcal{MF}^{wa}_{\mathscr{K}}(\varphi)$, and another contravariant functor $\underline{V}^*_{\mathrm{cris}}$ from $\mathcal{MF}_{\mathscr{K}}(\varphi)$ to such (not necessarily finite-dimensional) $\mathbb{Q}_p[\boldsymbol{\mathcal{G}}_{\mathscr{K}}]$-modules. Similarly we get $\underline{D}^*_{\mathrm{st}}$: $\mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}}) \to \mathcal{MF}^{wa}_{\mathscr{K}}(\varphi, N)$ and V^*_{st} from $\mathcal{MF}_K(\varphi)$ to (not necessarily finite-dimensional) $\mathbb{Q}_p[\boldsymbol{\mathcal{G}}_{\mathscr{K}}]$-modules such that $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ acts continuously on any $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable subspaces of finite $\mathbb{Q}_p$-dimension. See [33] for the definitions. There are at least four proofs of the following fundamental theorem: [18], [17], [5], and [52].

Theorem 2.4.2 (Colmez-Fontaine)**.** *The contravariant functor* $\underline{D}^*_{\mathrm{cris}}$ *(respectively,* $\underline{D}^*_{\mathrm{st}}$*) is an anti-equivalence of categories, and* $\underline{V}^*_{\mathrm{cris}}$ *(respectively,* $\underline{V}^*_{\mathrm{st}}$*) restricted to weakly admissible objects is its quasi-inverse.*

For each $n \in \mathbb{Z}$, the *Tate object* $\mathbf{1}_{\mathcal{MF}}(n)$ is a filtered isocrystal defined as follows:

the underlying isocrystal is $(\mathscr{K}_0\mathbf{e}, \varphi(\mathbf{e}) = p^n\mathbf{e})$ and the associated grading is concentrated in degree n. Clearly, $\mathbf{1}_{\mathcal{MF}}(n)$ is weakly admissible. For any filtered isocrystal D, we put $D(n) := D \otimes \mathbf{1}_{\mathcal{MF}}(n)$ and call it the *n-fold Tate twist* of D. One can check without difficulty that a filtered isocrystal D is weakly admissible if and only if its Tate twist $D(n)$ for some $n \in \mathbb{Z}$ is weakly admissible. Later in Remark 4.1.3, we will see that if the residue field k is *algebraically closed*, then any rank-1 weakly admissible filtered isocrystal is isomorphic to some Tate object $\mathbf{1}(n)$. This follows from Dieudonné-Manin classification (Theorem 4.1.2).

2.4.3 Filtered isocrystals and isocrystals with Hodge-Pink structure

For a Hodge-Pink structure on a finite dimensional $\mathscr{K}_0$-vector space D, we obtain a filtration on $\widehat{\mathcal{D}}_{x_0} := \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \otimes_{\mathscr{K}_0} D$, as discussed in §2.3.3. And by reducing the associated filtration $\mathrm{Fil}^\bullet_\Lambda \widehat{\mathcal{D}}_{x_0}$ on $\widehat{\mathcal{D}}_{x_0}$ modulo $\mathcal{P}(u) \cdot \widehat{\mathcal{D}}_{x_0}$, we obtain a filtration $\mathrm{Fil}^\bullet_\Lambda D_{\mathscr{K}})$ on $D_{\mathscr{K}}$ since $D_{\mathscr{K}} \cong \widehat{\mathcal{D}}_{x_0}/\mathcal{P}(u)\widehat{\mathcal{D}}_{x_0}$. More precisely,

(2.4.3.1)
$$\mathrm{Fil}^w_\Lambda D_{\mathscr{K}} := \frac{\mathrm{Fil}^w_\Lambda \widehat{\mathcal{D}}_{x_0}}{\mathrm{Fil}^w_\Lambda \widehat{\mathcal{D}}_{x_0} \cap \mathcal{P}(u)\cdot\widehat{\mathcal{D}}_{x_0}} = \frac{\widehat{\mathcal{D}}_{x_0} \cap \mathcal{P}(u)^w\Lambda}{\mathcal{P}(u)\widehat{\mathcal{D}}_{x_0} \cap \mathcal{P}(u)^w\Lambda} \subset \frac{\widehat{\mathcal{D}}_{x_0}}{\mathcal{P}(u)\cdot\widehat{\mathcal{D}}_{x_0}} \cong D_{\mathscr{K}}.$$

The assignment $(D, \varphi, \Lambda) \mapsto (D, \varphi, \mathrm{Fil}^\bullet_\Lambda D_{\mathscr{K}})$ defines a functor $\underline{\mathcal{F}} : \mathcal{HP}_K(\varphi) \to \mathcal{MF}_{\mathscr{K}}(\varphi)$.

This functor $\underline{\mathcal{F}}$ has a "section" $\underline{\mathrm{res}} : \mathcal{MF}_{\mathscr{K}}(\varphi) \to \mathcal{HP}_K(\varphi)$, in the sense that there exists a natural isomorphism $\underline{\mathcal{F}} \circ \underline{\mathrm{res}} \cong \mathrm{id}_{\mathcal{MF}_{\mathscr{K}}(\varphi)}$. Namely, for $(D, \varphi, \mathrm{Fil}^\bullet D_{\mathscr{K}}) \in \mathcal{MF}_{\mathscr{K}}(\varphi)$, we put $\underline{\mathrm{res}}(D, \varphi, \mathrm{Fil}^\bullet D_{\mathscr{K}}) := (D, \varphi, \Lambda)$, where

$$\Lambda := \mathrm{Fil}^0(D_{\mathscr{K}} \otimes_{\mathscr{K}} \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}[1/\mathcal{P}(u)]) = \sum_{w\in\mathbb{Z}} (\mathrm{Fil}^w D_{\mathscr{K}}) \otimes_{\mathscr{K}} (\mathcal{P}(u)^{-w}\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}).$$

The natural isomorphism $\underline{\mathcal{F}} \circ \underline{\mathrm{res}} \cong \mathrm{id}_{\mathcal{MF}_{\mathscr{K}}(\varphi)}$ is immediate from the construction.

Here is the motivation for introducing the functor $\underline{\mathrm{res}}$. By Theorem 2.4.2, the category $\mathcal{MF}^{wa}_{\mathscr{K}}(\varphi)$ is equivalent (or anti-equivalent) to the category of crystalline representations of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$. On the other hand, we will see later in Corollary 5.2.4 that there exists a fully faithful (contravariant) functor $\underline{V}^*_{\mathcal{HP}} : \mathcal{HP}^{wa}_K(\varphi) \to \mathrm{Rep}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$. Kisin's work [52, §2.1] shows that the functor $\underline{D}^*_{\mathrm{st}} \circ \underline{\mathrm{res}} \circ \underline{V}^*_{\mathcal{HP}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}}) \to \mathrm{Rep}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$ induced by $\underline{\mathrm{res}}$ is naturally isomorphic to the functor obtained by restricting the $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-action to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$. (See §5.2.12 for more discussion.)

We now record some properties of $\underline{\mathcal{F}}$ and $\underline{\mathrm{res}}$ which directly fall out of the definition. The functors $\underline{\mathcal{F}}$ and $\underline{\mathrm{res}}$ commute with quotients, tensor products (hence, symmetric and alternating products), internal homs, and duality. Clearly, both functors $\underline{\mathcal{F}}$ and $\underline{\mathrm{res}}$ preserve the Newton numbers t_N on both sides, since each does nothing on the underlying isocrystal (D, φ). They also preserve the Hodge numbers t_H on both sides. In fact, the functors $\underline{\mathcal{F}}$ and $\underline{\mathrm{res}}$, by construction, "respect" Hodge type for $\mathcal{MF}_{\mathscr{K}}(\varphi)$ and Hodge-Pink type for $\mathcal{HP}_K(\varphi)$ in the following sense: for a fixed $\mathbf{v} := \{m_w\}$, if $(D, \varphi, \mathrm{Fil}^\bullet D_{\mathscr{K}})$ is of Hodge type $\mathbf{v}$, then $\underline{\mathrm{res}}(D, \varphi, \mathrm{Fil}^\bullet D_{\mathscr{K}})$ is of Hodge-Pink type $\mathbf{v}$ and similarly for $\underline{\mathcal{F}}$.

Now we show that *the functors $\underline{\mathcal{F}}$ and $\underline{\mathrm{res}}$ take weakly admissible objects in one category to weakly admissible objects in the other.* One can directly show that $\underline{\mathcal{F}}$ takes a saturated subobject in $\mathcal{HP}_K(\varphi)$ to a saturated subobject in $\mathcal{MF}_{\mathscr{K}}(\varphi)$. In other words, for a Hodge-Pinks structure (D, Λ) and a $\mathscr{K}_0$-subspace D', the Hodge-Pink structure $\Lambda' := \Lambda \cap \widehat{\mathcal{D}}'_{x_0}[\frac{1}{\mathcal{P}(u)}]$ for D' induces the subspace filtration $\mathrm{Fil}^w_{\Lambda'} D'_{\mathscr{K}} = D'_{\mathscr{K}} \cap \mathrm{Fil}^w_{\Lambda} D_{\mathscr{K}}$ for each w. Since $\underline{\mathcal{F}}$ preserves Hodge and Newton numbers, we have that $(D, \varphi, \Lambda) \in \mathcal{HP}_K(\varphi)$ is weakly admissible if and only if $\underline{\mathcal{F}}(D, \varphi, \Lambda)$ is. The claim for $\underline{\mathrm{res}}$ also follows from the natural isomorphism $\underline{\mathcal{F}} \circ \underline{\mathrm{res}} \cong \mathrm{id}_{\mathcal{MF}_{\mathscr{K}}(\varphi)}$.

Even though $\underline{\mathcal{F}}$ and $\underline{\mathrm{res}}$ are *not* quasi-inverse equivalences of categories in general, they are quasi-inverses on rank-1 objects. Indeed, a Hodge-Pink structure on 1-dimensional $\mathscr{K}_0$-vector space is uniquely determined by its Hodge number, and the same holds for a filtration on 1-dimensional $\mathscr{K}_0$-vector space. Note also that this functor $\mathcal{HP}_K(\varphi) \to \mathcal{MF}_{\mathscr{K}}(\varphi)$ sends the Tate object $\mathbf{1}(n)$ in $\mathcal{HP}_K(\varphi)$ to $\mathbf{1}_{\mathcal{MF}}(n)$ in $\mathcal{MF}_{\mathscr{K}}(\varphi)$. This explains our notations for Tate objects in $\mathcal{HP}_K(\varphi)$ and Tate objects in $\mathcal{MF}_{\mathscr{K}}(\varphi)$.

2.4.4

For the rest of this section, we outline the results from [52] which are relevant to this work. Let $N_\nabla = -u\lambda\frac{d}{du}$, a $\mathscr{K}_0$-linear derivation on $\mathcal{O}_{\mathbf{\Delta}}$. We have an equality $N_\nabla \circ \sigma = p\frac{\mathcal{P}(u)}{\mathcal{P}(0)}\cdot(\sigma \circ N_\nabla)$. For a vector bundle $\mathcal{M}$ on $\mathbf{\Delta}$ (i.e., a finite free $\mathcal{O}_{\mathbf{\Delta}}$-module $\mathcal{M}$), a *differential operator* $N_\nabla^{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ *over* N_∇ is a $\mathscr{K}_0$-linear map such that for any $f \in \mathcal{O}_{\mathbf{\Delta}}$ and $m \in \mathcal{M}$ we have the "Leibnitz rule" $N_\nabla^{\mathcal{M}}(f\cdot m) = N_\nabla(f)\cdot m + f\cdot N_\nabla^{\mathcal{M}}(m)$. Giving such an $N_\nabla^{\mathcal{M}}$ is equivalent to giving a logarithmic connection $\nabla^{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_{\mathbf{\Delta}}} \Omega_{\mathbf{\Delta}}[\frac{1}{u\lambda}]$, as follows: for a given $N_\nabla^{\mathcal{M}}$, set $\nabla^{\mathcal{M}}(m) := N_\nabla^{\mathcal{M}}(m) \otimes \left(-\frac{du}{u\lambda}\right)$; for a given $\nabla^{\mathcal{M}}$, define

$$N_\nabla^{\mathcal{M}} : \mathcal{M} \xrightarrow{\nabla^{\mathcal{M}}} M \otimes_{\mathcal{O}_{\mathbf{\Delta}}} \Omega_{\mathbf{\Delta}}[1/u\lambda] \xrightarrow{\mathrm{id}\otimes N_\nabla} \mathcal{M},$$

where $N_\nabla : \Omega_{\mathbf{\Delta}} \to \mathcal{O}_{\mathbf{\Delta}}$ denotes the map $\omega \mapsto -\omega\lambda\frac{\omega}{du}$ induced from the derivation $N_\nabla = -u\lambda\frac{d}{du}$ by the universal property of $\Omega_{\mathbf{\Delta}}$.

Now, we consider $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ equipped with a differential operator $N_\nabla^{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ over N_∇ which satisfies $N_\nabla^{\mathcal{M}} \circ \varphi_{\mathcal{M}} = p\frac{\mathcal{P}(u)}{\mathcal{P}(0)}\cdot(\varphi_{\mathcal{M}} \circ N_\nabla^{\mathcal{M}})$; or equivalently, a logarithmic connection $\nabla^{\mathcal{M}}$ which commutes with $\varphi_{\mathcal{M}}$. Now, it follows from the "Leibnitz rule" that $N_\nabla^{\mathcal{M}}(u\cdot m) \in u\cdot\mathcal{M}$ for any $m \in \mathcal{M}$, so the reduction of $N_\nabla^{\mathcal{M}}$ modulo $u\cdot\mathcal{M}$ makes sense. We put $N := N_\nabla^{\mathcal{M}} \bmod u\cdot\mathcal{M}$, and clearly it satisfies $N \circ \bar{\varphi} = p\bar{\varphi} \circ N$, where $\bar{\varphi} : \mathcal{M}/u\mathcal{M} \to \mathcal{M}/u\mathcal{M}$ is the reduction of φ modulo $u\mathcal{M}$. Let $\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi, N_\nabla)$ be the category of such "(φ, N_∇)-modules" $(\mathcal{M}, \varphi_{\mathcal{M}}, N_\nabla^{\mathcal{M}})$, and $\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi, N_\nabla; N = 0)$ the full subcategory of $\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi, N_\nabla)$ whose objects satisfy

$N = 0$. In terms of the logarithmic connection, $N = 0$ means that the pole of $\nabla^{\mathcal{M}}$ at $u = 0$ can be removed.

Theorem 2.4.5. *[52, §1.2] There exist quasi-inverse equivalences of* $\otimes$*-categories* $\underline{\mathcal{M}}^{\mathcal{MF}} : \mathcal{MF}_{\mathscr{K}}^{\geqslant 0}(\varphi, N) \to \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi, N_{\nabla})$ *and* $\underline{D}^{\mathcal{MF}} : \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi, N_{\nabla}) \to \mathcal{MF}_{\mathscr{K}}^{\geqslant 0}(\varphi, N)$, *which restricts to equivalences of categories* $\underline{\mathcal{M}}^{\mathcal{MF}} : \mathcal{MF}_{\mathscr{K}}^{\geqslant 0}(\varphi) \to \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi, N_{\nabla}; N = 0)$ *and* $\underline{D}^{\mathcal{MF}} : \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi, N_{\nabla}; N = 0) \to \mathcal{MF}_{\mathscr{K}}^{\geqslant 0}(\varphi)$. *Under these equivalences of categories, filtered* (φ, N)*-modules (respectively, filtered isocrystals) with Hodge-Tate weights in* $[0, h]$ *corresponds to the* (φ, N_{∇})*-vector bundles (respectively, with* $N = 0$*) of* $\mathcal{P}$*-height* $\leqslant h$.

In order for this equivalence of categories to be useful, we need to be able to identify the essential image of weakly admissible objects in $\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi, N_{\nabla}; N = 0)$ and $\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi, N_{\nabla})$.

Theorem 2.4.6. *[52, §1.3] A filtered* (φ, N)*-module* $D \in \mathcal{MF}_{\mathscr{K}}^{\geqslant 0}(\varphi, N)$ *is weakly admissible if and only if there exists* $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ *such that* $\mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathfrak{S}} \mathfrak{M} \cong \underline{\mathcal{M}}^{\mathcal{MF}}(D)$.

The proof makes a crucial use of Kedlaya's slope filtration theorem. The proofs can be found in [46], [48], and [49]. The notion of slope for an étale φ-module over $\mathcal{R}$ is reviewed in §4.1 below.

One can improve the statement of the theorem, using the following results.

1. The functor $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}] \to \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$, $\mathfrak{M} \twoheadrightarrow \mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathbf{\Delta}}$ is fully faithful (and the essential image exactly consists of the object which are "pure of slope 0" in the sense of Kedlaya). In other words, the φ-stable $\mathfrak{S}$-lattice[11] $\mathfrak{M}$ in $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ is unique up to isogeny if exists. See [52, Lemma 1.3.13], which is also proved in Proposition 4.3.3 in this paper.

[11] In this paper, a lattice is always assumed to be locally free of constant rank.

2. The forgetful functor $\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi, N_\nabla; N = 0) \to \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$; $(\mathcal{M}, \varphi_{\mathcal{M}}, N_\nabla^{\mathcal{M}}) \mapsto (\mathcal{M}, \varphi_{\mathcal{M}})$ is fully faithful, and the essential image has a description in terms of a certain singular connection given by a concrete formula being logarithmic. See [52, Lemma 1.3.10] for the proof. We comment on this in more detail later in §5.2.12.

Combining above results, we obtain the following corollary.

Corollary 2.4.7. *Let D be a weakly admissible filtered (φ, N)-module with non-negative Hodge-Tate weights, and let $\underline{\mathfrak{M}}(D) := \mathfrak{M}[\frac{1}{p}]$ where $\mathfrak{M}$ is a φ-stable $\mathfrak{S}$-lattice in $\underline{\mathcal{M}}^{\mathcal{MF}}(D)$, whose existence is guaranteed by Theorem 2.4.6. This assignment defines a functor of $\otimes$-categories $\underline{\mathfrak{M}} : \mathcal{MF}_{\mathscr{K}}^{wa,\geqslant 0}(\varphi, N) \to \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}]$, which restricts to a fully faithful functor on $\mathcal{MF}_{\mathscr{K}}^{wa,\geqslant 0}(\varphi)$.*

Furthermore, $\underline{\mathfrak{M}}$ induces an equivalence of categories between objects of rank 1 *and between objects "of Barsotti-Tate type," i.e., $\mathcal{MF}_{\mathscr{K}}^{wa,[0,1]}(\varphi) \xrightarrow{\sim} \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}[\frac{1}{p}]$.*

The failure of the full faithfulness of $\underline{\mathfrak{M}}$ on $\mathcal{MF}_{\mathscr{K}}^{wa,\geqslant 0}(\varphi, N)$ is exactly because $\mathfrak{M}$ "forgets" the monodromy operator N. See [52, Corollary 1.3.15]. The failure of the essential surjectivity, if it occurs, comes from the step where we forgets the differential operator $N_\nabla^{\mathcal{M}}$. In fact, it is hard to expect to have any more general essential surjectivity result than the above corollary.[12] But the essential image of $\mathcal{MF}_{\mathscr{K}}^{wa,\geqslant 0}(\varphi)$ under $\underline{\mathfrak{M}}$ has a simple discription. See [52, Lemma 1.3.10] and Proposition 5.2.13 of this paper.

While it is hard to associate to a filtered isocrystal (or a filtered (φ, N)-module) an integral structure which corresponds to a $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable $\mathbb{Z}_p$-lattice in a crystalline representation, an object in the target category $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}]$ has an obvious notion

[12] See §11.3.13 which indicates that $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-deformation spaces of $\mathcal{P}$-height $\leqslant h$ usually have bigger dimension than crystalline or semi-stable deformation spaces with Hodge-Tate weights in $[0, h]$.

of "integral structure," namely a choice of φ-stable $\mathfrak{S}$-lattice $\mathfrak{M}$ in the isogeny class $\mathfrak{M}[\frac{1}{p}] \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}]$. To interpret the meaning of this integral structure, we now return to the "Galois representation" side. We first need the following result.

Proposition 2.4.8. *[52, proposition 2.1.12] The functor $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi) \to \underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)$, defined by $\mathfrak{M} \mapsto \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$, is fully faithful.*

The proof of this innocent-looking proposition requires all the equivalences of categories we discussed above.

We have an anti-equivalence of categories $\underline{T}^*_{\mathcal{E}}$ from the category of étale φ-modules free over $\mathfrak{o}_{\mathcal{E}}$ into the category of finite free $\mathbb{Z}_p$-modules with continuous $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty} \cong \boldsymbol{\mathcal{G}}_K$-action. (See [31, §A.1.2] or Proposition 5.1.7 of this paper.) So we can associate to $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ a lattice $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representation $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) := \underline{T}^*_{\mathcal{E}}(\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M})$. The previous proposition shows that the contravariant functor $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi) \to \mathrm{Rep}^{\mathrm{free}}_{\mathbb{Z}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$ is fully faithful.

Let $\mathrm{Rep}^{\geqslant 0}_{\mathbb{Z}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$ denote the essentially image of $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ under the fully faithful functor $\underline{T}^*_{\mathfrak{S}}$, and $\mathrm{Rep}^{\geqslant 0}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$ denote the isogeny category of $\mathrm{Rep}^{\geqslant 0}_{\mathbb{Z}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$; i.e., the category of $\mathbb{Q}_p$-representations V of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ such that there exists a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable lattice $T \in \mathrm{Rep}^{\geqslant 0}_{\mathbb{Z}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$. Clearly, we have an anti-equivalence of categories $\underline{V}^*_{\mathfrak{S}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}] \to \mathrm{Rep}^{\geqslant 0}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$. It can be seen, with some work, that if $V \in \mathrm{Rep}^{\geqslant 0}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$, then any $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable lattice in V belongs to $\mathrm{Rep}^{\geqslant 0}_{\mathbb{Z}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$. More precisely, we have the following proposition which is proved in Proposition 5.2.9.

Proposition 2.4.9. *If $V = \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})[\frac{1}{p}]$, then the set of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable lattices T' in V is naturally in inclusion reversing bijection with φ-stable $\mathfrak{S}$-lattices $\mathfrak{M}' \subset \mathfrak{M}[\frac{1}{p}]$, and $\mathfrak{M}'$ is automatically of $\mathcal{P}$-height $\leqslant h$ if $\mathfrak{M}$ is.*

We now discuss applications of Corollary 2.4.7 to semi-stable and crystalline $\mathcal{G}_{\mathscr{K}}$-representations. Consider the composition of functors

$$\mathrm{Rep}^{\mathrm{st},\geqslant 0}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}}) \xrightarrow[\cong]{\underline{D}^*_{\mathrm{st}}} \mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi, N) \xrightarrow{\underline{\mathfrak{M}}} \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[1/p] \tag{2.4.9.1}$$

$$\xrightarrow[\cong]{\underline{V}^*_{\mathfrak{S}}} \mathrm{Rep}^{\geqslant 0}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}_\infty}) \hookrightarrow \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}_\infty}),$$

where $\mathrm{Rep}^{\mathrm{st},\geqslant 0}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}})$ is the category of semistable representations with non-negative Hodge-Tate weights, and the second arrow $\underline{\mathfrak{M}}$ is as defined in Corollary 2.4.7. All the arrows become fully faithful[13] when we replace $\underline{D}^*_{\mathrm{st}}$ by $\underline{D}^*_{\mathrm{cris}} : \mathrm{Rep}^{\mathrm{cris},\geqslant 0}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}}) \xrightarrow{\sim} \mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi)$, hence the composition is a fully faithful functor $\mathrm{Rep}^{\mathrm{cris},\geqslant 0}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}}) \rightarrow \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}_\infty})$.

On the other hand, we also have another functor $\mathrm{Rep}^{\mathrm{st},\geqslant 0}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}}) \rightarrow \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}_\infty})$ obtained by restricting a semi-stable $\mathcal{G}_{\mathscr{K}}$-representation to a $\mathcal{G}_{\mathscr{K}_\infty}$-representation.

Theorem 2.4.10. *[52, Proposition 2.1.5] The two functors*

$$\mathrm{Rep}^{\mathrm{st},\geqslant 0}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}}) \rightarrow \mathrm{Rep}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}_\infty}),$$

one of which is the restriction to $\mathcal{G}_{\mathscr{K}_\infty}$ and the other of which is the composition of functors from (2.4.9.1)*, are naturally isomorphic. In particular, the functor obtained by restricting to a $\mathcal{G}_{\mathscr{K}_\infty}$-representation is fully faithful on $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}})$. Furthermore, the restriction to $\mathcal{G}_{\mathscr{K}_\infty}$ of a semi-stable $\mathbb{Q}_p$-representation of $\mathcal{G}_{\mathscr{K}}$ with non-negative Hodge-Tate weights belongs to $\mathrm{Rep}^{\geqslant 0}_{\mathbb{Q}_p}(\mathcal{G}_{\mathscr{K}_\infty})$.*

To digress, note also that Theorem 2.4.2 follows from above; it has been well-known that the proof of Theorem 2.4.2 reduces to showing that a certain inequality of dimensions is in fact an equality, which directly follows from above.

[13] It is not a deep theorem that $\underline{D}^*_{\mathrm{cris}}$ and $\underline{D}^*_{\mathrm{st}}$ are fully faithful; the hard part is the essential surjectivity, which requires Theorem 2.4.2

Let D be a weakly admissible filtered (φ, N)-module with non-negative Hodge-Tate weights, and choose $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ so that $\mathfrak{M}[\frac{1}{p}] = \underline{\mathfrak{M}}(D)$(Corollary 2.4.7). The above theorem, combined with Proposition 2.4.9, tells us that the choice of $\mathfrak{M}$ exactly corresponds to the $\mathcal{G}_{\mathscr{K}_\infty}$-stable $\mathbb{Z}_p$-lattice of the semi-stable representation $V_{\mathrm{st}}^*(D)$.

In using the fully faithful functor $\underline{\mathfrak{M}} : \mathcal{MF}_{\mathscr{K}}^{wa,\geqslant 0}(\varphi) \to \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}]$ to study crystalline representations, we face two major roadblocks. First, $\underline{\mathfrak{M}}$ is not essentially surjective. Second, a choice of $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ in the isogeny class $\mathfrak{M}[\frac{1}{p}] = \underline{\mathfrak{M}}(D)$ corresponds to a $\mathcal{G}_{\mathscr{K}_\infty}$-stable lattice of $V_{\mathrm{cris}}^*(D)$ which is not necessarily $\mathcal{G}_{\mathscr{K}}$-stable. On the other hand, for crystalline $\mathcal{G}_{\mathscr{K}}$-representations with Hodge-Tate weights in $[0, 1]$, we have the following result which completely removes these roadblocks when $p > 2$.

Theorem 2.4.11. *[52, §2.2]*

1. *(Kisin's classification of Barsotti-Tate groups) If $p > 2$, then there exists an anti-equivalence of categories $\underline{G}^*$ from $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}$ to the category of Barsotti-Tate groups over $\mathfrak{o}_{\mathscr{K}}$. Furthermore, for any $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}$ we have a $\mathcal{G}_{\mathscr{K}_\infty}$-equivariant isomorphism $T_p\left(\underline{G}^*(\mathfrak{M})\right) \cong \underline{T}_{\mathfrak{S}}^*(\mathfrak{M})$.*

2. *There exists an anti-equivalence of categories between the isogeny category of Barsotti-Tate groups over $\mathfrak{o}_{\mathscr{K}}$ and $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}[\frac{1}{p}]$. Furthermore, for an object $\mathfrak{M}[\frac{1}{p}] \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}[\frac{1}{p}]$ and an isogeny class $[G]$ containing a Barsotti-Tate group $G_{/\mathfrak{o}_{\mathscr{K}}}$ which correspond to each other under the anti-equivalence of categories $\underline{G}^*$, we have a $\mathcal{G}_{\mathscr{K}_\infty}$-equivariant isomorphism $V_p(G) \cong \underline{V}_{\mathfrak{S}}^*(\mathfrak{M}[\frac{1}{p}])$. In particular, for any crystalline $\mathcal{G}_{\mathscr{K}}$-representation V, there exists a Barsotti-Tate group $G_{/\mathfrak{o}_{\mathscr{K}}}$ such that $V \cong V_p(G)$ as a $\mathcal{G}_{\mathscr{K}}$-representations.*

One also has a covariant version of Kisin's classification, by taking duality on Barsotti-Tate groups (or equivalently, by taking suitable duality on $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}$, which will be defined in Definition 8.3.2). Theorem 2.4.11(1) was originally conjectured by Breuil in [11] for *all* primes p including $p = 2$. For $p > 2$ Kisin [52, §2.2, § A] proved the conjecture. Allowing $p = 2$, Kisin [53] proves this conjecture for *connected* Barsotti-Tate groups using a certain full subcategory of $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}$; his proof rests on Zink's theory of windows and displays. (Under the *contravariant* correspondences, $\underline{G}^*(\mathfrak{M})$ is connected if $\varphi_{\mathfrak{M}}$ is "topologically nilpotent.") It is conjectured that Kisin's classification of Barsotti-Tate groups should hold for $p = 2$ without the connectedness assumption.

As a consequence, if $p > 2$ then any $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable $\mathbb{Z}_p$-lattice of crystalline representation with Hodge-Tate weights in $[0, 1]$ is $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable. Therefore $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}$ classifies $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable $\mathbb{Z}_p$-lattices crystalline representations with Hodge-Tate weights in $[0, 1]$.

2.4.12 Overview of §3–§7

In this work, we shall study $\mathrm{Rep}^{\geqslant 0}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$, which is classified by $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}]$. As stated above, we have a fully faithful functor $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}] \to \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$, defined by the scalar extension $\mathfrak{S} \to \mathcal{O}_{\boldsymbol{\Delta}}$, where the essential image is the full subcategory of φ-vector bundles "pure of slope 0" in the sense of Kedlaya. At the first part of what follows, we shall prove results analogous to [52, §1], but without using the differential operator $N^{\mathcal{M}}_{\nabla}$ (which is not available in the equi-characteristic case).

The role of $N^{\mathcal{M}}_{\nabla}$ is quite limited in [52, §1]. There are two places where $N^{\mathcal{M}}_{\nabla}$ is used, one of which is avoidable and the other not. One place where $N^{\mathcal{M}}_{\nabla}$ is used is the "Dwork's trick" argument in the proof of Theomem 2.4.6. We carry out this step only using the Frobenius map φ; see Proposition 4.2.1. (An analogous situation can be found in [47], which carries out the "Dwork's trick" step [20] in the proof of de Jong's theorem only using the Frobenius structure.)

Kisin [52, (1.2)] crucially used $N_\nabla^{\mathcal{M}}$ in order to show that $\underline{D}^{\mathcal{MF}}$ and $\underline{\mathcal{M}}^{\mathcal{MF}}$ are quasi-inverse equivalences of categories between $\mathcal{MF}_{\mathscr{K}}^{\geqslant 0}(\varphi)$ and $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi, N_\nabla; N = 0)$. In fact, we should not get equivalences of categories between $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$ and $\mathcal{MF}_{\mathscr{K}}^{\geqslant 0}(\varphi)$, because the forgetful functor $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi, N_\nabla; N = 0) \to \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$ is not an equivalences of categories. On the other hand, the construction of $\underline{D}^{\mathcal{MF}}$ does *not* involve $N_\nabla^{\mathcal{M}}$ (more precisely, the construction only uses that $N_\nabla^{\mathcal{M}} \bmod u{\cdot}\mathcal{M} = 0$), and the construction of the filtration from $\mathcal{M} \in \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$ suggests that one may be able to factor $\underline{D}^{\mathcal{MF}}$ as $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi) \to \mathcal{HP}_K^{\geqslant 0}(\varphi) \xrightarrow{\mathcal{F}} \mathcal{MF}_{\mathscr{K}}^{\geqslant 0}(\varphi)$ where the second map is defined in §2.4.3. See [52, (1.2.7)] for the construction. In fact, this idea works and we obtain quasi-inverse equivalences of categories $\underline{D}$ and $\underline{\mathcal{M}}$ between $\mathcal{HP}_K^{\geqslant 0}(\varphi)$ and $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$. This is proved in Propositions 3.2.1 and 3.2.5.

Next, we will interpret the weak admissibility of $(D, \varphi, \Lambda) \in \mathcal{HP}_K^{\geqslant 0}(\varphi)$ in terms of $\underline{\mathcal{M}}(D, \varphi, \Lambda)$ being pure of slope 0 in the sense of Kedlaya [46, 48, 49]. But recall that this full subcategory of pure slope 0 objects is equivalent to $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}]$, so we obtain an equivalence of categories $\underline{\mathbb{H}} : \mathcal{HP}_K^{wa,\geqslant 0}(\varphi) \xrightarrow{\sim} \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}]$. By composing with the anti-equivalence of categories $\underline{T}_{\mathfrak{S}}^* : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}] \to \mathrm{Rep}_{\mathbb{Q}_p}^{\geqslant 0}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$ we obtain an anti-equivalence of categories $\underline{V}_{\mathcal{HP}}^* : \mathcal{HP}_K^{wa,\geqslant 0}(\varphi) \xrightarrow{\sim} \mathrm{Rep}_{\mathbb{Q}_p}^{\geqslant 0}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$. This anti-equivalence of categories plays an important role in the study of deformations later in §XI.

Having eliminated the differential operators $N_\nabla^{\mathcal{M}}$, we now have a reasonable analogue for $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ by replacing various φ-modules with the analogous constructions for $\mathfrak{o}_0 = \mathbb{Z}_p$. In fact, most of the proofs work in this equi-characteristic analogue with few modifications. This equi-characteristic theory may be thought of as an "equi-characteristic analogue of Fontaine's p-adic Hodge theory," as observed by Genestier-Lafforgue [35] and Hartl [39, 41].

Remark 2.4.13 (The case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$). Instead of considering Hodge-Pink structures, one might want to consider the filtration on $D \otimes_{\mathscr{K}_0} \mathscr{K}$ obtained by reducing (2.3.3.1) modulo $\mathcal{P}(u)$, as in the p-adic case. In fact, one obtains the same Hodge-Pink weights and multiplicities using this filtration on $D \otimes_{\mathscr{K}_0} \mathscr{K}$. With the absence of the differential operator $N_\nabla^{\mathcal{M}}$ as in [52], however, it turns out that the category of isocrystals D with filtration on $D_{\mathscr{K}}$ does not have enough information to build an equivalence of categories with $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$. See also [39, Rmk 2.2.3] for more discussion on the inadequacy of "filtered φ-module" in the equi-characteristic setting.

CHAPTER III

Hodge-Pink theory and rigid analytic φ-vector bundles

In this chapter, we give construct a vector bundle over the open unit disk from an isocrystal with Hodge-Pink structure. The construction is closely related to Kisin's work [52, (1.2)] which was motivated by Berger's work [5, §II, III] in the (φ, Γ)-module setting. Our construction differs from Kisin's in that we work with Hodge-Pink structures instead of filtered (φ, N)-modules (hence the theory works in the equi-characteristic setting), and we use Frobenius structure but avoid differential structure.

3.1 Construction

Let $D := (D, \varphi, \Lambda) \in \mathcal{HP}_K^{\geqslant 0}(\varphi)$ throughout this chapter; i.e., we assume that all the Hodge-Pink weights for D are non-negative. We would like to construct a vector bundle $\underline{\mathcal{M}}(D) \in \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$ such that $\underline{\mathcal{M}}(D)/u\underline{\mathcal{M}}(D) \cong D$ and $\underline{\mathcal{M}}(D)^{\widehat{}}_{x_0} \cong \Lambda$.

We state the following classical lemma without proof, which will be useful:

Lemma 3.1.1. *Let $I \subset [0, 1)$ be a sub-interval, $\mathcal{M}$ be a finite free $\mathcal{O}_{\boldsymbol{\Delta}_I}$-module and $\mathcal{N} \subset \mathcal{M}$ be an $\mathcal{O}_{\boldsymbol{\Delta}_I}$-submodule. Then $\mathcal{N} \subset \mathcal{M}$ is closed if and only if $\mathcal{N}$ is finite free.*

Proof. The hard part is "only if" direction, which is reduced to the case when $\mathcal{M}$

is free of rank 1 by [46, Lemma 2.4]. This case is handled by [57, (7.3)]. The proof crucially uses that $\mathcal{O}_{\mathbf{\Delta}_I}$ is a Bézout domain, which uses the discrete valuation of $\mathscr{K}_0$ (or more generally, the spherical completeness). $\square$

3.1.2

By §2.2.4, the scalar extension $\mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} D$ is an étale φ-module over $\mathcal{O}_{\mathbf{\Delta}}$. For each non-negative integer n, define

$$\begin{aligned} \iota_n \quad : \quad & \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} D \overset{\mathrm{id} \otimes \varphi_D^{-n}}{\longrightarrow} \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} (\sigma^{*n} D) \hookrightarrow \mathcal{O}_{\widehat{\mathbf{\Delta}, x_n}} \otimes_{\mathscr{K}_0} (\sigma^{*n} D) \\ \cong \quad & \mathcal{O}_{\widehat{\mathbf{\Delta}, x_n}} \otimes_{\sigma^n, \mathcal{O}_{\widehat{\mathbf{\Delta}, x_0}}} (\mathcal{O}_{\widehat{\mathbf{\Delta}, x_0}} \otimes_{\mathscr{K}_0} D), \end{aligned}$$

where $\sigma^n : \mathcal{O}_{\widehat{\mathbf{\Delta}, x_0}} \hookrightarrow \mathcal{O}_{\widehat{\mathbf{\Delta}, x_n}}$ is induced by $\sigma^n : \mathcal{O}_{\mathbf{\Delta}} \to \mathcal{O}_{\mathbf{\Delta}}$, as discussed at (2.1.3.3). We set $\mathcal{D} := \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} D$.

Now we extend ι_n to the following map:

(3.1.2.1) $$\iota_n : \mathcal{D}[1/\lambda] \to \mathcal{O}_{\widehat{\mathbf{\Delta}, x_n}}[1/\lambda] \otimes_{\sigma^n, \mathcal{O}_{\widehat{\mathbf{\Delta}, x_0}}} \widehat{\mathcal{D}}_{x_0}.$$

The target of this map carries the tensor product filtration, where the second factor $\widehat{\mathcal{D}}_{x_0}$ carries the filtration coming from the Hodge-Pink structure Λ, as defined in (2.3.3.1), and the first factor $\mathcal{O}_{\widehat{\mathbf{\Delta}, x_n}}[\frac{1}{\lambda}]$ has a decreasing filtration defined by $\lambda^i \cdot \mathcal{O}_{\widehat{\mathbf{\Delta}, x_n}} = (\sigma^n \mathcal{P}(u))^i \cdot \mathcal{O}_{\widehat{\mathbf{\Delta}, x_n}}$. Also, observe that the target of this map is naturally isomorphic to $\widehat{\mathcal{D}}_{x_n}[\frac{1}{\lambda}]$ using $\varphi_D^{-n} : D \xrightarrow{\sim} \sigma^* n D$ over $\mathscr{K}_0$ (i.e., *not* respecting how D "naturally" sits in each if $n > 0$), where $\widehat{\mathcal{D}}_{x_n}$ is the completed stalk of $\mathcal{D}$ at $x_n \in \mathbf{\Delta}$.

3.1.3

Set

$$\begin{aligned} \underline{\mathcal{M}}_n(D) \quad &:= \quad (\iota_n)^{-1}\big(\operatorname{Fil}^0(\widehat{\mathcal{D}}_{x_n}[1/\lambda])\big) \\ \underline{\mathcal{M}}(D) \quad &:= \quad \bigcap_{n \geq 0} \underline{\mathcal{M}}_n(D) = \{x \in \mathcal{D}[1/\lambda] : \ \iota_n(x) \in \operatorname{Fil}^0(\widehat{\mathcal{D}}_{x_n}[1/\lambda]), \ \forall n \geq 0\}, \end{aligned}$$

Let h be the maximum among Hodge-Pink weights for (D,Λ). Then we have $\mathcal{D}\subset\underline{\mathcal{M}}(D)\subset\underline{\mathcal{M}}_n(D)\subset(\lambda^{-h}\cdot\mathcal{D})$. Clearly each $\underline{\mathcal{M}}_n(D)$, hence $\underline{\mathcal{M}}(D)$, is closed in $\lambda^{-h}\cdot\mathcal{D}$, so by Lemma 3.1.1,both $\underline{\mathcal{M}}_n(D)$ and $\underline{\mathcal{M}}(D)$ are finite free $\mathcal{O}_{\boldsymbol{\Delta}}$-modules.

The inclusions induce isomorphisms $\mathcal{D}[\frac{1}{\lambda}]\xrightarrow{\sim}\underline{\mathcal{M}}(D)[\frac{1}{\lambda}]\xrightarrow{\sim}\underline{\mathcal{M}}_n(D)[\frac{1}{\lambda}]$. In other words, $\mathcal{D}$, $\underline{\mathcal{M}}_n(D)$, and $\underline{\mathcal{M}}(D)$, viewed as coherent sheaves on $\boldsymbol{\Delta}$, are naturally isomorphic outside the zero locus $\{x_n\}_{n\geq 0}$ of λ. To study the local behavior of $\underline{\mathcal{M}}(D)$ near x_n, we look at the completed stalks and make use of the following fact: the inclusion $\underline{\mathcal{M}}(D)\subset\underline{\mathcal{M}}_n(D)$ induces an isomorphism near x_n, and ι_n induces the isomorphism below, which can be seen from the definition.

$$\underline{\mathcal{M}}(D)\widehat{_{x_n}}\xrightarrow{\sim}\underline{\mathcal{M}}_n(D)\widehat{_{x_n}}\xrightarrow{\sim}\mathrm{Fil}^0(\widehat{\mathcal{D}}_{x_n}[1/\lambda]). \tag{3.1.3.1}$$

In particular $\underline{\mathcal{M}}(D)\widehat{_{x_0}}=\Lambda$ inside of $\widehat{\mathcal{D}}_{x_0}[\frac{1}{\lambda}]$, by Lemma 2.3.5.

By §2.2.4(2), the natural φ-module structure on $\mathcal{D}:=\mathcal{O}_{\boldsymbol{\Delta}}\otimes_{\mathscr{K}_0}D$ is étale since D is an étale φ-module. So the $\mathcal{O}_{\boldsymbol{\Delta}}$-linear isomorphism $\varphi_{\mathcal{D}}:\sigma^*\mathcal{D}\to\mathcal{D}$ induces an $\mathcal{O}_{\boldsymbol{\Delta}}[\frac{1}{\lambda}]$-linear isomorphism $\varphi_{\mathcal{D}}[\frac{1}{\lambda}]:(\sigma^*\mathcal{D})[\frac{1}{\lambda}]\xrightarrow{\sim}\mathcal{D}[\frac{1}{\lambda}]$. We will prove that the $\mathcal{O}_{\boldsymbol{\Delta}}$-submodule $\underline{\mathcal{M}}(D)\subset\mathcal{D}[\frac{1}{\lambda}]$ is $\varphi_{\mathcal{D}}$-stable; i.e., $\sigma^*\big(\underline{\mathcal{M}}(D)\big)$ is carried into $\underline{\mathcal{M}}(D)$. Once this is done, we show that the induced φ-structure on $\underline{\mathcal{M}}(D)$ over $\mathcal{O}_{\boldsymbol{\Delta}}$ is of finite $\mathcal{P}$-height by "analytic-local" argument.

3.1.4 Rank-1 example

Before we move on, let us work out $\underline{\mathcal{M}}(D,\Lambda)$ when D is of rank 1 and the Hodge-Pink structure Λ is effective. We choose a $\mathscr{K}_0$-basis $\mathbf{e}\in D$, and write $\varphi(\sigma^*\mathbf{e})=\alpha_{\mathbf{e}}\cdot\mathbf{e}$ for some $\alpha_{\mathbf{e}}\in\mathscr{K}_0^{\times}$. Since $\Lambda=\mathcal{P}(u)^{-h}\widehat{\mathcal{D}}_{x_0}$ for some $h\geqslant 0$, we obtain $\underline{\mathcal{M}}_n(D,\Lambda)=\big(\sigma^n(\mathcal{P}(u))\big)^{-h}\mathcal{D}$ for all $n\geqslant 0$. Therefore, $\underline{\mathcal{M}}(D,\Lambda)=\lambda^{-h}\mathcal{D}$, which is stable under $\varphi_{\mathcal{D}[\frac{1}{\lambda}]}:\sigma^*\mathcal{D}[\frac{1}{\lambda}]\to\mathcal{D}[\frac{1}{\lambda}]$. We can also compute $\varphi_{\mathcal{D}}[\frac{1}{\lambda}]$ on $\underline{\mathcal{M}}(D,\Lambda)=\lambda^{-h}\mathcal{D}$ for the

$\mathcal{O}_{\mathbf{\Delta}}$-generator $\lambda^{-h}\mathbf{e}$, as follows (using the definition of λ in §2.1.3):

$$\varphi(\sigma^*(\lambda^{-h}\mathbf{e})) = \alpha_{\mathbf{e}} \left(\frac{\mathcal{P}(u)}{\mathcal{P}(0)}\right)^h \cdot(\lambda^{-h}\mathbf{e}). \tag{3.1.4.1}$$

If $\dim_{\mathscr{K}_0} D > 1$, then it may be much harder to compute $\underline{\mathcal{M}}(D,\Lambda)$ explicitly; $\underline{\mathcal{M}}(D,\Lambda)$ may not have a simple expression such as $\lambda^{-h}\mathcal{D}$.

Proposition 3.1.5. *Let $D := (D, \varphi_D, \Lambda) \in \mathcal{HP}_K^{\geqslant 0}(\varphi)$. Then, $\varphi_{\mathcal{D}}[\frac{1}{\lambda}] : (\sigma^*\mathcal{D})[\frac{1}{\lambda}] \xrightarrow{\sim} \mathcal{D}[\frac{1}{\lambda}]$ restricts to $\varphi : \sigma^*\underline{\mathcal{M}}(D) \to \underline{\mathcal{M}}(D)$. Furthermore, we have an isomorphism*

$$\operatorname{coker}\varphi \cong \Lambda/(\widehat{\mathcal{D}}_{x_0}) \cong \bigoplus_{w>0} (\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}/\mathcal{P}(u)^w)^{m_w}, \tag{3.1.5.1}$$

where the right side is a finite direct sum.

Upon verifying the proposition, we would obtain a *functor* $\underline{\mathcal{M}} : \mathcal{HP}_K^{\geqslant 0}(\varphi) \to \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ because one can check that if a $\mathscr{K}_0$-linear map $f : D \to D'$ respects Hodge-Pink structures on both sides then $\mathcal{O}_{\mathbf{\Delta}}[\frac{1}{\lambda}] \otimes f : \mathcal{D}[\frac{1}{\lambda}] \to \mathcal{D}'[\frac{1}{\lambda}]$ takes $\underline{\mathcal{M}}_n(D)$ into $\underline{\mathcal{M}}_n(D')$ for each $n \geq 0$, hence $\underline{\mathcal{M}}(D)$ into $\underline{\mathcal{M}}(D')$. The proposition also says that we can recover the Hodge-Pink type of an effective Hodge-Pink structure D from $\underline{\mathcal{M}}(D)$, since $\operatorname{coker}\varphi \cong \Lambda/\widehat{\mathcal{D}}_{x_0}$. (See Corollary 2.3.6.)

Proof. When $\mathfrak{o}_0 = \mathbb{Z}_p$, this proposition can be read off from the proof of [52, Lemma 1.2.2]. The same argument goes through with few modifications when $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. The statement can be checked locally at each point on $\mathbf{\Delta}$. Having that $\mathcal{D}[\frac{1}{\lambda}] = \underline{\mathcal{M}}(D)[\frac{1}{\lambda}]$, it is enough to verify the result locally at x_n, for each $n \geq 0$.

Let h be the maximum of the Hodge-Pink weights for D so that we have $\underline{\mathcal{M}}(D) \subset \underline{\mathcal{M}}_n(D) \subset \lambda^{-h}\cdot\mathcal{D}$. In (2.1.3.4) we have seen that $\sigma : \mathcal{O}_{\mathbf{\Delta}} \to \mathcal{O}_{\mathbf{\Delta}}$ induces an $\mathcal{O}_{\mathbf{\Delta}}$-isomorphism $\gamma_{n,1} : \sigma^*\mathcal{O}_{\widehat{\mathbf{\Delta}},x_n} \xrightarrow{\sim} \mathcal{O}_{\widehat{\mathbf{\Delta}},x_{n+1}}$, where $\sigma^*\mathcal{O}_{\widehat{\mathbf{\Delta}},x_n} := \mathcal{O}_{\mathbf{\Delta}} \otimes_{\sigma,\mathcal{O}_{\mathbf{\Delta}}} \mathcal{O}_{\widehat{\mathbf{\Delta}},x_n}$. We have the following commutative diagram which shows how ι_n and φ interact.

$$
(3.1.5.2)\qquad
\begin{array}{ccc}
\lambda^{-h}\cdot(\sigma^*\mathcal{D}) & \xrightarrow{\ \sigma^*\iota_n\ } & \sigma^*\Big((\sigma^n\mathcal{P}(u))^{-h}\cdot\big(\mathcal{O}_{\widehat{\mathbf{\Delta},x_n}}\otimes_{\mathscr{K}_0}(\sigma^{*n}D)\big)\Big) \\
 & & \big\downarrow \simeq \\
{\scriptstyle \varphi_{\mathcal{D}}=\mathrm{id}\otimes\varphi_D}\ \big\downarrow \simeq & & (\sigma^{n+1}\mathcal{P}(u))^{-h}\cdot\Big(\big(\sigma^*\mathcal{O}_{\widehat{\mathbf{\Delta},x_n}}\big)\otimes_{\mathscr{K}_0}\big(\sigma^{*n}D\big)\Big) \\
 & & \simeq \big\downarrow {\scriptstyle \gamma_{n,1}\otimes\mathrm{id}} \\
\lambda^{-h}\cdot\mathcal{D} & \xrightarrow{\ \iota_{n+1}\ } & (\sigma^{n+1}\mathcal{P}(u))^{-h}\cdot\Big(\sigma^{*n+1}\big(\mathcal{O}_{\widehat{\mathbf{\Delta},x_{n+1}}}\otimes_{\mathscr{K}_0}D\big)\Big)
\end{array}\quad .
$$

Choose an interval I_n, so that $\mathbf{\Delta}_{I_n}$ contains x_n but does not contain x_m for $m \neq n$. We can further assume that $(I_n)^{1/q} = I_{n+1}$ for all $n \geq 0$, so that we have $\sigma : \mathcal{O}_{\mathbf{\Delta}_{I_n}} \to \mathcal{O}_{\mathbf{\Delta}_{I_{n+1}}}$ for each n. Then, since $(\lambda^{-h}\cdot\mathcal{D})/\underline{\mathcal{M}}_n(D)$ is supported on the (discrete) zero locus of λ, we have the following exact sequence

$$
(3.1.5.3)\qquad 0 \to \underline{\mathcal{M}}_n(D)_{I_n} \xrightarrow{\jmath_n} \lambda^{-h}\cdot\mathcal{D}_{I_n} \xrightarrow{\iota_n} \frac{(\sigma^n\mathcal{P}(u))^{-h}\cdot\widehat{\mathcal{D}}_{x_n}}{\mathrm{Fil}^0(\widehat{\mathcal{D}}_{x_n}[1/\lambda])} \to 0
$$

Indeed, the cokernel of $\jmath_n$ is supported at x_n from the choice of I_n, so the right exactness follows from the isomorphism (3.1.3.1) and the definition of λ. Let us denote by Q_n the cokernel of $\jmath_n$.

Combining (3.1.5.2) and (3.1.5.3), we obtain the following commutative diagram of coherent sheaves on $\mathbf{\Delta}_{I_{n+1}}$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \big(\sigma^*\underline{\mathcal{M}}_n(D)\big)_{I_{n+1}} & \longrightarrow & \big(\lambda^{-h}\cdot(\sigma^*\mathcal{D})\big)_{I_n} & \xrightarrow{\ \sigma^*\iota_n\ } & (\sigma^*Q_n)_{I_{n+1}} & \longrightarrow & 0 \\
 & & \exists\,\big\downarrow & & {\scriptstyle (\varphi_{\mathcal{D}})_{I_{n+1}}}\big\downarrow\simeq & & {\scriptstyle \gamma_{n,1}\otimes\mathrm{id}}\big\downarrow\simeq & & \\
0 & \longrightarrow & \underline{\mathcal{M}}_n(D)_{I_{n+1}} & \longrightarrow & \lambda^{-h}\cdot\mathcal{D}_{I_{n+1}} & \xrightarrow{\ \iota_{n+1}\ } & (Q_n)_{I_{n+1}} & \longrightarrow & 0
\end{array}
$$

Hence, the left vertical arrow induced by $\varphi_{\mathcal{D}}[\frac{1}{\lambda}]$ exists and is an isomorphism. Since the inclusion $\underline{\mathcal{M}}(D) \subset \underline{\mathcal{M}}_n(D)$ induces an isomorphism at x_n (as may be seen on completed stalks using (3.1.3.1)), we conclude that $\varphi_{\mathcal{D}} : \sigma^*(\lambda^{-h}\cdot\mathcal{D}) \to \lambda^{-h}\cdot\mathcal{D}$ restricts to an isomorphism $\varphi_{x_{n+1}} : (\sigma^*\underline{\mathcal{M}}(D))_{x_{n+1}} \xrightarrow{\sim} \underline{\mathcal{M}}(D)_{x_{n+1}}$ for all $n \geq 0$.

Now, it is left to verify the lemma at x_0. From the isomorphism $\mathcal{D}[\frac{1}{\lambda}] \xrightarrow{\sim} \underline{\mathcal{M}}(D)[\frac{1}{\lambda}]$ induced by the natural inclusion, we obtain $(\sigma^*\mathcal{D})[\frac{1}{\sigma(\lambda)}] \xrightarrow{\sim} (\sigma^*\underline{\mathcal{M}}(D))[\frac{1}{\sigma(\lambda)}]$. Since $\sigma(\lambda)$ does *not* vanish at $x_0 \in \mathbf{\Delta}$ (by definition or by (2.1.3.2)), the natural map $(\sigma^*\mathcal{D})_{x_0} \to \big(\sigma^*\underline{\mathcal{M}}(D)\big)_{x_0}$ induced by the natural inclusion is an isomorphism. So we have the following maps:

$$\big(\sigma^*\underline{\mathcal{M}}(D)\big)_{x_0} \xleftarrow{\simeq} (\sigma^*\mathcal{D})_{x_0} \xrightarrow[\cong]{\varphi_{\mathcal{D}}} (\mathcal{D})_{x_0} \hookrightarrow \underline{\mathcal{M}}(D)_{x_0}$$

This proves that $\varphi_{\mathcal{D}}[\frac{1}{\lambda}] : (\sigma^*\mathcal{D})[\frac{1}{\lambda}] \to \mathcal{D}[\frac{1}{\lambda}]$ restricts to a map $\varphi : \sigma^*\underline{\mathcal{M}}(D) \to \underline{\mathcal{M}}(D)$, and that

$$\operatorname{coker}\varphi \xrightarrow{\sim} \underline{\mathcal{M}}(D)_{\widehat{x_0}} / \widehat{\mathcal{D}}_{x_0} \xrightarrow{\sim} \Lambda/\widehat{\mathcal{D}}_{x_0},$$

where the second isomorphism follows from $n = 0$ case of (3.1.3.1) and from Lemma 2.3.5. This proves the isomorphism (3.1.5.1). □

Proposition 3.1.6. *The functor $\underline{\mathcal{M}} : \mathcal{HP}_K^{\geqslant 0}(\varphi) \to \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ is an exact functor of $\otimes$-categiries. In other words, $\underline{\mathcal{M}}$ satisfies the following properties.*

1. *$\underline{\mathcal{M}}$ commutes with $\otimes$-products.*
2. *$\underline{\mathcal{M}}$ takes a short exact sequence of the source category into that of the target category.*

Proof. Since $\underline{\mathcal{M}}(D)$ is a coherent sheaf on $\mathbf{\Delta}$, it suffices to check these properties on completed stalks at each point of $\mathbf{\Delta}$. We also have $\mathcal{D}[\frac{1}{\lambda}] = \underline{\mathcal{M}}(D)[\frac{1}{\lambda}]$.

For $D, D' \in \mathcal{HP}_K^{\geqslant 0}(\varphi)$, we obtain a natural map $\underline{\mathcal{M}}(D) \otimes \underline{\mathcal{M}}(D') \to \underline{\mathcal{M}}(D \otimes D')$ from the universal property of $\otimes$-product, which is clearly an isomorphism outside $\{x_n\}$. Now we use (3.1.3.1) to conclude that this natural map is an isomorphism at x_n for each n.

For a short exact sequence $0 \to (D', \Lambda') \to (D, \Lambda) \to (D'', \Lambda'') \to 0$ in $\mathcal{HP}_K^{\geqslant 0}(\varphi)$, one gets a sequence of maps $0 \to \underline{\mathcal{M}}(D') \to \underline{\mathcal{M}}(D) \to \underline{\mathcal{M}}(D'') \to 0$. It is enough to check the exactness completed stalks at x_n, for which we use (3.1.3.1). □

3.2 Equvalence of categories

In this section, we construct a functor $\underline{D} : \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi) \to \mathcal{HP}_K^{\geqslant 0}(\varphi)$, which will shown to be a quasi-inverse to the functor $\underline{\mathcal{M}}$ constructed in the previous section.

Let $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$; i.e., a $(\varphi, \mathcal{O}_{\mathbf{\Delta}})$-module of finite $\mathcal{P}$-height, and consider the φ-module $\mathcal{M}/u\mathcal{M}$, which is an isocrystal (i.e., an étale φ-module over $\mathscr{K}_0$) since $\mathcal{P}(0)$ is a unit in $\mathscr{K}_0$. Hence the scalar extension $\mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} (\mathcal{M}/u\mathcal{M})$ is an étale φ-module on $\mathbf{\Delta}$ by §2.2.4(2). We set

$$\underline{D}(\mathcal{M}) := (\mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} (\mathcal{M}/u\mathcal{M}),\ \mathcal{O}_{\mathbf{\Delta}} \otimes \varphi) \tag{3.2.0.1}$$

To give a Hodge-Pink structure on $\mathcal{M}/u\mathcal{M}$, we need the following lemma. The case $\mathfrak{o}_0 = \mathbb{Z}_p$ can be extracted from the proof of [52, Lemma 1.2.6] (except the functorial property; i.e., (2) in the statement below). The same proof also works if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$.

Proposition 3.2.1. *For $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$, there exists a unique $\mathcal{O}_{\mathbf{\Delta}}$-linear "$\varphi$-compatible section" $\xi : \underline{D}(\mathcal{M}) \to \mathcal{M}$. In other words, there exists a unique ξ which reduces to the identity map modulo u and commutes with φ-structures on both sides. Furthermore, ξ enjoys the following properties:*

1. *The section ξ induces an isomorphism $\underline{D}(\mathcal{M})[1/\lambda] \xrightarrow{\sim} \mathcal{M}[1/\lambda]$. Furthermore, on any $\mathbf{\Delta}_I$ which contains x_0 and does not contain x_n for $n \neq 0$, the images of ξ and $\varphi_{\mathcal{M}}$ coincide.*

2. *Consider $\mathcal{M}, \mathcal{M}' \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$. Let ξ and ξ' be the unique φ-compatible sections for $\mathcal{M}$ and $\mathcal{M}'$, respectively. Then, for any morphism $f : \mathcal{M} \to \mathcal{M}'$ of*

$\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$, *the following diagram commutes:*

(3.2.1.1)
$$\begin{array}{ccc} \underline{\mathcal{D}}(\mathcal{M}) & \xrightarrow{\xi} & \mathcal{M} \\ {\scriptstyle \mathcal{O}_{\mathbf{\Delta}}\otimes\bar{f}}\downarrow & & \downarrow{\scriptstyle f} \\ \underline{\mathcal{D}}(\mathcal{M}') & \xrightarrow{\xi'} & \mathcal{M}', \end{array}$$

where $\bar{f} : \mathcal{M}/u\mathcal{M} \to \mathcal{M}'/u\mathcal{M}'$ *is the reduction of* f *modulo* u.

Remark 3.2.2. Before we begin the proof, let us discuss a consequence of the lemma. We view $\widehat{\mathcal{M}}_{x_0}$ as an effective Hodge-Pink structure for the isocrystal $\mathcal{M}/u\mathcal{M}$. We define a functor $\underline{D} : \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi) \to \mathcal{HP}_K^{\geqslant 0}(\varphi)$, as follows:

(3.2.2.1)
$$\underline{D}(\mathcal{M},\, \varphi_{\mathcal{M}}) = \left(\mathcal{M}/u\mathcal{M}, \varphi_{\mathcal{M}} \bmod u\mathcal{M},\, \widehat{\mathcal{M}}_{x_0}\right),$$

The functor $\underline{D}$ carries a morphism $f : \mathcal{M} \to \mathcal{M}'$ of $\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ to a morphism (f mod $u\mathcal{M}$) : $\mathcal{M}/u\mathcal{M} \to \mathcal{M}'/u\mathcal{M}'$. This defines a morphism of $\mathcal{HP}_K^{\geqslant 0}(\varphi)$ (i.e., takes the Hodge-Pink structure of the source into that of the target) essentially because of the functoriality of the φ-compatible section (Proposition 3.2.1(2)).

3.2.3 Rank-1 Example

Before we prove Proposition 3.2.1, we work out the rank-1 case. Let $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ be of rank 1 over $\mathcal{O}_{\mathbf{\Delta}}$ and set $D := \mathcal{M}/u\mathcal{M}$ equipped with $\bar{\varphi} := \varphi_{\mathcal{M}}$ mod $u\mathcal{M}$. We choose a $\mathcal{O}_{\mathbf{\Delta}}$-basis $\mathbf{e} \in \mathcal{M}$, and denote by $\bar{\mathbf{e}} \in D$ the image of $\mathbf{e}$ under the natural projection. Since $\varphi_{\mathcal{M}}(\sigma^*\mathbf{e})$ spans $\mathcal{P}(u)^h\mathcal{M}$ for a suitable $h \geqslant 0$, we may write $\varphi_{\mathcal{M}}(\sigma^*\mathbf{e}) = \alpha_{\mathbf{e}} \left(\frac{\mathcal{P}(u)}{\mathcal{P}(0)}\right)^h \cdot \mathbf{e}$ for some $\alpha_{\mathbf{e}} \in \mathscr{K}_0^{\times}$. Then we have $\bar{\varphi}(\sigma^*\bar{\mathbf{e}}) = \alpha_{\mathbf{e}} \cdot \bar{\mathbf{e}}$. Therefore, we have $\varphi_{\mathcal{M}}(\sigma^*(\lambda^h \cdot \mathbf{e})) = \alpha_{\mathbf{e}} \cdot (\lambda^h \cdot \mathbf{e})$, and $\lambda(0) = 1$ (or rather, $\lambda \equiv 1 \bmod u$), so $\lambda^h \cdot \mathbf{e}$ reduces to $\bar{\mathbf{e}}$ modulo $u\mathcal{M}$. This shows that $\bar{\mathbf{e}} \mapsto \lambda^h \mathbf{e}$ induces a φ-compatible map $\xi : \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} D \to \mathcal{M}$. By Proposition 3.2.1, this is the unique such map. Following the recipe of Remark 3.2.2, we obtain a Hodge-Pink structure $\Lambda = \lambda^{-h}\widehat{\mathcal{D}}_{x_0} = \mathcal{P}(u)^{-h}\widehat{\mathcal{D}}_{x_0}$. This defines $\underline{D}(\mathcal{M}) \in \mathcal{HP}_K^{\geqslant 0}(\varphi)$ of rank 1.

From the above discussion and §3.1.4, it is not difficult to show that $\underline{\mathcal{M}}$ and $\underline{D}$ are quasi-inverse equivalences between categories of rank-1 objects. The equivalence will be generalized to an arbitrary rank later in Proposition 3.2.5(3).

Proof of Proposition 3.2.1. We proceed in four steps.

(1) *existence of* ξ

Recall that $\mathcal{O}_{\boldsymbol{\Delta}}$ is a Fréchet space with respect to norms $\|\cdot\|_r$ for $r \in q^{\mathbb{Q}_{<0}}$. See §2.1.1 for the definition of the norms. By choosing a $\mathcal{O}_{\boldsymbol{\Delta}}$-basis $\{\mathbf{e}_i\}_{i=1,\cdots d}$ for $\mathcal{M}$, one can define a norm $\|\cdot\|_r$ on $\mathcal{M}$ by taking the maximum of $\|\cdot\|_r$ on coefficients, which makes $\mathcal{M}$ a Fréchet space. The topology on $\mathcal{M}$ generated by $\|\cdot\|_r$ is independent of the choice of basis for $\mathcal{M}$. Likewise, $\sigma^{*n}(\mathcal{M})$ is a Fréchet space for all $n \geqslant 0$.

Starting from any $\mathscr{K}_0$-linear section $s_0 : \mathcal{M}/u\mathcal{M} \to \mathcal{M}$, which does not have to be φ-compatible, we would like to construct a new section $s : \mathcal{M}/u\mathcal{M} \to \mathcal{M}$ such that $s \circ \bar{\varphi} = \varphi_{\mathcal{M}} \circ \sigma^* s$. Here we give a formula for s, and will show that it is well-defined.

(3.2.3.1)

$$s := s_0 + \sum_{i\geq 0}(\varphi_{\mathcal{M}}^{i+1} \circ \sigma^{*i+1} s_0 \circ \bar{\varphi}^{-(i+1)} - \varphi_{\mathcal{M}}^{i} \circ \sigma^{*i} s_0 \circ \bar{\varphi}^{-i}) = \text{“} \lim_{i\to\infty}(\varphi_{\mathcal{M}}^{i} \circ \sigma^{*i} s_0 \circ \bar{\varphi}^{-i}) \text{”}$$

Since $\bar{\varphi} : \sigma^*(\mathcal{M}/u\mathcal{M}) \to \mathcal{M}/u\mathcal{M}$ is bijective, $\bar{\varphi}^{-1}$ makes sense. If the right side is well defined, then it clearly satisfies $s \circ \bar{\varphi} = \varphi_{\mathcal{M}} \circ \sigma^* s$. Since $\mathcal{M}$ is a Fréchet space, it is enough to check the convergence for each norm $\|\cdot\|_r$.

We have uniquely $\varphi_{\mathcal{M}}(\sigma^*\mathbf{e}_j) = \sum_{i=1}^{d} a_{ij}\mathbf{e}_i$ where $\sigma^*\mathbf{e}_j := 1 \otimes \mathbf{e}_j \in \sigma^*(\mathcal{M}/u\mathcal{M})$ and $a_{ij} \in \mathcal{O}_{\boldsymbol{\Delta}}$. Take a non-negative integer b such that $q^b \geq \max_{i,j}\{\|a_{ij}\|_r\}$. (Note that b depends on r.) Then we have $\|\varphi_{\mathcal{M}}(\sigma^*\mathbf{e}_i)\|_r \leq q^b \|\mathbf{e}_i\|_r$, and it follows that $\|\varphi_{\mathcal{M}}(\sigma^* m)\|_r \leq q^b \|m\|_r$ for any $m \in \mathcal{M}$ by using the inequality $\|\sigma(f)\|_r = \|f\|_{r^{1/q}} \geq \|f\|_r$ (which follows from the maximum modulus principle).

Take any $\mathfrak{o}_{\mathscr{K}_0}$-lattice $\mathcal{L} \subset \mathcal{M}/u\mathcal{M}$. Increase b so that we have $\varphi^{-1}(\mathcal{L}) \subset \pi_0^{-b}(\sigma^*\mathcal{L})$. (So now, b depends on both r and $\mathcal{L}$.) Since $\operatorname{im}(\varphi_{\mathcal{M}} \circ \sigma^* s_0 \circ \bar{\varphi}^{-1} - s_0) \subset u\mathcal{M}$, we set $\widetilde{\mathcal{L}} := u^{-1}(\varphi_{\mathcal{M}} \circ \sigma^* s_0 \circ \bar{\varphi}^{-1} - s_0)(\mathcal{L}) \subset \mathcal{M}$. Now we have

$$\begin{aligned}\left\| \left(\varphi_{\mathcal{M}}^{i+1} \circ (\sigma^*)^{i+1} s_0 \circ \bar{\varphi}^{-(i+1)} - \varphi_{\mathcal{M}}^{i} \circ (\sigma^*)^{i} s_0 \circ \bar{\varphi}^{-i}\right)(\mathcal{L}) \right\|_r \\ \leq q^{ib} \left\| u^{q^i} \varphi_{\mathcal{M}}^i ((\sigma^*)^i \widetilde{\mathcal{L}}) \right\|_r \leq q^{2ib} r^{q^i} \left\| \widetilde{\mathcal{L}} \right\|_r,\end{aligned}$$

where $\left\| \widetilde{\mathcal{L}} \right\|_r := \sup_{m \in \widetilde{\mathcal{L}}} \{ \|m\|_r \}$, which is clearly finite. (We normalized the absolute value so that $|\pi_0| = \frac{1}{q}$.) Observe that $q^{2ib} r^{q^i} \to 0$ as $i \to \infty$ for any $r \in (0,1)$ and any non-negative b (hence for any choice of $\mathcal{L}$). For any $x \in \mathcal{M}/u\mathcal{M}$, choosing $\mathcal{L}$ to contain x proves that the formula for $s(x)$ makes sense. Now let $\xi := \operatorname{id} \otimes s : \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} (\mathcal{M}/u\mathcal{M}) \to \mathcal{M}$.

(2) *uniqueness of* ξ *and diagram* (3.2.1.1)

Consider $\mathcal{M}, \mathcal{M}' \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ and an $\mathcal{O}_{\mathbf{\Delta}}$-linear φ-compatible map $f : \mathcal{M} \to \mathcal{M}'$. Let $\bar{f} : \mathcal{M}/u\mathcal{M} \to \mathcal{M}'/u\mathcal{M}'$ be the reduction of f modulo u. Consider some φ-compatible sections $s : \mathcal{M}/u\mathcal{M} \to \mathcal{M}$ and $s' : \mathcal{M}'/u\mathcal{M}' \to \mathcal{M}'$, and we show that $f \circ s = s' \circ \bar{f}$. This shows the commutative diagram (3.2.1.1), and the uniqueness of ξ also follows from the case when $\mathcal{M} = \mathcal{M}'$ and $f = \operatorname{id}_{\mathcal{M}}$.

Observe that both $f \circ s$ and $s' \circ \bar{f}$ are φ-compatible map $\mathcal{M}/u\mathcal{M} \to \mathcal{M}'$ such that the post-composition of both with the natural projection $\mathcal{M}' \twoheadrightarrow \mathcal{M}'/u\mathcal{M}'$ is $\bar{f}$. So we have $\operatorname{im}(f \circ s - s' \circ \bar{f}) \subset u\mathcal{M}'$. From the φ-compatibility, we obtain:

$$\varphi_{\mathcal{M}'}^i \circ \left(\sigma^{*i}(f \circ s - s' \circ \bar{f})\right) = (f \circ s - s' \circ \bar{f}) \circ \bar{\varphi}^i,$$

for any positive integer i. Since $\bar{\varphi} : \sigma^*(\mathcal{M}/u\mathcal{M}) \to \mathcal{M}/u\mathcal{M}$ is an isomorphism, we deduce from above equality that $\operatorname{im}(f \circ s - s' \circ \bar{f}) \subset u^{q^i}\mathcal{M}'$ for any positive integer

i, so we have $f \circ s - s' \circ \bar{f} = 0$.

(3) *claims on* $\mathrm{im}(\xi)$

Since ξ is an isomorphism modulo u, ξ induces an isomorphism on stalks at the origin, so it is an isomorphism on some neighborhood of the origin. Let $\mathbf{\Delta}_{\leqslant r}$ denote the rigid analytic closed disk of radius r centered at 0 over $\mathscr{K}_0$. Take r such that $\mathbf{\Delta}_{\leqslant r}$ contains x_0 and not x_n for $n \neq 0$, and choose i such that $\xi_{\leqslant r^{q^i}}$ is an isomorphism. Since ξ is φ-compatible, we have the following commutative diagram

$$\begin{array}{ccc} \sigma^*\underline{D}(\mathcal{M}) & \xrightarrow{\sigma^*\xi} & \sigma^*\mathcal{M} \\ \downarrow{\cong} & & \downarrow{\varphi_{\mathcal{M}}} \\ \underline{D}(\mathcal{M}) & \xrightarrow{\xi} & \mathcal{M} \end{array}$$

If $i > 1$, the right vertical arrow is an isomorphism on $\mathbf{\Delta}_{\leqslant r^{q^{i-1}}}$ by the finite $\mathcal{P}$-height condition. So we get that $\xi_{\leqslant r^{q^{i-1}}}$ is an isomorphism. And when $i = 1$, the above diagram exactly tells that the image of $\xi_{\leqslant r}$ coincides with the image of $\varphi_{\mathcal{M},\leqslant r}$. Hence the cokernel of $\xi_{\leqslant r}$ is killed by some power of $\mathcal{P}(u)$, say $\mathcal{P}(u)^h$.

By repeating this argument for $\mathbf{\Delta}_{\leqslant r^{q^{-n}}}$ with $n \geqslant 0$, we obtain that the cokernel of $\xi_{\leqslant r^{q^{-n}}}$ is killed by $\left(\prod_{i=0}^{n} \sigma^i\left(\frac{\mathcal{P}(u)}{\mathcal{P}(0)}\right)\right)^h$ for all $n \geqslant 0$. Therefore, $\operatorname{coker}\xi$ is killed by λ^h. □

Remark 3.2.4. In this remark, we show that $(D, \varphi_D, \Lambda) := \underline{D}(\mathcal{M})$ can be easily computed if $\varphi_{\mathcal{M}}$ is explicitly given (with respect to a basis). The only possibly non-trivial part is to compute the Hodge-Pink structure Λ, which can be done as follows.

Choose $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ and fix an $\mathcal{O}_{\mathbf{\Delta}}$-basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ of $\mathcal{M}$. We let $\mathbf{e}_i$ also denote its image in $\Lambda := \widehat{\mathcal{M}}_{x_0}$. Let $\bar{\mathbf{e}}_i$ to denote the image of $\mathbf{e}_i$ in $D := \mathcal{M}/u\mathcal{M}$ and view it as an element in $\widehat{\mathcal{D}}_{x_0}$. We want to give a basis of Λ in terms of $\widehat{\xi}_{x_0}(\mathbf{e}_i)$.

So Proposition 3.2.1(1) shows that the $\widehat{\xi}_{x_0}(\mathbf{e}_i)$ and the $\varphi_{\mathcal{M}}(\sigma^*\mathbf{e}_i)$ generate the same submodule in Λ, so $\sum_i a^{ij}\widehat{\xi}_{x_0}(\mathbf{e}_i)$ generates Λ, where $(a^{ij}) = (a_{ij})^{-1}$ with $(a_{ij}) \in \mathrm{GL}_n(\mathcal{O}_{\mathbf{\Delta}}[\frac{1}{\mathcal{P}(u)}])$ is the matrix representation of $\varphi_{\mathcal{M}}$ for the chosen basis; i.e., $\mathbf{e}_j = \sum_j a^{ij}\varphi_{\mathcal{M}}(\sigma^*\mathbf{e}_i)$.

Having defined $\underline{\mathcal{M}} : \mathcal{HP}_K^{\geqslant 0}(\varphi) \to \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ and $\underline{D} : \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi) \to \mathcal{HP}_K^{\geqslant 0}(\varphi)$, it is quite straightforward to check the following:

Proposition 3.2.5. *The functor $\underline{D} : \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi) \to \mathcal{HP}_K^{\geqslant 0}(\varphi)$ is an exact equivalence of $\otimes$-categories. More precisely, we have the following properties:*

1. *$\underline{D}$ commutes with $\otimes$-products.*
2. *$\underline{D}$ takes a short exact sequence of the source category into that of the target category.*
3. *$\underline{\mathcal{M}}$ and $\underline{D}$ are quasi-inverse to each other.*

Since the functors $\underline{\mathcal{M}}$ and $\underline{D}$ commute with $\otimes$-products (in particular, with Tate twists), we can extend them to quasi-inverse equivalences of $\otimes$-categories between $\mathcal{HP}_K(\varphi)$ and generalized φ-modules over $\mathcal{O}_{\mathbf{\Delta}}$ defined at §2.2.11.

Proof. First two claims are straightforward from Proposition 3.2.1, especially from the uniqueness of ξ. By construction, the underlying isocrystal for $(\underline{D} \circ \underline{\mathcal{M}})(D, \Lambda)$ is naturally isomorphic to D. That this isomorphism takes the Hodge-Pink structure for $(\underline{D} \circ \underline{\mathcal{M}})(D, \Lambda)$ isomorphically onto Λ follows from the isomorphism (3.1.3.1) and Lemma 2.3.5. This shows the natural isomorphism $\underline{D} \circ \underline{\mathcal{M}} \xrightarrow{\sim} \mathrm{id}$.

Recall that $(\underline{\mathcal{M}} \circ \underline{D})(\mathcal{M})$ is constructed as a submodule of $\mathcal{D}[1/\lambda]$, where $\mathcal{D} := \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} (M/u\mathcal{M})$. We view $\mathcal{M}$ as a submodule of $\mathcal{D}[1/\lambda]$ via $\mathcal{M} \subset \mathcal{M}[1/\lambda] \xleftarrow{\sim} \mathcal{D}[1/\lambda]$ where the isomorphism is induced from the unique φ-compatible section $\xi : \mathcal{D} \to \mathcal{M}$ (Proposition 3.2.1). To obtain a functorial isomorphism $(\underline{\mathcal{M}} \circ \underline{D})(\mathcal{M}) \cong \mathcal{M}$, we show that both sides defines the same $\mathcal{O}_{\mathbf{\Delta}}$-submodule of $\mathcal{D}[1/\lambda]$. It is enough to check locally at x_n for each n.

The completed stalks of both at x_0 define the same $\widehat{\mathcal{O}_{\mathbf{\Delta},x_0}}$-lattice Λ inside $\widehat{\mathcal{D}}_{x_0}[1/\lambda]$. So for $\mathbf{\Delta}_{\leqslant r}$ which contains x_0 but not x_n for $n \neq 0$, we have an equality $(\underline{\mathcal{M}} \circ \underline{D})(\mathcal{M})_{\leqslant r} = \mathcal{M}_{\leqslant r}$ inside $\underline{\mathcal{D}}(\mathcal{M})_{\leqslant r}[1/\lambda]$. By pulling back $(\underline{\mathcal{M}} \circ \underline{D})(\mathcal{M})_{\leqslant r} = \mathcal{M}_{\leqslant r}$ by σ^n, we obtain $\big(\sigma^n(\underline{\mathcal{M}} \circ \underline{D})(\mathcal{M})\big)_{\leqslant r^{1/q^n}} = (\sigma^n \mathcal{M})_{\leqslant r^{1/q^n}}$. Since $\mathcal{M}$ is of finite $\mathcal{P}$-height, $\varphi^n_{\mathcal{M}}$ is an isomorphism outside x_0 and the same holds for $(\underline{\mathcal{M}} \circ \underline{D})(\mathcal{M})$. Therefore we have $(\underline{\mathcal{M}} \circ \underline{D})(\mathcal{M}) = \mathcal{M}$. □

3.2.6 Relation with $(\varphi, \mathfrak{S})$-modules of finite $\mathcal{P}$-height

Let us define the following functor of $\otimes$-categories:

(3.2.6.1)

$$\underline{\mathbb{H}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[1/\pi_0] \to \mathcal{HP}_K^{\geqslant 0}(\varphi), \quad \underline{\mathbb{H}}(\mathfrak{M}[1/\pi_0]) = \underline{D}(\mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathfrak{S}[1/\pi_0]} \mathfrak{M}[1/\pi_0]).$$

One can directly see that the functor $\underline{\mathbb{H}}$ preserves the Hodge-Pink type; more precisely, $\mathfrak{M}[\frac{1}{\pi_0}] \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}]$ is of Hodge-Pink type $\mathbf{v}$ if and only if $\underline{\mathbb{H}}(\mathfrak{M}[\frac{1}{\pi_0}])$ is of Hodge-Pink type $\mathbf{v}$. This follows from the definitions of Hodge-Pink type, together with Proposition 3.1.5. (Note that $\mathfrak{S}[\frac{1}{\pi_0}]/(\mathcal{P}(u)^w) \cong \mathcal{O}_{\mathbf{\Delta}}/(\mathcal{P}(u)^w)$.)

In next chapter, we show that $\underline{\mathbb{H}}$ is fully faithful (or equivalently, the scalar extension functor $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ is fully faithful) and the essential image is exactly the full subcategory of weakly admissible objects. Similarly, we may extend $\underline{\mathbb{H}}$ to a fully faithful functor from the isogeny category of generalized φ-modules over $\mathcal{O}_{\mathbf{\Delta}}$ to $\mathcal{HP}_K(\varphi)$, with an essential image $\mathcal{HP}_K^{wa}(\varphi)$.

For the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, it is proved by Genestier-Lafforgue [35, Lemma 2.8] that $\underline{\mathbb{H}}$ induces an equivalence of categories $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \mathcal{HP}_K^{wa,\geqslant 0}(\varphi)$. (A proof can be found in Hartl [39, Theorem 2.5.3].) In the next chapter, we give a slightly different proof which is closely related to Kisin's proof for [52, Thm 1.3.8]. Our proof also works for the case of $\mathfrak{o}_0 = \mathbb{Z}_p$, which has not been studied as far as the author is aware of.

CHAPTER IV

Weakly admissible Hodge-Pink structure

In this chapter, we prove that the functor $\underline{\mathbb{H}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \mathcal{HP}_K^{\geqslant 0}(\varphi)$ defined in (3.2.6.1) is fully faithful and that the essential image is exactly $\mathcal{HP}_K^{wa,\geqslant 0}(\varphi)$. (See Theorem 4.3.4 for the precise statement.) The key step is to show that weak admissibility on $\mathcal{HP}_K^{\geqslant 0}(\varphi)$ is equivalent to the "pure-of-slope-0" condition on $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$, under the equivalences of categories $\underline{\mathcal{M}}$ and $\underline{D}$. The main technical ingredient for the key step is the slope filtration theorem, which was proved by Kedlaya [46, 48, 49] in the case of $\mathfrak{o}_0 = \mathbb{Z}_p$, and by Hartl [39, Theorem 1.7.7.] in the case of $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. Below we review the theory of slopes and state relevant properties without proof.

The idea of relating the "pure-of-slope-0" condition and weak admissibility of filtered (φ, N)-modules originally came from Berger [5, §IV]. Our approach is more akin to Kisin's variant [52, (1.3)]. In the p-adic setting, the difference with Kisin's approach and ours is that Kisin used a logarithmic connection [52, Lemma 1.3.5] for the "Dwork's trick" step, while we solely work with Frobenius structure so the same argument works in the analogous equi-characteristic setting; see Proposition 4.2.1. Note that there is no good analogue of the logarithmic connections in the equi-characteristic setting.

4.1 Review of slopes

For completeness of the exposition, we give a definition of slope and state the slope filtration theorem of Kedlaya in the p-adic setting and Hartl in the equi-characteristic setting.

4.1.1 Simple objects

We define the slope using the "Dieudonné-Manin classification" over $\mathcal{R}^{\mathrm{alg}}$. (See §6.1.10 for the definition of $\mathcal{R}^{\mathrm{alg}}$.) For these, we need to define basic "building blocks."

Let $\bar{k}/k$ be an algebraic closure, and recall that $F_0 := \mathfrak{o}_0[\frac{1}{\pi_0}]$. Let R be an F_0-algebra, equipped with an endomorphism $\sigma : R \to R$ that fixes F_0. In the intended applications R will be one of the following:

1. (The case $\mathfrak{o}_0 = \mathbb{Z}_p$) a complete field extension $\mathscr{K}_0(\bar{k})$ over $\mathscr{K}_0$ where $\mathscr{K}_0(\bar{k}) := W(\bar{k})[\frac{1}{p}]$, equipped with the Witt vector Frobenius endomorphism σ.
2. (The case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$) a complete field extension $\mathscr{K}_0(\bar{k})$ over $\mathscr{K}_0$ where $\mathscr{K}_0(\bar{k}) := \bar{k}((\pi_0))$, equipped with the unique continuous endomorphism σ such that $\sigma(\pi_0) = \pi_0$ and $\sigma(\alpha) = \alpha^q$ for all $\alpha \in \bar{k}$. (If k is not perfect, which is allowed when $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then $\mathscr{K}_0(\bar{k})$ is *not* the completion of the maximal unramified extension of $\mathscr{K}_0$.)
3. the Robba rings $\mathcal{R}^{\mathrm{alg}}$ and $\mathcal{R}^{\mathrm{alg},bd}$ equipped with the natural Frobenius endomorphism σ, introduced in §6.1.10.

We define the following étale φ-module $(\mathcal{M}_{d,n}, \varphi) \in \underline{\mathrm{Mod}}^{\text{ét}}_R(\varphi)$ for any $d, n \in \mathbb{Z}_{>0}$:

$$\begin{aligned} \mathcal{M}_{d,n} &:= \bigoplus_{i=1}^{n} (R \cdot \mathbf{e}_i) \\ \varphi(\sigma^* \mathbf{e}_i) &= \mathbf{e}_{i+1},\ i \neq n \\ \varphi(\sigma^* \mathbf{e}_n) &= \pi_0^d \cdot \mathbf{e}_1 \end{aligned}$$

In particular, since $\sigma(\pi_0) = \pi_0$, for any $m \in \mathcal{M}_{d,n}$ we have $\varphi^n(\sigma^{*n} m) = \pi_0^d \cdot m$. (We define slopes and slope filtrations so that $M_{d,n}$ is "pure of slope d/n.") Observe that $\mathcal{M}_{d,n}$ has a nontrivial proper φ-stable subobject if d and n are *not* coprime.

Theorem 4.1.2 (Dieudonné-Manin Classification)**.** *Let* (R, σ) *be either* $(\mathscr{K}_0(\bar{k}), \sigma)$ *or* $(\mathcal{R}^{\text{alg}}, \sigma)$. *Then any* $\mathcal{M} \in \underline{\mathrm{Mod}}^{\text{ét,free}}_R(\varphi)$ *is isomorphic to a direct sum* $\bigoplus_{j=1}^{c} \mathcal{M}_{(d_j, n_j)}$, *where* $d_j \in \mathbb{Z}$ *and* $n_j \in \mathbb{Z}_{>0}$ *satisfy* $(d_j, n_j) = 1$ *for each* j. *The pairs* (d_j, n_j) *are uniquely determined up to permutation.*

Proof. If $\mathfrak{o}_0 = \mathbb{Z}_p$, then Kedlaya [48, Theorem 4.5.7] proves the theorem simultaneously for both $R = \mathscr{K}_0(\bar{k}) = W(\bar{k})[\frac{1}{p}]$ and $R = \mathcal{R}^{\text{alg}}$. Simpler proofs for the case $R = W(\bar{k})[\frac{1}{p}]$ can be found in [23], [61], [44], and [46, Theorem 5.6]. If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then the theorem for $R = \mathscr{K}_0(\bar{k}) = \bar{k}((\pi_0))$ is proved in [56, §A 2.1]. The theorem for the case $R = \mathcal{R}^{\text{alg}}$ is proved in to [42, Theorem 11.1, Corollary 11.8]. □

Remark 4.1.3. We record the following special case of the theorem: for any rank-1 étale φ-module $D \in \underline{\mathrm{Mod}}^{\text{ét,free}}_{\mathscr{K}(\bar{k})}(\varphi)$, one can find a basis $\mathbf{e} \in \mathcal{M}$ so that $\varphi(\sigma^* \mathbf{e}) = \pi_0^d \cdot \mathbf{e}$ for some $d \in \mathbb{Z}$. This gives a classification of rank-1 isocrystals with weakly admissible Hodge-Pink structure, and rank-1 weakly admissible filtered isocrystals if $\mathfrak{o}_0 = \mathbb{Z}_p$.

4.1.4 Slope

Let $\mathcal{M}$ be an étale φ-module of rank n over $\mathcal{R}^{\text{alg}}$. The *degree*[1] of $\mathcal{M}$, which is denoted by $\deg(\mathcal{M})$, is the unique integer d such that $\det \mathcal{M} \cong \mathcal{M}_{d,1}$, which is always well-defined by Theorem 4.1.2. The ratio $\text{sl}(\mathcal{M}) := d/n$ of $d := \deg(\mathcal{M})$ and $n := \text{rank}_{\mathcal{R}}(\mathcal{M})$ is called the *slope* of $\mathcal{M}$. In more concrete terms, if $\mathcal{M} \cong \bigoplus_{j=1}^{c} \mathcal{M}_{(d_j,n_j)}$, then we have $\deg(\mathcal{M}) = \sum_j d_j$ and $\text{sl}(\mathcal{M}) = \left(\sum_j d_j\right) / \left(\sum_j n_j\right)$. Clearly, we have $\deg(\mathcal{M}) = \text{sl}(\det \mathcal{M})$.

We say that $\mathcal{M}$ is *pure*[2] *of slope* $s = d/n$, where d/n is a reduced fraction, if $\mathcal{M} \cong \mathcal{M}_{d,n}^{\oplus c}$ for some c. The full subcategories of étale φ-modules pure of slope s will be denoted by $\underline{\text{Mod}}_{\mathcal{R}^{\text{alg}}}^{\text{sl}=s}(\varphi)$.

For a φ-module $\mathcal{M}$ free over a base ring contained in $\mathcal{R}^{\text{alg}}$ (for example, over $\mathcal{R}^{\text{alg},bd}$, $\mathcal{R}$, $\mathcal{R}^{bd}$, or $\mathcal{O}_{\mathbf{\Delta}}$), the *degree* and the *slope* of $\mathcal{M}$ are defined to be the degree and the slope of $\mathcal{R}^{\text{alg}} \otimes \mathcal{M}$, respectively. One can check that the degree for $\mathcal{M} \in \underline{\text{Mod}}_{\mathcal{R}^{bd}}(\varphi)$ or $\mathcal{M} \in \underline{\text{Mod}}_{\mathcal{R}^{\text{alg},bd}}(\varphi)$ coincides with the valuation of the determinant of any Frobenius matrix.

We say that $\mathcal{M}$ is *pure of slope* s if $\mathcal{R}^{\text{alg}} \otimes \mathcal{M}$ is so. We use superscript $\text{sl} = s$ to denote the full subcategories of étale φ-modules pure of slope s, for example, $\underline{\text{Mod}}_{\mathcal{R}}^{\text{sl}=s}(\varphi)$, $\underline{\text{Mod}}_{\mathcal{R}^{bd}}^{\text{sl}=s}(\varphi)$, $\underline{\text{Mod}}_{\mathbf{\Delta}}^{\text{sl}=s}(\varphi)$, and so on.

We state the following proposition without proof, which will be used later in proving Theorem 4.3.4.

Proposition 4.1.5. *The φ-modules $M_{d,n}$ over $\mathcal{R}^{\text{alg}}$ satisfy* $\text{Hom}_{\varphi}(\mathcal{M}_{d,n}, \mathcal{M}_{d',n'}) = 0$ *if and only if $d/n > d'/n'$. In particular, any φ-submodule of $\mathcal{M}_{d,n}$ has slope $\leqslant d/n$.*

If $\mathfrak{o}_0 = \mathbb{Z}_p$, then the proposition is just [48, Proposition 4.1.3(a)]. If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$,

[1] This definition of degree differs by sign from Hartl's definition [39, Def 1.5.1]. As Hartl remarked, Hartl's definition follows the "geometric" convention whereas this definition follows the "arithmetic" convention.

[2] Sometimes, it is called *isoclinic* of slope s.

then by a standard argument (e.g. [39, Proposition 1.4.1]) we are reduced to [42, Proposition 8.5].

For any $\mathcal{M}^{\mathrm{alg}} \in \underline{\mathrm{Mod}}_{\mathcal{R}^{\mathrm{alg}}}(\varphi)$, we have an isomorphism $\mathcal{M}^{\mathrm{alg}} \cong \bigoplus_{i=1}^{c}(\mathcal{M}_{d_i,n_i})^{\oplus a_i}$ from the Dieudonné-Manin decomposition. By re-indexing if necessary, one can arrange to have $d_1/n_1 < d_2/n_2 < \cdots < d_c/n_c$. The following filtration is called the *slope filtration* for $\mathcal{M}^{\mathrm{alg}}$:

(4.1.5.1)

$$0 = \mathcal{M}_0^{\mathrm{alg}} \subset \mathcal{M}_1^{\mathrm{alg}} \subset \cdots \subset \mathcal{M}_c^{\mathrm{alg}} = \mathcal{M}^{\mathrm{alg}}, \quad \text{where } \mathcal{M}_j^{\mathrm{alg}} := \bigoplus_{i \leq j} (\mathcal{M}_{d_i,n_i})^{\oplus a_i}$$

If $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathcal{R}}(\varphi)$, then the following (very difficult) theorem asserts that the slope filtration for $\mathcal{R}^{\mathrm{alg}} \otimes_{\mathcal{R}} \mathcal{M}$ descends over $\mathcal{R}$.

Theorem 4.1.6 (Slope Filtration Theorem)**.**

1. *The scalar extension functor $\underline{\mathrm{Mod}}_{\mathcal{R}^{bd}}^{\mathrm{sl}=s}(\varphi) \to \underline{\mathrm{Mod}}_{\mathcal{R}}^{\mathrm{sl}=s}(\varphi)$ is an equivalence of categories. In particular, any $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathcal{R}}^{\mathrm{sl}=s}(\varphi)$ uniquely descends to $\mathcal{M}^{bd} \in \underline{\mathrm{Mod}}_{\mathcal{R}^{bd}}^{\mathrm{sl}=s}(\varphi)$*

2. *For any $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathcal{R}}(\varphi)$, there exists a unique and canonical filtration (called the* slope filtration*) $0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_c = \mathcal{M}$ by saturated φ-stable $\mathcal{R}$-submodules such that each subquotient $\mathcal{M}_i/\mathcal{M}_{i-1}$ is pure of slope s_i and $s_1 < s_2 < \cdots < s_c$. Furthermore, the slope filtration for $\mathcal{R}^{\mathrm{alg}} \otimes_{\mathcal{R}} \mathcal{M}$ is exactly $\{\mathcal{R}^{\mathrm{alg}} \otimes_{\mathcal{R}} \mathcal{M}_i\}$.*

Proof. If $\mathfrak{o}_0 = \mathbb{Z}_p$, then the first part is [48, Theorem 6.3.3] and the second part is [46, Theorem 6.10]. If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then the first part is [39, Corollary 1.7.6] and the second part is [39, Theorem 1.7.7]. □

For the future reference, we give a useful characterization of étale φ-module (over $\mathcal{R}^{bd}$ or $\mathcal{R}$) pure of slope 0.

Lemma 4.1.7. *An étale φ-module $\mathcal{M}^{bd}$ finite free over $\mathcal{R}^{bd}$ is pure of slope 0 if and only if there exists a φ-compatible isomorphism $\mathcal{M}^{bd} \cong \mathcal{R}^{bd} \otimes_{\mathfrak{o}_{\mathcal{R}^{bd}}} \mathcal{M}^{\mathrm{int}}$ for some étale φ-module $\mathcal{M}^{\mathrm{int}}$ over $\mathfrak{o}_{\mathcal{R}^{bd}}$. Similarly, an étale φ-module $\mathcal{M}$ finite free over $\mathcal{R}$ is pure of slope 0 if and only if there exists a φ-compatible isomorphism $\mathcal{M} \cong \mathcal{R} \otimes_{\mathfrak{o}_{\mathcal{R}^{bd}}} \mathcal{M}^{\mathrm{int}}$ for some étale φ-module $\mathcal{M}^{\mathrm{int}}$ over $\mathfrak{o}_{\mathcal{R}^{bd}}$.*

Proof. The claim for étale $(\varphi, \mathcal{R})$-modules is reduced to the claim for étale $(\varphi, \mathcal{R}^{bd})$-modules by Theorem 4.1.6(1). Let $\mathcal{M}^{bd} \in \underline{\mathrm{Mod}}^{\mathrm{sl}=0}_{\mathcal{R}^{bd}}(\varphi)$ be pure of slope 0 and of $\mathcal{R}^{bd}$-rank n. By definition, we have a φ-compatible isomorphism

$$\mathcal{M}^{\mathrm{alg},bd} := \mathcal{R}^{\mathrm{alg},bd} \otimes_{\mathcal{R}^{bd}} \mathcal{M}^{bd} \cong (\mathcal{M}_{0,1})^{\oplus n},$$

where $\mathcal{M}_{0,1}$ is the simple object over $\mathcal{R}^{\mathrm{alg},bd}$ defined in §4.1.1. In particular, $\mathcal{M}^{\mathrm{alg},bd}$ has a φ-stable $\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}}$-lattice $\mathcal{M}^{\mathrm{alg,int}}$ which is an étale φ-module over $\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}}$. (Indeed, this claim holds for $\mathcal{M}_{0,1}$, hence for any finite direct sum thereof.) We put $\mathcal{M}^{\mathrm{int}} := \mathcal{M}^{\mathrm{alg,int}} \cap \mathcal{M}^{bd}$, where the intersection is taken inside $\mathcal{M}^{\mathrm{alg},bd}$. Clearly, $\mathcal{M}^{\mathrm{int}}$ is a φ-stable $\mathfrak{o}_{\mathcal{R}^{bd}}$-lattice of $\mathcal{M}^{bd}$. Furthermore, $\mathcal{M}^{\mathrm{int}}$ is an étale φ-module over $\mathfrak{o}_{\mathcal{R}^{bd}}$, which can be seen by taking the faithfully flat scalar extension $\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}} \otimes_{\mathfrak{o}_{\mathcal{R}^{bd}}} M^{\mathrm{int}} \cong \mathcal{M}^{\mathrm{alg,int}}$.

Conversely, assume that we have a φ-compatible isomorphism $\mathcal{M}^{bd} \cong \mathcal{R} \otimes_{\mathfrak{o}_{\mathcal{R}^{bd}}} \mathcal{M}^{\mathrm{int}}$ for some étale φ-module $\mathcal{M}^{\mathrm{int}}$ over $\mathfrak{o}_{\mathcal{R}^{bd}}$. Let $\mathcal{M}_1$ be the smallest non-zero slope filtration of $\mathcal{M} := \mathcal{R} \otimes_{\mathfrak{o}_{\mathcal{R}^{bd}}} \mathcal{M}^{\mathrm{int}}$ which is pure of slope d_1/n_1 where d_1 and n_1 are coprime. We put $\mathcal{M}_1^{\mathrm{int}} := \mathcal{M}_1 \cap \mathcal{M}^{\mathrm{int}}$, where the intersection is taken inside $\mathcal{M}$. Then $\mathcal{M}_1^{\mathrm{int}}$ is a φ-stable étale $\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}}$-lattice in $\mathcal{M}_1^{\mathrm{alg}}$, which cannot happen if the slope s_1 is negative. This shows that any successive quotient $\mathcal{M}_j/\mathcal{M}_{j-1}$ of the slope filtration for $\mathcal{M}$ is pure of some slope $s_j \geqslant 0$ for each $j \geqslant 1$. On the other hand, the top exterior power $\det \mathcal{M}$ is pure of slope 0, since $\mathcal{R}^{\mathrm{alg}} \otimes_{\mathcal{R}} \det \mathcal{M} \cong \mathcal{M}_{d,1}$ for some $d \geqslant 0$ and admits an étale $\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}}$-lattice $\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}} \otimes_{\mathfrak{o}_{\mathcal{R}^{bd}}} \det \mathcal{M}^{\mathrm{int}}$. (Note that

$\mathcal{M}_{d,1}$ admits an étale $\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}}$-lattice only when $d = 0$.) Since we showed that each successive quotient $\mathcal{M}_j/\mathcal{M}_{j-1}$ of the slope filtration is pure of some non-negative slope, that $\det \mathcal{M}$ is pure of slope 0 implies that $\mathcal{M}$ is pure of slope 0 (so in turn, $\mathcal{M}^{bd}$ is pure of slope 0). □

4.2 "Dwork's trick" for φ-modules

The aim of this section is to prove the following: for any $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$, the slope filtration for $\mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}}} \mathcal{M}$ extends uniquely to a filtration of $\mathcal{M}$ by φ-stable saturated $\mathcal{O}_{\mathbf{\Delta}}$-submodules of $\mathcal{M}$. The crucial difference with [52, Lemma 1.3.5] is that our proof only uses the Frobenius map φ, not a logarithmic connection. The argument works for both cases $\mathfrak{o}_0 = \mathbb{Z}_p$ and $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. A similar situation can be found in the proof of de Jong's theorem: Dwork's trick [20, Prop 6.4] can be carried out without a connection. See [47, §5].

Let R be a Bézout domain, and let M be a finite free R-module. We say that an R-submodule $N \subset M$ is *saturated* if N is finitely presented (or equivalently, finite free) and the quotient M/N has no nontrivial R-torsion. Since flatness and torsionfree-ness coincide over a Bézout domain, it is equivalent to require that M/N is free over R. In particular, if $N \subset M$ is saturated, then an R-basis of N extends to an R-basis of M.

Proposition 4.2.1. *Let $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$, and let $\mathcal{N}_{\mathcal{R}} \subset \mathcal{M}_{\mathcal{R}}$ be a φ-stable saturated submodule over $\mathcal{R}$. Then there exists a φ-stable saturated submodule $\mathcal{N} \subset \mathcal{M}$ such that $\mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}}} \mathcal{N} \cong \mathcal{N}_{\mathcal{R}}$.*

Corollary 4.2.2. *Let $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ and $0 = \mathcal{M}_{\mathcal{R},0} \subset \mathcal{M}_{\mathcal{R},1} \subset \cdots \subset \mathcal{M}_{\mathcal{R},c} = \mathcal{M}_{\mathcal{R}}$ be the slope filtration for $\mathcal{M}_{\mathcal{R}} := \mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}}} \mathcal{M}$. Then for each $\mathcal{M}_{\mathcal{R},i}$, there exists a saturated φ-stable submodule $\mathcal{M}_i \subset \mathcal{M}$ such that $\mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}}} \mathcal{M}_i \cong \mathcal{M}_{\mathcal{R},i}$.*

Proof of Proposition. We show the existence of $\mathcal{N}$ in the following steps (4.2.3)–(4.2.6).

4.2.3 Uniqueness

Let I be either $(r,1)$ or $[r,1)$ for some $0 \leq r < 1$, and assume that there exists a saturated submodule $\mathcal{N}_I \subset \mathcal{M}_I$ such that $\mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}_I}} \mathcal{N}_I = \mathcal{N}_{\mathcal{R}}$ as a submodule of $\mathcal{M}_{\mathcal{R}}$. Then we have an equality $\mathcal{N}_I = \mathcal{M}_I \cap \mathcal{N}_{\mathcal{R}}$ inside $M_{\mathcal{R}}$. This can be seen, for example, by choosing a $\mathcal{O}_{\mathbf{\Delta}_I}$-basis for $\mathcal{N}_I$ and extending it to a $\mathcal{O}_{\mathbf{\Delta}_I}$-basis for $\mathcal{M}_I$. Therefore, such $\mathcal{N}_I$ is unique if exists. By taking $I = [0,1)$, we obtain the uniqueness assertion of the proposition.

4.2.4 Reduction to the case when $\mathrm{rank}_{\mathcal{R}}(\mathcal{N}_{\mathcal{R}}) = 1$

This can be done by the following well-known trick. If the proposition holds for rank-1 submodules, then $\det \mathcal{N}_{\mathcal{R}}$ of extends to "$\det \mathcal{N}$" over $\mathcal{O}_{\mathbf{\Delta}}$. (Note that $\mathcal{N}$ is finite free since it is closed in $\mathcal{M}$.) Now one can check that $\mathcal{N} := \{m \in \mathcal{M} | m \wedge x = 0,\ \forall x \in \text{"}\det \mathcal{N}\text{"}\}$ extends $\mathcal{N}_{\mathcal{R}}$.

From now on, assume that $\mathrm{rank}_{\mathcal{R}}(\mathcal{N}_{\mathcal{R}}) = 1$ and let I be either $(r,1)$ or $[r,1)$ for some $0 \leq r < 1$. Consider the submodule $\mathcal{N}_I := \mathcal{M}_I \cap \mathcal{N}_{\mathcal{R}}$ in $\mathcal{M}_I$, which can be seen to be saturated inside $\mathcal{M}_{\mathcal{R}}$. Therefore we have $\mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}_I}} \mathcal{N}_I = \mathcal{N}_{\mathcal{R}}$ if and only if $\mathcal{N}_I \neq \{0\}$. In particular, if $\mathcal{N} := \mathcal{M} \cap \mathcal{N}_{\mathcal{R}} \neq \{0\}$, then $\mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}}} \mathcal{N} = \mathcal{N}_{\mathcal{R}}$.

Claim 4.2.5. *There exists a unique saturated submodule $\mathcal{N}_{(0,1)} \subset \mathcal{M}_{(0,1)}$, such that $\mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}_{(0,1)}}} \mathcal{N}_{(0,1)} = \mathcal{N}_{\mathcal{R}}$.*

This claim is exactly [52, Lemma 1.3.4] if $\mathfrak{o}_0 = \mathbb{Z}_p$, and the same proof works for $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$-case. We give a proof below, closely following the argument of [52, Lemma 1.3.4].

Since $\mathcal{N}_{\mathcal{R}}$ is finitely presented, there exists $r \in (0,1)$ and a saturated $\mathcal{O}_{\mathbf{\Delta}_{(r,1)}}$-submodule $\mathcal{N}_{(r,1)} \subset \mathcal{M}_{(r,1)}$ such that $\mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}_{(r,1)}}} \mathcal{N}_{(r,1)} = \mathcal{N}_{\mathcal{R}}$. The Frobenius map φ on $\mathcal{N}_{\mathcal{R}}$ induces $\varphi : \sigma^*(\mathcal{N}_{(r,1)}) \to \mathcal{N}_{(r^{1/q},1)}$, where $\sigma^*(\mathcal{N}_{(r,1)})$ is the scalar extension by $\sigma : \mathcal{O}_{\mathbf{\Delta}_{(r,1)}} \to \mathcal{O}_{\mathbf{\Delta}_{(r^{1/q},1)}}$.

We set $\mathcal{N}_{(r^q,1)} := \mathcal{M}_{(r^q,1)} \cap \mathcal{N}_{(r,1)}$, which is a saturated submodule of $\mathcal{M}_{(r^q,1)}$. As mentioned in §4.2.4, in order to show that $\mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}_{(r^q,1)}}} \mathcal{N}_{(r^q,1)} = \mathcal{N}_{(r,1)}$, it is enough to show that $\mathcal{N}_{(r^q,1)}$ is non-zero. For this, we look at the following diagram with left exact rows.

$$\begin{array}{ccccccc}
0 \longrightarrow & \sigma^*\left(\mathcal{N}_{(r^q,1)}\right) & \longrightarrow & \sigma^*\left(\mathcal{M}_{(r^q,1)}\right) \oplus \sigma^*\left(\mathcal{N}_{(r,1)}\right) & \longrightarrow & \sigma^*\left(\mathcal{M}_{(r,1)}\right) \\
& \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi\oplus\varphi} & & \downarrow{\scriptstyle\varphi} \\
0 \longrightarrow & \mathcal{N}_{(r,1)} & \longrightarrow & \mathcal{M}_{(r,1)} \oplus \mathcal{N}_{(r^{1/q},1)} & \longrightarrow & \mathcal{M}_{(r^{1/q},1)},
\end{array}$$

where the left horizontal maps are diagonal inclusions and the right horizontal maps are defined by $(a,b) \mapsto a-b$. The top row is left exact since σ is flat. (Recall that a torsion free module over a Bézout domain is always flat.) Furthermore, the central and right vertical maps are injective, so the cokernels of both maps are torsion modules. It follows that the cokernel of the left vertical map is also torsion, which proves that $\mathcal{N}_{(r^q,1)}$ is nonzero.

By repeating this process, we obtain a vector bundle $\mathcal{N}_{(r^{q^n},1)}$ of rank-1 for each n, which glues to give a vector bundle $\mathcal{N}_{(0,1)}$ of rank-1. By construction, $\mathcal{N}_{(0,1)} \subset \mathcal{M}_{(0,1)}$ is saturated and we have $\mathcal{R} \otimes_{\mathcal{O}_{\mathbf{\Delta}_{(0,1)}}} \mathcal{N}_{(0,1)} = \mathcal{N}_{\mathcal{R}}$. The uniqueness of such $\mathcal{N}_{(0,1)}$ follows from (4.2.3). (Here, we identify a vector bundle of rank n on $\mathbf{\Delta}_I$ with its global sections, which is necessarily a free $\mathcal{O}_{\mathbf{\Delta}_I}$-module of rank n. See §6.1.5 for more discussions.)

4.2.6 Extending $\mathcal{N}_{(0,1)}$ to $\mathcal{N}$

This is the key step. Roughly speaking, we extend a saturated $\mathcal{O}_{\mathbf{\Delta}_{(0,1)}}[\frac{1}{\lambda}]$-submodule $\mathcal{N}_{(0,1)}[\frac{1}{\lambda}] \subset \mathcal{M}_{(0,1)}[\frac{1}{\lambda}]$ to a saturated $\mathcal{O}_{\mathbf{\Delta}}[\frac{1}{\lambda}]$-submodule $\mathcal{N}[\frac{1}{\lambda}] \subset \mathcal{M}[\frac{1}{\lambda}]$, and glue $\mathcal{N}[\frac{1}{\lambda}]$ and $\mathcal{N}_{(0,1)}$ to obtain $\mathcal{N}$. The point is that we have the φ-compatible section $\xi : \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} (\mathcal{M}/u\mathcal{M}) \hookrightarrow \mathcal{M}$, whose cokernel is killed by some power of λ (Proposition 3.2.1). We use this to find a basis for $\mathcal{M}[\frac{1}{\lambda}]$ which makes the "φ-matrix" very simple.

Let $\bar{\mathbf{e}}_1, \cdots, \bar{\mathbf{e}}_n$ be a $\mathscr{K}_0$ basis for $\mathcal{M}/u\mathcal{M}$, and we put $\mathbf{e}_i := \xi(1 \otimes \bar{e}_i)$. Then $\{\mathbf{e}_i\}$ is a $\mathcal{O}_{\mathbf{\Delta}}[\frac{1}{\lambda}]$-basis for $\mathcal{M}[\frac{1}{\lambda}]$. By construction, the matrix for $\varphi_{\mathcal{M}}[\frac{1}{\lambda}]$ with respect to the basis $\{\mathbf{e}_i\}$ is the same as the matrix for $\bar{\varphi} := \varphi_{\mathcal{M}/u\mathcal{M}}$ with respect to the basis $\{\bar{\mathbf{e}}_i\}$. In particular, all the entries of this matrix lie in $\mathscr{K}_0$. (In fact, if $\bar{\varphi}(\sigma^*\bar{\mathbf{e}}_j) = \sum_i \alpha_{ij}\bar{\mathbf{e}}_i$ with $\alpha_{ij} \in \mathscr{K}_0$, then we have $\varphi(\sigma^*\mathbf{e}_j) = \xi\left(\varphi(\sigma^*\bar{\mathbf{e}}_j)\right) = \sum_i \alpha_{ij}\mathbf{e}_i$.)

Let $\mathbf{\Delta}_{\mathscr{K}_0(\bar{k})}$ be the open unit disk over $\mathscr{K}_0(\bar{k})$, and let $\mathcal{M}_{\mathscr{K}_0(\bar{k})}$ denote $\mathcal{O}_{\mathbf{\Delta}_{\mathscr{K}_0(\bar{k})}} \otimes_{\mathcal{O}_{\mathbf{\Delta}}} \mathcal{M}$. By the Dieudonné-Manin decomposition over $\mathscr{K}_0(\bar{k})$ (Theorem 4.1.2), we can find a $\mathscr{K}_0(\bar{k})$-basis $\{\bar{\mathbf{e}}'_j\}$ for $\mathcal{M}_{\mathscr{K}_0(\bar{k})}/u\mathcal{M}_{\mathscr{K}_0(\bar{k})} \cong \mathscr{K}_0(\bar{k}) \otimes_{\mathscr{K}_0} (\mathcal{M}/u\mathcal{M})$ so that $\varphi^n(\sigma^{*n}\bar{\mathbf{e}}'_j) = \pi_0^{d_j}\bar{\mathbf{e}}'_j$. We put $\mathbf{e}'_j := \xi_{\mathscr{K}_0(\bar{k})}(1 \otimes \bar{\mathbf{e}}'_j)$, where $\xi_{\mathscr{K}_0(\bar{k})} = \mathcal{O}_{\mathbf{\Delta}_{\mathscr{K}_0(\bar{k})}} \otimes \xi$. By construction the bases $\{\mathbf{e}'_j\}$ and $\{\mathbf{e}_i\}$ are related by $\mathrm{GL}_n(\mathscr{K}_0(\bar{k}))$. Since $\xi_{\mathscr{K}_0(\bar{k})}$ is φ-compatible by construction, the matrix for $\varphi_{\mathcal{M}_{\mathscr{K}_0(\bar{k})}}[\frac{1}{\lambda}]$ with respect to the basis $\{\mathbf{e}'_i\}$ is in $\mathrm{GL}_n(\mathscr{K}_0(\bar{k}))$ by the same argument as above.

Let $\mathbf{\Delta}_{\mathscr{K}_0(\bar{k}),(0,1)}$ be the punctured open unit disk over $\mathscr{K}_0(\bar{k})$. Choose a $\mathcal{O}_{\mathbf{\Delta}_{\mathscr{K}_0(\bar{k}),(0,1)}}$-basis $\mathbf{e} \in \mathcal{N}_{(0,1)}$, and express it as a linear combination of $\mathbf{e}_i$ as follows:

$$\mathbf{e} = \frac{1}{g}\sum_{i=1}^{n} f_i\mathbf{e}_i = \frac{1}{g}\sum_{i=1}^{n} f'_i\mathbf{e}'_i, \quad f_i, g \in \mathcal{O}_{\mathbf{\Delta}_{(0,1)}},$$

where g divides $\leqslant a$ for $a \geq 0$. We choose f_i and g so that f_i and λ generate the unit ideal in $\mathcal{O}_{\mathbf{\Delta}_{(0,1)}}$. As above, $f'_j \in \mathcal{O}_{\mathbf{\Delta}_{\mathscr{K}_0(\bar{k}),(0,1)}}$ are $\mathscr{K}_0(\bar{k})$-linear combinations of

f_i and conversely. Then the proposition can be reduced to the following claim.

Claim 4.2.6.1. *There exists* $f' \in \mathcal{O}_{\mathbf{\Delta}_{\mathscr{K}_0(\bar{k}),(0,1)}}$ *such that* $f'_j = c'_j \cdot f'$ *where* $c'_j \in \mathscr{K}_0(\bar{k})$*.*

Let us grant the claim for a moment. Since f_i are $\mathscr{K}_0(\bar{k})$-linear combinations of f'_j, we can write $f_i = c''_i \cdot f'$ for $c''_i \in \mathscr{K}_0(\bar{k})$. Hence, the ratio for nonzero f_i and f_j satisfies $f_i/f_j = c''_i/c''_j \in \mathscr{K}_0(\bar{k}) \cap \mathrm{Frac}(\mathcal{O}_{\mathbf{\Delta}_{(0,1)}}) = \mathscr{K}_0$, so we may write $f_i = c_i \cdot f$, for some $c_i \in \mathscr{K}_0$ and some $f \in \mathcal{O}_{\mathbf{\Delta}_{(0,1)}}$ that is coprime to λ by our choice of f_i. Set

$$\mathbf{e}_0 := \frac{1}{f}\mathbf{e} = \frac{1}{g}\sum_{i=1}^{n} c_i \mathbf{e}_i.$$

Observe that $\mathbf{e}_0$ is an element in $\mathcal{N}_{(0,1)} = \mathcal{N}_{(0,1)}[\frac{1}{f}] \cap \mathcal{N}_{(0,1)}[\frac{1}{\lambda}]$ and generates $\mathcal{N}_{(0,1)}$ over $\mathcal{O}_{\mathbf{\Delta}_{(0,1)}}$. Furthermore, $\mathbf{e}_0$ belongs to $\mathcal{M} = \mathcal{M}[\frac{1}{\lambda}] \cap \mathcal{M}_{(0,1)}$. Now, $\mathcal{N} := \mathcal{O}_{\mathbf{\Delta}} \cdot \mathbf{e}_0 \subset \mathcal{M}$ is the submodule which extends $\mathcal{N}_{\mathcal{R}}$.

It is left to prove Claim 4.2.6.1. Let $\alpha \in \mathcal{O}_{\mathbf{\Delta}_{(0,1)}}$ be such that $\varphi^n(\sigma^{*n}\mathbf{e}) = \alpha \mathbf{e}$, where n is the rank of $\mathcal{M}$. Since $\varphi^n(\sigma^{*n}\mathbf{e}'_j) = \pi_0^{d_j}\mathbf{e}'_j$, we obtain, for each j,

$$\alpha \frac{f'_j}{g} = \pi_0^{d_j} \cdot \sigma^n(\frac{f'_j}{g}) = \pi_0^{d_j} \cdot \frac{\sigma^n(f'_j)}{\sigma^n(g)},$$

Here, the divisions are performed inside $\mathrm{Frac}\left(\mathcal{O}_{\mathbf{\Delta}_{\mathscr{K}_0(\bar{k}),(0,1)}}\right)$. So we get that $\alpha\sigma^n(g) \cdot f'_j = \pi_0^{d_j} g \sigma^n(f'_j)$. Hence, for any pair of nonzero f'_i and f'_j, we have $\sigma^n(\frac{f'_i}{f'_j}) = \pi_0^{d_i - d_j}\frac{f'_i}{f'_j}$. By lemma 4.2.6.3, we are reduced to the following claim:

Claim 4.2.6.2. *Let* $f = \sum_{i\in\mathbb{Z}} a_n u^n \in \mathscr{K}_0(\bar{k})[[u, \frac{1}{u}]]$*, and assume that* $\pi_0^d \sigma^n(f) = f$ *for some* $d \in \mathbb{Z}$*. Then* $d = 0$ *and* $f \in \mathscr{K}_0(\bar{k})$*, which is fixed by* σ^n*. (In other words,* $f \in W(\mathbb{F}_{p^n})$ *if* $\mathfrak{o}_0 = \mathbb{Z}_p$*, and* $f \in \mathbb{F}_{q^n}(\!(\pi_0)\!)$ *if* $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$*.)*

The equation $\sum_{i\in\mathbb{Z}} \pi_0^d \sigma^n(a_n) u^{qn} = \sum_{i\in\mathbb{Z}} a_n u^n$ forces that $a_n = 0$ for $n \neq 0$. Since $\sigma : \mathscr{K}_0(\bar{k}) \to \mathscr{K}_0(\bar{k})$ preserves π_0-order, $d = 0$ and $\sigma^n(a_0) = a_0$.

To complete the proof of the proposition. it is left to show the following lemma:

□

Lemma 4.2.6.3. *Let F be a complete discretely valued field with residue characteristic p. For any subinterval $I \in [0,1)$ with endpoints in $\{0\} \cup p^{\mathbb{Q}_{\leqslant 0}}$, let $\mathbf{\Delta}_{F,I}$ be the subdomain of the open unit disk over F with coordinate u which is defined by the "suitable" boundary condition corresponding to I. Then the natural map* $\mathrm{Frac}(\mathcal{O}_{\mathbf{\Delta}_{F,I}}) \to F[[u, \frac{1}{u}]]$ *of* F-vector spaces*, which sends a "meromorphic" function f to its formal infinite-tailed Laurent expansion in u, is injective.*

Note that $F[[u, \frac{1}{u}]]$ does not have a natural ring structure; the expression

$$(\sum_{i\in\mathbb{Z}} \alpha_i u^i)\cdot(\sum_{j\in\mathbb{Z}} \beta_j u^j) = \sum_{n\in\mathbb{Z}}(\sum_{i+j=n} \alpha_i\beta_j)u^n$$

for $\alpha_i, \beta_j \in F$ does not make sense without any convergence assumption on (possibly infinite) sums $\sum_{i+j=n} \alpha_i\beta_j$ for each $n \in \mathbb{Z}$. Therefore the natural inclusion $\mathcal{O}_{\mathbf{\Delta}_{F,I}} \hookrightarrow F[[u, \frac{1}{u}]]$ does not imply the lemma.

Proof. Choose $f, g \in \mathcal{O}_{\mathbf{\Delta}_{F,I}}$ so that the formal Laurent expansion of f/g is zero. Then we want to show that $f = 0$. We first handle the case when $I = [r,r]$ for some $r \in p^{\mathbb{Q}_{\leqslant 0}}$. Then for any point $x \in \mathbf{\Delta}_{F,[r,r]}$ such that $g(x) \neq 0$, $(f/g)(x)$ makes sense and is zero. In particular, $f(x) = 0$ for all but finitely many points $x \in \mathbf{\Delta}_{F,[r,r]}$. But since the zero locus of f is a "closed" affinoid subdomain, $f(x) = 0$ for any point $x \in \mathbf{\Delta}_{F,[r,r]}$. Therefore $f = 0$ (since the sup norm on $\mathbf{\Delta}_{F,[r,r]}$ is a *norm*, not just a semi-norm.)

Now assume that I has a non-zero length. Then we can find a closed subinterval $J \subset I$ such that g does not vanish in $\mathbf{\Delta}_{F,J}$. This implies that g is a unit in $\mathcal{O}_{\mathbf{\Delta}_{F,J}}$ by Remark 6.2.3(1) and Proposition 6.2.6.1 (or by some direct computation), so f/g is a rigid-analytic function on $\mathbf{\Delta}_{F,J}$. But since the natural map $\mathcal{O}_{\mathbf{\Delta}_{F,J}} \to F[[u, \frac{1}{u}]]$ is injective, we obtain $f = 0$ in $\mathcal{O}_{\mathbf{\Delta}_{F,J}}$. Since the natural "restriction" map $\mathcal{O}_{\mathbf{\Delta}_{F,I}} \to \mathcal{O}_{\mathbf{\Delta}_{F,J}}$ is injective, so $f = 0$ in $\mathcal{O}_{\mathbf{\Delta}_{F,I}}$. $\square$

4.3 φ-vector bundle pure of slope 0 and weak admissibility

Recall that $\underline{\mathrm{Mod}}^{\mathrm{sl}=0}_{\mathbf{\Delta}}(\varphi) \subset \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ denotes the full subcategory of φ-vector bundles pure of slope 0; i.e., $\mathcal{M}$ such that $\mathcal{R}^{\mathrm{alg}} \otimes_{\mathcal{O}_{\mathbf{\Delta}}} \mathcal{M}$ is pure of slope 0. We first show that the scalar extension functor induces an equivalence of categories $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \xrightarrow{\sim} \underline{\mathrm{Mod}}^{\mathrm{sl}=0}_{\mathbf{\Delta}}(\varphi)$, up to a certain technical lemma whose proof will be given later in §6.3. Next, we show that the weak admissibility on $\mathcal{HP}^{\geqslant 0}_K(\varphi)$ is equivalent to the "pure-of-slope-0" condition on $\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ under the equivalences of categories $\underline{\mathcal{M}}$ and $\underline{D}$. The proof uses the slope filtration on $\mathcal{M} \in \underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ by φ-stable saturated $\mathcal{O}_{\mathbf{\Delta}}$-submodules (Corollary 4.2.2). Combining these two results, we see that the functor $\underline{\mathbb{H}}$, defined in (3.2.6.1), induces an equivalence of categories $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \xrightarrow{\sim} \mathcal{HP}^{wa,\geqslant 0}_K(\varphi)$.

We start with the following well-known lemma, which we call the "extension lemma."

Lemma 4.3.1. *Let $\mathcal{M}^{bd}$ be a finite free $\mathfrak{S}[\frac{1}{\pi_0}]$-module*[3]*, and M a finite free $\mathfrak{o}_{\mathcal{E}}$-module such that there exists an $\mathcal{E}$-isomorphism $\alpha : \mathcal{E} \otimes_{\mathfrak{S}[\frac{1}{\pi_0}]} \mathcal{M}^{bd} \xrightarrow{\sim} \mathcal{E} \otimes_{\mathfrak{o}_{\mathcal{E}}} M$. Then there exists a finite free $\mathfrak{S}$-module $\mathfrak{M}$, and isomorphisms $\beta : \mathfrak{S}[\frac{1}{\pi_0}] \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\sim} \mathcal{M}^{bd}$ and $\gamma : \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\sim} M$ over $\mathfrak{S}[\frac{1}{\pi_0}]$ and $\mathfrak{o}_{\mathcal{E}}$, respectively, such that $\alpha \circ \beta = \gamma$; the triple $(\mathfrak{M}, \beta, \gamma)$ is unique up to unique isomorphism.*

If M and $\mathcal{M}^{bd}$ are φ-modules over their respective base rings and α is a φ-compatible isomorphism, then one can give a unique φ structure on $\mathfrak{M}$ so that β and γ are φ-compatible. If, furthermore, M is an étale φ-module and the cokernel of

[3]We use the notation $\mathcal{M}^{bd}$ because $\mathfrak{S}[\frac{1}{\pi_0}]$ is the ring of bounded global rigid-analytic functions on the open unit disk.

$\varphi_{\mathcal{M}^{bd}}:\sigma^*\mathcal{M}^{bd}\to\mathcal{M}^{bd}$ is annihilated by $\mathcal{P}(u)^h$ then the cokernel of $\varphi_{\mathfrak{M}}:\sigma^*\mathfrak{M}\to\mathfrak{M}$ is annihilated by $\mathcal{P}(u)^h$; i.e., $\mathfrak{M}\in\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$.

Therefore, the above lemma can be viewed as an analogue of the result that on a smooth surface, a vector bundle defined outside a closed point uniquely extends over the point.

Proof. Let us first handle the case *without* φ-structure. Let $\mathfrak{S}_{(\pi_0)}$ be the localization of $\mathfrak{S}[\frac{1}{u}]$ at the prime ideal $\pi_0\mathfrak{S}[\frac{1}{u}]$. Note that $\mathfrak{o}_{\mathcal{E}}$ is the π_0-adic completion of $\mathfrak{S}_{(\pi_0)}$. We first observe the following general fact whose proof is immediate:

Claim. *Let R be a discrete valuation ring with maximal ideal $\mathfrak{m}_R$, and $\widehat{R}$ the $\mathfrak{m}_R$-adic completion of R. Let $F:=\mathrm{Frac}\,R$ and $\widehat{F}:=\mathrm{Frac}\,\widehat{R}$. Let V be a finite-dimensional vector space over F. Then there exists a natural bijective correspondence between the set of R-lattices M in V and the set of $\widehat{R}$-lattices $\widehat{M}$ in $\widehat{F}\otimes_F V$, as follows: $M\mapsto\widehat{R}\otimes_R M$ and $\widehat{M}\mapsto V\cap\widehat{M}$ where the intersection is taken inside $\widehat{F}\otimes_F V$.*

Applying this claim to $R=\mathfrak{S}_{(\pi_0)}$ and $V:=\mathrm{Frac}\,\mathfrak{S}\otimes_{\mathfrak{S}[\frac{1}{\pi_0}]}\mathcal{M}^{bd}$, we obtain a unique $\mathfrak{S}_{(\pi_0)}$-lattice $\mathfrak{M}_{(\pi_0)}$ in V such that $\mathfrak{o}_{\mathcal{E}}\otimes_{\mathfrak{S}_{(\pi_0)}}\mathfrak{M}_{(\pi_0)}=M$ in $\mathcal{E}\otimes_{\mathfrak{S}[\frac{1}{\pi_0}]}\mathcal{M}^{bd}$. (Note that we view M as an $\mathfrak{o}_{\mathcal{E}}$-lattice in $\mathcal{E}\otimes_{\mathfrak{S}[\frac{1}{\pi_0}]}\mathcal{M}^{bd}$ via the isomorphism $\alpha:\mathcal{E}\otimes_{\mathfrak{S}[\frac{1}{\pi_0}]}\mathcal{M}^{bd}\xrightarrow{\sim}\mathcal{E}\otimes_{\mathfrak{o}_{\mathcal{E}}}M$.) Now $\mathfrak{M}_{(\pi_0)}$ "smears out" to a vector bundle over some open neighborhood of $(\pi_0)\in\mathrm{Spec}\,\mathfrak{S}[\frac{1}{u}]$. Gluing this with $\mathcal{M}^{bd}$ (a vector bundle on $\mathrm{Spec}\,\mathfrak{S}[\frac{1}{\pi_0}]$) we obtain a vector bundle $\mathfrak{M}^{(*)}$ on $(\mathrm{Spec}\,\mathfrak{S})-V(\mathfrak{m}_{\mathfrak{S}})$ where $V(\mathfrak{m}_{\mathfrak{S}})$ is the closed point of $\mathrm{Spec}\,\mathfrak{S}$. By [73, 2, Exp XI, Corollaire 3.8] we obtain a unique vector bundle $\mathfrak{M}$ on $\mathrm{Spec}\,\mathfrak{S}$ which extends $\mathfrak{M}^{(*)}$. By construction, we are naturally given isomorphisms $\beta:\mathfrak{S}[\frac{1}{\pi_0}]\otimes_{\mathfrak{S}}\mathfrak{M}\xrightarrow{\sim}\mathcal{M}^{bd}$ and $\gamma:\mathfrak{o}_{\mathcal{E}}\otimes_{\mathfrak{S}}\mathfrak{M}\xrightarrow{\sim}M$ over $\mathfrak{S}[\frac{1}{\pi_0}]$ and $\mathfrak{o}_{\mathcal{E}}$, respectively, as asserted in the statement. Furthermore, we have by construction that $\mathfrak{M}=\mathcal{M}^{bd}\cap M$ inside $\mathcal{E}\otimes_{\mathfrak{S}[\frac{1}{\pi_0}]}\mathcal{M}^{\mathrm{bd}}$ (which is identified with $\mathcal{E}\otimes_{\mathfrak{o}_{\mathcal{E}}}M$ via α).

Let us prove the claim regarding φ-structure. Clearly, $\mathfrak{M} = \mathcal{M}^{bd} \cap M$ is a φ-stable $\mathfrak{S}$-submodule of both $\mathcal{M}^{bd}$ and M. Now, assume that M is an étale φ-module and $\mathcal{P}(u)^h$ annihilates the cokernel of $\varphi_{\mathcal{M}^{bd}} : \sigma^*\mathcal{M}^{bd} \to \mathcal{M}^{bd}$. Using β, $\varphi_{\mathfrak{M}} : \sigma^*\mathfrak{M} \to \mathfrak{M}$ has cokernel killed by $\mathcal{P}(u)^h$ after inverting π_0. But $\operatorname{coker} \varphi^{\mathfrak{M}}$ vanishes after scalar extention to $\mathfrak{S}_{\widehat{(\pi_0)}} = \mathfrak{o}_{\mathcal{E}}$ due to γ, so $\operatorname{coker} \varphi_{\mathfrak{M}}$ has no nontrivial π_0-torsion. In other words, $\operatorname{coker} \varphi_{\mathfrak{M}}$ is killed by $\mathcal{P}(u)^h$. □

We need another auxiliary lemma, which we call a "gluing lemma" or "matrix factorization lemma." We give the full proof later in §6.3.

Proposition 4.3.2. *For any $A \in \mathrm{GL}_n(\mathcal{R})$, there exists $U \in \mathrm{GL}_n(\mathcal{O}_{\boldsymbol{\Delta}})$ and $V \in \mathrm{GL}_n(\mathcal{R}^{bd})$ such that $A = UV$.*

Proof. If $\mathfrak{o}_0 = \mathbb{Z}_p$, then the proposition is exactly [46, Prop 6.5]. The discussion in [46, §6] carries over word-by-word to the case of $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. For interested readers, see §6.3 of this paper. □

Now we are ready to prove the following:

Proposition 4.3.3. *The scalar extension $\mathfrak{M}[\frac{1}{\pi_0}] \mapsto \mathcal{O}_{\boldsymbol{\Delta}} \otimes_{\mathfrak{S}[\frac{1}{\pi_0}]} \mathfrak{M}[\frac{1}{\pi_0}]$ induces an equivalence of $\otimes$-categories $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \xrightarrow{\sim} \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}^{\mathrm{sl}=0}(\varphi)$. Furthermore, a three-term complex $(\dagger) :\ 0 \to \mathfrak{M}'[\frac{1}{\pi_0}] \to \mathfrak{M}[\frac{1}{\pi_0}] \to \mathfrak{M}''[\frac{1}{\pi_0}] \to 0$ is short exact in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}]$ if and only if $\mathcal{O}_{\boldsymbol{\Delta}} \otimes (\dagger)$ is short exact in $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}^{\mathrm{sl}=0}(\varphi)$.*

Proof. For any $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$, the scalar extension $\mathcal{O}_{\boldsymbol{\Delta}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is necessarily pure of slope 0. In fact, $\mathfrak{o}_{\mathcal{R}^{bd}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is an étale φ-module since $\mathcal{P}(u) \in (\mathfrak{o}_{\mathcal{R}^{bd}})^{\times}$, and is a φ-stable $\mathfrak{o}_{\mathcal{R}^{bd}}$-lattice of $\mathcal{R} \otimes_{\mathfrak{S}} \mathfrak{M}$. Now, the claim follows from the discussion in Lemma 4.1.7. The exactness assertion follows since $\mathcal{O}_{\boldsymbol{\Delta}}$ is faithfully flat over $\mathfrak{S}[\frac{1}{\pi_0}]$ by Proposition 6.2.8.

Fix any $\mathcal{M} \in \underline{\mathrm{Mod}}^{\mathrm{sl}=0}_{\mathbf{\Delta}}(\varphi)$, free of rank n. By Theorem 4.1.6(1), there exists $\mathcal{M}_{\mathcal{R}^{bd}} \in \underline{\mathrm{Mod}}^{\mathrm{sl}=0}_{\mathcal{R}^{bd}}(\varphi)$ such that $\mathcal{R} \otimes_{\mathcal{R}^{bd}} \mathcal{M}_{\mathcal{R}^{bd}} \cong \mathcal{M}_{\mathcal{R}}$. Hence $\mathcal{M}_{\mathcal{R}}$ carries two $\mathcal{R}$-bases: one from $\mathcal{O}_{\mathbf{\Delta}}$-basis for $\mathcal{M}$ and the other from $\mathcal{R}^{bd}$-basis for $\mathcal{M}_{\mathcal{R}^{bd}}$. They are related by a matrix in $\mathrm{GL}_n(\mathcal{R})$, but the preceeding "gluing lemma" (Proposition 4.3.2) implies that one can modify the chosen bases so that they coincide in $\mathcal{M}_{\mathcal{R}}$.

Let $\mathcal{M}^{bd}$ be the $\mathfrak{S}[\frac{1}{\pi_0}]$-span of this common basis. Since $\mathfrak{S}[\frac{1}{\pi_0}] = \mathcal{R}^{bd} \cap \mathcal{O}_{\mathbf{\Delta}}$, we have an equality $\mathcal{M}^{bd} = \mathcal{M}_{\mathcal{R}^{bd}} \cap \mathcal{M}$ as a submodule of $\mathcal{M}_{\mathcal{R}}$. Therefore $\mathcal{M}^{bd}$ is a φ-stable $\mathfrak{S}[\frac{1}{\pi_0}]$-submodule of both $\mathcal{M}_{\mathcal{R}^{bd}}$ and $\mathcal{M}$. Now we obtain the full faithfulness as follows. Assuming $\mathcal{M} = \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathfrak{S}} \mathfrak{M}$ for some $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$, the construction above gives $\mathcal{M}^{bd} = \mathfrak{M}[\frac{1}{\pi_0}]$. And thanks to Theorem 4.1.6(1), any morphisms $\mathfrak{M} \otimes_{\mathfrak{S}} \mathcal{O}_{\mathbf{\Delta}} \to \mathfrak{M}' \otimes_{\mathfrak{S}} \mathcal{O}_{\mathbf{\Delta}}$ of $\underline{\mathrm{Mod}}_{\mathbf{\Delta}}(\varphi)$ restrict to $\mathfrak{M}[\frac{1}{\pi_0}] \to \mathfrak{M}'[\frac{1}{\pi_0}]$.

For the essential surjectivity, the "extension lemma" (Lemma 4.3.1) produces the φ-stable $\mathfrak{S}$-lattice $\mathfrak{M}$ of both $\mathcal{M}^{bd}$ and $\mathfrak{o}_{\mathcal{E}} \otimes_{\mathcal{R}^{bd}} \mathcal{M}_{\mathcal{R}^{bd}}$, which is of $\mathcal{P}$-height $\leqslant h$ if $\mathcal{M}^{bd}$ is. On the other hand, if $\mathcal{M} \cong \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathfrak{S}[\frac{1}{\pi_0}]} \mathcal{M}^{bd}$ is of $\mathcal{P}$-height $\leqslant h$, then so is $\mathcal{M}^{bd}$ by the faithful flatness of $\mathcal{O}_{\mathbf{\Delta}}$ over $\mathfrak{S}[\frac{1}{\pi_0}]$ (Proposition 6.2.8). □

Theorem 4.3.4. *Let $D \in \mathcal{HP}^{\geqslant 0}_K(\varphi)$. Then D is weakly admissible if and only if $\underline{\mathcal{M}}(D)$ is pure of slope 0. In particular, $\underline{\mathbb{H}} : \mathfrak{M}[\frac{1}{\pi_0}] \mapsto \underline{D}\left(\mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathfrak{S}[\frac{1}{\pi_0}]} \mathfrak{M}[\frac{1}{\pi_0}]\right)$ induces an equivalence of categories $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \mathcal{HP}^{wa,\geqslant 0}_K(\varphi)$. Furthermore, a three-term complex $(\dagger)$: $0 \to \mathfrak{M}'[\frac{1}{\pi_0}] \to \mathfrak{M}[\frac{1}{\pi_0}] \to \mathfrak{M}''[\frac{1}{\pi_0}] \to 0$ is short exact in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}]$ if and only if $\underline{\mathbb{H}}(\dagger)$ is short exact in $\mathcal{HP}^{wa,\geqslant 0}_K(\varphi)$.*

Proof. Granting that D is weakly admissible if and only if $\underline{\mathcal{M}}(D)$ is pure of slope 0, it follows from Propositions 4.3.3 and 3.2.5(3) that $\underline{\mathbb{H}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \mathcal{HP}^{\geqslant 0}_K(\varphi)$ is fully faithful with essential image $\mathcal{HP}^{wa,\geqslant 0}_K(\varphi)$. The exactness assertion follows from Propositions 3.1.6(2) and 3.2.5(2), and the exactness assertion of Proposition 4.3.3.

We first verify that

$$\deg(\underline{\mathcal{M}}(D)) = t_N(D) - t_H(D) \tag{4.3.4.1}$$

Since $\underline{\mathcal{M}}(\cdot)$ commutes with $\otimes$-product (Proposition 3.1.6), one can replace D with its determinant and reduce the verification of the equality (4.3.4.1) to the rank-1 case. In the rank-1 case, (4.3.4.1) can be directly read off from the computation of $\underline{\mathcal{M}}(D)$ which is done in §3.1.4, especially from (3.1.4.1). This verifies (4.3.4.1), and proves the theorem for the rank-1 case.

Now, assume that $\underline{\mathcal{M}}(D)$ is pure of slope 0 and of any rank. Then for any subobject D', we have $\deg(\underline{\mathcal{M}}(D')) \geq 0$. In fact, this can be checked after extending scalars to $\mathcal{R}^{\mathrm{alg}}$, and then the claim follows from Proposition 4.1.5. By (4.3.4.1), it implies that D is weakly admissible.

Now, assume that D is an isocrystal with weakly admissible effective Hodge-Pink structure. By Corollary 4.2.2, we have the following "slope filtration" for $\underline{\mathcal{M}}(D)$ by φ-stable saturated modules on $\mathbf{\Delta}$:

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_c = \underline{\mathcal{M}}(D)$$

Let s_i be the unique slope for $\mathcal{M}_i/\mathcal{M}_{i-1}$ and n_i be the rank of $\mathcal{M}_i/\mathcal{M}_{i-1}$. Put $D_i := \underline{D}(\mathcal{M}_i)$. By extending scalars to $\mathcal{R}^{\mathrm{alg}}$ and applying the Dieudonné-Manin classification (Theorem 4.1.2), one can see that $\deg(\underline{\mathcal{M}}(D)) = \sum s_i n_i$ and $\deg(\mathcal{M}_1) = s_1 n_1$. The weak admissibility implies $\sum s_i n_i = \deg(\underline{\mathcal{M}}(D)) = t_N(D) - t_H(D) = 0$ and $\deg(\mathcal{M}_1) = t_N(D_1) - t_H(D_1) \geq 0$, so $s_i \geq 0$. But since $s_1 < s_i$ for any $i \neq 1$, we must have $c = 1$ and $s_1 = 0$; i.e., $\underline{\mathcal{M}}(D)$ is pure of slope 0. □

Remark 4.3.5. Since $\underline{\mathbb{H}}$ commutes with $\otimes$-products (in particular, with Tate twists), we may immediately extend the above theorem, as follows: there exists an equivalence of categories $\underline{\mathbb{H}}$ from generalized φ-modules over $\mathfrak{S}$ as in §2.2.11 to isocrystals with weakly admissible Hodge-Pink structures, which commutes with all the natural operations, such as $\otimes$-products, internal homs, and duality.

4.3.6 Rank-1 example

Let D be a rank-1 isocrystal with weakly admissible effective Hodge-Pink structure, and we put $\widehat{\mathcal{D}}_{x_0} := \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \otimes_{\mathscr{K}_0} D$. We choose a $\mathscr{K}_0$-basis $\mathbf{e} \in D$ and write $\varphi_D(\sigma^*\mathbf{e}) = (\alpha\pi_0^h)\cdot\mathbf{e}$ for some $\alpha \in W^\times$ and $h \geqslant 0$. By weak admissibility the Hodge-Pink structure is $\Lambda = \mathcal{P}(u)^{-h}\widehat{\mathcal{D}}_{x_0}$.

In §3.1.4, we have seen that $\underline{\mathcal{M}}(D) = \lambda^{-h}\mathcal{D} \subset \mathcal{D}[\frac{1}{\lambda}]$, where $\mathcal{D} := \mathcal{O}_{\mathbf{\Delta}} \otimes_{\mathscr{K}_0} D \cong \mathcal{O}_{\mathbf{\Delta}}\mathbf{e}$ by choosing a $\mathscr{K}_0$-basis $\mathbf{e}$ for D. We choose the following $\mathcal{O}_{\mathbf{\Delta}}$-basis $\mathbf{e}' := \lambda^{-h}\mathbf{e} \in \underline{\mathcal{M}}(D)$ of $\underline{\mathcal{M}}(D)$, so we have by (3.1.4.1) that $\varphi_{\underline{\mathcal{M}}(D)}(\sigma^*\mathbf{e}') = \alpha\pi_0^h\left(\frac{\mathcal{P}(u)}{\mathcal{P}(0)}\right)^h\mathbf{e}' = \alpha\mathcal{P}(u)^h\mathbf{e}'$, using our normalization $\mathcal{P}(0) = \pi_0$.

Clearly $\mathfrak{M} := \mathfrak{S}\cdot\mathbf{e}'$ is a φ-stable $\mathfrak{S}$-lattice in $\underline{\mathcal{M}}(D)$. By Proposition 4.3.3, such a $\mathfrak{S}$-lattice $\mathfrak{M}$ is unique up to isogeny. Therefore $\underline{\mathbb{H}}(\mathfrak{M}[\frac{1}{\pi_0}]) = D$ where $\mathfrak{M} \cong \mathfrak{S}\mathbf{e}'$ with $\varphi_{\mathfrak{M}}(\sigma^*\mathbf{e}') = (\alpha\mathcal{P}(u)^h)\cdot\mathbf{e}'$ and D is as above. Applying this to the case $\alpha = 1$, we obtain $\underline{\mathbb{H}}(\mathfrak{S}(h)[\frac{1}{\pi_0}]) = \mathbf{1}(h)$ where $\mathfrak{S}(h)$ is the Tate object as defined in Definition 2.2.6 and is the Tate object $\mathbf{1}(h)$ as defined in (2.3.2.1). Note that we used the normalization $\mathcal{P}(0) = \pi_0$ for getting $\underline{\mathbb{H}}(\mathfrak{S}(h)) = \mathbf{1}(h)$; otherwise, the formula would involve some suitable "unramified twist" corresponding to $\mathcal{P}(0)/\pi_0 \in W^\times$.

CHAPTER V

π_0-adic $\mathcal{G}_K$-representation of finite $\mathcal{P}$-height

Let $\mathrm{Rep}_{\mathfrak{o}_0}(\mathcal{G}_K)$ denote the category of finitely generated (*not* necessarily free) $\mathfrak{o}_0$-modules with continuous linear $\mathcal{G}_K$-action, with the obvious notion of morphism. We also let $\mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\mathcal{G}_K)$ (respectively, $\mathrm{Rep}^{\mathrm{tor}}_{\mathfrak{o}_0}(\mathcal{G}_K)$) denote the full subcategory of $\mathrm{Rep}_{\mathfrak{o}_0}(\mathcal{G}_K)$, whose objects have free (respectively, torsion) underlying $\mathfrak{o}_0$-modules. We have obvious notions of $\otimes$-product, internal hom, and duality for this category.

In this chapter, we construct a contravariant functor $T^*_{\mathfrak{S}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi) \to \mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\mathcal{G}_K)$, and show that it is fully faithful. The construction of $T^*_{\mathfrak{S}}$ uses Fontaine's theory of étale φ-modules (or its variant for $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$). To show the full faithfulness, we use equivalences of categories discussed in §III–§IV. The essential image of $T^*_{\mathfrak{S}}$ will be the main object of study in the later part of our work.

5.1 Étale φ-modules and π_0-adic representations of $\mathcal{G}_K$

Fontaine's theory of étale φ-moduless [31, §A1.2] gives a classification of $\mathbb{Z}_p$-lattice $\mathcal{G}_K$-representations via étale φ-modules over $\mathfrak{o}_{\mathcal{E}}$; in other words, an equivalence of categories between $\mathrm{Rep}_{\mathfrak{o}_0}(\mathcal{G}_K)$ and $\underline{\mathrm{Mod}}^{\text{ét}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)$ when $\mathfrak{o}_0 = \mathbb{Z}_p$. But in fact, Fontaine's argument carries over to prove the "same" equivalence of categories for $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. In this section, we reproduce [31, §A1.2] in a way that works for both cases $\mathfrak{o}_0 = \mathbb{Z}_p$ and $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. In this section (§5.1), we do *not* assume that K has a finite p-basis.

This will come up later in §8.1.12.

5.1.1 More Rings

We first define some more rings we need. Recall that $K = k((u))$ where k is a field of characteristic $p > 0$.

$\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$	the maximal unramified extension (i.e., strict henselization) of $\mathfrak{o}_{\mathcal{E}}$
$\mathcal{E}^{\mathrm{ur}}$	the fraction field of $\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$
$\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$	the π_0-adic completion of $\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$
$\widehat{\mathcal{E}}^{\mathrm{ur}}$	fraction field of $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$

By the universal property of strict henselization, there exists a unique map $\sigma : \mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}} \to \mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$ over $\sigma : \mathfrak{o}_{\mathcal{E}} \to \mathfrak{o}_{\mathcal{E}}$ which reduces to the qth power map on the residue field K^{sep}. Since this σ on $\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$ is an isometry for the valuation topology, it continuously extends to $\sigma : \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \to \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$. Using this σ, all the rings above become σ-flat.

If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, we can write $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \cong K^{\mathrm{sep}}[[\pi_0]]$ and $\widehat{\mathcal{E}}^{\mathrm{ur}} \cong K^{\mathrm{sep}}((\pi_0))$, and σ acts as the qth power on the coefficients of π_0-adic expansions (i.e., on K^{sep}) and the identity on π_0.

The natural action of $\boldsymbol{\mathcal{G}}_K \cong \mathrm{Gal}(\mathcal{E}^{\mathrm{ur}}/\mathcal{E})$ on $\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$ extends to $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ and $\widehat{\mathcal{E}}^{\mathrm{ur}}$ via isometry, and this action commutes with the Frobenius σ (by the universal property of the strict henselization). Also, we have $(\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{\boldsymbol{\mathcal{G}}_K} = \mathfrak{o}_{\mathcal{E}}$; this can be seen from Krasner's lemma (or by noting that $\boldsymbol{\mathcal{G}}_K$ acts only on "coefficients" in the p-adic Teichmüller expansions if $\mathfrak{o}_0 = \mathbb{Z}_p$, or in the formal power series expansion via $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \cong K^{\mathrm{sep}}[[\pi_0]]$ if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$).

5.1.2 Duality

The categories $\mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ and $\underline{\mathrm{Mod}}^{\mathrm{\acute{e}t}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)$ are equipped with $\otimes$-products and internal homs which satisfy all the "natural" compatibilities. We also have "duality" for these categories, but since we allowed torsion objects we need to treat free objects and torsion objects separately. We define duality on free and torsion objects in $\mathrm{Rep}_{\mathfrak{o}_0}(\mathcal{G}_K)$ as follows:

$$T^* := \begin{cases} \mathrm{Hom}_{\mathfrak{o}_0}(T, \mathfrak{o}_0), & \text{for } T \in \mathrm{Rep}_{\mathfrak{o}_0}^{\text{free}}(\mathcal{G}_K) \\ \mathrm{Hom}_{\mathfrak{o}_0}(T, F_0/\mathfrak{o}_0), & \text{for } T \in \mathrm{Rep}_{\mathfrak{o}_0}^{\text{tor}}(\mathcal{G}_K), \end{cases}$$

where $\mathfrak{o}_0$ and $F_0/\mathfrak{o}_0$ are given the trivial $\mathcal{G}_K$-action. Even though $F_0/\mathfrak{o}_0$ is not finitely generated (hence not an element of $\mathrm{Rep}_{\mathfrak{o}_0}(\mathcal{G}_K)$), any $\mathfrak{o}_0$-linear map from a torsion object T into $F_0/\mathfrak{o}_0$ factors through some finite submodule $\frac{1}{\pi_0^N}\mathfrak{o}_0/\mathfrak{o}_0 \subset F_0/\mathfrak{o}_0$ for $N \gg 0$ depending on T. So T^* can always be written as some internal hom, whether T is torsion or free.

Similarly, we define duality on free and torsion objects in $\underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét}}(\varphi)$:

$$M^* := \begin{cases} \mathrm{Hom}_{\mathfrak{o}_{\mathcal{E}}}(T, \mathfrak{o}_{\mathcal{E}}), & \text{for } M \in \underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét,free}}(\varphi) \\ \mathrm{Hom}_{\mathfrak{o}_{\mathcal{E}}}(T, \mathcal{E}/\mathfrak{o}_{\mathcal{E}}), & \text{for } M \in \underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét,tor}}(\varphi), \end{cases}$$

where the φ-module structures on $\mathfrak{o}_{\mathcal{E}}$ and $\mathcal{E}/\mathfrak{o}_{\mathcal{E}}$ are given by linearizing the σ on $\mathfrak{o}_{\mathcal{E}}$ and $\mathcal{E}$, respectively. Again, even though $\mathcal{E}/\mathfrak{o}_{\mathcal{E}}$ is not finitely generated, any $\mathfrak{o}_{\mathcal{E}}$-linear maps from any torsion object M into $\mathcal{E}/\mathfrak{o}_{\mathcal{E}}$ factor through some finite submodule $\frac{1}{\pi_0^N}\mathfrak{o}_{\mathcal{E}}/\mathfrak{o}_{\mathcal{E}} \subset \mathcal{E}/\mathfrak{o}_{\mathcal{E}}$ for $N \gg 0$. So M^* can be written as some internal hom, whether M is torsion or free.

For the rest of this section, we will construct quasi-inverse equivalences between $\mathrm{Rep}_{\mathfrak{o}_0}(\mathcal{G}_K)$ and $\underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét}}(\varphi)$, which *respects* all the natural operations, such as $\otimes$-products, internal homs, and duality.

5.1.3

For $T \in \mathrm{Rep}_{\mathfrak{o}_0}(\mathcal{G}_K)$, we define

$$\underline{D}_{\mathcal{E}}(T) := (\mathfrak{o}_{\widehat{\mathcal{E}^{\text{ur}}}} \otimes_{\mathfrak{o}_0} T)^{\mathcal{G}_K}, \tag{5.1.3.1}$$

where $\boldsymbol{\mathcal{G}}_K$ acts on the both factors of $T \otimes_{\mathfrak{o}_0} \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$. Since σ and the natural $\boldsymbol{\mathcal{G}}_K$-action on $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ commute, the $\otimes$-product Frobenius structure $\varphi : \sigma^*\big(\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_0} T\big) \to \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_0} T$ restricts to $\varphi : \sigma^*\underline{D}_{\mathcal{E}}(T) \to \underline{D}_{\mathcal{E}}(T)$. The following lemma tells that $\underline{D}_{\mathcal{E}}(V)$ is in fact an étale φ-module over $\mathfrak{o}_{\mathcal{E}}$.

Lemma 5.1.4. *For any $T \in \mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$, the natural map*

$$\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} \underline{D}_{\mathcal{E}}(T) \to \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_0} T \tag{5.1.4.1}$$

is a $\boldsymbol{\mathcal{G}}_K$-equivariant isomorphism of φ-modules.

Remark 5.1.5. Before we begin the proof, let us discuss formal consequences of the isomorphism (5.1.4.1), together with the faithful flatness of $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ over $\mathfrak{o}_{\mathcal{E}}$. All the properties below can be checked after some faithfully flat scalar extension, namely by applying $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} (\cdot)$, and then one can use the isomorphism (5.1.4.1).

1. $\underline{D}_{\mathcal{E}}(T)$ is a finitely generated $\mathfrak{o}_{\mathcal{E}}$-module, so it is an étale φ-module. In particular, we obtain a functor $\underline{D}_{\mathcal{E}} : \mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K) \to \underline{\mathrm{Mod}}^{\text{ét}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)$.

2. A $\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K]$-module T is free of $\mathfrak{o}_0$-rank n (respectively, a finite torsion $\mathfrak{o}_0$-module of length n) if and only if $\underline{D}_{\mathcal{E}}(T)$ is so as an $\mathfrak{o}_{\mathcal{E}}$-module

3. A complex $(*)$ in $\mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ is exact if and only if $\underline{D}_{\mathcal{E}}(*)$ is exact in $\underline{\mathrm{Mod}}^{\text{ét}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)$.

4. For any $T, T' \in \mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$, the natural map $\underline{D}_{\mathcal{E}}(T)\otimes_{\mathfrak{o}_{\mathcal{E}}}\underline{D}_{\mathcal{E}}(T') \to \underline{D}_{\mathcal{E}}\left(T \otimes_{\mathfrak{o}_0} T'\right)$ is a φ-compatible isomorphism.

5. For any $T, T' \in \mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$, the natural map

$$\underline{D}_{\mathcal{E}}\left(\mathrm{Hom}_{\mathfrak{o}_0}(T, T')\right) \to \mathrm{Hom}_{\mathfrak{o}_{\mathcal{E}}}\left(\underline{D}_{\mathcal{E}}(T), \underline{D}_{\mathcal{E}}(T')\right)$$

   is a φ-compatible isomorphism. In particular, trivially $\underline{D}_{\mathcal{E}}(\mathfrak{o}_0) = \mathfrak{o}_{\mathcal{E}}$ (respectively, with the natural $\boldsymbol{\mathcal{G}}_K$-action and φ-structure) and $\underline{D}_{\mathcal{E}}(F_0/\mathfrak{o}_0) = \mathcal{E}/\mathfrak{o}_{\mathcal{E}}$ (or rather, $\underline{D}_{\mathcal{E}}$ sends the direct system $\{\frac{1}{\pi_0^n}\mathfrak{o}_{\mathcal{E}}/\mathfrak{o}_{\mathcal{E}}\}$ to $\{\frac{1}{\pi_0^n}\mathfrak{o}_0/\mathfrak{o}_0\}$), we conclude that the natural map $\underline{D}_{\mathcal{E}}(T^*) \to (\underline{D}_{\mathcal{E}}(T))^*$ is a φ-compatible isomorphism.

Using the duality we can define a contravariant version of the functor $\underline{D}_{\mathcal{E}}^*(\cdot)$, which is often more useful. But for this, we need to treat torsion and free cases separately:

$$(5.1.5.1) \quad \underline{D}_{\mathcal{E}}^*(T) := \underline{D}_{\mathcal{E}}(T^*) \cong \begin{cases} \mathrm{Hom}_{\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K]}(T, \mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}), & \text{for } T \in \mathrm{Rep}_{\mathfrak{o}_0}^{\mathrm{tor}}(\boldsymbol{\mathcal{G}}_K) \\ \mathrm{Hom}_{\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K]}(T, \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}), & \text{for } T \in \mathrm{Rep}_{\mathfrak{o}_0}^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K). \end{cases}$$

Since $\underline{D}_{\mathcal{E}}$ commutes with the duality by (5) above, we have $\underline{D}_{\mathcal{E}}^*(T) \cong (\underline{D}_{\mathcal{E}}(T))^*$. One can also formulate Lemma 5.1.4 using $\underline{D}_{\mathcal{E}}^*(\cdot)$, and show the properties listed above assuming that all $\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K]$-modules involved are either all finite free over $\mathfrak{o}_0$ or all finite torsion $\mathfrak{o}_0$-modules.

Proof of Lemma 5.1.4. First, it can be seen that the map (5.1.4.1) is $\boldsymbol{\mathcal{G}}_K$-equivariant and φ-compatible, so we only need to show it is an isomorphism as $\mathfrak{o}_{\mathcal{E}}$-modules.

If $\pi_o \cdot T = 0$, then the map (5.1.4.1) being an isomorphism basically follows from classical Galois descent theory. If $\pi_0^N \cdot T = 0$, then we use the induction on N; consider the exact sequence $0 \to \pi_0^{N-1}T \to T \to T/\pi_0^{N-1}T \to 0$, and since the statement is true for the flanking terms, it is true for the middle term.

For the general case, we use the "dictionary" between $\mathfrak{o}_{\mathcal{E}}$-modules M and projective systems $\{M/\pi_0^n M\}_n$ (Proposition 7.4.1). For any $T \in \mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$, observe that $\varprojlim_n \underline{D}_{\mathcal{E}}(T/\pi_0^n T) \cong \underline{D}_{\mathcal{E}}(T)$; in other words, the natural map $\left(\varprojlim_n (\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}/\pi_0^n) \otimes T\right)^{\boldsymbol{\mathcal{G}}_K} \to \varprojlim_n \left[((\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}/\pi_0^n) \otimes T)^{\boldsymbol{\mathcal{G}}_K}\right]$ is an isomorphism which can be seen directly by the explicit description of $\boldsymbol{\mathcal{G}}_K$-action on $\varprojlim_n (\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}/\pi_0^n) \otimes T$.

Since we proved Lemma 5.1.4 for torsion representations, it follows from Remark 5.1.5 that the functor $\underline{D}_{\mathcal{E}}$ is exact for torsion representations. So we have the following right exact sequence for any integers n and N:

$$\underline{D}_{\mathcal{E}}(T/\pi_0^{n+N}T) \xrightarrow{\pi_0^n} \underline{D}_{\mathcal{E}}(T/\pi_0^{n+N}T) \to \underline{D}_{\mathcal{E}}(T/\pi_0^n T) \to 0.$$

(One can check the exactness after applying $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} (\cdot)$, and then use that the natural map (5.1.4.1) is an isomorphism for torsion $\boldsymbol{\mathcal{G}}_K$-representation, which we have already proved.) In particular, each transition map induces an isomorphism $(\mathfrak{o}_{\mathcal{E}}/\pi_0^n) \otimes \underline{D}_{\mathcal{E}}(T/\pi_0^{n+1}T) \xrightarrow{\sim} \underline{D}_{\mathcal{E}}(T/\pi_0^n T)$. Moreover, we have already seen that $\underline{D}_{\mathcal{E}}(T/\pi_0 T)$ is finite-dimensional over $\mathfrak{o}_{\mathcal{E}}/(\pi_0)$. Therefore by passing to the projective limit over N, we conclude that $\underline{D}_{\mathcal{E}}(T)$ is finitely generated over $\mathfrak{o}_{\mathcal{E}}$ such that the natural map $(\mathfrak{o}_{\mathcal{E}}/\pi_0^n) \otimes_{\mathfrak{o}_{\mathcal{E}}} \underline{D}_{\mathcal{E}}(T) \hookrightarrow \underline{D}_{\mathcal{E}}(T/\pi_0^n T)$ is an isomorphism. We finally conclude the map (5.1.4.1) is an isomorphism by the "dictionary" between $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$-modules M and projective systems $\{M/\pi_0^n M\}_n$. (See Proposition 7.4.1.)

The étale-ness can be checked after a faithfully flat scalar extension $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} (\cdot)$, and the target of the isomorphism (5.1.4.1) is clearly an étale φ-module. □

5.1.6

Next, we construct a functor $\underline{T}_{\mathcal{E}}(\cdot) : \underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét}}(\varphi) \to \mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$, which will be shown to be a quasi-inverse to the functor $\underline{D}_{\mathcal{E}}$. For any $M \in \underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét}}(\varphi)$, we let

$$\underline{T}_{\mathcal{E}}(M) := (\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} M)^{\varphi=1} = \{x \in \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} M |\ \varphi(\sigma^* x) = x\}. \tag{5.1.6.1}$$

The $\boldsymbol{\mathcal{G}}_K$-action on $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} M$ via the the first factor restricts to an action on $\underline{T}_{\mathcal{E}}(M)$ since the Frobenius map and $\boldsymbol{\mathcal{G}}_K$-action commute.

As previously, we can use the duality to define a contravariant version of the functor $\underline{T}_{\mathcal{E}}(M)$, for which we should treat torsion and free cases separately:

$$\underline{T}^*_{\mathcal{E}}(M) := \underline{T}_{\mathcal{E}}(M^*) = \begin{cases} \mathrm{Hom}_{\mathfrak{o}_{\mathcal{E}},\varphi}(M, \mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}), & \text{for } M \in \underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét,tor}}(\varphi) \\ \mathrm{Hom}_{\mathfrak{o}_{\mathcal{E}},\varphi}(M, \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}), & \text{for } M \in \underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét,free}}(\varphi), \end{cases} \tag{5.1.6.2}$$

In fact, we will see below that $\underline{T}_{\mathcal{E}}$ will also commute with the duality; i.e., there exists a natural isomorphism $\underline{T}_{\mathcal{E}}(M^*) \cong (\underline{T}_{\mathcal{E}}(M))^*$. We leave it to readers to formulate the next proposition (Proposition 5.1.7) using the contravariant functor $\underline{T}_{\mathcal{E}}^*$ and assuming that all étale φ-modules involved are either all finite free over $\mathfrak{o}_{\mathcal{E}}$ or all finite torsion $\mathfrak{o}_{\mathcal{E}}$-modules.

Proposition 5.1.7.

1. *For any $M \in \underline{\mathrm{Mod}}^{\text{ét}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)$ the natural map*

   (5.1.7.1) $$\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_0} \underline{T}_{\mathcal{E}}(M) \to \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} M$$

   is a $\mathcal{G}_K$-equivariant isomorphism of φ-modules. In particular, $\underline{T}_{\mathcal{E}}(M)$ is finitely generated as an $\mathfrak{o}_0$-module, and M is free of $\mathfrak{o}_{\mathcal{E}}$-rank n (respectively, a finite torsion $\mathfrak{o}_{\mathcal{E}}$-module of length n) if and only if $\underline{T}_{\mathcal{E}}(M)$ is so as an $\mathfrak{o}_0$-module.

2. *The functors $\underline{D}_{\mathcal{E}}$ and $\underline{T}_{\mathcal{E}}$ are quasi-inverse anti-equivalences between $\underline{\mathrm{Mod}}^{\text{ét}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)$ and $\mathrm{Rep}_{\mathfrak{o}_0}(\mathcal{G}_K)$, which are exact and commute with $\otimes$-products, internal homs, and duality. Moreover, $\underline{D}_{\mathcal{E}}$ and $\underline{T}_{\mathcal{E}}$ restrict to quasi-inverse anti-equivalences between $\underline{\mathrm{Mod}}^{\text{ét,free}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)$ and $\mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\mathcal{G}_K)$ (respectively, between $\underline{\mathrm{Mod}}^{\text{ét,tor}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)$ and $\mathrm{Rep}^{\mathrm{tor}}_{\mathfrak{o}_0}(\mathcal{G}_K)$).*

The proposition for the case $\mathfrak{o}_0 = \mathbb{Z}_p$ is proved in [31, A, §1.2]. When $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, the proposition for objects killed by π_0 can be obtained from [45, Proposition 4.1.1].

Proof. Using the same argument as before, one can show (1) implies (2), aside from the quasi-inverse claim. In order to construct a natural isomorphism $\underline{T}_{\mathcal{E}} \circ \underline{D}_{\mathcal{E}} \cong \mathrm{id}$, it is enough to show the image of the $\mathcal{G}_K$-equivariant injective map $T \hookrightarrow \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_0} T$ is exactly $(\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_0} T)^{\varphi=1}$. Since this inclusion has an $\mathfrak{o}_0$-linear section (as $\mathfrak{o}_0 \to \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ does, via successive approximation) and the image is contained in $(\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_0} T)^{\varphi=1}$, it is enough to show $(\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_0} T)^{\varphi=1} \cong T$ as abstract $\mathfrak{o}_0$-modules (i.e., forgetting their embeddings into $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_0} T$). By the structure theorem for finitely generated modules over a principal ideal domain, we are reduced to showing $(\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{\varphi=1} = \mathfrak{o}_0$ and $\left(\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}/(\pi_0^d)\right)^{\varphi=1} = \mathfrak{o}_0/(\pi_0^d)$. The other natural isomorphism $\underline{D}_{\mathcal{E}} \circ \underline{T}_{\mathcal{E}} \cong \mathrm{id}$ can be obtained by applying $(\cdot)^{\mathcal{G}_K}$ to the natural isomorphism (5.1.7.1).

Now, let us give a proof of (1). By the same argument as in the proof of the Lemma 5.1.4, it is enough to handle the case when $\pi_0 \cdot M = 0$, which we assume from now on. (For the limit argument, we have $\varprojlim_n (M/\pi_0^n M)^{\varphi=1} \xrightarrow{\sim} \left(\varprojlim_n M/\pi_0^n M\right)^{\varphi=1}$ by the $\mathfrak{o}_0$-linearity of φ, and so the rest of the argument goes unchanged.)

Let M be an étale φ-module over $K \cong \mathfrak{o}_{\mathcal{E}}/(\pi_0)$. We would like to show that the natural map

$$K^{\mathrm{sep}} \otimes_{\mathbb{F}_q} \underline{T}_{\mathcal{E}}(M) \to K^{\mathrm{sep}} \otimes_K M$$

is a $\mathcal{G}_K$-equivariant isomorphism of φ-modules. This statement for $q = p$ is proved in [31, A, Proposition 1.2.6], which carries over for any q, as follows.

We will in fact prove the contravariant version of the statement, namely for $\pi_0 \cdot M = 0$, the natural map

$$K^{\mathrm{sep}} \otimes_{\mathbb{F}_q} \underline{T}^*_{\mathcal{E}}(M) \to K^{\mathrm{sep}} \otimes_K M^* \cong \mathrm{Hom}_K(M, K^{\mathrm{sep}}), \tag{5.1.7.2}$$

is a $\mathcal{G}_K$-equivariant isomorphism of φ-modules, where M^* is the dual étale φ-module in the sense of §5.1.2.

Define

$$A_M := \frac{\mathrm{Sym}_K(M)}{\langle m^q - \varphi(\sigma^* m) | \forall m \in M\rangle},$$

which is clearly a finite étale algebra over K of rank $q^{\mathrm{rank}_K M}$. Observe that $\underline{T}^*_{\mathcal{E}}(M) = \mathrm{Hom}_{\mathrm{alg}\,/K}(A_M, K^{\mathrm{sep}})$. So by counting, we conclude that $\dim_{\mathbb{F}_q} \underline{T}^*_{\mathcal{E}}(M) = \mathrm{rank}_K M$. (In fact, one can naturally give $\mathrm{Spec}\, A_M$ a structure of group scheme with $\mathfrak{o}_0/(\pi_0)$-

action in such a way that $(\operatorname{Spec} A_M)(K^{\mathrm{sep}}) \cong \underline{T}^*_{\mathcal{E}}(M)$ is a $\boldsymbol{\mathcal{G}}_K$-equivariant isomorphism of $\mathfrak{o}_0$-modules. See §7.2 for more discussions.)

Now since the both sides of (5.1.7.2) have the same K^{sep}-dimension, it is enough to show the injectivity. Assume $m_1, \cdots, m_r \in \underline{T}^*_{\mathcal{E}}(M)$ are linearly independent over $\mathbb{F}_q$ but not over K^{sep}. Assume, furthermore, that $r > 1$ is the minimum cardinality of a set with this property. We may assume $\sum_{i=1}^r c_i m_i = 0$ for some $c_i \in K^{\mathrm{sep}}$ with $c_1 = 1$. By applying φ, we also obtain $\sum_{i=1}^r c_i^q m_i = 0$, so by subtracting we get a K^{sep}-linear dependence relation $\sum_{i=2}^r (c_i^q - c_i) m_i = 0$ with fewer than r elements. By our choice of r we get $c_i^q = c_i$ for all i, which contradicts to the $\mathbb{F}_q$-linear independence of $\{m_i\}$. □

5.1.8 Contravariant Theory

It is often much more convenient to work with the contravariant functors $\underline{T}^*_{\mathcal{E}}$ and $\underline{D}^*_{\mathcal{E}}$. It is a formal consequence of Lemma 5.1.4 and Proposition 5.1.7 that $\underline{T}^*_{\mathcal{E}}$ and $\underline{D}^*_{\mathcal{E}}$ are quasi-inverse exact anti-equivalences of categories between suitable source and target categories; commute with $\otimes$-products, internal homs, and duality; and satisfy various other properties as asserted in Lemma 5.1.4 and Proposition 5.1.7.

When working with these contravariant functors, one often needs the fact that $\underline{T}^*_{\mathcal{E}}$ and $\underline{D}^*_{\mathcal{E}}$ "commute" with the reduction mod π_0^n. The following lemma shows that this is indeed the case, but it is not completely trivial because the functors are defined differently for torsion and finite free objects.

Lemma 5.1.9. *Let $f : M' \to M$ be an "isogeny" of étale φ-modules finite free over $\mathfrak{o}_{\mathcal{E}}$; i.e., $f[\frac{1}{\pi_0}] : M'[\frac{1}{\pi_0}] \to M[\frac{1}{\pi_0}]$ is an isomorphism. Then we have a natural isomorphism $\underline{T}^*_{\mathcal{E}}(\operatorname{coker} f) \cong \operatorname{coker}(\underline{T}^*_{\mathcal{E}}(f))$, where $\underline{T}^*_{\mathcal{E}}(f) : \underline{T}^*_{\mathcal{E}}(M) \to \underline{T}^*_{\mathcal{E}}(M')$ is the map induced from f. In particular, if M is an étale φ-module finite free over $\mathfrak{o}_{\mathcal{E}}$ then* *we have a natural isomorphism* $(\mathfrak{o}_0/\pi_0^n)\otimes_{\mathfrak{o}_0}\underline{T}^*_{\mathcal{E}}(M)\cong\underline{T}^*_{\mathcal{E}}((\mathfrak{o}_0/\pi_0^n)\otimes_{\mathfrak{o}_0}M)$*.*

Similarly for any isogeny $f:T'\to T$ *of* $\mathfrak{o}_0$*-lattice* $\mathcal{G}_K$*-representations, we have a natural isomorphism* $\underline{D}^*_{\mathcal{E}}(\operatorname{coker} f)\cong\operatorname{coker}(\underline{D}^*_{\mathcal{E}}(f))$*.*

Proof. We view both M' and M as submodules of $M'[\frac{1}{\pi_0}]$ via the isomorphism $f[\frac{1}{\pi_0}]$, and replace f with the natural inclusion. We also view $\underline{T}^*_{\mathcal{E}}(M)$ and $\underline{T}^*_{\mathcal{E}}(M')$ as submodules of $\mathrm{Hom}_{\mathfrak{o}_{\mathcal{E}},\varphi}(M',\widehat{\mathcal{E}}^{\mathrm{ur}})$ via the natural inclusion $\mathfrak{o}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}\hookrightarrow\widehat{\mathcal{E}}^{\mathrm{ur}}:=\mathfrak{o}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}[\frac{1}{\pi_0}]$. Then $\underline{T}^*_{\mathcal{E}}(f)$ is the natural inclusion $\underline{T}^*_{\mathcal{E}}(M)\hookrightarrow\underline{T}^*_{\mathcal{E}}(M')$, whose cokernel is isomorphic to $\mathrm{Hom}_{\mathfrak{o}_{\mathcal{E}},\varphi}(M/M',\mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}})$. The same argument also shows the claim for $\underline{D}^*_{\mathcal{E}}$. □

5.1.10

We comment on the classifications of F_0-representation of $\mathcal{G}_K$. Let $\mathrm{Rep}_{F_0}(\mathcal{G}_K)$ be the category of finite-dimensional F_0-vector spaces with continuous $\mathcal{G}_K$-action. For any $(\rho,V)\in\mathrm{Rep}_{F_0}(\mathcal{G}_K)$, there exists an $\mathcal{G}_K$-stable $\mathfrak{o}_0$-lattice $T\subset V$. (This follows from the compactness of $\mathcal{G}_K$.) In other words, the category $\mathrm{Rep}_{F_0}(\mathcal{G}_K)$ is equivalent to the isogeny category $\mathrm{Rep}_{\mathfrak{o}_0}(\mathcal{G}_K)[\frac{1}{\pi_0}]\cong\mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\mathcal{G}_K)[\frac{1}{\pi_0}]$. Therefore, the quasi-inverse equivalences of categories $\underline{T}_{\mathcal{E}}$ and $\underline{D}_{\mathcal{E}}$ induce quasi-inverse equivalences of categories $\underline{V}_{\mathcal{E}}:\underline{\mathrm{Mod}}^{\text{ét,free}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)[\frac{1}{\pi_0}]\xrightarrow{\sim}\mathrm{Rep}_{F_0}(\mathcal{G}_K)$ and $\underline{D}_{\mathcal{E}}:\mathrm{Rep}_{F_0}(\mathcal{G}_K)\xrightarrow{\sim}\underline{\mathrm{Mod}}^{\text{ét,free}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)[\frac{1}{\pi_0}]$. The same statement holds for the contravariant versions, so we obtain quasi-inverse anti-equivalences of categories $\underline{V}^*_{\mathcal{E}}$ and $\underline{D}^*_{\mathcal{E}}$.

5.2 Main theorem and $\mathcal{G}_K$-representations of finite $\mathcal{P}$-height

Consider the functor $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)\to\underline{\mathrm{Mod}}^{\text{ét,free}}_{\mathfrak{o}_{\mathcal{E}}}(\varphi)$ defined by scalar extension $\mathfrak{M}\mapsto\mathfrak{o}_{\mathcal{E}}\otimes_{\mathfrak{S}}\mathfrak{M}$. In this section, we show that this functor is *fully faithful* (Theorem 5.2.3). Since the target category has an anti-equivalence of categories with $\mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\mathcal{G}_K)$ via $\underline{T}^*_{\mathcal{E}}$, this implies the full faithfulness of the contra-variant functor $\underline{T}^*_{\mathfrak{S}}:\mathfrak{M}\mapsto\underline{T}^*_{\mathcal{E}}(\mathfrak{o}_{\mathcal{E}}\otimes_{\mathfrak{S}}\mathfrak{M})$ from $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ to $\mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\mathcal{G}_K)$. This theorem was first proved by Kisin

[52, Proposition 2.1.12] for $\mathfrak{o}_0 = \mathbb{Z}_p$, and our proof is closed related to his. In the case of $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, it is known that $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ and $\underline{\mathrm{Mod}}_{\mathfrak{o}_\mathcal{E}}(\varphi)$ classify certain kind of π_0-divisible groups over $\mathfrak{o}_K$ and K, respectively. (See §7.3 for the precise statement and a proof.) Therefore, the full faithfulness of $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi) \to \underline{\mathrm{Mod}}_{\mathfrak{o}_\mathcal{E}}(\varphi)$ can be viewed as an equi-characteristic analogue of Tate's theorem [75, §4.2].

5.2.1

For $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$, we associate a $\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K]$-module

$$\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) := T^*_{\mathcal{E}}(\mathfrak{o}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}) \cong \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}),$$

which defines a contravariant exact functor $\underline{T}^*_{\mathfrak{S}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi) \to \mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ compatible with $\otimes$-products.

We need one more lemma for the proof of the main theorem. Compare with [52, Lemma 2.1.9].

Lemma 5.2.2. *Let $f : \mathfrak{M} \to \mathfrak{M}'$ be a morphism in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ such that $\mathfrak{o}_\mathcal{E} \otimes f$: $\mathfrak{o}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M} \to \mathfrak{o}_\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}'$ is an isomorphism. Then f is an isomorphism.*

Proof. Since f is a morphism of free $\mathfrak{S}$-modules of same (finite) rank, it is an isomorphism if its determinant is. Hence, we may assume that $\mathfrak{M}$ and $\mathfrak{M}'$ are free of rank 1. Since $\mathfrak{o}_\mathcal{E} \otimes f$ is an isomorphism in $\underline{\mathrm{Mod}}^{\text{ét,free}}_{\mathfrak{o}_\mathcal{E}}(\varphi)$ and $\mathfrak{o}_\mathcal{E}$ is the π_0-adic completion of $\mathfrak{S}[\frac{1}{u}]$, it is enough to show that f is an isogeny – in other words, f is an isomorphism in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}]$. For this claim, we use the equivalence of categories $\underline{\mathbb{H}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \mathcal{HP}^{wa,\geqslant 0}_K(\varphi)$ (Theorem 4.3.4)[1].

We set $(D, \Lambda) := \underline{\mathbb{H}}(\mathfrak{M})$ and $(D', \Lambda') := \underline{\mathbb{H}}(\mathfrak{M}')$. Note that $\underline{\mathbb{H}}(f)$ is a non-zero morphism of isocrystals with weakly admissible Hodge-Pink structures. Since D and D' are 1 dimensional, $\underline{\mathbb{H}}(f) : D \to D'$ induces an isomorphism of isocrystals,

[1] In fact, we only need the full faithfulness of the functor $\underline{\mathbb{H}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \mathcal{HP}^{wa,\geqslant 0}_K(\varphi)$.

so $t_N(D) = t_N(D')$. Let h denote this common Newton number. By the weak admissibility, we have $h = t_H(D) = t_H(D')$. Hence $\Lambda = \mathcal{P}(u)^{-h}\cdot(\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \otimes_{\mathscr{K}_0} D)$ and $\Lambda' = \mathcal{P}(u)^{-h}\cdot(\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \otimes_{\mathscr{K}_0} D')$, so $\underline{\mathbb{H}}(f) : \Lambda \to \Lambda'$ is visibly an isomorphism in $\mathcal{HP}_K^{wa,\geqslant 0}(\varphi)$, which shows that f is an isogeny. □

Now we are ready to prove the main theorem. Compare with [52, Proposition 2.1.12].

Theorem 5.2.3. *The functor* $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi) \to \underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét,free}}(\varphi)$ *defined by* $\mathfrak{M} \to \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ *is fully faithful. Equivalently, the contravariant functor* $\underline{T}^*_{\mathfrak{S}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi) \to \mathrm{Rep}^{\text{free}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ *is fully faithful.*

Proof. Let $\mathfrak{M}_0$ be a finitely generated torsion-free (not necessarily free) $\mathfrak{S}$-module equipped with a map $\varphi_{\mathfrak{M}_0} : \sigma^*\mathfrak{M}_0 \to \mathfrak{M}_0$ such that $\mathrm{coker}(\varphi_{\mathfrak{M}_0})$ is killed by $\mathcal{P}(u)^h$ for some h. Then, we can "saturate" $\mathfrak{M}_0$ to get another φ-module $\mathfrak{M}_0^{\text{sat}}$, which is finite free over $\mathfrak{S}$ and contains $\mathfrak{M}_0$ with $\mathrm{coker}\,\varphi_{\mathfrak{M}_0^{\text{sat}}}$ killed by $\mathcal{P}(u)^h$. Indeed, define $\mathfrak{M}_0^{\text{sat}} := (\mathfrak{M}_0[\frac{1}{\pi_0}]) \cap (\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_0)$ with its evident φ-structure, where the intersection is taken inside $\mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}_0$. Both $\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_0$ and $\mathfrak{M}_0[\frac{1}{\pi_0}]$ are torsion-free, hence free over $\mathfrak{o}_{\mathcal{E}}$ and $\mathfrak{S}[\frac{1}{\pi_0}]$, respectively. By the proof of Lemma 4.3.1, $\mathfrak{M}_0^{\text{sat}}$ is finite free over $\mathfrak{S}$ and it recovers $\mathfrak{M}_0[\frac{1}{\pi_0}]$ and $\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_0$. Lemma 4.3.1 also shows that since $\mathrm{coker}\,\varphi_{\mathfrak{M}_0}$ is killed by $\mathcal{P}(u)^h$, the same holds for $\mathrm{coker}\,\varphi_{\mathfrak{M}_0^{\text{sat}}}$.

Now suppose that $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ and put $M_i := \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_i$ for $i = 1, 2$. Given a morphism $f : M_1 \to M_2$ in $\underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét,free}}(\varphi)$, we would like to show that it restricts to $\mathfrak{M}_1 \to \mathfrak{M}_2$.

Let us first handle the case when $M = M_1 = M_2$ and $f = \mathrm{id}$; i.e., $\mathfrak{M}_i$ $(i = 1, 2)$ are φ-stable $\mathfrak{S}$-lattices in M and we seek to prove $\mathfrak{M}_1 = \mathfrak{M}_2$ if they are both of finite $\mathcal{P}$-height. Clearly $\mathfrak{M}_1 + \mathfrak{M}_2$ defines a φ-stable submodule of M of finite $\mathcal{P}$-height, and it is finitely presented over $\mathfrak{S}$, so the inclusion $\mathfrak{M}_i \hookrightarrow (\mathfrak{M}_1+\mathfrak{M}_2)^{\mathrm{sat}}$ is an equality by Lemma 5.2.2. Therefore $\mathfrak{M}_1 = \mathfrak{M}_2$.

Now we handle the general case. By replacing f by $(1, f) : M_1 \to M_1 \oplus M_2$ and M_2 by $M_1 \oplus M_2$, we may assume that f is injective, so we can regard $\mathfrak{M}_i$ $(i = 1, 2)$ as $(\varphi, \mathfrak{S})$-submodules of M_2. As in the special case treated above, $(\mathfrak{M}_1 + \mathfrak{M}_2)^{\mathrm{sat}} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ is another φ-stable $\mathfrak{S}$-lattice of M_2, so the inclusion $\mathfrak{M}_2 \hookrightarrow (\mathfrak{M}_1 + \mathfrak{M}_2)^{\mathrm{sat}}$ is an equality by Lemma 5.2.2. Therefore, $\mathfrak{M}_1 \subset (\mathfrak{M}_1 + \mathfrak{M}_2)^{\mathrm{sat}} = \mathfrak{M}_2$. □

Corollary 5.2.4. *The contravariant functor* $\underline{V}^*_{\mathfrak{S}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \mathrm{Rep}_{F_0}(\boldsymbol{\mathcal{G}}_K)$ *is fully faithful, and there exists a fully faithful exact functor* $\underline{V}^*_{\mathcal{HP}} : \mathcal{HP}^{wa}_K(\varphi) \to \mathrm{Rep}_{F_0}(\boldsymbol{\mathcal{G}}_K)$ *which commutes with* $\otimes$*-products and such that we have a natural isomorphism* $\underline{V}^*_{\mathfrak{S}} \cong \underline{V}^*_{\mathcal{HP}} \circ \underline{\mathbb{H}}$ *of functors* $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \to \mathrm{Rep}_{F_0}(\boldsymbol{\mathcal{G}}_K)$.

Proof. The first claim directly follows from the above theorem. In order to prove the second claim, consider the following contravariant functor $\underline{V}^*_{\mathfrak{S}} \circ \underline{\mathbb{H}}^{-1} : \mathcal{HP}^{wa,\geqslant 0}_K(\varphi) \to \mathrm{Rep}_{F_0}(\boldsymbol{\mathcal{G}}_K)$ which commutes with $\otimes$-products (in particular, with Tate twists), where $\underline{\mathbb{H}}^{-1} : \mathcal{HP}^{wa,\geqslant 0}_K(\varphi) \to \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}]$ is a quasi-inverse of $\underline{\mathbb{H}}$ defined by $\underline{\mathbb{H}}^{-1}(D) = \mathfrak{M}[\frac{1}{\pi_0}]$ where $\mathfrak{M}[\frac{1}{\pi_0}]$ is the unique φ-stable $\mathfrak{S}[\frac{1}{\pi_0}]$-lattice of finite $\mathcal{P}$-height in $\underline{\mathcal{M}}(D)$. Now, we set $\underline{V}^*_{\mathcal{HP}}(D) \cong \big((\underline{V}^*_{\mathfrak{S}} \circ \underline{\mathbb{H}}^{-1})(D(N))\big)(-N)$ with N big enough so that $D(N)$ is effective. This definition is independent of N, and the functor $\underline{V}^*_{\mathcal{HP}}$ satisfies all the desired properties. □

Lemma 5.2.5. *A three-term complex* $D^\bullet :\ 0 \to D' \to D \to D'' \to 0$ *of isocrystals with weakly admissible Hodge-Pink structures is short exact if and only if* $\underline{V}^*_{\mathcal{HP}}(D^\bullet)$ *is short exact in* $\mathrm{Rep}_{F_0}(\boldsymbol{\mathcal{G}}_K)$.

Similarly, a three-term complex $\mathfrak{M}^\bullet :\ 0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ *in* $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ *is short exact if and only if* $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}^\bullet)$ *is short exact in* $\mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$.

Proof. By Proposition 5.1.7(2) and $\mathfrak{S}$-flatness of $\mathfrak{o}_{\mathcal{E}}$, $\underline{T}^*_{\mathfrak{S}}$ is an exact functor (i.e., $\underline{T}^*_{\mathfrak{S}}$ takes a short exact sequence in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ to a short exact sequence in $\mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$). Using the exactness assertion of Theorem 4.3.4, $\underline{V}^*_{\mathcal{HP}}$ is an exact functor. So it suffices to prove the "if" assertions.

Now let us assume that $\underline{V}^*_{\mathcal{HP}}(D^\bullet)$ is short exact in $\mathrm{Rep}_{F_0}(\boldsymbol{\mathcal{G}}_K)$ and show that $D^\bullet$ is short exact. By assumption, we have $\dim_{\mathscr{K}_0} D = \dim_{\mathscr{K}_0} D' + \dim_{\mathscr{K}_0} D''$ since $\underline{V}^*_{\mathcal{HP}}$ is rank-preserving. It immediately follows that $D^\bullet$ is short exact for the underlying isocrystals (*without* Hodge-Pink structures).

Let Λ', Λ, and Λ'' be the (weakly admissible) Hodge-Pink structures for D', D, and D'', respectively. It remains to show that the natural inclusions $\Lambda' \hookrightarrow (\Lambda')^{\mathrm{sat}} := \Lambda \cap \mathcal{O}_{\widehat{\boldsymbol{\Delta}},x_0}[\frac{1}{\mathcal{P}(u)}] \otimes_{\mathscr{K}_0} D'$ and $\Lambda/\Lambda' \hookrightarrow \Lambda''$ of Hodge-Pink structures on D' and D'', respectively, are isomorphisms. This claim can be checked after passing to the determinants. Let us first replace D' with its determinant and put $h' := t_N(D')$. By weak admissibility, $\Lambda' = \mathcal{P}(u)^{-h'}\widehat{\mathcal{D}}'_{x_0}$ where $\widehat{\mathcal{D}}'_{x_0} = \mathcal{O}_{\widehat{\boldsymbol{\Delta}},x_0} \otimes_{\mathscr{K}_0} D'$, and $(\Lambda')^{\mathrm{sat}} \cong \mathcal{P}(u)^{-h'_s}\widehat{\mathcal{D}}'_{x_0}$ for some $h'_s \geq h'$ (since $\Lambda' \subset (\Lambda')^{\mathrm{sat}}$). On the other hand, by weak admissibility of (D, Λ) we have $h'_s \leq t_N(D') = h'$. This shows that $D^\bullet$ is left exact. Now we replace D'' with its determinant and put $h'' := t_N(D'')$. Since both Λ/Λ' and Λ'' are weakly admissible by Proposition 2.3.8 and by assumption, we obtain that $\Lambda/\Lambda' \xrightarrow{\sim} \Lambda'' = \mathcal{P}(u)^{-h''}\widehat{\mathcal{D}}''_{x_0}$. This shows that $D^\bullet$ is exact.

Now let show the lemma for $\underline{T}^*_{\mathfrak{S}}$. Assume that $T^*_{\mathfrak{S}}(\mathfrak{M}^\bullet)$ is a short exact sequence. It follows from Corollary 5.2.4 that we have $\underline{V}^*_{\mathcal{HP}}\left(\underline{\mathbb{H}}(\mathfrak{M}^\bullet[\frac{1}{\pi_0}])\right) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}^\bullet)[\frac{1}{\pi_0}]$, and that $\underline{\mathbb{H}}(\mathfrak{M}^\bullet[\frac{1}{\pi_0}])$ is a short exact sequence in $\mathcal{HP}^{wa,\geqslant 0}_K(\varphi)$. By the exactness assertions of Theorem 4.3.4, $\mathfrak{M}^\bullet[\frac{1}{\pi_0}]$ is a short exact sequence, so $\mathfrak{M}^\bullet$ is left exact. Furthermore, the natural map $\mathfrak{M}/\mathfrak{M}' \to \mathfrak{M}''$ is an isomorphism since the natural map $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}'') \to \ker[\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \twoheadrightarrow \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}')]$ is an isomorphism and $\underline{T}^*_{\mathfrak{S}}$ is fully faithful (Theorem 5.2.3).

Note that we have a natural isomorphism $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}/\mathfrak{M}') \cong \ker[\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \twoheadrightarrow \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}')]$. □

Remark 5.2.6. Since the functor $\underline{T}^*_{\mathfrak{S}}$ commutes with $\otimes$-products (in particular, with Tate twists), we may extend $\underline{T}^*_{\mathfrak{S}}$ to a functor on generalized φ-modules over $\mathfrak{S}$ (see §2.2.11), and the theorem implies that this $\underline{T}^*_{\mathfrak{S}}$ is fully faithful. Unlike $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$, the category of generalized φ-modules have duality and internal hom. It is not hard to show that the functor $\underline{T}^*_{\mathfrak{S}}$ commutes with these operations.

5.2.7

From now on, we focus on the essential image of $\underline{T}^*_{\mathfrak{S}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi) \to \mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$. But this subcategory is not stable under the natural duality in $\mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$, while any "good" class of representations should be stable under the natural operations such as $\otimes$-product, duality, and internal hom. So we consider a slightly larger full subcategory which is stable under all these operations.

As suggested in Remark 5.2.6, one possible solution is to consider the essential image of generalized φ-modules over $\mathfrak{S}$ under $\underline{T}^*_{\mathfrak{S}}$. This full subcategory has the following alternative description. We put $\mathfrak{o}_0(r) := \underline{T}^*_{\mathfrak{S}}(\mathfrak{S}(r))$ if $r \geq 0$ and $\mathfrak{o}_0(r) := (\mathfrak{o}(-r))^*$ if $r < 0$. For any $T \in \mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$, we put $T(r) := T \otimes_{\mathfrak{o}_0} \mathfrak{o}_0(r)$. If $\mathfrak{o}_0 = \mathbb{Z}_p$ then $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ acts on $\mathfrak{o}_0(1)$ by the restriction of the p-adic cyclotomic character to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty} \cong \boldsymbol{\mathcal{G}}_K$; and if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ then $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ acts on $\mathfrak{o}_0(1)$ by the Lubin-Tate character; i.e., the character obtained by the Lubin-Tate formal group (as is verified in Example 7.3.7(3)).

Definition 5.2.8. A $\mathfrak{o}_0$-lattice $\boldsymbol{\mathcal{G}}_K$-representation $T \in \mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ is *of finite $\mathcal{P}$-height* if for some $r \in \mathbb{Z}$, there exists $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ such that $T(r) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$. We say that T is *of $\mathcal{P}$-height* $\leqslant h$ if there exists $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$ of $\mathcal{P}$-height $\leqslant h$, such that $T \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$.

We say that $V \in \mathrm{Rep}_{F_0}(\boldsymbol{\mathcal{G}}_K)$ is *of finite $\mathcal{P}$-height* if there exists a $\boldsymbol{\mathcal{G}}_K$-stable $\mathfrak{o}_0$-lattice $T \subset V$ which is of finite $\mathcal{P}$-height. Similarly, we say that V is *of $\mathcal{P}$-height $\leqslant h$* if there exists a $\boldsymbol{\mathcal{G}}_K$-stable $\mathfrak{o}_0$-lattice $T \subset V$ which is of $\mathcal{P}$-height $\leqslant h$.

We let $\mathrm{Rep}_{\mathfrak{o}_0}^{\mathrm{free},\mathcal{P}}(\boldsymbol{\mathcal{G}}_K)$ and $\mathrm{Rep}_{F_0}^{\mathcal{P}}(\boldsymbol{\mathcal{G}}_K)$ denote the full subcategories of $\boldsymbol{\mathcal{G}}_K$-representations of finite $\mathcal{P}$-height. We let $\mathrm{Rep}_{\mathfrak{o}_0}^{\mathrm{free},\leqslant h}(\boldsymbol{\mathcal{G}}_K)$ and $\mathrm{Rep}_{F_0}^{\leqslant h}(\boldsymbol{\mathcal{G}}_K)$ denote the full subcategories of representations of $\mathcal{P}$-height $\leqslant h$.

The full subcategories $\mathrm{Rep}_{\mathfrak{o}_0}^{\mathrm{free},\mathcal{P}}(\boldsymbol{\mathcal{G}}_K)$ and $\mathrm{Rep}_{F_0}^{\mathcal{P}}(\boldsymbol{\mathcal{G}}_K)$ are stable under $\otimes$-product, duality, and internal hom of the ambient categories. But the $\mathcal{P}$-height $\leqslant h$ condition is not stable under any of these operations. Note also that $\mathrm{Rep}_{F_0}^{\mathcal{P}}(\boldsymbol{\mathcal{G}}_K)$ is exactly the essential image of $\mathcal{HP}_K^{wa}(\varphi)$ by $\underline{V}^*_{\mathcal{HP}}$.

The following proposition says that for an F_0-representation of $\mathcal{P}$-height $\leqslant h$, any $\boldsymbol{\mathcal{G}}_K$-stable $\mathfrak{o}_0$-lattice is of $\mathcal{P}$-height $\leqslant h$. Compare with [52, Lemma 2.1.15].

Proposition 5.2.9. *Let $V \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})[\frac{1}{\pi_0}]$, and assume that $\mathfrak{M}$ is of $\mathcal{P}$-height $\leqslant h$. Then the map $\mathfrak{M}' \mapsto \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}')$ is a bijection between φ-stable $\mathfrak{S}$-lattices $\mathfrak{M}' \subset \mathfrak{M}[\frac{1}{\pi_0}]$ which are of $\mathcal{P}$-height $\leqslant h$ and $\boldsymbol{\mathcal{G}}_K$-stable lattices $T' \subset V$.*

Proof. We need to produce, for a given $\boldsymbol{\mathcal{G}}_K$-stable lattice $T' \subset V$, a φ-stable $\mathfrak{S}$-lattice $\mathfrak{M}' \subset \mathfrak{M}[\frac{1}{\pi_0}]$ which is of $\mathcal{P}$-height $\leqslant h$. By Proposition 5.1.7, we have a φ-stable $\mathfrak{o}_{\mathcal{E}}$-lattice $M' \subset \mathcal{E} \otimes_{\mathfrak{S}} \mathfrak{M}$ such that $\underline{T}^*_{\mathcal{E}}(M') \cong T'$. Now, it follows from the proof of Lemma 4.3.1 that there exists a common φ-stable $\mathfrak{S}$-lattice $\mathfrak{M}'$ of both M' and $\mathfrak{M}[\frac{1}{\pi_0}]$, which is of $\mathcal{P}$-height $\leqslant h$. □

We digress to study the case of $\mathcal{P}$-heights $\leqslant 0$.

Proposition 5.2.10. *Any $T \in \mathrm{Rep}_{\mathfrak{o}_0}^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$ is unramified if and only if there exists an étale $(\varphi, \mathfrak{S})$-module $\mathfrak{M}$ such that $T \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ as $\boldsymbol{\mathcal{G}}_K$-representations. In* *particular, any* unramified $\mathfrak{o}_0$*-lattice* $\mathcal{G}_K$*-representation is of* $\mathcal{P}$*-height* $\leqslant h$ *for any* $h \geqslant 0$.

This proposition can be thought of as an analogue of the fact that a p-adic $\mathcal{G}_{\mathscr{K}}$-representation V is crystalline of Hodge-Tate weight 0 if and only if V is unramified. From this together with [52, Proposition 2.1.5] one can also deduce the proposition for the case $\mathfrak{o}_0 = \mathbb{Z}_p$. (Note that $\mathcal{G}_{\mathscr{K}_\infty}/I_{\mathscr{K}_\infty} \xrightarrow{\sim} \mathcal{G}_{\mathscr{K}}/I_{\mathscr{K}}$.)

Proof. First, assume that $T \in \mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\mathcal{G}_K)$ is unramified and we seek an étale $\mathfrak{S}$-lattice in the étale φ-module $\underline{D}^*_{\mathcal{E}}(T) := \mathrm{Hom}_{\mathfrak{o}_0[\mathcal{G}_K]}(T, \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})$. Since I_K acts trivially on T, any $\mathfrak{o}_0[\mathcal{G}_K]$-map $l : T \to \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ factors through $(\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})^{I_K} \cong (\widehat{W^{sh}}[[u]][\frac{1}{u}])^\wedge \cong \mathfrak{o}_{\mathcal{E}}\widehat{\otimes}_W \widehat{W^{sh}}$, where $\widehat{(\cdot)}$ denotes the π_0-adic completion and $\widehat{W^{sh}}$ denotes the π_0-adic completion of the strict henselization of W. (Recall that $W = W(k)$ if $\mathfrak{o}_0 = \mathbb{Z}_p$, and $W = k[[\pi_0]]$ if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$.) So we have a natural isomorphism of φ-modules:

$$(5.2.10.1) \qquad \underline{D}^*_{\mathcal{E}}(T) \cong \mathfrak{o}_{\mathcal{E}}\widehat{\otimes}_W \underline{U}^*(T) \xleftarrow{\sim} \mathfrak{o}_{\mathcal{E}} \otimes_W \underline{U}^*(T),$$

where $\underline{U}^*(T) := \mathrm{Hom}_{\mathfrak{o}_0[\mathcal{G}_K]}(T, \widehat{W^{sh}})$ equipped with the φ-structure induced from the natural Frobenius endomorphism $\sigma : \widehat{W^{sh}} \to \widehat{W^{sh}}$.[2] We can deduce from the first isomorphism in (5.2.10.1) that $\underline{U}^*(T)$ is finitely generated over W since it is π_0-adically separated and complete, so we obtain the second isomorphism in (5.2.10.1). Furthermore, it follows from (5.2.10.1) that $\underline{U}^*(T)$ is an étale (φ, W)-module (using that $\mathfrak{o}_{\mathcal{E}}$ is fully faithful over W). So $\mathfrak{M} := \mathfrak{S} \otimes_W \underline{U}^*(T)$ is an étale $(\varphi, \mathfrak{S})$-module, and we have $T \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ by construction.

Now, let us show that $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ is unramified if $\mathfrak{M}$ is an étale $(\varphi, \mathfrak{S})$-module. Consider an étale (φ, W)-module $\mathfrak{M}/u\mathfrak{M}$ where the φ-structure is given by the reduction $\bar{\varphi}$ of $\varphi : \sigma^*\mathfrak{M} \to \mathfrak{M}$ modulo $u\mathfrak{M}$. We first show that the natural projection

[2] The Frobenius endomorphism $\sigma : \widehat{W^{sh}} \to \widehat{W^{sh}}$ can be obtained by restricting $\sigma : \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \to \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$. By the universal property of strict henselization, σ is a unique endomorphism $\sigma : \widehat{W^{sh}} \to \widehat{W^{sh}}$ which extends $\sigma : W \to W$ and reduces to the qth power map $\sigma : k^{\mathrm{sep}} \to k^{\mathrm{sep}}$ modulo π_0.

$\mathfrak{M} \twoheadrightarrow \mathfrak{M}/u\mathfrak{M}$ has a unique φ-compatible section, so it gives a natural isomorphism $\mathfrak{M} \xleftarrow{\sim} \mathfrak{S} \otimes_W (\mathfrak{M}/u\mathfrak{M})$ of φ-modules. The proof is analogous to Proposition 3.2.1 (but easier). Let $s_0 : \mathfrak{M}/u\mathfrak{M} \to \mathfrak{M}$ be a section which is not necessarily φ-compatible, and consider

(5.2.10.2)

$$s := s_0 + \sum_{i\geq 0}(\varphi^{i+1} \circ \sigma^{*i+1} s_0 \circ \bar{\varphi}^{-(i+1)} - \varphi^i \circ \sigma^{*i} s_0 \circ \bar{\varphi}^{-i}) = \text{“} \lim_{i\to\infty} (\varphi^i \circ \sigma^{*i} s_0 \circ \bar{\varphi}^{-i}) \text{”}$$

If the right side is well-defined, then it clearly satisfies $s \circ \bar{\varphi} = \varphi \circ \sigma^* s$. Since s_0 is a section, the image of $\varphi \circ \sigma^* s_0 \circ \bar{\varphi}^{-1} - s_0$ is contained in $u\mathfrak{M}$. By induction we obtain

$$\text{(5.2.10.3)} \qquad \operatorname{im}(\varphi^{i+1} \circ \sigma^{*i+1} s_0 \circ \bar{\varphi}^{-(i+1)} - \varphi^i \circ \sigma^{*i} s_0 \circ \bar{\varphi}^{-i}) \subset u^{q^i}\mathfrak{M}.$$

Therefore the right side of (5.2.10.2) converges (u-adically). The proof of uniqueness is identical as in the proof of Proposition 3.2.1.

Now, let us consider $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M})[\frac{1}{\pi_0}] \cong \operatorname{Hom}_{\mathscr{K}_0,\varphi}(D, \widehat{\mathcal{E}}^{\mathrm{ur}})$ where $D := (\mathfrak{M}/u\mathfrak{M})[\frac{1}{\pi_0}]$. (Recall that $\mathscr{K}_0 = W[\frac{1}{\pi_0}]$.) We claim that any φ-compatible map $l : D \to \widehat{\mathcal{E}}^{\mathrm{ur}}$ factors through $\widehat{W}^{sh}[\frac{1}{\pi_0}]$. (This shows that $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ is unramified since I_K acts trivially on $\widehat{W}^{sh}$.) To show the claim, it is enough to show that any map $l : W^{sh} \otimes_W D \to \widehat{\mathcal{E}}^{\mathrm{ur}}$ of $(\varphi, \widehat{W}^{sh}[\frac{1}{\pi_0}])$-modules factors through $\widehat{W}^{sh}[\frac{1}{\pi_0}]$. In the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, we may further assume that the residue field k^{sep} of $\widehat{W}^{sh}$ is algebraically closed; if any φ-compatible map $\bar{k}(\!(\pi_0)\!) \otimes_{\mathscr{K}_0} D \to \big(\bar{k}(\!(u)\!)\big)^{\mathrm{sep}}(\!(\pi_0)\!)$ factors through $\bar{k}(\!(\pi_0)\!)$ then any φ-compatible map $k^{\mathrm{sep}}(\!(\pi_0)\!) \otimes_{\mathscr{K}_0} D \to K^{\mathrm{sep}}(\!(\pi_0)\!)$ factors through $k^{\mathrm{sep}}(\!(\pi_0)\!)$, because $\bar{k}(\!(\pi_0)\!) \cap \widehat{\mathcal{E}}^{\mathrm{ur}} = \widehat{W}^{sh}[\frac{1}{\pi_0}]$ where the intersection is taken inside $\big(\bar{k}(\!(u)\!)\big)^{\mathrm{sep}}(\!(\pi_0)\!)$. (Recall that $\widehat{W}^{sh} \cong k^{\mathrm{sep}}[[\pi_0]]$ and $\widehat{\mathcal{E}}^{\mathrm{ur}} \cong K^{\mathrm{sep}}(\!(\pi_0)\!)$.)

Now, we rename $\widehat{W}^{sh}[\frac{1}{\pi_0}]$ as $\mathscr{K}_0$, $W^{sh}[\frac{1}{\pi_0}] \otimes_{\mathscr{K}_0} D$ as D, and $\widehat{\mathcal{E}}^{\mathrm{ur}}$ as $\mathcal{E}$ if $\mathfrak{o}_0 = \mathbb{Z}_p$; and we rename $\bar{k}(\!(\pi_0)\!)$ as $\mathscr{K}_0$, $\bar{k}(\!(\pi_0)\!) \otimes_{\mathscr{K}_0} D$ as D, and $\big(\bar{k}(\!(u)\!)\big)^{\mathrm{sep}}(\!(\pi_0)\!)$ as $\mathcal{E}$ if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. By Dieudonné-Manin decomposition (Theorem 4.1.2), we can find a $\mathscr{K}_0$-basis $\{\mathbf{e}_i\}$

for D such that $\varphi_D(\sigma^*\mathbf{e}_i) = \mathbf{e}_i$ for each i. For any φ-compatible map $l : D \to \mathcal{E}$, $l(\mathbf{e}_i) \in \mathcal{E}$ satisfies $\sigma(l(\mathbf{e}_i)) = l(\mathbf{e}_i)$ for each i (i.e., $l(\mathbf{e}_i) \in \mathfrak{o}_0[\frac{1}{\pi_0}]$ for each i), so clearly the image of l lies in $\mathscr{K}_0$. $\square$

We record the following corollary of the proof. Define an $\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K/I_K]$-module $\underline{T}_W(U) := (\widehat{W}^{sh} \otimes_W U)^{\varphi=1}$ and $\underline{T}^*_W(U) := \underline{T}_W(U^*)$ for any finite free étale (φ, W)-module U; and (φ, W)-modules $\underline{U}(T) := (\widehat{W}^{sh} \otimes_W T)^{\boldsymbol{\mathcal{G}}_K}$ and $\underline{U}^*(T) := \underline{U}(T^*)$ for any unramified $\mathfrak{o}_0$-lattice $\boldsymbol{\mathcal{G}}_K$-representation.

Corollary 5.2.11. *The assignments $\underline{T}_W$ and $\underline{U}$ define quasi-inverse rank-preserving exact equivalences of categories between $\mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K/I_K)$ and the category of finite free étale (φ, W)-modules which respects $\otimes$-products, internal homs, and duality. Furthermore, we have a natural isomorphism $\underline{D}_{\mathcal{E}}(T) \cong \mathfrak{o}_{\mathcal{E}} \otimes_W \underline{U}(T)$ of étale $(\varphi, \mathfrak{o}_{\mathcal{E}})$-modules for any $T \in \mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K/I_K)$ and a natural $\boldsymbol{\mathcal{G}}_K$-equivariant isomorphism $\underline{T}_W(U^*) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{S} \otimes_W U)$ for any finite free étale (φ, W)-module U.*

5.2.12 Relation with Weakly Admissible Filtered Isocrystals

This subsection is a continuation of §2.4; throughout this paragraph, we assume that $\mathfrak{o}_0 = \mathbb{Z}_p$ and we identify $\boldsymbol{\mathcal{G}}_K$ with $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$. In §2.4.3, we defined a functor $\underline{\mathrm{res}} : \mathcal{MF}_K(\varphi) \to \mathcal{HP}_K(\varphi)$. We extend this functor to $\underline{\mathrm{res}} : \mathcal{MF}_K(\varphi, N) \to \mathcal{HP}_K(\varphi)$ so that $\underline{\mathrm{res}}(D)$ is weakly admissible if and only if D is weakly admissible. We define this functor via the rigid analytic technique we discussed in §III–§IV. By theorem of Colmez-Fontaine (Theorem 2.4.2) and Corollary 5.2.4, the natural functors $\underline{V}^*_{\mathrm{st}} : \mathcal{MF}^{wa}_K(\varphi, N) \to \mathrm{Rep}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}})$ and $\underline{V}^*_{\mathcal{HP}} : \mathcal{HP}^{wa}_K(\varphi) \to \mathrm{Rep}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$ are fully faithful with expected essential images. We interpret the functor $\underline{\mathrm{res}}$ in terms of the associated Galois representations.

We have the following diagrams of functors which commute up to natural isomorphisms.

$$\begin{array}{ccc} \mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi,N) & \underset{\underline{D}^{\mathcal{MF}}}{\overset{\underline{\mathcal{M}}^{\mathcal{MF}}}{\rightleftarrows}} & \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi,N_\nabla) \\ \downarrow & & \downarrow \\ \mathcal{MF}^{\geqslant 0}_{\mathscr{K}}(\varphi,N) & \underset{\underline{D}^{\mathcal{MF}}}{\overset{\underline{\mathcal{M}}^{\mathcal{MF}}}{\rightleftarrows}} & \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi,N_\nabla) \end{array} \qquad \begin{array}{ccc} \mathcal{HP}^{wa,\geqslant 0}_{K}(\varphi) & \underset{\underline{D}}{\overset{\underline{\mathcal{M}}}{\rightleftarrows}} & \underline{\mathrm{Mod}}^{\mathrm{sl}=0}_{\boldsymbol{\Delta}}(\varphi) \\ \downarrow & & \downarrow \\ \mathcal{HP}^{\geqslant 0}_{K}(\varphi) & \underset{\underline{D}}{\overset{\underline{\mathcal{M}}}{\rightleftarrows}} & \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi) \end{array}$$

The first commutative diagram was obtained by Kisin [52, §1], and the second commutative diagram was obtained from the results in §III-§IV. The top row of the first square restricts to equivalences of categories $\mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi) \cong \underline{\mathrm{Mod}}^{\mathrm{sl}=0}_{\boldsymbol{\Delta}}(\varphi, N_\nabla; N=0)$ and similarly for the bottom row.

Now, by passing to the φ- or (φ, N_∇)- vector bundles on $\boldsymbol{\Delta}$ using the equivalences of categories, we can define the covariant functor $\underline{\mathrm{res}} : \mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi, N) \to \mathcal{HP}^{wa,\geqslant 0}_{K}(\varphi)$ as the composition across the top in the following diagram which commutes up to isomorphism:

(5.2.12.1)

$$\begin{array}{ccccccc} \underline{\mathrm{res}} : \mathcal{MF}^{\geqslant 0}_{\mathscr{K}}(\varphi,N) & \xrightarrow[\cong]{\underline{\mathcal{M}}^{\mathcal{MF}}} & \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi,N_\nabla) & \longrightarrow & \underline{\mathrm{Mod}}^{\mathrm{sl}=0}_{\boldsymbol{\Delta}}(\varphi) & \xrightarrow[\cong]{\underline{D}} & \mathcal{HP}^{\geqslant 0}_{K}(\varphi), \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \underline{\mathrm{res}} : \mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi,N) & \xrightarrow{\cong} & \underline{\mathrm{Mod}}^{\mathrm{sl}=0}_{\boldsymbol{\Delta}}(\varphi,N_\nabla) & \longrightarrow & \underline{\mathrm{Mod}}^{\mathrm{sl}=0}_{\boldsymbol{\Delta}}(\varphi) & \xrightarrow{\cong} & \mathcal{HP}^{wa,\geqslant 0}_{K}(\varphi) \\ & \searrow^{\underline{\mathfrak{M}}} & & & \uparrow\cong & \nearrow^{\underline{\mathbb{H}}}_{\cong} & \\ & & & & \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[1/p] & & \end{array}$$

where the functors in the middle in both rows are defined by forgetting the differential operator N_∇, and $\underline{\mathfrak{M}}$ is defined in Corollary 2.4.7. The natural isomorphism in the left in the second row was obtained by Kisin [52, Theorem 1.3.8] (see also Theorem 2.4.6 and the discussion that follows), and the natural isomorphism in the right in the second row is obtained from Theorem 4.3.4. (In particular, for any $D \in \mathcal{MF}^{\geqslant 0}_{K}(\varphi, N)$, $\underline{\mathrm{res}}(D)$ is weakly admissible if and only if D is weakly admissible.) Since each arrow commutes with $\otimes$-products (in particular, with Tate twists), we can extend it to $\underline{\mathrm{res}} : \mathcal{MF}_{\mathscr{K}}(\varphi, N) \to \mathcal{HP}_K(\varphi)$. One can check without difficulty that the restriction $\underline{\text{res}}$ to the objects with $N = 0$ coincides with the functor $\mathcal{MF}_K(\varphi) \to \mathcal{HP}_K(\varphi)$ that is defined in §2.4.3, by unwinding the construction of $\underline{\mathcal{M}}^{\mathcal{MF}}$. (See the beginning of [52, (1.2)] for the construction of $\underline{\mathcal{M}}^{\mathcal{MF}}$.) Furthermore, the functor $\underline{\text{res}} : \mathcal{MF}^{wa}_{\mathscr{K}}(\varphi) \to \mathcal{HP}^{wa}_K(\varphi)$ is *fully faithful* by Kisin's theorem (stated in Corollary 2.4.7).

The functor $\underline{\text{res}} : \mathcal{MF}_{\mathscr{K}}(\varphi, N) \to \mathcal{HP}_K(\varphi)$ is exact and commutes with all the natural operations, such as $\otimes$-products, internal homs, and duality. Also, $\underline{\text{res}}$ preserves the Newton number t_N and the Hodge number t_H. (It is enough to check on rank-1 objects, so $N = 0$ and the claim follows from §2.4.3.) Furthermore, for $D \in \mathcal{MF}^{wa}_{\mathscr{K}}(\varphi, N)$ and for a collection $\mathbf{v} := \{m_w\}_{w\in\mathbb{Z}}$ of non-negative integers, D is of Hodge type $\mathbf{v}$ if and only if $\underline{\text{res}}(D)$ is of Hodge-Pink type $\mathbf{v}$. This can be seen from [52, Lemma 1.2.1].

Recall that we have the following anti-equivalences of categories:

$$\begin{aligned}
\underline{V}^*_{\text{st}} &: \mathcal{MF}^{wa}_{\mathscr{K}}(\varphi, N) \xrightarrow{\sim} \text{Rep}^{\text{st}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}}), \\
\underline{V}^*_{\text{cris}} &: \mathcal{MF}^{wa}_{\mathscr{K}}(\varphi) \xrightarrow{\sim} \text{Rep}^{\text{cris}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}}) \\
\underline{V}^*_{\mathcal{HP}} &: \mathcal{HP}^{wa}_K(\varphi) \xrightarrow{\sim} \text{Rep}^{\mathcal{P}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}).
\end{aligned}$$

(See Theorem 2.4.2 and comments to it for the statement and the bibliographic note for the former, and Corollary 5.2.4 for the latter.) Thus $\underline{\text{res}} : \mathcal{MF}^{wa}_{\mathscr{K}}(\varphi, N) \to \mathcal{HP}^{wa}_K(\varphi)$ induces a functor $\text{Rep}^{\text{st}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}}) \to \text{Rep}^{\mathcal{P}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})$, which is naturally isomorphic to the functor obtained by restricting the $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-action to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ by [52, Corollary 2.1.14]. Furthermore, this functor is fully faithful when restricted to the full subcategory of crystalline representation. We summarize the discussion by the following diagram of functors which commutes up to isomorphism.

(5.2.12.2)

$$\begin{array}{ccccc}
\mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi,N) & & \xrightarrow{\quad\underline{\mathrm{res}}\quad} & & \mathcal{HP}^{wa,\geqslant 0}_{K}(\varphi) \\
& \searrow^{\underline{\mathfrak{M}}} & & \nearrow^{\underline{\mathbb{H}}}_{\cong} & \\
\underline{V}^{*}_{\mathrm{st}}\ \Big\downarrow\ \cong & & \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}] & & \cong\ \Big\downarrow\ V^{*}_{\mathcal{HP}} \\
& & & \searrow^{\cong}_{\underline{V}^{*}_{\mathfrak{S}}} & \\
\mathrm{Rep}^{\mathrm{st},\geqslant 0}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}}) & & \xrightarrow{\qquad\qquad} & & \mathrm{Rep}^{\geqslant 0}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi) & & \xhookrightarrow{\quad\underline{\mathrm{res}}\quad} & & \mathcal{HP}^{wa,\geqslant 0}_{K}(\varphi) \\
& \searrow^{\underline{\mathfrak{M}}} & & \nearrow^{\underline{\mathbb{H}}}_{\cong} & \\
\underline{V}^{*}_{\mathrm{cris}}\ \Big\downarrow\ \cong & & \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}] & & \cong\ \Big\downarrow\ V^{*}_{\mathcal{HP}} \\
& & & \searrow^{\cong}_{\underline{V}^{*}_{\mathfrak{S}}} & \\
\mathrm{Rep}^{\mathrm{cris},\geqslant 0}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}}) & & \xhookrightarrow{\qquad\qquad} & & \mathrm{Rep}^{\geqslant 0}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty})
\end{array}$$

We end this discussion by giving a criterion for a given weakly admissible Hodge-Pink structure to be in the essential image of $\mathcal{MF}^{wa}_{\mathscr{K}}(\varphi)$ by the functor $\underline{\mathrm{res}}$. Let $\mathcal{M} \in \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$ and set $\mathcal{D} := \mathcal{O}_{\boldsymbol{\Delta}} \otimes_{\mathscr{K}_0} (\mathcal{M}/u\mathcal{M})$. Recall from Proposition 3.2.1 that we have a φ-compatible isomorphism $\xi[\frac{1}{\lambda}] : \mathcal{D}[\frac{1}{\lambda}] \xrightarrow{\sim} \mathcal{M}[\frac{1}{\lambda}]$, and the target is equipped with a natural connection which commutes with φ as follows:

$$\mathrm{id}_{\mathcal{M}/u\mathcal{M}} \otimes d_{\boldsymbol{\Delta}} : (\mathcal{M}/u\mathcal{M}) \otimes_{\mathscr{K}_0} \mathcal{O}_{\boldsymbol{\Delta}}[1/\lambda] \to (\mathcal{M}/u\mathcal{M}) \otimes_{\mathscr{K}_0} \Omega_{\boldsymbol{\Delta}}[1/\lambda],$$

where $d_{\boldsymbol{\Delta}}$ is the "universal derivation" on $\mathcal{O}_{\boldsymbol{\Delta}}$. Transporting through the isomorphism $\xi[\frac{1}{\lambda}]$, we obtain a singular connection $\nabla^{\mathcal{M}} : \mathcal{M} \to \mathcal{M} \otimes_{\mathcal{O}_{\boldsymbol{\Delta}}} \Omega_{\boldsymbol{\Delta}}[1/\lambda]$ which commutes with $\varphi_{\mathcal{M}}$. By the construction of $\underline{\mathcal{M}}^{\mathcal{MF}}$ (or by [52, Lemma 1.3.10]), $\mathcal{M}$ is in the essential image of the forgetful functor $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi, N_\nabla; N=0) \to \underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi)$ if and only if this specific construction of $N^{\mathcal{M}}_{\nabla}$ on $\mathcal{M}[\frac{1}{\lambda}]$ maps $\mathcal{M}$ into $\mathcal{M}$ (so $(\mathcal{M}, N^{\mathcal{M}}_{\nabla})$ is an object in $\underline{\mathrm{Mod}}_{\boldsymbol{\Delta}}(\varphi, N_\nabla; N=0)$); or equivalently, if and only if the connection $\nabla^{\mathcal{M}} : \mathcal{M}[\frac{1}{\lambda}] \to \mathcal{M} \otimes \Omega_{\boldsymbol{\Delta}}[\frac{1}{\lambda}]$ corresponding to $N^{\mathcal{M}}_{\nabla}$ (as defined in §2.4.4) has at most a simple pole at $\{x_n\}_{n\geqslant 0}$. But since this $\nabla^{\mathcal{M}}$ commutes with φ, it is enough to check that the pole of $\nabla^{\mathcal{M}}$ at x_0 is of order at most 1. (Indeed, by transporting by φ^n, we see that the order of pole or $\nabla^{\mathcal{M}}$ at x_n is equal to order of pole of $\nabla^{\mathcal{M}}$ at x_0.) The order of pole of $\nabla^{\mathcal{M}}$ at x_0 can be checked after passing to the completed stalk $\widehat{\mathcal{M}}_{x_0} = \Lambda$. Now the following proposition follows.

Proposition 5.2.13. *Let* $N_\nabla = -u\lambda\frac{d}{du} : \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \to \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}$ *be the* $\mathscr{K}_0$*-linear differential operator. Then* $(D,\Lambda) \in \mathcal{HP}^{\geqslant 0}_K(\varphi)$ *is in the essential image of* $\underline{\mathrm{res}} : \mathcal{MF}^{\geqslant 0}_{\mathscr{K}}(\varphi) \to \mathcal{HP}^{\geqslant 0}_K(\varphi)$ *if and only if the lattice* Λ *in* $\widehat{\mathcal{D}}_{x_0}[\frac{1}{\lambda}]$ *is stable under the differential operator* $N_\nabla : \widehat{\mathcal{D}}_{x_0}[\frac{1}{\lambda}] \to \widehat{\mathcal{D}}_{x_0}[\frac{1}{\lambda}]$*. Moreover, if* (D,Λ) *is weakly admissible and stable under* N_∇*, then it is in the essential image of* $\underline{\mathrm{res}} : \mathcal{MF}^{wa,\geqslant 0}_{\mathscr{K}}(\varphi) \to \mathcal{HP}^{wa,\geqslant 0}_K(\varphi)$*.*

Let $(D,\Lambda) := \underline{\mathbb{H}}(\mathfrak{M}[\frac{1}{p}])$ for some $\mathfrak{M}[\frac{1}{p}] \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{p}]$. If the matrix representation of $\varphi_{\mathfrak{M}}$ for some $\mathfrak{S}$-basis of $\mathfrak{M}$ is known, then one can write a basis of Λ in terms of a $\mathscr{K}_0$-basis of D (viewed as a basis of $\widehat{\mathcal{D}}_{x_0}$), whose computation just involves inverting the $\varphi_{\mathfrak{M}}$-matrix (Remark 3.2.4). So the above proposition gives a computable criterion to check whether $\mathfrak{M}[\frac{1}{p}]$ comes from a weakly admissible filtered isocrystal. On the other hand, even when $\mathfrak{M}[\frac{1}{p}] = \underline{\mathfrak{M}}(D)$ for some weakly admissible filtered isocrystal, the choice of $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ does not have to correspond to $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable $\mathbb{Z}_p$-lattice of $V^*_{\mathrm{cris}}(D)$, but just a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable lattice.

Finally, we comment on the functor $\underline{\mathcal{F}} : \mathcal{HP}^{wa}_K(\varphi) \to \mathcal{MF}^{wa}_{\mathscr{K}}(\varphi)$ that we defined earlier in §2.4.3. Recall that $\underline{\mathrm{res}}$ is a "section"[3] to $\underline{\mathcal{F}}$, in the sense that there is a natural isomorphism $\underline{\mathcal{F}} \circ \underline{\mathrm{res}} \cong \mathrm{id}_{\mathcal{MF}_{\mathscr{K}}(\varphi)}$. By the equivalence of categories with Galois representations, we obtain a "mysterious" functor $\mathrm{Rep}^{\mathcal{P}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}) \to \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}})$ which has the restriction to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-functor $\underline{\mathrm{res}}$ as a "section."

5.2.14 Rank-1 examples: Tate objects

Consider the Tate object $\mathfrak{S}(h)$ for some $h \geqslant 0$ as defined in Definition 2.2.6; i.e., $\mathfrak{S}(h) \cong \mathfrak{S}\cdot\mathbf{e}$ equipped with $\varphi(\sigma^*\mathbf{e}) = \mathcal{P}(u)^h\mathbf{e}$. In the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, we will show later in §7.3.7 that $\underline{T}^*(\mathfrak{S}(h)) \cong \chi^h_{\mathcal{LT}}$ for any $h \geqslant 0$, where $\chi_{\mathcal{LT}}$ is the Lubin-Tate character. We now show an analogue of this fact for the case $\mathfrak{o}_0 = \mathbb{Z}_p$: identifying

[3] In general, $\underline{\mathrm{res}} \circ \underline{\mathcal{F}} \cong \mathrm{id}_{\mathcal{HP}_K(\varphi)}$ does *not* hold, so $\underline{\mathrm{res}}$ and $\underline{\mathcal{F}}$ cannot be quasi-inverse (unless we restrict to "Barsotti-Tate" objects or rank 1-objects.)

$\boldsymbol{\mathcal{G}}_K$ with $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ as in §1.3.1.2, we have $\underline{T}^*_{\mathfrak{S}}(\mathfrak{S}(h) \cong \chi^h_{\mathrm{cyc}}|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}$ for any $h \geqslant 0$, where χ_{cyc} is the p-adic cyclotomic character.

Recall that $\chi^h_{\mathrm{cyc}} \cong \underline{V}^*_{\mathrm{cris}}(\mathbf{1}_{\mathcal{MF}}(h))$ where $\mathbf{1}_{\mathcal{MF}}(h)$ is the Tate object in $\mathcal{MF}_K(\varphi)$; i.e., $\mathbf{1}_{\mathcal{MF}}(h)$ is the weakly admissible filtered isocrystal with the underlying isocrystal $(\mathscr{K}_0\mathbf{e}, \varphi(\sigma^*\mathbf{e}) = p^h\mathbf{e})$. (By weak admissibility, the associated grading to the filtration is concentrated in degree h.) We have seen in §2.4.3 that $\underline{\mathrm{res}}(\mathbf{1}_{\mathcal{MF}}(h)) = \mathbf{1}(h)$ where $\mathbf{1}(h)$ is the Tate object in $\mathcal{HP}_K(\varphi)$ as defined in (2.3.2.1). Therefore we have $\underline{V}^*_{\mathrm{cris}}(\mathbf{1}_{\mathcal{MF}}(h))|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}} \cong \underline{V}^*_{\mathcal{HP}}(\mathbf{1}(h))$ by (5.2.12.2). On the other hand, we have seen that $\underline{\mathbb{H}}(\mathfrak{S}(h)[\frac{1}{p}]) = \mathbf{1}(h)$ in §4.3.6 so by definition of $\underline{V}^*_{\mathcal{HP}}$ (Corollary 5.2.4) we have $\underline{V}^*_{\mathcal{HP}}(\mathbf{1}(h) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{S}(h))[\frac{1}{p}]$. This shows that the desired $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-isomorphism $\underline{T}^*_{\mathfrak{S}}(\mathfrak{S}(h)) \cong \chi^h_{\mathrm{cyc}}|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}$ for any $h \geqslant 0$.

CHAPTER VI

Some non-archimedean functional analysis

The aim of this chapter is to prove Proposition 4.3.2. When $\mathfrak{o}_0 = \mathbb{Z}_p$, Proposition 4.3.2 is proved in [46, Prop 6.5], and the same proof also works in the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. We also review basic properties of the analytic rings $\mathcal{O}_{\boldsymbol{\Delta}}$, $\mathcal{R}$, etc., and the theory of Newton polygons which will be used in the proof of Proposition 4.3.2.

6.1 Rigid-analytic disks

In this section, we review basic properties of $\mathcal{O}_{\boldsymbol{\Delta}}$ and $\mathcal{R}$, and give a precise definition of $\mathcal{R}^{\mathrm{alg}}$.

Definition 6.1.1. For each $r \in q^{\mathbb{Q}_{<0}}$, we define the following multiplicative[1] norm on $\mathfrak{S}[\frac{1}{\pi_0}, \frac{1}{u}]$:

$$\|f\|_r = \max_{i \gg -\infty} \{|a_i|\, r^i\} = \max_x \{\, |f(x)|\, \} \tag{6.1.1.1}$$

where $f(u) = \sum_{i \gg -\infty} a_i u^i \in \mathfrak{S}[\frac{1}{\pi_0}, \frac{1}{u}]$ and the second maximum is taken among $x \in \mathbb{C}_{\mathscr{K}_0}$ such that $|x| = r$.

By taking logarithm, we obtain the following valuation w_γ for $\mathfrak{S}[\frac{1}{\pi_0}]$:

$$w_\gamma(f) = \min_{i \geq 0} \{v(a_i) + \gamma \cdot i\} = \min_x \{\, v(f(x))\, \}, \tag{6.1.1.2}$$

[1] This is obviously submultiplicative, and can be seen to be multiplicative. See [48, Lemma 2.1.7], for example.

where $\gamma = -\log_q r$ and all the other notations are as above.

If $f \in \mathfrak{S}[\frac{1}{\pi_0}]$, then by the maximum modulus principle $\|f\|_r$ is the maximum among $|f(x)|$ for all $x \in \mathbb{C}_{\mathscr{K}_0}$ which satisfy $|x| \leq r$.

6.1.2 Closed disks and annuli

Let $T_{\leqslant r}$ be the following affinoid $\mathscr{K}_0$-algebra:

(6.1.2.1) $$T_{\leqslant r} := \{\sum_{i\geq 0} a_i u^i \in \mathscr{K}_0[[u]], \text{ such that } |a_i|\, r^i \to 0 \text{ as } i \to \infty\},$$

(In the valuation language, the above condition translates to $v(a_i) + \gamma \cdot i \to \infty$ as $i \to \infty$, where $\gamma = -\log_q r$.) This condition is nothing but convergence on the closed disk of radius r in $\mathbb{C}_{\mathscr{K}_0}$. One can check without difficulty that $T_{\leqslant r}$ is the completion of $\mathfrak{S}[\frac{1}{\pi_0}]$ with respect to the norm $\|\cdot\|_r$, and with this norm $T_{\leqslant r}$ becomes an affiniod $\mathscr{K}_0$-algebra. Note that $\|\cdot\|_r$ is precisely the "sup norm" over the closed disk of radius r (by the maximum modulus principle). We set $\mathbf{\Delta}_{\leqslant r} := \mathrm{Sp}\,(T_{\leqslant r})$, and call it the *rigid-analytic closed disk of radius* r.

Let $I := [r_1, r_2] \subset (0, 1)$ be a closed subinterval away from 0 and 1, with endpoints in $q^{\mathbb{Q}}$ (allowing $r_1 = r_2$), and let T_I be the following affiniod $\mathscr{K}_0$-algebra:

(6.1.2.2)
$$T_{[r_1,r_2]} := \{\sum_{i\in\mathbb{Z}} a_i u^i \in \mathscr{K}_0[[u, \frac{1}{u}]], \text{ such that } \lim_{i\to-\infty} |a_i|\, r_1^i = 0 \text{ and } \lim_{i\to\infty} |a_i|\, r_2^i = 0\}.$$

One can check without difficulty that T_I is the completion of $\mathfrak{S}[\frac{1}{u}, \frac{1}{\pi_0}]$ with respect to the following submultiplicative "sup norm":

(6.1.2.3) $$\left\| \sum_{i\geq -N} a_i u^i \right\|_{[r_1,r_2]} = \max_i\{|a_i|\, r_1^i,\ |a_i|\, r_2^i\} = \max\{\, \|f\|_{r_1}, \|f\|_{r_2} \}.$$

By maximum modulus principle, this is same as the maximum of $|f(x)|$ for $x \in \mathbb{C}_{\mathscr{K}_0}$ with $|x| \in [r_1, r_2]$, and with this norm T_I becomes an affiniod $\mathscr{K}_0$-algebra. We define the *rigid-analytic closed annulus* $\mathbf{\Delta}_I := \mathrm{Sp}(T_I)$. (If $r := r_1 = r_2$ then we get a *rigid-analytic circle* of radius r.)

To allow $I := [0, r]$, we often write $T_{[0,r]} := T_{\leqslant r}$ and $\mathbf{\Delta}_{[0,r]} := \mathbf{\Delta}_{\leqslant r}$. It is well known that T_I, for any closed subinterval $I \subset [0, 1)$ with endpoints in $q^{\mathbb{Q}} \cup \{0\}$, is a principal ideal domain. We make a further remark on this later at §6.2.7.

6.1.3 Open disks, annuli and punctured disks

As before, the endpoints of any subinterval $I \subset [0, 1)$ that we consider are always assumed to lie in $q^{\mathbb{Q}_{<0}} \cup \{0\}$. For any subinterval $I \subset [0, 1)$, we define a rigid-analytic space $\mathbf{\Delta}_I := \bigcup_{J \in \mathscr{J}} \mathbf{\Delta}_J$ with $\{\mathbf{\Delta}_J\}_{J\in\mathscr{J}}$ as an *admissible* affinoid chart, where $\mathscr{J}$ is a set of closed subintervals $J \subset I$ with endpoints in $q^{\mathbb{Q}_{<0}} \cup \{0\}$, such that $\bigcup_{J\in\mathscr{J}} J = I$. Concretely, the set of $\mathbb{C}_{\mathscr{K}_0}$ points of $\mathbf{\Delta}_I$ is exactly $\{x \in \mathbb{C}_{\mathscr{K}_0} : |x| \in I\}$, and the structure sheaf $\mathscr{O}_{\mathbf{\Delta}_I}$ is obtained by "gluing" $\mathscr{O}_{\mathbf{\Delta}_J}$. We call $\mathbf{\Delta}_{<r} := \mathbf{\Delta}_{[0,r)}$ the *rigid-analytic open disk of radius* r, and we denote by $\mathbf{\Delta} := \mathbf{\Delta}_{<1}$ the rigid-analytic open unit disk. We write $\dot{\mathbf{\Delta}} := \mathbf{\Delta}_{(0,1)}$ to denote the rigid-analytic punctured open unit disk. Note that distinct choices of $\mathscr{J}$ yield the same rigid-analytic space [8, 9.1]. In particular, if I is already a closed interval, then the above construction yields the affinoid variety $\mathbf{\Delta}_I := \mathrm{Sp}(T_I)$. If $I = [0, r)$, then we may choose $\mathscr{J} := \{[0, r'] : r' < r\}$ so we regard $\mathbf{\Delta}_{<r}$ as a rising union of closed disks $\mathbf{\Delta}_{\leqslant r'}$ for $0 < r' < r$. Similarly, if $0 \notin I$, then we may choose a suitable $\mathscr{J}$ so that $\mathbf{\Delta}_I$ is a rising union of closed annuli. From now on, we always choose such $\mathscr{J}$.

For closed subintervals $J' \subset J \subset [0, 1)$, we have the natural continuous inclusion $T_J \hookrightarrow T_{J'}$ of affinoid $\mathscr{K}_0$-algebras. Furthermore, if both J and J' contain 0, then the inclusion has the *dense* image since T_J contain $\mathfrak{S}[\frac{1}{\pi_0}]$ which is dense in $T_{J'}$. The same holds if both J and J' are away from 0, since T_J contains $\mathfrak{S}[\frac{1}{\pi_0}, \frac{1}{u}]$ which is dense in $T_{J'}$. So choosing $\mathscr{J}$ for $\mathbf{\Delta}_I$ as above, we obtain a projective system $\{T_J\}_{J\in\mathscr{J}}$ such that each transition map is continuous with dense image[2], which can be thought of as the "Mittag-Leffler" condition for Banach modules. Now applying the sheaf axioms, we obtain that the ring of global sections is $\Gamma(\mathcal{O}_{\mathbf{\Delta}_I}) = \varprojlim_{J\in\mathcal{J}} T_J \, (= \bigcap_{J\in\mathcal{J}} T_J)$, where the transition maps are as above. This is a Fréchet space[3] for the topology generated by the sup-norms on $\mathbf{\Delta}_J$ for $J \in \mathcal{J}$. (Recall that $\mathcal{J}$ is always countable.) It follows from the denseness of the image of each transition map that the image of the natural map $\Gamma(\mathcal{O}_{\mathbf{\Delta}_I}) \hookrightarrow T_J$ has a dense image.[4]

The rings of rigid analytic functions $\Gamma(\mathcal{O}_{\mathbf{\Delta}_I})$ naturally sits inside $\mathscr{K}_0[[u, \frac{1}{u}]]$ as a $\mathscr{K}_0$-subspace, and we have that $f(u) \in \mathscr{K}_0[[u, \frac{1}{u}]]$ is an element of $\Gamma(\mathcal{O}_{\mathbf{\Delta}_I})$ if and only if $f(x)$ converges for any $x \in \mathbb{C}_{\mathscr{K}_0}$ with $|x| \in I$. so an element of $\Gamma(\mathcal{O}_{\mathbf{\Delta}_I})$ can be characterized by the absolute values of the coefficients of its (infinite-tailed) Laurent expansion in u. We leave the precise formulation to interested readers.

Lastly, it is well-known that $\Gamma(\mathcal{O}_{\mathbf{\Delta}_I})$ is a Bézout domain for any subinterval I. It also follows that the Robba ring $\mathcal{R}$ (Definition 2.1.2) is a Bézout domain. We make a further remark on this later at §6.2.7.

Remark 6.1.4. As remarked earlier, $\Gamma(\mathcal{O}_{\mathbf{\Delta}_{<r}})$ contains $\mathfrak{S}[\frac{1}{\pi_0}]$ as a dense subring, so it can be constructed as the *Fréchet completion* of $\mathfrak{S}[\frac{1}{\pi_0}]$ for the sup-norms $\|\cdot\|_{r'}$ for $0 < r' < r$. Similarly, $\Gamma(\mathcal{O}_{\mathbf{\Delta}_I})$ for $0 \notin I$ can be constructed as the Fréchet completion of $\mathfrak{S}[\frac{1}{\pi_0}, \frac{1}{u}]$ for the sup-norms $\|\cdot\|_J$ on $\mathbf{\Delta}_J$ for $J \in \mathcal{J}$. This "purely analytic" point of view also works when constructing such analytic rings as $\mathcal{R}^{\mathrm{alg}}$ (if $\mathfrak{o}_0 = \mathbb{Z}_p$) for which it is hard to give a precise geometric meaning. (If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then see §2.1.5 for a "geometric" interpretation of $\mathcal{R}^{\mathrm{alg}}$.)

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[2]This says that the rising union $\mathbf{\Delta}_I = \bigcup_{j\in\mathcal{J}} \mathbf{\Delta}_J$, where $\mathcal{J}$ is as above, is a (non-archimedean analogue of) "Stein exhaustion" relative to $\mathcal{O}_{\mathbf{\Delta}_I}$ in the sense of [36, IV.§1, Definition 6].

[3]Concretely, this means that any sequence $\{f_n\}$ in $\Gamma(\mathbf{\Delta}_I, \mathcal{O}_{\mathbf{\Delta}_I})$ converges if and only if $\{f_n\}$ is Cauchy with respect to the norm $\|\cdot\|_r$ for each $r \in I \cap q^{\mathbb{Q}_{<0}}$.

[4]This can be seen from the containment $\mathfrak{S}[\frac{1}{\pi_0}] \subset \Gamma(\mathcal{O}_{\mathbf{\Delta}_{<r}})$, and $\mathfrak{S}[\frac{1}{\pi_0}, \frac{1}{u}] \subset \Gamma(\mathcal{O}_{\mathbf{\Delta}_I})$ if $0 \notin I$.

6.1.5 Coherent sheaves and vector bundles

For the definition of coherent sheaves on $\mathbf{\Delta}_I$ (or rather, coherent sheaves on any rigid-analytic space), we refer to [8, §9.4]. We say a coherent sheaf $\mathcal{M}$ on $\mathbf{\Delta}_I$ (or rather, on any rigid-analytic space) if $\mathcal{M}$ becomes a finite free module over some admissible covering.

For a coherent sheaf $\mathcal{M}$ on $\mathbf{\Delta}_I$, we can express the global sections of a coherent sheaf $\mathcal{M}$ on $\mathbf{\Delta}_I$ as the following projective limit $\Gamma(\mathbf{\Delta}_I, \mathcal{M}) = \varprojlim_{J\in\mathcal{J}} \mathcal{M}_J$. Furthermore, each transition map has a dense image since $\mathcal{M}_{J'} = T_{J'} \otimes_{T_J} \mathcal{M}_J$ for $\mathbf{\Delta}_J \supseteq \mathbf{\Delta}_{J'}$ with T_J dense in $T_{J'}$. (Thus the projective system $\mathcal{M}$ satisfies the "Mittag-Leffler" condition for Banach modules.) So the global sections functor $\mathcal{M} \mapsto \Gamma(\mathcal{M})$ is an exact and fully faithful functor from the category of coherent sheaves on $\mathbf{\Delta}_I$ to the category of $\Gamma(\mathcal{O}_{\mathbf{\Delta}_I})$-modules and induces an equivalence between vector bundles of rank n over $\mathbf{\Delta}_I$ and (locally) free $\Gamma(\mathbf{\Delta}_I, \mathcal{O}_{\mathbf{\Delta}})$-modules of rank n.[5] A quasi-inverse from the essential image to the category of coherent sheaves is given as follows: if $M \cong \Gamma(\mathbf{\Delta}_I, \mathcal{M})$ for some coherent sheaf $\mathcal{M}$ then associate the projective system $\{M \otimes_{\Gamma(\mathcal{O}_{\mathbf{\Delta}_I})} T_{[r,r']}\}_{[r,r']\subset I}$ recovers M. See [38, §V] which gives a proof over an open polydisks (in particular, an open disk), but the argument can be adapted to $\mathbf{\Delta}_I$. The upshot is that we can recover a coherent sheave $\mathcal{M}$ from its global sections $\Gamma(\mathcal{M})$. From now on, we do not strictly distinguish a coherent sheaf $\mathcal{M}$ from $\Gamma(\mathcal{M})$.

6.1.6 Remark on Frobenius morphism

We define (the standard) Frobenius map $\sigma : T_{[r,r']} \to T_{[r^{1/q},r'^{1/q}]}$ over $\sigma_{\mathscr{K}_0} : \mathscr{K}_0 \to \mathscr{K}_0$ by $\sigma(u) = u^q$. (Recall that $q = p$ if $\mathfrak{o}_0 = \mathbb{Z}_p$.) By passing to the inverse limit, we also get $\sigma : \mathcal{O}_{\mathbf{\Delta}_I} \hookrightarrow \mathcal{O}_{\mathbf{\Delta}_{I^{1/q}}}$. where $I^{1/q} \subset [0,1)$ is the subinterval whose endpoints

[5]The global section $\Gamma(\mathcal{M})$ for a coherent sheaf $\mathcal{M}$ may *not* be finitely generated modules. It takes an extra work to show that if $\mathcal{M}$ is a vector bundle on $\mathbf{\Delta}_I$ then $\Gamma(\mathcal{M})$ is finite locally free over $\Gamma(\mathcal{O}_{\mathbf{\Delta}_I})$. See [38, §V, Théorème 1].

are qth root of the endpoints of I. This construction actually gives endomorphisms $\sigma : \mathcal{O}_{\boldsymbol{\Delta}} \to \mathcal{O}_{\boldsymbol{\Delta}}$ and $\sigma : \mathcal{O}_{\dot{\boldsymbol{\Delta}}} \to \mathcal{O}_{\dot{\boldsymbol{\Delta}}}$.

Since $\sigma : T_{[r,r']} \to T_{[r^{1/q},r'^{1/q}]}$ is not $\mathscr{K}_0$-linear but $\sigma_{\mathscr{K}_0}$-semilinear, we need to take its linearization $\sigma^* T_{[r,r']} \to T_{[r^{1/q},r'^{1/q}]}$ to get a map on affinoid spaces $\sigma : \boldsymbol{\Delta}_{[r^{1/q},r'^{1/q}]} \to \sigma^* \boldsymbol{\Delta}_{[r,r']}$ over $\mathscr{K}_0$. Similarly, one gets the Frobenius map $\sigma : \boldsymbol{\Delta}_{I^{1/q}} \to \sigma^* \boldsymbol{\Delta}_I$ by gluing these.

For a coherent sheaf $\mathcal{M}$ on $\boldsymbol{\Delta}_{I \cup I^{1/q}}$ (or for its global sections), the *Frobenius structure*, or the *φ-structure* is a $\mathcal{O}_{\boldsymbol{\Delta}_{I^{1/q}}}$-linear map $\varphi : \sigma^* (\mathcal{M}|_{\boldsymbol{\Delta}_I}) \to \mathcal{M}|_{\boldsymbol{\Delta}_{I^{1/q}}}$, where $\sigma^* (\mathcal{M}|_{\boldsymbol{\Delta}_I}) := \mathcal{O}_{\boldsymbol{\Delta}_{I^{1/q}}} \otimes_{\sigma, \mathcal{O}_{\boldsymbol{\Delta}_I}} (\mathcal{M}|_{\boldsymbol{\Delta}_I})$.

6.1.7

We define the following subalgebras of bounded (respectively, "integral") functions in $\mathcal{O}_{\boldsymbol{\Delta}_{[r,1)}}$:

$$\begin{aligned} \mathcal{O}^{bd}_{\boldsymbol{\Delta}_{[r,1)}} &:= \{ f(u) \in \mathcal{O}_{\boldsymbol{\Delta}_{[r,1)}} : \ |f(x)| \leqslant C, \text{ for all } x \in \boldsymbol{\Delta}_{[r,1)} \text{ and for some } C \} \\ \mathcal{O}^{\mathrm{int}}_{\boldsymbol{\Delta}_{[r,1)}} &:= \{ f(u) \in \mathcal{O}_{\boldsymbol{\Delta}_{[r,1)}} : \ |f(x)| \leqslant 1, \quad \text{for all } x \in \boldsymbol{\Delta}_{[r,1)} \}. \end{aligned}$$

Clearly we have $\mathcal{O}^{bd}_{\boldsymbol{\Delta}_{[r,1)}} = \mathcal{O}^{\mathrm{int}}_{\boldsymbol{\Delta}_{[r,1)}}[\frac{1}{\pi_0}]$. It is useful that $\mathcal{O}^{\mathrm{int}}_{\boldsymbol{\Delta}_{[r,1)}}$ is a complete normed W-algebra with respect to the norm $\| \cdot \|_r$ (or equivalently, with respect to the valuation w_γ where $\gamma = -\log_q r$). Furthermore, the above rings are principal ideal domains by [48, §2.6]. We make further comments on this later in §6.2.7.

If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, we have an interesting alternative description of $\mathcal{O}^{\mathrm{int}}_{\boldsymbol{\Delta}_{[r,1)}}$: namely, we have an equality $\mathcal{O}^{\mathrm{int}}_{\boldsymbol{\Delta}_{[r,1)}} = \mathcal{O}_{\boldsymbol{\Delta}_{K, \leqslant r'}}$ of k-subspaces of $k[[u, \pi_0, \frac{1}{u}, \frac{1}{\pi_0}]]$, where $\boldsymbol{\Delta}_{K, \leqslant r'}$ is a rigid-analytic closed disk of radius $r' = q^{-1/\gamma}$ over K with coordinate π_0. One can check that the sup-norm on $\mathcal{O}_{\boldsymbol{\Delta}_{K, \leqslant r'}}$ is exactly $\| \cdot \|_r^{1/\gamma}$ on $\mathcal{O}^{\mathrm{int}}_{\boldsymbol{\Delta}_{[r,1)}}$. The "additive" version of this claim is that the valuation corresponding to the sup norm on $\mathcal{O}_{\boldsymbol{\Delta}_{K, \leqslant r'}}$ is exactly $\frac{1}{\gamma} w_\gamma(\cdot)$, which we will verify. Take an element $f = \sum_{i \in \mathbb{Z}} a_i u^i =$

$\sum_{i\in\mathbb{Z},j\in\mathbb{Z}_{\geq 0}} c_{ij}u^i\pi_0^j$, where $a_i = \sum_j c_{ij}\pi_0^j \in W$ and $c_{ij} \in k$. Then we can check by hand that

$$\min_j \left\{ \mathrm{ord}_u \left(\sum_i c_{ij}u^i \right) + (1/\gamma)\cdot j \right\} = \min_{c_{ij}\neq 0} \{ i + (1/\gamma)\cdot j \} = \min_i \{ i + (1/\gamma)\cdot \mathrm{ord}_{\pi_0}(a_i) \},$$

where the term on the left end is the definition of the valuation on $\mathcal{O}_{\mathbf{\Delta}_{K,[0,r']}}$ and the the term on the right end is visibly $\frac{1}{\gamma}\cdot w_\gamma(f)$. (In fact, the normalization of this partial valuation used in [48, §2] is $\frac{1}{\gamma}\cdot w_\gamma(f)$, not $w_\gamma(f)$.) Also, for such $f \in \mathcal{O}_{\mathbf{\Delta},[r,1)}$ the condition $|f(x)| \leq 1$ for all $r \leqslant |x| < 1$ says $|a_i|\,\rho^i \leqslant 1$ for all $r \leqslant \rho < 1$ and $i \in \mathbb{Z}$, which forces $|a_i| \leq 1$ for all i (i.e., $a_i \in W$).

6.1.8 More analytic rings

Roughly speaking, we repeat all the above constructions of analytic rings with K replaced by $\mathbb{C}_K$. To provide intuition, we start with the case when $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. As pointed out in §2.1.5, we could carry out all the previous constructions using the rigid-analytic open unit disk $\mathbf{\Delta}_K$ over K with coordinate π_0. Then we repeat the constructions of the analytic rings (such as $\mathcal{R}$) with $\mathbf{\Delta}_K$ replaced by $\mathbf{\Delta}_{\mathbb{C}_K}$. In the case when $\mathfrak{o}_0 = \mathbb{Z}_p$, we should give a purely analytic construction due to the lack of the "open unit disk over $\mathbb{C}_K$ with coordinate p," working with the valuation $\mathrm{ord}_u(\cdot)$ on $\mathbb{C}_K$ induced from the normalized valuation on $K = k(\!(u)\!)$.

If $\mathfrak{o}_0 = \mathbb{Z}_p$, then set $\mathfrak{S}^{\mathrm{alg}} := W(\mathfrak{o}_{\mathbb{C}_K})$ and $\mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}} := W(\mathbb{C}_K)$, where $W(\cdot)$ is the ring of Witt vectors[6]. Let σ be the Witt vector Frobenius map on $\mathfrak{S}^{\mathrm{alg}}$ and $\mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}$. Similarly if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then set $\mathfrak{S}^{\mathrm{alg}} := \mathfrak{o}_{\mathbb{C}_K}[[\pi_0]]$ and $\mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}} := \mathbb{C}_K[[\pi_0]]$. Let σ be the continuous "partial q-Frobenius endomorphism," i.e., $\sigma(\pi_0) = \pi_0$ and $\sigma(\alpha) = \alpha^q$ for any $\alpha \in \mathbb{C}_K$. Note that in both cases $\mathfrak{S}^{\mathrm{alg}}/(\pi_0^n) \to \mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}/(\pi_0^n)$ is injective for all $n \geq 1$. (In fact, it

[6] Identifying K with the field of norms for $\mathscr{K}_\infty/\mathscr{K}$ as discussed in §1.3.1.2, we have an isomorphism $\mathfrak{R} \cong \mathfrak{o}_{\mathbb{C}_K}$, where $\mathfrak{R} := \varprojlim_{x^p \leftarrow x} \mathfrak{o}_{\bar{\mathscr{K}}}/(p)$. The readers who are familiar with the p-adic Hodge theory may recognize the ring $\mathfrak{S}^{\mathrm{alg}} \cong W(\mathfrak{R})$. See, for example, [78, §4.3] and [32].

suffices to check the case $n = 1$, which is obvious.)

We have a natural σ-compatible embedding $\mathfrak{o}_{\mathcal{E}} \hookrightarrow \mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}$ which restricts to $\mathfrak{S} \hookrightarrow \mathfrak{S}^{\mathrm{alg}}$. If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ then it is clear. If $\mathfrak{o}_0 = \mathbb{Z}_p$ then the completed direct limit of $\{\mathfrak{o}_{\mathcal{E}} \xrightarrow{\sigma} \mathfrak{o}_{\mathcal{E}} \xrightarrow{\sigma} \cdots\}$ induces the system of p-power maps on $k(\!(u)\!)$ modulo p, or equivalently the tower of fields $\{k(\!(u)\!)^{p^{-n}}\}$, so this completion is naturally isomorphic to $W(K^{\mathrm{perf}})$. We define the $\mathscr{K}_0$-linear map $\mathfrak{o}_{\mathcal{E}} \to \mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}$ using the functoriality of the Witt vector ring construction. Furthermore, since $\sigma(u) = u^p$ the image of u in $W(K^{\mathrm{perf}})$ is "p-divisible" (in the *multiplicative* sense) it is the Teichmüller lift of the image of its reduction in K^{perf}. Hence $u \in \mathfrak{o}_{\mathcal{E}}$ maps to the Teichmüller lift $[u] \in \mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}$ of $u \in \mathbb{C}_K$. This shows that $\mathfrak{S}$ lands in $\mathfrak{S}^{\mathrm{alg}}$. Using these natural embeddings, we view $\mathfrak{S}$, $\mathfrak{S}^{\mathrm{alg}}$ and $\mathfrak{o}_{\mathcal{E}}$ as subrings of $\mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}$.

For any $\alpha \in \mathbb{C}_K$, we denote by $[\alpha] \in \mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}$ the Teichmüller lift if $\mathfrak{o}_0 = \mathbb{Z}_p$, and the image of α under the natural inclusion $\mathbb{C}_K \hookrightarrow \mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}$ if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. (In both cases $[\alpha] \in (\mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}})^{\times}$ if $\alpha \neq 0$.) Any element $f \in \mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}[\frac{1}{\pi_0}]$ can be uniquely expressed as $f = \sum_{j \gg -\infty} [\alpha_j]\pi_0^j$, where $\alpha_j \in \mathbb{C}_K$, and one can directly check that $f \in \mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}]$ if and only if all α_j are in $\mathfrak{o}_{\mathbb{C}_K}$ (i.e., $\mathrm{ord}_u(\alpha_j) \geq 0$ for all j.); and $f \in \mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}, \frac{1}{[u]}]$ if and only if the $\mathrm{ord}_u(\alpha_i)$'s are bounded below.

Now let us extend the valuations $w_\gamma(\cdot)$ from $\mathfrak{S}[\frac{1}{\pi_0}]$ to $\mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}, \frac{1}{[u]}]$ for $\gamma \in \mathbb{Q}_{>0}$ as follows:

$$w_\gamma(f) := \min_j \{j + \gamma \cdot \mathrm{ord}_u(\alpha_j)\}, \tag{6.1.8.1}$$

where $f = \sum_{j \gg -\infty} [\alpha_j]\pi_0^j \in \mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}, \frac{1}{[u]}]$. This *a priori* sub-multiplicative valuation w_γ is in fact multiplicative, by [48, Lemma 2.1.7]. Note also that $w_\gamma(\sigma(f)) = w_{q\gamma}(f)$.

Remark 6.1.9. To prove properties on w_γ such as the strict triangule inequality and multiplicativity, the following "coordinate-free" description of w_γ can be useful, especially when $\mathfrak{o}_0 = \mathbb{Z}_p$. For $f \in \mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}[\frac{1}{\pi_0}]$ and $n \in \mathbb{Z}$, we define

$$w(f;n) := \min\{i \in \mathbb{Z} |\, u^{-i} f \in \mathfrak{S}^{\mathrm{alg}}[1/\pi_0] + \pi_0^{n+1} \mathfrak{o}_{\mathcal{E}}^{\mathrm{alg}}\}.$$

More concretely, if $f = \sum_{j \gg -\infty} [\alpha_j] \pi_0^j$, then $w(f;n) = \min_{j \leq n}\{\mathrm{ord}_u(\alpha_j)\}$ (which could be infinite even if $f \neq 0$). Now, we can see that whenever $w_\gamma(f)$ is defined, we have $w_\gamma(f) = \min_n\{n + \gamma \cdot w(f;n)\}$. In fact, if $w_\gamma(f) = n + \gamma \cdot \mathrm{ord}_u(\alpha_n)$ for some n, then we have $w(f;n) = \mathrm{ord}_u(\alpha_n)$.

As a corollary of this alternative definition of w_γ, we can check that w_γ restricted to $\mathfrak{S}[\frac{1}{\pi_0}, \frac{1}{u}]$ coincides with the previous definition of w_γ for $\mathfrak{S}[\frac{1}{\pi_0}, \frac{1}{u}]$, which is defined in (6.1.1.2).

6.1.10 More Robba rings

For a subinterval $I \subset [0,1)$ with $0 \in I$, we define $\mathcal{O}_{\mathbf{\Delta}_I}^{\mathrm{alg}}$ to be the Fréchet completion of $\mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}]$ for w_γ with $q^{-\gamma} \in I$. Similarly for a subinterval $I \subset (0,1)$, we define $\mathcal{O}_{\mathbf{\Delta}_I}^{\mathrm{alg}}$ to be the Fréchet completion of $\mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}, \frac{1}{[u]}]$ for w_γ with $q^{-\gamma} \in I$. For any two subintervals $I' \subset I$, we have a natural continuous injective map $\mathcal{O}_{\mathbf{\Delta}_I}^{\mathrm{alg}} \to \mathcal{O}_{\mathbf{\Delta}_{I'}}^{\mathrm{alg}}$, which has a dense image if the subintervals either both contain 0 or are both away from 0. If $I = [0, r]$, then $\mathcal{O}_{\mathbf{\Delta}_{\leqslant r}}^{\mathrm{alg}}$ is complete for the valuation w_γ where $\gamma := -\log_q r$. Similarly if $I = [r_1, r_2]$, then $\mathcal{O}_{\mathbf{\Delta}_I}^{\mathrm{alg}}$ is complete for the submultiplicative valuation $w_I(\cdot) := \min\{w_{\gamma_1}(\cdot), w_{\gamma_2}(\cdot)\}$, where $\gamma_i := -\log_q r_i$. So if I is closed, then $\mathcal{O}_{\mathbf{\Delta}_I}^{\mathrm{alg}}$ is a Banach $\mathscr{K}_0$-algebra. We leave the verification to readers.

One can directly check that the Frobenius endomorphism $\sigma : \mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}, \frac{1}{[u]}] \to \mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}, \frac{1}{[u]}]$, introduced in §6.1.8, continuously extends to a map $\sigma : \mathcal{O}_{\mathbf{\Delta}_I}^{\mathrm{alg}} \to \mathcal{O}_{\mathbf{\Delta}_{I^{1/q}}}^{\mathrm{alg}}$, where $I^{1/q} \subset [0,1)$ is the subinterval whose endpoints are qth root of the endpoints of I.

For $0 < r < 1$ and $\gamma := -\log_q r$, we put

$$\begin{aligned}
\mathcal{O}^{\mathrm{alg},bd}_{\mathbf{\Delta}_{[r,1)}} &:= \left\{ \sum_{j \gg -\infty} [\alpha_j]\pi_0^j \in \mathfrak{o}^{\mathrm{alg}}_{\mathcal{E}}[\frac{1}{\pi_0}], \text{ such that } j + \gamma \cdot \mathrm{ord}_u(\alpha_j) \to \infty \text{ as } j \to \infty \right\} \\
\mathcal{O}^{\mathrm{alg,int}}_{\mathbf{\Delta}_{[r,1)}} &:= \left\{ \sum_{j \geq 0} [\alpha_j]\pi_0^j \in \mathfrak{o}^{\mathrm{alg}}_{\mathcal{E}}, \text{ such that } j + \gamma \cdot \mathrm{ord}_u(\alpha_j) \to \infty \text{ as } j \to \infty \right\}.
\end{aligned}$$

For $r = 0$, we put $\mathcal{O}^{\mathrm{alg},bd}_{\mathbf{\Delta}_{[0,1)}} := \mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}]$ and $\mathcal{O}^{\mathrm{alg,int}}_{\mathbf{\Delta}_{[0,1)}} = \mathfrak{S}^{\mathrm{alg}}$. (Note that $\mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}] = \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}} \cap \mathcal{O}^{\mathrm{alg},bd}_{\mathbf{\Delta}_{[r,1)}}$ and this convention is consistent with $\mathfrak{S}[\frac{1}{\pi_0}] = \mathcal{O}^{bd}_{\mathbf{\Delta}}$.) For any $0 \leq r < 1$, we have $\mathcal{O}^{\mathrm{alg},bd}_{\mathbf{\Delta}_{[r,1)}} = \mathcal{O}^{\mathrm{alg,int}}_{\mathbf{\Delta}_{[r,1)}}[\frac{1}{\pi_0}]$, and $\mathcal{O}^{\mathrm{alg,int}}_{\mathbf{\Delta}_{[r,1)}}$ is complete for the valuation w_γ where $\gamma = -\log_q r$. Also, $\sigma : \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I} \to \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_{I^{1/q}}}$ restricts to the subalgebras of bounded functions (respectively, integral functions).

Now, we are ready to define the Robba rings:

$$\begin{aligned}
\mathcal{R}^{\mathrm{alg}} &:= \varinjlim_r \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_{[r,1)}} \\
\mathcal{R}^{\mathrm{alg},bd} &:= \varinjlim_r \mathcal{O}^{\mathrm{alg},bd}_{\mathbf{\Delta}_{[r,1)}} \\
\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}} &:= \varinjlim_r \mathcal{O}^{\mathrm{alg,int}}_{\mathbf{\Delta}_{[r,1)}}
\end{aligned}$$

Just as $\mathcal{R}^{bd}$, $\mathcal{R}^{\mathrm{alg},bd}$ has the discrete π_0-adic valuation ord_{π_0} for which $\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}}$ is the valuation ring. In other words, for $f = \sum_{j \gg -\infty} [\alpha_j]\pi_0^j$, we define $\mathrm{ord}_{\pi_0}(f)$ as the minimal j such that $\alpha_j \neq 0$. We leave to readers the verification that this is a valuation. And precisely the same argument that shows that $\mathfrak{o}_{\mathcal{R}^{bd}}$ is a discretely valuation ring with a uniformizer p shows the same claim for $\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}}$. (See [20, §4.3] for more details.)

Since the inclusion $\mathfrak{S}[\frac{1}{\pi_0}, \frac{1}{u}] \hookrightarrow \mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}, \frac{1}{[u]}]$ respects all w_γ (Remark 6.1.9), we obtain a continuous embedding $\mathcal{O}_{\mathbf{\Delta}_I} \hookrightarrow \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ and $\mathcal{R} \hookrightarrow \mathcal{R}^{\mathrm{alg}}$, and similarly for their bounded counterparts. It turns out that all of them are faithfully flat ring extensions, by Proposition 6.2.8.

The Frobenius maps $\sigma : \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_{[r,1)}} \to \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_{[r^{1/q},1)}}$ induce a Frobenius endomorphism σ

on each of $\mathcal{R}^{\mathrm{alg}}$, $\mathcal{R}^{\mathrm{alg},bd}$, and $\mathfrak{o}_{\mathcal{R}^{\mathrm{alg},bd}}$. With this choice of σ, these "Robba rings" are σ-rings over $(\mathfrak{S}, \sigma)$.

The following table is for those who would like to compare this exposition with [48, §2].

Notations in [48, §2]	Γ	$\Gamma_{1/\gamma}$	Γ_{con}	$\Gamma_{\mathrm{an},1/\gamma}$	$\Gamma_{\mathrm{an,con}}$
Notations from this paper	$\mathfrak{o}_{\mathcal{E}}$	$\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}}$	$\mathfrak{o}_{\mathcal{R}^{bd}}$	$\mathcal{O}_{\mathbf{\Delta}_{[r,1)}}$	$\mathcal{R}$

The superscript $(\cdot)^{\mathrm{alg}}$ has the same meaning in both sets of notations. Kedlaya [48, §2] normalizes the additive valuation differently; he works with $(1/\gamma)w_\gamma$ instead of w_γ.

6.2 Newton polygon

The Newton polygon for a rigid-analytic function is often useful in the study of rigid-analytic functions. For example, the theory of Newton polygons play an important role in Lazard's work [57], and in the proof of Proposition 4.3.2 which will be seen in the next section. From now on, we will primarily work with w_γ instead of $\|\cdot\|_r$; the graphs of piecewise linear functions are easier to handle than those of piecewise exponential function.

Even though we introduce the theory only for subrings of $\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$, the original paper [48, §2] handles more general analytic rings.

6.2.1 Newton polygon for a polynomial

In order to provide intuition for our discussion, let us first discuss the following simple case, which will be generalized later. Let $f(u) = \sum_{i \leq h} a_i u^i \in \mathscr{K}_0[u]$ be a *nonzero* polynomial of degree d.

Definition 6.2.1.1. The *Newton polygon* for $f(u)$ is the lower convex hull of the set of points $(i, v(a_i))$, where $v(\cdot) = \mathrm{ord}_{\pi_0}(\cdot)$ is the normalized valuation on $\mathscr{K}_0$.

The *slopes* of $f(u)$ are the negatives of the slopes of the (line segments of) Newton polygon for $f(u)$. For a slope γ of $f(u)$, we define the *multiplicity* of the slope γ as the difference of the x-coordinates of the end points of the line segment with slope $-\gamma$ in the Newton polygon. If γ does not occur as a slope, then we define the multiplicity for γ to be zero.

This notion of slopes has nothing to do with the slope of a φ-module introduced in 4.1.4. Also, the Newton polygon here is not directly related to the Newton polygon[7] for a φ-module over $\mathcal{R}$ (which we do not define), or anything of this sort.

Remark 6.2.1.2. Let $\{\alpha_i\}$ be the set of zeroes of $f(u)$ in a splitting field for $f(u)$ over $\mathscr{K}_0$ (or in $\mathbb{C}_{\mathscr{K}_0}$). Then one can show that the set of slopes for $f(u)$ coincides with the set $\{\mathrm{ord}_{\pi_0}(\alpha_i)\}$. The multiplicity for the slope s is exactly the number of zeroes α_i (counted with multiplicities) such that $\mathrm{ord}_{\pi_0}(\alpha_i) = s$.

Example 6.2.1.3.

1. Let $f(u) = (u-\pi_0)^2 \cdot (u-\pi_0^2) = u^3 + (-2\pi_0 - \pi_0^2)u^2 + (\pi_0^2 + 2\pi_0^3)u - \pi_0^4$. Then the Newton polygon for $f(u)$ is $\{(3,0),\ (1,2),\ (0,4)\}$. The slope 1 appears with multiplicity 2 and the slope 2 with multiplicity 1. (We get the same result even in characteristic 2.)

2. Let $f(u) = u^p - \pi_0^{p-1}u + \pi_0$. The Newton polygon for $f(u)$ is $\{(p,0),\ (0,1)\}$, so the unique slope $1/p$ appears with multiplicity p in the Newton polygon. It is also possible to see directly that all the zeroes of $f(u)$ have π_0-order $1/p$. For example, if $\alpha \in \mathbb{C}_{\mathscr{K}_0}$ is a zero of $f(u)$, then $\alpha + i \cdot \pi_0$ for $i \in \mathbb{F}_p$ are also zeroes of $f(u)$. In order for their product to have π_0-order 1, α should satisfy $\mathrm{ord}_{\pi_0}(\alpha) = 1/p$.

[7] Hartl [39, Definition 1.5.5] calls it the Harder-Narasimhan polygon.

Remark 6.2.1.4. Let $f(u), g(u) \in \mathscr{K}_0[u]$ be nonzero polynomials. Let NW_f (respectively, NW_g) be the set of all the vertices in the Newton polygon for $f(u)$ (respectively, for $g(u)$). Then the following statements are immediate:

1. The Newton polygon for $f(u)+g(u)$ "lies over" the lower convex hull of $NW_f \cup NW_g$.

2. It is possible to describe NW_{fg} in terms of NW_f and NW_g. (We will carry this out in more general setup later.) The set of slopes for $f(u) \cdot g(u)$ is the union of the set of slopes for $f(u)$ and the set of slopes for $g(u)$, and the multiplicities add up.

For $\gamma \in \mathbb{Q}_{>0}$, we call $f(u) \in \mathscr{K}_0[u]$ *pure of slope* γ if the Newton polygon for f consists of one line segment with slope γ. It follows that if $f(u)$ is pure of slope γ, then the multiplicity for the slope γ is necessarily equal to the degree of $f(u)$. Lazard [57, §4, Théorème 1] showed that if the base field is discretely valued then any $f \in \mathcal{O}_{\mathbf{\Delta}_I}$ can be expressed as a convergent product $f = g \cdot u^a \cdot (\prod_\gamma P_\gamma)$, where $g \in \mathcal{O}^\times_{\mathbf{\Delta}_I}$ and P_γ is a polynomial pure of slope γ with $P_\gamma(0) = 1$. (*c.f.* Weierstrass factorization theorem for entire functions.) See §6.2.7 for further discussions.

6.2.2 Newton polygon for a rigid-analytic function

Fix a subinterval $I \subset [0,1)$, and let $f = \sum_{j\in\mathbb{Z}}[\alpha_j]\pi_0^j \in \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$, where $\alpha_j \in \mathbb{C}_K$. Assume always that f is nonzero. Set $I_o := \{\gamma \in \mathbb{R} \,:\, q^{-\gamma} \in I\} \subset \mathbb{R}_{>0}$.

Definition. The *Newton polygon* NW_f for a nonzero $f \in \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ is the sub-polygon of the lower convex hull of the set of points $(\mathrm{ord}_u(\alpha_j), j)$, which consists of all line segments whose slopes lie in $-I_o$. Equivalently, NW_f is the sub-polygon of the lower convex hull of the points $(w(f;n), n)$ with the same condition on the slopes of line segments. The *slopes* of f are the negatives of the slopes of the line segments of Newton polygon for f. (The slopes belong to I_o by the definition of the Newton polygon.) For a slope γ of f, we define the *multiplicity* of the slope γ as the difference of the x-coordinates of the end points of the line segment with slope γ of the Newton polygon. If γ does not occur as a slope (for example, when $\gamma \notin I_o$), then we say that the multiplicity for γ is zero.

For a nonzero rigid-analytic function $f(u) \in \mathcal{O}_{\mathbf{\Delta}_I}$, we can give the following equivalent definition of the Newton polygon: write $f(u) = \sum_{i \in \mathbb{Z}} a_i u^i$ where $a_i \in \mathscr{K}_0$, then NW_f coincides with the sub-polygon of the lower convex hull of the points $(i, v(a_i))$ which consists of the line segments whose slopes lie in $-I_o$. This polygon is the same as the sub-polygon of the lower convex hull of the points $(w(f;n), n)$ with the same slope condition.

Remark 6.2.3.

1. We can make a correspondence between π_0-orders of the zeroes of $f(u) \in \mathcal{O}_{\mathbf{\Delta}_I}$ in $\mathbf{\Delta}_I$ and the slopes of the Newton polygon for $f(u)$, and can interpret the multiplicity of a slope in terms of zeroes as in Remark 6.2.1.2. We will make a precise statement in §6.2.7.

2. Let $f(u) \in \mathscr{K}_0[u]$ be a nonzero polynomial. Then the Newton polygon for $f(u)$ viewed as a section of $\mathcal{O}_{\mathbf{\Delta}_I}$ (or an element of $\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$) can be obtained by truncating the line segments of slope outside I_o from the previous Newton polygon for a polynomial $f(u)$. The factors of $f(u)$ which contribute to the slopes outside I_o have no zeroes in $\mathbf{\Delta}_I$, and in fact are units in $\mathcal{O}_{\mathbf{\Delta}_I}$ as we will see later, so it makes sense to ignore the contribution from these factors.

3. If $I \subset [0,1)$ is closed on the left (respectively, on the right), then the Newton polygon for any $f \in \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ is bounded on the left (respectively, on the right).

In particular if I is closed, then any Newton polygons are finite (i.e., any Newton polygons consist of finitely many vertices and line segments). This follows from the explicit description of $\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ in terms of valuation of coefficients of the "Laurent expansion". (We leave the verification to readers.)

As a consequence, the zero locus of $f \in \mathcal{O}_{\mathbf{\Delta}_I}$ is "discrete" (so *finite* if I is a closed subinterval). In fact, for any closed subinterval $J \subset I$ the Newton polygon for f viewed as an element in $\mathcal{O}_{\mathbf{\Delta}_J}$ is finite, and we use the correspondence between the zeroes of f in $\mathbf{\Delta}_J$ and the Newton polygon for $f \in \mathcal{O}_{\mathbf{\Delta}_J}$ (as explained above in (1)) to conclude that the zero locus of f in $\mathbf{\Delta}_J$ is finite.

On the other hand, the Newton polygon does not have to be finite if I is *not* a closed interval. For example, the rigid-analytic function $\lambda \in \mathcal{O}_{\mathbf{\Delta}}$, defined in §2.1.3, has the following Newton polygon: $\{(0,0), (qe,-1), (qe+q^2e,-2), \cdots\}$, where e is the degree of the point $x_0 \in \mathbf{\Delta}$ cut out by $\mathcal{P}(u)$. The set of slopes is $\{\frac{1}{q^n e}\}_{n\in\mathbb{Z}_{\geq 0}}$ and the slope $\frac{1}{q^n e}$ appears with multiplicity $q^n e$. Furthermore, if $\mathbf{\Delta}_I$ is a punctured open disk or an open annulus, then one can also find an example such that the Newton polygon is unbounded on both sides.

4. The nonzero elements of the subrings $\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}} \subset \mathcal{O}_{\mathbf{\Delta}_{[r,1)}}$ and $\mathcal{O}^{\mathrm{alg},bd}_{\mathbf{\Delta}_{[r,1)}} \subset \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_{[r,1)}}$ are exactly those with *finite* Newton polygon. This can be seen as follows. Let $f = \sum_{j\in\mathbb{Z}}[\alpha_j]\pi_0^j \in \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_{[r,1)}}$, where $\alpha_j \in \mathbb{C}_K$. By (3), the Newton polygon for f is always bounded on the left, and it is bounded on the right if and only if the y-coordinates of the Newton polygon are bounded below by some integer N, which means that $\alpha_j = 0$ for all $j < N$ (i.e., $f \in \mathfrak{o}^{\mathrm{alg}}_{\mathcal{E}}[\frac{1}{\pi_0}]$) so . Furthermore, if f is bounded, then the y-coordinates of the lower right endpoint of the Newton polygon for f is precisely the minimum among j such that $\alpha_j \neq 0$.

6.2.4 Newton polygons and the valuation w_γ

For $f = \sum_{j\in\mathbb{Z}}[\alpha_j]\pi_0^j \in \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ and for $\gamma \in I_o$ (i.e., $q^{-\gamma} \in I$), we have defined the following valuation earlier in (6.1.1.2)

$$w_\gamma(f) = \min_j\{j + \gamma\cdot\mathrm{ord}_u(\alpha_j)\}.$$

We can also show that for $f \in \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ as above, we have $j + \gamma\cdot\mathrm{ord}_u(\alpha_j) \to \infty$ as $j \to \pm\infty$. For a nonzero f, we define,

$$\begin{aligned} N_\gamma(f) &:= \max\{\,\mathrm{ord}_u(\alpha_j) \text{ such that } w_\gamma(f) = j + \gamma\cdot\mathrm{ord}_u(\alpha_j)\,\} \\ n_\gamma(f) &:= \min\{\,\mathrm{ord}_u(\alpha_j) \text{ such that } w_\gamma(f) = j + \gamma\cdot\mathrm{ord}_u(\alpha_j)\,\}. \end{aligned}$$

The following proposition is immediate.

Proposition 6.2.5.

1. *Assume that $N_\gamma(f) \neq n_\gamma(f)$. Then, $N_\gamma(f)$ (respectively, $n_\gamma(f)$) is the x-coordinate of the right end point (respectively, the left end point) of the line segment with slope γ in the Newton polygon for f. In particular, γ is a slope for f with multiplicity $N_\gamma(f) - n_\gamma(f) > 0$.*

2. *Assume that $N_\gamma(f) = n_\gamma(f)$. Then the Newton polygon for f does not contain any line segment of slope $-\gamma$ (i.e., γ is not a slope for f), and $N_\gamma(f) = n_\gamma(f)$ is the x-coordinate of the vertex of the Newton polygon whose adjacent line segments have one slope larger than $-\gamma$ and the other slope smaller than $-\gamma$.*

In either case, the multiplicity for γ is $N_\gamma(f) - n_\gamma(f)$

We sketch the idea of proof. For fixed γ consider a family of lines $l_w :\ y + \gamma\cdot x = w$ where the parameter w is chosen so that l_w passes through some vertex of the Newton polygon $(\mathrm{ord}_u(\alpha_j),\, j)$. Then the smallest value among those w occurs exactly when the vertex $(\mathrm{ord}_u(\alpha_j), j)$ that l_w passes through lies in the line segment of slope $-\gamma$ of the Newton polygon if γ is a slope for f, or when $(\mathrm{ord}_u(\alpha_j), j)$ is the vertex as described in (2) of the proposition if γ is not a slope. Proposition 6.2.5 follows from this consideration.

Proposition 6.2.6. *Let $f, f' \in \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ be non-zero elements and let us fix $\gamma \in I_o$ (i.e., $q^{-\gamma} \in I$). Let $N := N_\gamma(f)$, $N' := N_\gamma(f')$ and $n := n_\gamma(f)$, $n' := n_\gamma(f')$, and let NW_f (respectively, $NW_{f'}$) be the set of vertices of the Newton polygon for f (respectively, for f').*

1. *The Newton polygon for $f + f'$, if nonzero, "lies over" the lower convex hull of $NW_f \cup NW_{f'}$.*

2. *We have $N_\gamma(f{\cdot}f') = N+N'$ and $n_\gamma(f{\cdot}f') = n+n'$. Furthermore, if $(n, j_n), (N, j_N)$ are the vertices of NW_f and $(n', j_{n'}), (N', j_{N'})$ are the vertices of $NW_{f'}$ as in Proposition 6.2.5, then $(n + n', j_n + j_{n'}), (N + N', j_N + j_{N'})$ are the vertices of $NW_{f{\cdot}f'}$ as in Proposition 6.2.5.*

*In particular, the (*a priori *submultiplicative) valuation w_γ is multiplicative.*

The proof is quite elementary. See [48, Lemma 2.1.7] for the proof in the case $\mathfrak{o}_0 = \mathbb{Z}_p$, which also works in the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$.

As a corollary, we have the following interesting criterion for $f \in \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ to be a unit in terms of its Newton polygon.

Corollary 6.2.6.1. *The Newton polygon for $f \in \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ consists of a single vertex if and only if $f = [u]^c{\cdot}g$ for some $c \in \mathbb{Q}$ and $g \in \left(\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}\right)^\times$. (If $f \in \mathcal{O}_{\mathbf{\Delta}_I}$, then c is an integer.) Furthermore, if $0 \notin I$ (so $[u] \in \left(\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}\right)^\times$), then elements in $\left(\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}\right)^\times$ are exactly those whose Newton polygons consist of a single vertex.*

Proof. Let $f = [u]^c \cdot g$ for some $g \in \left(\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}\right)^\times$. By applying Proposition 6.2.6(2) to $g \cdot g^{-1} = 1$, we know that the Newton polygon for g consists of a single vertex (since the constant function 1 has this property.) And because the Newton polygon for $[u]^c$ consists of a single vertex, we conclude that the product $[u]^c \cdot g$ has the Newton polygon which consists of a single vertex, by Proposition 6.2.6(2).

For the "if" direction, assume that the Newton polygon for $f = \sum_{j\in\mathbb{Z}}[\alpha_j]\pi_0^j$ consists of a single point (c, n). In particular, we have $\mathrm{ord}_u(\alpha_n) = c$, so $\alpha_n \neq 0$. First, we reduce to the case when $(c, n) = (0, 0)$, and $\alpha_0 = 1$. If $0 \notin I$, we can do this by multiplying f by $([\alpha_n]\pi_0^n)^{-1}$. If $0 \in I$, then we show that $[u]^c$ divides f. If there exists $\alpha_j \neq 0$ such that $c_0 := \mathrm{ord}_u(\alpha_{j_0}) < c$, then the point $(c_0,\, j_0)$ appears in the Newton polygon for f. But this contradicts to the assumption that the Newton polygon for f is a single point (c, n). Therefore, we may replace f by $([\alpha_n]\pi_0^n)^{-1} \cdot f$ in call cases.

Now, it is enough to show that if the Newton polygon for f is $\{(0, 0)\}$ and $a_0 = 1$, then f is a unit. By assumptions and the proposition in (6.2.5), we have $w_\gamma(f - 1) > 0$, so $w_\gamma((f - 1)^i) \to \infty$ as $i \to \infty$, for any $\gamma \in I_o \cap \mathbb{Q}_{>0}$. On the other hand, $\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ is a Fréchet space for the valuations w_γ for $\gamma \in I_o \cap \mathbb{Q}_{>0}$. Therefore, the infinite sum $\sum_{i\in\mathbb{Z}_{>0}}(f - 1)^i$ converges in $\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$, and we have $\big(1 + (f - 1)\big) \cdot \big(\sum_{i\in\mathbb{Z}_{>0}}(f - 1)^i\big) = 1$. □

The following is a corollary to both the statement and the proof of Corollary 6.2.6.1, and will be used in the proof of Proposition 4.3.2.

Corollary 6.2.6.2. *Let $I \subset (0, 1)$ be a subinterval (so we have $u \in \mathcal{O}^\times_{\mathbf{\Delta}_I}$). Then, for any $f(u) \in \mathcal{O}^\times_{\mathbf{\Delta}_I}$, there exists a unit $g \in (\mathcal{O}^{bd}_{\mathbf{\Delta}_I})^\times$ such that the Newton polygon for $g \cdot f$ consists of a single vertex $\{(0, 0)\}$ and $w_\gamma(g \cdot f - 1) > 0$ for all $\gamma \in I_o$.*

Proof. By Corollary 6.2.6.1, we know that the Newton polygon for $f(u) = \sum_{i\in\mathbb{Z}} a_i u^i$

consists of a single point, say $\{(j, j')\}$. Now take $g(u) := (a_j u^j)^{-1}$. Then clearly the Newton polygon for $g{\cdot}f$ is $\{(0,0)\}$. And since the constant term for $g{\cdot}f$ is 1, we have seen in the proof of Corollary 6.2.6.1 that $w_\gamma(g{\cdot}f - 1) > 0$ for all $\gamma \in I_o$. $\square$

In fact, we will prove the GL_n version of this corollary by induction on n. Hence, this corollary serves as the base case to initiate the induction.

We digress to record nice corollaries to Corollary 6.2.6.1.

Corollary 6.2.6.3. *All the units of* $\mathcal{O}_{\mathbf{\Delta}_{[r,1)}}$ *and* $\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_{[r,1)}}$ *are bounded for any* $0 \le r < 1$*; i.e., we have* $\mathcal{O}^{\times}_{\mathbf{\Delta}_{[r,1)}} = \left(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}}\right)^{\times}$ *and* $\left(\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_{[r,1)}}\right)^{\times} = \left(\mathcal{O}^{\mathrm{alg},bd}_{\mathbf{\Delta}_{[r,1)}}\right)^{\times}$*. In particular, we have* $\mathcal{O}^{\times}_{\mathbf{\Delta}} = \left(\mathfrak{S}[\frac{1}{\pi_0}]\right)^{\times}$, $\left(\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}}\right)^{\times} = \left(\mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}]\right)^{\times}$, $\mathcal{R}^{\times} = \left(\mathcal{R}^{bd}\right)^{\times}$ *and* $\left(\mathcal{R}^{\mathrm{alg}}\right)^{\times} = \left(\mathcal{R}^{\mathrm{alg},bd}\right)^{\times}$.

Proof. Since $\mathcal{O}^{bd}_{\mathbf{\Delta}} = \mathfrak{S}[\frac{1}{\pi_0}]$ and $\mathcal{O}^{\mathrm{alg},bd}_{\mathbf{\Delta}} = \mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}]$, it is enough to prove the first two equalities. One inclusion is obvious, so we prove $\mathcal{O}^{\times}_{\mathbf{\Delta}_{[r,1)}} \subset (\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})^{\times}$. For $f \in \mathcal{O}^{\times}_{\mathbf{\Delta}_{[r,1)}}$, the Newton polygon for $f(u)$ is a single point by Corollary 6.2.6.1, in particular finite. But as remarked earlier (Remark 6.2.3(4)), it follows that $f \in \mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}}$. Since f^{-1} also has the Newton polygon consisting of a single point as well, we have $f^{-1} \in \mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}}$. The case of $\mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_{[r,1)}}$ is similar. $\square$

6.2.7 Remarks on Bézout property

We record the following proposition which gives an interpretation of slopes and multiplicities analogous to Remark 6.2.1.2 and Remark 6.2.3(1). The statement can be regarded as a version of "Weierstrass preparation", and the proof as an analogue of "Weierstrass division algorithm" and "approximate Euclid's algorithm." See [57, §2,3] for a proof. We will not use this proposition later.

Proposition 6.2.7.1. *[57, §3, Proposition 2] Let* $f(u) \in \mathcal{O}_{\mathbf{\Delta}_I}$*, and assume that* $f(u)$ *has a slope* γ *with multiplicity* d*. Then there exists a polynomial* $P_\gamma(u) \in \mathscr{K}_0[u]$ *of* *degree d and pure of slope γ which divides $f(u)$. Furthermore, $f(u)/P_\gamma(u)$ does not have γ as its slope, and $P_\gamma(u)$ is unique up to scalar multiple (so it is unique if we require $P(0) = 1$).*

If we write $f(u) = P_\gamma(u) \cdot g(u)$, then γ is *not* a slope for $g(u)$, by Proposition 6.2.6(2). Therefore, we can immediately deduce the following statement by induction on the number of slopes: if $I \subset [0, 1)$ is a closed subinterval (so the Newton polygon is finite), then any $f(u) \in \mathcal{O}_{\boldsymbol{\Delta}_I}$ can be written as a product of a polynomial and a unit in $\mathcal{O}^\times_{\boldsymbol{\Delta}_I}$. In particular, $\mathcal{O}_{\boldsymbol{\Delta}_I}$ is a principal ideal domain if I is closed. With more work, we can prove the following for any subinterval $I \subset [0, 1)$: any $f \in \mathcal{O}_{\boldsymbol{\Delta}_I}$ can be expressed as a convergent product $f = g \cdot u^a \cdot (\prod_\gamma P_\gamma)$, where the (possibly infinite) product is over all slopes γ of $f(u)$, P_γ is a polynomial pure of slope γ with $P_\gamma(0) = 1$, and $g \in \mathcal{O}^\times_{\boldsymbol{\Delta}_I}$. Moreover, $\mathcal{O}_{\boldsymbol{\Delta}_I}$ is a *Bézout domain*. (See [57, §4] for a proof. The key step is to prove the convergence of certain infinite products, which can be handled if the base field is discretely valued.)

Recall that the ring $\mathcal{O}^{\mathrm{int}}_{\boldsymbol{\Delta}_{[r,1)}}$ is a complete with respect to $\|\cdot\|_r$. A similar argument which proves that $\mathcal{O}_{\boldsymbol{\Delta}_I}$ is a principal ideal domain when I is closed shows that $\mathcal{O}^{\mathrm{int}}_{\boldsymbol{\Delta}_{[r,1)}}$ is a principal ideal domain. See [48, §2.6] for more details. If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, this is easier to prove due to the identity $\mathcal{O}^{\mathrm{int}}_{\boldsymbol{\Delta}_{[r,1)}} = \mathcal{O}_{\boldsymbol{\Delta}_{K,\leqslant r'}}$ as a subspace of $k[[\pi_0, u, \frac{1}{\pi_0}, \frac{1}{u}]]$ with the same ring structure, where $\boldsymbol{\Delta}_{K,\leqslant r'}$ is the closed disk over $K = k(\!(u)\!)$ with coordinate π_0 of radius $r' = q^{-1/(\log_q r)}$. To summarize, we have the following proposition:

Proposition 6.2.7.2.

1. *For a closed interval $I \subset [0, 1)$, the ring $\mathcal{O}_{\boldsymbol{\Delta}_I}$ is a principal ideal domain.*
2. *For any interval $I \subset [0, 1)$, the ring $\mathcal{O}_{\boldsymbol{\Delta}_I}$ is a Bézout domain.*

3. *The ring of bounded functions* $\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}}$ *is a principal ideal domain.*

We end this section by the following faithful flatness result.

Proposition 6.2.8. *The natural inclusions* $\mathfrak{S}[\frac{1}{\pi_0}] \hookrightarrow \mathcal{O}_{\mathbf{\Delta}}$ *and* $\mathfrak{S}^{\mathrm{alg}}[\frac{1}{\pi_0}] \hookrightarrow \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}}$ *are faithfully flat. The natural continuous maps* $\mathcal{O}_{\mathbf{\Delta}_I} \hookrightarrow \mathcal{O}^{\mathrm{alg}}_{\mathbf{\Delta}_I}$ *and* $\mathcal{R} \hookrightarrow \mathcal{R}^{\mathrm{alg}}$ *are faithfully flat.*

Proof. First of all, note that the source of any map in the statement is a Bézout domain by Proposition 6.2.7.2. The flatness is clear since for modules over a Bézout domain, flatness is equivalent to having no nonzero torsion. To see the faithful flatness, we first observe any non-unit element in the source cannot become a unit in the target, which is clear from Corollaries 6.2.6.3 and 6.2.6.1. The following claim asserts that this suffices to show the full faithfulness of ring extensions of Bézout domains.

Claim. *Let* A *be a Bézout domain and* B *a flat* A*-algebra. Then* B *is faithfully flat over* A *if and only if any non-unit element* $a \in A$ *does not become to a unit in* B*.*

The "only if" direction is trivial. Now, assume that any non-unit element in A does not become a unit in B, and show that any map of A-module $M' \to M$ is injective if and only if $B \otimes_A M' \to B \otimes_A M$ is injective. For this, it is enough to show that the composite $\langle m' \rangle \hookrightarrow M' \to M$ is injective for any $m' \in M'$, since by flatness $B \otimes_A \langle m' \rangle \to B \otimes_A M'$ is injective. By replacing M' with $\langle m' \rangle$ and M with the image of $\langle m' \rangle$, it is enough to handle the case when both M' and M are generated by one element and the map $M' \to M$ is surjective.

Now we can write $M' \cong A/J$ and $M \cong A/I$ for (not necessarily finitely generated) ideals $J \subseteq I$ of A. Since $B \otimes_A M' \xrightarrow{\sim} B \otimes_A M$, we have $JB = IB$. We are reduced to showing that $J = I$. Assume that $J \subsetneq I$ and choose an element $x \in I \setminus J$.

Then $x = \sum_{i=1}^{n} b_i y_i$ for $b_i \in B$ and $y_i \in J$. Let $J' \subset A$ be the ideal generated by $\{y_1, \cdots, y_n\}$ and $I' \subset A$ the ideal generated by $\{y_1, \cdots, y_n, x\}$. Since A is a Bézout domain, J' and I' are principally generated. Let $y' \in J'$ and $x' \in I'$ be principal generators, respectively, and we have $x'|y'$. Since $J' \subsetneq I'$ by construction, y'/x' is a non-unit element in A. On the other hand, we have $J'B = I'B$ by construction, which implies that y'/x' is a unit in B. This contradicts to our assumption that any non-unit element in A does not become a unit in B. □

6.3 Proof of Proposition 4.3.2

Now we are ready to prove Proposition 4.3.2. For a subinterval $I \subset [0, 1)$ and $r \in I \setminus \{0\}$, we extend the norm $\|\cdot\|_r$ to $n \times n$ matrices $A = (A_{ij}) \in \mathrm{Mat}_n(\mathcal{O}_{\mathbf{\Delta}_I})$ by $\|A\|_r := \max_{i,j}\{\|A_{ij}\|_r\}$. Similarly, define the additive valuation $w_\gamma(A) := \min_{i,j}\{w_\gamma(A_{ij})\}$. This satisfies the strict triangular inequality and the submultiplicativity:

- $w_\gamma(A + B) \geq \min\{w_\gamma(A), w_\gamma(B)\}$ and the equality holds if $w_\gamma(A) \neq w_\gamma(B)$.
- $w_\gamma(AB) \geq w_\gamma(A) + w_\gamma(B)$.
- If $w_\gamma(A) > 0$ then $w_\gamma(\det(A)) > 0$. Similarly if $w_\gamma(A - \mathrm{Id}_n) > 0$, then $w_\gamma(\det(A) - 1) > 0$. (Indeed, write $A = \mathrm{Id}_n + X$ for some $X = (x_{ij})$ with $w_\gamma(x_{ij}) > 0$, and $\det A - 1$ can be written as a sum of terms only involving x_{ij}.)

Now let us restate Proposition 4.3.2 as follows:

Proposition 6.3.1 (Proposition 4.3.2 restated)**.**

1. *For any $A \in \mathrm{GL}_n(\mathcal{R})$, there exists $U \in \mathrm{GL}_n(\mathcal{O}_{\mathbf{\Delta}})$ and $V \in \mathrm{GL}_n(\mathcal{R}^{bd})$ such that $A = UV$.*

2. *If* $A \in \mathrm{GL}_n(\mathcal{O}_{\mathbf{\Delta}_{[r,1)}})$ *with* $0 < r < 1$ *and if* $w_\gamma(A - \mathrm{Id}_n) > 0$ *for* $\gamma = -\log_q r$*, then there exist matrices* $U \in \mathrm{GL}_n(\mathcal{O}_{\mathbf{\Delta}})$ *and* $V \in \mathrm{GL}_n(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ *such that* $A = UV$*. This pair* U *and* V *is can be chosen to satisfy the following additional conditions:*

   - $U - \mathrm{Id}_n$ *involves only positive powers of* u *and* V *involves no positive powers of* u*.*
   - *We have* $w_\gamma(U - \mathrm{Id}_n) > 0$ *and* $w_\gamma(V - \mathrm{Id}_n) > 0$*.*
   - $V \in \mathrm{GL}_n(\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}})$*.*

   Such U *and* V *are unique and also satisfy inequalities* $w_\gamma(U - \mathrm{Id}_n) \geq w_\gamma(A - \mathrm{Id}_n)$ *and* $w_\gamma(V - \mathrm{Id}_n) \geq w_\gamma(A - \mathrm{Id}_n)$*.*

For the proof, we closely follow [46, Prop 6.5]. The proof is roughly divided into two steps:

Step 1: Reduce (1) to (2)

Step 2: Produce the unique matrices U and V in (2) by approximation.

The following lemma takes care of *Step 1*:

Lemma 6.3.2. *Fix* $\gamma \in \mathbb{Q}_{>0}$ *and let* $r = q^{-\gamma}$*. Then for any* $A \in \mathrm{GL}_n(\mathcal{O}_{\mathbf{\Delta}_{[r,1)}})$*, there exists an invertible matrix* $B \in \mathrm{GL}_n(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ *such that* $w_\gamma(AB - \mathrm{Id}_n) > 0$*. Furthermore, if* $w_\gamma(\det(A) - 1) > 0$*, then we may choose* B *such that* $\det(B) = 1$*.*

To handle *Step 1*, first apply this lemma to $A \in \mathrm{GL}_n(\mathcal{O}_{\mathbf{\Delta}_{[r,1)}})$, to obtain $AB \in \mathrm{GL}_n(\mathcal{O}_{\mathbf{\Delta}_{[r,1)}})$ with $B \in \mathrm{GL}_n(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ and $w_\gamma(AB - \mathrm{Id}_n) > 0$. Granting both the lemma and Proposition 6.3.1 (2), one can apply Proposition 6.3.1 (2) to AB to get a factorization $AB = UV$. This gives a factorization $A = U \cdot (VB^{-1})$ where $U \in \mathrm{GL}_n(\mathcal{O}_{\mathbf{\Delta}})$ and $(VB^{-1}) \in \mathrm{GL}_n(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$. Now, for any $A \in \mathrm{GL}_n(\mathcal{R})$ there exists some $r \in (0,1)$ such that A converges on $\mathbf{\Delta}_{[r,1)}$. Take this r and let $\gamma := -\log_q r$.

Then the above factorization $A = U\cdot(VB^{-1})$ proves Proposition 6.3.1 (1), so the lemma completes *Step 1.*

Remark. With little extra work, one can prove this lemma with γ replaced by any closed sub interval $I_o \subset \mathbb{R}_{>0}$. Compare with [48, Lemma 2.7.1] and [46, Lemma 6.2]. We do not need this generalization.

Proof of Lemma 6.3.2. The case $n = 1$ is handled by Corollary 6.2.6.2. Also from $n = 1$ case, we can find a unit $g \in (\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})^\times$ so that $w_\gamma(g\cdot\det(A) - 1) > 0$. Therefore by replacing A by A·diag$(g, 1, \cdots, 1)$, for example, we may and will assume that $w_\gamma(\det(A) - 1) > 0$. We will carry out the induction on n with this extra hypothesis on the determinant. We assume by induction (with $n > 1$) that for any $A \in \mathrm{GL}_{n-1}(\mathcal{O}_{\mathbf{\Delta}_{[r,1)}})$ such that $w_\gamma(\det(A) - 1) > 0$, there exists a matrix $B \in \mathrm{SL}_{n-1}(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ such that $w_\gamma(AB - \mathrm{Id}_n) > 0$.

Let us outline the strategy of the proof:

1. For any $A \in \mathrm{GL}_n(\mathcal{O}_{\mathbf{\Delta}_{[r,1)}})$, find $B_0 \in \mathrm{SL}_n(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ such that the upper left $(n-1)\times(n-1)$-minor of AB_0 satisfies the induction hypothesis. The induction hypothesis produces $B_1 \in \mathrm{SL}_n(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ such that $\|(AB_0B_1)_{ij} - \delta_{ij}\| > 0$ for $1 \le i, j \le n-1$, where δ_{ij} is the Kronecker delta.
2. Find a series of elementary column operations so that n-th column and n-th row satisfy the same inequality.

(1) Finding B_0 and applying the induction hypothesis.

Let c_i denote the ni-cofactor of A, so we have $\det(A) = \sum_{i=1}^n c_i\cdot A_{ni}$, and $c_i = (A^{-1})_{in}\det(A)$. If we put $\alpha_i := \det(A)^{-1}A_{ni}$, then we have $\sum_{i=1}^n \alpha_i c_i = 1$. In order to get an idea for how to find B_0, let us assume that we have $B_0 \in \mathrm{SL}_n(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ such that the nn-cofactor c'_n of AB_0 satisfy $w_\gamma(c'_n - 1) > 0$ (so c'_n is necessarily a unit by our criterion via Newton polygon: Corollary 6.2.6.1).

The cofactor c'_n satisfies:

$$\begin{aligned} c'_n &= (B_0^{-1}A^{-1})_{nn}\det(A) \\ &= \sum_{i=1}^{n}(B_0^{-1})_{ni}(A^{-1})_{in}\det(A) \\ &= \sum_{i=1}^{n}(B_0^{-1})_{ni}\cdot c_i \\ &= 1+\sum_{i=1}^{n}\big((B_0^{-1})_{ni}-\alpha_i\big)\cdot c_i \end{aligned}$$

Let $\beta_i := (B_0^{-1})_{ni} \in \mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}}$. Then $\{\beta_i\}$ generates a unit ideal in $\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}}$ since B_0^{-1} is invertible. Conversely, if we can find $\{\beta_i\}$ which generates the unit ideal and satisfies $w_\gamma(\beta_i - \alpha_i) > -w_\gamma(c_i)$ for all i, then we can find B_0 that works; indeed, since $n > 1$ and $\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}}$ is a principal ideal domain (Proposition 6.2.7.2), one can find an invertible matrix $B_0^{-1} \in \mathrm{SL}_n(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ whose n-th row is (β_i), and the above calculation shows that this B_0 works.

To find such $\{\beta_i\}$, we first take $\beta'_i \in \mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}}$ such that $w_\gamma(\beta'_i - \alpha_i) > -w_\gamma(c_i)$ for all i. This is possible because $\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}} \subset \mathcal{O}_{\mathbf{\Delta}_{[r,1)}}$ is a dense subalgebra. But $\{\beta'_i\}$ may not generate the unit ideal, so we modify β'_n as follows. Observe that the elements $\{\beta'_n + \pi_0^j\}_j$ are pairwise coprime (i.e., any two elements generate the unit ideal) in $\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}}$. If $j \gg 0$ (namely, if $j > -w_\gamma(c_i)$), then we still have $w_\gamma(\beta'_n + \pi_0^j - \alpha_n) > -w_\gamma(c_i)$. Since $\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}}$ is a principal ideal domain, the ideal generated by $\{\beta'_1, \cdots, \beta'_{n-1}\}$ is principal, say generated by β. Since β cannot have infinitely many prime factors (being an element in a principal ideal domain), we conclude that there exists an integer $j \gg 0$ such that $\{\beta'_1, \cdots, \beta'_{n-1}, \beta'_n + \pi_0^j\}$ generates the unit ideal and the inequality $w_\gamma(\beta'_n + \pi_0^j - \alpha_n) > -w_\gamma(c_i)$ holds. We set $\beta_n := \beta'_n + \pi_0^j$ for the above choice of j, and $\beta_i := \beta'_i$ for $i \neq n$.

To summarize, if we choose a matrix $B_0^{-1} \in \mathrm{SL}_n(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ whose n-th row is (β_i), then the upper left $(n-1)\times(n-1)$-minor of AB_0 satisfies the induction hypothesis. Then the induction hypothesis gives a $B_1' \in \mathrm{SL}_{n-1}(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ which "works" for the upper left $(n-1)\times(n-1)$-minor of AB_0. Now, extend this matrix to $B_1 \in \mathrm{SL}_n(\mathcal{O}^{bd}_{\mathbf{\Delta}_{[r,1)}})$ by setting $(B_1)_{nn} = 1$, $(B_1)_{in} = (B_1)_{ni} = 0$ for $i \neq n$ and the upper left $(n-1)\times(n-1)$-minor of B_1 to be equal to B_1'. Then AB_0B_1 still satisfies the following:

- our running hypothesis $w_\gamma(\det(AB_0B_1) - 1) = w_\gamma(\det(A) - 1) > 0$, (because the determinant of B_0 and B_1 are both 1)
- $w_\gamma\left((AB_0B_1)_{ij} - \delta_{ij}\right) > 0$ for $1 \leq i, j \leq n-1$.

Since it is enough to prove the statement for AB_0B_1, we rename AB_0B_1 to be A.

Now that we have the inequalities $w_\gamma\left(A_{ij} - \delta_{ij}\right) > 0$ for $1 \leq i, j \leq n-1$ (so $w_\gamma(A_{ij}) \geq 0$ for $1 \leq i, j \leq n-1$), our next goal is to perform elementary column operations on A (which correspond to multiplying A by elementary matrices on the right) so that in the resulting matrix, the same inequalities hold for all i and j. This process will look like Gaussian elimination, except that instead of eliminating the off-diagonal terms in n-th row and column we make them close to 0. For this reason we may call this process "approximate Gaussian elimination".

(2) "Clearing" the nth column

We first "clear" off-diagonal entries from the nth column. Let $A^{(0)} := A$ and put

$A^{(h+1)} := A^{(h)} \cdot B_2^{(h)}$, where

$$(B_2^{(h)})_{mj} := \begin{cases} \delta_{mj} & j < n \text{ or } j = m = n \\ -A_{mn}^{(h)} & j = n \text{ and } m < n \end{cases} .$$

Concretely, we subtract $A_{mn}^{(h)}$ times the mth column from the nth column for each $m = 1, \cdots, n-1$. (Note also that $\det A^{(h+1)} = \det A^{(h)} = \det A$ for any $h \geq 0$.) Therefore we have:

$$A_{ij}^{(h+1)} := \begin{cases} A_{ij}^{(h)} & j < n \\ A_{in}^{(h)} - \sum_{m=1}^{n-1} A_{im}^{(h)} \cdot A_{mn}^{(h)} & j = n \end{cases} .$$

At each step, the minimum valuation $\min_{1 \leq i \leq n-1}\{w_\gamma(A_{in}^{(h)})\}$ increases by at least $\min_{1 \leq i,j \leq n-1}\{w_\gamma(A_{ij} - \delta_{ij})\}$ which is positive and independent of h. To see this, we just rewrite $A_{in}^{(h+1)}$ for $i < n$:

$$\begin{aligned} A_{in}^{(h+1)} &= A_{in}^{(h)} - \sum_{m=1}^{n-1} A_{im} \cdot A_{mn}^{(h)} \\ &= A_{in}^{(h)} \cdot (1 - A_{ii}) - \sum_{m \neq i,n} A_{im} \cdot A_{mn}^{(h)} \\ &= A_{in}^{(h)} \cdot (\delta_{ii} - A_{ii}) - \sum_{m \neq i,n} (A_{im} - \delta_{im}) \cdot A_{mn}^{(h)}, \end{aligned}$$

and the claim is immediate from the last expression.

Since $\min_{1 \leq i \leq n-1}\{w_\gamma(A_{in}^{(h)})\}$ increases at each step by at least some fixed positive number, we may choose $h \gg 0$ so that the following inequality holds:

$$w_\gamma(A_{in}^{(h)}) > \max\left\{0, \max_{1 \leq j \leq n-1}\left\{-w_\gamma(A_{nj}^{(h)})\right\}\right\} \qquad (i = 1, \cdots, n-1)$$

(Recall that $A_{nj}^{(h)} = A_{nj}$, so the right side is independent of h.) Therefore we have $w_\gamma(A_{ij}^{(h)}) > 0$ for all $i < n$ and all j; and $w_\gamma(A_{in}^{(h)} \cdot A_{nj}^{(h)}) > 0$ for any $1 \leq i, j \leq n-1$. Because $\det(A) = \det(A^{(h)})$, we still have the inequality $w_\gamma(\det(A^{(h)}) - 1) > 0$.

Furthermore, we also have $w_\gamma(A_{nn}^{(h)} - 1) > 0$. To see this, it follows from the inequality $w_\gamma(A_{in}^{(h)} \cdot A_{nj}^{(h)}) > 0$ for $i, j < n$ and $o_\gamma(A_{ij}^{(h)}) = w_\gamma(A_{ij})$ for all $i, j < n$ that

$w_\gamma\left(\sum_{j\neq n} c_j^{(h)} A_{nj}^{(h)}\right) > 0$ where $c_j^{(h)}$ is the nj-cofactor of $A^{(h)}$. But since $\det(A^{(h)}) = \sum_{j=1}^{n} c_j^{(h)} A_{nj}^{(h)}$ and $w_\gamma(\det(A^{(h)}) - 1) > 0$, we get $w_\gamma\left(c_n^{(h)} \cdot A_{nn}^{(h)} - 1\right) > 0$. But $A_{ij}^{(h)} = A_{ij}$ for $i, j < n$, so by our initial arrangements for A we have $w_\gamma(c_n^{(h)} - 1) > 0$ (so $w_\gamma(c_n^{(h)}) = 0$). Since $c_n^{(h)} \cdot A_{nn}^{(h)} - 1 = c_n^{(h)} \cdot (A_{nn}^{(h)} - 1) + (c_n^{(h)} - 1)$, we deduce $w_\gamma(A_{ij}^{(h)} - 1)$ as claimed.

Let us list all (relevant) properties we have arranged for $A^{(h)}$ to satisfy:

- $w_\gamma(\det(A^{(h)}) - 1) > 0$.
- $w_\gamma(A_{ij}^{(h)} - \delta_{ij}) > 0$ if $1 \leq i, j \leq n-1$ or if $j = n$.
- $w_\gamma(A_{in}^{(h)} \cdot A_{nj}^{(h)}) > 0$ for any $1 \leq i, j \leq n-1$.

(3) "Clearing" the off-diagonal entries in the nth row

Now that $A_{nn}^{(h)}$ satisfies the desired inequality $w_\gamma(A_{nn}^{(h)} - 1) > 0$, we can "clear" the remaining entries in the nth row. Starting from $A^{(h)}$, define $A^{(l+1)} := A^{(l)} \cdot B_2^{(l)}$ for $l \geq h$, where

$$(B_2^{(l)})_{ij} := \begin{cases} \delta_{ij} & i < n \text{ or } i = j = n \\ -A_{nj}^{(l)} & i = n \text{ and } j < n \end{cases} .$$

Concretely, we subtract $A_{nj}^{(l)}$ times the nth column from the jth column for each $j = 1, \cdots, n-1$:

$$A_{ij}^{(l+1)} := \begin{cases} A_{ij}^{(l)} - A_{in}^{(l)} \cdot A_{nj}^{(l)} & j < n \\ A_{in}^{(l)} & j = n \end{cases} .$$

First, observe that for $j < n$, the valuation $w_\gamma(A_{nj}^{(l)})$ increases by at least $w_\gamma(A_{nn}^{(h)} - 1)$ which is a positive number independent of h. (Note that $A_{nn}^{(h)} = A_{nn}^{(l)}$, so the above statement is clear from the recursive formula.) Thus for $l \gg h$, we have the inequalities

$$w_\gamma(A_{nj}^{(l)}) > 0 \qquad (j = 1, \cdots, n-1)$$

Now, we need to check that these column operations preserve the inequality $w_\gamma(A^{(l)}_{ij} - \delta_{ij}) > 0$ for $1 \leq i, j \leq n-1$, so it suffices that $w_\gamma(A^{(l)}_{in} \cdot A^{(l)}_{nj}) > 0$ for $1 \leq i, j \leq n-1$ and all $l \geq h$. In fact, we have $w_\gamma(A^{(h)}_{in} \cdot A^{(h)}_{nj}) > 0$ for $1 \leq i, j \leq n-1$, and $A^{(h)}_{in} = A^{(l)}_{in}$ while $w_\gamma(A^{(l)}_{nj}) > w_\gamma(A^{(h)}_{nj})$ for $l > h$ (since $w_\gamma(1 - A^{(l)}_{nn}) = w_\gamma(1 - A^{(h)}_{nn}) > 0$ for all $l \geq h$), hence the claim is clear.

To sum up, we have the inequality $w_\gamma(A^{(l)}_{ij} - \delta_{ij}) > 0$ for all i and j, in other words $w_\gamma(A^{(l)} - \mathrm{Id}_n) > 0$. This finally concludes the proof of the lemma. □

We have reduced the Proposition 6.3.1 to proving the second part of its statement.This follows from the lemma below, which roughly says that one can uniquely factor a matrix A over $\mathbf{\Delta}_{[r,1)}$ into a "holomorphic part" U and a "polar part" V, with some "boundedness" condition if A is close enough to Id_n:

Lemma 6.3.3. *Assume that $A \in \mathrm{Mat}_n(\mathcal{O}_{\mathbf{\Delta}_{[r,1)}})$ satisfies $w_\gamma(A - \mathrm{Id}_n) \geq c$ for some $\gamma := -\log_q r \in \mathbb{Q}_{>0}$, and $c > 0$. Then there exists a unique pair of matrices $U = \mathrm{Id}_n + \sum_{i \in \mathbb{Z}_{>0}} U_i u^i \in \mathrm{Mat}_n(\mathcal{O}_{\mathbf{\Delta}})$ and $V = \sum_{i \in \mathbb{Z}_{\geq 0}} V_i u^{-i} \in \mathrm{Mat}_n(\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}})$, where $U_i, V_i \in \mathrm{Mat}_n(\mathscr{K}_0)$, such that $A = UV$, $w_\gamma(U - \mathrm{Id}_n) > 0$ and $w_\gamma(V - \mathrm{Id}_n) > 0$. Moreover, these matrices U and V satisfy $w_\gamma(U - \mathrm{Id}_n) \geq c$ and $w_\gamma(V - \mathrm{Id}_n) \geq c$.*

Since $\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}}$ is a complete normed algebra for the valuation w_γ, it follows from $w_\gamma(\det(V) - 1) > 0$ that $\det(V)$ is invertible, so $V \in \mathrm{GL}_n(\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}})$.

Reduction of Proposition 6.3.1 (2) to Lemma 6.3.3. Assuming that A is invertible in addition to all the hypotheses in the lemma, it is enough to show that U and V given from the lemma are invertible. This statement only involves the determinants of U and V, hence we are reduced to $n = 1$ case.

Assume that $A \in \mathcal{O}^{\times}_{\mathbf{\Delta}_{[r,1)}}$ satisfies $w_\gamma(A - 1) > 0$. Then by lemma, we obtain $U \in \mathcal{O}_{\mathbf{\Delta}}$ with constant term 1 and $V \in \left(\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}}\right)^{\times}$ in $1 + u^{-1}\mathscr{K}_0[[u^{-1}]]$ such that

$A = UV$. (This is not the end of the proof because we need U to be invertible in $\mathcal{O}_{\mathbf{\Delta}}$, not just in $\mathcal{O}_{\mathbf{\Delta}_{[r,1)}}$.) Since A^{-1} also satisfies $w_\gamma(A^{-1}-1) > 0$ (because $w_\gamma(A(A^{-1}-1)) > 0$ and $w_\gamma(A) = 0$), we also have $A^{-1} = U'V'$. Since V and V' are invertible, we obtain $U \cdot U' = (V \cdot V')^{-1}$, which is an element of $\mathcal{O}_{\mathbf{\Delta}} \cap \mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}} = \mathfrak{S}$. But $U \cdot U'$ has the constant term 1, therefore is a unit in $\mathfrak{S}$. This shows $U \in \mathcal{O}^{\times}_{\mathbf{\Delta}}$. □

Proof of Lemma 6.3.3. We first make the following observations:

1. If $f(u) \in \mathcal{O}_{\mathbf{\Delta}_{[r,1)}}$ has no nontrivial "principal part" (i.e., no nonzero terms with negative powers of u) in its Laurent expansion, then $f(u)$ can be extended to a section of $\mathcal{O}_{\mathbf{\Delta}}$.

2. If the Laurent expansion of $g(u) \in \mathcal{O}_{\mathbf{\Delta}_{[r,1)}}$ has no terms with positive powers of u, then $g(u)$ is automatically bounded; in fact, $g(\frac{1}{u})$ is bounded on $\mathbf{\Delta}_{(1,\frac{1}{r}]}$ because $g(\frac{1}{u})$ has no negative powers of u so it extends to the closed disk $\mathbf{\Delta}_{\leqslant \frac{1}{r}}$. Furthermore, if $w_\gamma(g) > 0$ where $\gamma = -\log_q r$, then $g(u) \in \mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}}$.

Now, let A be as in the statement of the lemma. It can be seen from the observations above that once we find the factorization $A = UV$ for $U, V \in \mathrm{Mat}_n(\mathcal{O}_{\mathbf{\Delta}_{[r,1)}})$ where $U = \mathrm{Id}_n + \sum_{i \in \mathbb{Z}_{>0}} U_i u^i$ and $V = \sum_{i \in \mathbb{Z}_{\geq 0}} V_i u^{-i}$ with $w_\gamma(U - \mathrm{Id}_n) > 0$ and $w_\gamma(V - \mathrm{id}_n) > 0$, then automatically U and V belong to where they should: i.e., $U \in \mathrm{Mat}_n(\mathcal{O}_{\mathbf{\Delta}})$ and $V \in \mathrm{Mat}_n(\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}})$.

We first show the uniqueness. Assume that there exist two desired factorizations $A = UV = U'V'$. Since we required all these matrices to be "close" to Id_n with respect to the valuation w_γ (i.e., $w_\gamma(A - \mathrm{Id}_n) > 0$, $w_\gamma(U - \mathrm{Id}_n) > 0$, etc.) they become invertible over $\mathcal{O}_{\mathbf{\Delta}_{[r,r]}}$. (The inequalities forces $w_\gamma(\det(A) - 1) > 0$, etc., and that $\mathcal{O}_{\mathbf{\Delta}_{[r,r]}}$ is the completion of $\mathcal{O}_{\mathbf{\Delta}_{[r,1)}}$ with respect to w_γ.) So we have $(U')^{-1}U = V'V^{-1}$ in $\mathrm{GL}_n(\mathcal{O}_{\mathbf{\Delta}_{[r,r]}})$. But $(U')^{-1}U - \mathrm{Id}_n$ has only terms with positive powers of u while

$V'V^{-1} - \mathrm{Id}_n$ has no terms with positive powers of u. This can happen only when $(U')^{-1}U = \mathrm{Id}_n$ and $V'V^{-1} = \mathrm{Id}_n$ from the beginning.

Now we show the existence of such a factorization. We define a sequence of invertible matrices $\{V^{(h)}\}_{h\in\mathbb{Z}_{\geq 0}}$ over $\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}}$ by the following recursion formula. Let $V^{(0)} := \mathrm{Id}_n$. Given $V^{(h)}$, we set $A(V^{(h)})^{-1} = P^{(h)} + H^{(h)}$, where $H^{(h)}$ consists of terms with positive powers of u and $P^{(h)}$ consists of terms with non-positive powers of u. By the second observation made at the beginning of the proof, $P^{(h)} \in \mathrm{Mat}_n(\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}})$ for all $h \geq 0$. Define $V^{(h+1)} := P^{(h)}V^{(h)}$, and we need to show that $P^{(h)}$ lies in $\mathrm{GL}_n(\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}})$, hence in turn $V^{(h+1)}$ is. Since $\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}}$ is complete with respect to w_γ, it suffices to show $w_\gamma(P^{(h)} - \mathrm{Id}_n) > 0$.

Observe first that $w_\gamma\left(A(V^{(h)})^{-1} - \mathrm{Id}_n\right) = \min\left\{w_\gamma(P^{(h)} - \mathrm{Id}_n), w_\gamma(H^{(h)})\right\}$ because we defined $P^{(h)}$ and $H^{(h)}$ by "chopping" the Laurent series for $A(V^{(h)})^{-1}$. So it is enough to show $w_\gamma\left(A(V^{(h)})^{-1} - \mathrm{Id}_n\right) \geq c$. If $h = 0$ then we have $w_\gamma(A - \mathrm{Id}_n) \geq c$ by assumption, so it follows that $w_\gamma(P^{(0)} - \mathrm{Id}_n) \geq c$ and $w_\gamma(H^{(0)}) \geq c$. Now assume that we have $w_\gamma\left(A(V^{(h)})^{-1} - \mathrm{Id}_n\right) \geq c$, hence $w_\gamma(P^{(h)} - \mathrm{Id}_n) \geq c$ and $w_\gamma(H^{(h)}) \geq c$. In particular $P^{(h)}$ is invertible and $w_\gamma((P^{(h)})^{-1} - \mathrm{Id}_n) \geq w_\gamma(\mathrm{Id}_n - P^{(h)}) + w_\gamma((P^{(h)})^{-1}) \geq c$. Now the claim for $h+1$ follows since

$$\begin{aligned} A(V^{(h+1)})^{-1} - \mathrm{Id}_n &= A(V^{(h)})^{-1}(P^{(h)})^{-1} - \mathrm{Id}_n \\ &= (A(V^{(h)})^{-1} - \mathrm{Id}_n)(P^{(h)})^{-1} + ((P^{(h)})^{-1} - \mathrm{Id}_n). \end{aligned}$$

We now digress to prove the following stronger estimates:

Claim 6.3.3.1.

1. $w_\gamma(H^{(h+1)} - H^{(h)}) \geq (h+2)c$
2. $w_\gamma(P^{(h)} - \mathrm{Id}_n) \geq (h+1)c$

We begin with the following observation:

$$\begin{aligned}(P^{(h+1)}-\mathrm{Id}_n)+H^{(h+1)} &= A(V^{(h+1)})^{-1}-\mathrm{Id}_n\\ &= A(V^{(h)})^{-1}(P^{(h)})^{-1}-\mathrm{Id}_n\\ &= H^{(h)}(P^{(h)})^{-1}\\ &= H^{(h)}+H^{(h)}((P^{(h)})^{-1}-\mathrm{Id}_n).\end{aligned}$$

Now observe that $(P^{(h+1)}-\mathrm{Id}_n)+(H^{(h+1)}-H^{(h)})=H^{(h)}((P^{(h)})^{-1}-\mathrm{Id}_n)$, and that $P^{(h+1)}-\mathrm{Id}_n$ involves only negative powers of u and $H^{(h+1)}-H^{(h)}$ involves only positive powers of u. Therefore, we have

$$\min\left\{w_\gamma(P^{(h+1)}-\mathrm{Id}_n),\, w_\gamma(H^{(h+1)}-H^{(h)})\right\}=w_\gamma\left(H^{(h)}\left((P^{(h)})^{-1}-\mathrm{Id}_n\right)\right).$$

Claim 6.3.3.1(2) for the case $h=0$ is clear since by construction of $P^{(0)}$ and $H^{(0)}$, we have $\min\{w_\gamma(P^{(0)}-\mathrm{id}_n),\, w_\gamma(H^{(0)})\}=w_\gamma(A-\mathrm{id}_n)\geq c$. To prove 6.3.3.1(2), we proceed by induction on h. Assuming $w_\gamma(P^{(h)}-\mathrm{Id}_n)\geq(h+1)c$, we have

$$\begin{aligned}\min\left\{w_\gamma(P^{(h+1)}-\mathrm{Id}_n),\, w_\gamma(H^{(h+1)}-H^{(h)})\right\} &= w_\gamma\left(H^{(h)}\right)+w_\gamma\left((P^{(h)})^{-1}-\mathrm{Id}_n\right)\\ &\geq c+(h+1)c\end{aligned}$$

As a byproduct, we also get $w_\gamma(H^{(h+1)}-H^{(h)})\geq(h+2)c$. This proves Claim 6.3.3.1.

Now we can conclude the proof of the lemma. It follows from Claim 6.3.3.1 that $w_\gamma(V^{(h)}-\mathrm{Id}_n)\geq c$ for $h\geq 1$. (The case $h=1$ is clear since $V^{(1)}=P^{(0)}$. Now use induction on h and $V^{(h+1)}-\mathrm{Id}_n=(P^{(h)}-\mathrm{Id}_n)V^{(h)}+(V^{(h)}-\mathrm{Id}_n)$.) We have also seen that $\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}}$ is complete with respect to the valuation w_γ, so the estimate $w_\gamma(P^{(h)}-\mathrm{Id}_n)\geq(h+1)c$ implies that $P^{(h)}\to\mathrm{Id}_n$ in $\mathrm{Mat}_n(\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}})$ as $h\to\infty$. The convergence of $\{V^{(h)}\}$ in $\mathrm{Mat}_n(\mathcal{O}^{\mathrm{int}}_{\mathbf{\Delta}_{[r,1)}})$ follows from the estimate:

$$w_\gamma(V^{(h+1)}-V^{(h)})\geq w_\gamma(V^{(h)})+w_\gamma(P^{(h)}-\mathrm{Id}_n)>(h+1)c$$

Let V denote the limit. Then, V involves no positive powers of u and satisfies $w_\gamma(V - \mathrm{id}_n) \geq c$ because all $V^{(h)}$ have these properties.

Furthermore $H^{(h)} = A(V^{(h)})^{-1} - P^{(h)}$ converges to $AV^{-1} - \mathrm{Id}_n$ as $h \to \infty$ for the topology generated by the valuation w_γ. So $AV^{-1} - \mathrm{Id}_n$ only involves positive powers of u and satisfies $w_\gamma(AV^{-1} - \mathrm{Id}_n) \geq c$; we can check these properties by viewing the matrices as elements of $\mathrm{Mat}_n(\mathcal{O}_{\mathbf{\Delta}_{[r,r]}})$ which is the completion of $\mathrm{Mat}_n(\mathcal{O}_{\mathbf{\Delta}_{[r,1)}})$ for the valuation w_γ, and all $H^{(h)}$ have these properties. So $U := AV^{-1}$ and V satisfy all the desired properties. $\square$

CHAPTER VII

Effective local shtukas and π_0-divisible groups

Throughout this chapter, we put $\mathfrak{o}_0 := \mathbb{F}_q[[\pi_0]]$. Recall our setup in this case: $\mathfrak{o}_K = k[[u]]$ where k contains $\mathbb{F}_q$ and has a finite p-basis, and we fix a local injection $\mathfrak{o}_0 \to \mathfrak{o}_K$ which sends π_0 to $u_0 \neq 0$ (and we put $\mathcal{P} := \pi_0 \widehat{\otimes} 1 - 1 \widehat{\otimes} u_0 \in \mathfrak{o}_0 \widehat{\otimes}_{\mathbb{F}_q} \mathfrak{o}_K \cong \mathfrak{S}$). One of the main purposes of this chapter is to show that in the case of $\mathfrak{o}_0 := \mathbb{F}_q[[\pi_0]]$, $(\varphi, \mathfrak{S})$-modules of finite $\mathcal{P}$-height naturally come up as the semi-linear algebra structure that classifies a certain type of π_0-divisible groups over $\mathfrak{o}_K$, namely π_0-divisible groups of finite $\mathcal{P}$-height (Definition 7.3). In fact, this classification works not just over $\mathfrak{o}_K$ but over any base (formal) scheme over $\operatorname{Spf} \mathfrak{o}_0$, in which case the relevant semi-linear algebra structures called "effective local shtukas" were introduced and studied by Genestier-Lafforgue [35] and Hartl [39, 41]. See Theorem 7.3.2 for a more precise statement, and for now we content ourselves with mentioning that the statement resembles contravariant Dieudonné theory for Barsotti-Tate groups. This justifies viewing $\boldsymbol{\mathcal{G}}_K$-representations of finite $\mathcal{P}$-height as equi-characteristic analogues of crystalline representations. This result was announced by Hartl [40], but since the proof was not available to the author, we work out the proof here.

Convention

Let S be a scheme, and $\mathfrak{M}$ a sheaf on S. By $f \in \mathfrak{M}$, we mean $f \in \Gamma(U, \mathfrak{M})$ for some open $U \subset S$.

7.1 Local shtukas

Throughout the section, S is either a scheme over $\operatorname{Spec} \mathfrak{o}_0$ or a formal scheme over $\operatorname{Spf} \mathfrak{o}_0$[1] and $\sigma_S : S \to S$ is the *absolute* q-Frobenius endomorphism (i.e., σ_S induces identity on the underlying topological space and qth power map on the structure sheaf). We let $u_0 \in \Gamma(S, \mathcal{O}_S)$ denote the image of π_0 under the structure morphism $\mathfrak{o}_0 \to \Gamma(S, \mathcal{O}_S)$. The examples to keep in mind are $S = \operatorname{Spf} \mathfrak{o}_K$, $\operatorname{Spec} \mathfrak{o}_K$, and $\operatorname{Spec} K$.

On $\mathcal{O}_S[[\pi_0]] \cong \mathcal{O}_S \widehat{\otimes}_{\mathbb{F}_q} \mathfrak{o}_0$, we use the partial Frobenius endomorphism $\sigma := \sigma_S \widehat{\otimes} \mathfrak{o}_0 : \mathcal{O}_S \widehat{\otimes}_{\mathbb{F}_q} \mathfrak{o}_0 \to \mathcal{O}_S \widehat{\otimes}_{\mathbb{F}_q} \mathfrak{o}_0$. Concretely, for a section $f := \sum_i a_i \pi_0^i$ where $a_i \in \Gamma(U, \mathcal{O}_S)$ for some open $U \subset S$, we define $\sigma(f) := \sum_i a_i^q \pi_0^i$. If $S = \operatorname{Spec} \mathfrak{o}_K$ or $S = \operatorname{Spec} K$, this recovers the natural σ on $\mathfrak{S}$ and $\mathfrak{o}_{\mathcal{E}}$, respectively.

Definition 7.1.1. A *local shtuka of rank n over S* is a pair $(\mathfrak{M}, \varphi)$ where $\mathfrak{M}$ is a sheaf of (topological) $\mathcal{O}_S \widehat{\otimes}_{\mathbb{F}_q} \mathfrak{o}_0 \cong \mathcal{O}_S[[\pi_0]]$-modules together with a $\mathcal{O}_S[[\pi_0]][\frac{1}{\pi_0 - u_0}]$-linear map $\varphi : \sigma^* \mathfrak{M}[\frac{1}{\pi_0 - u_0}] \xrightarrow{\sim} \mathfrak{M}[\frac{1}{\pi_0 - u_0}]$ such that the following condition holds.

- There exists a Zariski covering $\{U\}$ of S such that $\mathfrak{M}|_U$ is a free $\mathcal{O}_U[[\pi_0]]$-module of rank n for each U. Equivalently, by Corollary 7.4.3, $\mathfrak{M}$ is a locally free $\mathcal{O}_S[[\pi_0]]$-module.

- There exists an integer N such that $\varphi(\sigma_S^* \mathfrak{M}) \subset (\pi_0 - u_0)^{-N} \mathfrak{M}$.

We call $\operatorname{rk}_{\mathcal{O}_S[[\pi_0]]} \mathfrak{M}$ the *rank* of the shtuka $\mathfrak{M}$.

[1] In Hartl's original definition, the base S is assumed to be a formal scheme over $\operatorname{Spf} \mathfrak{o}_0$, in which case our definition of local shtukas (Definition 7.1.1) will coincide with Hartl's, thanks to Proposition 7.1.9. But since it is convenient to include the case $S = \operatorname{Spec} K$, we allow S to be any scheme over $\operatorname{Spec} \mathfrak{o}_0$.

A local shtuka $\mathfrak{M}$ is called *effective* if one can take $N = 0$. In other words, an effective local shtuka is nothing but a φ-module of finite $\mathcal{P}$-height which is locally free over $\mathcal{O}_S[[\pi_0]]$. An effective local shtuka $\mathfrak{M}$ is called *étale* (respectively, *strict*, or *of* $\mathcal{P}$*-height* $\leqslant h$) if φ is an isomorphism (respectively, if $(\pi_0 - u_0)\cdot\operatorname{coker}\varphi = 0$, or if $(\pi_0 - u_0)^h\cdot\operatorname{coker}\varphi = 0$).

We let $\underline{\mathrm{Sh}}_{\mathfrak{o}_0}(S)$ denote the category of local shtukas over S with the obvious notion of morphisms. Let $\underline{\mathrm{Sh}}^{\geqslant 0}_{\mathfrak{o}_0}(S)$, $\underline{\mathrm{Sh}}^{\text{ét}}_{\mathfrak{o}_0}(S)$, $\underline{\mathrm{Sh}}^{\leqslant 1}_{\mathfrak{o}_0}$, and $\underline{\mathrm{Sh}}^{\leqslant h}_{\mathfrak{o}_0}$ denote the full subcategories of effective local shtukas, étale local shtukas, strict local shtukas, and local shtukas of $\mathcal{P}$-height $\leqslant h$, respectively.

Example 7.1.2. If $u_0 \in \Gamma(S, \mathcal{O}_S)$ is invertible, then any local shtuka over S is étale. In particular, if $K \cong k(\!(u)\!)$ is a field extension of $\mathbb{F}_q(\!(\pi_0)\!)$ via $\pi_0 \mapsto u_0$, then local shtukas over $\operatorname{Spec} K$ is precisely étale φ-modules free over $\mathfrak{o}_{\mathcal{E}} \cong K[[\pi_0]]$.

We can also see that effective local shtukas over $\mathfrak{o}_K$ are precisely $(\varphi, \mathfrak{S})$-modules of finite $\mathcal{P}$-height. (Recall that $\mathfrak{S} \cong \mathfrak{o}_K[[\pi_0]]$.) More generally, local shtukas over $\mathfrak{o}_K$ are precisely generalized $(\varphi, \mathfrak{S})$-modules as in §2.2.11. Then Theorem 5.2.3 asserts that the base change for local shtukas by $\mathfrak{o}_K \hookrightarrow K$ is fully faithful. This can be generalized, by the argument given in [75, §4.2], to the following statement: for a connected normal noetherian $\mathbb{F}_q[[u_0]]$-scheme S such that $u_0 \in \Gamma(\mathcal{O}_S)$ is not zero, associating the generic fiber defines a fully faithful functor from the category of local shtukas over S to the category local shtukas over the function field of S.

Example 7.1.3. Let C be a (geometrically integral) curve over some finite field $\mathbb{F}$ of characteristic p. Pick a closed point $P \in C$ and let $\widehat{\mathcal{O}_{C,P}}$ be the completed local ring at the place P. By choosing a uniformizer $\pi_0 \in \mathcal{O}_{C,P}$ at P, we identify $\widehat{\mathcal{O}_{C,P}} \cong \mathbb{F}(P)[[\pi_0]]$, where $\mathbb{F}(P)$ is the residue field at P. Let S be a formal scheme over $\operatorname{Spf}(\widehat{\mathcal{O}_{C,P}})$, and let $u_0 \in \Gamma(S, \mathcal{O}_S)$ be the image of π_0. (So u_0 is locally topologically nilpotent.) Under this setting, local shtukas over S can arise from the following sources:

"Localization" of a (global) "shtuka" over S Let $\mathscr{E}$, $\mathscr{E}'$ be vector bundles over $C \times_{\mathbb{F}} S$, equipped with the following structure:

$$\text{either (i)} \quad \begin{array}{ccc} \mathscr{E} & \xrightarrow{j} & \mathscr{E}' \\ & \nearrow_{t} & \\ \sigma^*\mathscr{E} & & \end{array} \quad \text{, or (ii)} \quad \begin{array}{ccc} & & \mathscr{E} \\ & \nearrow^{t} & \\ \mathscr{E}' & \xrightarrow[j]{} & \sigma^*\mathscr{E} \end{array} \ ,$$

where $\sigma = \mathrm{id}_C \times \sigma_S$ is the partial q-Frobenius and the following conditions are satisfied.

- t and j are injective.
- The support of $\operatorname{coker} t$ is exactly the graph $\Gamma_o \subset C \times_{\mathbb{F}} S$ of the morphism $o : S \to \operatorname{Spf}(\mathcal{O}_{\widehat{C,P}}) \to C$, where the first map is the structure morphism and the second is the natural map. (Compare with Remark 1.3.4.)
- The support of $\operatorname{coker} j$ is the graph of some morphism $\infty : S \to C$ and is disjoint from the graph of o.
- The sheaves $\operatorname{coker} t$ and $\operatorname{coker} j$ are invertible[2] over their respective supports.

This gives an example of a *right shtuka* over S in the case of (i) (respectively, a *left shtuka* over S in the case of (ii)). Now let $\mathfrak{M} := \widehat{\mathscr{E}}$ be the completion of $\mathscr{E}$ at Γ_o, and view it as an $\mathcal{O}_S[[\pi_0]]$-module via the isomorphism $\mathcal{O}_S[[\pi_0]] \xrightarrow{\sim} \mathcal{O}_{\widehat{C \times S, \Gamma_o}} \cong \mathcal{O}_S[[\pi_0 - u_0]]$, defined by $\pi_0 \mapsto u_0 + (\pi_0 - u_0)$. (This makes sense since S is a formal scheme over $\operatorname{Spf}(\mathcal{O}_{\widehat{C,P}})$ so u_0 is locally topologically nilpotent.) Now set $\varphi := j^{-1} \circ t : \sigma^*\mathfrak{M} \to \mathfrak{M}$ if $\mathscr{E}$ is a right shtuka; and set $\varphi : t \circ j^{-1}$ if $\mathscr{E}$ is a left shtuka, respectively. Observe that j becomes an isomorphism after

[2] The example still works fine if we just assume $\operatorname{coker} t$ and $\operatorname{coker} j$ are *locally free* of finite rank over their respective support, but the definition of Drinfeld's shtuka [26, §I] requires them to be invertible.

completion because $\operatorname{coker} j$ is supported disjointly from Γ_o, and that $\operatorname{coker}\varphi = \operatorname{coker} t$ is an invertible sheaf on Γ_o, which is cut out by $\pi_0 - u_0$. So $(\mathfrak{M},\varphi)$ is an effective local shtuka over S.

π_0-divisible group associated to a Drinfeld module Let $\infty \in C$ be a closed point distinct from P, and let A be the coordinate ring for the affine curve $C \setminus \{\infty\}$. We let P also denote the maximal ideal of A which corresponds to the closed point $P \in C$. For a Drinfeld A-module $\mathcal{L}$ over S, one can associates "a π_0-divisible group" $G := \varinjlim_n \mathcal{L}[P^n]_S$. Since the Verschiebung for $\mathcal{L}[P^n]_S$ vanishes for each n, one has the "Dieudonné-type" anti-equivalence

$$G \rightsquigarrow \mathscr{H}om_S(G,\mathbb{G}_a) \cong \varprojlim_n \mathscr{H}om(\mathcal{L}[P^n]_S,\mathbb{G}_a) =: \mathfrak{M}.$$

(See Theorems 7.2.6 and 7.3.2 for the precise statement.) Under the Frobenius structure $\varphi_{\mathfrak{M}}$ induced from the relative Frobenius on G, $\mathfrak{M}$ becomes a strict local shtuka. This example is worked out later in §7.3 with more generality.

7.1.4 Formal properties

Let $(\mathfrak{M},\varphi)$ and $(\mathfrak{M}',\varphi')$ be local shtukas over S.

1. (*Base Change*) Let $S' \xrightarrow{f} S$ be a morphism of (formal) schemes over $\mathfrak{o}_0$. We set $f^*\mathfrak{M} := \mathcal{O}_{S'}[[\pi_0]] \otimes_{f^{-1}(\mathcal{O}_S[[\pi_0]])} f^{-1}\mathfrak{M}$ together with the induced Frobenius structure $f^*\varphi := \mathcal{O}_{S'}[[\pi_0]] \otimes f^{-1}\varphi$, which makes sense as below:

$$\sigma_{S'}^*(f^*\mathfrak{M}) \cong f^*(\sigma_{S'}^*\mathfrak{M}) \xrightarrow{f^*\varphi} \mathfrak{M}.$$

Then the "pullback" $(f^*\mathfrak{M}, f^*\varphi)$ is again a local shtuka over S'. Moreover, if $\mathfrak{M}$ is effective, of $\mathcal{P}$-height $\leqslant h$, strict, or étale, respectively, then so is its pullback $f^*\mathfrak{M}$.

2. The tensor product $(\mathfrak{M}\otimes_{\mathcal{O}_S[[\pi_0]]}\mathfrak{M}', \varphi\otimes\varphi')$ is again a local shtuka; $\varphi\otimes\varphi'$ makes sense as a Frobenius as shown below: $\sigma_S^*(\mathfrak{M}\otimes\mathfrak{M}') \xleftarrow{\sim} (\sigma_S^*\mathfrak{M})\otimes(\sigma_S^*\mathfrak{M}') \xrightarrow{\varphi\otimes\varphi'} \mathfrak{M}\otimes\mathfrak{M}'$.

   Let $\mathfrak{L}_S$ be a local shtuka whose underlying module is $\mathcal{O}_S[[\pi_0]]\mathbf{e}$ with the "natural" choice of the Frobenius structure φ, i.e., $\varphi(\sigma^*\mathbf{e}) = \mathbf{e}$. Then $\mathfrak{L}_S$ is the "left and right identity" for $\otimes$-product.

3. Internal hom is defined in local shtukas. We put $\mathfrak{N} := \mathscr{H}om_{\mathcal{O}_S[[\pi_0]]}(\mathfrak{M}, \mathfrak{M}')$, and define a Frobenius structure $\varphi_{\mathfrak{N}} : (\sigma_S^*\mathfrak{N})[\frac{1}{\pi_0-u_0}] \xrightarrow{\sim} \mathfrak{N}[\frac{1}{\pi_0-u_0}]$, as follows: $\varphi_{\mathfrak{N}}(f) := \varphi'\circ f\circ\varphi^{-1} \in \mathfrak{N}$ where $f \in (\sigma_S^*\mathfrak{N})[\frac{1}{\pi_0-u_0}]$ is viewed as a map $f : (\sigma_S^*\mathfrak{M})[\frac{1}{\pi_0-u_0}] \to (\sigma_S^*\mathfrak{M}')[\frac{1}{\pi_0-u_0}]$. One can directly check that $(\mathfrak{N}, \varphi_{\mathfrak{N}})$ is a local shtuka over S.

4. One can define duality by $\mathfrak{M}^* := \mathscr{H}om_{\mathcal{O}_S[[\pi_0]]}(\mathfrak{M}, \mathfrak{L}_S)$ on $\underline{\mathrm{Sh}}_{\mathfrak{o}_0}(S)$.

Now we define *Tate objects* and *Tate twists.*

Definition 7.1.5. For *any* integer n, the *Tate object* $\mathfrak{L}_S(n)$ is a local shtuka whose underlying sheaf is $\mathcal{O}_S[[\pi_0]]\cdot\mathbf{e}$, and the Frobenius structure is defined by $\sigma_S^*\mathbf{e} \mapsto (\pi_0-u_0)^n\cdot\mathbf{e}$. For a local shtuka $(\mathfrak{M}, \varphi)$, the *Tate twist* $\mathfrak{M}(n)$ *by* n is the local shtuka $\mathfrak{M}(n) := \mathfrak{M}\otimes_{\mathcal{O}_S[[\pi_0]]}\mathfrak{L}_S(n) \cong (\mathfrak{M},\, (\pi_0-u_0)^n\cdot\varphi)$.

We record some immediate properties:

1. For any positive integer n, we have $\mathfrak{L}_S(n) \cong \mathfrak{L}_S(1)^{\otimes n}$. For any integer n, we have $\mathfrak{L}_S(-n) \cong \mathfrak{L}_S(n)^*$, so we also have $(\mathfrak{M}(n))^* \cong \mathfrak{M}^*(-n)$.

2. For a local shtuka $\mathfrak{M}$ over S, let N be an integer such that $\varphi(\sigma^*\mathfrak{M}) \subset (\pi_0 - u_0)^{-N}\mathfrak{M}$. Then the Tate twist $\mathfrak{M}(N)$ is an *effective* local shtuka.

3. Any rank-1 local shtuka $(\mathfrak{M}, \varphi)$ over S is, Zariski-locally on S, a Tate twist of a rank-1 étale shtuka. Indeed, by restricting to some Zariski-open of S, we may assume that $\mathfrak{M}$ is a free $\mathcal{O}_S[[\pi_0]]$-module of rank 1. Let us take a basis $\mathbf{e} \in \Gamma(S, \mathfrak{M})$. Then by definition, $\varphi(\sigma^*\mathbf{e}) = \alpha\cdot(\pi_0 - u_0)^n$ for some $\alpha \in \Gamma(S, \mathcal{O}_S[[\pi_0]]^\times)$ and $n \in \mathbb{Z}$.

7.1.6 Isogenies of local shtukas

A morphism of local shtukas $f : \mathfrak{M} \to \mathfrak{M}'$ is called an *isogeny* if f is injective and $\operatorname{coker} f$ is killed by some power of π_0, say by π_0^N. Then, there exists $g : \mathfrak{M}' \to \mathfrak{M}$ such that $f \circ g = \pi_0^N$ and $g \circ f = \pi_0^N$; consider the following commutative diagram

$$\begin{array}{ccccc} \mathfrak{M} & \xrightarrow{f} & \mathfrak{M}' & \twoheadrightarrow & \operatorname{coker} f \\ \downarrow{\scriptstyle \pi_0^N} & \swarrow{\scriptstyle g} & \downarrow{\scriptstyle \pi_0^N} & & \downarrow{\scriptstyle 0} \\ \mathfrak{M} & \xrightarrow[f]{} & \mathfrak{M}' & \twoheadrightarrow & \operatorname{coker} f. \end{array}$$

Therefore, we can define isogeny categories $\underline{\mathrm{Sh}}_{\mathfrak{o}_0}(S)[\frac{1}{\pi_0}]$, $\underline{\mathrm{Sh}}^{\geqslant 0}_{\mathfrak{o}_0}(S)[\frac{1}{\pi_0}]$, and $\underline{\mathrm{Sh}}^{\leqslant h}_{\mathfrak{o}_0}(S)[\frac{1}{\pi_0}]$ by formally inverting π_0 in the morphisms.

7.1.7

Our definition of local shtukas (Definition 7.1.1) slightly differs from Hartl's original definition in [39, §2.1]: Hartl additionally required that the quotient $(\pi_0 - u_0)^{-N}\mathfrak{M}/\varphi(\sigma_S^*\mathfrak{M})$ is locally free over S for any $N \gg 0$. We show that this additional assumption is automatic if either $u_0 \in \Gamma(S, \mathcal{O}_S)$ is locally topologically nilpotent (i.e., S is a formal scheme over $\operatorname{Spf} \mathfrak{o}_0$) or S is locally noetherian. For this we first need the following lemma.

Lemma 7.1.8. *Let S be a (formal) scheme over $\mathfrak{o}_0$ which satisfies one of the following assumptions: (1) S is locally noetherian; (2) $u_0 \in \Gamma(S, \mathcal{O}_S)$ is locally topologically nilpotent (i.e., S is a formal scheme over* $\operatorname{Spf} \mathfrak{o}_0$*)); or (3) the natural map $\mathcal{O}_S \to$*

$\mathcal{O}_S[\frac{1}{u_0}]$ *injective (i.e., when* u_0 *is nowhere a zero divisor on* S*; for example when* S *is integral). Then for any effective local shtuka* $\mathfrak{M}$ *over* S*, the Frobenius map* $\varphi_{\mathfrak{M}} : \sigma_S^*(\mathfrak{M}) \to \mathfrak{M}$ *is* injective.

Proof. The claim is local in S – more precisely, the claim is local in the relative formal spectrum $\operatorname{Spf}_S \mathcal{O}_S[[\pi_0]]$ which shares the underlying topological space with S. So we may assume $S = \operatorname{Spec} R$ for some $\mathfrak{o}_0$-algebra R, or $S = \operatorname{Spf} R$ for some admissible $\mathfrak{o}_0$-algebra R [27, 0_{I}, 7.1.2].

Let us consider the case (3) first. We may formulate the problem purely algebraically using R-modules (i.e., working over $\operatorname{Spec} R$, not $\operatorname{Spf} R$). If the natural map $R \to R[\frac{1}{u_0}]$ is injective, then the natural map $\mathfrak{M} \to R[\frac{1}{u_0}][[\pi_0]] \otimes_{R[[\pi_0]]} \mathfrak{M}$ is injective. So by Lemma 2.2.3.1, we are reduced to the case when u_0 is a unit in R. Let us assume this. Then $\pi_0 - u_0$ is a unit in $R[[\pi_0]]$, so it follows that any local shtuka over R is an étale φ-module over $R[[\pi_0]]$; i.e., φ is an isomorphism.

Let us consider the other two cases. First, it is enough to handle the case when S is a scheme; i.e., $S = \operatorname{Spec} R$ where R is either noetherian or such that $u_0 \in R$ is nilpotent. In fact, if $S = \operatorname{Spf} R$ where R is an admissible $\mathfrak{o}_0$-algebra and $\{I_\alpha\}$ is a fundamental system of open ideals in R, then it is enough to verify the lemma for R/I_α for each α (by the left exactness of inverse limit). So we rename R/I_α as R.

Second, we can even assume that R is local; indeed, once the lemma is known when R is local, then since the natural map $R[[\pi_0]] \to \prod (R_{\mathfrak{P}}[[\pi_0]])$ is injective (as $\mathfrak{P}$ varies over $\operatorname{Spec} R$) we may apply Lemma 2.2.3.1.

To summarize, it is enough to consider the case when $S = \operatorname{Spec} R$ where R is local and such that either (1) R is noetherian; or (2) $u_0 \in R$ is nilpotent.

Now, we show that the natural map $R[[\pi_0]] \to R[[\pi_0]][\frac{1}{\pi_0 - u_0}]$ is injective; once this is shown, it follows from the $R[[\pi_0]]$-flatness of $\mathfrak{M}$ that the natural map $\mathfrak{M} \to$

$\mathfrak{M}[\frac{1}{\pi_0 - u_0}]$ is injective, so we can use Lemma 2.2.3.1 to conclude the proof.

If $u_0 \in R$ is invertible then $\pi_0 - u_0 \in R[[\pi_0]]$ is invertible, so we may assume that $u_0 \in \mathfrak{m}_R$ where $\mathfrak{m}_R$ is the maximal ideal of R. We want to show that if $f \in R[[\pi_0]]$ satisfies $(\pi_0 - u_0)\cdot f = 0$ then $f = 0$. Since $R[[\pi_0]]$ injects into $R[[\pi_0]][\frac{1}{\pi_0}]$, we may regard f as an element of $R[[\pi_0]][\frac{1}{\pi_0}]$ in order to show $f = 0$ in $R[[\pi_0]]$.

If $u_0 \in R$ is nilpotent then $\pi_0 - u_0$ is a unit in $R[\pi_0][\frac{1}{\pi_0}]$ (hence in $R[[\pi_0]][\frac{1}{\pi_0}]$), since the infinite series $\frac{1}{\pi_0}(1 + \frac{u_0}{\pi_0} + (\frac{u_0}{\pi_0})^2 + \cdots)$ is a finite sum and gives the inverse of $\pi_0 - u_0$.

Now, consider the remaining case where R is a noetherian local ring. The assumption $(\pi_0 - u_0)\cdot f = 0$ implies that $f = \frac{u_0}{\pi_0}\cdot f$ in $R[[\pi_0]][\frac{1}{\pi_0}]$, so we have $f = \left(\frac{u_0}{\pi_0}\right)^n \cdot f$ for any positive integer n. Therefore, $f \in \bigcap_{n\geq 0} \left(\frac{u_0}{\pi_0}\right)^n \cdot R[[\pi_0]][\frac{1}{\pi_0}] = \{0\}$, by Krull's intersection theorem. □

Proposition 7.1.9. *Let S be a formal scheme over $\mathfrak{o}_0$, and assume that either (1) S is locally noetherian; or (2) $u_0 \in \mathcal{O}_S$ is locally topologically nilpotent (i.e., S is a formal scheme over* Spf $\mathfrak{o}_0$*). Let $\mathfrak{M}$ be a local shtuka over S. Then for any $N \gg 0$, the quotient $(\pi_0 - u_0)^{-N}\mathfrak{M}/\varphi(\sigma_S^*\mathfrak{M})$ is locally free over S. In particular, if $\mathfrak{M}$ is an effective local shtuka, then* $\mathrm{coker}(\varphi) := \mathfrak{M}/\varphi(\sigma^*\mathfrak{M})$ *is locally free over S.*

Proof. Let N be a positive integer such that the N-fold Tate twist $\mathfrak{M}(N)$ of $\mathfrak{M}$ (Definition 7.1.5) is effective. Observe that $(\pi_0 - u_0)^{-N}\mathfrak{M}/\varphi(\sigma_S^*\mathfrak{M}) \cong \mathrm{coker}(\varphi_{\mathfrak{M}(N)})$, so the first claim follows from the second claim. In order to show the second claim, consider the following short exact sequence

$$0 \to \sigma^*(\mathfrak{M}) \xrightarrow{\varphi_{\mathfrak{M}}} \mathfrak{M} \to \mathrm{coker}(\varphi) \to 0,$$

which remains exact after the base change to arbitrary closed (formal) subscheme of S, thanks to Lemma 7.1.8. Since the first two terms are flat over S, we can deduce that the last term is flat over S, from [9, Ch.I §2.5 Prop 4] or from an argument using Tor_1. The following exact sequence for $h \gg 0$ shows that $\mathrm{coker}(\varphi)$ is finitely presented over S (and hence is locally free):

$$0 \to \varphi(\sigma^*\mathfrak{M})/(\pi_0 - u_0)^h\mathfrak{M} \to \mathfrak{M}/(\pi_0 - u_0)^h\mathfrak{M} \to \mathrm{coker}(\varphi) \to 0.$$

□

Example 7.1.10. The following example due to Urs Hartl shows that Lemma 7.1.8 is false without assumptions on the base S. Let $R := k[[u_0]][t_0, t_1, \cdots]/(u_0t_0, t_i - u_0t_{i+1}|i = 0, 1, \cdots)$. Let $\mathfrak{M} := R[[\pi_0]]{\cdot}\mathbf{e}$ equipped with $\varphi_{\mathfrak{M}}(\sigma^*\mathbf{e}) = (\pi_0 - u_0){\cdot}\mathbf{e}$. Then $\mathfrak{M}$ is an effective local shtuka, but $\varphi_{\mathfrak{M}}$ is not injective since $(\sum_{i=0}^{\infty} t_i\pi_0^i){\cdot}(\sigma^*\mathbf{e})$ is in the kernel. The proof of Lemma 7.1.8 fails because $\pi_0 - u_0$ is not a regular element in $R[[\pi_0]]$.

7.2 Classification of finite locally free group schemes with trivial Verschiebung

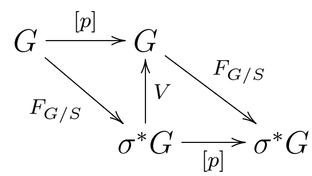
We digress to discuss the "Dieudonné-type" classification of finite locally free commutative group schemes with trivial Verschiebung. This is the main technical tool for the rest of this chapter. Most of the results in this section are also proved in [73, 3, Exp $\mathrm{VII_A}$, §7], where statements differ from ours by Cartier duality. The discussion of this section is also inspired by Abrashkin's study of Faltings's strict modules [1, §2], although we take a slightly different approach.

7.2.1 Preliminaries on group schemes

Let S be a scheme of characteristic p and let $\sigma : S \to S$ be the (absolute) p-Frobenius. Let G be a finite locally free group scheme over S. Let $\mathcal{A}_G$ denote the push-forward of $\mathcal{O}_G$ by the structure sheaf, and $\mathscr{I}_G \subset \mathcal{A}_G$ the augmentation ideal.

Put $\sigma^*G := G \times_{S,\sigma} S$ and let $F_{G/S} : G \to \sigma^*G$ be the relative Frobenius map, which is a group homomorphism thanks to the functorial properties of the Frobenius map.

In addition to this, there is a canonical S-group map $V_{G/S} : \sigma^*G \to G$ (called *Verschiebung*) which is functorial in G, commutes with base changes, and makes the following diagram commute:



The Verschiebung map is defined in [73, 3, Exp $\mathrm{VII_A}$, 4.3], and in [73, 3, Exp $\mathrm{VII_A}$, 4.3.3] it is shown that $V_{G/S} = (F^{\vee}_{G^{\vee}/S}$ (where $(\cdot)^{\vee}$ denotes Cartier dual). We will later concentrate on finite locally free commutative group schemes with *vanishing* Verschiebung.

Now, we associate to a finite locally free[3] group scheme G a finitely presented over $\mathcal{O}_S$, which will be endowed with a Frobenius structure φ, as follows:

$$\underline{\mathfrak{M}}^*(G) := \mathscr{H}om_{\mathrm{gp}/S}(G, \mathbb{G}_a) \cong \{f \in \mathcal{A}_G |\ \mu_G^*(f) = f \otimes 1 + 1 \otimes f\}$$

where $\mu_G : G \times_S G \to G$ is the group multiplication map. The absolute Frobenius endomorphism $F_{\mathbb{G}_a} : \mathbb{G}_a \to \mathbb{G}_a$ induces a σ_S-semilinear endomorphism on $\underline{\mathfrak{M}}^*(G)$. (The same map can be obtained from the absolute Frobenius $F_G : G \to G$.) We denote its linearization by $\varphi : \sigma^*\underline{\mathfrak{M}}^*(G) \to \underline{\mathfrak{M}}^*(G)$.

7.2.2 p-Lie algebras

Our next goal is to show that the φ-module $\underline{\mathfrak{M}}^*(G)$ is isomorphic to the "p-Lie algebra" of the Cartier dual $G^{\vee}$ of G (where the p-operation defines the φ-structure). We digress to recall the definition of p-Lie algebras, and later in in Lemma 7.2.3 we show that it are isomorphic to $\underline{\mathfrak{M}}^*(G)$ as a φ-module.

[3] We always assume that the rank of a finite locally free module is *constant*, not just locally constant.

We write $\omega_{G^\vee} := e^*\Omega_{G^\vee} \cong \mathscr{I}_{G^\vee}/\mathscr{I}_{G^\vee}^2$ for the "co-Lie algebra" of $G^\vee$. We let $\mathscr{L}ie_S(G^\vee)$ denote the "Lie algebra" of $G^\vee$; i.e., $\mathscr{L}ie_S(G^\vee) := \mathscr{D}er_S(\mathcal{A}_{G^\vee}, \mathcal{A}_{G^\vee}/\mathscr{I}_{G^\vee}) \cong \mathscr{H}om_{\mathcal{O}_S}(\omega_{G^\vee}, \mathcal{O}_S)$ where the isomorphism is given by the universal property of Kähler differential.

The Lie algebra $\mathscr{L}ie_S(G^\vee)$ is naturally equipped with an σ_S-semilinear endomorphism $l \mapsto l^{(p)}$ for any $l \in \mathscr{L}ie_S(G^\vee)$, called the *p-operation.* We recall the definition. For any $l \in \mathscr{L}ie_S(G^\vee)$, consider the following $\mathcal{O}_S$-linear map:

$$l^{(p)} := \mathcal{A}_{G^\vee} \xrightarrow{\mu^*_{G^\vee}} (\mathcal{A}_{G^\vee})^{\otimes p} \xrightarrow{l^{\otimes p}} \mathcal{O}_S, \tag{7.2.2.1}$$

where $\mu^*_{G^\vee} : \mathcal{A}_{G^\vee} \to (\mathcal{A}_{G^\vee})^{\otimes p}$ is the p-fold comultiplication map. That $l^{(p)}$ is an $\mathcal{O}_S$-derivation is proved in [73, 3, Exp VII$_\mathrm{A}$, 6.2]. (This can also be deduced from the proof of Lemma 7.2.3.) For any $a \in \mathcal{O}_S$ and $l \in \mathscr{L}ie_S(G^\vee)$, we have $(al)^{(p)} = (a^p)l^{(p)}$, so the p-operation defines a σ_S-semilinear endomorphism on $\mathscr{L}ie_S(G^\vee)$. We let $\varphi : \sigma_S^*(\mathscr{L}ie_S(G^\vee)) \to \mathscr{L}ie_S(G^\vee)$ denote the linearlzation of the p-operation. (Note also that $\mathscr{L}ie_S(G^\vee)$ together with this p-operation defines a commutative *p-Lie algebra* in the sense of [73, 3, Exp VII$_\mathrm{A}$, 5.2].)

Lemma 7.2.3. *We have a natural φ-compatible isomorphism $\underline{\mathfrak{M}}^*(G) \cong \mathscr{L}ie_S(G^\vee)$.*

Proof. Since $\mathcal{A}_G$ is finite locally free $\mathcal{O}_S$-module, we view a section $l \in \Gamma(U, \mathcal{A}_G) \cong \mathrm{Hom}_U(\mathcal{A}^*_G|_U, \mathcal{O}_U)$ over an open $U \subset S$ as a O_U-linear map $l : \mathcal{A}^*_G|_U \to \mathcal{O}_U$, where $\mathcal{A}^*_G \cong \mathscr{H}om_S(\mathcal{A}_G, \mathcal{O}_S)$ is the $\mathcal{O}_S$-linear dual of $\mathcal{A}_G$. Note that $\mathcal{A}^*_G$, together with the well-known Hopf algebra structure, is precisely $\mathcal{A}_{G^\vee}$. The condition for l to be in $\Gamma(U, \underline{\mathfrak{M}}^*(G))$ is exactly the Leibnitz rule: for any $\alpha, \beta \in \Gamma(U, \mathcal{A}_{G^*})$, the definition of $\underline{\mathfrak{M}}^*(G)$ can be re-written as

$$l(\alpha{\cdot}\beta) = l\left((\alpha \otimes \beta) \circ \mu_G\right) = (\mu^*_G l)\,(\alpha \otimes \beta) = (l \otimes 1 + 1 \otimes l)(\alpha \otimes \beta) = l(\alpha){\cdot}\beta + \alpha{\cdot}l(\beta),$$

where the first equality follows from the definition of multiplication $\alpha{\cdot}\beta = (\alpha \otimes \beta) \circ \mu^*_G$.

Now, we show the claim on φ on $\underline{\mathfrak{M}}^*(G)$. Viewing $\underline{\mathfrak{M}}^*(G)$ as a submodule of $\mathcal{A}_G$, we have $\varphi(\sigma^* l) = l^p$ for any $l \in \underline{\mathfrak{M}}^*(G)$, where the p-th power takes place in $\mathcal{A}_G$. This is exactly the linearization of the p-operation of the p-Lie algebra $\mathscr{L}ie_S(G^\vee)$, because for any $\alpha \in \Gamma(U, \mathcal{A}_{G^*})$, we have

$$l^p(\alpha) = (l^{\otimes p})(\mu_{G^\vee}^{(p)*}\alpha) = l^{(p)}(\alpha),$$

where $l^{(p)} := (l^{\otimes p}) \circ \mu_{G^\vee}^{(p)*}$ is the p-operation as defined in (7.2.2.1). □

It follows from this alternative definition of $\underline{\mathfrak{M}}^*(G)$ that the formation of $\underline{\mathfrak{M}}^*(G)$ commutes with any base change: i.e., for any $T \xrightarrow{f} S$ we have a natural isomorphism $\underline{\mathfrak{M}}^*(G_T) \xleftarrow{\sim} f^*\left(\underline{\mathfrak{M}}^*(G)\right)$. In particular, the Frobenius structure $\varphi : \sigma^*\underline{\mathfrak{M}}^*(G) \to \underline{\mathfrak{M}}^*(G)$, which was described earlier using $F_{\mathbb{G}_a/S} : \mathbb{G}_a \to \mathbb{G}_a$, can also be obtained from the relative Frobenius map $F_{G/S} : G \to \sigma^* G$ by functoriality.

7.2.4

Now we will "reverse" the construction of $\underline{\mathfrak{M}}^*(G)$ from G. Let $\mathfrak{M}$ be a finite *locally free* $\mathcal{O}_S$-module (of constant rank) endowed with an $\mathcal{O}_S$-linear map $\varphi : \sigma^*\mathfrak{M} \to \mathfrak{M}$. From this, we define a finite locally free group scheme $\underline{G}^*(\mathfrak{M}, \varphi) = \operatorname{Spec}_S \mathcal{A}_{\mathfrak{M}}$ as follows:

(7.2.4.1) $$\mathcal{A}_{\mathfrak{M}} := \frac{\operatorname{Sym}\mathfrak{M}}{\langle \varphi(\sigma^* m) - m^p |\ m \in \mathfrak{M}\rangle},$$

with the comultiplication map $\mu^*(m) := m \otimes 1 + 1 \otimes m$ for any $m \in \mathfrak{M}$. From the construction we have a natural φ-compatible isomorphism $(\mathfrak{M}, \varphi) \xrightarrow{\sim} \underline{\mathfrak{M}}^*\left(\underline{G}^*(\mathfrak{M}, \varphi)\right)$, which is induced from the natural map $\mathfrak{M} \to \mathcal{A}_{\mathfrak{M}}$. Also, the construction of $\underline{G}^*(\mathfrak{M}, \varphi)$ naturally commutes with any base change on S. By using a local $\mathcal{O}_S$-basis of $\mathfrak{M}$ we see that if $\mathfrak{M}$ has $\mathcal{O}_S$-rank n then $\underline{G}^*(\mathfrak{M}, \varphi)$ is finite locally free with order p^n, and

$\mathfrak{M} \to \mathcal{A}_{\mathfrak{M}}$ is a subbundle (i.e., an injection with image locally a direct factor) with $\mathcal{A}_{\mathfrak{M}}/\mathscr{I}_{\mathfrak{M}}^2 \cong \mathcal{O}_S \oplus \operatorname{coker}\varphi_{\mathfrak{M}}$ where $\mathscr{I}_{\mathfrak{M}}$ is the augmentation ideal of $\mathcal{A}_{\mathfrak{M}}$.

Moreover, the injective map $\mathfrak{M} \to \mathcal{A}_{\mathfrak{M}}$ has a natural splitting (not just a local splitting) which identifies $\mathfrak{M}$ as a direct factor of $\mathcal{A}_{\mathfrak{M}}$. We define this splitting locally and show that the local splittings glue to a global splitting. Let us choose a local basis $\mathbf{e}_1, \cdots, \mathbf{e}_n$ of $\mathfrak{M}$ over some open $U \subset S$. Then $\{\mathbf{e}_1^{i_1} \cdots \mathbf{e}_n^{i_n}\}$ for $0 \leq i_i, \cdots, i_n \leq p-1$ form a $\mathcal{O}_U$-basis of $\mathcal{A}_G|_U$. Let $\mathfrak{N}_U$ denote the submodule of $\mathcal{A}_G$ generated by $\mathbf{e}_1^{i_1} \cdots \mathbf{e}_n^{i_n}$ with $i_1 + \cdots + i_n > 1$. Clearly $\mathcal{A}_{\mathfrak{M}}|_U \cong \mathcal{O}_U \oplus \mathfrak{M}|_U \oplus \mathfrak{N}_U$, and this direct sum decomposition is independent of the choice of local basis and commutes with localization in S. So we obtain $\mathcal{A}_{\mathfrak{M}} \cong \mathcal{O}_S \oplus \mathfrak{M} \oplus \mathfrak{N}$ by gluing these local splittings. In particular, we obtain a natural injective map $\mathfrak{M}^* \hookrightarrow (\mathcal{A}_{\mathfrak{M}})^* \cong \mathcal{A}_{\underline{G}^*(\mathfrak{M},\varphi)^\vee}$ where $(\cdot)^*$ denotes the $\mathcal{O}_S$-linear dual.

Now, let us show that the Verschiebung for $\underline{G}^*(\mathfrak{M},\varphi)$ *vanishes*. We can view $\underline{\operatorname{Spec}}(\operatorname{Sym}\mathfrak{M})$ as a group scheme via the comultiplication map $\mu^*(m) := m \otimes 1 + 1 \otimes m$ for any $m \in \mathfrak{M} = \operatorname{Sym}^1 \mathfrak{M}$. Then $\underline{G}^*(\mathfrak{M},\varphi) \subset \underline{\operatorname{Spec}}(\operatorname{Sym}\mathfrak{M})$ becomes a closed subgroup scheme. But $\underline{\operatorname{Spec}}(\operatorname{Sym}\mathfrak{M})$ is, locally on S, isomorphic to the product of $\operatorname{rank}_S \mathfrak{M}$ copies of $\mathbb{G}_a$, so the claim follows. In particular, $\underline{G}^*(\mathfrak{M},\varphi)^\vee$ has vanishing relative Frobenius map.

In fact, much more is true: any finite locally free group scheme G over S with *vanishing Verschiebung* is isomorphic to $\underline{G}^*(\mathfrak{M},\varphi)$ for some locally free $\mathcal{O}_S$-module $\mathfrak{M}$ and φ. To prove this, we need (the second part of) the following lemma.

Lemma 7.2.5 ([73, 3, Exp $\mathrm{VII_A}$, Thm7.2])**.** *Let $\mathfrak{M}$ be a finite locally free $\mathcal{O}_S$-module, endowed with a Frobenius structure $\varphi : \sigma^*\mathfrak{M} \to \mathfrak{M}$. Put $G_{\mathfrak{M}} := \underline{G}^*(\mathfrak{M},\varphi)$. Then $G_{\mathfrak{M}}$ satisfies the following properties:*

1. *For any $T \to S$, we have a natural group isomorphism $G_{\mathfrak{M}}(T) \cong \operatorname{Hom}_{T,\varphi}(\mathfrak{M}_T, \mathcal{O}_T)$,*

where the φ-structure on $\mathcal{O}_T$ is induced (by p-th power map $\sigma_T : \mathcal{O}_T \to \mathcal{O}_T$).

2. *For any finite locally free commutative group scheme G over S, we have a natural group isomorphism* $\mathrm{Hom}_{\mathrm{gp}/\mathrm{S}}(G, G_{\mathfrak{M}}) \cong \mathrm{Hom}_{S,\varphi}(\mathfrak{M}, \underline{\mathfrak{M}}^*(G))$.

Proof. We first give a proof of (2). Let us consider the following isomorphisms:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{gp}/\mathrm{S}}(G, G_{\mathfrak{M}}) &\cong \mathrm{Hom}_{\mathrm{Hopf}/\mathcal{O}_\mathrm{S}}(\mathcal{A}_{\mathfrak{M}}, \mathcal{A}_G) \\ &\cong \{\, l \in \mathrm{Hom}_S(\mathfrak{M}, \mathcal{A}_G) \mid \mu_G^*(l(m)) = l(m) \otimes 1 + 1 \otimes l(m), \\ &\quad l(m)^p = l(\varphi(\sigma^* m)), \forall m \in \mathfrak{M}\,\}\end{aligned}$$

But the last term is precisely $\mathrm{Hom}_{S,\varphi}(\mathfrak{M}, \underline{\mathfrak{M}}^*(G))$; the first condition for l means that $l(m) \in \underline{\mathfrak{M}}^*(G)$ for any $m \in \mathfrak{M}$, and the second means that $l : \mathfrak{M} \to \underline{\mathfrak{M}}^*(G)$ commutes with the φ's since by the proof of Lemma 7.2.3 $\varphi_{\underline{\mathfrak{M}}^*(G)}(\sigma^* m') = (m')^p$ for any $m' \in \underline{\mathfrak{M}}^*(G)$ (where the pth power is taken inside $\mathcal{A}_G$).

The proof of (1) is quite similar but simpler, so we leave the it to readers. □

Theorem 7.2.6.

1. *[73, 3, Exp VII$_\mathrm{A}$, 7.4] The functors $\underline{\mathfrak{M}}^*(\cdot)$ and $\underline{G}^*(\cdot)$ induce quasi-inverse anti-equivalences of categories between the category of locally free $\mathcal{O}_S$-modules $\mathfrak{M}$ of rank n together with an $\mathcal{O}_S$-linear map $\varphi : \sigma^*\mathfrak{M} \to \mathfrak{M}$ and the category of finite locally free commutative group schemes of order p^n with vanishing Verschiebung (respectively, of order p^n).*

2. *$\underline{\mathfrak{M}}^*(\cdot)$ and $\underline{G}^*(\cdot)$ are "exact" in the sense that they send a short exact sequence in the source category to a short exact sequence in the target category.*

Proof. Let G be a finite locally free group scheme over S with vanishing Verschiebung. To prove (1) we need to prove that $\underline{\mathfrak{M}}^*(G)$ is locally free over $\mathcal{O}_S$, and that we have functorial isomorphisms $G \cong \underline{G}^*(\underline{\mathfrak{M}}^*(G))$ and $\mathfrak{M} \cong \underline{\mathfrak{M}}^*\left(\underline{G}^*(\mathfrak{M})\right)$. By definition, the $\mathcal{O}_S$-rank of $\mathfrak{M}$ is n if and only if and the order of G is p^n. So it remains to show that $\underline{\mathfrak{M}}^*(\cdot)$ and $\underline{G}^*(\cdot)$ are quasi-inverse anti-equivalences of categories.

For a φ-module $(\mathfrak{M}, \varphi)$ which is locally free of rank n over $\mathcal{O}_S$, we put $G_{\mathfrak{M}} := \underline{G}^*(\mathfrak{M}) = \operatorname{Spec} \mathcal{A}_{\mathfrak{M}}$ where $\mathcal{A}_{\mathfrak{M}}$ is the $\mathcal{O}_S$-bialgebra defined in (7.2.4.1). Let $\mathfrak{M}^* := \mathscr{H}om_{\mathcal{O}_S}(\mathfrak{M}, \mathcal{O}_S)$ be the $\mathcal{O}_S$-linear dual of $\mathfrak{M}$. We start with the following claim.

Claim 7.2.6.1. *There exists a natural isomorphism* $\mathcal{A}_{(G_{\mathfrak{M}})^\vee} \xleftarrow{\sim} \operatorname{Sym}(\mathfrak{M}^*)/\langle \alpha^p |\ \alpha \in \mathfrak{M}^* \rangle$ *as augmented* $\mathcal{O}_S$*-algebras.*

Observe that the natural projection $\mathscr{I}_{(G_{\mathfrak{M}})^\vee} \twoheadrightarrow \mathscr{I}_{(G_{\mathfrak{M}})^\vee}/\mathscr{I}^2_{(G_{\mathfrak{M}})^\vee} \cong \mathfrak{M}^*$ naturally splits, which follows from dualizing the natural splitting of the natural injection $\mathfrak{M} \to \mathscr{I}_{\mathfrak{M}}$. The image of $\mathfrak{M}^*$ in $\mathcal{A}_{(G_{\mathfrak{M}})^\vee}$ by this natural splitting generates $\mathcal{A}_{(G_{\mathfrak{M}})^\vee}$ as an $\mathcal{O}_S$-algebra, by Nakayama's lemma. Furthermore, any $\alpha \in \mathscr{I}_{(G_{\mathfrak{M}})^\vee}$ satisfies $\alpha^p = 0$ since the relative Frobenius map for $G_{\mathfrak{M}}$ is trivial. So we get a surjection of $\mathcal{O}_S$-algebras from the right side onto the left side. Since both terms are locally free of the same finite rank we have the claim.

From Claim 7.2.6.1, it follows that $\omega_{(G_{\mathfrak{M}})^\vee} \cong \mathfrak{M}^*$ as $\mathcal{O}_S$-modules so $\omega_{(G_{\mathfrak{M}})^\vee}$ is locally free of rank n over $\mathcal{O}_S$ (since $\mathfrak{M}^*$ is so). Furthermore, $\mathfrak{M} = \underline{\mathfrak{M}}^*\left(\underline{G}^*(\mathfrak{M})\right)$ as $\mathcal{O}_S$-submodules of $\mathcal{A}_{\mathfrak{M}}$, where $\underline{\mathfrak{M}}^*\left(\underline{G}^*(\mathfrak{M})\right) \subset \mathcal{A}_{\mathfrak{M}}$ is the submodule of elements m such that $\mu^*_{\underline{G}^*(\mathfrak{M})}(m) = 1 \otimes m + m \otimes 1$ (*c.f.* the proof of Lemma 7.2.3). In fact, applying Claim 7.2.6.1, we obtain a natural isomorphism $\omega_{(G_{\mathfrak{M}})^\vee} \cong \mathfrak{M}^*$ respecting the surjections from $\mathscr{I}_{(G_{\mathfrak{M}})^\vee}$, and apply the natural isomorphism $\underline{\mathfrak{M}}^*\left((G_{\mathfrak{M}})^\vee\right) \cong (\omega_{(G_{\mathfrak{M}})^\vee})^* = \mathscr{L}ie_S((G_{\mathfrak{M}})^\vee)$ (Lemma 7.2.3).

Now, let us compare the φ-structures on $\mathfrak{M}$ and $\underline{\mathfrak{M}}^*\left(\underline{G}^*(\mathfrak{M})\right)$. By construction of $\mathcal{A}_{\mathfrak{M}}$, we have $\varphi_{\mathfrak{M}}(\sigma^* m) = m^p$ for any $m \in \mathfrak{M}$ where the pth power is taken inside

$\mathcal{A}_{\mathfrak{M}}$. This coincides with the φ-structure on $\underline{\mathfrak{M}}^*\left(\underline{G}^*(\mathfrak{M})\right) := \mathscr{H}om_{\mathrm{gp}_{/S}}(\underline{G}^*(\mathfrak{M}), \mathbb{G}_a)$ where the φ-structure is induced from the (relative) Frobenius on $\mathbb{G}_a$. Therefore we obtain a natural φ-compatible isomorphism $\mathfrak{M} \cong \underline{\mathfrak{M}}^*\left(\underline{G}^*(\mathfrak{M})\right)$.

In order to prove the first part of the theorem, we proceed in two steps.

Step 1. Let G be a finite locally free commutative group scheme with vanishing Verschiebung, such that $\omega_{G^\vee}$ is locally free over $\mathcal{O}_S$. We put $\mathfrak{M} := \underline{\mathfrak{M}}^*(G) \cong (\omega_{G^\vee})^*$. Then, there is a natural isomorphism $G \xrightarrow{\sim} G_{\mathfrak{M}}$.

Step 2. If G be a finite locally free commutative group scheme with vanishing Verschiebung, then $\omega_{G^\vee}$ is locally free over $\mathcal{O}_S$.

We carry out **Step 1**. Since $\mathfrak{M} \cong (\omega_{G^\vee})^*$ by Lemma 7.2.3, we have a $\mathcal{O}_S$-linear isomorphism $\omega_{G^\vee} \cong \mathfrak{M}^*$ by double duality. By Lemma 7.2.5(2), we have $\mathrm{Hom}_{\mathrm{gp}_{/S}}(G, G_{\mathfrak{M}}) \cong \mathrm{Hom}_{S,\varphi}(\mathfrak{M}, \mathfrak{M})$. Therefore, we have a group homomorphism $f : G \to G_{\mathfrak{M}}$ which corresponds to $\mathrm{id}_{\mathfrak{M}}$. To show that this is an isomorphism, it suffices to show that the Cartier dual $f^\vee : (G_{\mathfrak{M}})^\vee \to G^\vee$ is an isomorphism.

The map $\mathscr{I}_{G^\vee}/\mathscr{I}^2_{G^\vee} \to \mathscr{I}_{(G_{\mathfrak{M}})^\vee}/\mathscr{I}^2_{(G_{\mathfrak{M}})^\vee}$ induced by $f^\vee$ is exactly $\mathrm{id}_{\mathfrak{M}^*} : \mathfrak{M}^* \to \mathfrak{M}^*$ with the identification of the source and the target with $\mathfrak{M}^*$ as discussed above. Therefore by Nakayama's lemma, $\mathcal{A}_{G^\vee} \to \mathcal{A}_{(G_{\mathfrak{M}})^\vee}$ induced by $f^\vee$ is surjective. On the other hand, since the relative Frobenius for $G^\vee$ is trivial by assumption, we have a surjective map $\mathcal{A}_{(G_{\mathfrak{M}})^\vee} \cong \mathrm{Sym}(\mathfrak{M}^*)/\langle \alpha^p = 0 |\ \alpha \in \mathfrak{M}^* \rangle \twoheadrightarrow \mathcal{A}_{G^\vee}$, which forces $f^\vee$ to be an isomorphism on structure sheaves over $\mathcal{O}_S$ and hence an isomorphism. Clearly, this construction is functorial, so we complete **Step 1**.

Now, we carry out **Step 2**.[4] We may assume that $S = \mathrm{Spec}\, R$ where R is a local ring with residue field k. Applying what we have just proved, we obtain an isomorphism $G_k \cong \underline{G}^*\left(\underline{\mathfrak{M}}^*(G_k)\right)$. On the other hand, we have the following natural

[4]The idea for this argument is sketched in the footnote to the théorème in [73, 3 Exp $\mathrm{VII_A}$ 7.4].

isomorphism $\underline{\mathfrak{M}}^*(G_k) \cong \underline{\mathfrak{M}}^*(G) \otimes_R k$ since $\underline{\mathfrak{M}}^*(\cdot)$ commutes with any base change.

Now consider a φ-module $(\mathfrak{M}', \varphi')$ which is finite free over R, such that there is a surjective φ-compatible map $\mathfrak{M}' \twoheadrightarrow \underline{\mathfrak{M}}^*(G)$ which reduces to an isomorphism $\mathfrak{M}' \otimes_R k \xrightarrow{\sim} \underline{\mathfrak{M}}^*(G) \otimes_R k$. By Lemma 7.2.5(2), the map $\mathfrak{M}' \twoheadrightarrow \underline{\mathfrak{M}}^*(G)$ corresponds to an S-group map $G \to \underline{G}^*(\mathfrak{M}', \varphi')$, which induces an isomorphism $G_k \xrightarrow{\sim} \underline{G}^*(\mathfrak{M}', \varphi')_k$ at the closed fiber. Hence by Nakayama's lemma, we conclude that $G \xrightarrow{\sim} \underline{G}^*(\mathfrak{M}', \varphi')$. This completes **Step 2** by the consequence of Claim 7.2.6.1 recorded above, hence the proof of the first part of the theorem.

For the second part of the theorem, we need to prove that any short exact sequence $(*):\ 1 \to G_1 \to G_2 \to G_3 \to 1$ induces a short exact sequence $\underline{\mathfrak{M}}^*(*):\ 0 \to \underline{\mathfrak{M}}^*(G_3) \to \underline{\mathfrak{M}}^*(G_2) \to \underline{\mathfrak{M}}^*(G_1) \to 0$ and conversely. The exactness of $(*)$ (respectively, $\underline{\mathfrak{M}}^*(*)$) is equivalent to the exactness of fibers at each $s \in S$ by the fiberwise flatness criterion [27, IV_3, (11.3.11)] Thus, we are immediately reduced to the case when $S = \operatorname{Spec} k$ where k is a field.

Let n_i be the k-rank of $\underline{\mathfrak{M}}^*(G_i)$. Assuming $(*)$ is exact, it is clear from the construction that we have the left exactness of $\underline{\mathfrak{M}}^*(*)$, since $\underline{\mathfrak{M}}^*(G_i) \cong \mathscr{H}om_k(\omega_{G_i^\vee}, k) = \mathscr{L}ie_S(G_i^\vee)$. But since $\mathcal{O}_S = k$ is a field, the equality $n_2 = n_1 + n_3$ forces the exactness of $\underline{\mathfrak{M}}^*(*)$. The same numerology proves the converse. $\square$

We record the following useful corollary:

Corollary 7.2.7. *Let G be a finite locally free commutative group scheme with trivial Verschiebung. Then naturally $\omega_G \cong \operatorname{coker}(\varphi_{\underline{\mathfrak{M}}^*(G)})$ as $\mathcal{O}_S$-modules.*

Proof. By the above theorem, we know that $G \cong \underline{G}^*(\underline{\mathfrak{M}}^*(G))$, and we have an explicit description of the coordinate ring of the right side, namely (7.2.4.1). $\square$

7.3 Effective local shtukas and π_0-divisible groups

Let S be either a scheme over $\operatorname{Spec} \mathfrak{o}_0$ or a formal scheme over $\operatorname{Spf} \mathfrak{o}_0$, and let $u_0 \in \Gamma(S, \mathcal{O}_S)$ be the image of π_0 under the structure morphism $\mathfrak{o}_0 \to \Gamma(S, \mathcal{O}_S)$. (An example to keep in mind is $S = \operatorname{Spec} \mathfrak{o}_K$.) In this section, we define a special kind of ind-representable *fppf*-sheaves of $\mathfrak{o}_0$-modules (namely, π_0-divisible groups of "finite $\mathcal{P}$-height"), which play the same[5] role in the equi-characteristic setting as Barsotti-Tate groups do in the p-adic setting.

Definition 7.3.1. Let G be an *fppf*-sheaf of $\mathfrak{o}_0$-modules over S. We say that G is a π_0*-divisible group of finite* $\mathcal{P}$*-height* if the following conditions are satisfied.

1. G is π_0^∞-torsion; i.e., $G \cong \varinjlim_n G[\pi_0^n]$.

2. G is π_0-divisible; i.e., $\pi_0 : G \to G$ is an epimorphism. Granting (1) and (2), the π_0-divisibility is equivalent to the exactness of

   $$(7.3.1.1) \quad (\dagger)_{m,n}: \qquad 1 \to G[\pi_0^m] \to G[\pi_0^{n+m}] \xrightarrow{[\pi_0^m]} G[\pi_0^n] \to 1, \qquad \forall n, m \geq 1,$$

   where the first map is the natural inclusion. (For a proof, one can adapt the argument presented in [65, I, §2].)

3. $G_1 := G[\pi_0]$ is representable by a finite locally free group scheme. (Assuming (2), this is equivalent to requiring that $G_n := G[\pi_0^n]$ are representable for all $n \geq 1$.)

4. The Verschiebung map for G vanishes (or rather, the Verschiebung map for G_n vanishes for all $n \geq 1$).

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[5]While Barsotti-Tate groups over a p-adic integer ring only give rise to crystalline representations with Hodge-Tate weights in $\{0, 1\}$, π_0-divisible groups of finite $\mathcal{P}$-height over $\mathfrak{o}_K$ give rise to "crystalline representations" of any non-negative "Hodge-Pink" weights.

5. The action of $\mathbb{F}_q$ on $\omega_G := \varprojlim_n \omega_{G_n}$ via functoriality of the $\mathcal{O}_S$-module structure on G is given by the "scalar multiplication" via the structure morphism $\mathbb{F}_q \to \Gamma(S, \mathcal{O}_S)$.

6. There exists a constant $h \in \mathbb{Z}_{\geq 0}$, such that $(\pi_0 - u_0)^h$ acts trivially on $\omega_G := \varprojlim_n \omega_{G_n}$.

We say that G is *of $\mathcal{P}$-height $\leqslant h$* if $(\pi_0 - u_0)^h \cdot \omega_G = 0$. A π_0-divisible group of $\mathcal{P}$-height 0 or ≤ 1 is called *étale* or *strict*, respectively. One can check that a π_0-divisible group G is étale if and only if all $G[\pi_0^n]$ are étale, and is strict if and only if π_0 acts via scalar multiplication by u_0 on ω_G.

The following theorem is the motivation for the above definition. This theorem can be viewed as an analogue of contravariant Dieudonné theory for π_0-divisible groups of finite $\mathcal{P}$-height.

Theorem 7.3.2. *There exist quasi-inverse anti-equivalences of categories $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}$ and $\underline{G}^*_{\mathfrak{o}_0}$ between the category of π_0-divisible groups of $\mathcal{P}$-height $\leqslant h$ over S and the category of effective local shtukas of $\mathcal{P}$-height $\leqslant h$ over S. The functors $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}$ and $\underline{G}^*_{\mathfrak{o}_0}$ enjoy the following additional properties.*

1. *The formation of $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}$ and $\underline{G}^*_{\mathfrak{o}_0}$ commute with any base change. More precisely, for any π_0-divisible groups $G_{/S}$ of $\mathcal{P}$-height $\leqslant h$ and any $S' \to S$, we have a natural isomorphism $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}(G_{S'}) \cong \underline{\mathfrak{M}}^*_{\mathfrak{o}_0}(G)_{S'}$; and similarly for $\underline{G}^*_{\mathfrak{o}_0}$.*

2. *Let $(*) : 0 \to G' \to G \to G'' \to 0$ be a sequence of morphisms of π_0-divisible groups of $\mathcal{P}$-height $\leqslant h$. Then $(*)$ is exact if and only if $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}(*)$ is exact. A similar statement is true for $\underline{G}^*_{\mathfrak{o}_0}$.*

3. *The rank of $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}(G)$ is n if and only if the order of $G[\pi_0]$ is of q^n (or equivalently, the order of $G[\pi_0^i]$ is of q^{in} for all i).*

We prove the theorem later in §7.3.5–§7.3.6. After we prove the theorem, we will usually suppress the subscript $(\cdot)_{\mathfrak{o}_0}$ since $\mathfrak{o}_0$ will be fixed throughout the discussion. (In the proof we need to vary the coefficient ring $\mathfrak{o}_0$, hence we specify this in the notation.)

Assume that u_0 is a unit in $\Gamma(S, \mathcal{O}_S)$. The main example is $S = \operatorname{Spec} K$. Then it follows that $\pi_0 - u_0$ is a unit in $\Gamma(S, \mathcal{O}_S[[\pi_0]])$, so all local shtukas over S are *étale*. Combining this with Theorem 7.3.2, we obtain the following corollary.

Corollary 7.3.3. *If u_0 is a unit in $\Gamma(S, \mathcal{O}_S)$, then any π_0-divisible group of finite $\mathcal{P}$-height over S is* étale.

Now, set $S = \operatorname{Spec} \mathfrak{o}_K$. Recall that effective local shtukas over $\mathfrak{o}_K$ are exactly $(\varphi, \mathfrak{S})$-modules of finite height, where $\mathfrak{S} = \mathfrak{o}_K[[[\pi_0]]$. For any effective local shtuka $\mathfrak{M}$ over $\mathfrak{o}_K$, we shall show that the $\mathfrak{o}_0$-linear $\boldsymbol{\mathcal{G}}_K$-representation $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ is isomorphic to the π_0-adic "Tate module" of the associated π_0-divisible group $\underline{G}^*_{\mathfrak{o}_0}(\mathfrak{M})$. We first define the π_0-adic Tate module $T_{\pi_0}(G)$ where G is a π_0-divisible group of finite $\mathcal{P}$-height over $\mathfrak{o}_K$, in a similar fashion as one defines the Tate module for a Barsotti-Tate group:

$$T_{\pi_0}(G) := \varprojlim_n G_n(K^{\mathrm{sep}}), \tag{7.3.3.1}$$

where the transition maps are $[\pi_0] : G_{n+1} \to G_n$. The following proposition essentially follows from Lemma 7.2.5(1).

Proposition 7.3.4. *For each effective local shtuka $\mathfrak{M}$ over $\mathfrak{o}_K$, there exists a natural $\mathfrak{o}_0$-linear $\boldsymbol{\mathcal{G}}_K$-equivariant isomorphism*

$$T_{\pi_0}\left(\underline{G}^*_{\mathfrak{o}_0}(\mathfrak{M})\right) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) = \operatorname{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}),$$

where $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \cong K^{\mathrm{sep}}[[\pi_0]]$ is the π_0-adic completion of the strict henselization of $\mathfrak{o}_{\mathcal{E}} \cong$

$K[[\pi_0]]$. *In particular, any* $\mathfrak{o}_0$*-lattice* $\boldsymbol{\mathcal{G}}_K$*-representation of* $\mathcal{P}$*-height* $\leqslant h$ *comes from the* π_0*-adic Tate module of some* π_0*-divisible group of* $\mathcal{P}$*-height* $\leqslant h$.

Proof. We have the following $\mathfrak{o}_0$-linear $\boldsymbol{\mathcal{G}}_K$-equivariant maps, which commute with the natural inclusions which define the direct system $\{G_n\}_n$:

$$\begin{aligned} G_n(K^{\text{sep}}) &\cong \mathrm{Hom}_{\mathfrak{o}_K,\varphi}(\underline{\mathfrak{M}}^*(G_n), K^{\text{sep}}) \\ &\xleftarrow{tr} \mathrm{Hom}_{\mathfrak{o}_K[[\pi_0]],\varphi}(\underline{\mathfrak{M}}^*(G_n), \mathcal{E}^{\text{ur}}/\mathfrak{o}_{\mathcal{E}^{\text{ur}}}) \\ &\xleftarrow{\sim} \mathrm{Hom}_{\mathfrak{o}_K[[\pi_0]],\varphi}(\underline{\mathfrak{M}}^*(G_n), \mathfrak{o}_{\mathcal{E}^{\text{ur}}}/(\pi_0^n)), \end{aligned}$$

where $\mathfrak{o}_0$ acts on $\mathrm{Hom}_{\mathfrak{o}_K,\varphi}(\underline{\mathfrak{M}}^*(G_n), K^{\text{sep}})$ through $\underline{\mathfrak{M}}^*(G_n)$. The first isomorphism is from Lemma 7.2.5(1) and the second map tr is induced by the "trace map" tr : $\sum_{i=1}^n a_i\pi_0^{-i} \mapsto \sum_{i=1}^n a_i$. One can construct the inverse of tr as follows: for a given $f \in \mathrm{Hom}_{\mathfrak{o}_K,\varphi}(\underline{\mathfrak{M}}^*(G_n), K^{\text{sep}})$, define recursively $a_i(f;m) := f(\pi_0^{i-1}m) - f(\pi_0^i m)$ for $i = n, \cdots, 1$, and check that $(tr^{-1}f)(m) := \sum_{i=1}^n a_i(f;m)\pi_0^{-i}$ works. Now by taking the projective limit, the proposition follows. $\square$

7.3.5 Proof of Theorem 7.3.2: the case $q = p$.

We first assume $q = p$, and we will use this case to handle general q. If $q = p$, then the $\mathfrak{o}_0$-action on $G[\pi_0^n]$ is determined by the action of π_0, and we do not have to worry about the action of $\mathbb{F}_q = \mathbb{F}_p$.

Let G be an ind-group scheme which satisfies (1), (3) and (4) of Definition 7.3.1. We put $G_n := G[\pi_0^n]$. Let us extend the construction of $\underline{\mathfrak{M}}^*(G)$ to such ind-group schemes G as follows:

$$\underline{\mathfrak{M}}^*(G) := \varprojlim_n \underline{\mathfrak{M}}^*(G_n)\,, \tag{7.3.5.1}$$

where the transition maps are induced from the natural inclusions $G_n \hookrightarrow G_{n+1}$. By the universal property of direct limit, we have a natural φ-compatible isomorphism

$\underline{\mathfrak{M}}^*(G) \cong \mathscr{H}om_{\mathbb{F}_p}(G, \mathbb{G}_a)$, where the right side means the sheaf of $\mathbb{F}_p$-linear homomorphisms of *fppf*-sheaves. By functoriality, $\mathfrak{o}_0 = \mathbb{F}_p[[\pi_0]]$ acts on $\underline{\mathfrak{M}}^*(G)$, which makes it into a module over $\mathcal{O}_S\widehat{\otimes}_{\mathbb{F}_p}\mathfrak{o}_0 \cong \mathcal{O}_S[[\pi_0]]$. We define $\varphi : \sigma^*\underline{\mathfrak{M}}^*(G) \to \underline{\mathfrak{M}}^*(G)$ by taking the limit of $\varphi_n : \sigma^*\underline{\mathfrak{M}}^*(G_n) \to \underline{\mathfrak{M}}^*(G_n)$, where $\sigma = \sigma_S\widehat{\otimes}\mathfrak{o}_0 : \mathcal{O}_S\widehat{\otimes}_{\mathbb{F}_p}\mathfrak{o}_0 \to \mathcal{O}_S\widehat{\otimes}_{\mathbb{F}_p}\mathfrak{o}_0$. Alternatively, one can directly construct φ out of the absolute Frobenius endomorphism $F_{\mathbb{G}_a} : \mathbb{G}_a \to \mathbb{G}_a$ just as we did in §7.2.1.

Applying $\underline{\mathfrak{M}}^*(\cdot)$ to the exact sequence $(\dagger)_{m,n}$ for the π_0-divisibility (i.e., the sequence (7.3.1.1) in (2) of Definition 7.3.1) is equivalent to the following exact sequence of $\mathcal{O}_S[[\pi_0]]$-modules with Frobenius structure φ:

$$(7.3.5.2) \qquad \underline{\mathfrak{M}}^*(\dagger)_{m,n} : \qquad 0 \to \underline{\mathfrak{M}}^*(G_n) \xrightarrow{[\pi_0^m]} \underline{\mathfrak{M}}^*(G_{n+m}) \to \underline{\mathfrak{M}}^*(G_m) \to 0$$

for each $m, n \geq 1$, identifying $\underline{\mathfrak{M}}^*(G_m)$ with $\underline{\mathfrak{M}}^*(G_{n+m})/(\pi_0^m)$. It is a standard fact that having the exact sequences $\underline{\mathfrak{M}}^*(\dagger)_{m,n}$ is equivalent to the local freeness of $\underline{\mathfrak{M}}^*(G)$ over $\mathcal{O}_S[[\pi_0]]$, and the $\mathcal{O}_S[[\pi_0]]$-rank of $\underline{\mathfrak{M}}^*(G)$ is precisely the $\mathcal{O}_S$-rank of $\underline{\mathfrak{M}}^*(G_1)$. (See Proposition 7.4.2, for example.) To summarize, the π_0-divisibility of G is equivalent to the local freeness of $\underline{\mathfrak{M}}^*(G)$ over $\mathcal{O}_S[[\pi_0]]$.

For ind-group schemes G over S satisfying (1)–(4) of Definition 7.3.1, $\underline{\mathfrak{M}}^*(\cdot)$ satisfies the following properties. First, $\underline{\mathfrak{M}}^*(\cdot)$ takes an exact sequence of such ind-group schemes into an exact sequence of $(\varphi, \mathcal{O}_S[[\pi_0]])$-modules, since the projective system $\{\underline{\mathfrak{M}}^*(G_n)\}$ satisfies the Mittag-Leffler condition over open affines in S. Second, the formation of $\underline{\mathfrak{M}}^*(\cdot)$ commutes with the base change in the following sense. For any map $f : T \to S$, we have a natural isomorphism

$$\underline{\mathfrak{M}}^*(G_T) = \varprojlim_n \underline{\mathfrak{M}}^*\big((G_n)_T\big) \cong \varprojlim_n f^*\big(\underline{\mathfrak{M}}^*(G_n)\big) \cong \mathcal{O}_T[[\pi_0]] \otimes_{f^{-1}\mathcal{O}_S[[\pi_0]]} f^{-1}\underline{\mathfrak{M}}^*(G),$$

where the last isomorphism uses that $\mathfrak{M}$ is locally free of finite rank over $\mathcal{O}_S[[\pi_0]]$. Recall that for local shtuka $\mathfrak{M}$ over S and a map $f : T \to S$, we defined in §7.1.4(1) the pullback as $f^*\mathfrak{M} := \mathcal{O}_T[[\pi_0]] \otimes_{f^{-1}\mathcal{O}_S[[\pi_0]]} f^{-1}\mathfrak{M}$.

Now, Corollary 7.2.7 asserts that $\omega_G \cong \varprojlim_n \operatorname{coker}(\varphi_{\underline{\mathfrak{M}}^*(G_n)}) \cong \operatorname{coker}(\varphi_{\underline{\mathfrak{M}}^*(G)})$. So condition (6) of Definition 7.3.1 is equivalent to requiring that $(\pi_0 - u_0)^h$ annihilates $\operatorname{coker}(\varphi_{\underline{\mathfrak{M}}^*(G)})$; i.e., $\underline{\mathfrak{M}}^*(G)$ is an effective local shtuka. Observe that the condition (5) of Definition 7.3.1 is automatic if $q = p$. We put $\underline{\mathfrak{M}}^*_{\mathbb{F}_p[[\pi_0]]} := \underline{\mathfrak{M}}^*$.

For any effective local shtuka $\mathfrak{M}$, we define $\underline{G}^*_{\mathbb{F}_p[[\pi_0]]}(\mathfrak{M})$ as follows.

$$\underline{G}^*_{\mathbb{F}_p[[\pi_0]]}(\mathfrak{M}) := \varinjlim_{n\geqslant 1} \underline{G}^*(\mathfrak{M}/\pi_0^n\mathfrak{M}), \tag{7.3.5.3}$$

where the limit is taken as a *fppf*-sheaf of $\mathbb{F}_p[[\pi_0]]$-modules with respect to transition maps induced by the natural projections $\mathfrak{M}/\pi_0^{n+1}\mathfrak{M} \twoheadrightarrow \mathfrak{M}/\pi_0^n\mathfrak{M}$. Observe that $\underline{G}^*(\mathfrak{M}/\pi_0^n\mathfrak{M}) \to \underline{G}^*_{\mathbb{F}_q[[\pi_0]]}(\mathfrak{M})$ is an isomorphism onto the π_0^n-torsion of the target. By construction, $\underline{G}^*_{\mathbb{F}_p[[\pi_0]]}(\mathfrak{M})$ satisfies the conditions (1), (3), and (4) of Definition 7.3.1. The π_0-divisibility (i.e., the condition (2) of Definition 7.3.1) is satisfied because we have the short exact sequence (7.3.5.2) by the $\mathcal{O}_S[[\pi_0]]$-local freeness (of finite rank) for $\mathfrak{M}$, which is in turn equivalent to the short exact sequence (7.3.1.1). The condition (6) of Definition 7.3.1 is satisfied thanks to Corollary 7.2.7. This shows that $\underline{G}^*_{\mathbb{F}_p[[\pi_0]]}(\mathfrak{M})$ is a π_0-divisible group of finite $\mathcal{P}$-height. That $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}$ and $\underline{G}^*_{\mathbb{F}_p[[\pi_0]]}$ are quasi-inverse and satisfy all the desired properties follows from Theorem 7.2.6. This completes the proof of Theorem 7.3.2 for the case $q = p$.

7.3.6 Proof of Theorem 7.3.2: the case $q = p^r$.

Put $q = p^r$. Let $\sigma : S \to S$ be the absolute Frobenius and $\sigma_q := \sigma^r$ the absolute q-Frobenius. Let G be a π_0-divisible group of $\mathcal{P}$-height $\leqslant h$ with the action of $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. We can restrict the action of $\mathfrak{o}_0$ to $\mathbb{F}_p[[\pi_0]]$ to view G as a π_0-divisible group of $\mathcal{P}$-height $\leqslant h$ with the action of $\mathbb{F}_p[[\pi_0]]$, so by the discussion of §7.3.5 we obtain an effective local shtuka $\mathfrak{M} := \underline{\mathfrak{M}}^*_{\mathbb{F}_p[[\pi_0]]}(G)$ with $\mathbb{F}_p[[\pi_0]]$-coefficients equipped with an action of $\mathbb{F}_q$ which commutes with φ and π_0-action. So we have the isotypic decomposition

$$\mathfrak{M} \cong \bigoplus_{i\in\mathbb{Z}/r\mathbb{Z}} \mathfrak{M}_{\chi_i},$$

where $\chi_i := \chi_0^{p^i}$, and $\chi_0 : \mathbb{F}_q^\times \to \Gamma(S, \mathcal{O}_S)^\times$ is obtained by restricting the structure morphism $\mathbb{F}_q \to \Gamma(S, \mathcal{O}_S)$. (The isotypic components for other characters $\chi : \mathbb{F}_q^\times \to \Gamma(S, \mathcal{O}_S)^\times$ vanish since the $\mathbb{F}_p^\times$-action on $\mathfrak{M}$ is given by $\chi_0|_{\mathbb{F}_p^\times}$.)

The natural p-Frobenius structure $\varphi_{\mathfrak{M}}$ on $\mathfrak{M}$ restricts to $\varphi_{\mathfrak{M},i} : \sigma^*(\mathfrak{M}_{\chi_i}) \to \mathfrak{M}_{\chi_{i+1}}$ for each i. So the q-Frobenius map $(\varphi_{\mathfrak{M}})^r := ((\sigma^{r-1})^*\varphi_{\mathfrak{M}} \circ \cdots \circ \sigma^*\varphi_{\mathfrak{M}} \circ \varphi_{\mathfrak{M}}) : \sigma_q^*\mathfrak{M} \to \mathfrak{M}$ restricts to $\varphi_q : \sigma_q^*(\mathfrak{M}_{\chi_0}) \to \mathfrak{M}_{\chi_0}$, which gives a q-Frobenius structure on $\mathfrak{M}_{\chi_0}$. We put $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}(G) := (\mathfrak{M}_{\chi_0}, \varphi_q)$.

Recall that $\omega_G \cong \operatorname{coker}(\varphi_{\mathfrak{M}})$, where $\omega_G := e_G^*\Omega^1_{G/S}$ is the co-Lie algebra for G. So the condition (5) of Definition 7.3.1 implies that $\varphi_{\mathfrak{M},i} : \sigma^*(\mathfrak{M}_{\chi_i}) \to \mathfrak{M}_{\chi_{i+1}}$ is surjective unless $i + 1 \equiv 0 \bmod r$, and that $\operatorname{coker}(\varphi_{\mathfrak{M}}) \cong \operatorname{coker}\left[\varphi_q : \sigma_q^*(\mathfrak{M}_{\chi_0}) \to \mathfrak{M}_{\chi_0}\right]$. This shows that $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}(G)$ is an effective local shtuka with $\mathfrak{o}_0$-coefficients. The exactness and base change assertions of the theorem (i.e., the claims (1) and (2) of Theorem 7.3.2) follow because the isotypic decomposition behaves well under base change and exact sequences.

In order to show that $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}$ is an anti-equivalence of categories, we need the following claim.

Claim A. *The map $\varphi_{\mathfrak{M},i} : \sigma^*(\mathfrak{M}_{\chi_i}) \to \mathfrak{M}_{\chi_{i+1}}$ is bijective unless $i + 1 \equiv 0 \bmod r$.*

This claim implies the rank assertion of the theorem (i.e., claim (3) of Theorem 7.3.2). In order to prove Claim A, we can assume S is local. Since we already showed that $\varphi_{\mathfrak{M},i}$ is a surjective map between finite locally free $\mathcal{O}_S[[\pi_0]]$-modules unless $i + 1 \equiv 0 \bmod r$, it is enough to show the source and the target have the same rank. But this immediately follows because $\pi_0 - u_0$ is not nilpotent in $\mathcal{O}_S[[\pi_0]]$ and $\varphi_{\mathfrak{M}}[\frac{1}{\pi_0-u_0}] : (\sigma^*\mathfrak{M})[\frac{1}{\pi_0-u_0}] \to \mathfrak{M}[\frac{1}{\pi_0-u_0}]$ is an isomorphism, so $\varphi_{\mathfrak{M},i}[\frac{1}{\pi_0-u_0}]$ is an isomorphism.

Claim B. *One can recover* $\mathfrak{M} := \underline{\mathfrak{M}}^*_{\mathbb{F}_p[[\pi_0]]}(G)$ *with its* $\mathbb{F}_q$*-action from* $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}(G) = (\mathfrak{M}_{\chi_0}, \varphi_q)$ *functorially and uniquely up to unique isomorphism.*

Combining this claim with the theorem for the case $q = p$, it follows that $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}$ is an anti-equivalence of categories.

To prove Claim B, first observe that by Claim A, $(\varphi_{\mathfrak{M}})^i : (\sigma^i)^*\mathfrak{M} \to \mathfrak{M}$ induces an $\mathbb{F}_q^\times$-equivariant isomorphism

$$(\sigma^i)^*\mathfrak{M}_{\chi_0} \xrightarrow{\sim} \mathfrak{M}_{\chi_i}$$

for each $0 \leq i < r$. Let us put $\mathfrak{M}'_i := \sigma^{i*}\left(\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}(G)\right)$ and $\mathfrak{M}' := \bigoplus_{i=0}^{r-1}\mathfrak{M}'_i$. We define a p-Frobenius structure $\varphi_{\mathfrak{M}'} : \sigma^*\mathfrak{M}' \to \mathfrak{M}'$ by $\mathrm{id}_{\mathfrak{M}'_{i+1}} : \sigma^*\mathfrak{M}'_i = \mathfrak{M}'_{i+1} \to \mathfrak{M}'_{i+1}$ for $i+1 \neq r$ and $\varphi_q : \sigma^*\mathfrak{M}'_{r-1} = \sigma_q^*\mathfrak{M}'_0 \to \mathfrak{M}'_0$, where φ_q is the q-Frobenius structure on $\mathfrak{M}'_0 := \underline{\mathfrak{M}}^*_{\mathfrak{o}_0}(G)$. One can directly see that $\mathfrak{M}$ and $\mathfrak{M}'$ are naturally isomorphic as φ-modules. This proves Claim B.

For any effective local shtuka $\mathfrak{M}_q$ (i.e., a finite locally free $\mathcal{O}_S[[\pi_0]]$-module equipped with a q-Frobenius structure $\varphi_q : \sigma_q^*\mathfrak{M}_q \to \mathfrak{M}_q$) let us define $\underline{G}^*_{\mathfrak{o}_0}(\mathfrak{M}_q)$ as follows. Following the recipe in Claim B, one obtains an effective local shtuka $\mathfrak{M}$ with $\mathbb{F}_p[[\pi_0]]$-coefficients equipped with an $\mathbb{F}_q$-action which is compatible with the p-Frobenius structure $\varphi_{\mathfrak{M}} : \sigma_p^*\mathfrak{M} \to \mathfrak{M}$. Therefore, by functoriality $\mathbb{F}_q$ acts on the π_0-divisible group $\underline{G}^*_{\mathbb{F}_p[[\pi_0]]}(\mathfrak{M})$ (which *a priori* comes equipped with $\mathbb{F}_p[[\pi_0]]$-action). Clearly, the $\mathbb{F}_q$-action and π_0-action commute, so $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ acts on $\underline{G}^*_{\mathbb{F}_p[[\pi_0]]}(\mathfrak{M})$, and this action satisfies the condition (5) of Definition 7.3.1. In other words, $\underline{G}^*_{\mathbb{F}_p[[\pi_0]]}(\mathfrak{M})$ is a π_0-divisible group of finite height with $\mathfrak{o}_0$-coefficients. We let $\underline{G}^*_{\mathfrak{o}_0}(\mathfrak{M})$ denote this

π_0-divisible group. Claims A and B, together with the case $q = p$ proved in §7.3.5, show that $\underline{\mathfrak{M}}^*_{\mathfrak{o}_0}$ and $\underline{G}^*_{\mathfrak{o}_0}$ are quasi-inverses and satisfy all the desired properties. This completes the proof of Theorem 7.3.2. □

7.3.7 Examples of π_0-divisible groups of finite $\mathcal{P}$-height

At first glance, the definition of π_0-divisible groups of finite $\mathcal{P}$-height involves many technical conditions such as having trivial Verschiebung. But the examples below show that *strict* π_0-divisible groups (i.e., π_0-divisible groups of $\mathcal{P}$-height $\leqslant 1$) occur quite naturally. One may regard the non-strict ones as a generalization to higher Hodge-Pink weights.

π_0-divisible group associated to a Drinfeld module: let $\operatorname{Spec} A = C \setminus \{\infty\}$, where C is a smooth projective geometrically connected curve over some finite field of characteristic p. Fix a closed point $P \in \operatorname{Spec} A$ (also view P as a maximal ideal of A) and choose a local parameter π_0 at P. Let S be a scheme over $\widehat{A}_P$, and $\mathcal{L}_{/S}$ a Drinfeld A-module[6]. Then the "π_0-divisible group" $G := \varinjlim_n \mathcal{L}[P^n]$ associated to $\mathcal{L}$ is a strict π_0-divisible group.

Strict formal $\mathbb{F}_q[[\pi_0]]$-module: Let S be a $\mathfrak{o}_0$-scheme on which u_0 (i.e., the image of π_0 in $\Gamma(S, \mathcal{O}_S)$) is locally nilpotent (or more generally, a formal scheme over $\operatorname{Spf} \mathfrak{o}_0$). Let $G_{/S}$ be a formal Lie group[7], equipped with an action of $\mathfrak{o}_0$. It follows from the Cartier theory that a formal Lie group which is killed by p always has trivial Verschiebung [34, Ch.I, Prop 2.1.1]. If we further assume that $\pi_0 - u_0$ acts trivially on ω_G, then G is automatically π_0^∞-torsion by the argument similar to [65, Ch.II, Lemma 4.2] or [75, (2.4) Lemma 0]. So if G is

[6] For the definition, see Drinfeld's original article [24] or Deligne-Husemöller [21].
[7] i.e., a formally smooth, ind-infinitesimal group with tangent space finitely generated over $\mathcal{O}_S$. See [65, II, (1.1)].

a formal Lie group with $\mathfrak{o}_0$-action, then G is a strict π_0-divisible group if and only if G is π_0-divisible and $\mathfrak{o}_0$ acts on ω_G via "scalar multiplication" through the structure morphism $\mathfrak{o}_0 \to \Gamma(S, \mathcal{O}_S)$.

Lubin-Tate formal group: Now, we define the "Lubin-Tate formal group" $\mathcal{LT}_S$ which corresponds to the local shtuka $\mathfrak{L}_S(1)$ via the anti-equivalence in Theorem 7.3.2. (See Definition 7.1.5 for the definition of $\mathfrak{L}_S(1)$.) This computation is also done in Hartl's dictionary [41, §3.4].

Let $\mathcal{LT}_S$ be $\widehat{\mathbb{G}_a} \cong \mathrm{Spf}_S\, \mathcal{O}_S[[X]]$ as a formal Lie group, equipped with π_0-action given by $[\pi_0]^*X = u_0X + X^q$. Clearly, $\mathcal{LT}_S$ is a strict π_0-divisible group. Now we compute $\underline{\mathfrak{M}}_q^*(\mathcal{LT}) \cong \mathscr{H}om_{\mathbb{F}_q}(\mathcal{LT}, \mathbb{G}_a)$. The right side is a rank-1 free module over $\mathscr{E}nd_{\mathbb{F}_q}(\widehat{\mathbb{G}_a}) \cong \mathcal{O}_S\{\{\tau\}\}$, where $\tau \in \mathrm{End}_{\mathbb{F}_q}(\widehat{\mathbb{G}_a})$ is defined by $\tau^*(X) = X^q$ and $\tau \cdot a = a^q \cdot \tau$ for $a \in \mathcal{O}_S$. Also, π_0 acts on $\underline{\mathfrak{M}}_q^*(\mathcal{LT})$ via the natural action of $(u_0 + \tau) \in \mathscr{E}nd_{\mathbb{F}_q}(\widehat{\mathbb{G}_a})$, and $\varphi : \sigma_q^*\big(\underline{\mathfrak{M}}_q^*(\mathcal{LT})\big) \to \underline{\mathfrak{M}}_q^*(\mathcal{LT})$ is given by $\varphi(\sigma_q^* m) = \tau \cdot m = (\pi_0 - u_0) \cdot m$ for any $m \in \underline{\mathfrak{M}}_q^*(\mathcal{LT})$. This shows that $\underline{\mathfrak{M}}_q^*(\mathcal{LT}) \cong \mathfrak{L}_S(1)$.

Now we work over $S = \mathrm{Spec}\, \mathfrak{o}_K$ and let $\mathfrak{o}_0(1) := T_{\pi_0}(\mathcal{LT})$ be the rank-1 lattice representation of $\boldsymbol{\mathcal{G}}_K$ given by the "Lubin-Tate character". Then we have $T_{\mathfrak{S}}^*\left(\mathfrak{L}_{\mathfrak{o}_K}(1)\right) \cong T_{\pi_0}(\mathcal{LT}) =: \mathfrak{o}_0(1)$ by Proposition 7.3.4, hence the notation $\mathfrak{L}_{\mathfrak{o}_K}(1)$.

Motivated by the example, we make the following definition:

Definition 7.3.8. We define $\mathcal{LT}_S^{\otimes h}$ for a non-negative integer h to be the π_0-divisible group which corresponds to the Tate object $\mathfrak{L}_S(h)$ via the anti-equivalence in Theorem 7.3.2

For $S = \operatorname{Spec} \mathfrak{o}_K$, we have

$$T_{\pi_0}(\mathcal{LT}^{\otimes h}) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{L}_{\mathfrak{o}_K}(h)) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{L}_{\mathfrak{o}_K}(1))^{\otimes h} \cong T_{\pi_0}(\mathcal{LT})^{\otimes h} \cong \mathfrak{o}_0(h)$$

by Proposition 7.3.4, hence the notation.

7.3.9 Duality

We now define a duality operation for local shtukas of $\mathcal{P}$-height $\leqslant h$ for any $h \geq 0$, or equivalently for π_0-divisible groups of $\mathcal{P}$-height $\leqslant h$.

Definition 7.3.9.1. For an effective local shtuka $\mathfrak{M}$ of $\mathcal{P}$-height $\leqslant h$, the *Faltings dual of $\mathcal{P}$-height h* is the effective local shtuka $\mathfrak{M}^\vee := \mathfrak{M}^*(h)$ of $\mathcal{P}$-height $\leqslant h$.

For a π_0-divisible group G of $\mathcal{P}$-height $\leqslant h$, the *Faltings dual of $\mathcal{P}$-height $\leqslant h$* is the π_0-divisible group $G^\vee$ which corresponds to $\underline{\mathfrak{M}}^*(G)^\vee$ via the anti-equivalence in Theorem 7.3.2.

One can check that Faltings dual is an exact anti-equivalence of categories which commutes with any base change and satisfies all the usual axioms for a good duality theory. The Faltings duality depends on the choice of the $\mathcal{P}$-height bound h, though we do not specify this in the notation.

7.3.10 Lubin-Tate type π_0-divisible group of $\mathcal{P}$-height $\leqslant h$

Note that the constant étale π_0-divisible group $\underline{F_0/\mathfrak{o}_0}$ and $\mathcal{LT}^{\otimes h}$ are each other's Faltings dual (of $\mathcal{P}$-height $\leqslant h$). Thus, by working on geometric fibers we get:

Lemma 7.3.10.1. *Let G be a π_0-divisible group of $\mathcal{P}$-height $\leqslant h$. Then the following are equivalent.*

1. *The geometric fiber $G_{\bar{s}}$ at each geometric point $\bar{s}$ of S is isomorphic to $(\mathcal{LT}^{\otimes h})^{\oplus n}$ for some n.*

2. The Faltings dual of $\mathcal{P}$-height $\leqslant h$ for G is étale.

We call a π_0-divisible group G is *of Lubin-Tate type of $\mathcal{P}$-height h* if G satisfies the equivalent conditions of the lemma. Lubin-Tate type π_0-divisible groups of $\mathcal{P}$-height h play the same role, in the equal characteristic arithmetic, as Barsotti-Tate groups of multiplicative type do in the mixed characteristic arithmetic.

7.4 Some commutative algebra over an adic ring

In this section, we state some standard facts about commutative algebra over an adic ring, which are used in this chapter (and elsewhere). Readers may skip this section and use this as the "reference sheet" for the standard facts when they are used.

Let A be an "$\mathfrak{a}$-adic ring", in other words, $A \cong \varprojlim_n A/\mathfrak{a}^n$ for some finitely generated ideal $\mathfrak{a} \subset A$. The example to keep in mind is $A = R[[t]]$ for any ring R and $\mathfrak{a} = tA$.

Proposition 7.4.1. *[37, Prop 7.2.10][8] The functors $\mathfrak{M} \mapsto \{\mathfrak{M}/\mathfrak{a}^n\}_n$ and $\{\mathfrak{M}_n\}_n \mapsto \varprojlim_n \mathfrak{M}_n$ are quasi-inverse* equivalences of categories *between the category of finitely generated A-modules and the category of projective systems $\{\mathfrak{M}_n\}_{n\geq 1}$ where each $\mathfrak{M}_n$ is an $A/\mathfrak{a}^n$-module, $\mathfrak{M}_1$ is a finitely generated $A/\mathfrak{a}$-module and each transition map induces $\mathfrak{M}_{n+1} \otimes A/\mathfrak{a}^n \xrightarrow{\sim} \mathfrak{M}_n$. Moreover, $\mathfrak{M}$ is locally free of finite A-rank if and only if each $\mathfrak{M}/\mathfrak{a}^n$ is locally free of finite $A/\mathfrak{a}^n$-rank.*

Now, we specialize to the case when $A = R[[t]]$ endowed with the t-adic topology. In this case, we have a simpler criterion for the local freeness over $R[[t]]$.

[8] This proposition is also stated in [27, I, Prop 7.2.9], except the local freeness assertion. But local freeness can be read off from [27, I, Cor 7.2.10], because locally free modules of finite rank are exactly finitely generated projective modules.

Proposition 7.4.2. *Let $\mathfrak{M}$ be a finitely generated $R[[t]]$-module (or more generally, t-adically separated and complete topological $R[[t]]$-module). Then $\mathfrak{M}$ is finite locally free over $R[[t]]$ if and only if $\mathfrak{M}$ has no nontrivial t-torsion and $\mathfrak{M}/t\mathfrak{M}$ is finite locally free over $R \cong R[[t]]/(t)$.*

Sketch of the Proof. The "only if" direction is obvious, so we sketch the "if" direction. The t-adic separatedness and completeness assumption implies by successive approximation[9] that $\mathfrak{M}/t^n\mathfrak{M}$ is finitely generated over $R[[t]]/(t^n)$ for each n, so in turn it implies that $\mathfrak{M}$ is finitely generated over $R[[t]]$.

Since there is no nontrivial t-torsion, we have the short exact sequences

$$0 \to \mathfrak{M}/t^n\mathfrak{M} \xrightarrow{t^m} \mathfrak{M}/t^{m+n}\mathfrak{M} \to \mathfrak{M}/t^m\mathfrak{M} \to 0,$$

for each $m, n \geqslant 1$. Then it follows from the local flatness criterion[10] that $\mathfrak{M}/t^n\mathfrak{M}$ is a flat $R[[t]]/(t^n)$-module for each n. This implies our claim by Proposition 7.4.1. □

We record the following interesting consequence, which roughly says that any finite locally free $R[[t]]$-module can be trivialized by "localizing" R.

Corollary 7.4.3. *Let $\mathfrak{M}$ be finite locally free over $R[[t]]$. Then there exists a (finite) Zariski-open covering $\{R[1/f]\}$ of R such that $\mathfrak{M} \otimes (R[1/f])[[t]]$ is free over $(R[1/f])[[t]]$ for each f.*

Proof. Take an open covering $\{R[1/f]\}$ which trivializes $\mathfrak{M}/t\mathfrak{M}$. This covering works, by successive approximation and the proof of the previous proposition. □

The following statement and the proof are taken from [34, Lemma 2.2.8].

Lemma 7.4.4. *Let $\mathfrak{M}$ be a locally free $R[[t]]$-module of rank r. Let $\mathfrak{M}' \subset \mathfrak{M}$ be an $R[[t]]$-submodule which satisfies the following properties:*

[9] or by "Nakayama's lemma" for nilpotent ideals

[10] See, for example, [62, Thm 22.3], especially the equivalence of (1) and (4′). Since the ideal $(t) \subset R[[t]]/(t^n)$ is nilpotent, we can apply the local flatness criterion without requiring R be noetherian.

1. *There exists an integer N such that $t^N\mathfrak{M} \subset \mathfrak{M}' \subset \mathfrak{M}$.*

2. *The quotient $\mathfrak{M}/\mathfrak{M}'$ is locally free over R.*

Then $\mathfrak{M}'$ is finite locally free as an $R[[t]]$-module.

Proof. Note that $\mathfrak{M}'/t^N\mathfrak{M} = \ker[\mathfrak{M}/t^N\mathfrak{M} \to \mathfrak{M}/\mathfrak{M}']$ where $\mathfrak{M}/t^N\mathfrak{M} \to \mathfrak{M}/\mathfrak{M}'$ is a surjection between finite locally free R-modules, so $\mathfrak{M}'/t^N\mathfrak{M}$ is finite locally free as an R-module. Hence $\mathfrak{M}'$ is $R[[t]]$-finite since $\mathfrak{M}$ is. Thus by Proposition 7.4.2, to show that $\mathfrak{M}'$ is finite locally free over $R[[t]]$ it is necessary and sufficient to show that $\mathfrak{M}'$ has no nontrivial t-torsion and that $\mathfrak{M}'/t\mathfrak{M}'$ is locally free as an R-module. But being a submodule of $\mathfrak{M}$, $\mathfrak{M}'$ is torsionfree, so it remains to show the local freeness of $\mathfrak{M}'/t\mathfrak{M}'$.

For any integer $n \geqslant N$, we have the following short exact sequence.

$$(*) \qquad 0 \to \mathfrak{M}'/t^n\mathfrak{M} \to \mathfrak{M}/t^n\mathfrak{M} \to \mathfrak{M}/\mathfrak{M}' \to 0$$

Since $\mathfrak{M}/\mathfrak{M}'$ is locally free (so projective) over R, this exact sequence is split and $R' \otimes_R (*)$ remains exact for any R-algebra R'. In particular, $\mathfrak{M}'/t^n\mathfrak{M}$ is finite locally free over R for any $n \geqslant N$.

Now, we have the following short exact sequence.

$$(**) \qquad 0 \to \mathfrak{M}'/t^N\mathfrak{M} \xrightarrow{t} \mathfrak{M}'/t^{N+1}\mathfrak{M} \to \mathfrak{M}'/t\mathfrak{M}' \to 0$$

The exactness follows from the injectivity of $\mathfrak{M}/t^N\mathfrak{M} \xrightarrow{t} \mathfrak{M}/t^{N+1}\mathfrak{M}$ and the exactness of $(*)$. Similarly, $R' \otimes_R (**)$ remains exact for any R-algebra R', using the exactness of $R' \otimes_R (*)$. This implies, by standard facts about flatness[11], that $\mathfrak{M}'/t\mathfrak{M}'$ is flat over R. It is clear from the exact sequence $(**)$ that $\mathfrak{M}'/t\mathfrak{M}'$ is finitely presented over R. □

[11] See, for example, [9, Ch.I §2.5 Prop 4].

The following lemma is a "partial converse" to the previous lemma.

Lemma 7.4.5. *Let $\mathfrak{M}$ and $\mathfrak{M}'$ be locally free $R[[t]]$-modules of the same finite constant rank. Assume that we have $R[[t]]$-linear map $f : \mathfrak{M}' \to \mathfrak{M}$ such that the image of f contains $t^N\mathfrak{M}$ for some integer N. Then* $\operatorname{coker}(f)$ *is locally free over R.*

Proof. Note first that f is necessarily injective, since $f[\frac{1}{t}]$ is surjective and hence an isomorphism. Similarly, for any ideal $I \subset R$, the reduction $(f \bmod I) : \mathfrak{M}'/I\mathfrak{M}' \to \mathfrak{M}/I\mathfrak{M}$ is injective.

Now, consider the following short exact sequence.

$$(\dagger) \qquad 0 \to \mathfrak{M}' \xrightarrow{f} \mathfrak{M} \to \operatorname{coker}(f) \to 0$$

The R-flatness of $\operatorname{coker}(f)$ follows because $(\dagger) \bmod t^N$ is short exact and for any ideal $I \subset R$ containing t^N, the right exact sequence $(\dagger) \bmod I$ is left exact. (Recall that any R'-module M' is flat provided $\operatorname{Tor}_1^{R'}(I, M') = 0$ for all ideals I of R.) Furthermore, $\operatorname{coker}(f)$ is finitely presented over R, thanks to the right exact sequence $(\dagger) \bmod (t^N)$. $\square$

CHAPTER VIII

Torsion $\mathcal{G}_K$-representations of $\mathcal{P}$-height $\leqslant h$

In this chapter, we introduce "torsion $(\varphi, \mathfrak{S})$-modules" of $\mathcal{P}$-height $\leqslant h$, which play a central role for the rest of this work. For the purpose of studying deformation theory in §XI, it is useful to consider torsion $(\varphi, \mathfrak{S})$-modules with various "coefficients," which will be made precise and studied in §8.2.

8.1 Torsion φ-modules and torsion $\mathcal{G}_K$-representation of $\mathcal{P}$-height $\leqslant h$

We begin with defining a torsion $(\varphi, \mathfrak{S})$-module of finite $\mathcal{P}$-height. One can immediately verify that a $(\varphi, \mathfrak{S})$-module obtained as a cokernel of an isogeny in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ satisfies the following definition. (In fact, we will also prove its converse in Proposition 8.1.4.)

Definition 8.1.1. A $(\varphi, \mathfrak{S})$-module $\mathfrak{M}$ is called a *torsion* $(\varphi, \mathfrak{S})$*-module of finite* $\mathcal{P}$*-height* if the following conditions are satisfied.

1. There exists an integer N such that $\pi_0^N \mathfrak{M} = 0$.
2. As a $\mathfrak{S}$-module, $\mathfrak{M}$ is of projective dimension $\leqslant 1$.
3. There exists an integer $h \geq 0$ such that $\mathcal{P}(u)^h \cdot \mathrm{coker}(\varphi_{\mathfrak{M}}) = 0$.

We say that such $\mathfrak{M}$ is *of* $\mathcal{P}$*-height* $\leqslant h$ if $\mathcal{P}(u)^h \cdot \mathrm{coker}(\varphi_{\mathfrak{M}}) = 0$. We let $(\mathrm{Mod}/\mathfrak{S})$ denote the category of torsion φ-modules over $\mathfrak{S}$ of finite $\mathcal{P}$-height, and $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$

the full subcategory of $(\mathrm{Mod}\,/\mathfrak{S})$ whose objects are of $\mathcal{P}$-height $\leqslant h$. We let $(\mathrm{ModFI}\,/\mathfrak{S})$ denote the full subcategory of $(\mathrm{Mod}\,/\mathfrak{S})$ whose objects are isomorphic to $\bigoplus_i(\mathfrak{S}/\pi_0^{n_i}\mathfrak{S})$ as $\mathfrak{S}$-modules[1], and $(\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}$ the full subcategory of $(\mathrm{ModFI}\,/\mathfrak{S})$ whose objects are of $\mathcal{P}$-height $\leqslant h$.

In the case $\mathfrak{o}_0 = \mathbb{Z}_p$, basic properties of torsion $(\varphi,\mathfrak{S})$-module of $\mathcal{P}$-height $\leqslant 1$ are studied in [51, §1.1], and this is easily adapted to the equi-characteristic case, as we now show.

In Definition 8.1.1, the condition on the projective dimension can be "simplified" as follows.

Proposition 8.1.2. *Let $\mathfrak{M}$ be a finitely generated $\mathfrak{S}$-module such that $\pi_0^N\mathfrak{M} = 0$ for some N. Then the following are equivalent.*

1. *As a $\mathfrak{S}$-module, $\mathfrak{M}$ is of projective dimension $\leqslant 1$. (So we allow $\mathfrak{M} = 0$.)*
2. *There exists one element $\alpha \in \mathfrak{m}_{\mathfrak{S}}\setminus\pi_0{\cdot}\mathfrak{S}$ such that $\mathfrak{M}$ has no nonzero α-torsion.*
3. *For any element $\alpha \in \mathfrak{m}_{\mathfrak{S}} \setminus \pi_0{\cdot}\mathfrak{S}$, $\mathfrak{M}$ has no nonzero α-torsion. In particular, $\mathfrak{M}$ has no nonzero u-torsion and $\mathcal{P}(u)$-torsion.*

For the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ (so $\mathfrak{S} = \mathfrak{o}_K[[\pi_0]]$), $\mathfrak{M}$ is of projective dimension $\leqslant 1$ as a $\mathfrak{S}$-module if and only if $\mathfrak{M}$ is finite free over $\mathfrak{o}_K$.

Proof. The last claim for the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ follows from the equivalence between (1) and (3) (using $\alpha = u$). The implication (3)⇒(2) is trivial, and the equivalence (2)⇔(1) is just the theorem of Auslander-Buchsbaum [62, Thm 19.1]. In order to show (1)⇒(3), assume that $\mathfrak{M}$ has nonzero α-torsion for some $\alpha \in \mathfrak{m}_{\mathfrak{S}} \setminus \pi_0{\cdot}\mathfrak{S}$. Then there exists an element $x \in \mathfrak{M}$ whose annihilator is exactly $\mathfrak{m}_{\mathfrak{S}}$, since $(\pi_0,\alpha)\mathfrak{S}$ is an

[1] The notation $(\mathrm{ModFI}\,/\mathfrak{S})$ stands for "Modules à Facteurs Invariants."

$\mathfrak{m}_{\mathfrak{S}}$-primary ideal. Then we have a short exact sequence

$$0 \longrightarrow \mathfrak{S}/\mathfrak{m}_{\mathfrak{S}} \xrightarrow{1\mapsto x} \mathfrak{M} \longrightarrow \mathfrak{M}/(x) \longrightarrow 0.$$

Since the projective dimension of $\mathfrak{S}/\mathfrak{m}_{\mathfrak{S}}$ is exactly 2 and the projective dimension of $\mathfrak{M}/(x)$ is at most 2 (by the homological criterion of regularity and the fact that $\mathfrak{S}$ is a regular local ring of dimension 2), we conclude that $\mathfrak{M}$ is of projective dimension 2. □

Next, we show that any $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ can be written as a cokernel of some isogeny $f : \mathfrak{M}_1 \to \mathfrak{M}_0$ in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$. We first need the following lemma.

Lemma 8.1.3. *For $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})$, the Frobenius structure $\varphi : \sigma^*\mathfrak{M} \to \mathfrak{M}$ is injective.*

Proof. By Proposition 8.1.2, the natural map $\mathfrak{M} \to \mathfrak{M}[\frac{1}{u}] \cong \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is injective. Now apply Lemma 2.2.3.1 for $R = \mathfrak{S}$ and $R' = \mathfrak{o}_{\mathcal{E}}$, keeping in mind that $\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ is an étale φ-module. □

Proposition 8.1.4. *For $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, there exist $\mathfrak{M}_0, \mathfrak{M}_1 \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$ and an isogeny $f : \mathfrak{M}_1 \to \mathfrak{M}_0$, such that $\mathfrak{M} \cong \mathrm{coker}(f)$ as a φ-module.*

Sketch of Proof. In the case $\mathfrak{o}_0 = \mathbb{Z}_p$, this proposition is exactly [52, Proposition 2.3.4] which can be adapted to the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. We sketch the proof.

It is enough to find $\mathfrak{M}_0 \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ and a surjective map $\mathfrak{M}_0 \twoheadrightarrow \mathfrak{M}$ of φ-modules. In fact, the kernel $\mathfrak{M}_1$ of this map is automatically free over $\mathfrak{S}$ since the projective dimension of $\mathfrak{M}$ is $\leqslant 1$, and we have $\mathcal{P}(u)^h{\cdot}\mathrm{coker}(\varphi_{\mathfrak{M}_1}) = 0$ thanks to the *injectivity* of $\varphi_{\mathfrak{M}}$ and the snake lemma. The construction of $\mathfrak{M}_0$ is identical to the one given in the proof of [52, Prop 2.3.4]. □

8.1.5

From now on, we write $(\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$ to denote torsion étale φ-modules over $\mathfrak{o}_{\mathcal{E}}$, which used to be denoted as $\underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét,tor}}(\varphi)$. From §5.1, we have quasi-inverse anti-equivalences of categories $\underline{D}_{\mathcal{E}}^*$ and $\underline{T}_{\mathcal{E}}^*$ between $(\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$ and $\mathrm{Rep}_{\mathfrak{o}_0}^{\mathrm{tor}}(\boldsymbol{\mathcal{G}}_K)$.

Since the scalar extension $\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} (\cdot)$ induces a functor $(\mathrm{Mod}/\mathfrak{S}) \to (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$, we can define a functor $\underline{T}_{\mathfrak{S}}^* : (\mathrm{Mod}/\mathfrak{S}) \to \mathrm{Rep}_{\mathfrak{o}_0}^{\mathrm{tor}}(\boldsymbol{\mathcal{G}}_K)$ by $\underline{T}_{\mathfrak{S}}^*(\mathfrak{M}) := \underline{T}_{\mathcal{E}}^*(\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}) \cong \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}})$ for $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})$. Note also that $\mathfrak{M} \to \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} = \mathfrak{M}[\frac{1}{u}]$ is injective by Proposition 8.1.4(3). The functor $\underline{T}_{\mathfrak{S}}^*(\cdot)$ may *not* be fully faithful on torsion objects.

To define $\underline{T}_{\mathfrak{S}}^*$ for $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$, we can use a "smaller" ring than $\mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$ with "integral" structure. We first introduce more rings:

- $\mathfrak{S}^{\mathrm{ur}}$ the integral closure of $\mathfrak{S}$ inside $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$.
- $\widehat{\mathfrak{S}}^{\mathrm{ur}}$ the closure of $\mathfrak{S}^{\mathrm{ur}} \subset \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ under the π_0-adic topology.

The endomorphism $\sigma : \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \to \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}$ restricts to flat endomorphisms of $\mathfrak{S}^{\mathrm{ur}}$ and $\widehat{\mathfrak{S}}^{\mathrm{ur}}$. The Galois group $\boldsymbol{\mathcal{G}}_K$ acts by isometries (with respect to the π_0-adic norm) and commute with σ.

Lemma 8.1.6. *For $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})$, the natural map*

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{S}^{\mathrm{ur}}[1/\pi_0]/\mathfrak{S}^{\mathrm{ur}}) \hookrightarrow \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}) =: \underline{T}_{\mathfrak{S}}^*(\mathfrak{M}),$$

which is induced by the natural inclusion of the second argument, is an $\boldsymbol{\mathcal{G}}_K$-isomorphism.

For $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$, the natural map

$$\mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}}^{\mathrm{ur}}) \hookrightarrow \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) =: \underline{T}_{\mathfrak{S}}^*(\mathfrak{M}),$$

which is induced by the natural inclusion of the second argument, is a $\boldsymbol{\mathcal{G}}_K$-isomorphism.

Proof. The statement for $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ follows from the statement for $\mathfrak{M}/\pi_0\mathfrak{M}$, thanks to Lemma 5.1.9. Therefore it is enough to prove the lemma for $\mathfrak{M} \in$

(Mod$/\mathfrak{S}$). If $\mathfrak{o}_0 := \mathbb{Z}_p$, then this follows from [31, §B. Propositions 1.8.3]. We give a proof when $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, so $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}} \cong K^{\mathrm{sep}}[[\pi_0]]$ and $\widehat{\mathfrak{S}}^{\mathrm{ur}} \cong \mathfrak{o}_{K^{\mathrm{sep}}}[[\pi_0]]$.

For any $f \in \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}})$, the image $f(\mathfrak{M})$ is finitely generated over $\mathfrak{S}$ and is stable under $\sigma : \mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}} \to \mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$. Now, consider an element $\alpha := \sum_{i=1}^{n} a_i \pi_0^{-i} \in \mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$ where $a_i \in K^{\mathrm{sep}}$. If not all of a_i are in $\mathfrak{o}_{K^{\mathrm{sep}}}$, then the $\mathfrak{S}$-span of $\{\sigma^j(\alpha)\}_{j\geq 0}$ cannot be finitely generated over $\mathfrak{S}$. Therefore, in order to have $\alpha \in f(\mathfrak{M})$, all a_i must lie in $\mathfrak{o}_{K^{\mathrm{sep}}}$. This shows that $\alpha \in \mathfrak{S}^{\mathrm{ur}}[\frac{1}{\pi_0}]/\mathfrak{S}^{\mathrm{ur}}$. □

Now we can make the following definition:

Definition 8.1.7. Let $M \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$. By a $\mathfrak{S}$-*lattice of* $\mathcal{P}$-*height* $\leqslant h$ in M we mean a φ-stable $\mathfrak{S}$-submodule $\mathfrak{M} \subset M$ such that $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ and $\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} \xrightarrow{\sim} M$.

We say that $T \in \mathrm{Rep}^{\mathrm{tor}}_{\mathfrak{o}_0}(\mathcal{G}_K)$ is *of* $\mathcal{P}$-*height* $\leqslant h$ if there exists $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ of $\mathcal{P}$-height $\leqslant h$ such that $T \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$, or equivalently, if $\underline{D}^*_{\mathcal{E}}(T)$ admits a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$. We say that $T \in \mathrm{Rep}^{\mathrm{tor}}_{\mathfrak{o}_0}(\mathcal{G}_K)$ is *of finite* $\mathcal{P}$-*height* if for some $r, h \in \mathbb{Z}$, the Tate twist $T(r)$ is of $\mathcal{P}$-height $\leqslant h$. We let $\mathrm{Rep}^{\mathrm{tor},\mathcal{P}}_{\mathfrak{o}_0}(\mathcal{G}_K)$ and $\mathrm{Rep}^{\mathrm{tor},\leqslant h}_{\mathfrak{o}_0}(\mathcal{G}_K)$ denote the categories of torsion representations of finite $\mathcal{P}$-height and of $\mathcal{P}$-height $\leqslant h$, respectively.

By Proposition 8.1.4, a torsion $\mathcal{G}_K$-representation T is of finite $\mathcal{P}$-height if and only if T is isomorphic to the cokernel of some isogeny $T_1 \hookrightarrow T_0$ of $\mathfrak{o}_0$-lattice $\mathcal{G}_K$-representations of finite $\mathcal{P}$-height, and T is of $\mathcal{P}$-height $\leqslant h$ if and only if one can find such T_0 and T_1 which are of $\mathcal{P}$-height $\leqslant h$.

Unlike the case of free étale φ-modules (*c.f.* Theorem 5.2.3), $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ in $M \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$ do not have to be unique. See §9.3 for more discussion. We will see later that if for $T \in \mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\mathcal{G}_K)$, $T/\pi_0^n T$ is of $\mathcal{P}$-height $\leqslant h$

for all $n \geq 1$, then $\underline{D}^*(T)$ has a (necessarily unique) $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$, so T is of $\mathcal{P}$-height $\leqslant h$ in the sense of Definition 5.2.8. (The converse is trivial.) This is not entirely trivial since the $\mathfrak{S}$-lattice in Definition 8.1.7 (applied to $\underline{D}^*(T/\pi_0^n T)$) is not unique, and this is proved in Proposition 9.2.6.

Consider $M \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$. In order for a φ-stable $\mathfrak{S}$-submodule $\mathfrak{M} \subset M$ to be a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$, $\mathfrak{M}$ has to be of projective dimension $\leqslant 1$ as a $\mathfrak{S}$-module, in addition to the condition $\mathcal{P}(u)^h \operatorname{coker}(\varphi|_{\mathfrak{M}}) = 0$. But in fact, the projective dimension condition is satisfied thanks to Proposition 8.1.2; because $\mathfrak{M}$ is a submodule of M which has no nontrivial u-torsion (so the same is true for $\mathfrak{M}$). So we obtain the following lemma.

Lemma 8.1.8. *Let M be a finitely generated torsion $\mathfrak{o}_{\mathcal{E}}$-module and $\mathfrak{M} \subset M$ be a finitely generated $\mathfrak{S}$-submodule. Then $\mathfrak{M}$ is of projective dimension $\leqslant 1$ as a $\mathfrak{S}$-module.*

Remark 8.1.9. A striking result is that once we formulate a deformation problem for $\boldsymbol{\mathcal{G}}_K$-representations of $\mathcal{P}$-height $\leqslant h$, the tangent space of the deformation functor is *finite-dimensional* if k is finite. This allows us to prove the existence of the universal deformation ring for $\boldsymbol{\mathcal{G}}_K$-representations of $\mathcal{P}$-height $\leqslant h$ (Theorem 11.1.2), similar to the classical theorem by Mazur for absolute Galois groups for a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$ [63, 64]. Note that without the $\mathcal{P}$-height $\leqslant h$ condition, the deformation functor has an infinite-dimensional tangent space even when k is finite, so there is no (complete noetherian) universal deformation ring. See §11.7.1.

Let $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. We saw in §7.3 that there exists a "Dieudonné-type" anti-equivalence of categories between $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ and certain kinds of π_0-divisible groups over $\mathfrak{o}_K$. Under this anti-equivalence, the functor $\underline{T}^*_{\mathfrak{S}}$ was interpreted as associating the "Tate module". (See Theorem 7.3.2 and Proposition 7.3.4.) For this reason, the representations of finite $\mathcal{P}$-height can be viewed as an equi-characteristic analogue of crystalline representation.

We digress to study the case of $\mathcal{P}$-heights $\leqslant 0$.

Proposition 8.1.10. *Any $T \in \mathrm{Rep}^{\mathrm{tor}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ is unramified if and only if there exists a π_0^∞-torsion étale $(\varphi, \mathfrak{S})$-module $\mathfrak{M}$ of projective dimension $\leqslant 1$ such that $T \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ as $\boldsymbol{\mathcal{G}}_K$-representations. In particular, any* unramified *$\mathfrak{o}_0$-torsion $\boldsymbol{\mathcal{G}}_K$-representation is of $\mathcal{P}$-height $\leqslant h$ for any $h \geqslant 0$.*

Proof. We first show that for any π_0^∞-torsion étale $(\varphi, \mathfrak{S})$-module $\mathfrak{M}$, $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ is unramified. Choose a "minimal" finite free $\mathfrak{S}$-module $\mathfrak{M}_0$ equipped with an $\mathfrak{S}$-linear surjection $\mathfrak{M}_0 \twoheadrightarrow \mathfrak{M}$ (i.e., the surjection induces a k-isomorphism $\mathfrak{M}_0/\mathfrak{m}_{\mathfrak{S}}\mathfrak{M}_0 \xrightarrow{\sim} \mathfrak{M}/\mathfrak{m}_{\mathfrak{S}}\mathfrak{M}$). Since $\mathfrak{M}$ is of projective dimension $\leqslant 1$, $\mathfrak{M}_1 := \ker[\mathfrak{M}_0 \twoheadrightarrow \mathfrak{M}]$ is also finite free over $\mathfrak{S}$. Choose any lift $\varphi_0 : \sigma^*\mathfrak{M}_0 \to \mathfrak{M}_0$ of $\varphi : \sigma^*\mathfrak{M} \to \mathfrak{M}$, and by Nakayama's lemma φ_0 is an isomorphism. This makes $\mathfrak{M}_1$ into an étale $(\varphi, \mathfrak{S})$-module. By Lemma 5.1.9 we have $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}_1)/\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}_0)$, and the right side is unramified by Proposition 5.2.10.

Now, assume that $T \in \mathrm{Rep}^{\mathrm{tor}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ is unramified and we seek an étale $\mathfrak{S}$-lattice in the étale φ-module $\underline{D}^*_{\mathcal{E}}(T) := \mathrm{Hom}_{\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K]}(T, \mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}})$. The idea of the proof is similar to the case when T is an unramified $\mathfrak{o}_0$-lattice $\boldsymbol{\mathcal{G}}_K$-representation (Proposition 5.2.10). Since I_K acts trivially on T, any $\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K]$-map $l : T \to \mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$ factors through $(\mathcal{E}^{\mathrm{ur}}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}})^{I_K} \cong \mathfrak{o}_{\mathcal{E}} \otimes_W (\mathrm{Frac}\, W^{sh}/W^{sh})$, where W^{sh} denotes the strict henselization of W. (Recall that $W = W(k)$ if $\mathfrak{o}_0 = \mathbb{Z}_p$, and $W = k[[\pi_0]]$ if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$.) So we have a natural isomorphism of φ-modules:

$$\underline{D}^*_{\mathcal{E}}(T) \cong \mathfrak{o}_{\mathcal{E}} \otimes_W \underline{U}^*(T), \tag{8.1.10.1}$$

where $\underline{U}^*(T) := \mathrm{Hom}_{\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K]}(T, \mathrm{Frac}\, W^{sh}/W^{sh})$ equipped with the φ-structure induced from the natural Frobenius endomorphism $\sigma : \operatorname{Frac} W^{sh}/W^{sh} \to \operatorname{Frac} W^{sh}/W^{sh}$.[2] Since $\mathfrak{o}_{\mathcal{E}}$ is faithfully flat over W, we can deduce from (8.1.10.1) that $\underline{U}^*(T)$ is (finitely generated over W and) a π_0^∞-torsion étale (φ, W)-module. So $\mathfrak{M} := \mathfrak{S} \otimes_W \underline{U}^*(T)$ is a π_0^∞-torsion étale $(\varphi, \mathfrak{S})$-module, and we have $T \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ by construction. □

We record the following corollary of the proof, which will be used later in the proof of Proposition 11.4.2. Let us define an $\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K/I_K]$-module

$$\underline{T}_W(U) := (W^{sh} \otimes_W U)^{\varphi=1} \text{ and } \underline{T}^*_W(U) := \operatorname{Hom}_{\mathfrak{o}_0,\varphi}(U, \operatorname{Frac} W^{sh}/W^{sh})$$

for any finite torsion étale (φ, W)-module U; and (φ, W)-modules

$$\underline{U}(T) := (W^{sh} \otimes_W T)^{\boldsymbol{\mathcal{G}}_K} \text{ and } \underline{U}^*(T) := \operatorname{Hom}_{\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K/I_K]}(T, \operatorname{Frac} W^{sh}/W^{sh})$$

for any unramified π_0^∞-torsion $\boldsymbol{\mathcal{G}}_K$-representation.

Corollary 8.1.11. *The assignments $\underline{T}_W$ and $\underline{U}$ define quasi-inverse length-preserving exact equivalences of categories between $\operatorname{Rep}^{\mathrm{tor}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K/I_K)$ and the category of finite torsion étale (φ, W)-modules which respects $\otimes$-products, internal homs, and duality. Furthermore, we have a natural isomorphism $\underline{D}_{\mathcal{E}}(T) \cong \mathfrak{o}_{\mathcal{E}} \otimes_W \underline{U}(T)$ of étale $(\varphi, \mathfrak{o}_{\mathcal{E}})$-modules for any $T \in \operatorname{Rep}^{\mathrm{tor}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K/I_K)$ and a natural $\boldsymbol{\mathcal{G}}_K$-equivariant isomorphism $\underline{T}_W(U^*) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{S} \otimes_W U)$ for any finite torsion étale (φ, W)-module U, where $U^* := \operatorname{Hom}_W(U, \mathscr{K}_0/W)$ is the dual torsion étale (φ, W)-module.*

8.1.12

We now show that the notion of $\mathcal{P}$-height $\leqslant h$ for $\mathfrak{o}_0$-torsion $\boldsymbol{\mathcal{G}}_K$-representations is insensitive to unramified extension of K (i.e., it only depends on the action of I_K). We first set up some notations. Let $\widehat{K}^{\mathrm{ur}} \cong k^{\mathrm{sep}}(\!(u)\!)$ denote the completed maximal unramified extention of K. For any complete "unramified" extension $K' := k'(\!(u)\!)$

[2]The Frobenius endomorphism $\sigma : \operatorname{Frac} W^{sh}/W^{sh} \to \operatorname{Frac} W^{sh}/W^{sh}$ can be obtained by restricting $\sigma : \mathcal{E}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}} \to \mathcal{E}/\mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}}$. Equivalently, one can obtain σ from the universal property of strict henselization.

of K (with k' perfect if $\mathfrak{o}_0 = \mathbb{Z}_p$), we let $\mathfrak{S}_{K'}$ and $\mathfrak{o}_{\mathcal{E}_{K'}}$ denote rings defined in a similar way to $\mathfrak{S}$ and $\mathfrak{o}_{\mathcal{E}}$ with K and k replaced with K' and k'. We also define endomorphisms $\sigma : \mathfrak{S}_{K'} \to \mathfrak{S}_{K'}$ and $\sigma : \mathfrak{o}_{\mathcal{E}_{K'}} \to \mathfrak{o}_{\mathcal{E}_{K'}}$ in a similar way we defined σ on $\mathfrak{S}$ and $\mathfrak{o}_{\mathcal{E}}$. So $(\mathfrak{S}_{K'}, \sigma)$ and $(\mathfrak{o}_{\mathcal{E}_{K'}}, \sigma)$ become σ-rings over $(\mathfrak{S}, \sigma)$.

In the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, we do not necessarily assume that K' has a finite p-basis, since we want to allow $K' = \widehat{K}^{\mathrm{ur}}$ and this does not have a finite p-basis unless k is perfect. Note that the theory of étale φ-modules (as discussed in §5.1 does not use the assumption of having a finite p-basis, and the definitions of $(\mathrm{Mod}/\mathfrak{S}_{K'})^{\leqslant h}$ and $\underline{T}^*_{\mathfrak{S}_{K'}}$ make sense as defined without assuming that K' has a finite p-basis. We say a $\mathfrak{o}_0$-torsion representation T of $I_K \cong \boldsymbol{\mathcal{G}}_{\widehat{K}^{\mathrm{ur}}}$ is *of $\mathcal{P}$-height $\leqslant h$* if there exists $\mathfrak{M}_{\widehat{K}^{\mathrm{ur}}} \in (\mathrm{Mod}/\mathfrak{S}_{\widehat{K}^{\mathrm{ur}}})^{\leqslant h}$ such that $T \cong \underline{T}^*_{\mathfrak{S}\widehat{K}^{\mathrm{ur}}}(\mathfrak{M}_{\widehat{K}^{\mathrm{ur}}})$ as $\boldsymbol{\mathcal{G}}_{\widehat{K}^{\mathrm{ur}}}$-representations.

Proposition 8.1.13. *An $\mathfrak{o}_0$-torsion $\boldsymbol{\mathcal{G}}_K$-representation T is of $\mathcal{P}$-height $\leqslant h$ in the sense of Definition 8.1.7 if and only if its restriction to I_K is of $\mathcal{P}$-height $\leqslant h$ in the above sense.*

Proof. The "only if" direction is trivial; if $T \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ as $\boldsymbol{\mathcal{G}}_K$-representations for some $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$, then we have a natural isomorphism $T \cong \underline{T}^*_{\mathfrak{S}_{\widehat{K}^{\mathrm{ur}}}}(\mathfrak{S}_{\widehat{K}^{\mathrm{ur}}} \otimes_{\mathfrak{S}} \mathfrak{M})$ as I_K-representations and clearly $\mathfrak{S}_{\widehat{K}^{\mathrm{ur}}} \otimes_{\mathfrak{S}} \mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S}_{\widehat{K}^{\mathrm{ur}}})^{\leqslant h}$.

To show the "if" direction, we assume that $T \in \mathrm{Rep}^{\mathrm{tor}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ is isomorphic to $\underline{T}^*_{\mathfrak{S}_{\widehat{K}^{\mathrm{ur}}}}(\mathfrak{M}_{\widehat{K}^{\mathrm{ur}}})$ as an I_K-representation for some $\mathfrak{M}_{\widehat{K}^{\mathrm{ur}}} \in (\mathrm{Mod}/\mathfrak{S}_{\widehat{K}^{\mathrm{ur}}})^{\leqslant h}$. Let $M := \underline{D}^*_{\mathcal{E}}(T)$ denote the étale $(\varphi, \mathfrak{o}_{\mathcal{E}})$-module corresponding to T, and we have a natural isomorphism $(\mathfrak{M}_{\widehat{K}^{\mathrm{ur}}})[\frac{1}{u}] \cong \mathfrak{o}_{\mathcal{E}_{\widehat{K}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} M$ of étale φ-modules. Let $\mathfrak{M} := M \cap \mathfrak{M}_{\widehat{K}^{\mathrm{ur}}}$, where the intersection is taken inside $\mathfrak{o}_{\mathcal{E}_{\widehat{K}^{\mathrm{ur}}}} \otimes_{\mathfrak{o}_{\mathcal{E}}} M$. Since $\mathfrak{M}$ is a $\mathfrak{S}$-submodule of $M = \mathfrak{M}[\frac{1}{u}]$ and has no non-zero infinitely u-divisible element, $\mathfrak{M}$ is finitely generated over $\mathfrak{S}$. Clearly, we have $\mathfrak{M}[\frac{1}{u}] = M$ and $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$. $\square$

The following theorem serves as a motivation for introducing torsion φ-modules and $\boldsymbol{\mathcal{G}}_K$-representations of finite $\mathcal{P}$-height.

Theorem 8.1.14 (Kisin [52])**.** *Let $\mathfrak{o}_0 = \mathbb{Z}_p$, and follows the notations as introduced in §1.3.1.2.*

1. *Let T be a torsion $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representation which can be obtained as a cokernel of an isogeny of Galois-stable lattices in semi-stable representations with Hodge-Tate weights in $[0,h]$. Then T as a representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty} \cong \boldsymbol{\mathcal{G}}_K$ is of $\mathcal{P}$-height $\leqslant h$.*

2. *(Breuil-Kisin classification of finite flat group schemes) If $p > 2$, then there exists an anti-equivalence of categories $\underline{G}^*$ from $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant 1}$ to the category of finite flat group schemes of p-power order over $\mathfrak{o}_{\mathscr{K}}$. Furthermore, for $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant 1}$ we have a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-equivariant isomorphism $\underline{G}^*(\mathfrak{M})(\overline{\mathscr{K}}) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$.*

3. *[15, Theorem 3.4.3] If $p > 2$, then "restricting the $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-action to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$" defines an equivalence of categories from the category of finite flat $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations (i.e., torsion $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations which are obtained from the generic fibers of finite flat group schemes over $\mathfrak{o}_{\mathscr{K}}$) to the category of torsion $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representations of $\mathcal{P}$-height $\leqslant 1$.*

Proof. The claim (1) follows from Proposition 8.1.4 and the fact that any Galois stable lattice in a semi-stable representation with Hodge-Tate weights in $[0,h]$ is automatically of $\mathcal{P}$-height $\leqslant h$ as a representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty} \cong \boldsymbol{\mathcal{G}}_K$ (Proposition 2.4.9 and Theorem 2.4.10).

The claim (2) follows from Proposition 8.1.4, Kisin's classification of Barsotti-Tate group (Theorem 2.4.11(1)), and Raynaud's theorem [6, Theorem 3.1.1] which asserts that any finite flat group scheme over $\mathfrak{o}_{\mathscr{K}}$ can be written as the kernel of an isogeny of Barsotti-Tate groups over $\mathfrak{o}_{\mathscr{K}}$.

The essential surjectivity of the claim (3) follows from the second statement of the theorem. We sketch the proof of the full faithfulness, which can be found in [15, Theorem 3.4.3]. Let T_1, T_2 be finite flat $\mathcal{G}_{\mathscr{K}}$-representations and let $f : T_1 \to T_2$ be a $\mathcal{G}_{\mathscr{K}_\infty}$-equivariant map. Taking the anti-equivalence of categories $\underline{D}^*_{\mathcal{E}}$, we obtain a map $\gamma : M_2 \to M_1$ of torsion étale φ-modules, and we can find some $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant 1$, say $\mathfrak{M}_i \subset M_i$, such that γ takes $\mathfrak{M}_2$ into $\mathfrak{M}_1$. (Compare with §9.2.3.) By the claim (2) of the theorem, γ corresponds to a map of finite flat group scheme models for T_1 and T_2, so f is $\mathcal{G}_{\mathscr{K}}$-equivariant. □

Theorem 8.1.14(2) was originally conjectured by Breuil in [11] for *all* primes p including $p = 2$, and he proved the special case when $p > 2$ and the finite flat group schemes killed by p. The case $p > 2$ (i.e., Theorem 8.1.14(2)) was proved by Kisin [52, (2.3)]. For $p = 2$, Kisin [53] proved the classification of *connected* finite flat group schemes using his classification of connected Barsotti-Tate groups. (Under the *contravariant* correspondences, the connectedness of finite flat group schemes corresponds to the condition that $\varphi_{\mathfrak{M}}$ is "topologically nilpotent.")

Remark 8.1.15. For the case $\mathfrak{o}_0 = \mathbb{Z}_p$, one can think of $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ as a "higher-weight analogue" of finite flat group schemes. Torsion étale φ-modules can be thought of as an "analogue" of the generic fibers, and $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ plays a role analogous to finite flat group scheme models or prolongations of a generic fiber. This point of view is supported by the Breuil-Kisin classification of finite flat group schemes for the case $h = 1$. On the other hand, torsion $(\varphi, \mathfrak{S})$-modules of finite $\mathcal{P}$-height only give rise to $\mathcal{G}_{\mathscr{K}_\infty}$-representations, and for $h > 1$ it seems to be hard to handle the gap between torsion semi-stable (or crystalline) $\mathcal{G}_{\mathscr{K}}$-representations and their restrictions to $\mathcal{G}_{\mathscr{K}_\infty}$.

8.1.16 φ-nilpotent objects

A torsion φ-module $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ is called *φ-nilpotent* if $\varphi^r(\sigma^*\mathfrak{M}) \subset \mathfrak{m}_{\mathfrak{S}} \cdot \mathfrak{M}$ for all sufficiently large r, or equivalently, if for any $x \in \mathfrak{M}$ the sequence $\varphi^r(\sigma^{*r}x)$ converges to 0 for the $\mathfrak{m}_{\mathfrak{S}}$-adic topology as $r \to \infty$. Note that this is the same as the u-adic topology on $\mathfrak{M}$.

The notion of φ-nilpotentness for such $\mathfrak{M}$ is "well-behaved" under subobjects, quotients, extensions, direct sums and tensor products. More precisely, for a short exact sequence $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ of torsion φ-modules, if two of them are φ-nilpotent then so is the third. If torsion φ-modules $\mathfrak{M}$ and $\mathfrak{M}'$ are φ-nilpotent, then so are their tensor product $\mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{M}'$ and direct sum $\mathfrak{M} \oplus \mathfrak{M}'$.

8.1.17 Analogue of connected-étale sequence

For $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ we define the *maximal étale submodule* $\mathfrak{M}^{\text{ét}} \subset \mathfrak{M}$ as follows:

$$\mathfrak{M}^{\text{ét}} := \bigcap_{r=1}^{\infty} \varphi^r(\sigma^{*r}\mathfrak{M}). \tag{8.1.17.1}$$

By the theorem of Auslander-Buchsbaum, $\mathfrak{M}^{\text{ét}}$ is of projective dimension $\leqslant 1$ as a $\mathfrak{S}$-module; we have seen that $\mathfrak{M}$ has no non-trivial u-torsion, so the same is true for $\mathfrak{M}^{\text{ét}}$ (using Lemma 8.1.3). Therefore, $\mathfrak{M}^{\text{ét}} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$. Clearly, $\mathfrak{M}^{\text{ét}}$ is an étale φ-module which contains all étale submodules of $\mathfrak{M}$, and any φ-compatible map $f : \mathfrak{M} \to \mathfrak{N}$ in $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ takes $\mathfrak{M}^{\text{ét}}$ into $\mathfrak{N}^{\text{ét}}$.

We now show that the quotient $\mathfrak{M}/\mathfrak{M}^{\text{ét}}$ also lies in $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$. Then we can say that $\mathfrak{M}/\mathfrak{M}^{\text{ét}}$ is a maximal φ-nilpotent quotient of $\mathfrak{M}$. The issue is to show that $\mathfrak{M}/\mathfrak{M}^{\text{ét}}$ is of projective dimension $\leqslant 1$ as a $\mathfrak{S}$-module. By the theorem of Auslander-Buchsbaum, it is enough to show that $\mathfrak{M}/\mathfrak{M}^{\text{ét}}$ has no nonzero u-torsion.

Since $\varphi : \sigma^*\mathfrak{M}^{\text{ét}} \to \mathfrak{M}^{\text{ét}}$ is an isomorphism, we obtain a surjective map $\bar{\varphi}$: $\sigma^*(\mathfrak{M}^{\text{ét}}/u\mathfrak{M}^{\text{ét}}) \to \mathfrak{M}^{\text{ét}}/u\mathfrak{M}^{\text{ét}}$ between modules of the same finite length, hence $\bar{\varphi}$ is an isomorphism. In particular, $x \in u\cdot\mathfrak{M}^{\text{ét}}$ if and only if $\varphi^r(\sigma^{*r}x) \in u\cdot\mathfrak{M}^{\text{ét}}$.

Now, let $y \in \mathfrak{M}$ be such that $uy \in \mathfrak{M}^{\text{ét}}$. Since the sequence $\varphi^r(\sigma^{*r}(uy))$ converges to 0 in $\mathfrak{M}$ as $r \to \infty$, the same is true in $\mathfrak{M}^{\text{ét}}$ by the Artin-Rees lemma. So there exists an r such that $\varphi^r(\sigma^{*r}(uy))$ is a u-multiple of some element in $\mathfrak{M}^{\text{ét}}$, hence $y \in \mathfrak{M}^{\text{ét}}$. This shows that $\mathfrak{M}/\mathfrak{M}^{\text{ét}} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$.

Let us summarize what we have proved:

Proposition 8.1.18. *For any* $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$*, we have a short exact sequence in* $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$

$$0 \to \mathfrak{M}^{\text{ét}} \to \mathfrak{M} \to \mathfrak{M}/\mathfrak{M}^{\text{ét}} \to 0, \tag{8.1.18.1}$$

where $\mathfrak{M}^{et}$ *is maximal among étale submodules of* $\mathfrak{M}$*, and* $\mathfrak{M}/\mathfrak{M}^{\text{ét}}$ *is maximal among* φ*-nilpotent quotients of* $\mathfrak{M}$*. The sequence* (8.1.18.1) *is functorial in* $\mathfrak{M}$ *in the sense that any map* $f : \mathfrak{M} \to \mathfrak{N}$ *in* $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ *takes* $\mathfrak{M}^{\text{ét}}$ *into* $\mathfrak{N}^{\text{ét}}$ *(hence induces* $\mathfrak{M}/\mathfrak{M}^{\text{ét}} \to \mathfrak{N}/\mathfrak{N}^{\text{ét}}$*). We call this exact sequence* connected-étale sequence *for* $\mathfrak{M}$.

We record the following facts.

1. Clearly, $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ is étale if and only if $\mathfrak{M}^{\text{ét}} = \mathfrak{M}$, and $\mathfrak{M}$ is φ-nilpotent if and only if $\mathfrak{M}^{\text{ét}} = 0$.

2. If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then the connected-étale sequence for $\mathfrak{M}$ exactly corresponds to the connected-étale sequence for $\underline{G}^*(\mathfrak{M})$.

3. If $\mathfrak{o}_0 = \mathbb{Z}_p$ and $h = 1$, then we have the anti-equivalence of categories $\underline{G}^*$ from $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant 1}$ to the category of finite flat group schemes of p-power order over $\mathfrak{o}_{\mathscr{K}}$, by the Breuil-Kisin classification (Theorem 8.1.14(2)). Under this correspondence, $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant 1}$ is étale if and only if $\underline{G}^*(\mathfrak{M})$ is étale, and $\mathfrak{M}$ is φ-nilpotent if and only if $\underline{G}^*(\mathfrak{M})$ is connected. Furthermore, for any $\mathfrak{M} \in$

$(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant 1}$, one obtains the connected-étale sequence for $\underline{G}^*(\mathfrak{M})$ by applying $\underline{G}^*$ to the connected-étale sequence for $\mathfrak{M}$ (which justifies the name).

8.2 φ-modules with coefficients

For an $\mathfrak{o}_0$-algebra A, we introduce a class of φ-modules "with A-coefficients," which will play an important role in deformation theory in §XI. Whenever possible we avoid restricting our choice of A to complete local noetherian $\mathfrak{o}_0$-algebras, since they actually occur in the arguments.

8.2.1

Let A be a continuous $\mathfrak{a}$-adic $\mathfrak{o}_0$-algebra (i.e., $\mathfrak{a} \subset A$ contains some power of π_0 and $A \cong \varprojlim_n A/\mathfrak{a}^n$). Two main examples which arise later are discrete $\mathfrak{o}_0$-algebras where π_0 is nilpotent and complete noetherian local $\mathfrak{o}_0$-algebras (in which case $\mathfrak{a} = \mathfrak{m}_A$). We often do not specify $\mathfrak{a}$ if there is no risk of confusion.

Let $(A, \mathfrak{a})$ be as above. For any $\mathfrak{o}_0$-algebra R, we set $R_A := \varprojlim_n (A/\mathfrak{a}^n \otimes_{\mathfrak{o}_0} R)$. If A is a discrete $\mathfrak{o}_0$-algebra where π_0 is nilpotent, then $R_A = A \otimes_{\mathfrak{o}_0} R$. If A is a complete noetherian local $\mathfrak{o}_0$-algebra, then $R_A = \varprojlim_n (A/\mathfrak{m}_A^n \otimes_{\mathfrak{o}_0} R)$. For any σ-ring (R, σ_R) over $(\mathfrak{o}_0, \mathrm{id})$, we A-linearly extend σ_R to R_A. In particular, if σ_R is finite flat, then so is σ_{R_A}. This is the case when $R = \mathfrak{S}$ and $R = \mathfrak{o}_{\mathcal{E}}$. (In the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ we use the assumption that the residue field k of $\mathfrak{o}_K$ has a finite p-basis.)

We let $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ denote the category of φ-modules of $\mathcal{P}$-height $\leqslant h$ which are finite locally free[3] over $\mathfrak{S}_A$. Similarly, we let $(\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$ denote the category of étale φ-modules which are finite locally free over $\mathfrak{o}_{\mathcal{E},A}$. If $A = \mathfrak{o}_0$ then $(\mathrm{ModFI}\,/\mathfrak{S})_{\mathfrak{o}_0}^{\leqslant h}$ is just $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$ and $(\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_{\mathfrak{o}_0}^{\text{ét}}$ is just $\underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét,free}}(\varphi)$, because $\mathfrak{S}_{\mathfrak{o}_0} \cong \mathfrak{S}$ and $\mathfrak{o}_{\mathcal{E},\mathfrak{o}_0} \cong \mathfrak{o}_{\mathcal{E}}$. If $\#(A) < \infty$ (i.e., if A is a finite artinian $\mathfrak{o}_0$-algebra), then

[3] A locally free module is always assumed to be of constant rank.

an object of $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ can be regarded, by forgetting A-action, as an object of $(\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}$. But coefficient rings A that are not artinian do appear in the later arguments (see §11.1.5).

Let $(A, \mathfrak{a})$ and $(B, \mathfrak{b})$ be continuous adic $\mathfrak{o}_0$-algebras. Consider a continuous $\mathfrak{o}_0$-map $A \to B$. Then, for $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$, the "completed" scalar extension $B\widehat{\otimes}_A\mathfrak{M}_A := \varprojlim_n (B/\mathfrak{b}^n \otimes_A \mathfrak{M}_A) \cong \mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{M}_A$, together with the Frobenius structure defined by B-linearly extending $\varphi_{\mathfrak{M}_A}$, is an object of $(\mathrm{ModFI}\,/\mathfrak{S})_B^{\leqslant h}$. This defines the "change-of-coefficients" functors $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h} \to (\mathrm{ModFI}\,/\mathfrak{S})_B^{\leqslant h}$, and similarly one can define $(\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}} \to (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_B^{\text{ét}}$.

The following result can be obtained from Proposition 7.4.2: for a continuous $\mathfrak{a}$-adic $\mathfrak{o}_0$-algebra A, the functors $\mathfrak{M}_A \mapsto \{A/\mathfrak{a}^n \otimes_A \mathfrak{M}_A\}_n$ and $\{\mathfrak{M}_n\}_n \mapsto \varprojlim_n \mathfrak{M}_n$ are quasi-inverse equivalences of categories between $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ and the category of projective systems $\{\mathfrak{M}_n\}_n$ such that $\mathfrak{M}_n \in (\mathrm{ModFI}\,/\mathfrak{S})_{A/\mathfrak{a}^n}^{\leqslant h}$ and $A/\mathfrak{a}^n \otimes_{A/\mathfrak{a}^{n+1}} \mathfrak{M}_{n+1} \xrightarrow{\sim} \mathfrak{M}_n$ for each n. We often apply this result when $(A, \mathfrak{a}) = (A, \mathfrak{m}_A)$ is a complete noetherian local $\mathfrak{o}_0$-algebra.

The scalar extension $\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} (\cdot) \cong \mathfrak{o}_{\mathcal{E},A} \otimes_{\mathfrak{S}_A} (\cdot)$ induces the functor $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h} \to (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$. We can immediately see that for any $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$, the Frobenius map $\varphi_{\mathfrak{M}_A}$ is *injective*, since we have a φ-compatible injective map $\mathfrak{M}_A \hookrightarrow \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_A$.

Definition 8.2.2. Let A be a continuous $\mathfrak{a}$-adic $\mathfrak{o}_0$-algebra, and consider $M \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$. By a *$\mathfrak{S}_A$-lattice of $\mathcal{P}$-height* $\leqslant h$ in M, we mean a φ-stable $\mathfrak{S}_A$-submodule $\mathfrak{M} \subset M$ such that $\mathfrak{M} \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ and $\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} = M$.

Beware that for $A = \mathbb{F}_q[\epsilon]/(\epsilon^2)$ there is an example of $M_A \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$ which does not admit any $\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$, whereas there exists a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ in M_A viewed as a torsion étale φ-module in the sense of Definition

8.1.7. See Remark 11.1.7 for the example.

Proposition 8.2.3. *Let A be a continuous $\mathfrak{a}$-adic $\mathfrak{o}_0$-algebra. For any object $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$, the cokernel of $\varphi_{\mathfrak{M}_A}$ is a flat A-module. If the residue field $k = \mathfrak{o}_K/(u)$ is finite then $\mathrm{coker}(\varphi_{\mathfrak{M}_A}) = \mathfrak{M}_A/\varphi(\sigma^*\mathfrak{M}_A)$ and $\varphi(\sigma^*\mathfrak{M}_A)/\mathcal{P}(u)^h\mathfrak{M}_A$ are finite projective A-modules.*

Proof. We showed that $\varphi_{\mathfrak{M}_A}$ is injective for any coefficient ring A. Therefore, the exact sequence

$$0 \to \sigma^*\mathfrak{M}_A \xrightarrow{\varphi_{\mathfrak{M}_A}} \mathfrak{M}_A \to \mathrm{coker}(\varphi_{\mathfrak{M}_A}) \to 0$$

stays short exact after applying $A/I \otimes_A (\cdot)$ for any ideal $I \subset A$. Hence, the first claim follows from standard facts about flatness (e.g. by [9, Ch.I §2.5 Prop 4], or by an argument using Tor_1^A.) If $k = \mathfrak{o}_K/(u)$ is finite then $\mathfrak{S}_A/\mathcal{P}(u)^h$ is finite free over A, so $\mathrm{coker}(\varphi_{\mathfrak{M}_A})$ is finite and projective over A, and hence the following short exact sequence of A-modules splits.

$$0 \to \varphi(\sigma^*\mathfrak{M}_A)/\mathcal{P}(u)^h\mathfrak{M}_A \to \mathfrak{M}_A/\mathcal{P}(u)^h\mathfrak{M}_A \to \mathrm{coker}(\varphi_{\mathfrak{M}_A}) \to 0.$$

□

8.2.4 Étale φ-modules with A-coefficients and $A[\mathcal{G}_K]$-modules

Assume $\#(A) < \infty$. For $T_A \in \mathrm{Rep}_A^{\mathrm{free}}(\mathcal{G}_K)$ (which can be viewed as a torsion $\mathcal{G}_K$-representation by forgetting the A-action), we let $\underline{D}_{\mathcal{E},A}(T_A)$ denote $\underline{D}_{\mathcal{E}}(T_A)$ viewed as an étale $(\varphi, \mathfrak{o}_{\mathcal{E},A})$-module. Similarly, for $M_A \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$ (which can be viewed as a torsion étale $(\varphi, \mathfrak{o}_{\mathcal{E}})$-module by forgetting the A-action), we write $\underline{T}_{\mathcal{E},A}(M_A)$ to denote $\underline{T}_{\mathcal{E}}(M_A)$ viewed as an $A[\mathcal{G}_K]$-module. From the definition it is clear that $\underline{D}_{\mathcal{E},A}$ and $\underline{T}_{\mathcal{E},A}$ are exact and commute with $\otimes$-products, internal homs, and duality. (Note that for $M_A \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$, we have a natural isomorphism $\mathrm{Hom}_{\mathfrak{o}_{\mathcal{E},A}}(M_A, \mathfrak{o}_{\mathcal{E},A}) \cong$

$\mathrm{Hom}_{\mathfrak{o}_{\mathcal{E}}}(M_A, \mathcal{E}/\mathfrak{o}_{\mathcal{E}})$ of étale $(\varphi, \mathfrak{o}_{\mathcal{E},A})$-modules, where the $\mathfrak{o}_{\mathcal{E},A}$-module of the right side is induced from M_A. A similar statement holds for $T_A \in \mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$.) Furthermore, one can directly check that $\underline{D}_{\mathcal{E},A}$ and $\underline{T}_{\mathcal{E},A}$ commutes with "change of coefficients" for any finite A-algebra B; i.e., for $T_A \in \mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$, we have a natural isomorphism $\underline{D}_{\mathcal{E},A}(T_A) \otimes_A B \xrightarrow{\sim} \underline{D}_{\mathcal{E},B}(T_A \otimes_A B)$, and similarly for $\underline{T}_{\mathcal{E},A}$.

We now show that $\underline{D}_{\mathcal{E},A}$ and $\underline{T}_{\mathcal{E},A}$ are quasi-inverse equivalence of categories between $\mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$ and $(\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$. The only non-trivial part is to show that $\underline{D}_{\mathcal{E},A}(T_A)$ is free over $\mathfrak{o}_{\mathcal{E},A}$ for $T_A \in \mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$, and $\underline{T}_{\mathcal{E},A}(M_A)$ is free over A for $M_A \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$. We will only prove $\mathfrak{o}_{\mathcal{E},A}$-freeness for $\underline{D}_{\mathcal{E},A}(T_A)$, and A-freeness of $\underline{T}_{\mathcal{E},A}(M_A)$ can be proved by essentially the same argument. It is enough to handle the case when A is local (with $\#(A) < \infty$). By applying the local flatness criterion to the free A-module T_A, one obtains an $\boldsymbol{\mathcal{G}}_K$-equivariant isomorphism $\mathrm{gr}^\bullet A \otimes_{A/\mathfrak{m}_A} (T_A/\mathfrak{m}_A T_A) \xrightarrow{\sim} \mathrm{gr}^\bullet T_A$, where we give $\mathfrak{m}_A$-adic filtrations on A and T_A. By applying $\underline{D}_{\mathcal{E}}$ to this isomorphism, we obtain the similar isomorphism for $\underline{D}_{\mathcal{E}}(T_A)$ with $\mathfrak{m}_A\mathfrak{o}_{\mathcal{E},A}$-adic filtration. Since $\mathfrak{o}_{\mathcal{E},A/\mathfrak{m}_A}$ is a product of fields, the local flatness criterion gives the $\mathfrak{o}_{\mathcal{E}.A}$-flatness of $\underline{D}_{\mathcal{E},A}(T_A)$.

We define the contravariant version of functors by composing with suitable duality; more precisely, $\underline{D}^*_{\mathcal{E},A}(-) := \mathrm{Hom}_{\mathfrak{o}_0[\boldsymbol{\mathcal{G}}_K]}(-, \mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}_A})$ and $\underline{T}^*_{\mathcal{E},A}(-) = \mathrm{Hom}_{\mathfrak{o}_{\mathcal{E},A},\varphi}(-, \mathfrak{o}_{\mathcal{E}^{\mathrm{ur}}_A})$. Clearly, $\underline{D}_{\mathcal{E},A}$ and $\underline{T}_{\mathcal{E},A}$ are exact quasi-inverse anti-equivalence of categories between $\mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$ and $(\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$, which commutes with $\otimes$-products, internal homs, duality, and "change of coefficients" for any finite A-algebra B. (Note that duality commutes with "change of coefficients.")

Let A be a complete local noetherian $\mathfrak{o}_0$-algebra with finite residue field (so that $\#(A/\mathfrak{m}_A^n) < \infty$ for each n). Using that $\underline{D}_{\mathcal{E},A/\mathfrak{m}_A^n}$ and $\underline{T}_{\mathcal{E},A/\mathfrak{m}_A^n}$ commute with "change of coefficients" for finite morphism, we define $\underline{D}_{\mathcal{E},A}(T_A) := \varprojlim_n \underline{D}_{\mathcal{E},A/\mathfrak{m}_A^n}(T_A \otimes_A A/\mathfrak{m}_A^n)$

and $\underline{T}_{\mathcal{E},A}(M_A) := \varprojlim_n \underline{T}_{\mathcal{E},A/\mathfrak{m}_A^n}(M_A \otimes_A A/\mathfrak{m}_A^n)$ for $T_A \in \mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$ and $M_A \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$, where the transition maps are induced from the natural projection $A/\mathfrak{m}_A^{n+1} \twoheadrightarrow A/\mathfrak{m}_A^n$; and we similarly define $\underline{D}^*_{\mathcal{E},A}$ and $\underline{T}^*_{\mathcal{E},A}$.[4] We similarly define $\underline{D}_{\mathcal{E},A}(T_A)$ and $\underline{T}_{\mathcal{E},A}(M_A)$.

By essentially the same "limit argument" as in the proofs of Lemma 5.1.4 and Proposition 5.1.7, we can show that $\underline{D}^*_{\mathcal{E},A}$ and $\underline{T}^*_{\mathcal{E},A}$ induce exact quasi-inverse equivalences of categories between $\mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$ and $(\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$ which commutes with $\otimes$-products, internal homs, duality, and "change of coefficients" for any A-algebra B with $\#(B/\mathfrak{m}_B) < \infty$; and a similar statement holds for $\underline{D}^*_{\mathcal{E},A}$ and $\underline{T}^*_{\mathcal{E},A}$. We leave the details to readers.

One can repeat the above discussion for $\underline{U}$ and $\underline{T}_W$ instead of $\underline{D}_{\mathcal{E}}$ and $\underline{T}_{\mathcal{E}}$ (using Corollary 8.1.11 instead of Proposition 5.1.7) and obtain quasi-inverse equivalences of categories between $\mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K/I_K)$ and the category of finite free étale $(\varphi, W\widehat{\otimes}_{\mathfrak{o}_0}A)$-modules (where A is a complete local noetherian $\mathfrak{o}_0$-algebra with finite residue field) which commutes with $\otimes$-products, internal homs, duality, and "change of coefficients" for any A-algebra B with $\#(B/\mathfrak{m}_B) < \infty$. We leave the details to readers.

8.2.5 φ-nilpotent objects

We generalized the notion of φ-nilpotent torsion φ-modules to $(\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ where A is as in §8.2.1. Namely, $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ is *φ-nilpotent* if for any sufficiently large integer N, the image $\varphi^N(\sigma^{N*}\mathfrak{M}_A)$ is contained in $\mathfrak{m}_{\mathfrak{S}}\cdot\mathfrak{M}_A$ (i.e., φ is topologically nilpotent for $\mathfrak{m}_{\mathfrak{S}}$-adic topology on $\mathfrak{M}_A$). If $\#(A) < \infty$, then $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ is φ-nilpotent if and only if $\mathfrak{M}_A$ is φ-nilpotent viewed as an object of $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ in the sense of §8.1.16.

[4] Alternatively, one may imitate the construction in §5.1, using $\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}},A} := \varprojlim_n (A/\mathfrak{m}_A^n \otimes_{\mathfrak{o}_0} \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})$.

Here we record some immediate formal properties.

1. For a short exact sequence $0 \to \mathfrak{M}'_A \to \mathfrak{M}_A \to \mathfrak{M}''_A \to 0$ in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$, if two of them are φ-nilpotent, then so is the third.

2. If $\mathfrak{M}_A, \mathfrak{M}'_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ are both φ-nilpotent, so are $\mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{M}'_A$ and $\mathfrak{M}_A \oplus \mathfrak{M}'_A$.

3. (*change of coefficients*) Let $(A, \mathfrak{a}) \to (B, \mathfrak{b})$ be a continuous map of adic $\mathfrak{o}_0$-algebras (where $\mathfrak{a}$ or $\mathfrak{b}$ can be trivial), and consider $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$. If $\mathfrak{M}_A$ is φ-nilpotent, then the "change of coefficients" $B\widehat{\otimes}_A\mathfrak{M}_A := \varprojlim_n (B/\mathfrak{b}^n \otimes_A \mathfrak{M}_A)$ is also φ-nilpotent. In particular, if A is complete local noetherian $\mathfrak{o}_0$-algebra, then $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ is φ-nilpotent if and only if $A/\mathfrak{m}_A^n \otimes_A \mathfrak{M}_A$ is φ-nilpotent for each n.

8.2.6 Analogue of connected-étale sequence

For $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ we have discussed the maximal étale submodule $\mathfrak{M}^{\text{ét}}$ and the maximal φ-nilpotent quotient $\mathfrak{M}^{\text{nilp}} := \mathfrak{M}/\mathfrak{M}^{\text{ét}}$ of $\mathfrak{M}$, which are in $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$. Now consider $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ for $\#(A) < \infty$. Viewing $\mathfrak{M}_A$ as an object in $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$, we obtain a short exact sequence

$$0 \to \mathfrak{M}_A^{\text{ét}} \to \mathfrak{M}_A \to \mathfrak{M}_A^{\text{nilp}} \to 0 \tag{8.2.6.1}$$

where $\mathfrak{M}_A^{\text{ét}}$ and $\mathfrak{M}_A^{\text{nilp}}$ are objects in $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$. By functoriality of connected-étale sequences (Proposition 8.1.18), the φ-compatible A-action on $\mathfrak{M}_A$ induces φ-compatible A-actions on $\mathfrak{M}_A^{\text{ét}}$ and $\mathfrak{M}_A^{\text{nilp}}$. We will show later in Proposition 8.2.7 that $\mathfrak{M}_A$ and $\mathfrak{M}_A^{\text{nilp}}$ are finite locally free $\mathfrak{S}_A$-modules, so they are objects in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$. (This is not *a priori* clear.)

Consider a finite A-algebra B (in particular, $\#(B) < \infty$). Let $\mathfrak{M}_B := \mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{M}_A$ for $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$. Let us grant that $\mathfrak{M}_A^{\text{ét}}$ and $\mathfrak{M}_A^{\text{nilp}}$ are finite locally free over

$\mathfrak{S}_A$, so objects in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$. By functoriality of connected-étale sequences in $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ (Proposition 8.1.18), we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{M}_A^{\text{ét}} & \longrightarrow & \mathfrak{M}_B & \longrightarrow & \mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{M}_A^{\text{nilp}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{M}_B^{\text{ét}} & \longrightarrow & \mathfrak{M}_B & \longrightarrow & \mathfrak{M}_B^{\text{nilp}} & \longrightarrow & 0. \end{array} \tag{8.2.6.2}$$

Note that $\mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{M}_A^{\text{ét}}$ and $\mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{M}_A^{\text{nilp}}$ are clearly étale and φ-nilpotent objects in $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$, respectively, since they have no non-zero u-torsion. By diagram chasing, the vertical arrow in the right end is surjective, but $\mathfrak{M}_B^{\text{nilp}}$ is the "biggest" quotient among φ-nilpotent quotients of $\mathfrak{M}_B$. Therefore, the natural map $\mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{M}_A^{\text{nilp}} \rightarrow \mathfrak{M}_B^{\text{nilp}}$ is an isomorphism, so the natural map $\mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{M}_A^{\text{ét}} \rightarrow \mathfrak{M}_B^{\text{ét}}$ is an isomorphism. This shows that the formation of connected-étale sequence (8.2.6.1) commutes with finite scalar extension.

Now, let A be a complete local noetherian $\mathfrak{o}_0$-algebra with finite residue field. Using the connected-étale sequence for $\mathfrak{M}_n := \mathfrak{M}_A \otimes_A A/\mathfrak{m}_A^n$ and the scalar extension $A/\mathfrak{m}_A^{n+1} \twoheadrightarrow A/\mathfrak{m}_A^n$ for $n \geq 1$, we obtain an exact sequence of $(\varphi, \mathfrak{S}_A)$-modules:

$$0 \rightarrow \mathfrak{M}_A^{\text{ét}} \rightarrow \mathfrak{M}_A \rightarrow \mathfrak{M}_A^{\text{nilp}} \rightarrow 0, \tag{8.2.6.3}$$

where $\mathfrak{M}_A^{\text{ét}} = \varprojlim_n (\mathfrak{M}_n)^{\text{ét}}$ and $\mathfrak{M}_A^{\text{nilp}} = \varprojlim_n (\mathfrak{M}_n)^{\text{nilp}}$. If $\#(A) < \infty$ then the exact sequence (8.2.6.3) recovers the connected-étale sequence for $\mathfrak{M}_A$ viewed as an object in $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ as in (8.2.6.1).

The next proposition shows that $\mathfrak{M}_A^{\text{ét}}$ and $\mathfrak{M}_A^{\text{nilp}}$ are finite locally free $\mathfrak{S}_A$-modules (so they are objects in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$) and satisfy various natural properties.

Proposition 8.2.7. *Let A be a complete local noetherian $\mathfrak{o}_0$-algebra with finite residue field. For $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$, $\mathfrak{M}_A^{\text{ét}}$ and $\mathfrak{M}_A/\mathfrak{M}_A^{\text{ét}}$ are finite locally free $\mathfrak{S}_A$-modules. Furthermore, the exact sequence* (8.2.6.3) *is functorial in $\mathfrak{M}_A$ and respects* *any scalar extension under any local (therefore continuous) map $A \to B$ of complete local noetherian $\mathfrak{o}_0$-algebras with finite residue fields (i.e., we have $B\widehat{\otimes}_A\mathfrak{M}_A^{\text{ét}} = (B\widehat{\otimes}_A\mathfrak{M}_A)^{\text{ét}}$ as a submodule of $B\widehat{\otimes}_A\mathfrak{M}_A$, where $B\widehat{\otimes}_A(-) := \varprojlim_n B/\mathfrak{m}_B^n \otimes_A (-)$ denotes the "completed" scalar extension).*

This proposition will later be generalized for some $\mathfrak{o}_0$-algebras A that are not complete local noetherian. See Proposition 11.4.2 for the precise statement.

Proof. By §8.2.6, the proposition follows if we show that $\mathfrak{M}_A^{\text{nilp}}$ is finite free over $\mathfrak{S}_A$ (in which case $\mathfrak{M}_A^{\text{ét}}$ is forced to be finite free over $\mathfrak{S}_A$). On the other hand, since $\mathfrak{S}_A \cong (W \otimes_{\mathfrak{o}_0} A)[[u]]$, Proposition 7.4.2 asserts that $\mathfrak{M}_A^{\text{nilp}}$ is finite free over $\mathfrak{S}_A$ if and only if $\mathfrak{M}_A^{\text{nilp}}$ has no nonzero u-torsion and $\mathfrak{M}_A^{\text{nilp}}/u\mathfrak{M}_A^{\text{nilp}}$ is finite free over $\mathfrak{S}_A/(u)$. But $\mathfrak{M}_A^{\text{nilp}} \in (\text{Mod}\,/\mathfrak{S})^{\leqslant h}$ implies that $\mathfrak{M}_A^{\text{nilp}}$ has no nonzero u-torsion by Proposition 8.1.2, so it suffices to show that $\mathfrak{M}_A^{\text{nilp}}/u\mathfrak{M}_A^{\text{nilp}}$ is finite free over $\mathfrak{S}_A/(u)$.

Consider $\overline{\mathfrak{M}}_A := \mathfrak{M}_A/u\mathfrak{M}_A$ viewed as a φ-module via $\overline{\varphi} := \varphi_{\mathfrak{M}_A} \bmod u$, and put $\overline{\mathfrak{M}}_A^{\text{ét}} := \bigcap_{r=1}^{\infty} \overline{\varphi}^r(\sigma^{*r}\overline{\mathfrak{M}}_A)$ and $\overline{\mathfrak{M}}_A^{\text{nilp}} := \overline{\mathfrak{M}}_A/\overline{\mathfrak{M}}_A^{\text{ét}}$. Clearly $\overline{\mathfrak{M}}_A^{\text{ét}}$ is an étale submodule of $\overline{\mathfrak{M}}_A$ which contains all étale subobjects of $\overline{\mathfrak{M}}_A$. We say a $(\varphi, \mathfrak{S}_A/(u))$-module $(\overline{\mathfrak{N}}_A, \overline{\varphi})$ is φ-*nilpotent* if $\overline{\varphi}^r$ is the zero map for any $r \gg 1$. Clearly $\overline{\mathfrak{M}}_A^{\text{nilp}}$ is φ-nilpotent and any φ-nilpotent quotient of $\overline{\mathfrak{M}}_A$ factors through $\overline{\mathfrak{M}}_A^{\text{nilp}}$.

In the proof of Proposition 8.1.18, we showed that $\mathfrak{M}_A^{\text{ét}}/u\mathfrak{M}_A^{\text{ét}} \subset \overline{\mathfrak{M}}_A^{\text{ét}}$, so we have a natural surjective map $\mathfrak{M}_A^{\text{nilp}}/u\mathfrak{M}_A^{\text{nilp}} \subset \overline{\mathfrak{M}}_A^{\text{nilp}}$. But since $\mathfrak{M}_A^{\text{nilp}}/u\mathfrak{M}_A^{\text{nilp}}$ is φ-nilpotent and $\overline{\mathfrak{M}}_A^{\text{nilp}}$ is "maximal" among φ-nilpotent quotients of $\overline{\mathfrak{M}}_A$, $\mathfrak{M}_A^{\text{nilp}}/u\mathfrak{M}_A^{\text{nilp}}$ and $\overline{\mathfrak{M}}_A^{\text{nilp}}$ are the same quotients of $\overline{\mathfrak{M}}_A$.

It is left to show that $\overline{\mathfrak{M}}_A^{\text{nilp}}$ is finite free over $\mathfrak{S}_A/(u)$. Let $\mathbb{F} := A/\mathfrak{m}_A$ and consider the $(\varphi, \mathfrak{S}_{\mathbb{F}}/(u))$-module $\overline{\mathfrak{M}}_A^{\text{nilp}} \otimes_A \mathbb{F}$. Since $\mathfrak{S}_{\mathbb{F}}/(u) \cong k \otimes_{\mathbb{F}_q} \mathbb{F}$ is a product of fields and $\sigma : \mathfrak{S}_{\mathbb{F}} \to \mathfrak{S}_{\mathbb{F}}$ permutes the orthogonal idempotents, $\overline{\mathfrak{M}}_A^{\text{nilp}} \otimes_A \mathbb{F}$ is finite free over $\mathfrak{S}_{\mathbb{F}}/(u)$. Now consider the natural map $\overline{\mathfrak{M}}_A^{\text{ét}} \otimes_A \mathbb{F} \to (\overline{\mathfrak{M}}_A \otimes_A \mathbb{F})^{\text{ét}}$. (Note that

$\overline{\varphi} : \sigma^*\overline{\mathfrak{M}}_A \to \overline{\mathfrak{M}}_A$ is A-linear.) So we have a natural map $\overline{\mathfrak{M}}_A^{\mathrm{nilp}} \otimes_A \mathbb{F} \to (\overline{\mathfrak{M}}_A \otimes_A \mathbb{F})^{\mathrm{nilp}}$ which is surjective by a diagram chasing similar to (8.2.6.2). Since $\overline{\mathfrak{M}}_A^{\mathrm{nilp}} \otimes_A \mathbb{F}$ is φ-nilpotent and $(\overline{\mathfrak{M}}_A \otimes_A \mathbb{F})^{\mathrm{nilp}}$ is maximal among φ-nilpotent quotient of $\overline{\mathfrak{M}}_A \otimes_A \mathbb{F}$, the natural maps $\overline{\mathfrak{M}}_A^{\mathrm{nilp}} \otimes_A \mathbb{F} \to (\overline{\mathfrak{M}}_A \otimes_A \mathbb{F})^{\mathrm{nilp}}$ and $\overline{\mathfrak{M}}_A^{\text{ét}} \otimes_A \mathbb{F} \to (\overline{\mathfrak{M}}_A \otimes_A \mathbb{F})^{\text{ét}}$ are isomorphisms. It follows from Nakayama's lemma and length consideration that $\overline{\mathfrak{M}}_A^{\mathrm{nilp}}$ is finite free over $\mathfrak{S}_A/(u)$. $\square$

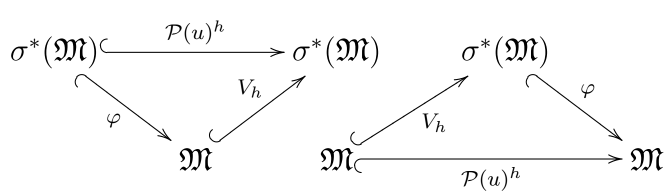
8.3 Duality

For any $h \geq 0$, we define a duality theory for $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, and $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ for a continuous adic $\mathfrak{o}_0$-algebra A as in §8.2.1. For $\mathfrak{o}_0 = \mathbb{Z}_p$, this duality for $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant 1}$ is induced from the Cartier duality of finite flat group schemes by the Breuil-Kisin classification, and similarly the duality for $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}$ is induced from the duality of Barsotti-Tate groups by the Breuil-Kisin classification. For $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, our duality coincides with the Faltings duality of $\mathcal{P}$-height h from §7.3.9.

8.3.1 π_0-Verschiebung of $\mathcal{P}$-height h

We consider $\mathfrak{M}$ in in one of the following categories: $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, and $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ for a continuous adic $\mathfrak{o}_0$-algebra A as in §8.2.1. Recall that $\mathfrak{M}$ has no nonzero $\mathcal{P}(u)^h$-torsion[5], and the image of $\varphi_{\mathfrak{M}}$ contains $\mathcal{P}(u)^h \cdot \mathfrak{M}$ by assumption. Now one will show below that there exists a unique map $V_h : \mathfrak{M} \to \sigma^*\mathfrak{M}$ which makes the following diagram commute.

(8.3.1.1)



[5]This is clear for $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, and for $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ this follows from Proposition 8.1.2. For $\mathfrak{M} \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$, we are reduced to showing that $\mathcal{P}(u)$ is $\mathfrak{S}_A$-regular, but the natural map $\mathfrak{S}_A \to \mathfrak{o}_{\mathcal{E},A}$ is injective and $\mathcal{P}(u)$ is a unit in $\mathfrak{o}_{\mathcal{E},A}$.

We first give a formula for $V_h : \mathfrak{M} \to \sigma_S^*(\mathfrak{M})$, as follows:

$$V_h(m) = \varphi^{-1}\left(\mathcal{P}(u)^h m\right) \text{ for } m \in \mathfrak{M}.$$

This formula is well-defined since φ is injective by Corollary 2.2.3.2 and Lemma 8.1.3. Clearly, V_h is the unique map which satisfies the commutative diagram on the right. To see that V_h satisfies the other commutative diagram, it is enough to check $\varphi \circ V_h \circ \varphi = \varphi \circ \mathcal{P}(u)^h \operatorname{id}_{\sigma^*\mathfrak{M}}$ since φ is injective. But both sides are equal to $\mathcal{P}(u)^h \cdot (\operatorname{id}_{\mathfrak{M}} \circ \varphi)$.

The (unique) $\mathfrak{S}$-linear map $V_h : \mathfrak{M} \to \sigma_S^*(\mathfrak{M})$ which satisfies the commutative diagrams (8.3.1.1) is called π_0-*Verschiebung of* $\mathcal{P}$*-height* h. When $\mathfrak{o}_0 = \mathbb{Z}_p$ and $h = 1$, see [53, §1] for the relation between V_1 and the Verschiebung map of Dieudonné crystals.

Definition 8.3.2. Let $\mathfrak{M}$ be an object in one of the following categories: $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$, and $(\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ for a continuous adic $\mathfrak{o}_0$-algebra A as in §8.2.1. We define another φ-module $\mathfrak{M}^\vee$, as follows.

- The underlying module for $\mathfrak{M}^\vee$ is $\mathfrak{M}^*$, where

$$\mathfrak{M}^* := \begin{cases} \operatorname{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}[\frac{1}{\pi_0}]/\mathfrak{S}), & \text{if } \mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h} \\ \operatorname{Hom}_{\mathfrak{S}_A}(\mathfrak{M}, \mathfrak{S}_A), & \text{if } \mathfrak{M} \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h} \\ \operatorname{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}), & \text{if } \mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}. \end{cases}$$

- We set $\varphi_{\mathfrak{M}^\vee} = (V_h)^* : \sigma^*(\mathfrak{M}^*) \cong (\sigma^*\mathfrak{M})^* \to \mathfrak{M}^*$, where V_h is the π_0-Verschiebung of $\mathcal{P}$-height h which is defined above in §8.3.1. Alternatively, we can construct $\varphi_{\mathfrak{M}^\vee}$ as follows. Consider $\varphi_{\mathfrak{M}} : (\sigma^*\mathfrak{M})[\frac{1}{\mathcal{P}(u)}] \xrightarrow{\sim} \mathfrak{M}[\frac{1}{\mathcal{P}(u)}]$, and we view $l \in \sigma^*(\mathfrak{M}^*)$ as a functional on $\sigma^*\mathfrak{M}$. Now we define

$$\varphi_{\mathfrak{M}^\vee}(l) := l \circ (\mathcal{P}(u)^h \cdot \varphi_{\mathfrak{M}}^{-1}) \in (\mathfrak{M}^*)[1/\mathcal{P}(u)],$$

which actually defines an element in $\mathfrak{M}^*$ since $\mathfrak{M}$ is of $\mathcal{P}$-height $\leqslant h$.

This φ-module $\mathfrak{M}^\vee$ is called the *dual of $\mathcal{P}$-height h* for $\mathfrak{M}$. This duality $\mathfrak{M}^\vee$ depends on h, even though we do not specify this in the notation.

It is straightforward that the duality of $\mathcal{P}$-height $\leqslant h$ for $(\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ commutes with the change of coefficients. If A is a finite artinian $\mathfrak{o}_0$-algebra, then this duality for $(\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ and $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ are compatible.

The following lemma, whose proof is immediate, may provide a motivation for the definition.

Lemma 8.3.3. *Let $\mathfrak{M}$ be an object in one of the following categories: $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$, and $(\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ for a continuous adic $\mathfrak{o}_0$-algebra A as in §8.2.1. Then we have a natural φ-compatible isomorphism $\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}^\vee \cong (\mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M})^*(h)$, where the right side is the Tate twist of the natural duality for (free or torsion) étale φ-modules.*

For a torsion or free étale φ-module over $\mathfrak{o}_{\mathcal{E}}$, we put $M^\vee := M^*(h)$ where M^* is the natural duality, and call $M^\vee$ the *dual of $\mathcal{P}$-height h*.

Although the duality of $\mathcal{P}$-height h is defined separately for $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ and $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$, they are compatible in the following sense.

Lemma 8.3.4. *For $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, there exists a natural isomorphism $\mathfrak{M}^\vee \cong \varprojlim_n (\mathfrak{M}/\pi_0^n\mathfrak{M})^\vee$, where $(\mathfrak{M}/\pi_0^n\mathfrak{M})^\vee$ is the dual as $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$. Furthermore, for any isogeny $\mathfrak{M}' \xrightarrow{f} \mathfrak{M}$ in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, there exists a natural isomorphism $\mathrm{coker}(f^\vee) \cong (\mathrm{coker}\, f)^\vee$, where $f^\vee : \mathfrak{M}^\vee \to (\mathfrak{M}')^\vee$ is the dual isogeny and $(\mathrm{coker}\, f)^\vee$ is the dual for $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$.*

Proof. The first claim is clear from the definition. The second claim can be seen by viewing both $\mathfrak{M}^*$ and $(\mathfrak{M}')^*$ as submodules of $\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}', \mathfrak{S}[\frac{1}{\pi_0}]) \cong \mathrm{Hom}_{\mathfrak{S}}(\mathfrak{M}, \mathfrak{S}[\frac{1}{\pi_0}])$.

(*c.f.* Lemma 5.1.9) □

8.3.5 Lubin-Tate type φ-modules and maximal Lubin-Tate quotients

Let $\mathfrak{M}$ be an object in one of the following categories: $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, and $(\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_A$ for a continuous adic $\mathfrak{o}_0$-algebra A as in §8.2.1. Then $\mathfrak{M}$ is called *of Lubin-Tate type of $\mathcal{P}$-height h* if the following (obviously) equivalent conditions are satisfied.

- The π_0-Verschiebung of $\mathcal{P}$-height h for $\mathfrak{M}$ is an isomorphism.
- The dual $\mathfrak{M}^\vee$ is étale (where $(\cdot)^\vee$ denotes the duality of $\mathcal{P}$-height h).

The notion of Lubin-Tate type φ-modules of $\mathcal{P}$-height h clearly depends on the choice of h.

Assume that $\mathfrak{M}$ is an object of one of the following categories: $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, and $(\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_A$ where A is a complete local noetherian $\mathfrak{o}_0$-algebra with finite residue field. From Propositions 8.1.18 and 8.2.7, there exists an maximal étale subobject $(\mathfrak{M}^\vee)^{\text{ét}} \subset \mathfrak{M}^\vee$. By passing to the duality of $\mathcal{P}$-height h, we see that $\mathfrak{M}^{\mathcal{LT}} := (\mathfrak{M}^{\vee,\text{ét}})^\vee$ is a quotient of $\mathfrak{M}$ which is maximal among quotients which are of Lubin-Tate type of $\mathcal{P}$-height h. We call $\mathfrak{M}^{\mathcal{LT}}$ the *maximal Lubin-Tate quotient (of $\mathcal{P}$-height h)*. Since the formation of both the maximal étale submodule and the duality of $\mathcal{P}$-height h commute with the change of coefficients, the formation of maximal Lubin-Tate quotient also commutes with the change of coefficients.

Later in Proposition 11.4.2, we show the existence of the maximal étale subobject for more general φ-modules $\mathfrak{M} \in (\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_A$ than the case when A is complete local noetherian. Our discussion of maximal Lubin-Tate quotient carries over word-by-word in that case as well.

8.3.6 Unipotent φ-modules of $\mathcal{P}$-height $\leqslant h$

Let $\mathfrak{M}$ be an object in one of $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, and $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ for a continuous adic $\mathfrak{o}_0$-algebra A as in §8.2.1. We say $\mathfrak{M} \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ is *unipotent of $\mathcal{P}$-height* $\leqslant h$ if the following (obviously) equivalent conditions are satisfied.

1. The π_0-Verschiebung of $\mathcal{P}$-height h for $\mathfrak{M}$ is topologically nilpotent. In other words, for any sufficiently large N, the composite $V_h^N = \sigma^{N-1*}(V_h)\circ\cdots\circ\sigma^*(V_h)\circ V_h : \mathfrak{M} \to \sigma^{N*}\mathfrak{M}$ has the image in $\mathfrak{m}_{\mathfrak{S}}\cdot(\sigma^{N*}\mathfrak{M})$.

2. $\mathfrak{M}^\vee$ is φ-nilpotent, where $(\cdot)^\vee$ denotes the duality of $\mathcal{P}$-height h.

3. (Under the extra assumption that A is a complete local noetherian $\mathfrak{o}_0$-algebra with finite residue field if $\mathfrak{M} \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$) The maximal Lubin-Tate quotient $\mathfrak{M}^{\mathcal{LT}}$ for $\mathfrak{M}$ is trivial.

The conditions (1) and (2) are equivalent to the condition (3) whenever maximal Lubin-Tate quotients are well-defined. (See Proposition 11.4.2 for more general case when maximal Lubin-Tate quotients are well-defined.) The notion of unipotent φ-module (of $\mathcal{P}$-height $\leqslant h$) clearly depends on the choice of h.

We emphasize that for a unipotent φ-module $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ or $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, it is *not* true that the associated $\mathcal{G}_K$-representation $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ is unipotent (i.e., an extension of trivial representations).

Remark 8.3.7 (Formal Properties). Here we record some immediate formal properties.

1. Consider a short exact sequence $0 \to \mathfrak{M}' \to \mathfrak{M} \to \mathfrak{M}'' \to 0$ in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, or $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$. If two of them are of Lubin-Tate type of $\mathcal{P}$-height $\leqslant h$ (respectively, unipotent of $\mathcal{P}$-height $\leqslant h$), then so is the third.

2. Let $\mathfrak{M}$ and $\mathfrak{M}'$ are objects in $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, or $(\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_A$. If both $\mathfrak{M}$ and $\mathfrak{M}'$ are of Lubin-Tate type of $\mathcal{P}$-height $\leqslant h$ (respectively, unipotent of $\mathcal{P}$-height $\leqslant h$), then so are their tensor product $\mathfrak{M} \otimes \mathfrak{M}'$ and direct sum $\mathfrak{M} \oplus \mathfrak{M}'$.

3. (*change of coefficients*) Let $(A, \mathfrak{a}) \to (B, \mathfrak{b})$ be a continuous map of adic $\mathfrak{o}_0$-algebras (where $\mathfrak{a}$ and/or $\mathfrak{b}$ is allowed to be trivial), and let $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_A$. If $\mathfrak{M}_A$ is of Lubin-Tate type of $\mathcal{P}$-height $\leqslant h$ (respectively, unipotent of $\mathcal{P}$-height $\leqslant h$), then so is the "change of coefficients" $B\widehat{\otimes}_A\mathfrak{M}_A := \varprojlim_n (B/\mathfrak{b}^n \otimes_A \mathfrak{M}_A)$. Furthermore, if A is complete local noetherian $\mathfrak{o}_0$-algebra with finite residue field, then $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_A$ is of Lubin-Tate type of $\mathcal{P}$-height $\leqslant h$ (respectively, unipotent of $\mathcal{P}$-height $\leqslant h$) if and only if $A/\mathfrak{m}_A^n \otimes_A \mathfrak{M}_A$ is so for each n.

Remark 8.3.8. We explain where the names "Lubin-Tate type" and "unipotent" come from. In the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ is of Lubin-Tate type of $\mathcal{P}$-height h if and only if the corresponding π_0-divisible group $\underline{G}^*(\mathfrak{M})$ is of Lubin-Tate type of $\mathcal{P}$-height h (i.e., $\underline{G}^*(\mathfrak{M}) \otimes_{\mathfrak{o}_K} \widehat{\mathfrak{o}}_{K^{\mathrm{ur}}}$ is isomorphic to a product of copies of $\mathcal{LT}^{\otimes h}$).

For the case $\mathfrak{o}_0 = \mathbb{Z}_p$ with $p > 2$, a φ-module $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}$ is of Lubin-Tate type of $\mathcal{P}$-height $\leqslant 1$ (respectively, unipotent of $\mathcal{P}$-height $\leqslant 1$) if and only if the corresponding Barsotti-Tate group $\underline{G}^*(\mathfrak{M})$ is multiplicative (respectively, unipotent). Similarly, $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant 1}$ is of Lubin-Tate type of $\mathcal{P}$-height $\leqslant 1$ (respectively, unipotent of $\mathcal{P}$-height $\leqslant 1$) if and only if the corresponding finite flat group scheme $\underline{G}^*(\mathfrak{M})$ is multiplicative (respectively, unipotent). A finite locally free group scheme G is called *multiplicative* if the Cartier dual of G is étale; and G is called *unipotent* if the Cartier dual of G is connected.

Remark 8.3.9. Kisin [53] works with the *covariant* correspondence between $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant 1}$ and the category of finite flat group schemes of p-power order, by post-composing the Cartier duality to the contravariant correspondence $\underline{G}^*$, and similarly for Barsotti-Tate groups. Under the covariant correspondence, unipotent torsion φ-modules correspond to connected finite flat group schemes, and similarly for formal (i.e., connected) Barsotti-Tate groups. So in [53], unipotent φ-modules are called "formal" or "connected."

CHAPTER IX

“Raynaud’s theory” for torsion φ-modules

In this chapter, we develop the analogue of Raynaud’s theory [69] for torsion φ-modules. If $\mathfrak{o}_0 = \mathbb{Z}_p$, $p > 2$, and the $\mathcal{P}$-height is $\leqslant 1$, then the discussions of this chapter *exactly* recovers Raynaud’s theory for finite flat group schemes over $\mathfrak{o}_{\mathscr{K}}$ by the Breuil-Kisin classification [52, Theorem 2.3.5].

9.1 Classification of rank-1 objects in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{\mathbb{F}}$

Fix a finite extension $\mathbb{F}/\mathbb{F}_q$ and put $q^d := \#(\mathbb{F})$. (Recall that $q = p$ if $\mathfrak{o}_0 = \mathbb{Z}_p$.) We view $\mathbb{F}$ as an $\mathfrak{o}_0$-algebra such that $\pi_0\mathbb{F} = 0$. In this section, we give a classification of rank-1 objects in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{\mathbb{F}}$ if k contains $\mathbb{F}$. Just as in Raynaud’s theory for group schemes of type $(p, \cdots, p)$, this classification is used to analyze the semisimplification of the inertia action on torsion $\boldsymbol{\mathcal{G}}_K$-representation of $\mathcal{P}$-height $\leqslant h$, later in §9.4. Compare with [69, §1].

9.1.1

We fix an embedding $\mathbb{F} \hookrightarrow k$. Let $\chi_0 : \mathbb{F}^\times \xrightarrow{\sim} \mu_{q^d-1}(\mathfrak{o}_K) \subset k^\times \subset \mathfrak{o}_K^\times$ be the character which is obtained by restricting the fixed inclusion $\mathbb{F} \hookrightarrow k$. Put $\chi_i := \chi_0^{q^i}$, for $i \in \mathbb{Z}/d\mathbb{Z}$, which plays the same role as the fundamental characters in Raynaud’s theory [69, §1.1]. In fact, χ_i are all the characters which can extend to a field embedding $\mathbb{F} \hookrightarrow k$, and different choices of the fixed embedding $\mathbb{F} \hookrightarrow k$ result in a cyclic permutation of the labeling of χ_i. (This can be seen from $\chi_i|_{\mathbb{F}_q} = \chi_0|_{\mathbb{F}_q}$ and $\chi_i^{q^d} = \chi_i$ for $i \in \mathbb{Z}/d\mathbb{Z}$.)

Choose $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, equipped with a φ-compatible $\mathbb{F}$-action. (In particular $\pi_0 \cdot \mathfrak{M} = 0$, so automatically $\mathfrak{M} \in (\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}$.) Consider the following isotypic decomposition of $\mathfrak{M}$ for the $\mathbb{F}$-action:

$$\mathfrak{M} \cong \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \mathfrak{M}_i,$$

where $\mathbb{F}$ acts on $\mathfrak{M}_i$ via the character χ_i. Clearly, φ restricts to $\sigma^*\mathfrak{M}_i \rightarrow \mathfrak{M}_{i+1}$. It follows that to give an $\mathfrak{M} \in (\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_{\mathbb{F}}$ is equivalent to give $\{\mathfrak{M}_i, \delta_i\}_{i \in \mathbb{Z}/d\mathbb{Z}}$, where each $\mathfrak{M}_i$ is finite free over $\mathfrak{S}/(\pi_0) \cong \mathfrak{o}_K$, and the image of each $\delta_i : \sigma^*\mathfrak{M}_i \rightarrow \mathfrak{M}_{i+1}$ contains $u^{eh} \cdot \mathfrak{M}_{i+1}$. (Observe that $\mathcal{P}(u) \equiv u^e \bmod (\pi_0)$, if $\mathfrak{o}_0 = \mathbb{Z}_p$; and $\mathcal{P}(u) \equiv -u_0 \bmod \pi_0$ where $\mathrm{ord}_u(u_0) = e$, if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$.) From this, we obtain the following lemma.

Lemma 9.1.2. *For $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ equipped with a φ-compatible $\mathbb{F}$-action we have $\mathfrak{M} \in (\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_{\mathbb{F}}$. In other words, $\mathfrak{M}$ is finite free over $\mathfrak{S}_{\mathbb{F}} \cong \mathfrak{o}_K \otimes_{\mathbb{F}_q} \mathbb{F}$.*

Proof. It is enough to prove that $\mathfrak{M}_i$ for each $i \in \mathbb{Z}/d\mathbb{Z}$ is of the same $\mathfrak{o}_K$-rank. But since $\mathfrak{M}$ is of $\mathcal{P}$-height $\leqslant h$ we have $u^{eh} \cdot \mathfrak{M}_{i+1} \subset \delta_i(\mathfrak{M}_i) \subset \mathfrak{M}_{i+1}$, and δ_i is injective for any i because φ is. $\square$

Let us further assume that $\mathfrak{M}$ is of $\mathfrak{S}_{\mathbb{F}}$-rank 1, so each $\mathfrak{M}_i$ is free of $\mathfrak{o}_K$-rank-1. By choosing a basis $\mathbf{e}_i \in \mathfrak{M}_i$ for each i, we may view the maps δ_i as elements in $\mathfrak{o}_K$ such that $\mathrm{ord}_u(\delta_i) \leq he$. Given $\{\delta_i |\ \mathrm{ord}_u(\delta_i) \leq he\}_{i \in \mathbb{Z}/d\mathbb{Z}}$, we can reconstruct $\mathfrak{M}$, as follows. Put $\mathfrak{M} := \bigoplus_{i \in \mathbb{Z}/d\mathbb{Z}} \mathfrak{o}_K \cdot \mathbf{e}_i$, and $\varphi_{\mathfrak{M}}(\sigma^*\mathbf{e}_i) := \delta_i \mathbf{e}_{i+1}$. For $a \in \mathbb{F}^{\times}$, we put $[a]\mathbf{e}_i := \chi_i(a) \cdot \mathbf{e}_i$.

If we choose a different set of bases, say $\alpha_i \mathbf{e}_i \in \mathfrak{M}_i$ where $\alpha_i \in \mathfrak{o}_K^\times$, then δ_i is replaced by $\alpha_{i+1}^{-1}\delta_i\alpha_i^q$. In particular, $\mathrm{ord}_u(\delta_i)$ is independent of the choice of bases. Note also that for given $\{\delta_i\}$ and $\{\delta'_i\}$ such that $\mathrm{ord}_u(\delta_i) = \mathrm{ord}_u(\delta'_i)$ for each i, the solutions α_i of the equations $\delta'_i = \alpha_{i+1}^{-1}\delta_i\alpha_i^q$ lie in some unramified extension of $\mathfrak{o}_K$. To summarize, we have proved the following:

Proposition 9.1.3. *Assume that $\mathbb{F}$ can embed into k. Then the assignment $\mathfrak{M} \mapsto \{\delta_i\}_{i\in\mathbb{Z}/d\mathbb{Z}}$ defines a bijection between the isomorphism classes of rank-1 objects in $(\mathrm{ModFI}\,/\mathfrak{S})_{\mathbb{F}}^{\leqslant h}$ and equivalence classes of $\{\delta_i |\ \mathrm{ord}_u(\delta_i) \leq he\}_{i\in\mathbb{Z}/d\mathbb{Z}}$ under the equivalence relation $\{\delta_i\} \sim \{\alpha_{i+1}^{-1}\delta_i\alpha_i^q\}$ for $\alpha_i \in \mathfrak{o}_K^\times$. If, furthermore, $\mathfrak{o}_K$ is strictly henselian, then the assignment $\mathfrak{M} \mapsto \{n_i = \mathrm{ord}_u(\delta_i)\}_{i\in\mathbb{Z}/d\mathbb{Z}}$ defines a bijection onto the families of r integers $0 \leq n_i \leq he$.*

We may improve our choice of δ_i as follows. By modifying the basis, we can arrange to have $\delta_i = u^{n_i}$ for all $i \neq d-1$. Now write $\delta_{d-1} := \bar{\alpha}\beta u^{n_{d-1}}$, where $\bar{\alpha} \in k^\times$, and $\beta \equiv 1 \bmod u$. If we replace $\mathbf{e}_0$ with $\beta\mathbf{e}_0$ and modify the rest of the basis so that $\delta_i = u^{n_i}$ for all $i \neq d-1$ (i.e., replace $\mathbf{e}_i$ with $\beta^{q^i}\mathbf{e}_i$ for all $i \in \mathbb{Z}/d\mathbb{Z}$), then δ_{d-1} is replaced with $\bar{\alpha}\beta^{q^d-1}u^{n_{d-1}}$. By repeating this process, we may assume that $\beta = 1$. For the similar reason, $\bar{\alpha}$ is unique up to $(k^\times)^{q^d-1}$-multiples.

For each $(\bar{\alpha}, \underline{n})$, where $\bar{\alpha} \in k^\times/(k^\times)^{q^d-1}$ and $\underline{n} = \{n_0, \cdots, n_{d-1}\}$ with $n_i \in [0, he]$, we define $\mathfrak{M}_{(\bar{\alpha},\underline{n})}$, as follows.

$$\begin{aligned}
\mathfrak{M}_{(\bar{\alpha},\underline{n})} &:= \bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}} \mathfrak{o}_K \cdot \mathbf{e}_i \\
\varphi(\sigma^*\mathbf{e}_i) &:= u^{n_i}\mathbf{e}_{i+1}, \quad \text{if } i \neq d-1 \\
\varphi(\sigma^*\mathbf{e}_{d-1}) &:= \bar{\alpha}u^{n_{d-1}}\mathbf{e}_0 \\
[a]\mathbf{e}_i &:= \chi_0(a)^{p^i}\cdot\mathbf{e}_i, \quad \forall a \in \mathbb{F}.
\end{aligned}$$

We have proved the following

Corollary 9.1.4. *Assume that $\mathbb{F}$ can embed into k with $\chi_0 : \mathbb{F} \hookrightarrow k$ such an embedding. Then for any $\mathfrak{M} \in (\mathrm{ModFI}/\mathfrak{S})_{\mathbb{F}}^{\leqslant h}$ of $\mathfrak{S}_{\mathbb{F}}$-rank 1, there exists $\bar{\alpha} \in k^{\times}$ unique up to $(k^{\times})^{q^d-1}$-multiple and unique$\underline{n} = \{n_0, \cdots, n_{d-1}\}$ with $n_i \in [0, he]$, such that $\mathfrak{M} \cong \mathfrak{M}_{(\bar{\alpha},\underline{n})}$.*

9.1.5 Duality, étale and Lubin-Tate type objects

Let $\mathfrak{M} \in (\mathrm{ModFI}/\mathfrak{S})_{\mathbb{F}}^{\leqslant h}$ be of $\mathfrak{S}_{\mathbb{F}}$-rank 1, which corresponds to $\{\delta_i\}_{i\in\mathbb{Z}/d\mathbb{Z}}$ under the bijection given in Proposition 9.1.3. In other words, $\mathfrak{M} \cong \bigoplus_{i\in\mathbb{Z}/d\mathbb{Z}} \mathfrak{o}_K \cdot \mathbf{e}_i$ with $\varphi(\sigma^*\mathbf{e}_i) = \delta_i \mathbf{e}_{i+1}$. It is straightforward to verify the following claims:

Duality The dual $\mathfrak{M}^{\vee}$ of $\mathcal{P}$-height $\leqslant h$ corresponds to $\{(\mathcal{P}(u)^h \bmod \pi_0)/\delta_i\}_{i\in\mathbb{Z}/d\mathbb{Z}}$.

Étale/Lubin-Tate type $\mathfrak{M}$ is étale if and only if $\mathrm{ord}_u(\delta_i) = 0$ for all i; $\mathfrak{M}$ is of Lubin-Tate type of $\mathcal{P}$-height h if and only if $\mathrm{ord}_u(\delta_i) = he$ for all i.

9.2 $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$

In this section, we study $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ in a fixed $M \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$. We also introduce an operation which plays a role similar to schematic closure of the generic fiber of a finite flat group scheme over $\mathfrak{o}_{\mathscr{K}}$. As an application, we show that $T \in \mathrm{Rep}_{\mathfrak{o}_0}^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$ is isomorphic to $\underline{T}_{\mathfrak{S}}^*(\mathfrak{M})$ for some $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)$ if and only if for each $n > 0$ there exists $\mathfrak{M}_n \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ killed by π_0^n such that $T/\pi_0^n T \cong \underline{T}_{\mathfrak{S}}^*(\mathfrak{M}_n)$ in $\mathrm{Rep}_{\mathfrak{o}_0}^{\mathrm{tor}}(\boldsymbol{\mathcal{G}}_K)$. Based on the analogy discussed in Remark 8.1.15, this can be thought of as an analogue of [69, §2].

9.2.1 Analogue of schematic closure

Choose $M, M' \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}} = \underline{\mathrm{Mod}}_{\mathfrak{o}_{\mathcal{E}}}^{\text{ét,tor}}(\varphi)$ and a φ-compatible $\mathfrak{o}_{\mathcal{E}}$-linear surjective map $f : M \to M'$. Let $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ be such that $M \cong \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$. We obtain a φ-compatible $\mathfrak{S}$-linear surjective map $f|_{\mathfrak{M}} : \mathfrak{M} \twoheadrightarrow f(\mathfrak{M})$ which recovers f by extending scalars to $\mathfrak{o}_{\mathcal{E}}$ (i.e., after inverting u, as $\mathfrak{M}$ and $f(\mathfrak{M})$ are killed by some power of π_0). Furthermore, $f(\mathfrak{M}) \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ by Lemma 8.1.8, so $f(\mathfrak{M}) \subset M'$ is a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ (Definition 8.1.7). Also, we have that $\ker(f) \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$ and $\ker(f|_{\mathfrak{M}}) \subset \ker(f)$ is a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$. Using the analogy with finite flat group schemes discussed in Remark 8.1.15, the φ-compatible surjection $f|_{\mathfrak{M}} : \mathfrak{M} \twoheadrightarrow f(\mathfrak{M})$ plays the role of schematic closure of a closed subgroup scheme of the generic fiber.

We record an immediate consequence for $\mathcal{G}_K$-representations of $\mathcal{P}$-height $\leqslant h$. Recall that $\mathrm{Rep}^{\mathrm{tor},\leqslant h}_{\mathfrak{o}_0}(\mathcal{G}_K)$ denotes the category of torsion $\mathfrak{o}_0$-representations of $\mathcal{G}_K$ with $\mathcal{P}$-height $\leqslant h$.

Proposition 9.2.2. *The category* $\mathrm{Rep}^{\mathrm{tor},\leqslant h}_{\mathfrak{o}_0}(\mathcal{G}_K)$ *is closed under finite direct products, subobjects, and quotients.*

Proof. The direct product aspect is obvious. Let $T = \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ for some $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, and set $M := \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M} \cong \underline{D}^*_{\mathcal{E}}(T)$. Any $\mathcal{G}_K$-stable submodule $T' \subset T$ corresponds to a φ-compatible surjection $f : M \twoheadrightarrow M'$ where $M' := \underline{D}^*_{\mathcal{E}}(T')$. Then $f(\mathfrak{M}) \subset M'$ is a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ by the discussions at §9.2.1, so T' is of $\mathcal{P}$-height $\leqslant h$. Similarly, $\underline{D}^*_{\mathcal{E}}(T/T') \cong \ker(f)$, and $\ker(f|_{\mathfrak{M}}) \subset \ker(f)$ is a $\mathfrak{S}$-lattice of $\mathcal{P}$-module $\leqslant h$. Thus, T/T' is of $\mathcal{P}$-height $\leqslant h$. $\square$

9.2.3 Partial ordering on $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$

We fix an étale φ module $M \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$. For any two $\mathfrak{S}$-lattices $\mathfrak{M}_1, \mathfrak{M}_2 \subset M$ of $\mathcal{P}$-height $\leqslant h$, there exists a $\mathfrak{S}$-lattice $\mathfrak{M} \subset M$ of $\mathcal{P}$-height $\leqslant h$ that contains both – for example, $\mathfrak{M} := \mathfrak{M}_1 + \mathfrak{M}_2$ does the job. Similarly, there exists a $\mathfrak{S}$-lattice $\mathfrak{M}' \subset M$ of $\mathcal{P}$-height $\leqslant h$ that is contained in both – for example, $\mathfrak{M}' := \mathfrak{M}_1 \cap \mathfrak{M}_2$ does the job. Therefore, one can define a partial ordering by inclusion on the set of

$\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ in M.

Lemma 9.2.4. *Suppose that $M \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$ has a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$. Then there exist a (maximal) $\mathfrak{S}$-lattice $\mathfrak{M}^+$ of $\mathcal{P}$-height $\leqslant h$ which contains any $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$, and a (minimal) $\mathfrak{S}$-lattice $\mathfrak{M}^-$ of $\mathcal{P}$-height $\leqslant h$ which is contained in any $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$. In particular, there are only finitely many $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ in a fixed $M \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$.*

Proof. The last claim follows from the existence of $\mathfrak{M}^+$ and $\mathfrak{M}^-$, because the set of $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ for M injects into the set of $\mathfrak{S}$-submodules of $\mathfrak{M}^+/\mathfrak{M}^-$, which is of finite length since $\mathfrak{M}^+[\frac{1}{u}] \cong M \cong \mathfrak{M}^-[\frac{1}{u}]$. In order to prove the lemma, it is enough to show the existence of the maximal element, by the duality of $\mathcal{P}$-height h.

Let $\mathfrak{M} \subset M$ be a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$. We first assume that either $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ or that $p{\cdot}M = 0$ if $\mathfrak{o}_0 = \mathbb{Z}_p$. In those cases, we can view $\mathfrak{M}$ as a finite free $\mathfrak{o}_K$-module. Consider the following algebras

$$(9.2.4.1)\quad A_M := \frac{\mathrm{Sym}_K M}{\langle m^q - \varphi(\sigma^* m) : m \in M\rangle}, \text{ and } \quad \mathcal{A}_{\mathfrak{M}} := \frac{\mathrm{Sym}_{\mathfrak{o}_K} \mathfrak{M}}{\langle m^q - \varphi(\sigma^* m) : m \in \mathfrak{M}\rangle}.$$

Clearly, A_M is an étale K-algebra, and $\mathcal{A}_{\mathfrak{M}}$ is finite flat over $\mathfrak{o}_K$ with $\mathcal{A}_{\mathfrak{M}} \otimes_{\mathfrak{o}_K} K \cong A_M$. (Note that $\mathfrak{M}$ is u-torsionfree, so is finite free over $\mathfrak{o}_K$.) If $\mathfrak{M}' \supset \mathfrak{M}$ is another $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$, then $\mathcal{A}_{\mathfrak{M}'}$ is finite over $\mathcal{A}_{\mathfrak{M}}$ and we have $\mathcal{A}_{\mathfrak{M}'} \otimes_{\mathfrak{o}_K} K = A_M$. But the integral closure of $\mathcal{A}_{\mathfrak{M}}$ in A_M is finite over $\mathcal{A}_{\mathfrak{M}}$ since A_M is étale[1], so the set of $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ is bounded above. This proves the lemma when $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, as well as when $\mathfrak{p}_0 = \mathbb{Z}_p$ and $p{\cdot}M = 0$.

Now, assume that $\mathfrak{o}_0 = \mathbb{Z}_p$. It is enough to show that for any two $\mathfrak{S}$-lattices $\mathfrak{M} \subset \mathfrak{M}' \subset M$ of $\mathcal{P}$-height $\leqslant h$, the length of $\mathfrak{M}'/\mathfrak{M}$ has an upper bound that only

[1]Since A_M is étale, the "generic trace pairing" $A_M \otimes_K A_M \to K$ is perfect. The integral closure of $\mathcal{A}_{\mathfrak{M}}$ is therefore contained in the $\mathfrak{o}_K$-linear dual of $\mathcal{A}_{\mathfrak{M}}$ embedded in A_M via the "generic trace pairing", and this is a finite $\mathcal{A}_{\mathfrak{M}}$-module.

depends on M. We reduce to the settled case when $p\cdot M = 0$, as follows. Consider the following commutative diagram with exact rows.

(9.2.4.2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{M}[p] & \longrightarrow & \mathfrak{M} & \longrightarrow & \mathfrak{M}/\mathfrak{M}[p] & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \mathfrak{M}'[p] & \longrightarrow & \mathfrak{M}' & \longrightarrow & \mathfrak{M}'/\mathfrak{M}'[p] & \longrightarrow & 0, \end{array}$$

where $\mathfrak{M}[p]$ denotes the submodule of $\mathfrak{M}$ that is killed by p. By the snake lemma, we get a short exact sequence

(9.2.4.3)
$$0 \to \frac{\mathfrak{M}'[p]}{\mathfrak{M}[p]} \to \frac{\mathfrak{M}'}{\mathfrak{M}} \to \frac{\mathfrak{M}'/\mathfrak{M}'[p]}{\mathfrak{M}/\mathfrak{M}[p]} \to 0.$$

By repeating this process for the $\mathfrak{S}$-lattices $\mathfrak{M}/\mathfrak{M}[p] \subset \mathfrak{M}'/\mathfrak{M}'[p]$ inside of $M/M[p]$ (see §9.2.1) and using the additivity of length on short exact sequences, we reduce the lemma to the case when $p\cdot M = 0$. But this case is already handled. $\square$

Remark 9.2.5. Consider $M \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$ and a $\mathfrak{S}$-lattice $\mathfrak{M} \subset M$ of $\mathcal{P}$-height $\leqslant h$. Assume that either $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ or $p\cdot M = 0$, and let $\mathcal{A}_{\mathfrak{M}}$ and A_M be as in (9.2.4.1). We can define comultiplications on $\mathcal{A}_{\mathfrak{M}}$ and A_M by $m \mapsto m \otimes 1 + 1 \otimes m$ for any $m \in \mathfrak{M}$ and $m \in M$, respectively. Let $G_{\mathfrak{M}} := \operatorname{Spec} \mathcal{A}_{\mathfrak{M}}$ and $G_M := \operatorname{Spec} A_M$ denote the corresponding finite flat group schemes over $\mathfrak{o}_K$ and K, respectively. (If $q = p$ then we have $G_{\mathfrak{M}} \cong \underline{G}^*(\mathfrak{M})$ and $G_M \cong \underline{G}^*(M)$, where $\underline{G}^*(\cdot)$ is as defined in §7.2.4.)

Note that $G_{\mathfrak{M}}$ is a prolongation of G_M, and the assignment $\mathfrak{M} \rightsquigarrow G_{\mathfrak{M}}$ preserves the natural partial orderings; i.e., if $\mathfrak{M}$ and $\mathfrak{M}'$ are two $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ in M, then $\mathfrak{M}' \subset \mathfrak{M}$ if and only if there exists a map $G_{\mathfrak{M}'} \to G_{\mathfrak{M}}$ which prolongs the identity map on the generic fiber G_M. (See [69, Definition 2.2.1].) Therefore, Lemma 9.2.4 for the case when either $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ or $p\cdot M = 0$ can be deduced from the existence of maximal and minimal prolongations of a finite flat group scheme [69, Corollaire 2.2.3].

We digress to record the following interesting fact. For any $\alpha \in \mathfrak{o}_0$ consider $[\alpha]^* : \mathcal{A}_{\mathfrak{M}} \to \mathcal{A}_{\mathfrak{M}}$ and $[\alpha]^* : A_M \to A_M$ induced from $m \mapsto \alpha \cdot m$ for any $m \in \mathfrak{M}$ and $m \in M$, respectively. This defines $\mathfrak{o}_0$-actions on $G_{\mathfrak{M}}$ and G_M, respectively. (This is also true when $\mathfrak{o}_0 = \mathbb{Z}_p$ and $p \cdot M = 0$. In particular, it follows that the group schemes $G_{\mathfrak{M}}$ and G_M are killed by p.) Therefore $G_M(K^{\mathrm{sep}}) \cong G_{\mathfrak{M}}(K^{\mathrm{sep}})$ is naturally an $\mathfrak{o}_0$-torsion $\boldsymbol{\mathcal{G}}_K$-representation. By an argument similar to the proof of Proposition 7.3.4, we can show that there exists a natural $\mathfrak{o}_0$-linear $\boldsymbol{\mathcal{G}}_K$-equivariant isomorphism $G_M(K^{\mathrm{sep}}) \cong \underline{T}^*_{\mathcal{E}}(M)$, and so $G_{\mathfrak{M}}(K^{\mathrm{sep}}) \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$.

Proposition 9.2.6. *Let M be an étale φ-module which is free over $\mathfrak{o}_{\mathcal{E}}$, and suppose that $M_n := M/\pi_0^n M$ has a $\mathfrak{S}$-lattice $\mathfrak{M}(n) \subset M_n$ of $\mathcal{P}$-height $\leqslant h$, for each n. Then M has a $\mathfrak{S}$-lattice $\mathfrak{M}$ of $\mathcal{P}$-height $\leqslant h$. Furthermore, the $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ is unique.*

This proposition shows that a $\mathfrak{o}_0$-lattice $\boldsymbol{\mathcal{G}}_K$-representation T is of $\mathcal{P}$-height $\leqslant h$ (Definition 5.2.8) if and only if $T/\pi_0^n T$ is of $\mathcal{P}$-height $\leqslant h$ (Definition 8.1.7) for all $n \geq 1$.

Proof. The proof will be quite similar to [69, Proposition 2.3.1], working with $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ and the analogue of schematic closures (from §9.2.1) in place of finite flat group scheme models and schematic closures. The uniqueness of $\mathfrak{M}$ follows from Theorem 5.2.3, so we only need to show the existence. We proceed in several steps.

9.2.6.1

For each n, we may modify $\mathfrak{M}(n)$ so that the natural projection $\mathrm{pr}_n : M_n \twoheadrightarrow M_{n-1}$ *restricts to $\mathfrak{M}(n) \to \mathfrak{M}(n-1)$. (We do not require this to be surjective.)*

We recursively modify $\mathfrak{M}(n)$ with n increasing. Suppose that the claim is true for each $j < n$ and we look for a $\mathfrak{S}$-lattice $\widetilde{\mathfrak{M}}(n) \subset M_n$ of $\mathcal{P}$-height $\leqslant h$ such that pr_n restricts to $\widetilde{\mathfrak{M}}(n) \to \mathfrak{M}(n-1)$.

By the duality of $\mathcal{P}$-height $\leqslant h$, we obtain $\mathrm{pr}_n^\vee : M_{n-1}^\vee \to M_n^\vee$. Consider the "graph morphism" $\mathrm{pr}_n^\vee \otimes \mathrm{id} : M_{n-1}^\vee \otimes M_n^\vee \twoheadrightarrow M_n^\vee$, and we let $\mathfrak{N}$ be the image of $\mathfrak{M}(n-1)^\vee \otimes \mathfrak{M}(n)^\vee$ by this morphism. Then $\mathfrak{N} \subset M_n^\vee$ is a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ (containing $\mathfrak{M}(n)^\vee$) and $\mathrm{pr}_n^\vee$ induces $\mathfrak{M}(n-1)^\vee \to \mathfrak{N}$. Now take $\widetilde{\mathfrak{M}}(n) := \mathfrak{N}^\vee$.

9.2.6.2

For $i \leqslant n$, let $\mathfrak{M}(n)_i \subset M_i$ be the image of $\mathfrak{M}(n)$ under the natural projection $M_n \twoheadrightarrow M_i$. Clearly, $\mathrm{pr}_i : M_i \twoheadrightarrow M_{i-1}$ restricts to $\mathfrak{M}(n)_i \to \mathfrak{M}(n)_{i-1}$ for all $i \leqslant n$. We put $\overline{\mathfrak{M}}_i^{(n)} := \ker[\mathfrak{M}(n)_i \to \mathfrak{M}(n)_{i-1}]$ for $1 \leqslant i \leqslant n$, which is viewed as a submodule of M_1 via $\overline{\mathfrak{M}}_i^{(n)} \subset \ker[\mathrm{pr}_i : M_i \twoheadrightarrow M_{i-1}] \cong M_1$ (where the isomorphism uses multiplication by π_0^{i-1}). Then $\overline{\mathfrak{M}}_i^{(n)}$ is a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ for M_1.

Now, $\mathfrak{M}(n+1) \to \mathfrak{M}(n)$ from the previous step produces a map $\overline{\mathfrak{M}}_i^{(n+1)} \to \overline{\mathfrak{M}}_i^{(n)}$ for all $n \geqslant i$, and this becomes the identity map on M_1 after tensoring with $\mathfrak{o}_{\mathcal{E}}$. So for each fixed i, we obtained a decreasing sequence $\{\overline{\mathfrak{M}}_i^{(n)}\}_{n \geqslant i}$ of $\mathfrak{S}$-lattices of $\mathcal{P}$-height for M_1. By Lemma 9.2.4, *there is a minimal element* $\overline{\mathfrak{M}}_i := \overline{\mathfrak{M}}_i^{(n_0)}$ *in the sequence, so we have an equality* $\overline{\mathfrak{M}}_i^{(n)} = \overline{\mathfrak{M}}_i$ *for all* $n \geqslant n_0$ *for some* $n_0 = n_0(i) \geq i$.

9.2.6.3

We claim that the sequence $\{\overline{\mathfrak{M}}_i\}_i$ *of* $\mathfrak{S}$*-lattices of* $\mathcal{P}$*-height* $\leqslant h$ *in* M_1 *is increasing, so there exists an integer* i_0 *such that the equality* $\overline{\mathfrak{M}}_{i_0} = \overline{\mathfrak{M}}_i$ *holds for any* $i \geqslant i_0$.

By the previous step and Lemma 9.2.4, it is enough to show that with n fixed (and arbitrarily large), the sequence $\{\overline{\mathfrak{M}}_i^{(n)}\}_{i \leqslant n}$ is *increasing* in i. In fact, the π_0-multiplication map induces an injective map $M_{i-1} \hookrightarrow M_i$, hence $\mathfrak{M}(n)_{i-1} \hookrightarrow \mathfrak{M}(n)_i$

for each $i \leqslant n$. This induces a map $\overline{\mathfrak{M}}_{i-1}^{(n)} \to \overline{\mathfrak{M}}_i^{(n)}$ on $\mathfrak{S}$-submodules which becomes the identity map on M_1 after tensoring with $\mathfrak{o}_{\mathcal{E}}$. The claim follows.

9.2.6.4

We are ready to conclude the proof. We may assume $i_0 = 1$ by replacing $\mathfrak{M}(n)$ with $\ker\left[\mathfrak{M}(n+i_0) \twoheadrightarrow \mathfrak{M}(n+i_0)_{i_0}\right]$. (Recall that $\mathfrak{M}(n+i_0)_{i_0}$ is the image of $\mathfrak{M}(n+i_0)$ under the natural projection $M_{n+i_0} \twoheadrightarrow M_{i_0}$.) So the previous step implies that the map induced by π_0-multiplication $\overline{\mathfrak{M}}_{i-1}^{(n)} \to \overline{\mathfrak{M}}_i^{(n)}$ is an isomorphism for all i and for $n \gg 0$. (More precisely, $n \geqslant n_0 = n_0(i)$ will be enough, where n_0, depending on i, is as in §9.2.6.2.) We deduce that for fixed i and for $n \gg 0$ depending on i, we have the following diagram with the horizontal sequence short exact:

$$(\dagger\dagger)\qquad \begin{array}{ccccccccc} & & \mathfrak{M}(n)_{i+1} & \xrightarrow{[\pi_0^i]} \searrow & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \mathfrak{M}(n)_1 & \longrightarrow & \mathfrak{M}(n)_{i+1} & \longrightarrow & \mathfrak{M}(n)_i & \longrightarrow & 0. \end{array}$$

Indeed, the content is that the inclusion $\mathfrak{M}(n)_1 = \mathfrak{M}_1^{(n)} \subseteq \mathfrak{M}_{i+1}^{(n)} = \ker[\mathfrak{M}(n)_{i+1} \to \mathfrak{M}(n)_i]$ is an equality for $n \gg 0$ (depending on i), and this is a consequence of having $\overline{\mathfrak{M}}_j \subset M_1$ the same for all j.

Now, for each n, let $\mathfrak{M}_n$ be the minimal element of the decreasing sequence

$$\mathfrak{M}(n) \supset \mathfrak{M}(n+1)_n \supset \cdots \supset \mathfrak{M}(n+r)_n \supset \cdots .$$

Since $\mathfrak{M}_1$ is torsion-free over $\mathfrak{S}/(\pi_0) = \mathbb{F}_q[[u]]$, it is free, and then by induction we infer that each $\mathfrak{M}_n$ is free over $\mathfrak{S}/(\pi_0^n)$ with $\mathfrak{M}_{n+1}/\pi_0^n\mathfrak{M}_{n+1} \xrightarrow{\sim} \mathfrak{M}_n$. And from the diagram ($\dagger\dagger$), we obtain the following diagram with the horizontal sequence short exact:

$$\begin{array}{ccccccccc} & & \mathfrak{M}_{i+j} & \xrightarrow{[\pi_0^i]} \searrow & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & \mathfrak{M}_j & \longrightarrow & \mathfrak{M}_{i+j} & \longrightarrow & \mathfrak{M}_i & \longrightarrow & 0. \end{array}$$

Hence $\mathfrak{M} = \varprojlim_n \mathfrak{M}_n$ is a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ in M. □

We record the following interesting application of Proposition 9.2.6, which is analogous to the fact that a p-adic $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representation is crystalline (respectively, semi-stable) if and only if its $I_{\mathscr{K}}$-restriction is so. We use the same notations as in §8.1.12. We say that an $\mathfrak{o}_0$-lattice representation T of $I_K \cong \boldsymbol{\mathcal{G}}_{\widehat{K}^{\mathrm{ur}}}$ is *of $\mathcal{P}$-height $\leqslant h$* if there exists a finite free $(\varphi, \mathfrak{S}_{\widehat{K}^{\mathrm{ur}}})$-module $\mathfrak{M}_{\widehat{K}^{\mathrm{ur}}}$ such that $T \cong \underline{T}^*_{\mathfrak{S}_{\widehat{K}^{\mathrm{ur}}}}(\mathfrak{M}_{\widehat{K}^{\mathrm{ur}}})$ as I_K-representations and $\mathrm{coker}(\varphi_{\mathfrak{M}_{\widehat{K}^{\mathrm{ur}}}})$ is annihilated by $\mathcal{P}(u)^h$.

Proposition 9.2.7. *An $\mathfrak{o}_0$-lattice $\boldsymbol{\mathcal{G}}_K$-representation T is of $\mathcal{P}$-height $\leqslant h$ in the sense of Definition 5.2.8 if and only if its restriction to I_K is of $\mathcal{P}$-height $\leqslant h$ in the above sense.*

Proof. As in the proof of Proposition 8.1.13, the "only if" direction is trivial. Now, assume that the restriction to I_K of $T \in \mathrm{Rep}^{\mathrm{free}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ is of $\mathcal{P}$-height $\leqslant h$. Clearly, the restriction to I_K of $T/\pi_0^n T$ is of $\mathcal{P}$-height $\leqslant h$ for each $n \geq 1$ in the sense of §8.1.12. By Proposition 9.2.6, it is enough to show that $T/\pi_0^n T$ is of $\mathcal{P}$-height $\leqslant h$ as an $\mathfrak{o}_0$-torsion $\boldsymbol{\mathcal{G}}_K$-representation, which follows from Proposition 8.1.13. □

9.3 The case of small h and small ramification

In this section, we show that if $he < q - 1$ then the scalar extension functor $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h} \to (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$ is fully faithful. The proof uses the classification of rank-1 objects in $(\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_{\mathbb{F}}$, proved in Proposition 9.1.3. For finite flat group schemes, the corresponding theory is discussed in [69, §3.3].

9.3.1

Let $T \in \mathrm{Rep}^{\mathrm{tor}}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ be a *semi-simple* torsion $\boldsymbol{\mathcal{G}}_K$-representation, where $\boldsymbol{\mathcal{G}}_K$ acts by $\rho : \boldsymbol{\mathcal{G}}_K \to \mathrm{Aut}_{\mathfrak{o}_0}(T)$. This forces $\pi_0 \cdot T = 0$. Under this assumption, we claim that T is *tame*; i.e., the wild inertia group I_K^w acts trivially on T. In fact, one may assume T is simple. Since the cardinality of any I_K^w-orbit is some power of p and the zero element is fixed by I_K^w, the $\boldsymbol{\mathcal{G}}_K$-submodule $T^{I_K^w}$ is non-trivial so it equals T by simplicity.

Now, we temporarily assume that $\mathfrak{o}_K$ is strictly henselian so that $\boldsymbol{\mathcal{G}}_K = I_K$ where I_K is the inertia group for K. Assume that T is *simple*. Then, the commutant $\mathrm{End}_{\mathbb{F}_q[I_t]}(T)$ is a finite-dimensional division algebra over $\mathbb{F}_q$, so it is a finite field extension of $\mathbb{F}_q$. We put $\mathbb{F} := \mathrm{End}_{\mathbb{F}_q[I_t]}(T)$, and view T as an $\mathbb{F}$-vector space via the natural action of its commutant. Since I_t is commutative, the image $\rho(I_t)$ is contained in the commutant. Therefore, by simplicity T is a 1-dimensional $\mathbb{F}$-vector space and the I_K-action on T is given by a (tame) character $\rho : I_K \twoheadrightarrow I_t \to \mathbb{F}^\times$.

To summarize, we have proved the following well-known proposition.

Proposition 9.3.1.1. *If T be a* semi-simple *torsion $\boldsymbol{\mathcal{G}}_K$-representation, then T is tame. If the residue field k of K is separably closed and T is simple, then there exists a finite extension $\mathbb{F}/\mathbb{F}_q$, which makes T a* 1*-dimensional $\mathbb{F}$-representation of $\boldsymbol{\mathcal{G}}_K = I_K$.*

We stop assuming that k is separably closed. Let T be an $\mathbb{F}$-representation of $\mathcal{P}$-height $\leqslant h$, and $M := \underline{D}^*_{\mathcal{E},\mathbb{F}}(T)$. Though it is not true in general that the φ-compatible $\mathbb{F}$-action on M preserves any $\mathfrak{S}$-lattice $\mathfrak{M} \subset M$ of $\mathcal{P}$-height $\leqslant h$, it is possible to find some $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ with this property, namely $\mathfrak{M}^+$ and $\mathfrak{M}^-$ from Lemma 9.2.4. Indeed, any automorphism of M restricts to an automorphism of its maximal $\mathfrak{S}$-lattice $\mathfrak{M}^+$ of $\mathcal{P}$-height $\leqslant h$, and the same is true for $\mathfrak{M}^-$ by duality of $\mathcal{P}$-height h. Furthermore, by Lemma 9.1.2, any torsion φ-module with a φ-compatible $\mathbb{F}$-action is in $(\mathrm{ModFI}\,/\mathfrak{S})_{\mathbb{F}}^{\leqslant h}$. We have proved the following proposition.

Proposition 9.3.1.2. *Consider $M \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}_{\mathbb{F}}$ which has a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$. Then there exists a $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$(e.g., the maximal and minimal $\mathfrak{S}$-lattice $\mathfrak{M}^+$ and $\mathfrak{M}^-$ of $\mathcal{P}$-height $\leqslant h$).*

The upshot of this discussion is that when $\mathfrak{o}_K$ is strictly henselian (i.e., k is separably closed), for any torsion representation T of $\mathcal{P}$-height $\leqslant h$, each Jordan-Hölder constituent of T comes from some rank-1 object in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{\mathbb{F}}$ for some finite $\mathbb{F}/\mathbb{F}_q$ (depending on the Jordan-Hölder constituent). This is one of the motivations for our classification of rank-1 objects in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{\mathbb{F}}$.

9.3.2 $\mathfrak{S}_{\mathbb{F}}$-lattices of $\mathcal{P}$-height $\leqslant h$

Assume that there exists an $\mathbb{F}_q$-embedding $\mathbb{F} \hookrightarrow k$ and fix it. Consider an étale φ-module $M \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}_{\mathbb{F}}$ of $\mathfrak{o}_{\mathcal{E},\mathbb{F}}$-rank 1 that admits a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$. We study its maximal and minimal $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$, using their φ-compatible $\mathbb{F}$-action (Proposition 9.3.1.2) and our classification result (Proposition 9.1.3).

Let $\mathfrak{M}, \mathfrak{M}' \subset M$ be $\mathfrak{S}_{\mathbb{F}}$-lattices of $\mathcal{P}$-height $\leqslant h$. We choose an $\mathfrak{o}_K$-basis $\{\mathbf{e}_i\}$ for $\mathfrak{M}$ and $\{\mathbf{e}'_i\}$ for $\mathfrak{M}'$, coming from the isotypic decomposition for $\mathbb{F}$-action. Then we have $\varphi_{\mathfrak{M}}(\sigma^*\mathbf{e}_i) = \delta_i\mathbf{e}_{i+1}$ and $\varphi_{\mathfrak{M}'}(\sigma^*\mathbf{e}'_i) = \delta'_i\mathbf{e}'_{i+1}$ for some $\delta_i, \delta'_i \in \mathfrak{o}_K$ of u-order $\leqslant he$. (See Proposition 9.1.3.)

By assumption, we have $\mathfrak{M}' \otimes_{\mathfrak{o}_K} K = \mathfrak{M} \otimes_{\mathfrak{o}_K} K = M$, and by the choice of the bases we have $\mathbf{e}'_i = \alpha_i\mathbf{e}_i$ for some $\alpha_i \in K^\times$. Since $\varphi_M = \varphi_{\mathfrak{M}}[\frac{1}{u}] = \varphi_{\mathfrak{M}'}[\frac{1}{u}]$, we get the following compatibility condition.

$$\delta'_i = \alpha_{i+1}^{-1}\cdot\delta_i\cdot\alpha_i^q \qquad (9.3.2.1)$$

If $\mathfrak{M}' \subset \mathfrak{M}$ (e.g., $\mathfrak{M} = \mathfrak{M}^+$ or $\mathfrak{M}' = \mathfrak{M}^-$) then $\alpha_i \in \mathfrak{o}_K$ for all i. Assume that we are in this case.

We give a criterion for $\mathfrak{M}$ to be maximal, in terms of $\{\mathrm{ord}_u(\delta_i)\}$. We proceed in the following steps.

Step (1)

If $\mathfrak{M}'$ is not maximal, then $\mathrm{ord}_u(\delta'_i) \geq q-1$ for some i. In particular, if $he < q-1$ then there exists at most one $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ in M.

The second claim follows from the first since $\mathrm{ord}_u(\delta'_i) \leq he$. To show the first claim, we may assume that $\mathfrak{M} \supsetneq \mathfrak{M}'$ (taking $\mathfrak{M} = \mathfrak{M}^+$), so we have $\alpha_i \notin \mathfrak{o}_K^\times$ for some i. Choose $i_0 \in \mathbb{Z}/d\mathbb{Z}$ so that $\mathrm{ord}_u(\alpha_{i_0}) \geq 1$ is maximal among $\mathrm{ord}_u(\alpha_i)$. So by (9.3.2.1), we know that $\mathrm{ord}_u(\delta'_{i_0}) \geq q-1$.

Step (2)

Assume that for some i, we have $\mathrm{ord}_u(\alpha_i) > \mathrm{ord}_u(\alpha_{i+1})$ (so necessarily, $d > 1$). For such i, we have $\mathrm{ord}_u(\delta'_i) \geq q$ by (9.3.2.1). In particular, this case can occur only when $he \geq q$. Conversely, starting with $\mathfrak{M}'$ such that there exists an i_0 with $\mathrm{ord}_u(\delta'_{i_0}) \geq q$, one may take $\alpha_{i_0} = u$ and $\alpha_i = 1$ for $i \neq i_0$. Then $\delta_{i_0} = u^{-q}\delta'_{i_0}$, $\delta_{i_0-1} = u\delta'_{i_0-1}$ and $\delta_i = \delta'_i$ for $i \neq i_0, i_0-1$ give the solution to the equations (9.3.2.1), hence another $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M} \subset M$ of $\mathcal{P}$-height $\leqslant h$ which contains $\mathfrak{M}'$.

Step (3)

For any $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M}' \subset M$ of $\mathcal{P}$-height $\leqslant h$, there exists a $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M} \subset M$ of $\mathcal{P}$-height which contains $\mathfrak{M}'$ and satisfies that $\mathrm{ord}_u(\delta_i) \leq q-1$ for all i.

If $d = 1$ then we may take $\alpha_1 := u^c$ where $c := \lfloor \frac{\mathrm{ord}_u(\delta'_1)}{q-1} \rfloor$ so that $\delta_1 = u^{-c(q-1)}\delta'_1$ has the u-order (strictly) less than $q-1$. So we may assume that $d > 1$ and $\mathrm{ord}_u(\delta'_{i_0}) \geq q$ for some i_0. As in **Step (2)**, we may take $\alpha_{i_0} = u$ and $\alpha_i = 1$ for $i \neq i_0$. Furthermore, one can check that $\sum_{i \in \mathbb{Z}/d\mathbb{Z}} \delta_i < \sum_{i \in \mathbb{Z}/d\mathbb{Z}} \delta'_i$. If $\mathrm{ord}_u(\delta_i) \geq q$ for some i then we apply this process to $\mathfrak{M}$ (instead of $\mathfrak{M}'$). This terminates after finitely many times because at each time the positive integer $\sum_{i\in\mathbb{Z}/d\mathbb{Z}}\delta_i$ decreases, and the resulting $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M}$ of $\mathcal{P}$-height $\leqslant h$ in M satisfies that $\mathrm{ord}_u(\delta_i)\leq q-1$ for all i.

Step (4)

Take $\mathfrak{M} := \mathfrak{M}^+$. By the previous step, we may assume $\mathrm{ord}_u(\delta'_i)\leq q-1$, for all i, in which case all α_i have the same valuation. Now, assume that the valuation of α_i is positive. Then by (9.3.2.1), this can only happen when $\mathrm{ord}_u(\alpha_i)=1$, $\mathrm{ord}_u(\delta'_i)=q-1$, and $\mathrm{ord}_u(\delta_i)=0$, for all i. In other words, $\mathfrak{M}$ is étale as a φ-module.

In the special case when $he=q-1$, the equalities $\mathrm{ord}_u(\delta'_i)=q-1$ mean that $\mathfrak{M}'$ is of Lubin-Tate type of $\mathcal{P}$-height h. In fact, $\mathrm{ord}_u(\delta'_i)=q-1=he=\mathrm{ord}_u(\mathcal{P}(u)^h \bmod \pi_0)$; see §9.1.5.

We now state the following proposition. Compare with [69, Proposition 3.3.2].

Proposition 9.3.3. *Consider $M\in(\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}_{\mathbb{F}}$ of $\mathfrak{o}_{\mathcal{E},\mathbb{F}}$-rank 1. Assume that M admits a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$. Let $\mathfrak{M}$ be a $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$ in M.*

1. *Consider a decomposition $\mathfrak{M} := \bigoplus_{\mathbb{Z}/d\mathbb{Z}}\mathfrak{o}_K\cdot\mathbf{e}_i$ with $\varphi(\sigma^*\mathbf{e}_i)=\delta_i\mathbf{e}_{i+1}$. Then $\mathfrak{M}$ is maximal among $\mathfrak{S}_{\mathbb{F}}$-lattices of $\mathcal{P}$-height $\leqslant h$ in M if and only if $\mathrm{ord}_u(\delta_i)\leq q-1$ for all i and this inequality is strict for some i.*

2. *If $he<q-1$ then M admits at most one $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$, which is always an $\mathfrak{S}_{\mathbb{F}}$-lattice.*

3. *Assume that $he=q-1$. Then either M has a unique $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$, or M has exactly two $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ where one of them is étale and the other is of Lubin-Tate type of $\mathcal{P}$-height h. In either case, any $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ in M is also a $\mathfrak{S}_{\mathbb{F}}$-lattice.*

Proof. It remains to establish (3). Under the assumptions of (3), it follows from **Steps (1) – (4)** above that if M does *not* have a unique $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$

then $\mathfrak{M}^+$ is étale and $\mathfrak{M}^-$ is of Lubin-Tate type, where $\mathfrak{M}^+$ and $\mathfrak{M}^-$ are the maximal and the minimal $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$. So it remains to show that if $\mathfrak{M}^+$ is étale and $\mathfrak{M}$ is a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ for M with $\mathfrak{M} \subsetneq \mathfrak{M}^+$ (but $\mathfrak{M} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ may *not a priori* be a $\mathfrak{S}_{\mathbb{F}}$-lattice in M), then it is of Lubin-Tate type. (Then the inclusion $\mathfrak{M} \supset \mathfrak{M}^-$ has to be an equality.) Note that this claim does not follow from **Steps (2)** and **(4)** because we do not know whether $\mathfrak{M}$ is a $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$ in M.

It follows from the assumption that $\mathfrak{M}$ is φ-nilpotent (i.e., $\mathfrak{M}^{\text{ét}} = 0$) since $\mathfrak{M}$ is not étale and is simple in $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$. (See Proposition 8.1.18.) It suffices to show $\mathfrak{M}$ is of Lubin-Tate type after the scalar extension by $\mathfrak{o}_K \to \widehat{\mathfrak{o}}_K^{sh}$, where $\widehat{\mathfrak{o}}_K^{sh}$ is the completion of the maximal unramified extension of $\mathfrak{o}_K$, since duality commutes with such scalar extension (and the étale and φ-nilpotent properties are insensitive to such scalar extension). Thus, the proposition is reduced to showing the following claim:

Claim. *Assume that $he = q - 1$ and k is separably closed. Assume that $\mathfrak{M}^+$ is étale and $\mathfrak{M}$ is φ-nilpotent. Then $\mathfrak{M}$ is of Lubin-Tate type of $\mathcal{P}$-height h.*

First observe that $\boldsymbol{\mathcal{G}}_K = I_K$ acts trivially on $\underline{T}_{\mathcal{E}}(M) = \underline{T}_{\mathfrak{S}}(\mathfrak{M}^+)$. Consider the following finite flat group scheme $G^+ := \operatorname{Spec} \mathcal{A}_{\mathfrak{M}^+}$ over $\mathfrak{o}_K$, as follows:

$$\mathcal{A}_{\mathfrak{M}^+} := \frac{\mathrm{Sym}_{\mathfrak{o}_K} \mathfrak{M}^+}{\langle m^q - \varphi_{\mathfrak{M}^+}(\sigma^* m) : m \in \mathfrak{M}^+ \rangle}, \tag{9.3.3.1}$$

where co-multiplication and co-action of $\mathfrak{o}_0$ are induced from $m \mapsto m \otimes 1 + 1 \otimes m$ and $m \mapsto \alpha \cdot m$ for any $m \in \mathfrak{M}^+$ and $\alpha \in \mathfrak{o}_0$. Since $\mathfrak{M}^+$ is étale (i.e., $\varphi_{\varphi_{\mathfrak{M}^+}}$ is an isomorphism), we can easily check that G^+ is finite étale over $\mathfrak{o}_K$ (with an action of $\mathbb{F}$). Furthermore since $\mathfrak{o}_K$ is strictly henselian, G^+ is isomorphic to a constant étale group scheme $\underline{\mathbb{F}}$ over $\mathfrak{o}_K$.

From this, one can find a $\mathfrak{S}/(\pi_0)$-basis $\{\mathbf{e}_i\}$ for the étale φ-module $\mathfrak{M}^+$ such that $\varphi(\sigma^*\mathbf{e}_i) = \mathbf{e}_i$ for all i. We can see this as follows. Since G^+ is a constant group scheme, $\boldsymbol{\mathcal{G}}_K = I_K$ acts trivially on $G^+(K^{\mathrm{sep}})$ which is isomorphic to $\underline{T}^*_{\mathcal{E}}(M)$ as noted in Remark 9.2.5. By choosing an $\mathbb{F}_q$-isomorphism $\underline{T}^*_{\mathcal{E}}(M) \cong \mathbb{F}_q^d$ (which is $\boldsymbol{\mathcal{G}}_K$-equivariant by giving the trivial $\boldsymbol{\mathcal{G}}_K$-action on the right), we obtain an $\mathfrak{o}_{\mathcal{E}}/(\pi_0)$-basis $\{\mathbf{e}_i\}$ for M such that $\varphi(\sigma^*\mathbf{e}_i) = \mathbf{e}_i$ for all i. Clearly $\mathfrak{S}/(\pi_0)$-span of $\{\mathbf{e}_i\}$ is a φ-stable étale $\mathfrak{S}$-lattice of M, so it has to equal $\mathfrak{M}^+$.

Now, consider a φ-compatible projection $f_i : M \twoheadrightarrow K\cdot\mathbf{e}_i$ for each i. Since $\mathfrak{M} \subset \prod_i f_i(\mathfrak{M})$, it is enough to show that $f_i(\mathfrak{M})$ is of Lubin-Tate type of $\mathcal{P}$-height h for each i; if we show this then $\prod_i f_i(\mathfrak{M})$ is the minimal $\mathfrak{S}$-lattice $\mathfrak{M}^-$ of $\mathcal{P}$-height $\leqslant h$ in M as $\mathfrak{S}$-lattices in M (being of Lubin-Tate type of $\mathcal{P}$-height $\leqslant h$), so the inclusion $\mathfrak{M} \subset \prod_i f_i(\mathfrak{M}) = \mathfrak{M}^-$ should be an equality. By replacing $\mathfrak{M}$ with $f_i(\mathfrak{M})$, we may assume $\mathbb{F} = \mathbb{F}_q$ (i.e., $\mathfrak{M}$ is of $\mathfrak{o}_K$-rank 1). Then we clearly see that $\mathfrak{M}$, being φ-nilpotent, has to be of Lubin-Tate type of $\mathcal{P}$-height h. □

Corollary 9.3.4. *Assume that $he < q-1$. Then for any torsion étale φ-module $M \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$, there exists at most one $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$.*

Proof. We need to show that for any two $\mathfrak{S}$-lattices $\mathfrak{M}, \mathfrak{M}' \subset M$ of $\mathcal{P}$-height $\leqslant h$, an inclusion $\mathfrak{M} \subset \mathfrak{M}'$ implies equality. This can be checked after a faithfully flat scalar extension, so we may assume that the residue field is separably closed. By considering Jordan-Hölder series and using §9.2.1, one can reduce the claim to the case when M is *simple*. Then by Proposition 9.3.1.1 and the previous proposition, we are done. □

Corollary 9.3.5. *Assume that $he < q-1$. For $\mathfrak{M}, \mathfrak{M}' \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$, we put $M := \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}$ and $M' := \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}'$. We view $\mathfrak{M}$ and $\mathfrak{M}'$ as submodules of M and*

M'.

1. *Any φ-compatible morphism $f_K : M \to M'$ restricts to $f : \mathfrak{M} \to \mathfrak{M}'$. In other words, the scalar extension functor $\mathfrak{M} \rightsquigarrow M$ is fully faithful.*

2. *For any φ-compatible morphism $f : \mathfrak{M} \to \mathfrak{M}'$ in $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, $\ker(f)$ and $\mathrm{coker}(f)$ are also objects of $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$. In other words, $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$ is an abelian category.*

3. *Let $\mathrm{Ext}^{\leqslant h}(\mathfrak{M}, \mathfrak{M}')$ be the group of extensions in $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, and let $\mathrm{Ext}^{\text{ét}}(M, M')$ be the group of extensions in $(\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}$. The natural homomorphism*

$$\mathrm{Ext}^{\leqslant h}_{\mathfrak{o}_K}(\mathfrak{M}, \mathfrak{M}') \to \mathrm{Ext}^{\text{ét}}_K(M, M')$$

is injective.

Proof. Put $C := \mathrm{coker}(f_K)$ and $I := \mathrm{im}(f_K)$. Let $\mathfrak{C} \subset C$ be the image of $\mathfrak{M}'$ under the natural projection $M' \twoheadrightarrow M'/f(M) = C$, and let $\mathfrak{I} \subset I$ be the image of $\mathfrak{M}$ under $M \twoheadrightarrow f_K(M) = I$. Then by §9.2.1 both $\mathfrak{I}$ and $\ker[\mathfrak{M}' \twoheadrightarrow \mathfrak{C}]$ are $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ in $I \xrightarrow{\sim} \ker[M' \twoheadrightarrow C]$, where the isomorphism is induced by f_K. So by Corollary 9.3.4, we have the isomorphism $\mathfrak{I} \xrightarrow{\sim} \ker[\mathfrak{M}' \twoheadrightarrow \mathfrak{C}]$ which extends the isomorphism $I \xrightarrow{\sim} \ker[M' \twoheadrightarrow C]$. Now define $f : \mathfrak{M} \to \mathfrak{M}'$ as follows:

$$f : \mathfrak{M} \twoheadrightarrow \mathfrak{I} \xrightarrow{\sim} \ker[\mathfrak{M}' \twoheadrightarrow \mathfrak{C}] \hookrightarrow \mathfrak{M}'.$$

Clearly, this morphism f extends f_K, and $\ker(f) = \ker[\mathfrak{M} \twoheadrightarrow \mathfrak{I}]$ and $\mathrm{coker}(f) = \mathfrak{C}$ are objects in $(\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$. This proves (1) and (2). Finally, (3) is a formal consequence of (1). □

9.4 Torsion Galois representations

In this section, we describe the $\boldsymbol{\mathcal{G}}_K$-action associated to a rank-1 objects in $(\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_{\mathbb{F}}$. We can use this result to analyze the semisimplification of the inertia action on torsion $\boldsymbol{\mathcal{G}}_K$-representation of $\mathcal{P}$-height $\leqslant h$. For finite flat group schemes, the corresponding theory is discussed in [69, §3.4].

9.4.1 Kummer theory

We assume that the residue field k contains $\mathbb{F} := \mathbb{F}_{q^d}$. This assumption is satisfied for all d if K is strictly henselian.

Pick an element $\delta \in K^\times$ and let $K_d^{(\delta)}/K$ be the Galois extension generated by the roots of $X^{q^d-1} - \delta$. Pick a root $\delta_d \in K_d^{(\delta)}$ to this polynomial. Then we get a continuous homomorphism

$$(9.4.1.1) \qquad \xi_d^{(\delta)} : \boldsymbol{\mathcal{G}}_K \twoheadrightarrow \mu_{q^d-1}(K); \qquad \xi_d^{(\delta)}(\gamma) := \frac{\gamma \cdot \delta_d}{\delta_d}, \quad \forall \gamma \in \boldsymbol{\mathcal{G}}_K,$$

which is independent of the choice of δ_d.

Following §9.1.1, we let $\chi_0 : \mathbb{F}_{q^d}^\times \xrightarrow{\sim} \mu_{q^d-1}(k) \cong \mu_{q^d-1}(K)$ be a character which extends to an $\mathbb{F}_q$-morphism of fields $\mathbb{F}_{q^d} \to k$, and we put $\chi_i = \chi_0^{q^i}$. Using the inverse isomorphism χ_0^{-1} (not the inverse character), we obtain a character $\omega_d^{(\delta)} := \chi_0^{-1} \circ \xi_d^{(\delta)} : \boldsymbol{\mathcal{G}}_K \to \mathbb{F}_{q^d}^\times$. If we have used χ_i^{-1}, instead of χ_0^{-1}, then we obtain $(\omega_d^{(\delta)})^{1/q^i}$.

The formation of $\omega_d^{(\delta)}$ is compatible with finite extension of K, as $\xi_d^{(\delta)}$ is. For any $\delta, \delta' \in K^\times$, one can directly check that $\omega_d^{(\delta\delta')} = \omega_d^{(\delta)} \omega_d^{(\delta')}$. By construction, $\omega_d^{(\delta)}$ factors through the quotient $\mathrm{Gal}(K_d^{(\delta)}/K)$, so $\omega_d^{(\delta)}$ is unramified if and only if $\delta \in \mathfrak{o}_K^\times$.

We put $\omega_d := \omega_d^{(u)}$, and $\xi_d := \xi_d^{(u)}$. *A priori*, the character ω_d depend on the choice of uniformizer $u \in \mathfrak{o}_K$, but $\omega_d|_{I_K}$ does not; more generally, one can check that $\omega_d^{(\delta)}|_{I_K} = (\omega_d)^{\mathrm{ord}_u(\delta)}|_{I_K}$. We call $\omega_d|_{I_K}$ a *fundamental character of level d.*

The formation of fundamental characters does not necessarily commute with finite extension of K (especially, ramified ones) because the construction involves a uniformizer u, but we have $\omega_{d/K}|_{I_{K'}} = (\omega_{d/K'})^{e(K'/K)}|_{I_{K'}}$ for any finite extension K'/K.

9.4.2 1-dimensional $\mathbb{F}$-representations of $\mathcal{P}$-height $\leqslant h$

Choose $\bar{\alpha} \in k^\times/(k^\times)^{q^d-1}$ and $\underline{n} = \{n_0, \cdots, n_{d-1}\}$ with $n_i \in [0, he]$, and let $\mathfrak{M} := \mathfrak{M}_{(\bar{\alpha},\underline{n})}$ (Corollary 9.1.4). We put $\delta_i := u^{n_i}$ if $i \neq d-1$ and $\delta_{d-1} := \bar{\alpha}u^{n_{d-1}}$ so that we have $\varphi_{\mathfrak{M}}(\sigma^*\mathbf{e}_i) = \delta_i \mathbf{e}_{i+1}$.

We would like to compute $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{o}_K,\varphi}(\mathfrak{M}, K^{\mathrm{sep}})$. Giving an element $l \in \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ is equivalent to giving $l(\mathbf{e}_i) = x_i \in K^{\mathrm{sep}}$ for each $i \in \mathbb{Z}/d\mathbb{Z}$, such that $x_i^q = \delta_i x_{i+1}$. In turn, it is equivalent to giving an element $x_0 \in K^{\mathrm{sep}}$ such that $x_0^{q^d} = \delta x_0$, where $\delta := \prod_{i=0}^{d-1}(\delta_i)^{q^{d-1-i}} = \bar{\alpha}u^n$ with $n := \sum_{i=0}^{d-1} n_i q^{d-1-i}$. So by identifying $l \in \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ with $x_0 = l(\mathbf{e}_0) \in K^{\mathrm{sep}}$, we will view $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ as an $\mathbb{F}_q$-submodule of K^{sep}. Under this identification, the natural $\mathbb{F}$-action translates to $[a] : x_0 \mapsto \chi_0(a)\cdot x_0$ for $a \in \mathbb{F}^\times$, and the $\boldsymbol{\mathcal{G}}_K$-action is via the natural action on K^{sep}. That is, for $\gamma \in \boldsymbol{\mathcal{G}}_K$, we have $\gamma \cdot x_0 = \xi_d^{(\delta)}(\gamma)\cdot x_0$. This proves the first part of the following proposition.

Proposition 9.4.3.

1. *The $\boldsymbol{\mathcal{G}}_K$-action on the 1-dimensional $\mathbb{F}$-vector space $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}_{(\bar{\alpha},\underline{n})})$ is given by the character $\omega_d^{(\delta)}$, where $\delta := \bar{\alpha}u^n$ and $n := \sum_{i=0}^{d-1} n_i q^{d-1-i}$. In particular, I_K acts on $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ by the character $(\omega_d)^n$.*

2. *In the case $\mathfrak{o}_0 = \mathbb{Z}_p$, the $\mathbb{F}$-valued $I_{\mathscr{K}_\infty}$-character $(\omega_1)^e|_{I_{\mathscr{K}_\infty}}$ is the mod-p cyclotomic character restricted to $I_{\mathscr{K}_\infty}$. In the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, the $\mathbb{F}$-valued I_K-character $(\omega_1)^e|_{I_K}$ is the mod-π_0 Lubin-Tate character restricted to I_K.*

Proof. It remains to prove the second part of the proposition. The computation in §9.4.2 shows that $\boldsymbol{\mathcal{G}}_K$ acts on $\underline{T}^*_{\mathfrak{S}}(\mathfrak{S}_{\mathbb{F}_p}(1))$ via $\omega_1^{(\mathcal{P}(u))}$. In the case $\mathfrak{o}_0 = \mathbb{Z}_p$, it follows from §5.2.14 that $\omega_1^{(\mathcal{P}(u))}$ is the $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-restriction to the mod p cyclotomic character. In the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, it follows from §7.3.7 that $\omega_1^{(\mathcal{P}(u))}$ is the mod π_0 Lubin-Tate character. On the other hand, since $\mathrm{ord}_u(\mathcal{P}(u) \bmod \pi_0) = e$ we have $\omega_1^e|_{I_K} = \omega_1^{(\mathcal{P}(u))}|_{I_K}$. □

The following theorem gives a classification of $\mathbb{F}^\times$-valued $\boldsymbol{\mathcal{G}}_K$-characters of $\mathcal{P}$-height $\leqslant h$.

Theorem 9.4.4. *Assume that $\mathbb{F}$ embeds into k, and let ψ be a $\mathbb{F}^\times$-valued character on $\boldsymbol{\mathcal{G}}_K$. Then ψ is of $\mathcal{P}$-height $\leqslant h$ (Definition 8.1.7) if and only if $\psi|_{I_K} = (\omega_d)^n$, where $n = \sum_{i=0}^{d-1} n_i q^{d-1-i}$ for some $0 \leq n_i \leq he$ for each $i \in \mathbb{Z}/d\mathbb{Z}$. Equivalently, ψ is of $\mathcal{P}$-height $\leqslant h$ if and only if $\psi = \omega_d^{(\delta)}$, where $\delta = \bar{\alpha} u^n$ for some $\bar{\alpha} \in k^\times$ and n as above.*

Proof. The "only if" direction is just Proposition 9.4.3(1). For the "if" direction, we first observe that $(\omega_d)^n$ makes sense as a character of $\boldsymbol{\mathcal{G}}_K$. So if $\psi|_{I_K} = (\omega_d)^n$, then we can write $\psi = \psi^{\mathrm{ur}} \cdot (\omega_d)^n$, where ψ^{ur} is an unramified character. Since any unramified $\mathfrak{o}_0$-torsion $\boldsymbol{\mathcal{G}}_K$-representation is of $\mathcal{P}$-height $\leqslant 0$ (Proposition 8.1.10) it follows from Corollary 9.1.4 and Proposition 9.4.3(1) that there exists $\bar{\alpha} \in k^\times$, well-defined up to $(k^\times)^{q^d-1}$-multiple, such that $\psi^{\mathrm{ur}} = \omega_d^{(\bar{\alpha})}$. Therefore by Proposition 9.4.3(1), we have $\psi = \omega_d^{(\bar{\alpha} u^n)} \cong \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}_{(\alpha, \underline{n})})$, where $\underline{n} = \{n_0, \cdots, n_{d-1}\}$. □

Remark 9.4.5. Using Proposition 9.3.3, one can improve the numerical condition in the statement as follows. an $\mathbb{F}^\times$-valued character ψ is of $\mathcal{P}$-height $\leqslant h$ if and only if $\psi|_{I_K} = (\omega_d)^n$, where $n = \sum_{i=0}^{d-1} n_i q^{d-1-i}$ for some $0 \leq n_i \leq \min\{he, q-1\}$ for each $i \in \mathbb{Z}/d\mathbb{Z}$, and not all n_i are $q-1$.

If k is finite, then we can remove the condition that $\mathbb{F}$ embeds in k by using local class field theory, and obtain the following result. Let $\mathbb{F}_0$ be the maximal subfield of $\mathbb{F}$ that embeds in k, and put $q^{d_0} := \#(\mathbb{F}_0)$. Then a character ψ on $\boldsymbol{\mathcal{G}}_K$ is of $\mathcal{P}$-height

$\leqslant h$ if and only if $\psi|_{I_K} = (\omega_{d_0})^n$ where $n = \sum_{i=0}^{d_0-1} n_i q^{d_0-1-i}$ for some $n_i \in [0, he]$ for each $i \in \mathbb{Z}/d_0\mathbb{Z}$. The "only if" direction is by Proposition 9.4.3(1) and local class field theory, and the "if" direction follows from Proposition 8.1.13. (Alternatively, note that ψ is of $\mathcal{P}$-height $\leqslant h$ if and only if $\psi|_{\mathcal{G}_{K'}}$ is so for some finite unramified extension K'/K by Proposition 8.1.10).

9.4.6 Relation with torsion crystalline representations.

For this paragraph, we assume that $\mathfrak{o}_0 = \mathbb{Z}_p$, (so $q = p$). We have a norm-field isomorphism $\mathcal{G}_{\mathscr{K}_\infty} \cong \mathcal{G}_K$ as explained in §1.3.1.2, and we assume that $\mathbb{F}$ embeds in k and fix an embedding $\chi_0 : \mathbb{F} \hookrightarrow k$. We start with the following observation.

Lemma 9.4.7. *The restriction of the $\mathcal{G}_{\mathscr{K}}$-action to $\mathcal{G}_{\mathscr{K}_\infty}$ defines an equivalence of categories from the category of mod p* semi-simple *representations of $\mathcal{G}_{\mathscr{K}}$ to the category of mod p* semi-simple *representations of $\mathcal{G}_{\mathscr{K}_\infty}$. Moreover, any irreducible mod p representation $\bar{\rho}_\infty$ of $\mathcal{G}_{\mathscr{K}_\infty}$ uniquely extends to a $\mathcal{G}_{\mathscr{K}}$-representation $\bar{\rho}$ which is necessarily irreducible.*

Proof. By Proposition 9.3.1.1 any semi-simple mod p representation $\bar{\rho}_\infty$ of $\mathcal{G}_{\mathscr{K}_\infty}$ is tame, and similarly, any semi-simple mod p representation $\bar{\rho}$ of $\mathcal{G}_{\mathscr{K}}$ is tame. On the other hand, $\mathscr{K}_\infty/\mathscr{K}$ is linearly disjoint from any tame extension, so we have $I^w_{\mathscr{K}} \cdot \mathcal{G}_{\mathscr{K}_\infty} = \mathcal{G}_{\mathscr{K}}$. In particular, we have $\bar{\rho}(\mathcal{G}_{\mathscr{K}_\infty}) = \bar{\rho}(\mathcal{G}_{\mathscr{K}})$. The lemma follows. $\square$

It follows from the lemma above that the character $\psi = \omega_d^{(\delta)}$ from Theorem 9.4.4 can be extended to an $\mathbb{F}^\times$-valued character of $\mathcal{G}_{\mathscr{K}}$. In fact, we can easily find a candidate for it. Recall that $\delta = \bar{\alpha} u^n$ where $\bar{\alpha} \in k^\times$ and $n = \sum_{i=0}^{d-1} n_i p^{d-1-i}$ for some $0 \leq n_i \leq he$ for each $i \in \mathbb{Z}/d\mathbb{Z}$. Now, we put $\tilde{\delta} := [\bar{\alpha}]\pi^n \in \mathscr{K}$, where $[\bar{\alpha}]$ denotes the Teichmüller lift and π is the fixed uniformizer for $\mathscr{K}$ such that $\mathcal{P}(\pi) = 0$. We define an $\mathbb{F}^\times$-valued character $\omega_d^{(\tilde{\delta})}$ on $\mathcal{G}_{\mathscr{K}}$ in the similar way that we defined $\omega_d^{(\delta)}$, but we use the (p^d-1)th root of $\tilde{\delta} \in \mathscr{K}$, instead of that of δ. More precisely:

$$\omega_d^{(\tilde{\delta})} : \gamma \mapsto \frac{\gamma \cdot \tilde{\delta}^{1/(p^d-1)}}{\tilde{\delta}^{1/(p^d-1)}} \mapsto \chi_0^{-1}\left(\frac{\gamma \cdot \tilde{\delta}^{1/(p^d-1)}}{\tilde{\delta}^{1/(p^d-1)}}\right) \in \mathbb{F}^\times, \quad \forall \gamma \in \boldsymbol{\mathcal{G}}_{\mathscr{K}}. \tag{9.4.7.1}$$

Let us first show that $\omega_d^{(\tilde{\delta})}|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}} \cong \omega_d^{(\delta)}$ under the norm-field isomorphism $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty} \cong \boldsymbol{\mathcal{G}}_K$. Recall from §1.3.2 that we have a natural embedding of $\mathfrak{o}_K \cong k[[u]]$ with its image in $\mathfrak{R} := \varprojlim_{x^p \leftarrow x} \mathfrak{o}_{\overline{\mathscr{K}}}/(p)$ under the natural embedding which sends u to $\underline{\pi} := \{\pi^{(n)} \bmod p\}_{n \geqslant 0}$ and $\bar{\alpha} \in k$ to $\{[\bar{\alpha}^{p^{-n}}] \bmod p\}_{n \geqslant 0}$. Identifying $\mathfrak{o}_K$ with its image in $\mathfrak{R}$, we have $\delta = \{[\alpha^{p^{-n}}]\pi^{(n)} \bmod p\}_{n \geqslant 0} \in \mathfrak{R}$. Now, choose a root δ_d of $X^{q^d-1} - \delta$ in $\mathfrak{R}$; or equivalently, choose a root $\delta_d^{(n)} \in \mathfrak{o}_{\overline{\mathscr{K}}}$ of $X^{q^d-1} - [\bar{\alpha}^{p^{-n}}]\pi^{(n)}$ for each $n \geqslant 0$ so that $(\tilde{\delta}^{(n+1)})^p = \tilde{\delta}^{(n)}$. We can directly see that for any $\gamma \in \boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ we have

$$\omega_d^{(\delta)}(\gamma) \cdot \delta_d = \gamma \cdot \delta_d = \left\{(\omega_d^{(\tilde{\delta})}(\gamma))^{p^{-n}} \cdot \delta_d^{(n)} \bmod p\right\}_{n \geqslant 0} = \omega_d^{(\tilde{\delta})}(\gamma) \cdot \delta_d, \tag{9.4.7.2}$$

where the first equality is by definition of $\omega_d^{(\delta)}$ as in (9.4.1.1), the second equality is obtained from computing $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-action on $\delta_d^{(n)}$, and the last equality follows since we embed $\mathbb{F}$ in $\mathfrak{R}$ via $\bar{\alpha} \mapsto \{[\bar{\alpha}^{p^{-n}}] \bmod p\}$. (Here, we identify $\mathbb{F}^\times \cong \mu_{p^d-1}(\mathfrak{o}_{\overline{\mathscr{K}}}) \cong \mu_{p^d-1}(\mathfrak{R})$, where the isomorphisms are induced from the fixed embedding $\chi_0 : \mathbb{F} \hookrightarrow k$.)

Furthermore, we can see that $\omega_d^{(\tilde{\delta})}$ can be obtained as the cokernel of some isogeny of lattice crystalline representations with Hodge-Tate weights in $[0, h]$. Indeed, $\omega_d^{(\tilde{\delta})}$ is the product of h characters which come from the generic fibers of some finite flat group schemes over $\mathfrak{o}_{\mathscr{K}}$, by partitioning each n_i into the sum of h integers between 0 and e and applying Raynaud's theorem [69, §3.4]. We have proved the following proposition.

Proposition 9.4.8. *If $\mathbb{F}$ embeds into k, then any $\mathbb{F}^\times$-valued character that is obtained as a $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable quotient of a lattice crystalline representation with Hodge-Tate weights in $[0, h]$ can be written as $\omega_d^{(\tilde{\delta})}$ for some $\widetilde{\delta} = [\bar{\alpha}]\pi^n$, where $\bar{\alpha} \in k^\times$ and*

$n = \sum_{i=0}^{d-1} n_i p^{d-1-i}$ *for some* $0 \leq n_i \leq he$ *for each* $i \in \mathbb{Z}/d\mathbb{Z}$. *This character* $\omega_d^{(\tilde{\delta})}$ *is the unique* $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$*-character whose* $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$*-restriction is* $\omega_d^{(\delta)} = \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}_{(\bar{\alpha},\underline{n})})$.

Remark 9.4.9. By taking $\delta = u$ and $\tilde{\delta} = \pi$, the computation (9.4.7.2) also shows that $\omega_d|_{I_{\mathscr{K}_\infty}} : I_{\mathscr{K}_\infty} \to \mathbb{F}_{p^d}^\times$ is the $I_{\mathscr{K}_\infty}$-restriction of a fundamental character of level d for $I_{\mathscr{K}}$.

Remark 9.4.10. For $p > 2$, it is not difficult to compute the Breuil module corresponding to $\mathfrak{M}_{(\bar{\alpha},\underline{n})} \in (\mathrm{ModFI}\,/\mathfrak{S})_{\mathbb{F}}^{\leqslant 1}$, so we can recovers the above results in [70, §2]. Furthermore, one can extend the results using torsion φ-modules with "tame descent datum" and obtain the higher-weight generalization of [70, §3].

CHAPTER X

Categories co-fibered in groupoids

The purpose of this chapter is to present the basic definitions and set up the notations. We mostly follow [51, §A]. More detailed discussion can be found in [77, §3], [73, 1, Exp VI] and the open source algebraic stack project [74, §4].

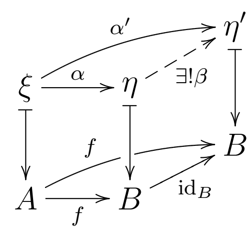
10.1 Basic definitions

Let $\mathscr{E}$ and $\mathscr{F}$ be categories and let $\Pi = \Pi_{\mathscr{F}/\mathscr{E}} : \mathscr{F} \to \mathscr{E}$ be a functor. For an object $A \in \mathrm{Ob}(\mathscr{E})$, we define the *fiber* of $\mathscr{F}$ over A as the subcategory $\mathscr{F}(A)$ of $\mathscr{F}$ such that $\mathrm{Ob}\big(\mathscr{F}(A)\big) = \{\xi \in \mathrm{Ob}(\mathscr{F}) \,:\, \Pi(\xi) = A\}$ (here, we do mean the *equality* $\Pi(\xi) = A$, not $\Pi(\xi) \cong A$), and arrows $\xi \to \eta$ in $\mathscr{F}(A)$ are the arrows in $\mathscr{F}$ which are mapped to id_A via Π. We say an object $\xi \in \mathrm{Ob}(\mathscr{F})$ *is over* $A \in \mathrm{Ob}(\mathscr{E})$ if $\Pi(\xi) = A$; i.e., if $\xi \in \mathrm{Ob}(\mathscr{F}(A))$. For objects $\xi \in \mathrm{Ob}(\mathscr{F}(A))$ and $\eta \in \mathrm{Ob}(\mathscr{F}(B))$ and a morphism $f : A \to B$, we say a morphism $\alpha : \xi \to \eta$ *covers* $f : A \to B$ if $\Pi(\alpha) = f$

The following definition is from §10 and (5.1) of [73, 1, Exp VI], which is weaker than [77, Def 3.1].

Definition 10.1.1. Consider $\xi \in \mathrm{Ob}(\mathscr{F}(A))$ and $\eta \in \mathrm{Ob}(\mathscr{F}(B))$, for $A, B \in \mathrm{Ob}(\mathscr{E})$. Let $f : A \to B$ be a morphism of $\mathscr{E}$. Then a morphism α, which covers f, is called *co-cartesian for* Π if for any $\eta' \in \mathrm{Ob}(\mathscr{F}(B))$ and any morphism $\alpha' : \xi \to \eta'$ with

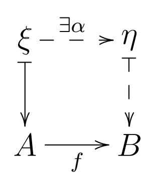
$\Pi(\alpha') = f$, there exists a unique morphism $\beta : \eta \to \eta'$ such that $\alpha' = \beta \circ \alpha$. If Π is understood, we say that f is *co-cartesian*.



Definition 10.1.2. We say that $\mathscr{F}$ is a *category co-fibered in groupoids* over $\mathscr{E}$ (or a *groupoid* over $\mathscr{E}$, or $\mathscr{E}$*-groupoid*) if the following conditions are satisfied.

(G1) Every morphism in $\mathscr{F}$ is co-cartesian

(G2) (*Existence of enough co-cartesian lifts*) For any $\xi \in \mathrm{Ob}(\mathscr{F}(A))$ and a morphism $f : A \to B$ be a morphism of $\mathscr{E}$, there exists a *co-cartesian* morphism $\alpha : \xi \to \eta$ which covers f.



Let $\mathscr{F}$ be an $\mathscr{E}$-groupoid and let $\mathscr{F}'$ be a subcategory of $\mathscr{F}$. We say that $\mathscr{F}'$ is an $\mathscr{E}$*-subgroupoid* if $\mathscr{F}'$ has enough co-cartesian lifts.

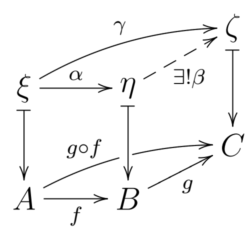
As a trivial example, the identity functor $\mathrm{id}_{\mathscr{E}} : \mathscr{E} \to \mathscr{E}$ is an $\mathscr{E}$-groupoid.

Under the condition (G2), the condition (G1) holds if and only if all the fibers $\mathscr{F}(A)$ are groupoids [77, Prop 3.22] – hence the terminology. In applications, the base category $\mathscr{E}$ is a certain category of rings (with extra structures) and the condition (G2) says that for any ξ over A and $f : A \to B$, we can always "extend scalars" to obtain η.

A functor $\Pi : \mathscr{F} \to \mathscr{E}$ is a category co-fibered in groupoids if and only if $\Pi^o : \mathscr{F}^o \to \mathscr{E}^o$ is a category "fibered in groupoids," in the sense of [77, (3.1.1)]. The results for categories fibered in groupoids also apply to categories *co*-fibered in groupoids by changing the direction of arrows.

Remark 10.1.3. Let $\mathscr{F}$ be an $\mathscr{E}$-groupoid. Then any morphism $\alpha : \xi \to \eta$ in $\mathscr{F}$ satisfies the following *strong co-cartesian property*:

> Let $f := \Pi(\alpha) : A \to B$ and $g : B \to C$ be morphisms in $\mathscr{E}$. For any $\zeta \in \mathrm{Ob}(\mathscr{F}(C))$ and a morphism $\gamma : \xi \to \zeta$ over $g \circ f$, there exists a unique morphism $\beta : \eta \to \zeta$ over g such that $\gamma = \beta \circ \alpha$.



In fact, by the existence of enough co-cartesian lifts (Definition 10.1.2(G2)), there exists a co-cartesian morphism $\beta' : \eta \to \zeta'$ over g. Since any morphism in $\mathscr{F}$ are co-cartesian, $\gamma : \xi \to \zeta$ and $\beta' \circ \alpha : \xi \to \zeta'$ are co-cartesian over $g \circ f$. So by the definition of co-cartesian morphism, we have a unique isomorphism $\delta : \zeta \xrightarrow{\sim} \zeta'$ over id_C such that $\delta \circ \gamma = \beta' \circ \alpha$. Now take $\beta := \delta^{-1} \circ \beta'$, and the uniqueness is clear from the construction

As a consequence, we can prove that if $\Pi : \mathscr{F} \to \mathscr{E}$ is an $\mathscr{E}$-groupoid and $\Pi' : \mathscr{F}' \to \mathscr{F}$ be an $\mathscr{F}$-groupoid, then $\Pi' \circ \Pi : \mathscr{F}' \to \mathscr{E}$ is an $\mathscr{E}$-groupoid. The existence of enough co-cartesian lifts is automatic, but to show that all morphisms in $\mathscr{F}'$ are co-cartesian for $\Pi' \circ \Pi$ we need the strong co-cartesian property, which will be left to readers.

Remark 10.1.4. The notion of $\mathscr{E}$-groupoid can be viewed as a generalization of covariant functor $\mathscr{E} \to (\mathbf{Sets})$ in the following sense: a covariant functor $F : \mathscr{E} \to (\mathbf{Sets})$

associates to each $A \in \mathrm{Ob}(\mathscr{E})$ a set $F(A)$, but an $\mathscr{E}$-groupoid $\mathscr{F}$ "associates[1]" to each $A \in \mathrm{Ob}(\mathscr{E})$ a groupoid $\mathscr{F}(A)$. For an $\mathscr{E}$-groupoid $\mathscr{F}$, we may associate a covariant functor $|\mathscr{F}| : \mathscr{E} \to (\mathbf{Sets})$ which assigns to $A \in \mathrm{Ob}(\mathscr{E})$ the set $|\mathscr{F}(A)|$ of isomorphism classes in $\mathscr{F}(A)$. To rephrase, an $\mathscr{E}$-groupoid $\mathscr{F}$ retains the isomorphisms between objects over A while the associated functor $|\mathscr{F}|$ does not.

We can view a covariant functor $\mathscr{E} \to (\mathbf{Sets})$ as a $\mathscr{E}$-groupoid with some special property, which is discussed in §10.2.1.

Now, we define "maps" between $\mathscr{E}$-groupoids. The fact that fibers $\mathscr{F}(A)$ are groupoids, not just sets, introduces many technical complications.

Definition 10.1.5. For two groupoids $\Pi : \mathscr{F} \to \mathscr{E}$ and $\Pi' : \mathscr{F} \to \mathscr{E}$, a functor $\Phi : \mathscr{F} \to \mathscr{F}'$ is called an *1-morphism over* $\mathscr{E}$ if Φ "preserves the base"[2]. In other words, we have an *equality* of functors $\Pi = \Pi' \circ \Psi$, not just an isomorphism.

For two 1-morphisms $\Phi, \Psi : \mathscr{F} \rightrightarrows \mathscr{F}'$, we say that a natural transformation $\psi : \Phi \to \Psi$ is a *2-morphism over* $\mathscr{E}$ if ψ is base preserving. In other words, for any $\xi \in \mathrm{Ob}(\mathscr{F}(A))$, the arrow $\psi_\xi : \Phi(\xi) \to \Psi(\xi)$ is a morphism in $\mathscr{F}'(A)$; i.e., $\Pi'(\psi_\xi) = \mathrm{id}_A$. Any 2-morphism is automatically an isomorphism and the inverse $\psi^{-1} : \Psi \to \Phi$ is forced to be a 2-morphism. To emphasize this, we often call it a *2-isomorphism*. We define a groupoid $\mathscr{H}om_{\mathscr{E}}(\mathscr{F}, \mathscr{F}')$ with 1-morphisms $\mathscr{F} \to \mathscr{F}'$ over $\mathscr{E}$ as objects and 2-isomorphisms as morphisms.

We say that a 1-morphism $\Phi : \mathscr{F} \to \mathscr{F}'$ is an *1-isomorphism* if there exists another 1-morphism $\Psi : \mathscr{F}' \to \mathscr{F}$ such that we have 2-isomorphisms $\Psi \circ \Phi \cong \mathrm{id}_{\mathscr{F}}$ and $\Phi \circ \Psi \cong \mathrm{id}_{\mathscr{F}'}$. We say that Ψ is *quasi-inverse* of Φ.

[1] More precisely, this means the following. By choosing a preferred "co-cartesian lift" for each $\xi \in \mathrm{Ob}(\mathscr{F}(A))$ under $A \to B$, (which is called *a cleavage* [77, Definition 3.9]), one gets a so-called "pseudo-functor" $A \mapsto \mathscr{F}(A)$ from $\mathscr{E}$ to the "category" of groupoids. We will not work with pseudo-functors. For more discussion on pseudo-functors, see [77, 3.1.2].

[2] In general, a "1-morphism" of co-fibered categories is also required to be *co-cartesian*, which means that it sends a co-cartesian morphism to a co-cartesian morphism. Any 1-morphism between categories co-fibered in groupoids is automatically cartesian.

Any 1-morphism $\Phi : \mathscr{F} \to \mathscr{F}'$ over $\mathscr{E}$ induces a functor $\Phi(A) : \mathscr{F}(A) \to \mathscr{F}'(A)$ on fibers for each $A \in \mathrm{Ob}(\mathscr{E})$. The following proposition gives a useful fiber-criterion for a 1-morphism to be fully faithful or 1-isomorphism. The proof can be found in Prop 3.36 and Lemma 3.37 of [77].

Proposition 10.1.6. *A 1-morphism $\Phi : \mathscr{F} \to \mathscr{F}'$ over $\mathscr{E}$ is a 1-isomorphism (respectively, fully faithful as a functor) if and only if $\Phi(A)$ is an equivalence of categories (respectively, fully faithful) for each $A \in \mathrm{Ob}(\mathscr{E})$.*

The equality of 1-morphisms is often too restrictive; it is more natural to allow 2-isomorphisms in place of equality. For example, we often need to consider *2-commutative diagrams* (instead of commutative diagrams) of 1-morphisms, which means a diagram of 1-morphisms with a *fixed* 2-isomorphism[3] for each two paths with the same source and target (in a "compatible" manner if there are more than two different paths with the same source and target[4]). This often makes the precise statements more complicated than the actual contents are.

We define 2-fiber product following [74, Def 2.2.7], which is *different* from the fiber product (or 1-fiber product) of categories as defined in [73, 1, Exp VI, §3] which requires the diagram below (10.1.7.3) to *commute* for a unique Φ.

Definition 10.1.7. Let $\mathscr{F}$, $\mathscr{F}_1$ and $\mathscr{F}_2$ be $\mathscr{E}$-groupoids, and let $\Phi_i : \mathscr{F}_i \to \mathscr{F}$ for $i = 1, 2$ be 1-morphisms over $\mathscr{E}$. Then by *2-fiber product*, we mean an $\mathscr{E}$-groupoid $\mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2$, equipped with 1-morphisms $\mathrm{pr}_i : \mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2 \to \mathscr{F}_i$, and a 2-isomorphism $\omega : \Phi_1 \circ \mathrm{pr}_1 \xrightarrow{\sim} \Phi_2 \circ \mathrm{pr}_2$, which satisfies the following "2-universal property."

(F1) For any $\mathscr{E}$-groupoid $\mathscr{G}$, 1-morphisms $\Psi_i : \mathscr{G} \to \mathscr{F}_i$ for $i = 1, 2$, and 2-isomorphism $\psi : \Phi_1 \circ \Psi_1 \xrightarrow{\sim} \Phi_2 \circ \Psi_2$, there exist a 1-morphism $\Psi : \mathscr{G} \to \mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2$ and 2-

[3] We always fix a 2-isomorphism between each pair of paths in a 2-commutative diagram, even though the 2-isomorphisms will be omitted from the notations.

[4] We will not be precise on this, but the diagram (10.1.7.3) is an example of this.

isomorphisms $\phi_i : \Psi_i \xrightarrow{\sim} \mathrm{pr}_i \circ \Psi$ for $i = 1, 2$, which makes the following diagram commute.

(10.1.7.1)

$$\begin{CD} \Phi_1 \circ \Psi_1 @>{\psi}>> \Phi_2 \circ \Psi_2 \\ @V{\text{“}\Phi_1 \circ \phi_1\text{”}}VV @VV{\text{“}\Phi_2 \circ \phi_2\text{”}}V \\ \Phi_1 \circ \mathrm{pr}_1 \circ \Psi @>>{\text{“}\omega \circ \Psi\text{”}}> \Phi_2 \circ \mathrm{pr}_2 \circ \Psi \end{CD}$$

Here, “$\Phi_i \circ \phi_i$”: $\Phi_i \circ \Psi_i \to \Phi_i \circ \mathrm{pr}_i \circ \Psi$ is the 2-isomorphism induced from the 2-isomorphism ϕ_i, etc.

(F2) For any (Ψ, ϕ_1, ϕ_2) and $(\Psi', \phi'_1, \phi'_2)$ which satisfies (F1), there exists a unique 2-isomorphism $\theta : \Psi \xrightarrow{\sim} \Psi'$, which makes the following diagrams commute for $i = 1, 2$.

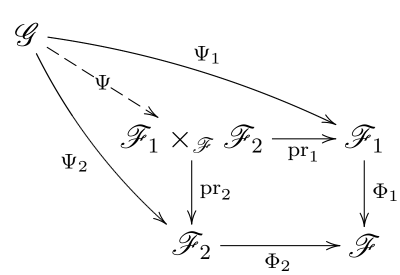
(10.1.7.2)

$$\begin{array}{ccc} \Psi_i & \xrightarrow{\phi_i} & \mathrm{pr}_i \circ \Psi \\ & \searrow^{\phi'_i} & \downarrow{\scriptstyle \text{“}\mathrm{pr}_i \circ \theta\text{”}} \\ & & \mathrm{pr}_i \circ \Psi' \end{array}$$

The fiber product $\mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2$ is unique up to 1-isomorphism, which is unique up to unique 2-isomorphism that makes the diagram (10.1.7.2) commute.

Roughly speaking, (F1) says that for each $(\mathscr{G}, \Psi_1, \Psi_2, \psi)$ as in (F1), we have a 1-morphism Ψ which makes the diagram below 2-commute in every possible way and in every possible sense, and (F2) says that such a Ψ is unique up to unique 2-isomorphism which respects all the 2-isomorphisms between any two compositions of 1-morphisms with the same source and target.

(10.1.7.3)



10.1.8

The 2-fiber product $\mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2$ always exists and provided by the following explicit construction:

1. An object over $A \in \mathrm{Ob}(\mathscr{E})$ is a triple (ξ_1, ξ_2, α) where $\xi_i \in \mathrm{Ob}(\mathscr{F}_i(A))$ for $i = 1, 2$ and $\alpha : \Phi_1(\xi_1) \xrightarrow{\sim} \Phi_2(\xi_2)$ is a morphism in $\mathscr{F}(A)$.

2. A morphism $(\xi_1, \xi_2, \alpha) \to (\eta_1, \eta_2, \beta)$ is a pair $(\gamma_i : \xi_i \to \eta_i)_{i=1,2}$ such that $\beta \circ \Phi_1(\gamma_1) = \Phi_2(\gamma_2) \circ \alpha$.

3. The functors pr_i is $(\xi_1, \xi_2, \alpha) \mapsto \xi_i$ and $(\gamma_1, \gamma_2) \mapsto \gamma_i$. We define the 2-isomorphism $\omega : \Phi_1 \circ \mathrm{pr}_1 \xrightarrow{\sim} \Phi_2 \circ \mathrm{pr}_2$ by $\omega_{(\xi_1, \xi_2, \alpha)} = \alpha$.

Remark 10.1.9. We record an immediate property of 2-fiber product. If $\Phi_1 : \mathscr{F}_1 \to \mathscr{F}$ is a 1-morphism which makes $\mathscr{F}_1$ an $\mathscr{F}$-groupoid (for example, if $\mathscr{F} = \mathscr{E}$), then $\mathrm{pr}_2 : \mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2 \to \mathscr{F}_2$ is an $\mathscr{F}_2$-groupoid. The proof uses the strong co-cartesian property (Remark 10.1.3). Combining this with the last paragraph of Remark 10.1.3, the functor $\mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2 \xrightarrow{\mathrm{pr}_2} \mathscr{F}_2 \xrightarrow{\Pi_2} \mathscr{E}$ is a groupoid over $\mathscr{E}$.

It is also useful to note that if Φ_2 is fully faithful (respectively, 1-isomorphism) then so is pr_1. Indeed, for two objects (ξ_1, ξ_2, α), (η_1, η_2, β) of $\mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2$ and a morphism $\gamma_1 : \xi_1 \to \eta_1$ in $\mathscr{F}_1$, we can always find a unique morphism $\gamma = (\gamma_1, \gamma_2) : (\xi_1, \xi_2, \alpha) \to (\eta_1, \eta_2, \beta)$, so that $\mathrm{pr}_1(\gamma) = \gamma_1$, as follows: considering the following diagram

$$\begin{array}{ccc} \Phi_1(\xi_1) & \xrightarrow[\alpha]{\sim} & \Phi_2(\xi_2) \\ {\scriptstyle \Phi_1(\gamma_1)} \downarrow & & \downarrow {\scriptstyle \beta \circ \Phi_1(\gamma_1) \circ \alpha^{-1}} \\ \Phi_1(\eta_1) & \xrightarrow[\beta]{\sim} & \Phi_2(\eta_2) \end{array}$$

and using the full faithfulness of Φ_2, we let $\gamma_2 : \xi_2 \to \eta_2$ be the unique morphism in $\mathscr{F}_2$ such that $\Phi_2(\gamma_2) = \beta \circ \Phi_1(\gamma_1) \circ \alpha^{-1}$. If, furthermore, Φ_2 is essentially surjective, then so is pr_1: for any $\xi_1 \in \mathrm{Ob}(\mathscr{F}_1(A))$, we may find $\xi_2 \in \mathrm{Ob}(\mathscr{F}_2(A))$ and $\alpha : \Phi_1(\xi_1) \xrightarrow{\sim} \Phi_2(\xi_2)$ in $\mathscr{F}(A)$, so we have $\mathrm{pr}_1(\xi_1, \xi_2, \alpha) = \xi_1$.

Remark 10.1.10. Consider two functors $|\mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2|$ and $|\mathscr{F}_1| \times_{|\mathscr{F}|} |\mathscr{F}_2|$ on $\mathscr{E}$. We have a natural transformation

$$(10.1.10.1) \qquad |\mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2| \to |\mathscr{F}_1| \times_{|\mathscr{F}|} |\mathscr{F}_2|; \qquad [(\xi_1, \xi_2, \alpha)] \mapsto ([\xi_1], [\xi_2]),$$

which is seen to be surjective. But this natural transformation does *not* have to be an isomorphism, that is to say, the formation of 2-fiber product does *not* commute with the passage to the associated functor. This is why we work with "deformation groupoids," rather than deformation functors. This is observed by [51, (A.6)].

Here is an example when the natural transformation (10.1.10.1) is not an isomorphism. We start with a non-trivial group G, and we will construct a "universal G-torsor" over a fixed category $\mathscr{E}$ as follows. Define a category $\mathscr{E}/G$ by $\mathrm{Ob}(\mathscr{E}/G) = \mathrm{Ob}(\mathscr{E})$ and $\mathrm{Hom}_{\mathscr{E}/G}(A, B) = \mathrm{Hom}_{\mathscr{E}}(A, B) \times G$, and define a functor $\Pi_{\mathscr{E}/G} : \mathscr{E}/G \to \mathscr{E}$ by the identity map on objects and the natural projection $\mathrm{Hom}_{\mathscr{E}}(A, B) \times G \twoheadrightarrow \mathrm{Hom}_{\mathscr{E}}(A, B)$ on morphisms. Clearly $\mathscr{E}/G$ is an $\mathscr{E}$-groupoid. Viewing $\mathscr{E}$ as an $\mathscr{E}$-groupoid via the identity functor, we have a 1-morphism $\Phi : \mathscr{E} \to \mathscr{E}/G$ defined by the identity map on objects and $\Phi(f : A \to B) = (f, e_G)$ on morphisms. Then both functors $|\mathscr{E}|$ and $|\mathscr{E}/G|$ maps any object $A \in \mathscr{E}$ to an one-element set $\{A\}$, and $|\Phi|$ is the "identity natural transform" between these functors. So $|\mathscr{E}| \times_{|\mathscr{E}/G|} |\mathscr{E}|$ map any object $A \in \mathscr{E}$ to an one-element set $\{A\}$.

Now, let us work out the 2-fiber product $\mathscr{E} \times_{\mathscr{E}/G} \mathscr{E}$. Using §10.1.8, objects of a fiber $(\mathscr{E} \times_{\mathscr{E}/G} \mathscr{E})(A)$ are of the form (A, A, α), where α is any element of G and all the morphisms in $(\mathscr{E} \times_{\mathscr{E}/G} \mathscr{E})(A)$ are identity morphisms. In other words, the groupoid $(\mathscr{E} \times_{\mathscr{E}/G} \mathscr{E})(A)$ is a set, and is in bijection with G. In particular the natural transformation $\left|\mathscr{E} \times_{\mathscr{E}/G} \mathscr{E}\right| = \mathscr{E} \times_{\mathscr{E}/G} \mathscr{E} \to |\mathscr{E}| \times_{|\mathscr{E}/G|} |\mathscr{E}|$ cannot be an isomorphism because $(\mathscr{E} \times_{\mathscr{E}/G} \mathscr{E})(A) \cong G$ and $|\mathscr{E}|(A) \times_{|\mathscr{E}/G|(A)} |\mathscr{E}|(A)$ is an one-element set.

Lastly, we remark that if either Φ_1 or Φ_2 is fully faithful as a functor, then the natural transformation $|\mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2| \to |\mathscr{F}_1| \times_{|\mathscr{F}|} |\mathscr{F}_2|$ is in fact isomorphism. This can be read off from the discussion in Remark 10.1.9.

We define one more operation which will be needed soon.

Definition 10.1.11. Let $\mathscr{E}' \to \mathscr{E}$ be any functor, which may not define a category co-fibered over $\mathscr{E}$. For an $\mathscr{E}$-groupoid $\Pi : \mathscr{F} \to \mathscr{E}$, we define a category $\mathscr{F}_{\mathscr{E}'}$ as follows: objects are pairs (ξ, A') where $\xi \in \mathrm{Ob}(\mathscr{F})$ and $A' \in \mathrm{Ob}(\mathscr{E}')$ map to the same object in $\mathscr{E}$ (not just isomorphic ones), and morphisms $(\xi, A') \to (\eta, B')$ are pairs $(\xi \to \eta, A' \to B')$ which map to the same morphism in $\mathscr{E}$. In [73, 1, Exp VI, §3], this category is called the fiber product and denoted by $\mathscr{F} \times_{\mathscr{E}} \mathscr{E}'$, but this is *not* the 2-fiber product even if $\mathscr{E}'$ happens to be co-fibered over $\mathscr{E}$.

There are natural "projection functors" $\mathscr{F}_{\mathscr{E}'} \to \mathscr{F}$ and $\mathscr{F}_{\mathscr{E}'} \to \mathscr{E}'$, and it is straightforward to check that the second projection makes $\mathscr{F}_{\mathscr{E}'}$ an $\mathscr{E}'$-groupoid. (This is stated, without proof, in [73, 1, Exp VI, Prop 6.6].) We call this $\mathscr{E}'$-groupoid the *base change* of $\mathscr{F}$ over $\mathscr{E}'$.

In the special case when $\mathscr{E}'$ is a subcategory (respectively, a full subcategory), one can show that $\mathscr{F}_{\mathscr{E}'}$ can be viewed as a subcategory of $\mathscr{F}$ (respectively, a full subcategory of $\mathscr{F}$) by the first projection. In this case, we often write $\mathscr{F}|_{\mathscr{E}'}$ instead of $\mathscr{F}_{\mathscr{E}'}$, and call it the *restriction* of $\mathscr{F}$ over $\mathscr{E}'$.

Remark 10.1.12. We end this section with a remark on "aesthetics." By *choosing* a cleavage [77, Def 3.9], in other words a preferred cartesian lift for each arrow in $\mathscr{E}$, we can associate to an $\mathscr{E}$-groupoid $\Pi : \mathscr{F} \to \mathscr{E}$ a "pseudo-functor" $A \mapsto \mathscr{F}(A)$

from $\mathscr{E}$ to a "category" of groupoids. This is called a pseudo-functor because the equalities in the axioms of functor are replaced by isomorphisms. The notions of pseudo-functor on $\mathscr{E}$ and groupoids over $\mathscr{E}$ with (a fixed) cleavage are equivalent.[5] See Prop 3.11 and §3.1.3 in [77].

This pseudo-functor description of $\mathscr{E}$-groupoids may appeal as more satisfactory one. For example to define a pseudo-functor, one just have to define a fiber $\mathscr{F}(A)$ for each $A \in \mathrm{Ob}(\mathscr{E})$ and specify how they "pull back." On the other hand, unless an $\mathscr{E}$-groupoid $\mathscr{F}$ is co-fibered in sets §10.2.1 or split [77, Def 3.12], there is no canonical or preferred choice of cleavage on $\mathscr{F}$. So we do not choose a cleavage, unless it does not sacrifice concreteness.

10.2 The 2-Yoneda lemma and representibility

The goal of this section is to define representability for an $\mathscr{E}$-groupoid. We first explain how to view a functor $\mathscr{E} \to (\mathbf{Sets})$ as an $\mathscr{E}$-groupoid, and identify the class of $\mathscr{E}$-groupoids which come from functors. Then we may define the representability of an $\mathscr{E}$-groupoid using the representability of a functor.

For the purpose of completeness, we state without proof the 2-Yoneda lemma, which plays the same role for $\mathscr{E}$-groupoids as Yoneda lemma does for functors. Roughly speaking, the 2-Yoneda lemma says that an object $A \in \mathrm{Ob}(\mathscr{E})$ can be viewed as an $\mathscr{E}$-groupoid. Even though it is not technically necessary to discuss 2-Yoneda lemma[6], it offers conceptual clarification.

10.2.1 Functors and categories co-fibered in sets

We view a set as a groupoid[7] where all morphisms are identities. We say a groupoid $\Pi : \mathscr{F} \to \mathscr{E}$ is *co-fibered in sets* if the fiber $\mathscr{F}(A)$ for each $A \in \mathrm{Ob}(\mathscr{E})$ is

[5] By the axiom of choice, any $\mathscr{E}$-groupoid has a cleavage.
[6] It is possible to define the representablilty of a functor without stating the Yoneda lemma.
[7] We always assume that the objects of a groupoid form a set.

a set. This is equivalent to requiring that for each $\xi \in \mathrm{Ob}(\mathscr{F}(A))$ and $f : A \to B$, there exists only one (co-cartesian) arrow $\xi \to \eta$ over f. See [77, Prop 3.25] for the proof.

It is not hard to check that if $\Pi : \mathscr{F} \to \mathscr{E}$ is co-fibered in sets, then the assignment $A \mapsto \mathscr{F}(A)$ defines a functor $F : \mathscr{E} \to (\mathbf{Sets})$. In fact, the converse is also true. Namely, for a given functor $F : \mathscr{E} \to (\mathbf{Sets})$, we can construct a category $\Pi : \mathscr{F} \to \mathscr{E}$ co-fibered in sets with $\mathscr{F}(A) = F(A)$ for each $A \in \mathrm{Ob}(\mathscr{E})$. We give the construction without proof. Define a category $\mathscr{F}$, so that an object is a pair (ξ, A) where $\xi \in F(A)$ and a morphism $(\xi, A) \to (\eta, B)$ is an arrow $f : A \to B$ in $\mathscr{E}$ such that $F(f) : F(A) \to F(B)$ takes ξ into η. By forgetting ξ, we obtain $\mathscr{F} \to \mathscr{E}$ which is co-fibered in sets.

From now, we often use the same letter F to denote the category co-fibered in sets which corresponds to a functor $F : \mathscr{E} \to (\mathbf{Sets})$. Note the groupoid $\mathscr{H}om_{\mathcal{E}}(F, F')$ of 1-morphisms of categories co-fibered in sets *is* a set. For any 1-morphism $\psi : F \to F'$ of categories co-fibered in sets over $\mathscr{E}$, one obtains a natural transformations of functors $F \to F'$ by putting $\psi_A : F(A) \to F'(A)$ for each $A \in \mathrm{Ob}(\mathscr{E})$. Conversely, for given functors $F, F' : \mathscr{E} \to (\mathbf{Sets})$ and a natural transformation $\psi : F \to F'$, one obtains a 1-morphism $F \to F'$ over $\mathscr{E}$ by putting $(\xi, A) \mapsto (\psi_A(\xi), A)$ and $\left[(\xi, A) \xrightarrow{f} (\eta, B)\right] \mapsto \left[(\psi_A(\xi), A) \xrightarrow{f} (\psi_B(\eta), B)\right]$, where $f : (\xi, A) \to (\eta, B)$ means the morphism defined by $f : A \to B$. Therefore, we conclude that the notions of category co-fibered in sets and functor are interchangeable, and the set $\mathscr{H}om_{\mathscr{E}}(F, F')$ of 1-morphisms is naturally in bijection with the set of natural transformations $F \to F'$ of functors.

For each $\mathscr{E}$-groupoid $\mathscr{F}$, we have associated a functor $|\mathscr{F}|$ (Remark 10.1.4). We denote by the same notation $|\mathscr{F}|$ the category co-fibered in sets which corresponds to the functor $|\mathscr{F}|$. Then we obtain a 1-morphism $\mathscr{F} \to |\mathscr{F}|$ by associating to each object ξ the "isomorphism class" of ξ over $\Pi(\xi)$.

10.2.2 Categories co-fibered in equivalence relations

The notion of category co-fibered in sets is *not* stable under 1-isomorphisms. In this subsection, we identify the class of $\mathscr{E}$-groupoids which are 1-isomorphic to categories co-fibered in sets.

We say a groupoid $\mathscr{C}$ is an *equivalence relation*[8] if there exists at most one morphism between any two objects of $\mathscr{C}$. A groupoid $\mathscr{C}$ is an equivalence relation if and only if the natural functor $\mathscr{C} \to |\mathscr{C}|$, which associates to $\xi \in \mathrm{Ob}(\mathscr{C})$ the isomorphism class of ξ, is an equivalence of categories. In other words, an equivalence relation is a groupoid which is equivalent to a set (viewed as a groupoid).

We say an $\mathscr{E}$-groupoid $\mathscr{F}$ is *co-fibered in equivalence of categories* if for each $A \in \mathrm{Ob}(\mathscr{E})$, the fiber $\mathscr{F}(A)$ is an equivalence relation. To rephrase, for any objects $\xi, \eta \in \mathrm{Ob}(\mathscr{F})$ and a morphism $f : \Pi(\xi) \to \Pi(\eta)$ in $\mathscr{E}$, there exists a unique morphism $\xi \to \eta$ over f. It follows from Proposition 10.1.6 that an $\mathscr{E}$-groupoid $\mathscr{F}$ is co-fibered in equivalence relations if and only if the natural 1-morphism $\mathscr{F} \to |\mathscr{F}|$ is a 1-isomorphism. In other words, an $\mathscr{E}$-groupoid $\mathscr{F}$ is co-fibered in equivalence relations if and only if it is 1-isomorphic to a category co-fibered in sets over $\mathscr{E}$.

Now to each $A \in \mathrm{Ob}(\mathscr{E})$, we associate a category $(\mathscr{E}/A)$ co-fibered in sets over $\mathscr{E}$.

Definition 10.2.3. Let $A \in \mathrm{Ob}(\mathscr{E})$. We denote by $(\mathscr{E}/A)$ the category co-fibered in sets which correspond to the functor $\mathrm{Hom}_{\mathscr{E}}(A, -) : \mathscr{E} \to (\mathbf{Sets})$. Explicitly, $(\mathscr{E}/A)$ can be described as follows.

1. An object is an arrow $f : A \to B$ in $\mathscr{E}$.

[8] For an equivalence relation $\mathscr{C}$, we obtain an "equivalence relation" on $\mathrm{Ob}(\mathscr{C})$ in the usual sense: $\xi \sim \eta$ for $\xi, \eta \in \mathrm{Ob}(\mathscr{C})$ if and only if $\mathrm{Hom}_{\mathscr{C}}(\xi, \eta)$ is non-empty. Conversely, for an "equivalence relation" $R \subset X \times X$, we can construct an equivalence relation $\mathscr{C}$ with $\mathrm{Ob}(sC) = X$, and for $\xi, \eta \in X$, set $\mathrm{Hom}_{\mathscr{C}}(\xi, \eta) = \{*\}$ if and only if $\xi \sim \eta$.

2. A morphism $\alpha : (A \xrightarrow{f_1} B_1) \to (A \xrightarrow{f_2} B_2)$ is an arrow $\alpha : B_1 \to B_2$ such that $\alpha \circ f_1 = f_2$.

3. The functor $\Pi_A : (\mathscr{E}/A) \to \mathscr{E}$ is defined by forgetting the morphism from A. In other words, $\Pi_A(A \to B) = B$ and $\Pi_A \left[(A \to B_1) \xrightarrow{\alpha} (A \to B_2) \right] = [B_1 \xrightarrow{\alpha} B_2]$.

For any $f : A' \to A$, we have a natural transformation $\mathrm{Hom}_{\mathscr{E}}(A, -) \xrightarrow{-\circ f} \mathrm{Hom}_{\mathscr{E}}(A', -)$ by pre-composing f. We let $(\mathscr{E}/f) : (\mathscr{E}/A) \to (\mathscr{E}/A')$ denote the corresponding 1-morphism. Explicitly, $(\mathscr{E}/f) : (A \to B) \mapsto (A' \xrightarrow{f} A \to B)$ on objects $(A \to B) \in \mathrm{Ob}(\mathscr{E}/A)$.

The Yoneda lemma and the discussion in §10.2.1 implies that the morphisms $A \to B$ in $\mathscr{E}$ and the 1-morphisms $(\mathscr{E}/B) \to (\mathscr{E}/A)$ are in bijection. In fact, we have the following stronger version of the "Yoneda lemma" for $\mathscr{E}$-groupoids.

Let $\mathscr{F}$ be an $\mathscr{E}$-groupoid. Define a functor $ev_A : \mathscr{H}om_{\mathscr{E}}((\mathscr{E}/A), \mathscr{F}) \to \mathscr{F}(A)$ by "evaluating" at the universal object $\mathrm{id}_A \in \mathrm{Ob}(\mathscr{E}/A)$. More precisely,

1. For any 1-morphism $\Phi : (\mathscr{E}/A) \to \mathscr{F}$, we define $ev_A(\Phi) := \Phi(\mathrm{id}_A) \in \mathrm{Ob}(\mathscr{F}(A))$ by evaluating at the "universal object" $(A \xrightarrow{\mathrm{id}_A} A) \in \mathrm{Ob}(\mathscr{E}/A)$.

2. For two 1-morphisms $\Phi, \Phi' : (\mathscr{E}/A) \to \mathscr{F}$ and a 2-isomorphism $\psi : \Phi \to \Phi'$, we put $ev_A(\psi) = \psi_{\mathrm{id}_A}$, which is a morphism in $\mathscr{F}(A)$.

Proposition 10.2.4 (2-Yoneda lemma)**.** *The functor* $ev_A : \mathscr{H}om_{\mathscr{E}}((\mathscr{E}/A), \mathscr{F}) \to \mathscr{F}(A)$ *is an equivalence of categories.*

If $\mathscr{F}$ is co-fibered in sets then 2-Yoneda lemma recovers the usual Yoneda lemma for functors.

Sketch of the proof. We indicate the idea how to construct a quasi-inverse of ev_A. For any object $\xi \in \mathrm{Ob}(\mathscr{F}(A))$, we can define a 1-morphism $\Phi_\xi : (\mathscr{E}/A) \to \mathscr{F}$ so that $\Phi_\xi(\mathrm{id}_A) = \xi$, as follows. For any $(A \xrightarrow{f} B) \in \mathrm{Ob}(\mathscr{E}/A)$, take a co-cartesian lift $\xi \to \eta$ over f and put $\Phi_\xi(A \xrightarrow{f} B) = \eta$. If $(A \xrightarrow{f} B) \xrightarrow{g} (A \xrightarrow{f'} B')$ is a morphism in $(\mathscr{E}/A)$, then the strong co-cartesian property (Remark 10.1.3) gives a morphism $\Phi_\xi(A \xrightarrow{f} B) \to \Phi_\xi(A \xrightarrow{f'} B')$ over g, which we take as $\Psi_\xi(g)$. One can check that Φ_ξ is well-defined and that $\xi \mapsto \Phi_\xi$ gives a quasi-inverse to ev_A. □

Before we define the notion of representability for $\mathscr{E}$-groupoids, we record the following useful fact. Let $\Pi : \mathscr{F} \to \mathscr{E}$ be a groupoid, and let $\xi \in \mathrm{Ob}(\mathscr{F}(A))$ for $A \in \mathrm{Ob}(\mathscr{E})$. We may define a groupoid $(\mathscr{F}/\xi)$ over $\mathscr{F}$, and by Remark 10.1.3, $(\mathscr{F}/\xi) \to \mathscr{F} \xrightarrow{\Pi} \mathscr{E}$ is a groupoid over $\mathscr{E}$. On the other hand, the functor Π induces a 1-morphism $\Pi|_A : (\mathscr{F}/\xi) \to (\mathscr{E}/A)$ over $\mathscr{E}$ in an obvious manner. The functors $(\mathscr{F}/\xi) \xrightarrow{\Pi|_A} (\mathscr{E}/A) \to \mathscr{E}$ and $(\mathscr{F}/\xi) \to \mathscr{F} \xrightarrow{\Pi} \mathscr{E}$ are identical, hence give the identical $\mathscr{E}$-groupoid structure on $(\mathscr{F}/\xi)$.

The following lemma is just a re-phrasing of the strong co-cartesian property (Remark 10.1.3).

Lemma 10.2.5. *The 1-morphism* $\Pi|_A : (\mathscr{F}/\xi) \to (\mathscr{E}/A)$ *over* $\mathscr{E}$ *is always a 1-isomorphism.*

Definition/Proposition 10.2.6. An $\mathscr{E}$-groupoid $\mathscr{F}$ is called *representable* if the following equivalent properties hold.

(R1) For some $A \in \mathrm{Ob}(\mathscr{E})$, there exists an 1-isomorphism $\Phi : (\mathscr{E}/A) \xrightarrow{\sim} \mathscr{F}$. In this case, we say that A *represents* $\mathscr{F}$, and the object $\xi := \Phi(\mathrm{id}_A) \in \mathrm{Ob}(\mathscr{F}(A))$ is called the *universal object.*

(R2) For some $\xi \in \mathrm{Ob}(\mathscr{F})$, there exists an 1-isomorphism $\widetilde{\Phi} : (\mathscr{F}/\xi) \xrightarrow{\sim} \mathscr{F}$ over $\mathscr{E}$. In this case, we say that $A := \Pi_{\mathscr{F}/\mathscr{E}}(\xi) \in \mathrm{Ob}(\mathscr{E})$ *represents* $\mathscr{F}$, and the object ξ is called the the *universal object.*

Furthermore, the objects A and ξ which satisfy one of (R1) and (R2), if exist, satisfy the other. The representing object $A \in \mathrm{Ob}(\mathscr{E})$ is unique up to canonical isomorphism in $\mathscr{E}$, and the universal object ξ is unique up to canonical isomorphism in $\mathscr{F}$.

Proof. The uniqueness aspect of the statement follows from 2-Yoneda lemma, like in the case of functors, and the rest of the claims follow from Lemma 10.2.5. □

Recall that the $\mathscr{E}$-groupoid $(\mathscr{E}/A)$ co-fibered in sets corresponds to the representable functor $\mathrm{Hom}_{\mathscr{E}}(A, -)$, therefore this notion, especially (R1), recovers the usual representability for functors if $\mathscr{F}$ is co-fibered in sets. Also (R2) (or Lemma 10.2.5) says that for some object $\xi \in \mathrm{Ob}(\mathscr{F})$, the $\mathscr{E}$-groupoid $(\mathscr{F}/\xi)$ is representable.

Even if $\mathscr{F}$ is representable, it does not have to be co-fibered in sets over $\mathscr{E}$ but is necessarily co-fibered in equivalence categories. Conversely, the $\mathscr{E}$-groupoid $\mathscr{F}$ is representable if and only if the functor $|\mathscr{F}|$ is representable and $\mathscr{F}$ is co-fibered in equivalence relations.

Definition 10.2.7.

1. A 1-morphism $\Phi : \mathscr{F}' \to \mathscr{F}$ over $\mathscr{E}$ is called *relatively representable*[9] if for each $\xi \in \mathrm{Ob}(\mathscr{F})$, the 2-fiber product $\mathscr{F}'_\xi := (\mathscr{F}/\xi) \times_{\mathscr{F},\Phi} \mathscr{F}'$, which is an $\mathscr{E}$-groupoid by Remarks 10.1.9, is representable over $\mathscr{E}$.

2. Assume that $\mathscr{E}$ is a subcategory of the category of rings. Then Φ is called *formally smooth* if the associated natural transformation $|\Phi| : |\mathscr{F}'| \to |\mathscr{F}|$ is formally smooth.

For a property $\mathbf{P}$ of objects of $\mathscr{E}$, we say a representable $\mathscr{E}$-groupoid $\mathscr{F}$ has the property $\mathbf{P}$ if the representing object $A \in \mathrm{Ob}(\mathscr{E})$ does. Similarly for a property $\mathbf{P}$

[9] If $\mathscr{F}$ and $\mathscr{F}'$ are co-fibered in sets and Φ is fully faithful as a functor (i.e., if Φ is a monomorphism of functors $|\mathscr{F}'| \to |\mathscr{F}|$), then this definition of relative representability coincides with seemingly more popular one, e.g. [64, §19], by Schlessinger's criterion.

of morphisms in $\mathscr{E}$, we say a relatively representable 1-morphism $\Phi : \mathscr{F}' \to \mathscr{F}$ has the property **P** if the morphism in $\mathscr{E}$ that represents $\Phi_\xi := \mathrm{pr}_1 : \mathscr{F}'_\xi \to (\mathscr{F}/\xi)$ has the property **P**. (By assumption, both $\mathscr{F}'_\xi$ and $(\mathscr{F}/\xi)$ are representable over $\mathscr{E}$.) On the other hand, we define formal smoothness for any 1-morphism Φ, not necessarily relatively representable.

One can check, from §10.1.8 and the definitions, that relative representability and formal smoothness stable under "2-categorical base change." More precisely, we have

Proposition 10.2.8. *Assume that we have the following "2-cocartesian diagram"*

$$\begin{array}{ccc} \mathscr{G}' & \xrightarrow{\Phi'} & \mathscr{G} \\ \downarrow & & \downarrow \\ \mathscr{F}' & \xrightarrow{\Phi} & \mathscr{F} \end{array},$$

in other words, the natural 1-morphism $\mathscr{G}' \to \mathscr{F}' \times_{\mathscr{F}} \mathscr{G}$, induced from the above diagram by 2-categorical universal property, is a 1-isomorphism. Then the following hold.

1. *If Φ is formally smooth, then so is Φ'.*

2. *If Φ is relatively representable, then so is Φ'. Furthermore, if $x \in \mathrm{Ob}(\mathscr{G}(A))$ maps to $\xi \in \mathrm{Ob}(\mathscr{F}(A))$ by the 1-morphism $\mathscr{G} \to \mathscr{F}$, then the 1-morphism $\mathscr{G}'_x \to \mathscr{F}'_\xi$ of $(\mathscr{E}/A)$-groupoids induced by the 2-categorical universal property is a 1-isomorphism, so the representing objects of both $(\mathscr{E}/A)$-groupoids are isomorphic.*

10.3 Deformation and framed deformation groupoids

We now define groupoids whose objects correspond to "deformations" or "framed deformations" of a residual $\boldsymbol{\mathcal{G}}_K$-representation. They are groupoids over the following "base categories" $\mathscr{E} = \mathfrak{AR}_{\mathfrak{o}}, \widehat{\mathfrak{AR}_{\mathfrak{o}}}, \mathfrak{Aug}_{\mathfrak{o}}, \widehat{\mathfrak{Aug}_{\mathfrak{o}}}$, which will now be defined.

10.3.1 Base categories

Let $\mathfrak{o}$ be a local domain that is a finite extension of $\mathfrak{o}_0$ with residue field $\mathbb{F}$, and put $F := \mathrm{Frac}(\mathfrak{o})$. Let $\mathfrak{AR}_\mathfrak{o}$ be the category of artin local $\mathfrak{o}$-algebras A whose residue field is $\mathbb{F}$ (via the natural map). Similarly, let $\widehat{\mathfrak{AR}}_\mathfrak{o}$ be the category of complete local noetherian $\mathfrak{o}$-algebras with residue field $\mathbb{F}$.

We often need to consider "deformations" over a ring which is not a complete local noetherian ring, so we introduce the category $\mathfrak{Aug}_\mathfrak{o}$ of pairs (A, I) where A is an $\mathfrak{o}$-algebra such that π_0 is nilpotent in A, and $I \subset A$ is an ideal containing $\mathfrak{m}_\mathfrak{o}A$ such that $I^N = 0$ for some N. Morphisms $(A, I) \to (B, J)$ in $\mathfrak{Aug}_\mathfrak{o}$ are $\mathfrak{o}$-algebra maps $A \to B$ which send I into J. Using the fully faithful functor $\mathfrak{AR}_\mathfrak{o} \to \mathfrak{Aug}_\mathfrak{o}$, $A \mapsto (A, \mathfrak{m}_A)$, we regard $\mathfrak{AR}_\mathfrak{o}$ as a full subcategory of $\mathfrak{Aug}_\mathfrak{o}$. Any $\mathfrak{o}/\mathfrak{m}_\mathfrak{o}$-algebra A can be viewed as an object in $\mathfrak{Aug}_\mathfrak{o}$ by setting $I = \{0\}$. Also, $A := (\mathfrak{o}/\mathfrak{m}_\mathfrak{o}^N)[t]$ together with $I := \mathfrak{m}_\mathfrak{o} \cdot A$ defines an object in $\mathfrak{Aug}_\mathfrak{o}$ that is not artinian with non-zero I. In many cases, the nilpotent ideal I does not play an important role and can be replaced by bigger nilpotent ideal, for example the nilradical of A if A is noetherian.

We may also define a category $\widehat{\mathfrak{Aug}}_\mathfrak{o}$ of pairs (A, I) where A is an topological $\mathfrak{o}$-algebra which is an admissible ring (so necessarily π_0 is topologically nilpotent), and I is an ideal which contains $\mathfrak{m}_\mathfrak{o}A$ and such that $I/\mathfrak{m}_\mathfrak{o}^n A \subset A/\mathfrak{m}_\mathfrak{o}^n A$ is nilpotent for each n. Morphisms $(A, I) \to (B, J)$ are continuous $\mathfrak{o}$-maps which send I into J. We have a fully faithful functor $\widehat{\mathfrak{AR}}_\mathfrak{o} \to \widehat{\mathfrak{Aug}}_\mathfrak{o}$, $A \mapsto (A, \mathfrak{m}_A)$, so we regard $\widehat{\mathfrak{AR}}_\mathfrak{o}$ as a full subcategory of $\widehat{\mathfrak{Aug}}_\mathfrak{o}$. We will not use this category very often.

10.3.2 Deformation groupoid

Let $T_\mathbb{F}$ be a finite-dimensional $\mathbb{F}$-vector space and let $\rho_\mathbb{F} : \boldsymbol{\mathcal{G}}_K \to \mathrm{GL}(T_\mathbb{F})$ be a continuous homomorphism. We define the category $\mathscr{D}_{\rho_\mathbb{F}}$ of *deformations* of $\rho_\mathbb{F}$, and the functor $\Pi : \mathscr{D}_{\rho_{\mathbb{F}}} \to \mathfrak{AR}_{\mathfrak{o}}$ which makes $\mathscr{D}_{\rho_{\mathbb{F}}}$ a groupoid over $\mathfrak{AR}_{\mathfrak{o}}$, as follows. An object over $A \in \mathfrak{AR}_{\mathfrak{o}}$ is (ρ_A, T_A, ι_A), where T_A is a finite free A-module with a continuous A-linear action of $\boldsymbol{\mathcal{G}}_K$ by ρ_A, and $\iota_A : T_{\mathbb{F}} \xrightarrow{\sim} T_A \otimes_A (A/\mathfrak{m}_A)$ is a $\boldsymbol{\mathcal{G}}_K$-equivariant isomorphism over the natural isomorphism $\mathbb{F} \xrightarrow{\sim} A/\mathfrak{m}_A$. Given a morphism $f : A \to B$ in $\mathfrak{AR}_{\mathfrak{o}}$, we define a *morphism* $\alpha : (\rho_A, T_A, \iota_A) \to (\rho'_B, T'_B, \iota'_B)$ *over* f to be an equivalence class of $\boldsymbol{\mathcal{G}}_K$-equivariant morphism $T_A \to T'_B$ over f which respects ι_A and ι'_B; i.e., α makes the following diagram commute:

(10.3.2.1)
$$\begin{array}{ccc} T_A \otimes_A (A/\mathfrak{m}_A) & \xrightarrow{\overline{\alpha}} & T'_B \otimes_B (B/\mathfrak{m}_B) \\ {\scriptstyle \iota_A}\nwarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\nearrow{\scriptstyle \iota'_B} \\ & T_{\mathbb{F}} & \end{array},$$

where $\overline{\alpha}$ is induced by α, and two morphisms α and α' are *equivalent* if one is a $(1 + \mathfrak{m}_B)$-multiple of the other. Since any morphism over id_A is necessarily an isomorphism by Nakayama's lemma, the the category $\mathscr{D}_{\rho_{\mathbb{F}}}(A)$ of objects over A and morphisms over id_A is a groupoid for any $A \in \mathfrak{AR}_{\mathfrak{o}}$. Furthermore, if $\mathrm{End}_{\boldsymbol{\mathcal{G}}_K}(\rho_{\mathbb{F}}) \cong \mathbb{F}$, then for any deformation ρ_A we have $\mathrm{End}_{\boldsymbol{\mathcal{G}}_K}(\rho_A) \cong A$ by Nakayama's lemma applied to $A \hookrightarrow \mathrm{End}_{\boldsymbol{\mathcal{G}}_K}(\rho_A)$. So the groupoid $\mathscr{D}_{\rho_{\mathbb{F}}}(A)$ is an equivalence relation for any $A \in \mathfrak{AR}_{\mathfrak{o}}$ when $\mathrm{End}_{\boldsymbol{\mathcal{G}}_K}(\rho_{\mathbb{F}}) \cong \mathbb{F}$.

One can check that the assignments $(\rho_A, T_A, \iota_A) \mapsto A$ and $\alpha \mapsto f$ define a functor $\Pi : \mathscr{D}_{\rho_{\mathbb{F}}} \to \mathfrak{AR}_{\mathfrak{o}}$, and the fiber over $A \in \mathfrak{AR}_{\mathfrak{o}}$ is exactly $\mathscr{D}_{\rho_{\mathbb{F}}}(A)$. By the universal property of tensor products, giving a morphism α in $\mathscr{D}_{\rho_{\mathbb{F}}}$ is equivalent to giving a morphism $T_A \otimes_{A,f} A' \xrightarrow{\sim} T'_{A'}$ in $\mathscr{D}_{\rho_{\mathbb{F}}}(A')$. This shows that any morphism in $\mathscr{D}_{\rho_{\mathbb{F}}}$ is co-cartesian, hence $\mathscr{D}_{\rho_{\mathbb{F}}}$ is a groupoid over $\mathfrak{AR}_{\mathfrak{o}}$.

We may repeat this construction by $\mathfrak{AR}_{\mathfrak{o}}$ replaced with $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ and requiring ρ_A to be continuous for the profinite topology on $\boldsymbol{\mathcal{G}}_K$ and the $\mathfrak{m}_A$-adic topology on $\mathrm{Aut}_A(T_A)$, obtaining a groupoid $\widehat{\Pi} : \widehat{\mathscr{D}}_{\rho_{\mathbb{F}}} \to \widehat{\mathfrak{AR}}_{\mathfrak{o}}$ such that we have an "equality" $\mathscr{D}_{\rho_{\mathbb{F}}} = \widehat{\mathscr{D}}_{\rho_{\mathbb{F}}}|_{\mathfrak{AR}_{\mathfrak{o}}}$ of $\mathfrak{AR}_{\mathfrak{o}}$-groupoids. Later in §10.4.1, we give a general recipe to extend a groupoid over $\mathfrak{AR}_{\mathfrak{o}}$ to a groupoid over $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$ via "projective limit", which recovers $\widehat{\mathscr{D}}_{\rho_{\mathbb{F}}}$ when applied to $\mathscr{D}_{\rho_{\mathbb{F}}}$.

Now, we define another groupoid $\widetilde{\Pi} : \widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}} \to \mathfrak{Aug}_{\mathfrak{o}}$ which "extends" $\Pi : \mathscr{D}_{\rho_{\mathbb{F}}} \to \mathfrak{AR}_{\mathfrak{o}}$, as follows. An object over $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$ is $(\rho_A, T_A, \iota_{(A,I)})$, where T_A is a free A-module with a continuous action of $\boldsymbol{\mathcal{G}}_K$ by ρ_A (for the discrete topology on A), and $\iota_{(A,I)} : T_{\mathbb{F}} \to T_A \otimes_A (A/I)$ is a $\boldsymbol{\mathcal{G}}_K$-equivariant morphism over $\mathbb{F} \to A/I$ which induces an isomorphism $T_{\mathbb{F}} \otimes_{\mathbb{F}} (A/I) \xrightarrow{\sim} T_A \otimes_A (A/I)$. A *morphisms* $\alpha : (\rho_A, T_A, \iota_{(A,I)}) \to (\rho'_B, T'_B, \iota'_{(B,J)})$ over $f : (A, I) \to (B, J)$ is an equivalence class of $\boldsymbol{\mathcal{G}}_K$-equivariant morphisms $\alpha : T_A \to T_B$ over f which respect $\iota_{(A,I)}$ and $\iota'_{(B,J)}$; i.e., α makes the following diagram commute:

(10.3.2.2)
$$\begin{array}{ccc} T_A \otimes_A (A/I) & \xrightarrow{\ \overline{\alpha}\ } & T'_B \otimes_B (B/J) \\ & {\scriptstyle \iota_{(A,I)}} \nwarrow \quad \nearrow {\scriptstyle \iota'_{(B,J)}} & \\ & T_{\mathbb{F}} & \end{array}$$

where $\overline{\alpha}$ is induced from α. We say such α_1 and α_2 are *equivalent* if they are $(1+J)$-multiples of each other.

For $A \in \mathfrak{AR}_{\mathfrak{o}}$, we have an "equality" of categories $\mathscr{D}_{\rho_{\mathbb{F}}}(A) = \widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}(A, \mathfrak{m}_A)$, therefore "equality" of $\mathfrak{AR}_{\mathfrak{o}}$-groupoids $\mathscr{D}_{\rho_{\mathbb{F}}} = \widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}|_{\mathfrak{AR}_{\mathfrak{o}}}$. Later in §10.4.4, we give a general recipe to extend a groupoid over $\mathfrak{AR}_{\mathfrak{o}}$ to a groupoid over $\mathfrak{Aug}_{\mathfrak{o}}$ via a "direct limit," recovering $\widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}$ when applied to $\mathscr{D}_{\rho_{\mathbb{F}}}$.

10.3.3 Framed deformation groupoid

Let $T_{\mathbb{F}}$ and $\rho_{\mathbb{F}}$ be as above, and we fix a *framing* $\beta_{\mathbb{F}} : \mathbb{F}^n \xrightarrow{\sim} T_{\mathbb{F}}$. We define the category $\mathscr{D}^{\square}_{\rho_{\mathbb{F}}} (= \mathscr{D}^{\square}_{\rho_{\mathbb{F}}, \beta_{\mathbb{F}}})$ of *framed deformations* of $\rho_{\mathbb{F}}$, and the functor $\Pi^{\square} : \mathscr{D}^{\square}_{\rho_{\mathbb{F}}} \to \mathfrak{AR}_{\mathfrak{o}}$ which makes $\mathscr{D}^{\square}_{\rho_{\mathbb{F}}}$ a groupoid over $\mathfrak{AR}_{\mathfrak{o}}$. (The groupoid $\mathscr{D}^{\square}_{\rho_{\mathbb{F}}}$ will depend on the choice of framing $\beta_{\mathbb{F}}$, but we do not specify this in the notation unless necessary.) Objects over $A \in \mathfrak{AR}_{\mathfrak{o}}$ are tuples $(\rho_A, T_A, \iota, \beta_A)$ where (ρ_A, T_A, ι) is an object in

$\mathscr{D}_{\rho_{\mathbb{F}}}(A)$, and $\beta_A : A^{\oplus n} \xrightarrow{\sim} T_A$ is a *framing* which lifts $\beta_{\mathbb{F}}$ via ι_A; i.e., β_A makes the following diagram commute.

(10.3.3.1)
$$\begin{array}{ccc} (A/\mathfrak{m}_A)^{\oplus n} & \xrightarrow[\cong]{\beta_A \otimes (A/\mathfrak{m}_A)} & T_A \otimes_A (A/\mathfrak{m}_A) \\ \cong \uparrow & & \cong \uparrow \iota_A \\ \mathbb{F}^{\oplus n} & \xrightarrow[\beta_{\mathbb{F}}]{\cong} & T_{\mathbb{F}} \end{array}$$

Given a morphism $f : A \to A'$ in $\mathfrak{AR}_{\mathfrak{o}}$, we define a *morphism* $\alpha : (\rho_A, T_A, \iota_A, \beta_A) \to (\rho'_{A'}, T'_{A'}, \iota'_{A'}, \beta'_{A'})$ *over* f to be a $\boldsymbol{\mathcal{G}}_K$-equivariant A-morphism $T_A raT'_{A'}$, which respects all the structures in the sense that we have the following commutative diagram in addition to (10.3.2.1).

(10.3.3.2)
$$\begin{array}{ccc} (A')^{\oplus n} & \xrightarrow[\cong]{\beta'_{A'}} & T'_{A'} \\ f^{\oplus n} \uparrow & & \uparrow \alpha \\ A^{\oplus n} & \xrightarrow[\beta_A]{\cong} & T_A \end{array}$$

Now, we can repeat the previous discussion to obtain groupoids $\Pi^{\square} : \mathscr{D}^{\square}_{\rho_{\mathbb{F}}} \to \mathfrak{AR}_{\mathfrak{o}}$, $\widehat{\Pi}^{\square} : \widehat{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}} \to \widehat{\mathfrak{AR}}_{\mathfrak{o}}$ and $\widetilde{\Pi}^{\square} : \widetilde{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}} \to \mathfrak{Aug}_{\mathfrak{o}}$. For $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$, an object $(\rho_A, T_A, \iota_A, \beta_A) \in \mathscr{D}^{\square}_{\rho_{\mathbb{F}}}(A, I)$ additionally satisfies the following commutative diagram.

(10.3.3.3)
$$\begin{array}{ccc} (A/I)^{\oplus n} & \xrightarrow[\cong]{\beta_A \otimes (A/I)} & T_A \otimes_A (A/I) \\ \uparrow & & \uparrow \iota_A \\ \mathbb{F}^{\oplus n} & \xrightarrow[\beta_{\mathbb{F}}]{\cong} & T_{\mathbb{F}} \end{array}$$

The 1-morphism $\mathscr{D}^{\square}_{\rho_{\mathbb{F}}} \to \mathscr{D}_{\rho_{\mathbb{F}}}$ defined by "forgetting the framing" is formally smooth in the sense of Definition 10.2.7(2). Furthermore, it makes $\mathscr{D}^{\square}_{\rho_{\mathbb{F}}}$ into a $\widehat{\underline{\mathrm{PGL}}}(n)$-*torsor* over $\mathscr{D}_{\rho_{\mathbb{F}}}$, where $\widehat{\underline{\mathrm{PGL}}}(n)$ is a functor $\widehat{\underline{\mathrm{PGL}}}(n) : A \mapsto \widehat{\mathrm{PGL}}(n, A) := \{g \in \mathrm{PGL}(n, A) |\ g \bmod \mathfrak{m}_A = \mathrm{Id}_n\}$ on $\mathfrak{AR}_{\mathfrak{o}}$ (or the corresponding category co-fibered in sets)[10]. More precisely, we have an 1-isomorphism

$$\Xi : \widehat{\underline{\mathrm{PGL}}}(n) \times_{\mathfrak{Aug}_{\mathfrak{o}}} \mathscr{D}^{\square}_{\rho_{\mathbb{F}}} \xrightarrow{\sim} \mathscr{D}^{\square}_{\rho_{\mathbb{F}}} \times_{\mathscr{D}_{\rho_{\mathbb{F}}}} \mathscr{D}^{\square}_{\rho_{\mathbb{F}}},$$

[10] In other words, $\widehat{\underline{\mathrm{PGL}}}(n)$ is a formal completion of the linear algebraic group $\underline{\mathrm{PGL}}(n)_{\mathfrak{o}}$ along the identity section.

defined by $\Xi(g_A, (\rho_A, T_A, \iota_A, \beta_A)) = ((\rho_A, T_A, \iota_A, \beta_A), (\rho_A, T_A, \iota_A, \beta_A \circ (\widetilde{g_A})^{-1}), \mathrm{id}_{T_A})$ for each $g_A \in \widehat{\mathrm{PGL}}(n, A)$, where $\widetilde{g_A} \in \widehat{\mathrm{GL}}(n, A)$ is a lift of g_A. This 1-morphism does not depend on the choice of lift $\widetilde{g_A}$ up to 2-isomorphism, since for any $a \in 1 + \mathfrak{m}_A$ we have an isomorphism $((\rho_A, T_A, \iota_A, \beta_A), (\rho_A, T_A, \iota_A, \beta_A \circ (a \cdot \widetilde{g_A})^{-1}), \mathrm{id}_{T_A}) \cong ((\rho_A, T_A, \iota_A, \beta_A), (\rho_A, T_A, \iota_A, \beta_A \circ (\widetilde{g_A})^{-1}), a \cdot \mathrm{id}_{T_A} \sim \mathrm{id}_{T_A})$. One can directly check that this 1-morphism is actually an 1-isomorphism.

As a consequence, the 1-morphism $\mathscr{D}^{\square}_{\rho_{\mathbb{F}}} \to \mathscr{D}_{\rho_{\mathbb{F}}}$ is relatively representable, namely for any $\xi \in \mathscr{D}_{\rho_{\mathbb{F}}}(A)$, the groupoid $\mathscr{D}^{\square}_{\rho_{\mathbb{F}}, \xi}$ is representable by $\widehat{\underline{\mathrm{PGL}}}(n)_A$. The same properties hold for the deformation groupoids over $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$ and $\mathfrak{Aug}_{\mathfrak{o}}$. We define $\widehat{\underline{\mathrm{PGL}}}(n) : (A, I) \mapsto \{g \in \mathrm{PGL}(n, A) |\ g \bmod I = \mathrm{Id}_n\}$ on $\mathfrak{Aug}_{\mathfrak{o}}$

10.4 2-categorical limits

In this section, we give a general recipe to extend a groupoid over $\mathfrak{AR}_{\mathfrak{o}}$ to a groupoid over $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$ via a 2-projective limit (respectively, to a groupoid over $\mathfrak{Aug}_{\mathfrak{o}}$ via a 2-direct limit). For the $\mathfrak{AR}_{\mathfrak{o}}$-groupoids $\mathscr{D}_{\rho_{\mathbb{F}}}$ and $\mathscr{D}^{\square}_{\rho_{\mathbb{F}}}$, we have already constructed $\widehat{\mathscr{D}}_{\rho_{\mathbb{F}}}$, $\widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}$ and $\widehat{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}}$, $\widetilde{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}}$, respectively, which are 1-isomorphic to the groupoids we obtain by the general recipe below. But the general recipe is needed when we work with subgroupoids of $\mathscr{D}_{\rho_{\mathbb{F}}}$ and $\mathscr{D}^{\square}_{\rho_{\mathbb{F}}}$ which can be naturally described only over $\mathfrak{AR}_{\mathfrak{o}}$, for example the full subcategory of deformations of $\mathcal{P}$-height $\leqslant h$, which is introduced in Definition 11.1.1.

For concreteness, we work with the restrictive choice of base categories which will come up in the application, but our definitions of 2-projective and direct limits can generalize to arbitrary base categories. We do not attempt to "explain" our definition, and refer to [73, 4, Exp VI, §6] for more general and complete discussions. Since [73, 4, Exp VI] works with *fibered categories*, not co-fibered categories, we often have to change the directions of arrows to translate the results for co-fibered categories.

10.4.1 2-projective limits

Recall that any functor on $\mathfrak{AR}_\mathfrak{o}$ can be uniquely extended to a functor on $\widehat{\mathfrak{AR}_\mathfrak{o}}$ by taking a projective limit. For a groupoid over $\mathfrak{AR}_\mathfrak{o}$, the same idea works, except that the definition of projective limit is more technical. Roughly speaking, to a $\mathfrak{AR}_\mathfrak{o}$-groupoid $\mathscr{F}$, we associate the $\widehat{\mathfrak{AR}_\mathfrak{o}}$-groupoid $\widehat{\mathscr{F}}$ so that the fiber $\widehat{\mathscr{F}}(A)$ over $A \in \widehat{\mathfrak{AR}_\mathfrak{o}}$ is the category of projective systems of objects in $\mathscr{F}(A/m_A^n)$. We refer to [73, 4, Exp VI, (6.10)] for interested readers.

For $A \in \widehat{\mathfrak{AR}_\mathfrak{o}}$, let $\mathfrak{AR}_\mathfrak{o}^A$ be the category where the objects are the $\mathfrak{o}$-algebras $A/\mathfrak{m}_A^n$ for $n > 0$ and the morphisms $A/\mathfrak{m}_A^n \to A/\mathfrak{m}_A^{n'}$ are the natural projections. Let $\mathscr{G}$ be a groupoid over $\mathfrak{AR}_\mathfrak{o}^A$. For example, given a groupoid $\mathscr{F}$ over $\mathfrak{AR}_\mathfrak{o}$ let $\mathscr{G} := \mathscr{F}|_{\mathfrak{AR}_\mathfrak{o}^A}$ be the sub-category of $\mathscr{F}$ whose objects and morphisms are over those of $\mathfrak{AR}_\mathfrak{o}^A$. Then we define a *2-projective limit* of $\mathscr{G}$ as follows:

$$\varprojlim_n \mathscr{G}(A/\mathfrak{m}_A^n) := \mathscr{H}om_{\mathfrak{AR}_\mathfrak{o}}(\mathfrak{AR}_\mathfrak{o}^A, \mathscr{G}), \tag{10.4.1.1}$$

where $\mathscr{H}om_{\mathfrak{AR}_\mathfrak{o}}(\cdot,\cdot)$ is the category of base-preserving[11] functors. “Evaluating at $A/\mathfrak{m}_A^n$” gives a functor $\varprojlim_n \mathscr{G}(A/\mathfrak{m}_A^n) \to \mathscr{G}(A/\mathfrak{m}_A^n)$ for each n, and we have a canonical 1-morphism $\mathfrak{AR}_\mathfrak{o}^A \times \varprojlim_n \mathscr{G}(A/\mathfrak{m}_A^n) \to \mathscr{G}$ of groupoids over $\mathfrak{AR}_\mathfrak{o}^A$. In fact, this 1-morphism is universal among 1-morphisms $\mathfrak{AR}_\mathfrak{o}^A \times C \to \mathscr{G}$ for any category C.

The groupoid $\varprojlim_n \mathscr{G}(A/\mathfrak{m}_A^n)$ has the following explicit description. The objects are projective systems $\{\xi_n | \ \xi_n \in \mathscr{G}(A/\mathfrak{m}_A^n)\}_n$ and morphisms $\{\xi_n\} \to \{\eta_n\}$ are collections $\{\xi_n \to \eta_n\}_n$ of morphisms in $\mathscr{G}$ which are compatible with the transition maps,

[11] We view $\mathfrak{AR}_\mathfrak{o}^A$ as a category over $\mathfrak{AR}_\mathfrak{o}$ via the natural inclusion functor. Base-preserving functors are defined in Definition 10.1.5, and morphisms of base-preserving functors are also required to be base-preserving in the sense of Definition 10.1.5. If $\mathscr{G}$ were a general co-fibered category, then we need to require that any functor in $\mathscr{H}om_{\mathfrak{AR}_\mathfrak{o}}(\mathfrak{AR}_\mathfrak{o}^A, \mathscr{G})$ sends any arrow in $\mathfrak{AR}_\mathfrak{o}^A$ to a cartesian arrow, but this is automatic since $\mathscr{F}$ is a groupoid over $\mathfrak{AR}_\mathfrak{o}$.

i.e., make the following diagram commute:

$$\begin{array}{ccc} \xi_{n+1} & \longrightarrow & \eta_{n+1} \\ \downarrow & & \downarrow \\ \xi_n & \longrightarrow & \eta_n \end{array}$$

If A is artin local with $(\mathfrak{m}_A)^{n_0} = 0$, then the functor $\{\xi_n\} \mapsto \xi_{n_0}$ defines an equivalence of categories $\varprojlim_n \mathscr{G}(A/\mathfrak{m}_A^n) \to \mathscr{G}(A)$. We can check that

$$\left| \varprojlim_n \mathscr{G}(A/\mathfrak{m}_A^n) \right| \cong \varprojlim_n |\mathscr{G}| \, (A/\mathfrak{m}_A^n). \tag{10.4.1.2}$$

In particular, for a category G co-fibered in sets (i.e. a functor), the 2-projective limit $\varprojlim_n G(A/\mathfrak{m}_A^n)$ is equivalent to the set-theoretic projective limit of the $G(A/\mathfrak{m}_A^n)$.

Now, let $\mathscr{F}$ be a groupoid over $\mathfrak{AR}_\mathfrak{o}$. We now define a groupoid $\widehat{\mathscr{F}}$ over $\widehat{\mathfrak{AR}}_\mathfrak{o}$, as follows. For any $A \in \widehat{\mathfrak{AR}}_\mathfrak{o}$, we set $\widehat{\mathscr{F}}(A) := \varprojlim_n \mathscr{F}(A/\mathfrak{m}_A^n)$. To a morphism $f : A \to B$ in $\widehat{\mathfrak{AR}}_\mathfrak{o}$, we can naturally associate a functor $f : \mathfrak{AR}_\mathfrak{o}^A \to \mathfrak{AR}_\mathfrak{o}^B$. For two objects $\xi \in \mathrm{Ob}(\widehat{\mathscr{F}}(A))$ and $\eta \in \mathrm{Ob}(\widehat{\mathscr{F}}(B))$, a *morphism* $\alpha : \xi \to \eta$ over f is a natural transformation $\xi \to \eta \circ f$. (We view ξ and η as functors into $\mathscr{F}$ via $\xi : \mathfrak{AR}_\mathfrak{o}^A \to \mathscr{F}|_{\mathfrak{AR}_\mathfrak{o}^A} \hookrightarrow \mathscr{F}$ and $\eta : \mathfrak{AR}_\mathfrak{o}^B \to \mathscr{F}|_{\mathfrak{AR}_\mathfrak{o}^B} \hookrightarrow \mathscr{F}$.) More concretely, a morphism $\{\xi_n\} \to \{\eta_n\}$ over f is a collection $\{\xi_n \to \eta_n\}$ of morphisms over $f_n : A/\mathfrak{m}_A^n \to B/\mathfrak{m}_B^n$, which are compatible with the transition maps.

This $\widehat{\mathfrak{AR}}_\mathfrak{o}$-groupoid $\widehat{\mathscr{F}}$ extends $\mathscr{F}$; i.e., we have a 1-isomorphism $\widehat{\mathscr{F}}|_{\mathfrak{AR}_\mathfrak{o}} \xrightarrow{\sim} \mathscr{F}$. (This amounts to the fact that the natural "projection functor" $\varprojlim_n \mathscr{F}(A/\mathfrak{m}_A^n) \to \mathscr{F}(A)$ is an equivalence of categories for each $A \in \mathfrak{AR}_\mathfrak{o}$.) Conversely, let $\mathscr{F}'$ be a $\widehat{\mathfrak{AR}}_\mathfrak{o}$-groupoid. We choose a cleavage (Remark 10.1.12) so that for any $A \twoheadrightarrow A/\mathfrak{m}_A^n$, we obtain a functor $\mathscr{F}'(A) \to \mathscr{F}'(A/\mathfrak{m}_A^n)$. Then we obtain a 1-morphism $\Xi : \mathscr{F}' \to \widehat{\mathscr{F}'|_{\mathfrak{AR}_\mathfrak{o}}}$ of $\widehat{\mathfrak{AR}}_\mathfrak{o}$-groupoids with cleavage, as follows: for each $A \in \widehat{\mathfrak{AR}}_\mathfrak{o}$, we define a functor $\Xi_A : \mathscr{F}'(A) \to \varprojlim_n \mathscr{F}'(A/\mathfrak{m}_A^n)$ by sending ξ to $\xi_{A/\mathfrak{m}_A^n}$, according to the choice of cleavage.

Definition 10.4.2. Let $\mathscr{F}'$ be an $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$-groupoid. We say that *the formation of $\mathscr{F}'$ commutes with 2-projective limits* if for some choice of cleavage (equivalently, for any choice of cleavage) on $\mathscr{F}'$, the 1-morphism $\Xi : \mathscr{F}' \to \widehat{\mathscr{F}'|_{\mathfrak{AR}_{\mathfrak{o}}}}$ is a 1-isomorphism.

Here is an example. Let $\mathscr{F} = \mathscr{D}_{\rho_{\mathbb{F}}}$ and $\mathscr{F}^{\square} = \mathscr{D}^{\square}_{\rho_{\mathbb{F}}}$, and we already defined $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$-groupoids $\widehat{\mathscr{D}}_{\rho_{\mathbb{F}}}$ and $\widehat{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}}$ in §10.3. We show that their formation commute with 2-projective limit. We first choose a cleavage so that $\widehat{\mathscr{D}}_{\rho_{\mathbb{F}}}(A) \to \widehat{\mathscr{D}}_{\rho_{\mathbb{F}}}(A/\mathfrak{m}_A^n)$ is given by $T \mapsto T/\mathfrak{m}_A^n T$, and similarly for $\widehat{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}}$. Let $\widehat{\mathscr{F}}$ and $\widehat{\mathscr{F}}^{\square}$ be the $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$-groupoids obtained by the 2-projective limits construction discussed above. Then, that $\Xi : \widehat{\mathscr{D}}_{\rho_{\mathbb{F}}} \xrightarrow{\sim} \widehat{\mathscr{F}}$ and $\Xi^{\square} : \widehat{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}} \xrightarrow{\sim} \widehat{\mathscr{F}}^{\square}$ are 1-isomorphisms follows from Proposition 7.4.1.

Definition 10.4.3. We say that an $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\mathscr{F}$ is *pro-representable* if $\widehat{\mathscr{F}}$ is representable.

10.4.4 2-direct limits

In this subsection, we explain how to extend $\mathscr{F} = \mathscr{D}_{\rho_{\mathbb{F}}}$ or $\mathscr{F} = \mathscr{D}^{\square}_{\rho_{\mathbb{F}}}$ over the bigger category $\mathfrak{Aug}_{\mathfrak{o}}$ by using a 2-direct limit.

For $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$, we form a category $\mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$ of pairs $(A', j_{A'} : A' \hookrightarrow A)$, where $A' \in \mathfrak{AR}_{\mathfrak{o}}$ and $j_{A'} : A' \hookrightarrow A$ maps $\mathfrak{m}_{A'}$ into I. We require that morphisms respect the injective map $j_{A'}$. We will often view A' as a $\mathfrak{o}$-subalgebra of A via $j_{A'}$, and will not mention $j_{A'}$ explicitly. For two objects A' and A'' in $\mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$, we can find another object $\mathrm{Im}[A' \otimes_{\mathfrak{o}} A'']$ which contains A' and A'' as a $\mathfrak{o}$-subalgebra of A. In other words, the category $\mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$ is *filtered*.[12]

To motivate the construction, consider a $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\mathscr{F}$ which is representable by $R \in \widehat{\mathfrak{AR}_{\mathfrak{o}}}$. Then for a noetherian $\mathfrak{o}$-algebra A where π_0 is nilpotent, consider the set of continuous $\mathfrak{o}$-maps $\mathrm{Hom}_{\mathfrak{o}}(R, A)$. Since by continuity any $R \to A$ factors

[12] If the base category is *not* filtered then the 2-direct limit can be counter-intuitive. For a more precise statement, see [73, 4, Exp VI, Exercice 6.8(1)].

through some $A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$ where $I \subset A$ is the nilradical, we have a natural bijection

$$\mathrm{Hom}_{\mathfrak{o}}(R, A) \cong \varinjlim_{A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}} \mathrm{Hom}_{\mathfrak{o}}(R, A') \cong \varinjlim_{A'} \mathscr{F}(A')$$

For an arbitrary $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\mathscr{F}$, it will be natural to define $\widetilde{\mathscr{F}}(A, I)$ as the direct limit of $\mathscr{F}(A')$ over $A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$. But since $\mathscr{F}(A')$ does not have to be equivalent to a set, we need to clarify what we mean by the "direct limit." Roughly speaking, to a $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\mathscr{F}$, we will associate the $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid $\widetilde{\mathscr{F}}$ so that the fiber $\widetilde{\mathscr{F}}(A, I)$ over $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$ is the category of direct systems of objects in $\mathscr{F}(A')$ for $A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$.

Let $\mathscr{G}$ be a groupoid over $\mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$. For example, we may take $\mathscr{G} := \mathscr{F}|_{\mathfrak{AR}_{\mathfrak{o}}^{(A,I)}}$ for some groupoid $\mathscr{F}$ over $\mathfrak{AR}_{\mathfrak{o}}$ as before. Define the *2-direct limit* $\varinjlim_{A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}} \mathscr{G}(A')$ as the category obtained from $\mathscr{G}$ by "formally inverting" all the co-cartesian morphisms, hence all morphisms, in $\mathscr{G}$. Since $\mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$ is filtered, the category $\varinjlim_{A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}} \mathscr{G}(A')$ is a "localization" of $\mathscr{G}$ in the following sense. The set of objects is exactly $\mathrm{Ob}(\mathscr{G})$, and the morphisms are equivalence classes of the following diagrams:

(10.4.4.1)
$$\begin{array}{ccc} \xi_{A'} & & \eta_{B'} \\ & \searrow^{\alpha} & \downarrow{\scriptstyle\beta} \\ & & \eta_{B''} \end{array},$$

where α and β are morphisms in $\mathscr{G}$. We write the above morphism as $\beta^{-1} \circ \alpha$, and the equivalence relation is generated by $\beta^{-1} \circ \alpha \sim (\gamma \circ \beta)^{-1} \circ (\gamma \circ \alpha)$ for any morphism $\gamma : \eta_{B''} \to \eta_{B'''}$ in $\mathscr{G}$. To rephrase, the set of morphisms can be written as follows:

(10.4.4.2)
$$\mathrm{Hom}_{\varinjlim \mathscr{G}}(\xi_{A'}, \eta_{B'}) = \varinjlim_{\eta_{B''} \in \mathrm{Ob}(\mathscr{G}/\eta_{B'})} \mathrm{Hom}_{\mathscr{G}}(\xi_{A'}, \eta_{B''}).$$

This gives a well-defined category (in particular, the composition of morphisms is well-defined) since $\mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$ is filtered and there are enough co-cartesian lifts in $\mathscr{G}$

(Definition 10.1.2(2)). See [73, 4, Exp VI, Prop 6.5] for more details, up to reversing the directions of arrows.

The natural inclusion define a functor $\mathscr{G} \to \varinjlim_{A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}} \mathscr{G}(A')$. We denote the image of $\xi \in \mathrm{Ob}(\mathscr{G})$ under this functor by $\{\xi\}$. For $A \in \mathfrak{AR}_{\mathfrak{o}}$ and a groupoid $\mathscr{G}$ over $\mathfrak{AR}_{\mathfrak{o}}^{(A,\mathfrak{m}_A)}$, the natural inclusion $\mathscr{G}(A) \to \varinjlim_{A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,\mathfrak{m}_A)}} \mathscr{G}(A')$ is an equivalence of categories since $A \in \mathfrak{AR}_{\mathfrak{o}}^{(A,\mathfrak{m}_A)}$ is the final object. In general, the 2-direct limit is equivalent to the "category of direct systems" by associating to each $\xi_{A'} \in \mathrm{Ob}(\mathscr{G})$ a direct system which has $\xi_{A'}$ in the A'-slot.[13] From this, we can check that

$$\left| \varinjlim_{A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}} \mathscr{G}(A') \right| \cong \varinjlim_{A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}} |\mathscr{G}|(A'). \tag{10.4.4.3}$$

In particular, for a category G co-fibered in sets (i.e. a functor), the 2-direct limit $\varinjlim_{A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}} G(A')$ is equivalent to the set-theoretic direct limit of $G(A')$ over $A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$. For more discussion of 2-direct limit, see [73, 4, Exp VI, §6], especially Proposition 6.2 and the discussion which follows.

Now, we can extend any $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\mathscr{F}$ to a groupoid $\widetilde{\mathscr{F}}$ over $\mathfrak{Aug}_{\mathfrak{o}}$ by declaring $\widetilde{\mathscr{F}}(A,I) := \varinjlim_{A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}} \mathscr{F}(A')$ for $(A,I) \in \mathfrak{Aug}_{\mathfrak{o}}$. A *morphism* $\{\xi_{A'}\} \to \{\eta_{B'}\}$ over $f : (A,I) \to (B,J)$ is defined in a similar fashion to (10.4.4.1). More precisely, we consider $B'' \in \mathfrak{AR}_{\mathfrak{o}}^{(B,J)}$ so that $f(A') \subset B''$ and $B' \subset B''$ as $\mathfrak{o}$-subalgebras of B. Then, a morphism $\{\xi_{A'}\} \to \{\eta_{B'}\}$ over f means an equivalence class of diagrams of the following form:

$$\begin{array}{ccc} \xi_{A'} & & \eta_{B'} \\ & \searrow^{\alpha} & \downarrow \beta \\ & & \eta_{B''} \end{array}, \tag{10.4.4.4}$$

where α is over $f|_{A'} : A' \to B''$, and β is over the inclusion $B' \hookrightarrow B''$ of $\mathfrak{o}$-subalgebras of B. We write this morphism as $\beta^{-1} \circ \alpha$ and the equivalence relation is generated

[13] The essential surjectivity is clear and the full faithfulness follows from (10.4.4.2).

by $\beta^{-1} \circ \alpha \sim (\gamma \circ \beta)^{-1} \circ (\gamma \circ \alpha)$ for any $\gamma : \eta_{B''} \to \eta_{B'''}$ over the inclusion $B'' \hookrightarrow B'''$ of $\mathfrak{o}$-subalgebras of B. To rephrase, the set $\mathrm{Hom}_f(\{\xi_{A'}\}, \{\eta_{B'}\})$ of morphisms over $f : (A, I) \to (B, J)$ can be written as follows:

$$\mathrm{Hom}_f(\{\xi_{A'}\}, \{\eta_{B'}\}) := \varinjlim_{B' \subset B''} \mathrm{Hom}_{f|_{A'}}(\xi_{A'}, \eta_{B''}).$$

It can be checked that $\widetilde{\mathscr{F}}$ is an $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid.[14]

This $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid $\widetilde{\mathscr{F}}$ extends $\mathscr{F}$; i.e., we have a 1-isomorphism $\mathscr{F} \xrightarrow{\sim} \widetilde{\mathscr{F}}|_{\mathfrak{AR}_{\mathfrak{o}}}$. (This amounts to the fact that the natural "inclusion functor"

$$\mathscr{F}(A) \to \varinjlim_{A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A, \mathfrak{m}_A)}} \mathscr{F}(A')$$

is an equivalence of categories for each $A \in \mathfrak{AR}_{\mathfrak{o}}$.) Conversely, let $\mathscr{F}'$ be a $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid. We choose a cleavage so that for any $A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$, we obtain a functor $\mathscr{F}'(A') \to \mathscr{F}'(A, I)$. Then we obtain a 1-morphism $\Xi : \widetilde{\mathscr{F}'|_{\mathfrak{AR}_{\mathfrak{o}}}} \to \mathscr{F}'$ of $\mathfrak{Aug}_{\mathfrak{o}}$-groupoids with cleavage, as follows: for each $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$, we define a functor $\Xi(A, I) : \varinjlim_{A'} \mathscr{F}'(A') \to \mathscr{F}'(A, I)$ by sending $\{\xi\}$ to $\xi_{(A,I)}$, according to the choice of cleavage.

Definition 10.4.5. Let $\mathscr{F}'$ be an $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid. We say that *the formation of* $\mathscr{F}'$ *commutes with 2-direct limits* if for some choice of cleavage (equivalently, for any choice of cleavage) on $\mathscr{F}'$, the 1-morphism $\Xi : \widetilde{\mathscr{F}'|_{\mathfrak{AR}_{\mathfrak{o}}}} \to \mathscr{F}'$ is a 1-isomorphism.

10.4.6

For $\mathscr{F} = \mathscr{D}_{\rho_{\mathbb{F}}}$ and $\mathscr{F}^{\square} = \mathscr{D}^{\square}_{\rho_{\mathbb{F}}}$ we already defined $\mathfrak{Aug}_{\mathfrak{o}}$-groupoids $\widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}$ and $\widetilde{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}}$ in §10.3. Let $\widetilde{\mathscr{F}}$ and $\widetilde{\mathscr{F}}^{\square}$ be the $\mathfrak{Aug}_{\mathfrak{o}}$-groupoids obtained by the 2-direct limit construction discussed above. In this subsection, we show that the formation of $\widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}$

[14]If we view the 2-direct limit as a category of direct systems instead of a localization, and define $\widetilde{\mathscr{F}}$ accordingly, then the set of morphisms $\{\xi_{A'}\} \to \{\eta_{B'}\}$ of direct systems over f is $\varprojlim_{A' \subset A''} \varinjlim_{B' \subset B''} \mathrm{Hom}_{f|_{A''}}(\xi_{A''}, \eta_{B''})$, but all the transition maps of the projective system are bijections, hence the notion of morphisms coincides.

and $\widetilde{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}}$ commutes with 2-direct limits, which provides 1-isomorphism $\widetilde{\mathscr{F}} \xrightarrow{\sim} \widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}$ and $\widetilde{\mathscr{F}}^{\square} \xrightarrow{\sim} \widetilde{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}}$.

The choice of cleavage is induced from the "choice[15]" of tensor product $T_{A'} \otimes_{A'} A$ among its isomorphism class. We make such a choice, and define 1-morphisms $\Xi : \widetilde{\mathscr{F}} \to \widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}$ and $\Xi^{\square} : \widetilde{\mathscr{F}}^{\square} \to \widetilde{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}}$, according to the choice of cleavage.

Showing that Ξ and $\Xi^{\square}$ are 1-isomorphisms is equivalent to showing that $\Xi(A, I)$ and $\Xi^{\square}(A, I)$ are equivalences of categories for each $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$. We carry out the proof as follows:

10.4.6.1

$\Xi(A, I)$ *and* $\Xi^{\square}(A, I)$ *are faithful.* This is clear since $T_{A'} \hookrightarrow T_A$.

10.4.6.2

$\Xi(A, I)$ *and* $\Xi^{\square}(A, I)$ *are essentially surjective.* Let A^+ be the preimage of $\mathbb{F}$ under the natural projection $A \twoheadrightarrow A/I$, so A^+ is local with nilpotent maximal ideal $I \cap A^+$. We first remark that each of ρ_A, T_A, ι_A and β_A "descends" to A^+, since each of them descends over $\mathbb{F}$ modulo I by definition. Now, by the compactness of $\boldsymbol{\mathcal{G}}_K$ and general properties of finitely presented modules and morphisms between them, we can find a finitely generated (hence finite artin local) $\mathfrak{o}$-subalgebra A' of A^+ over which each of ρ_A, T_A, ι_A and β_A descends. But any such A' is an object of $\mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$.

10.4.6.3

$\Xi(A, I)$ *and* $\Xi^{\square}(A, I)$ *are full.* Let $T_A = T_{A'} \otimes_{A'} A$ and $T_B = T_{B'} \otimes_{B'} B$ where $T_{A'}$ and $T_{B'}$ are free modules over $A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$ and $B' \in \mathfrak{AR}_{\mathfrak{o}}^{(B,J)}$, respectively. We

[15] Technically, tensor product is defined only up to unique isomorphism, not as a single object. "Choosing" a tensor product corresponds to choosing a cleavage for the category of modules co-fibered over the category of rings.

assume that $T_{A'} \otimes_{A'} (A'/\mathfrak{m}_{A'}) \cong T_{B'} \otimes_{B'} (B'/\mathfrak{m}_{B'}) \cong T_{\mathbb{F}}$, $T_A \otimes_A (A/I) \cong T_{\mathbb{F}} \otimes_{\mathbb{F}} (A/I)$ and $T_B \otimes_B (B/J) \cong T_{\mathbb{F}} \otimes_{\mathbb{F}} (B/J)$.

As before let A^+ and B^+ be the preimages of $\mathbb{F}$ under the natural projection $A \twoheadrightarrow A/I$ and $B \twoheadrightarrow B/J$, respectively. By the assumption, any morphism $\alpha : T_A \to T_B$ descends to a morphism $\alpha^+ : T_{A^+} \to T_{B^+}$. Hence, by general properties of morphisms between finitely generated modules, there exists $A'' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$ and $B'' \in \mathfrak{AR}_{\mathfrak{o}}^{(B,J)}$ such that the morphism α^+ descends to some $\alpha'' : T_{A''} \to T_{B''}$.

Now, assume that α has come from a morphism in $\widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}$ or in $\widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}^{\square}$. This essentially means that T_A and T_B carry some extra structures such as ρ_A, ρ_B, ι_A, ι_B, (or additionally β_A and β_B), and α satisfies some diagrams such as (10.3.2.2) (or additionally (10.3.3.2)). Then, by enlarging A'' and B'' by adding finitely many generators, we may ensure that α'' is a morphism in $\mathscr{D}_{\rho_{\mathbb{F}}}$ or $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square}$, which concludes the proof.

10.4.7 Properties of $\widehat{\mathscr{F}}$ and $\widetilde{\mathscr{F}}$

The following claims follow from our discussion of 2-categorical limits and Proposition 10.1.6. We skip the details and leave them to readers.

The construction of $\widehat{\mathscr{F}}$ (respectively, $\widetilde{\mathscr{F}}$) is "2-functorial" in the following sense. Any 1-morphism $\Phi : \mathscr{F}' \to \mathscr{F}'$ of $\mathfrak{AR}_{\mathfrak{o}}$-groupoids naturally extends to a 1-morphism $\widehat{\Phi} : \widehat{\mathscr{F}'} \to \widehat{\mathscr{F}}$ of $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$-groupoids (respectively, to a 1-morphism $\widetilde{\Phi} : \widetilde{\mathscr{F}'} \to \widetilde{\mathscr{F}}$ of $\mathfrak{Aug}_{\mathfrak{o}}$-groupoids), and any 2-isomorphism $\psi : \Phi \hookrightarrow \Phi'$ between 1-morphisms $\Phi, \Phi' : \mathscr{F} \to \mathscr{F}'$ naturally extends to a 2-isomorphism $\widehat{\psi} : \widehat{\Phi} \xrightarrow{\sim} \widehat{\Phi}'$ (respectively, to a 2-isomorphism $\widetilde{\psi} : \widetilde{\Phi} \xrightarrow{\sim} \widetilde{\Phi}'$). Note if Φ is a natural inclusion of an $\mathfrak{AR}_{\mathfrak{o}}$-subgroupoid (respectively, fully faithful, 1-isomorphism, formally smooth), then the same property holds for $\widehat{\Phi}$ and $\widetilde{\Phi}$.

The formation of $\widehat{\mathscr{F}}$ and $\widetilde{\mathscr{F}}$ commute with 2-fiber products in the following sense:

for $\mathfrak{AR}_{\mathfrak{o}}$-groupoids $\mathscr{F}_1$, $\mathscr{F}_2$ and $\mathscr{F}$, the natural 1-morphisms $\widehat{\mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2} \to \widehat{\mathscr{F}_1} \times_{\widehat{\mathscr{F}}} \widehat{\mathscr{F}_2}$ and $\widetilde{\mathscr{F}_1 \times_{\mathscr{F}} \mathscr{F}_2} \to \widetilde{\mathscr{F}_1} \times_{\widetilde{\mathscr{F}}} \widetilde{\mathscr{F}_2}$ are 1-isomorphisms, where the 1-morphisms are obtained by applying the "2-universal property of 2-fiber products" to $(\widehat{\mathrm{pr}}_1, \widehat{\mathrm{pr}}_2, \widehat{\omega})$ and $(\widetilde{\mathrm{pr}}_1, \widetilde{\mathrm{pr}}_2, \widetilde{\omega})$. That these 1-morphisms are 1-isomorphisms can be checked fiber-wise, which can be done using the explicit description of 2-fiber products stated in §10.1.8. See Definition 10.1.7 for the "2-universal property" and the notations used here.

Motivated by this discussion, we make the following definition.[16]

Definition 10.4.8. Let $\mathscr{F}'$ and $\mathscr{G}'$ be $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$-groupoids, whose formation commutes with 2-projective limits. Set $\mathscr{F} := \mathscr{F}'|_{\mathfrak{AR}_{\mathfrak{o}}}$ and $\mathscr{G} := \mathscr{G}'|_{\mathfrak{AR}_{\mathfrak{o}}}$, and fix 1-isomorphisms $\Xi^{\mathscr{F}} : \mathscr{F}' \xrightarrow{\sim} \widehat{\mathscr{F}}$ and $\Xi^{\mathscr{G}} : \mathscr{G}' \xrightarrow{\sim} \widehat{\mathscr{G}}$. We say that a 1-morphism $\Psi' : \mathscr{F}' \to \mathscr{G}'$ over $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$ *commutes with 2-projective limits* if there exists a 1-morphism $\Psi : \mathscr{F} \to \mathscr{G}$ such that $\Xi^{\mathscr{G}} \circ \Psi' \cong \widehat{\Psi} \circ \Xi^{\mathscr{F}}$.

Let $\mathscr{F}'$ and $\mathscr{G}'$ be $\mathfrak{Aug}_{\mathfrak{o}}$-groupoids, whose formation commute with 2-direct limits. Set $\mathscr{F} := \mathscr{F}'|_{\mathfrak{AR}_{\mathfrak{o}}}$ and $\mathscr{G} := \mathscr{G}'|_{\mathfrak{AR}_{\mathfrak{o}}}$, and fix 1-isomorphisms $\Xi^{\mathscr{F}} : \widetilde{\mathscr{F}} \xrightarrow{\sim} \mathscr{F}'$ and $\Xi^{\mathscr{G}} : \widetilde{\mathscr{G}} \xrightarrow{\sim} \mathscr{G}'$. We say that a 1-morphism $\Psi' : \mathscr{F}' \to \mathscr{G}'$ over $\mathfrak{Aug}_{\mathfrak{o}}$ *commutes with 2-direct limits* if there exists a 1-morphism $\Psi : \mathscr{F} \to \mathscr{G}$ such that $\Psi' \circ \Xi^{\mathscr{F}} \cong \Xi^{\mathscr{G}} \circ \widetilde{\Psi}$.

For example, the "forgetting the framing" functor $\widehat{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}} \to \widehat{\mathscr{D}}_{\rho_{\mathbb{F}}}$ or $\widetilde{\mathscr{D}}^{\square}_{\rho_{\mathbb{F}}} \to \widetilde{\mathscr{D}}_{\rho_{\mathbb{F}}}$ commutes with 2-projective or direct limits, respectively.

The following statement is a paraphrase of the discussion on 2-fiber products above: if both $\Psi_i : \widehat{\mathscr{F}_i} \to \widehat{\mathscr{F}}$ commute with 2-projective limits, then the 2-fiber product $\widehat{\mathscr{F}_1} \times_{\widehat{\mathscr{F}}} \widehat{\mathscr{F}_2}$ can be recovered from its restriction to $\mathfrak{AR}_{\mathfrak{o}}$-groupoid, and similarly if both $\Psi_i : \widetilde{\mathscr{F}_i} \to \widetilde{\mathscr{F}}$ commute with 2-direct limits, then the 2-fiber product $\widetilde{\mathscr{F}_1} \times_{\widetilde{\mathscr{F}}} \widetilde{\mathscr{F}_2}$ can be recovered from its restriction to $\mathfrak{AR}_{\mathfrak{o}}$-groupoid. Also it follows as a

[16] The author is not sure whether the following terminologies are standard.

consequence that the natural projections $\widehat{\mathscr{F}_1} \times_{\widehat{\mathscr{F}}} \widehat{\mathscr{F}_2} \to \widehat{\mathscr{F}_i}$ and and $\widetilde{\mathscr{F}_1} \times_{\widetilde{\mathscr{F}}} \widetilde{\mathscr{F}_2} \to \widetilde{\mathscr{F}_i}$ commute with 2-projective and direct limits, respectively.

The upshot is that the study of the groupoid $\mathscr{G}$ and 1-morphisms Ψ over $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$ or $\mathfrak{Aug}_{\mathfrak{o}}$ as above essentially reduces to that of the $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\mathscr{F}$ and 1-morphism Φ. Finally, we remark that the formation of any groupoids over $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$ that we consider commute with 2-projective limits, and all the 1-morphisms over $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$ commute with 2-projective limits. On the other hand, a $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid whose formation does not commute with 2-direct limits naturally arises in the study of deformations; see §11.1.5 for such an example.

CHAPTER XI

Deformations for $\mathcal{G}_K$-representations of $\mathcal{P}$-height $\leqslant h$

Throughout the chapter, we assume that *the residue field k of $\mathfrak{o}_K$ is finite.* This assumption is needed for the existence of universal deformation rings and universal framed deformation rings for $\mathcal{G}_K$-representations of $\mathcal{P}$-height $\leqslant h$ (Theorem 11.1.2, which is proved in §11.7). This theorem is not obvious at all, since the usual 'unrestricted' $\mathcal{G}_K$-deformation functor has infinite-dimensional tangent space (see §11.7.1), so there is no 'unrestricted' universal $\mathcal{G}_K$-deformation ring in the category of complete local noetherian rings. We study the local structure of the generic fibers of these deformation rings via suitable analogues of Kisin's techniques for analyzing potentially semi-stable deformation rings [55, §3]. This is done in §11.3.

From the definition of $\mathcal{G}_K$-representations of $\mathcal{P}$-height $\leqslant h$, Kisin's idea [51, §2] of "resolving flat deformation rings" works for $\mathcal{G}_K$-deformation rings of $\mathcal{P}$-height $\leqslant h$ (§11.1), and we can even perform Kisin's connected component analysis when $h = 1$ under a suitable separability assumption (§11.5). As an application, we give another proof of Kisin's connected component analysis of the generic fiber of certain flat deformation rings (Theorem 11.6.1) using $\mathcal{G}_{\mathscr{K}_\infty}$-deformation rings instead of the Breuil-Kisin classification of finite flat group schemes. We also point out that the 2-adic case of the theorem is handled in a more uniform manner this way.

We keep the notations from §10.3, with the following exception. For a groupoid $\mathscr{F}$ over $\mathfrak{AR}_{\mathfrak{o}}$, we use the same letter $\mathscr{F}$ to denote the extension of $\mathscr{F}$ to a groupoid over $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ or $\mathfrak{Aug}_{\mathfrak{o}}$. This is denoted by $\widehat{\mathscr{F}}$ or $\widetilde{\mathscr{F}}$ in §X.

11.1 Deformations and $\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$

In this section, we define groupoids of deformations (respectively, framed deformations) of $\mathcal{P}$-height $\leqslant h$, and construct "moduli of $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$" over deformation groupoids, which can be thought of as "resolutions." This was inspired by Kisin's resolution of flat deformation rings [51, §2.1].

Let $\rho_{\mathbb{F}}$ be a $\boldsymbol{\mathcal{G}}_K$-representation over $\mathbb{F}$, which is of $\mathcal{P}$-height $\leqslant h$ (Definition 8.1.7). That is to say, there exists $\mathfrak{M}_{\mathbb{F}} \in (\mathrm{ModFI}\,/\mathfrak{S})_{\mathbb{F}}^{\leqslant h}$ such that $T^*_{\mathfrak{S},\mathbb{F}}(\mathfrak{M}_{\mathbb{F}}) \cong \rho_{\mathbb{F}}$ as a $\mathbb{F}[\boldsymbol{\mathcal{G}}_K]$-module. (See Lemma 9.1.2.) We often use §9.2.1 without comment.

Definition 11.1.1. For $A \in \mathfrak{AR}_{\mathfrak{o}}$, we say that a deformation $(\rho_A, T_A, \iota) \in \mathscr{D}_{\rho_{\mathbb{F}}}(A)$ is *of $\mathcal{P}$-height $\leqslant h$* if (ρ_A, T_A) is of $\mathcal{P}$-height $\leqslant h$ as a torsion $\boldsymbol{\mathcal{G}}_K$-representation (Definition 8.1.7). We let $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h} \subset \mathscr{D}_{\rho_{\mathbb{F}}}$ denote the full subcategory whose objects are of $\mathcal{P}$-height $\leqslant h$. We say a framed deformation $(\rho_A, T_A, \iota, \beta_A) \in \mathscr{D}_{\rho_{\mathbb{F}}}^{\square}$ is *of $\mathcal{P}$-height $\leqslant h$*, if (ρ_A, T_A) is of $\mathcal{P}$-height $\leqslant h$ as a torsion $\boldsymbol{\mathcal{G}}_K$-representation. We let $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square,\leqslant h} \subset \mathscr{D}_{\rho_{\mathbb{F}}}^{\square}$ denote the full subcategory whose objects are of $\mathcal{P}$-height $\leqslant h$.

We can apply the discussion in §10.4.1 and §10.4.4 to extend $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}$ and $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square,\leqslant h}$ to $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$-groupoids and $\mathfrak{Aug}_{\mathfrak{o}}$-groupoids, respectively, and use §10.4.7 to extend all the relevant 1-morphisms over $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ and $\mathfrak{Aug}_{\mathfrak{o}}$, respectively. In particular, by §10.4.7 we can view $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}$ and $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square,\leqslant h}$ as subgroupoids of $\mathscr{D}_{\rho_{\mathbb{F}}}$ and $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square}$ over $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ and $\mathfrak{Aug}_{\mathfrak{o}}$, respectively. Also, $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square,\leqslant h}$ can be written as the 2-fiber product $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h} \times_{\mathscr{D}_{\rho_{\mathbb{F}}}} \mathscr{D}_{\rho_{\mathbb{F}}}^{\square}$, whether we view them as groupoids over $\mathfrak{AR}_{\mathfrak{o}}$, $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$, or $\mathfrak{Aug}_{\mathfrak{o}}$.

For $A \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$, a deformation $(\rho_A, T_A, \iota) \in \mathscr{D}_{\rho_{\mathbb{F}}}(A)$ is called *of $\mathcal{P}$-height $\leqslant h$* if

(ρ_A, T_A, ι) lies in (the essential image of) $\mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}}}(A)$. Concretely, this means that $T_A \otimes A/\mathfrak{m}_A^n$ is of $\mathcal{P}$-height $\leqslant h$ as a torsion $\boldsymbol{\mathcal{G}}_K$-representation (Definition 8.1.7) for all $n \geq 1$. For $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$, a deformation $(\rho_A, T_A, \iota) \in \mathscr{D}_{\rho_{\mathbb{F}}}(A, I)$ is called *of $\mathcal{P}$-height* $\leqslant h$ if (ρ_A, T_A, ι) lies in (the essential image of) $\mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}}}(A, I)$. Concretely, this means that there exists $A' \in \mathfrak{Aug}_{\mathfrak{o}}^{(A,I)}$ and a A'-deformation $(\rho_{A'}, T_{A'}, \iota)$ of $\mathcal{P}$-height $\leqslant h$ such that $T_A \cong T_{A'} \otimes_{A'} A$ as (A, I)-deformations of $\rho_{\mathbb{F}}$. We similarly define framed deformations of $\mathcal{P}$-height $\leqslant h$ with coefficients in $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ and $\mathfrak{Aug}_{\mathfrak{o}}$.

Having defined the $\mathfrak{AR}_{\mathfrak{o}}$-groupoids $\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ and $\mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}}}$, it is natural to ask if these groupoids or the associated functors are pro-representable. As remarked in §11.7.1 below, the tangent spaces $|\mathscr{D}_{\rho_{\mathbb{F}}}|(\mathbb{F}[\epsilon])$ and $\left|\mathscr{D}^{\square}_{\rho_{\mathbb{F}}}\right|(\mathbb{F}[\epsilon])$ are *not* finite-dimensional over $\mathbb{F}$, hence we cannot expect to have 'unrestricted' universal deformation rings and universal framed deformation rings. Later in (11.7), we will prove the following theorem, which asserts that we have finiteness of the tangent space via imposing the deformation condition of being of $\mathcal{P}$-height $\leqslant h$.

Theorem 11.1.2. *Assume that the residue field k is finite. Then the functor $\left|\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}\right|$ always has a hull. If $\mathrm{End}_{\boldsymbol{\mathcal{G}}_K}(\rho_{\mathbb{F}}) \cong \mathbb{F}$ then the $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$-groupoid $\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ is representable. The functor $\left|\mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}}}\right|$ is representable with no assumption on $\rho_{\mathbb{F}}$. Furthermore, the natural inclusions $\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}} \hookrightarrow \mathscr{D}_{\rho_{\mathbb{F}}}$ and $\mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}}} \hookrightarrow \mathscr{D}^{\square}_{\rho_{\mathbb{F}}}$ of $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$-groupoids are relatively representable by surjective maps in $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$.*

11.1.3 Topological convention

Let R and A be $\mathfrak{o}_0$-algebras. We set the following convention for the meaning of R_A:

1. If π_0 is nilpotent in a discrete $\mathfrak{o}_0$-algebra A, for example if $A \in \mathfrak{AR}_{\mathfrak{o}}$ or $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$ for some $I \subset A$, then we set $R_A := A \otimes_{\mathfrak{o}_0} R$. For example, $\mathfrak{S}_A := A \otimes_{\mathfrak{o}_0} \mathfrak{S}$

and $\mathfrak{o}_{\mathcal{E},A} := A \otimes_{\mathfrak{o}_0} \mathfrak{o}_{\mathcal{E}}$.

2. If A is a complete local noetherian $\mathfrak{o}_0$-algebra, then we set $R_A := \varprojlim_n (A/\mathfrak{m}_A^n) \otimes_{\mathfrak{o}_0} R$. For example, $\mathfrak{S}_A := \varprojlim_n (A/\mathfrak{m}_A^n) \otimes_{\mathfrak{o}_0} \mathfrak{S}$ and $\mathfrak{o}_{\mathcal{E},A} := \varprojlim_n (A/\mathfrak{m}_A^n) \otimes_{\mathfrak{o}_0} \mathfrak{o}_{\mathcal{E}}$.

3. If A is a finite F_0-algebra, then choose a finite flat $\mathfrak{o}_0$-subalgebra $A^o \in A$ with $A = A^o[\frac{1}{\pi_0}]$ and set $R_A := R_{A^o}[\frac{1}{\pi_0}]$. For example, $\mathfrak{S}_A := \mathfrak{S}_{A^o}[\frac{1}{\pi_0}]$ and $\mathfrak{o}_{\mathcal{E},A} := \mathfrak{o}_{\mathcal{E},A^o}[\frac{1}{\pi_0}]$. Note that R_A is independent of the choice of A^o; for any finite flat $\mathfrak{o}_0$-subalgebra $A^{o\prime} \subset A$ containing A^o, we have $R_{A^{o\prime}} \cong R_{A^o} \otimes_{A^o} A^{o\prime}$ (using A^o-finiteness of $A^{o\prime}$). Furthermore, for any finite A-algebra B, we have $R_B \cong R_A \otimes_A B$.

11.1.4

For $T_A \in \mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$ with $A \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$, we define $\underline{D}_{\mathcal{E}}^{\leqslant h}(T_A) \cong \underline{D}_{\mathcal{E},A}(T_A(-h))$. By the discussion in §8.2.4 $\underline{D}_{\mathcal{E}}^{\leqslant h}$ is an exact equivalence of categories $\mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K) \to (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$ which commutes with $\otimes$-products, internal homs, duality, and change of coefficients for $A \to B$ in $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$.

Let A be an $\mathfrak{o}_0$-module with $\pi_0^N \cdot A = 0$ for some N (e.g. $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$ for some $I \subset A$). For $T_A \in \mathrm{Rep}_A^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$, we define

$$\underline{D}_{\mathcal{E}}^{\leqslant h}(T_A) := \Big(T_A(-h) \otimes_A (\mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}})_A\Big)^{\boldsymbol{\mathcal{G}}_K}. \tag{11.1.4.1}$$

Note that there exists a finite $\mathfrak{o}_0$-subalgebra $A' \subset A$ and $T_{A'} \in \mathrm{Rep}_{A'}^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K)$ with $T_A \cong T_{A'} \otimes_{A'} A$ (because $\boldsymbol{\mathcal{G}}_K$ has a finite image in $\mathrm{Aut}_A(T_A)$). In this case, it easily follows that $\underline{D}_{\mathcal{E},A}^{\leqslant h}(T_A) \cong \underline{D}_{\mathcal{E}}^{\leqslant h}(T_{A'}) \otimes_{\mathfrak{o}_{\mathcal{E},A'}} \mathfrak{o}_{\mathcal{E},A}$. This shows that $\underline{D}_{\mathcal{E},A}^{\leqslant h}(T_A)$ is an étale φ-module which is finite free with $\mathfrak{o}_{\mathcal{E},A}$-rank equal to $\mathrm{rank}_A(T_A)$. Furthermore, $\underline{D}_{\mathcal{E},A}^{\leqslant h}$ is exact and commutes with change of coefficients for any $\mathfrak{o}_0$-map $A \to B$, which essentially reduces to the case when $\#(A) < \infty$ handled in §8.2.4.

The following is the reason for taking the Tate twist in the definition of $\underline{D}_{\mathcal{E}}^{\leqslant h}$. Choose $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$, and let $T := \underline{T}_{\mathfrak{S}}^{*}(\mathfrak{M})$. Then $\underline{D}_{\mathcal{E}}(T)$ does *not* have any $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ unless T is unramified. On the other hand, we have $\underline{D}_{\mathcal{E}}^{\leqslant h}(T) \cong \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} (\mathfrak{M}^{\vee})$, where $\mathfrak{M}^{\vee}$ is the dual of $\mathcal{P}$-height h. From now on, we work with $\underline{D}_{\mathcal{E}}^{\leqslant h}$ instead of the contravariant functor $\underline{D}_{\mathcal{E}}^{*}$, to associate an étale φ-module to a $\boldsymbol{\mathcal{G}}_K$-representation.

Now, let $(\rho_{\mathbb{F}}, T_{\mathbb{F}})$ be an $\mathbb{F}$-representation of $\mathcal{P}$-height $\leqslant h$, and let $M_{\mathbb{F}} := \underline{D}_{\mathcal{E}}^{\leqslant h}(T_{\mathbb{F}})$. Applying the functor $\underline{D}_{\mathcal{E}}^{\leqslant h}$ to a deformation (ρ_A, T_A, ι_A) of $\rho_{\mathbb{F}}$ over $A \in \widehat{\mathfrak{AR}_{\mathfrak{o}}}$, we obtain $M_A := \underline{D}_{\mathcal{E}}^{\leqslant h}(T_A)$, together with a φ-compatible isomorphism $M_{\mathbb{F}} \xrightarrow{\sim} M_A \otimes_A A/\mathfrak{m}_A$ obtained from ι_A. This motivates the following definition of the $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$-groupoid $\mathscr{D}_{M_{\mathbb{F}}}$.

- An object in $\mathscr{D}_{M_{\mathbb{F}}}(A)$ for $A \in \widehat{\mathfrak{AR}_{\mathfrak{o}}}$ is a pair (M_A, ι_A) where $M_A \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$, and $\iota_A : M_{\mathbb{F}} \xrightarrow{\sim} M_A \otimes_A (A/\mathfrak{m}_A)$ is a φ-compatible isomorphism.

- A morphism $(M_A, \iota_A) \to (M_B, \iota_B)$ over $f : A \to B$ is an equivalence class of φ-compatible maps $\alpha : M_A \to M_B$ over f, such that $\iota_B = \overline{\alpha} \circ \iota_A$ where $\overline{\alpha} : M_A \otimes_A (A/\mathfrak{m}_A) \to M_B \otimes_B (B/\mathfrak{m}_B)$ is induced by α. Two such maps are equivalent if one is a $(1 + \mathfrak{m}_B)$-multiple of the other. By Nakayama's lemma, α induces an isomorphism $M_A \otimes_{o_{\mathcal{E},A}} \mathfrak{o}_{\mathcal{E},B} \xrightarrow{\sim} M_B$.

Observe that the formation of $\mathscr{D}_{M_{\mathbb{F}}}$ commutes with 2-projective limits (Definition 10.4.2). By construction, we have a 1-isomorphism $\underline{D}_{\mathcal{E}}^{\leqslant h} : \mathscr{D}_{\rho_{\mathbb{F}}} \xrightarrow{\sim} \mathscr{D}_{M_{\mathbb{F}}}$ of groupoids over $\widehat{\mathfrak{AR}_{\mathfrak{o}}}$, which commutes with the 2-projective limits (Definition 10.4.8).

The following $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid extends the $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\mathscr{D}_{M_{\mathbb{F}}}$ via 2-direct limits, hence we denote this $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid by the same notation $\mathscr{D}_{M_{\mathbb{F}}}$.

- An object over $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$ is a pair (M_A, ι_A), where $M_A \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$; i.e., an étale φ-module which is free over $\mathfrak{o}_{\mathcal{E},A}$, and $\iota_A : M_{\mathbb{F}} \to M_A \otimes_A (A/I)$ is a

φ-compatible map which induces an isomorphism $M_{\mathbb{F}} \otimes_{\mathbb{F}} (A/I) \xrightarrow{\sim} M_A \otimes_A (A/I)$.

- A morphism $(M_A, \iota_{(A,I)}) \to (M_B, \iota_{(B,J)})$ over $f : (A, I) \to (B, J)$ is an equivalence class of φ-compatible maps $\alpha : M_A \to M_B$ over f such that $\iota_{(B,J)} = \overline{\alpha} \circ \iota_{(A,I)}$ where $\overline{\alpha} : M_A \otimes_A (A/I) \to M_B \otimes_B (B/J)$ is induced by α. Two such maps are equivalent if one is a $(1 + \mathfrak{m}_B)$-multiple of the other. By Nakayama's lemma for nilpotent ideals, α induces an isomorphism $M_A \otimes_{\mathfrak{o}_{\mathcal{E},A}} \mathfrak{o}_{\mathcal{E},B} \xrightarrow{\sim} M_B$.
- The assignment $\Xi : \{(M_{A'}, \iota)\} \mapsto (M_{A'} \otimes_{A'} A, \iota)$ defines a 1-isomorphism, $(\widetilde{\mathscr{D}_{M_{\mathbb{F}}}|_{\mathfrak{AR}_{\mathfrak{o}}}}) \to \mathscr{D}_{M_{\mathbb{F}}}$ where the left side is constructed in §10.4.4 and the right side is defined above.

We still need to prove that Ξ is a 1-isomorphism. Since essentially the same argument given in §10.4.6 works, we only sketch the proof. Having $\iota_{(A,I)}$, any object or a morphism of the above $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid always descends over $\mathfrak{o}_{\mathcal{E},A^+}$, where A^+ is the preimage of $\mathbb{F}$ under $A \twoheadrightarrow A/I$. So it descends over some finitely generated $\mathfrak{o}_{\mathcal{E}}$-subalgebra of $\mathfrak{o}_{\mathcal{E},A^+}$, which is necessarily of the form $\mathfrak{o}_{\mathcal{E},A'}$ for some $A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$.

The formula (11.1.4.1) defines a 1-morphism $\underline{D}_{\mathcal{E}}^{\leqslant h} : \mathscr{D}_{\rho_{\mathbb{F}}} \to \mathscr{D}_{M_{\mathbb{F}}}$ over $\mathfrak{Aug}_{\mathfrak{o}}$ since $\underline{D}_{\mathcal{E}}^{\leqslant h}$ commutes with change of coefficients for any map $(A, I) \to (B, J)$ in $\mathfrak{Aug}_{\mathfrak{o}}$. In fact, this 1-morphism commutes with 2-direct limits (Definition 10.4.8), which follows from the natural morphism $\underline{D}_{\mathcal{E}}^{\leqslant h}(T_{A'}) \otimes_{A'} A \to \underline{D}_{\mathcal{E}}^{\leqslant h}(T_{A'} \otimes_{A'} A)$ being an isomorphism for any $A' \in \mathfrak{AR}_{\mathfrak{o}}^{(A,I)}$. Since the 1-morphism $\underline{D}_{\mathcal{E}}^{\leqslant h}$ is a 1-isomorphism over $\mathfrak{AR}_{\mathfrak{o}}$, its extension $\underline{D}_{\mathcal{E}}^{\leqslant h}$ over $\mathfrak{Aug}_{\mathfrak{o}}$ is also a *1-isomorphism*, by the discussion in §10.4.7.

11.1.5 $\mathfrak{S}_A$-lattices of $\mathcal{P}$-height $\leqslant h$

Let A be either in $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ or in $\mathfrak{Aug}_{\mathfrak{o}}$. Consider $M_A \in (\mathrm{ModFI}\,/\mathfrak{o}_{\mathcal{E}})_A^{\text{ét}}$ and let $\mathfrak{M}_A \subset M_A$ be a $\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$ (Definition 8.2.2). For any $A \to B$, the scalar extension $\mathfrak{S}_B \otimes_{\mathfrak{S}_A} \mathfrak{M}_A \subset \mathfrak{o}_{\mathcal{E},B} \otimes_{\mathfrak{o}_{\mathcal{E},A}} M_A$ is a $\mathfrak{S}_B$-lattice of $\mathcal{P}$-height $\leqslant h$. Therefore, we can define a groupoid, whose fiber over A is the category of $\mathfrak{S}_A$-lattices of $\mathcal{P}$-height $\leqslant h$.

More precisely, we define a $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}$, as follows. Objects in $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}(A,I)$ are pairs $(\mathfrak{M}_A, \iota_{(A,I)})$ where $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ and $\iota_{(A,I)} : M_{\mathbb{F}} \to (\mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{o}_{\mathcal{E},A}) \otimes_A (A/I)$ is a φ-compatible map which induces an isomorphism $M_{\mathbb{F}} \otimes_{\mathbb{F}} (A/I) \xrightarrow{\sim} (\mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{o}_{\mathcal{E},A}) \otimes_A (A/I)$. A morphism is an equivalence class of φ-compatible maps which respect $\iota_{(A,I)}$, where two maps are equivalent if one is a $(1+I)$-multiple of the other. We warn that the formation of the $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}$ does *not* commute with 2-direct limits, since $\mathfrak{M}_A \otimes_A (A/I)$ is *not* required to be "constant."

We extend $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}$ (rather, its restriction to $\mathfrak{AR}_{\mathfrak{o}}$) to a $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$-groupoid by 2-projective limits. More concretely, objects in $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}(A)$ can be viewed as pairs $(\mathfrak{M}_A, \iota_A)$ where $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ and $\iota_A : M_{\mathbb{F}} \xrightarrow{\sim} (\mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{o}_{\mathcal{E},A}) \otimes_A (A/\mathfrak{m}_A)$ is a φ-compatible $\mathfrak{o}_{\mathcal{E},\mathbb{F}}$-linear isomorphism. A morphism is an equivalence class (under multiplication by $(1+\mathfrak{m}_A)$ of φ-compatible maps which respect ι_A.

We define a 1-morphism $\Upsilon : \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}} \to \mathscr{D}_{M_{\mathbb{F}}}$ over $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ and $\mathfrak{Aug}_{\mathfrak{o}}$ by $(\mathfrak{M}_A, \iota) \mapsto (M_A, \iota)$, where $M_A := \mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{o}_{\mathcal{E},A}$. We also have a 1-morphism $\underline{T}^{\leqslant h}_{\mathfrak{S}} : \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}} \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ over $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ defined by $(\mathfrak{M}_A, \iota) \mapsto (\underline{T}^*_{\mathfrak{S},A}(\mathfrak{M}^\vee_A), \underline{D}^{\leqslant h}_{\mathcal{E},A}(\iota))$. If $A \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$, then we have $\mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{o}_{\mathcal{E},A} \cong \underline{D}^{\leqslant h}_{\mathcal{E}}(\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}^\vee_A))$; i.e., we have a 2-isomorphism $\underline{D}^{\leqslant h}_{\mathcal{E}} \circ \underline{T}^{\leqslant h}_{\mathfrak{S}} \cong \Upsilon$ over $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$. All the 1-morphisms which appear in this paper will commute with 2-projective limits.

The following proposition shows that we can extend $\underline{T}^{\leqslant h}_{\mathfrak{S}}$ to a 1-morphism over $\mathfrak{Aug}_{\mathfrak{o}}$. The discussion in §10.4.7 does *not* apply because the formation of the $\mathfrak{Aug}_{\mathfrak{o}}$-groupoid $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}$ does not commute with 2-direct limits. Compare with [51, Proposition (2.1.3)].

Proposition 11.1.6. *There exists a 1-morphism* $\underline{T}_{\mathfrak{S}}^{\leqslant h} : \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h} \to \mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}$ *over* $\mathfrak{Aug}_{\mathfrak{o}}$ *which recovers the 1-morphism* $\underline{T}_{\mathfrak{S}}^{\leqslant h}$ *over* $\mathfrak{AR}_{\mathfrak{o}}$ *and makes the following diagram 2-commute:*

$$\begin{array}{ccc} \mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h} & \xrightarrow{\underline{D}_{\mathcal{E}}^{\leqslant h}} & \mathscr{D}_{M_{\mathbb{F}}} \\ & \nwarrow^{\underline{T}_{\mathfrak{S}}^{\leqslant h}} & \uparrow \Upsilon \\ & & \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h} \end{array}$$

Observe that this 2-commutative diagram determines $\underline{T}_{\mathfrak{S}}^{\leqslant h}$ uniquely up to 2-isomorphism since the horizontal 1-morphism $\underline{D}_{\mathcal{E}}^{\leqslant h}$ is fully faithful.

Proof. Let $(\mathfrak{M}_A, \iota) \in \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h}(A, I)$ and set $M_A := \mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{o}_{\mathcal{E},A} (= \mathfrak{M}_A[\frac{1}{u}])$. By the definition of $\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h}$ we have an isomorphism $\iota_{(A,I)} : M_{\mathbb{F}} \to (\mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{o}_{\mathcal{E},A}) \otimes_A (A/I)$, so M_A descends to a finite free étale $(\varphi, \mathfrak{o}_{\mathcal{E},A^+})$-module M_{A^+} where A^+ is the preimage of $\mathbb{F}$ under the natural projection $A \twoheadrightarrow A/I$. By the standard limit argument, there exists a $A' \in \mathfrak{Aug}_{\mathfrak{o}}^{(A,I)}$ such that $M_A := \mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{o}_{\mathcal{E},A} (= \mathfrak{M}_A[\frac{1}{u}])$ descends to a finite free étale $(\varphi, \mathfrak{o}_{\mathcal{E},A^+})$-module $M_{A'}$. For $A'' \in \mathfrak{Aug}_{\mathfrak{o}}^{(A,I)}$ containing A', we may repeat this process to obtain a finite étale $(\varphi, \mathfrak{o}_{\mathcal{E},A''})$-module $M_{A''}$ and we have a natural φ-compatible isomorphism $M_{A''} \cong M_{A'} \otimes_{\mathfrak{o}_{\mathcal{E},A'}} \mathfrak{o}_{\mathcal{E},A''}$ because both sides map onto same $\mathfrak{o}_{\mathcal{E},A''}$-submodule of M_{A^+} under the natural maps. Now we set $\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A, \iota) := (\underline{T}_{\mathcal{E}}(M_{A'}(-h)) \otimes_{A'} A,\, \iota)$, which is clearly an (A, I)-deformation of $\rho_{\mathbb{F}}$ and independent of the choice of $A' \in \mathfrak{Aug}_{\mathfrak{o}}^{(A,I)}$. It remains to show that $\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A, \iota)$ is of $\mathcal{P}$-height $\leqslant h$ as an (A, I)-deformation of $\rho_{\mathbb{F}}$.

We set $\mathfrak{M}_{A'} := M_{A'} \cap \mathfrak{M}_A \subset M_{A'}$. Since $\mathfrak{M}_{A'}$ is a $\mathfrak{S}$-submodule of finitely generated $\mathfrak{o}_{\mathcal{E}}$-module $M_{A'}$ with no nonzero infinitely u-divisible element, $\mathfrak{M}_{A'}$ is finitely generated over $\mathfrak{S}$. Clearly $\mathfrak{M}_{A'}$ is φ-stable submodule of $M_{A'}$ such that $\mathfrak{M}_A \otimes_{\mathfrak{S}_A} \mathfrak{o}_{\mathcal{E},A} = \mathfrak{M}_{A'}[\frac{1}{u}] = M_{A'} \cap \mathfrak{M}_A[\frac{1}{u}] = M_{A'}$. By construction, u is $\mathfrak{M}_{A'}$-regular, hence $\mathfrak{M}_{A'}$ is of projective dimension 1 as a $\mathfrak{S}$-module. To see that the cokernel of

φ on $\mathfrak{M}_{A'}$ is annihilated by $\mathcal{P}(u)^h$, we use the following φ-compatible right exact sequence

$$0 \to \mathfrak{M}_{A'} \to M_{A'} \oplus \mathfrak{M}_A \xrightarrow{(a,b)\mapsto b-a} M_A\,,$$

together with the injectivity of φ and the snake lemma. This shows that $\mathfrak{M}_{A'} \in (\mathrm{Mod}\,/\mathfrak{S})^{\leqslant h}$, Therefore, $\underline{T}_{\mathcal{E}}(M_{A'}(-h))$ is of $\mathcal{P}$-height $\leqslant h$ as an $\mathfrak{o}_0$-torsion $\boldsymbol{\mathcal{G}}_K$-representation, so $\underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}_A, \iota) := \underline{T}_{\mathcal{E}}(M_{A'}(-h)) \otimes_{A'} A$ is of $\mathcal{P}$-height $\leqslant h$, by definition. (We cannot conclude that $\mathfrak{M}_{A'} \in (\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_{A'}$, because we cannot show $\mathfrak{M}_{A'}$ is a projective $\mathfrak{S}_{A'}$-module.) □

Remark 11.1.7. It follows from Corollary 9.3.5 that if $he < q-1$ then $\underline{T}^{\leqslant h}_{\mathfrak{S}} : \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}} \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ is a *1-isomorphism.* But the assumption that $he < q-1$ is essential[1] because otherwise $\underline{T}^{\leqslant h}_{\mathfrak{S}}$ may not even be fully faithful. In fact, if $he \geq q-1$, then we have $\underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}_A) \cong \underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}_A(h))$ for any $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_A$.

If $he \geqslant q-1$ then $\underline{T}^{\leqslant h}_{\mathfrak{S}}$ may not be essentially surjective.[2] When $he \geqslant q$ we now give an example (with $A \in \mathfrak{AR}_{\mathfrak{o}}$) of a deformation T_A of $\mathcal{P}$-height $\leqslant h$ which cannot have any $\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$. Assume that $he \geqslant q$ and let $(\rho_{\mathbb{F}}, T_{\mathbb{F}})$ be the trivial 1-dimensional representation, so $M_{\mathbb{F}} := \underline{D}^{\leqslant h}_{\mathcal{E}}(T_{\mathbb{F}}) \cong \mathfrak{o}_{\mathcal{E},\mathbb{F}}(h)$. Take a deformation which corresponds, under $\underline{D}^{\leqslant h}_{\mathcal{E}}$, to $M_{\mathbb{F}[\epsilon]} \cong (\mathfrak{o}_{\mathcal{E},\mathbb{F}[\epsilon]})\cdot\mathbf{e}$ with $\varphi(\sigma^*\mathbf{e}) = (\mathcal{P}(u)^h + \frac{1}{u}\epsilon)\mathbf{e}$. Then, $\mathfrak{M}_{\mathbb{F}[\epsilon]} \cong \mathfrak{S}_F\cdot\mathbf{e} \oplus \mathfrak{S}_{\mathbb{F}}\cdot(\frac{1}{u}\epsilon\mathbf{e})$ is a $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$, so the deformation is of $\mathcal{P}$-height $\leqslant h$, but there is no $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-lattice of $\mathcal{P}$-height $\leqslant h$ in $M_{\mathbb{F}[\epsilon]}$. (One way to see this is by directly computing the "φ-matrix" for any $\mathfrak{o}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\mathbf{e}'$ of $M_{\mathbb{F}[\epsilon]}$, and show that it cannot divide $\mathcal{P}(u)^h$.)

[1] It sounds plausible to, but has not been verified by, the author that in the case of $he = q-1$, if we restrict $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}$ and $\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ to the full subcategories, whose non-zero subobjects or quotients are either never étale or never Lubin-Tate type, then $\underline{T}^{\leqslant h}_{\mathfrak{S}}$ induces a 1-isomorphism between these full subcategories.

[2] The author does not know if this bound $he < q-1$ is sharp for the essential surjectivity of $\underline{T}^{\leqslant h}_{\mathfrak{S}}$.

11.1.8

For $(R,J)\in\mathfrak{Aug}_{\mathfrak{o}}$, we write $\mathfrak{Aug}_{(R,J)}:=(\mathfrak{Aug}_{\mathfrak{o}}/(R,J))$ which is defined in Definition 10.2.3. Concretely, the objects of $\mathfrak{Aug}_{(R,J)}$ are pairs (A,I), where A is an R-algebra and $I\subset A$ is a nilpotent ideal such that $J\cdot A\subset I$.

For $(R,J)\in\mathfrak{Aug}_{\mathfrak{o}}$, any R-scheme X can be viewed as a functor $X:\mathfrak{Aug}_{(R,J)}\to$ (**Sets**) defined by $(A,I)\mapsto \mathrm{Hom}_R(\mathrm{Spec}\,A,X)$, hence as an $\mathfrak{Aug}_{(R,J)}$-groupoid which is co-fibered in sets. We use the same letter X to denote this $\mathfrak{Aug}_{(R,J)}$-groupoid. We say an $\mathfrak{Aug}_{(R,J)}$-groupoid $\mathscr{F}$ is *representable* by an R-scheme X, if we have a 1-isomorphism $X\xrightarrow{\sim}\mathscr{F}$.[3] We say that a 1-morphism $\mathscr{F}'\to\mathscr{F}$ over $\mathfrak{Aug}_{\mathfrak{o}}$ is *relatively representable by morphisms of scheme* if for any $\xi\in\mathscr{F}(A,I)$, the 2-fiber product $\mathscr{F}'_\xi$ is representable by a scheme X_ξ over A. We say that $\mathscr{F}'\to\mathscr{F}$ is is *relatively representable by projective morphisms* if X_ξ is projective over A for any $\xi\in\mathrm{Ob}(\mathscr{F}(A,I))$ and $(A,I)\in\mathfrak{Aug}_{\mathfrak{o}}$.

We now show that the 1-morphism $\underline{T}^{\leqslant h}_{\mathfrak{S}}:\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}\to\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ is relatively representable by projective morphisms in the above sense. In other words, we will show that the $\mathfrak{Aug}_{(R,J)}$-groupoid

$$\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}:=(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi)\times_{\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}}\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}} \tag{11.1.8.1}$$

for an object $\xi\in\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R,J)$ can be represented by a projective R-scheme. We first observe that it is enough to handle the case when $R\in\mathfrak{AR}_{\mathfrak{o}}$. Indeed, since any ξ over $(R,J)\in\mathfrak{Aug}_{\mathfrak{o}}$ "descends" to $\xi_{R'}$ over some $R'\in\mathfrak{AR}_{\mathfrak{o}}^{(R,J)}$, the $\mathfrak{AR}_{\mathfrak{o}}^{(R,J)}$-groupoid $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}$ can be represented by $X_{\xi_{R'}}\otimes_{R'}R$ if $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi_{R'}}$ can be represented by an R'-scheme $X_{\xi_{R'}}$. From now on, we will assume that ξ is an object over $R\in\mathfrak{AR}_{\mathfrak{o}}$.

Using the explicit description of 2-fiber products §10.1.8, objects in $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}(A,I)$

[3] If we can extend the groupoids over the category of $\mathfrak{o}$-schemes S equipped with a nilpotent quasi-coherent sheaf of ideal $\mathscr{I}\subset\mathcal{O}_S$ which contains $\mathfrak{m}_{\mathfrak{o}}\cdot\mathcal{O}_S$, then this notion recovers usual relative representability.

for $(A,I) \in \mathfrak{Aug}_R$ are of form $\Big(\mathfrak{M}_A, \iota, \alpha : \xi \otimes_R A \xrightarrow{\sim} \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A)\Big)$. Observe that the $\mathfrak{Aug}_R$-groupoid $\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}$ is co-fibered in equivalence relations; this is because $(\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}/\xi)$ is co-fibered in equivalence relations over $\mathfrak{Aug}_R$ and the natural map $\mathfrak{M}_A \to \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{S}} \mathfrak{M}_A$ is injective, so for any for any objects $(\mathfrak{M}_A, \iota, \alpha), (\mathfrak{M}'_A, \iota', \alpha') \in \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}(A,I)$ there can be at most one morphism $f : \mathfrak{M}_A \to \mathfrak{M}'_A$ which respects the isomorphisms α and α'. Now we may replace the $\mathfrak{Aug}_R$-groupoid by the associated functor $\big|\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}\big|$, and replace each fiber category with its set of isomorphism classes.

Let $M_\xi := \underline{D}_{\mathcal{E}}^{\leqslant h}(\xi)$ be the étale φ-module corresponding to ξ. (See §11.1.4 for the definition of $\underline{D}_{\mathcal{E}}^{\leqslant h}$.) Viewing $\mathfrak{M}_A$ as a $\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$ in $M_\xi \otimes_R A$, the set $\big|\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}(A,I)\big|$ for $(A,I) \in \mathfrak{Aug}_R$ can be identified with the set of $\mathfrak{S}_A$-lattices of $\mathcal{P}$-height $\leqslant h$ for $M_\xi \otimes_R A$, where $M_\xi = \underline{D}_{\mathcal{E}}^{\leqslant h}(\xi)$.

For any $\mathfrak{o}_0$-scheme X, we set $\mathfrak{S}_X := \mathfrak{S} \otimes_{\mathfrak{o}_0} \mathcal{O}_X$ and $\mathfrak{o}_{\mathcal{E},X} := \mathfrak{o}_{\mathcal{E}} \otimes_{\mathfrak{o}_0} \mathcal{O}_X$. We say a φ-stable $\mathfrak{S}_X$-lattice $\mathfrak{M}_X$ in a finite free étale $(\varphi, \mathfrak{o}_{\mathcal{E},X})$-module M_X is *of $\mathcal{P}$-height $\leqslant h$* if $\operatorname{coker} \varphi_{\mathfrak{M}_X}$ is annihilated by $\mathcal{P}(u)^h$. The following proposition asserts that the 1-morphism $\underline{T}_{\mathfrak{S}}^{\leqslant h} : \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h} \to \mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}$ is relatively representable by projective morphisms.

Proposition 11.1.9. *Assume that the residue field k of K is finite, and choose $\xi \in \mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}(R)$ for some $R \in \mathfrak{AR}_{\mathfrak{o}}$. Then there exists a projective R-scheme $\mathscr{GR}_\xi^{\leqslant h}$ and a $\mathfrak{S}_{\mathscr{GR}_\xi^{\leqslant h}}$-lattice $\underline{\mathfrak{M}}_\xi^{\leqslant h} \subset M_\xi \otimes_R \mathcal{O}_{\mathscr{GR}_\xi^{\leqslant h}}$ of $\mathcal{P}$-height $\leqslant h$ with the following property: $\underline{\mathfrak{M}}_\xi^{\leqslant h}$ defines a 1-isomorphism $\mathscr{GR}_\xi^{\leqslant h} \xrightarrow{\sim} \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}$, such that for any $(A,I) \in \mathfrak{Aug}_R$, an A-point $\eta \in \mathscr{GR}_\xi^{\leqslant h}(A)$ is mapped to $\eta^*(\underline{\mathfrak{M}}_\xi^{\leqslant h}) \in \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}(A,I)$.*

Any two pairs $(\mathscr{GR}_\xi^{\leqslant h}, \underline{\mathfrak{M}}_\xi^{\leqslant h})$ are related by a unique isomorphism. Moreover, the projective scheme $\mathscr{GR}_\xi^{\leqslant h}$ enjoys the following further properties:

1. *Let $\xi \to \xi'$ be a morphism in $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}$ over a morphism $R \to R'$ in $\mathfrak{AR}_{\mathfrak{o}}$. Then there exists a unique isomorphism $\mathscr{GR}_\xi^{\leqslant h} \otimes_R R' \xrightarrow{\sim} \mathscr{GR}_{\xi'}^{\leqslant h}$, which pulls back $\underline{\mathfrak{M}}_{\xi'}^{\leqslant h}$ to*

$\underline{\mathfrak{M}}_\xi^{\leqslant h} \otimes_R R'$ *inside of* $(M_\xi \otimes_R \mathcal{O}_{\mathscr{GR}_\xi^{\leqslant h}}) \otimes_R R'$.

2. $\mathscr{GR}_\xi^{\leqslant h}$ *is equipped with a canonical very ample line bundle, whose formation commutes with the base change described in (1).*

We call the $\mathfrak{S}_{\mathscr{GR}_\xi^{\leqslant h}}$-lattice $\underline{\mathfrak{M}}_\xi^{\leqslant h} \subset \mathcal{O}_{\mathscr{GR}_\xi^{\leqslant h}} \otimes_{\mathfrak{o}_0} M_\xi$ the *universal* $\mathfrak{S}$*-lattice of* $\mathcal{P}$*-height* $\leqslant h$ for ξ.

Idea of Proof. Since the proof is almost the same as that of [51, Proposition 2.1.7], we only indicate the idea. Let us first observe that for any $(A, I) \in \mathfrak{Aug}_R$ the natural injective map $\left|\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}(A, \mathfrak{m}_R{\cdot}A)\right| \hookrightarrow \left|\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}(A, I)\right|$ is *bijective*. Indeed, any $\mathfrak{M}_A \in \left|\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}(A, I)\right|$ satisfies

$$M_{\mathbb{F}} \otimes_{\mathbb{F}} (A/\mathfrak{m}_R{\cdot}A) \cong (M_\xi \otimes_R R/\mathfrak{m}_R) \otimes_{\mathbb{F}} (A/\mathfrak{m}_R{\cdot}A) \cong \mathfrak{M}_A[1/u] \otimes_A (A/\mathfrak{m}_R{\cdot}A),$$

and this means that $\mathfrak{M}_A \in \left|\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}(A, \mathfrak{m}_R{\cdot}A)\right|$.

Now, let $\widehat{\mathfrak{S}}_A \cong (W \otimes_{\mathfrak{o}_0} A)[[u]]$ be the u-adic completion of $\mathfrak{S}_A$, where $W := W(k)$ is the usual Witt vector ring if $\mathfrak{o}_0 = \mathbb{Z}_p$, and $W := k[[\pi_0]]$ if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. We write $\widehat{M}_A := M_A \otimes_{\mathfrak{S}_A} \widehat{\mathfrak{S}}_A$ where $M_A := M_\xi \otimes_R A$. By the main result of [4], the association $\mathfrak{M}_A \rightsquigarrow \mathfrak{M}_A \otimes_{\mathfrak{S}_A} \widehat{\mathfrak{S}}_A$ induces a natural bijection between the set of finite projective $\mathfrak{S}_A$-lattices $\mathfrak{M}_A$ in M_A and the set of finite projective $\widehat{\mathfrak{S}}_A$-lattices $\widehat{\mathfrak{M}}_A$ in $\widehat{M}_A$. Note that the latter is precisely the set of A-points of an ind-projective scheme over R: namely, the affine grassmanian for $\left(\mathrm{Res}_{W/\mathfrak{o}_0} \mathrm{GL}_d\right) \otimes_{\mathfrak{o}_0} R$, where d is the $\mathfrak{o}_{\mathcal{E},R}$-rank of M_ξ. (See [29, p42] for the definition of affine grassmanians.) It also follows that φ_{M_A} restricts to $\mathfrak{M}_A$ and the cokernel $\mathfrak{M}_A/\varphi_{M_A}(\sigma^*\mathfrak{M}_A)$ is annihilated by $\mathcal{P}(u)^h$ if and only if the corresponding conditions hold for $\widehat{\mathfrak{M}}_A \subset \widehat{M}_A$, so the functor $\left|\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}\right|$ can be represented by a closed ind-subscheme of the affine grassmanian for $\left(\mathrm{Res}_{W/\mathfrak{o}_0} \mathrm{GL}_d\right) \otimes_{\mathfrak{o}_0} R$. The argument in [51, Proposition 2.1.7] (with e replaced by he and p by q) shows that it is a projective R*-scheme*. We call this R-scheme $\mathscr{GR}_\xi^{\leqslant h}$.

(From the discussion in the paragraph right above, it follows that the "universal nilpotent coherent ideal" is $\mathfrak{m}_R\cdot\mathcal{O}_{\mathscr{G}\mathscr{R}_\xi^{\leqslant h}}$.)

To construct the universal lattice $\underline{\mathfrak{M}}_\xi^{\leqslant h}$ of $\mathcal{P}$-height $\leqslant h$, we first cover $\mathscr{G}\mathscr{R}_\xi^{\leqslant h}$ by affine open subschemes $\{\operatorname{Spec} A_i\}$. By the construction of $\mathscr{G}\mathscr{R}_\xi^{\leqslant h}$, each open affine subscheme $\operatorname{Spec} A_i$ carries the $\mathfrak{S}_{A_i}$-lattice $\mathfrak{M}_i$ of $\mathcal{P}$-height $\leqslant h$ which corresponds to the natural inclusion $\operatorname{Spec} A_i \hookrightarrow \mathscr{G}\mathscr{R}_\xi^{\leqslant h}$, and one can show that the $\mathfrak{M}_i$ glue to give $\underline{\mathfrak{M}}_\xi^{\leqslant h}$ which satisfies the properties claimed in the statement. Had we defined all the groupoids over the category of schemes X equipped with a nilpotent ideal sheaf $\mathscr{I}$ containing $\mathfrak{m}_\mathfrak{o}\cdot\mathcal{O}_X$, then $\underline{\mathfrak{M}}_\xi^{\leqslant h}$ would be the universal object. (This follows from the construction of $\mathscr{G}\mathscr{R}_\xi$ and $\mathfrak{M}_i$ over $\operatorname{Spec} A_i \hookrightarrow \mathscr{G}\mathscr{R}_\xi^{\leqslant h}$, as explained in the proof of [51, Proposition 2.1.7].)

To show that the formation commutes with base change, we observe that $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_\mathbb{F},\xi'} \cong (\mathscr{D}^{\leqslant h}_{\rho_\mathbb{F}}/\xi') \times_{(\mathscr{D}^{\leqslant h}_{\rho_\mathbb{F}}/\xi)} \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_\mathbb{F},\xi}$, so the same holds for the associated functors (because all the groupoids involved are fibered in *equivalence relations*). For the existence and construction of the canonical very ample line bundle, see [29, pp.42-43]. $\square$

11.1.10

We extend the proposition to allow $R \in \widehat{\mathfrak{A}\mathfrak{R}}_\mathfrak{o}$, because ultimately we would like to set R to be $R^{\square,\leqslant h}_{\rho_\mathbb{F}}$ or $R^{\leqslant h}_{\rho_\mathbb{F}}$ if such a deformation ring exists.

For $R \in \widehat{\mathfrak{A}\mathfrak{R}}_\mathfrak{o}$, let $\mathfrak{Aug}_R$ be the category whose objects are (A, I) where A is an R-algebra and $I \subset A$ is a nilpotent ideal such that $\mathfrak{m}_R\cdot A \subset I$. Note that a formal scheme X over $\operatorname{Spf} R$ gives rise to a category co-fibered in sets over $\mathfrak{Aug}_R$, so we may extend the notion of representability and relative representability allowing formal schemes, in the similar manner to §11.1.8.

For a fixed $\xi \in \mathscr{D}^{\leqslant h}_{\rho_\mathbb{F}}(R)$, we can define an $\mathfrak{Aug}_R$-groupoid $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_\mathbb{F},\xi}$ so that the fiber $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_\mathbb{F},\xi}(A, I)$ is the set of $\mathfrak{S}_A$-lattices of $\mathcal{P}$-height $\leqslant h$ in $M_\xi \otimes_R A$, where $M_\xi :=$

$\underline{D}_{\mathcal{E}}^{\leqslant h}(\xi)$. One way to define $\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}$ is by declaring $\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}(A,I) := \varprojlim_n \left(\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi_n}^{\leqslant h}(A,I)\right) \cong \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi_{n_0}}^{\leqslant h}(A,I)$, where $\xi_n := \xi \otimes_R R/\mathfrak{m}_R^n$ with n an integer such that $I^n = 0$.

By Proposition 11.1.9, we obtain a projective $R/\mathfrak{m}_R^n$-scheme $\mathscr{GR}_{\xi_n}^{\leqslant h}$ and a universal $\mathfrak{S}$-lattice $\underline{\mathfrak{M}}_{\xi_n}^{\leqslant h} \subset M_{\xi_n} \otimes_{R/\mathfrak{m}_R^n} \mathcal{O}_{\mathscr{GR}_{\xi_n}^{\leqslant h}}$ for each n, which is compatible with the base change under $R/\mathfrak{m}_R^n \twoheadrightarrow R/\mathfrak{m}_R^{n-1}$. On the other hand, we have a natural isomorphism $\left|\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}\right| \cong \varprojlim_n \left|\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi_n}^{\leqslant h}\right|$, by (10.4.1.2). Therefore it follows that the functor $\left|\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}\right|$ (hence the groupoid $\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}$) can be represented by the projective formal R-scheme $\widehat{\mathscr{GR}_{\xi}}^{\leqslant h} := \varinjlim_n \mathscr{GR}_{\xi_n}^{\leqslant h}$, and the $\mathfrak{S}_{\widehat{\mathscr{GR}_{\xi}}^{\leqslant h}}$-lattice $\widehat{\underline{\mathfrak{M}}_{\xi}^{\leqslant h}} := \varprojlim_n \underline{\mathfrak{M}}_{\xi_n}^{\leqslant h}$ satisfies the universal property similar to the one stated in Proposition 11.1.9. Furthermore, since each $\mathscr{GR}_{\xi_n}^{\leqslant h}$ is equipped with a (very) ample line bundle which is compatible with the direct system, it follows from Grothendieck's formal existence theorem that the formal scheme $\widehat{\mathscr{GR}_{\xi}}^{\leqslant h}$ comes from a projective scheme $\mathscr{GR}_{\xi}^{\leqslant h}$ over $\operatorname{Spec} R$ (which is unique up to unique isomorphism). Also, using the formal existence theorem for coherent sheaves on the projective formal scheme $\widehat{\mathscr{GR}_{\xi}}^{\leqslant h}\widehat{\otimes}_{\mathfrak{o}_0}\mathfrak{S}$ over $\mathfrak{S}_R := \mathfrak{S}\widehat{\otimes}_{\mathfrak{o}_0}R$, the "formal universal lattice" $\widehat{\underline{\mathfrak{M}}_{\xi}^{\leqslant h}}$ comes from a $\mathfrak{S}_{\mathscr{GR}_{\xi}^{\leqslant h}}$-lattice $\underline{\mathfrak{M}}_{\xi}^{\leqslant h}$ with the universal property similar to the one stated in Proposition 11.1.9. Here, $\mathfrak{S}_{\mathscr{GR}_{\xi}^{\leqslant h}} := \mathfrak{S} \otimes_{\mathfrak{o}_0} \mathcal{O}_{\mathscr{GR}_{\xi}^{\leqslant h}}$ and note that $\widehat{\mathscr{GR}_{\xi}}^{\leqslant h}\widehat{\otimes}_{\mathfrak{o}_0}\mathfrak{S}$ is the $\mathfrak{m}_R$-adic completion of $\mathscr{GR}_{\xi}^{\leqslant h} \otimes_{\mathfrak{o}_0} \mathfrak{S}$.

Let us assume furthermore that $\operatorname{End}_{\mathcal{G}_K}(\rho_{\mathbb{F}}) = \mathbb{F}$, in which case $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}$ is pro-representable (Theorem 11.1.2). Let $R_{\rho_{\mathbb{F}}}^{\leqslant h} \in \widehat{\mathfrak{AR}_{\mathfrak{o}}}$ be the universal deformation ring and $\xi_{\mathrm{univ}} \in \mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}(R_{\rho_{\mathbb{F}}}^{\leqslant h})$ be the universal object. Then the natural projection $\mathrm{pr}_2 : \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi_{\mathrm{univ}}}^{\leqslant h} \xrightarrow{\sim} \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h}$ is a 1-isomorphism, so $\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h}$ is representable by a projective $R_{\rho_{\mathbb{F}}}^{\leqslant h}$-scheme $\mathscr{GR}^{\leqslant h} := \mathscr{GR}_{\xi_{\mathrm{univ}}}^{\leqslant h}$. We put $\underline{\mathfrak{M}}^{\leqslant h} := \underline{\mathfrak{M}}_{\xi_{\mathrm{univ}}}^{\leqslant h}$.

To summarize, we have shown that Proposition 11.1.9 holds true even if we allow ξ to be over $R \in \widehat{\mathfrak{AR}_{\mathfrak{o}}}$. More precisely, we obtain the following corollary:

Corollary 11.1.11. *Assume that the residue field k of K is finite, and let $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$ for some $R \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$. Then there exists a projective R-scheme $\mathscr{GR}^{\leqslant h}_{\xi}$ and a $\mathfrak{S}_{\mathscr{GR}^{\leqslant h}_{\xi}}$-lattice $\underline{\mathfrak{M}}^{\leqslant h}_{\xi} \subset M_{\xi} \otimes_R \mathcal{O}_{\mathscr{GR}^{\leqslant h}_{\xi}}$ of $\mathcal{P}$-height $\leqslant h$, with the following property: $\underline{\mathfrak{M}}^{\leqslant h}_{\xi}$ defines a 1-isomorphism $\mathscr{GR}^{\leqslant h}_{\xi} \xrightarrow{\sim} \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}$, such that for any $(A, I) \in \mathfrak{Aug}_R$, an A-point $\eta \in \mathscr{GR}^{\leqslant h}_{\xi}(A)$ is mapped to $\eta^*(\underline{\mathfrak{M}}^{\leqslant h}_{\xi}) \in \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}(A, I)$. Any two pairs $(\mathscr{GR}^{\leqslant h}_{\xi}, \underline{\mathfrak{M}}^{\leqslant h}_{\xi})$ are related by a unique isomorphism, and the formation of this pair commutes with the base change in the sense of Proposition 11.1.9(1), but working in $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ instead of $\mathfrak{AR}_{\mathfrak{o}}$.*

If $\mathrm{End}_{\mathcal{G}_K}(\rho_{\mathbb{F}}) = \mathbb{F}$, then we have the following 2-commutative diagram of $\mathfrak{Aug}_{\mathfrak{o}}$-groupoids[4]*:*

$$\begin{array}{ccc} \mathscr{GR}^{\leqslant h} & \longrightarrow & \mathrm{Spec}\left(R^{\leqslant h}_{\rho_{\mathbb{F}}}\right) \\ \downarrow\cong & & \downarrow\cong \\ \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi_{\mathrm{univ}}} & \xrightarrow{\mathrm{pr}_1} & \left(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi_{\mathrm{univ}}\right) \\ \mathrm{pr}_2\downarrow\cong & & \downarrow\cong \\ \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}} & \xrightarrow{\underline{T}^{\leqslant h}_{\mathfrak{S}}} & \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}} \end{array},$$

where the upper left vertical arrow is induced by the $\mathfrak{S}_{\mathscr{GR}^{\leqslant h}_{\xi}}$-lattice $\underline{\mathfrak{M}}^{\leqslant h}$. In other words, $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}$ is representable by a projective $R^{\leqslant h}_{\rho_{\mathbb{F}}}$-scheme $\mathscr{GR}^{\leqslant h}$ together with the "universal object" $\underline{\mathfrak{M}}^{\leqslant h}$. Any two pairs $(\mathscr{GR}^{\leqslant h}, \underline{\mathfrak{M}}^{\leqslant h})$ are related by a unique isomorphism.

We call $\underline{\mathfrak{M}}^{\leqslant h}_{\xi}$ as in Corollary 11.1.11 the *universal $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$* for ξ, and $\underline{\mathfrak{M}}^{\leqslant h}$ the *universal $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$*.

11.1.12

In general, a universal deformation ring $R^{\leqslant h}_{\rho_{\mathbb{F}}}$ of $\mathcal{P}$-height $\leqslant h$ may not exist. Therefore we often work with the universal framed deformation ring $R^{\square,\leqslant h}_{\rho_{\mathbb{F}}}$ of $\mathcal{P}$-

[4] We identified schemes and the corresponding $\mathfrak{Aug}_{\mathfrak{o}}$-groupoids.

height $\leqslant h$. Let $\xi^{\square}_{\rm univ} \in \mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}}}(R^{\square,\leqslant h}_{\rho_{\mathbb{F}}})$ be a universal framed deformation of $\mathcal{P}$-height $\leqslant h$, and we denote the image of $\xi^{\square}_{\rm univ}$ under the "forgetful 1-morphism" $\mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}}} \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ by the same notation $\xi^{\square}_{\rm univ}$. Furthermore, the natural 1-morphism $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi^{\square}_{\rm univ}) \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ is formally smooth.

We put $\mathscr{GR}^{\square,\leqslant h} := \mathscr{GR}^{\leqslant h}_{\xi^{\square}_{\rm univ}}$ and $\underline{\mathfrak{M}}^{\square,\leqslant h} := \underline{\mathfrak{M}}^{\leqslant h}_{\xi^{\square}_{\rm univ}}$. This auxiliary space $\mathscr{GR}^{\square,\leqslant h}$ plays an important role in the study of the generic fiber $R^{\square,\leqslant h}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} F$.

11.2 Generic fibers of deformation rings

In the previous section, we have constructed projective morphisms $\mathscr{GR}^{\leqslant h} \to \operatorname{Spec}(R^{\leqslant h}_{\rho_{\mathbb{F}}})$ and $\mathscr{GR}^{\square,\leqslant h} \to \operatorname{Spec}(R^{\square,\leqslant h}_{\rho_{\mathbb{F}}})$. In this section, we show that $\mathscr{GR}^{\leqslant h} \otimes_{\mathfrak{o}} F \to \operatorname{Spec}(R^{\leqslant h}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} F)$ and $\mathscr{GR}^{\square,\leqslant h} \otimes_{\mathfrak{o}} F \to \operatorname{Spec}(R^{\square,\leqslant h}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} F)$ are isomorphisms (Proposition 11.2.6). This reduces the study of the generic fiber of deformation rings to the study of $\mathscr{GR}^{\leqslant h}$ and $\mathscr{GR}^{\square,\leqslant h}$ whose points have an interpretation in terms of $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$. Using this, we show that $R^{\leqslant h}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} F$ and $R^{\square,\leqslant h}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} F$ are formally smooth over F (Corollary 11.2.10).

As a first step, we need to give an interpretation of an A-point $\zeta_A : \operatorname{Spec} A \to \mathscr{GR}^{\leqslant h}_{\xi}$ for an $R \otimes_{\mathfrak{o}} F$-algebra A which is finite over F, which is done in Lemma 11.2.4. For this, we need a notion of $\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$ where A is a finite F-algebra; this will be introduced in §11.2.3.

11.2.1

As a motivation, we give an interpretation of the completions of $R^{\square,\leqslant h}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} F$ and $R^{\leqslant h}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} F$ at a maximal ideal, below in Proposition 11.2.2.

Let E be a finite extension of F, and let $\mathfrak{AR}_E$ denote the category of artin local E-algebras with residue field isomorphic to E. We put $\mathbb{F}' := \mathfrak{o}_E/\mathfrak{m}_E$ and $\rho_{\mathbb{F}'} := \rho_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$. We fix a deformation $\eta := (\rho_\eta, T_\eta) \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}'}}(\mathfrak{o}_E)$ and a framed deformation $\eta^{\square} \in \mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}'}}$.

We put $\eta_E :=$"$\eta \otimes_{\mathfrak{o}} E$" and $\eta_E^{\Box} :=$"$\eta^{\Box} \otimes_{\mathfrak{o}} E$." We let $\mathscr{D}_{\eta_E}^{\leqslant h} \subset \mathscr{D}_{\eta_E}$ denote the $\mathfrak{AR}_E$-groupoid of deformations of η_E which are of $\mathcal{P}$-height $\leqslant h$ as F_0-representations of $\boldsymbol{\mathcal{G}}_K$ in a similar way to §10.3.2 and Definition 11.1.1. We also let $\mathscr{D}_{\eta_E}^{\Box,\leqslant h} \subset \mathscr{D}_{\eta_E}^{\Box}$ denote the $\mathfrak{AR}_E$-groupoid of framed deformations of $\eta_E^{\Box}$ which are of $\mathcal{P}$-height $\leqslant h$ as F_0-representations of $\boldsymbol{\mathcal{G}}_K$ in a similar way to §10.3.2 and Definition 11.1.1. For simplicity, we often suppress the superscript $(\cdot)^{\Box}$ and let η_E denote either framed or unframed "E-deformation" of $\rho_{\mathbb{F}}$.

Proposition 11.2.2. *The framed deformation functor $\left|\mathscr{D}_{\eta_E}^{\Box,\leqslant h}\right|$ of $\mathcal{P}$-height $\leqslant h$ is prorepresentable by $(R_{\rho_{\mathbb{F}}}^{\Box,\leqslant h})^{\widehat{}}_{\eta_E}$ and the universal object is $\xi_{\mathrm{univ}}^{\Box} \otimes_{R_{\rho_{\mathbb{F}}}^{\Box,\leqslant h}} (R_{\rho_{\mathbb{F}}}^{\Box,\leqslant h})^{\widehat{}}_{\eta_E}$, where $(R_{\rho_{\mathbb{F}}}^{\Box,\leqslant h})^{\widehat{}}_{\eta_E}$ denotes the completion of $(R_{\rho_{\mathbb{F}}}^{\Box,\leqslant h}) \otimes_{\mathfrak{o}} E$ with respect to the kernel of $\eta_E : (R_{\rho_{\mathbb{F}}}^{\Box,\leqslant h}) \otimes_{\mathfrak{o}} E \twoheadrightarrow E$.*

If $R_{\rho_{\mathbb{F}}}^{\leqslant h}$ exists, then the deformation functor $\mathscr{D}_{\eta_E}^{\leqslant h}$ of $\mathcal{P}$-height $\leqslant h$ is prorepresentable by $(R_{\rho_{\mathbb{F}}}^{\leqslant h})^{\widehat{}}_{\eta_E}$ and the universal object is $\xi_{\mathrm{univ}} \otimes_{R_{\rho_{\mathbb{F}}}^{\leqslant h}} (R_{\rho_{\mathbb{F}}}^{\leqslant h})^{\widehat{}}_{\eta_E}$, where $(R_{\rho_{\mathbb{F}}}^{\leqslant h})^{\widehat{}}_{\eta_E}$ denotes the completion of $(R_{\rho_{\mathbb{F}}}^{\leqslant h}) \otimes_{\mathfrak{o}} E$ with respect to the kernel of $\eta_E : (R_{\rho_{\mathbb{F}}}^{\leqslant h}) \otimes_{\mathfrak{o}} E \twoheadrightarrow E$.

Proof. We only give a proof for the framed deformation part of the proposition, since the deformation part is essentially the same but easier.

For $A \in \mathfrak{AR}_E$, consider an $\mathfrak{o}$-map $\zeta_A : R_{\rho_{\mathbb{F}}}^{\Box,\leqslant h} \to A$ which reduces to η_E modulo $\mathfrak{m}_A$. Clearly, $\zeta_A^*(\xi_{\mathrm{univ}}^{\Box})$ is a framed deformation of $\eta_E^*(\xi_{\mathrm{univ}}^{\Box})$. On the other hand, ζ_A is factored by $\zeta_{A^o} : R_{\rho_{\mathbb{F}}}^{\Box,\leqslant h} \to A^o$ for some finite $\mathfrak{o}$-subalgebra $A^o \subset A$ with $A^o[\frac{1}{\pi_0}] = A$, and $\zeta_{A^o}^*(\xi_{\mathrm{univ}}^{\Box})$ is of $\mathcal{P}$-height $\leqslant h$ as an $\mathfrak{o}_0$-lattice representation by Proposition 9.2.6. It follows that $\zeta_A^*(\xi_{\mathrm{univ}}^{\Box})$ is of $\mathcal{P}$-height $\leqslant h$ as an F_0-lattice representation.

It is left to show that any framed deformation ζ_A of $\eta_E^*(\xi_{\mathrm{univ}}^{\Box})$ can be obtained as a pull back of $\xi_{\mathrm{univ}}^{\Box}$ under a unique map $R_{\rho_{\mathbb{F}}}^{\Box,\leqslant h} \to A$. Let A^+ be the preimage of $\mathfrak{o}_E$ under the natural projection $A \twoheadrightarrow E$. By definition of η_E, the framed deformation ζ_A descends to ζ_{A^+} over A^+, so also to ζ_{A^o} over some finite $\mathfrak{o}$-subalgebra $A^o \subset A$

with $A^o[\frac{1}{\pi_0}] = A$. By Proposition 2.4.9, ζ_{A^o} is of $\mathcal{P}$-height $\leqslant h$ as an $\mathfrak{o}_0$-lattice representation, so it corresponds to a unique $\mathfrak{o}$-map $R^{\square,\leqslant h}_{\rho_{\mathbb{F}}} \to A^o$. By composing it with the natural inclusion $A^o \hookrightarrow A$, we obtain the desired map $R^{\square,\leqslant h}_{\rho_{\mathbb{F}}} \to A$ as well as its uniqueness since the map $R^{\square,\leqslant h}_{\rho_{\mathbb{F}}} \to A$ is independent of the choice of A^o. □

11.2.3

For a finite algebra A over $F := \mathrm{Frac}(\mathfrak{o})$, we let $\mathrm{Int}(A)$ denote the set of finite $\mathfrak{o}$-subalgebras $A^o \subset A$ with $A^o[\frac{1}{\pi_0}] = A$. Since π_0 is not a zero-divisor in $A^o \in \mathrm{Int}(A)$, we have the notion of isogenies for $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{A^o}$ and $(\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}_{A^o}$, and the isogeny categories $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{A^o}[\frac{1}{\pi_0}]$ and $(\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}_{A^o}[\frac{1}{\pi_0}]$ are well-defined, just as in §2.2.7 and §7.1.6. We often denote the isogeny class containing $\mathfrak{M}_{A^o}$ as $\mathfrak{M}_{A^o}[\frac{1}{\pi_0}]$, and similarly for objects in $(\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}_{A^o}[\frac{1}{\pi_0}]$.

Let $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{A}$ be the category of φ-modules $\mathfrak{M}_A$ over $\mathfrak{S}_A$ such that $\mathfrak{M}_A = \mathfrak{M}_{A^o}[\frac{1}{\pi_0}]$ for some $\mathfrak{M}_{A^o} \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{A^o}$ where $A^o \in \mathrm{Int}(A)$. We similarly define $(\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}_{A}$. For example, if $A = F_0$, then $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{F_0}$ is exactly $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}[\frac{1}{\pi_0}]$, not $\underline{\mathrm{Mod}}_{\mathfrak{S}_{F_0}}(\varphi)^{\leqslant h}$. (Here, $\mathfrak{S}_{F_0} = \mathfrak{S}[\frac{1}{\pi_0}]$.)

For $M_A \in (\mathrm{ModFI}/\mathfrak{o}_{\mathcal{E}})^{\text{ét}}_{A}$, a φ-stable $\mathfrak{S}_A$-submodule $\mathfrak{M}_A \subset M_A$ is called a *$\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$* if $\mathfrak{o}_{\mathcal{E},A} \otimes_{\mathfrak{S}_A} \mathfrak{M}_A = M_A$ and $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{A}$.

Lemma 11.2.4. *Fix $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$ with $R \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$ and put $M_\xi := \underline{D}^{\leqslant h}_{\mathcal{E}}(\xi)$. For any R-algebra A which is finite over F : $\mathrm{Frac}(\mathfrak{o})$, the set of A-points $\mathscr{GR}^{\leqslant h}_{\xi}(A) = \mathrm{Hom}_R(\mathrm{Spec}\,A, \mathscr{GR}^{\leqslant h}_{\xi})$ is naturally in bijection with the set of $\mathfrak{S}_A$-lattices of $\mathcal{P}$-height $\leqslant h$ in $M_\xi \otimes_R A$.*

Proof. Let $\mathfrak{M}_A$ be a $\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$ in $M_\xi \otimes_R A$. Then by definition, there exists $A^o \in \mathrm{Int}(A)$ and $\mathfrak{M}_{A^o} \subset \mathfrak{M}_A$ such that $\mathfrak{M}_{A^o} \otimes_{A^o} A = \mathfrak{M}_A$ and $\mathfrak{M}_{A^o} \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{A^o}$. We may enlarge A^o so that the structure morphism $R \to A$ factors through A^o. Therefore, $\mathfrak{M}_A$ corresponds to an R-map $\zeta_A : \operatorname{Spec} A \to \operatorname{Spec} A^o \xrightarrow{\zeta_{A^o}} \mathscr{GR}_\xi^{\leqslant h}$, where ζ_{A^o} is the unique A^o-point that corresponds to $\mathfrak{M}_{A^o}$. This A-point ζ_A does not depend on the choice of A^o or $\mathfrak{M}_{A^o}$.

It remains to show that any R-map $\zeta_A : \operatorname{Spec} A \to \mathscr{GR}_\xi^{\leqslant h}$ comes from a $\mathfrak{S}_A$-lattice $\mathfrak{M}_A \subset M_\xi \otimes_R A$ of $\mathcal{P}$-height $\leqslant h$. We first handle the case when $A = E$ where E is a finite extension of F. Let ρ_ξ denote the deformation over R which corresponds to ξ. Since the structure morphism $R \to E$ factors through $\mathfrak{o}_E$ (which follows from [19, Lemma 7.1.9]), we obtain an $\mathfrak{o}_E$-representation $\rho_\xi \otimes_R \mathfrak{o}_E$ which is of $\mathcal{P}$-height $\leqslant h$ as an $\mathfrak{o}_0$-lattice representation. In other words, the étale φ-module $M_\xi \otimes_R \mathfrak{o}_E$ admits a $\mathfrak{S}$-lattice $\mathfrak{M}_{\mathfrak{o}_E}$ of $\mathcal{P}$-height $\leqslant h$ equipped with a φ-compatible $\mathfrak{o}_E$-action. By Lemma 11.2.5 below, we have $\mathfrak{M}_{\mathfrak{o}_E} \in (\mathrm{ModFI}/\mathfrak{S})_{\mathfrak{o}_E}^{\leqslant h}$, so $\mathfrak{M}_E := \mathfrak{M}_{\mathfrak{o}_E}[\frac{1}{\pi_0}]$ is an $\mathfrak{S}_E$-lattice of $\mathcal{P}$-height $\leqslant h$ in $M_\xi \otimes_R E$.

For the general case, it is enough to handle the case when A is local. Let $E := A/\mathfrak{m}_A$ be the residue field of A, and let $\eta_E : \operatorname{Spec} E \hookrightarrow \operatorname{Spec} A \xrightarrow{\zeta_A} \mathscr{GR}_\xi^{\leqslant h}$ be the underlying E-point. By the previous discussion for the case $A = E$, the E-point η_E is factored by an $\mathfrak{o}_E$-point $\eta : \operatorname{Spec} \mathfrak{o}_E \to \mathscr{GR}_\xi^{\leqslant h}$, so ζ_A is factored by $\zeta_{A^+} : \operatorname{Spec} A^+ \to \mathscr{GR}_\xi^{\leqslant h}$ where A^+ is the preimage of $\mathfrak{o}_E$ by the natural projection $A \twoheadrightarrow E$. But since $A^+ \cong \varinjlim_{A^o \in \mathrm{Int}(A)} A^o$, we see that ζ_{A^+} is factored by an R-map $\zeta_{A^o} : \operatorname{Spec} A^o \to \mathscr{GR}_\xi^{\leqslant h}$ for some R-subalgebra $A^o \in \mathrm{Int}(A)$. Now, let $\mathfrak{M}_{A^o} \subset M_\xi \otimes_R A^o$ denote the $\mathfrak{S}_{A^o}$-lattice of $\mathcal{P}$-height $\leqslant h$ which corresponds to ζ_{A^o}, and put $\mathfrak{M}_A := \mathfrak{M}_{A^o} \otimes_{A^o} A$. Clearly, the A-point ζ_A comes from $\mathfrak{M}_A$. □

Lemma 11.2.5. *Let $\mathfrak{M}_{\mathfrak{o}_E}$ be a $(\varphi, \mathfrak{S})$-module of $\mathcal{P}$-height $\leqslant h$ equipped with a φ-compatible action of $\mathfrak{o}_E$. Then $\mathfrak{M}_{\mathfrak{o}_E}$ is finite free over $\mathfrak{S}_{\mathfrak{o}_E}$, so $\mathfrak{M}_{\mathfrak{o}_E} \in (\mathrm{ModFI}/\mathfrak{S})_{\mathfrak{o}_E}^{\leqslant h}$.*

Proof. First observe that (i) $\mathfrak{S}_{o_E} \cong (W \otimes_{\mathfrak{o}_0} \mathfrak{o}_E)[[u]]$; (ii) $W \otimes_{\mathfrak{o}_0} \mathfrak{o}_E$ is a product of discrete valuation rings; and (iii) the (q-)Frobenius[5] endomorphism σ_W transitively permutes the primitive idempotents of $W \otimes_{\mathfrak{o}_0} \mathfrak{o}_E$. It follows that $\mathfrak{M}_{\mathfrak{o}_E}/u\mathfrak{M}_{\mathfrak{o}_E}$ is finite free over $W \otimes_{\mathfrak{o}_0} \mathfrak{o}_E$ since it is π_0-torsion free and is an étale φ-module. The $\mathfrak{S}_{\mathfrak{o}_E}$-freeness of $\mathfrak{M}_{\mathfrak{o}_E}$ follows from Proposition 7.4.2. □

Now, we are ready to prove the following proposition.

Proposition 11.2.6. *Let $R \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$. For any $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$, the $\mathfrak{o}$-morphism $\mathscr{GR}^{\leqslant h}_{\xi} \otimes_o F \to \operatorname{Spec}(R \otimes_{\mathfrak{o}} F)$ induced by the structure morphism for $\mathscr{GR}^{\leqslant h}_{\xi}$ is an isomorphism.*

Proof. Recall that the proper $\mathfrak{o}$-morphism $\mathscr{GR}^{\leqslant h}_{\xi} \otimes_o F \to \operatorname{Spec}(R \otimes_{\mathfrak{o}} F)$ is an isomorphism if and only if it is an étale monomorphism. (A morphism $X \to Y$ of schemes is called a *monomorphism* if it induces a monomorphism on the functors of points, or equivalently by [27, I, Proposition 5.3.8], if the diagonal map $Y \hookrightarrow Y \times_X Y$ is an isomorphism.) Note that $R \otimes_{\mathfrak{o}} F$ is a noetherian Jacobson ring by [62, pp.247 Lemma 1], so $\mathscr{GR}^{\leqslant h}_{\xi} \otimes_o F$ is a noetherian Jacobson scheme by [27, IV_3, Corollaire (10.4.7)]. So in order to check that $\mathscr{GR}^{\leqslant h}_{\xi} \otimes_o F \to \operatorname{Spec}(R \otimes_{\mathfrak{o}} F)$ is an étale monomorphism, it is enough to show that $\mathscr{GR}^{\leqslant h}_{\xi}(A) \to (\operatorname{Spec} R)(A)$ is a bijection for any finite F-algebra A.

Let A be a finite local F-algebra. We have $(\operatorname{Spec} R)(A) \cong \varinjlim_{A^o \in \mathrm{Int}(A)} (\operatorname{Spec} R)(A^o)$ by [19, Lemma 7.1.9]. Furthermore, we have $\mathscr{GR}^{\leqslant h}_{\xi}(A) \cong \varinjlim_{A^o \in \mathrm{Int}(A)} \mathscr{GR}^{\leqslant h}_{\xi}(A^o)$, which can be seen as follows. First, if $A = E$ is a field, then we have $\mathscr{GR}^{\leqslant h}_{\xi}(E) = \mathscr{GR}^{\leqslant h}_{\xi}(\mathfrak{o}_E)$ by the valuative criterion for properness. Now, if A is finite artin local F-algebra with residue field E, then $\mathscr{GR}^{\leqslant h}_{\xi}(A) = \mathscr{GR}^{\leqslant h}_{\xi}(A^+)$ where A^+ is the preimage of $\mathfrak{o}_E$ under the natural projection $A \twoheadrightarrow E$. Since $A^+ = \varinjlim_{A^o \in \mathrm{Int}(A)} A^o$, we have the claim. The case of general finite F-algebra A is immediate.

[5] Recall that $q = p$ if $\mathfrak{o}_0 = \mathbb{Z}_p$.

It immediately follows from Theorem 5.2.3 that $\mathscr{GR}_\xi^{\leqslant h}(A^o) \to (\operatorname{Spec} R)(A^o)$ is injective for any finite free $\mathfrak{o}$-algebra A^o, hence $\mathscr{GR}_\xi^{\leqslant h}(A) \to (\operatorname{Spec} R)(A)$ is injective for any finite F-algebra A. Now we show that this is also surjective for any finite F-algebra A. Let $A \in \mathfrak{AR}_E$ for some finite extension E/F and pick an A-point $\zeta_A : R \to A$. We let $\eta_E : R \xrightarrow{\zeta_A} A \twoheadrightarrow A/\mathfrak{m}_A \cong E$ be the E-point of $\operatorname{Spec} R$ on which ζ_A is supported. Choose a finite free $\mathfrak{o}_E$-subalgebra $A^o \subset A$ with $A = A^o[\frac{1}{\pi_0}]$ and an $\mathfrak{o}$-map $\zeta' : R \to A^o$ with $\zeta'[\frac{1}{\pi_0}] = \zeta_A$. (Such ζ' always exists for some A^o, as discussed at the beginning of the proof.) Since $\xi \otimes_R A^o$ is of $\mathcal{P}$-height $\leqslant h$, there exists a unique $\mathfrak{S}$-lattice $\mathfrak{M}_{A^o}$ of $\mathcal{P}$-height $\leqslant h$ for $M_{A^o} := M_\xi \otimes_R A^o$ which is equipped with a φ-compatible A^o-action. (As before, we put $M_\xi := \underline{D}_{\mathcal{E}}^{\leqslant h}(\xi)$).

Let $\mathfrak{M}'_{\mathfrak{o}_E}$ be the image of $\mathfrak{M}_{A^o}$ under the natural surjection $M_{A^o} \twoheadrightarrow M_{\mathfrak{o}_E} := M_{A^o} \otimes_{A^o} \mathfrak{o}_E$, and we put $\mathfrak{M}_{\mathfrak{o}_E} := \mathfrak{M}'_{\mathfrak{o}_E}[\frac{1}{\pi_0}] \cap M_{\mathfrak{o}_E}$ where the intersection is taken inside $M_{\mathfrak{o}_E}[\frac{1}{\pi_0}] \cong M_{\eta_E}$. Then $\mathfrak{M}_{\mathfrak{o}_E}$ is finite free over $\mathfrak{S}$ by the first paragraph of the proof of Theorem 5.2.3, hence is finite free over $\mathfrak{S}_{\mathfrak{o}_E}$ by Lemma 11.2.5.

By Lemma 11.2.7 below, $\mathfrak{M}_A := \mathfrak{M}_{A^o}[\frac{1}{\pi_0}]$ is free over $\mathfrak{S}_A$. Now, we choose a $\mathfrak{S}_{\mathfrak{o}_E}$-basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ for $\mathfrak{M}_{\mathfrak{o}_E}$ and lift it to a $\mathfrak{S}_A$-basis $\{\tilde{\mathbf{e}}_1, \cdots, \tilde{\mathbf{e}}_n\}$ for $\mathfrak{M}_A$. We choose $B^o \subset A$ which contains $\zeta_A(R) \subset A$ so that all the coefficients of $\varphi_{\mathfrak{M}_A}(\sigma^*\tilde{\mathbf{e}}_i)$ are contained in $\mathfrak{S}_{B^o}$. Let $\mathfrak{M}_{B^o}$ be the free $\mathfrak{S}_{B^o}$-submodule of $\mathfrak{M}_A$ spanned by $\{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$. Clearly $\mathfrak{M}_{B^o} \subset \mathfrak{M}_A$ is φ-stable and $\mathfrak{M}_{B^o}[\frac{1}{\pi_0}] = \mathfrak{M}_A$. Thus, $\mathfrak{M}_{B^o} \in (\operatorname{ModFI}/\mathfrak{S})_{B^o}^{\leqslant h}$. This shows that $\mathfrak{M}_{B^o}$ is a (unique) $\mathfrak{S}_{B^o}$-lattice of $\mathcal{P}$-height $\leqslant h$ in the étale $(\varphi, \mathfrak{o}_{\mathcal{E},B^o})$-module which corresponds to the map $R \to B^o$ that factors $\zeta_A : R \to A$. In other words, $\mathfrak{M}_{B^o}$ corresponds to a B^o-point of $\mathscr{GR}_\xi^{\leqslant h}$, so $\mathfrak{M}_A$ corresponds to an A-point of $\mathscr{GR}_\xi^{\leqslant h}$ which maps to $\zeta_A \in (\operatorname{Spec} R)(A)$.

Now, it is left to show the following lemma, which is exactly [55, Lemma 1.6.1] if $\mathfrak{o}_0 = \mathbb{Z}_p$. □

Lemma 11.2.7. *Let A be a finite F_0-algebra and let $\mathfrak{M}_A$ be a finitely generated $\mathfrak{S}_A$-module which is flat over $\mathfrak{S}[\frac{1}{\pi_0}]$ and equipped with a map $\varphi : \sigma^*\mathfrak{M}_A \to \mathfrak{M}_A$ whose cokernel is annihilated by $\mathcal{P}(u)^h$. Suppose that $M_A := \mathfrak{M}_A \otimes_{\mathfrak{S}[\frac{1}{\pi_0}]} \mathcal{E}$ is finite free over $\mathcal{E}_A$. Then $\mathfrak{M}_A$ is a finite projective $\mathfrak{S}_A$-module.*

Proof. The following proof is a lengthy way to say that the proof in [55, Lemma 1.6.1] also works if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. We prove the lemma by showing that the first nonzero Fitting ideal I for $\mathfrak{M}_A$ (i.e., the nth Fitting ideal, where n is the $\mathcal{E}_A$-rank of M_A) is equal to $\mathfrak{S}_A$. (See [28, §20.2] for Fitting ideals.)

Let $U \subset \operatorname{Spec} \mathfrak{S}[\frac{1}{\pi_0}]$ denote the largest open subscheme over which $\mathfrak{M}_A$ is $\mathfrak{S}_A$-flat, and let Z be its (reduced) complement. Since A is F_0-finite and $\mathfrak{M}_A \otimes_{\mathfrak{S}[\frac{1}{\pi_0}]} \mathcal{E}$ is free over $\mathcal{E}_A$ by assumption, Z is cut out by some non-zero $g \in \mathfrak{S}[\frac{1}{\pi_0}]$.

The isomorphism $(\sigma^*\mathfrak{M}_A)[\frac{1}{\mathcal{P}(u)}] \xrightarrow{\sim} \mathfrak{M}_A[\frac{1}{\mathcal{P}(u)}]$ implies that $g \in (\sigma(g) \cdot \mathcal{P}(u))$ and $\sigma(g) \in (g \cdot \mathcal{P}(u))$. Assume that g is not a unit, so there exists $x \in \overline{K}$ with $|x| < 1$ with $g(x) = 0$. Let x and y be such that $|x| < 1$ and $|y| < 1$ are smallest and largest among the nonzero roots of g, if any exist. Then all the nonzero roots of $\sigma(g)$ have absolute values between $|x|^{1/q}$ and $|y|^{1/q}$, which are strictly bigger than $|x|$ and $|y|$, respectively. Clearly x is a common root of g and $\sigma(g) \cdot \mathcal{P}(u)$. But since $\sigma(g)$ cannot have a root with absolute value $|x|$, x is a root of $\mathcal{P}(u)$. Similarly, a root w of $\sigma(g)$ with $|w| = |y|^{1/q}$ is also a root of $g \cdot \mathcal{P}(u)$, but g cannot have a root with absolute value $|y|^{1/q}$. Hence w is a root of $\mathcal{P}(u)$. But all roots of $\mathcal{P}(u)$ have same absolute value (being a $\mathfrak{S}^\times$-multiple of an Eisenstein polynomial), so $|x| = |w| = |y|^{1/q} > |y|$, which is a contradiction. This shows that g is either a unit or a unit multiple of a power of u.

In terms of Fitting ideals, we have shown that $u^i \in I$ for some $i \geq 0$. Therefore, in order to show that I is a unit ideal, it is enough to show this after taking u-adic completion. Let $\widehat{I} \subset \widehat{\mathfrak{S}_A} \cong \mathscr{K}_0[[u]]_A$ be the u-adic completion, so $\widehat{I} = (u^i)$ for some $i \geqslant 0$. Then σ extends continuously on $\mathscr{K}_0[[u]]_A$, and the u-adic completion $\widehat{\varphi} : \widehat{\sigma^*\mathfrak{M}_A} \to \widehat{\mathfrak{M}_A}$ is an isomorphism since $\mathcal{P}(u) \in (\mathscr{K}_0[[u]]_A)^\times$. Since the formation of Fitting ideals commutes with scalar extension [28, Corollary 20.5], it follows that $\sigma(\widehat{I})\cdot\mathscr{K}_0[[u]]_A = \widehat{I}$. This rules out $\widehat{I} = (u^i)$ with $i > 0$. $\square$

The following corollary is a re-interpretation of Proposition 11.2.6 using the interpretation of F-finite points of $\mathscr{GR}_\xi^{\leqslant h} \otimes_{\mathfrak{o}} F$ (Lemma 11.2.4).

Corollary 11.2.8. *Let A be a finite F_0-algebra, and let ρ_A be an A-representation of $\boldsymbol{\mathcal{G}}_K$ which is of $\mathcal{P}$-height $\leqslant h$ as an F_0-representation. Then there exists a unique $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ such that $\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A) \cong \rho_A$.*

Proof. The uniqueness of such $\mathfrak{M}_A$ is a consequence of full faithfulness of the functor $\underline{T}_{\mathfrak{S}}^{\leqslant h} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}[\frac{1}{\pi_0}] \to \mathrm{Rep}_{F_0}(\boldsymbol{\mathcal{G}}_K)$ (Theorem 5.2.3), so it suffices to show the existence. We may assume that A is also local, and let E denote its residue field. We put $\mathbb{F}' := \mathfrak{o}_E/\mathfrak{m}_E$ and F_0'/F_0 the unramified extension corresponding to the residue field extension $\mathbb{F}'/\mathbb{F}_q$. By choosing a $\boldsymbol{\mathcal{G}}_K$-stable $\mathfrak{o}_E$-lattice of $\rho_A \otimes_A E$ and reducing it modulo $\mathfrak{m}_E$, we obtain a residual representation $\rho_{\mathbb{F}'} := \rho_{\mathbb{F}} \otimes_{\mathbb{F}} \mathbb{F}'$. By essentially the same argument as the proof of Proposition 11.2.2, there exists a finite $\mathfrak{o}_{F_0'}$-subalgebra $A^o \subset A$ and a $\boldsymbol{\mathcal{G}}_K$-stable A^o-lattice ρ_{A^o} in ρ_A. Note that $A = A^o \otimes_{\mathfrak{o}_0} F_0'$. Let $\xi \in \mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}(A^o)$ be the deformation corresponding to ρ_{A^o}. By Proposition 11.2.6 we have an isomorphism $\mathscr{GR}_\xi^{\leqslant h} \otimes_{\mathfrak{o}_0} F_0' \xrightarrow{\sim} \operatorname{Spec} A$. By Lemma 11.2.4 this A-point of $\mathscr{GR}_\xi^{\leqslant h} \otimes_{\mathfrak{o}_0} F_0'$ corresponds to a unique $\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$, which we have been seeking. $\square$

Theorem 11.2.9. *Assume that k is finite. For any $\eta \in \mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}(\mathfrak{o}_E)$ for a finite extension E over F, the functor $\left|\mathscr{D}_{\eta_E}^{\leqslant h}\right|$ on $\mathfrak{AR}_E$ is formally smooth.*

Proof. Let $A \in \mathfrak{AR}_E$ with a nilpotent ideal $I \subset A$. We put $\overline{A} := A/I \in \mathfrak{AR}_E$. For $\overline{\zeta} \in \left|\mathscr{D}^{\leqslant h}_{\eta_E}(\overline{A})\right|$, we want to find $\zeta \in \left|\mathscr{D}^{\leqslant h}_{\eta_E}(A)\right|$ which reduces to $\overline{\zeta}$ modulo I. By Corollary 11.2.8, there exists $\mathfrak{M}_{\overline{A}} \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{\overline{A}}$ such that $\overline{\zeta} = \underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}_{\overline{A}})$. So it suffices to show that there exists $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{A}$ such that $\mathfrak{M}_A \otimes_A \overline{A} \cong \mathfrak{M}_{\overline{A}}$.

We first choose a finite flat $\mathfrak{o}_E$-subalgebra $A^o \subset A$ such that $A^o[\frac{1}{\pi_0}] = A$ and $I^o[\frac{1}{\pi_0}] = I$ where $I^o := I \cap A^o$. So we have $\overline{A} \cong (A^o/I^o)[\frac{1}{\pi_0}]$, and we put $\overline{A^o} := A^o/I^o$ and view it as a subring of $\overline{A}$. By enlarging A^o if necessary, we can assume that there exists $\mathfrak{M}_{\overline{A^o}} \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{\overline{A^o}}$ such that $\mathfrak{M}_{\overline{A^o}}[\frac{1}{\pi_0}] = \mathfrak{M}_{\overline{A}}$. By Proposition 8.2.3, $\omega_{\overline{A^o}} := \mathrm{coker}(\varphi_{\mathfrak{M}_{\overline{A^o}}})$ is finite free over $\overline{A^o}$. Let ω_{A^o} be a finite free A^o-module equipped with $\omega_{A^o} \otimes_{A^o} \overline{A^o} \cong \omega_{\overline{A^o}}$, and let $\mathfrak{M}_{A^o}$ be a finite free $\mathfrak{S}_{A^o}$-module equipped with $\mathfrak{M}_{A^o} \otimes_{A^o} \overline{A^o} \cong \mathfrak{M}_{\overline{A^o}}$. We can choose a $\mathfrak{S}_{A^o}/\mathcal{P}(u)^h$-linear surjective map $\mathfrak{M}_{A^o}/\mathcal{P}(u)^h\mathfrak{M}_{A^o} \twoheadrightarrow \omega_{A^o}$ which lifts the natural projection $\mathfrak{M}_{\overline{A^o}}/\mathcal{P}(u)^h\mathfrak{M}_{\overline{A^o}} \twoheadrightarrow \omega_{\overline{A^o}}$. Therefore, we obtain the following diagram with exact rows:

(11.2.9.1)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{N}_{A^o} & \longrightarrow & \mathfrak{M}_{A^o} & \longrightarrow & \omega_{A^o} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \sigma^*\mathfrak{M}_{\overline{A^o}} & \xrightarrow[\varphi_{\mathfrak{M}_{\overline{A^o}}}]{} & \mathfrak{M}_{\overline{A^o}} & \longrightarrow & \omega_{\overline{A^o}} & \longrightarrow & 0, \end{array}$$

where $\mathfrak{N}_{A^o}$ is the kernel of $\mathfrak{M}_{A^o} \twoheadrightarrow \mathfrak{M}_{A^o}/\mathcal{P}(u)^h\mathfrak{M}_{A^o} \twoheadrightarrow \omega_{A^o}$. Since ω_{A^o} is flat over A^o, the top row stays short exact after applying $(\cdot)\otimes_{A^o}\overline{A^o}$, so $\mathfrak{N}_{A^o}\otimes_{A^o}\overline{A^o} \xrightarrow{\sim} \sigma^*\mathfrak{M}_{\overline{A^o}}$ (i.e., the left vertical arrow in (11.2.9.1) is surjective). Therefore, we obtain a surjective map $r : \sigma^*\mathfrak{M}_{A^o} \twoheadrightarrow \mathfrak{N}_{A^o}$ which factors the natural projection $\sigma^*\mathfrak{M}_{A^o} \twoheadrightarrow \sigma^*\mathfrak{M}_{\overline{A^o}}$. Now, we define $\varphi_{\mathfrak{M}_{A^o}} : \sigma^*\mathfrak{M}_{A^o} \xrightarrow{r} \mathfrak{N}_{A^o} \hookrightarrow \mathfrak{M}_{A^o}$. Clearly $\varphi_{\mathfrak{M}_{A^o}}$ lifts $\varphi_{\mathfrak{M}_{\overline{A^o}}}$, and we have $\mathrm{coker}(\varphi_{\mathfrak{M}_{A^o}}) \cong \omega_{A^o}$ which is annihilated by $\mathcal{P}(u)^h$. So $\mathfrak{M}_A := \mathfrak{M}_{A^o}[\frac{1}{\pi_0}]$ is in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_{A}$ and lifts $\mathfrak{M}_{\overline{A}}$. (In fact, it also follows that $r : \sigma^*\mathfrak{M}_{A^o} \twoheadrightarrow \mathfrak{N}_{A^o}$ is an isomorphism by the injectivity of $\varphi_{\mathfrak{M}_{A^o}}$ (Corollary 2.2.3.2), but we do not need this in the proof.) □

Now we are ready to show the formal smoothness of the generic fiber of deformation rings of $\mathcal{P}$-height $\leqslant h$, as a corollary of Theorem 11.2.9.

Corollary 11.2.10. *Assume that k is finite. Let $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$ be such that the natural 1-morphism $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ of $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$-groupoids is formally smooth. Then $R[\frac{1}{\pi_0}]$ is formally smooth over F. In particular, the F-algebras $R^{\square,\leqslant h}_{\rho_{\mathbb{F}}}[\frac{1}{\pi_0}]$ and $R^{\leqslant h}_{\rho_{\mathbb{F}}}[\frac{1}{\pi_0}]$ (if it exists) are formally smooth over F.*

Proof. The second claim of the corollary follows from the first claim by taking $\xi = \xi^{\square}_{\mathrm{univ}}$ and $\xi = \xi_{\mathrm{univ}}$. To obtain the first claim, first note that a noetherian Jacobson F-algebra A (e.g., $A = R[\frac{1}{\pi_0}]$ for some complete local noetherian $\mathfrak{o}$-algebra R) is formally smooth over F if and only if its completion at each maximal ideal is geometrically regular, by [27, 0_{IV}, Théorème (20.5.8), Corollaires (22.6.5), (22.6.6)]. So it suffices to show that for any E-point $\eta_E : R \to E$ where E/F is some finite extension, the completion $R^{\wedge}_{\eta_E}$ of $R \otimes_{\mathfrak{o}} E$ with respect to $\ker[\eta_E \otimes E : R \otimes_{\mathfrak{o}} E \twoheadrightarrow E]$ is formally smooth over E. We use the same notation η_E to denote $\xi \otimes_{R,\eta_E} E$, and consider the 1-morphism $\operatorname{Spf} R^{\wedge}_{\eta_E} \to \mathscr{D}_{\eta_E}$ over $\mathfrak{AR}_E$ defined as follows: a (continuous) E-map $\zeta_A : R^{\wedge}_{\eta_E} \to A$ with $A \in \mathfrak{AR}_E$ is sent to $\xi \otimes_R A \in \mathscr{D}_{\eta_E}(A)$ where A is viewed as an R-algebra via $R \to R^{\wedge}_{\eta_E} \xrightarrow{\zeta_A} A$. Now using a similar argument to the proof of Proposition 11.2.2, one can show that the formal smoothness of the 1-morphism $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ implies the formal smoothness of the 1-morphism $\operatorname{Spf} R^{\wedge}_{\eta_E} \to \mathscr{D}_{\eta_E}$ over $\mathfrak{AR}_E$. The corollary then follows from Theorem 11.2.9. $\square$

11.2.11 Motivation: Relation with crystalline and semi-stable deformation rings

One can generalize Proposition 11.2.6 as follows. We may also consider the composition $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}} \xrightarrow{T^{\leqslant h}_{\mathfrak{S}}} \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}} \hookrightarrow \mathscr{D}_{\rho_{\mathbb{F}}}$ of 1-morphisms, where the latter is the natural inclusion. By Theorem 11.1.2, or rather Proposition 11.7.3, this inclusion $\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}} \hookrightarrow \mathscr{D}_{\rho_{\mathbb{F}}}$

is relatively representable by surjective maps of rings. For any $\xi \in \mathscr{D}_{\rho_{\mathbb{F}}}(R)$, let $R^{\leqslant h}$ be the universal quotient of R which represents $(\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h})_{\xi}$, and let $\xi^{\leqslant h}$ be the universal object[6] of $(\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h})_{\xi}$. Then the 1-morphism $\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h} \to \mathscr{D}_{\rho_{\mathbb{F}}}$ is relatively representable; $\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}},\xi}^{\leqslant h}$ is representable by a projective morphism $\mathscr{G}\mathscr{R}_{\xi^{\leqslant h}}^{\leqslant h} \to \operatorname{Spec} R$ which factors through the closed subscheme $\operatorname{Spec} R^{\leqslant h}$. By Proposition 11.2.6, this projective morphism induces an isomorphism $\mathscr{G}\mathscr{R}_{\xi^{\leqslant h}}^{\leqslant h} \otimes_{\mathfrak{o}} F \to \operatorname{Spec}(R^{\leqslant h} \otimes_{\mathfrak{o}} F)$. In the case $\mathfrak{o}_0 = \mathbb{Z}_p$, this proves [55, Proposition 1.6.4(2)]. Note that $R^{\leqslant h}$ may not equal the schematic image of $\mathscr{G}\mathscr{R}_{\xi^{\leqslant h}}^{\leqslant h}$ in $\operatorname{Spec} R$.

Now, we assume that $\mathfrak{o}_0 = \mathbb{Z}_p$. We fix an $\mathbb{F}$-representation $\bar{\rho}$ of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$, and put $\bar{\rho}_\infty := \bar{\rho}|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}$. We let $R_{\bar{\rho}}^{\square}$ be the universal framed deformation ring, and let ξ denote the restriction to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ of the universal framed deformation. Applying Theorem 11.1.2, or rather Proposition 11.7.3, we obtain the universal quotient $R_{\bar{\rho}}^{\square,\leqslant h}$ of $R_{\bar{\rho}}^{\square}$, over which ξ becomes of $\mathcal{P}$-height $\leqslant h$. So we obtain a map $\underline{\mathrm{res}} : R_{\bar{\rho}_\infty}^{\square,\leqslant h} \to R_{\bar{\rho}}^{\square,\leqslant h}$, where the source is the universal framed deformation ring of $\mathcal{P}$-height $\leqslant h$. From now on, we put $R_{\infty}^{\square,\leqslant h} := R_{\bar{\rho}_\infty}^{\square,\leqslant h}$, and often suppress the subscript $_{\bar{\rho}}$ on $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-deformation rings. (For example, we put $R^{\square} := R_{\bar{\rho}}^{\square}$.)

Let $R_{\mathrm{cris}}^{\square,\leqslant h}$ and $R_{\mathrm{st}}^{\square,\leqslant h}$ be the universal quotients of $R^{\square}$ whose artinian local points correspond to framed deformations that are torsion crystalline and torsion semi-stable, respectively, with Hodge-Tate weights in $[0,h]$. These quotients *a priori* factor through $R_{\bar{\rho}}^{\square,\leqslant h}$. Furthermore, Liu [59] shows that for any finite extension $E/\mathbb{Q}_p$, an E-point $x : R^{\square} \to E$ factors through the quotient $R_{\mathrm{cris}}^{\square,\leqslant h}$ or $R_{\mathrm{st}}^{\square,\leqslant h}$ if and only if the the corresponding E-representation V_x is crystalline or semi-stable with Hodge-Tate weights in $[0,h]$, respectively.[7] We call them *crystalline* and *semistable* framed deformation rings with Hodge-Tate weights in $[0,h]$, respectively. The generic

[6]In a more down-to earth manner, $R^{\leqslant h}$ is the biggest quotient of R such that the pull-back of ξ becomes $\mathcal{P}$-height $\leqslant h$, and $\xi^{\leqslant h}$ is the pull-back of ξ over $R^{\leqslant h}$.
[7]This result is also valid when $p = 2$.

fibers $R^{\square,\leqslant h}_{\mathrm{cris}}[\frac{1}{p}]$ and $R^{\square,\leqslant h}_{\mathrm{st}}[\frac{1}{p}]$ coincide with the crystalline and semi-stable quotients of $R^{\square}[\frac{1}{p}]$ constructed by Kisin [55].

The point is that "restricting to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty} \cong \boldsymbol{\mathcal{G}}_K$" defines the natural maps $\underline{\mathrm{res}}^{\mathrm{cris}} : R^{\square,\leqslant h}_{\infty} \to R^{\square,\leqslant h}_{\mathrm{cris}}$ and $\underline{\mathrm{res}}^{\mathrm{st}} : R^{\square,\leqslant h}_{\infty} \to R^{\square,\leqslant h}_{\mathrm{st}}$. Even though the map $\underline{\mathrm{res}}^{\mathrm{cris}}$ is quite mysterious in general (let alone $\underline{\mathrm{res}}^{\mathrm{st}}$), we give some applications later on of the maps $\underline{\mathrm{res}}^{\mathrm{cris}}$ and $\underline{\mathrm{res}}^{\mathrm{st}}$ in the study of crystalline and semi-stable frame deformation rings. See §11.4.17 and §11.6.

All the discussions above work for the universal deformation rings *if* both $R_{\bar\rho}$ and $R^{\leqslant h}_{\infty} := R^{\leqslant h}_{\bar\rho_\infty}$ exist. The author does not know whether $\mathrm{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}}}(\bar\rho) \cong \mathbb{F}$ guarantees $\mathrm{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}(\bar\rho_\infty) \cong \mathbb{F}$ (although he suspects that this may not be true). But we record the following cases where we do have the full faithfulness of restrictions to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ on residual representations:

1. If $\bar\rho$ is absolutely irreducible, then it is necessarily tame. Since the inclusion $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty} \hookrightarrow \boldsymbol{\mathcal{G}}_{\mathscr{K}}$ induces an isomorphism after quotienting out the wild inertia groups, we obtain that $\mathrm{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}(\bar\rho_\infty) \cong \mathbb{F}$ when $\mathrm{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}}}(\bar\rho) \cong \mathbb{F}$.

2. Under the following assumption, we have the full faithfulness of the restriction to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ for mod p crystalline representations: either $\mathscr{K}$ is absolutely unramified, $p > 2$, and $h < p-1$ [13]; or $h = 1$ and $p > 2$ (or any more general assumption for which one can prove the classification of finite flat group schemes over $\mathfrak{o}_{\mathscr{K}}$ [15, Theorem 3.4.3]).

11.3 Local structure of the generic fiber of deformation ring

The aim of this section is to compute the dimension of $R^{\leqslant h}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} F$ and $R^{\square,\leqslant h}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} F$ at a closed point of a given "Hodge-Pink type" (Corollary 11.3.11). We also show that fixing a "Hodge-Pink type" cuts out an equi-dimensional union of connected components when $\mathscr{K} \cong \mathfrak{S}[\frac{1}{\pi_0}]/(\mathcal{P}(u))$ is *separable* over $\mathscr{K}_0$ (i.e., $\mathcal{P}(u)$ is a $\mathfrak{S}^\times$-multiple of a *separable* Eisenstein polynomial). See Proposition 11.3.7 for a precise statement. Note that the separability condition is automatic if $\mathfrak{o}_0 = \mathbb{Z}_p$, but not automatic when $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. In the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, note that $K = k(\!(u)\!)$ is separable over $k(\!(u_0)\!)$ if and only if $\mathscr{K}/\mathscr{K}_0$ is so, since $\mathfrak{S}/(\pi_0 - u_0) = \mathfrak{o}_{\mathscr{K}} \cong \mathfrak{o}_K$ via $u \mapsto u$. Even though $\boldsymbol{\mathcal{G}}_K \cong \boldsymbol{\mathcal{G}}_{K'}$ for any finite purely inseparable extension K'/K, the notions of $\boldsymbol{\mathcal{G}}_K$-representations of $\mathcal{P}$-height $\leqslant h$ and $\boldsymbol{\mathcal{G}}_{K'}$-representations of $\mathcal{P}$-height $\leqslant h$ are not the same because the construction of $\mathfrak{S}$ and the choice of $\mathcal{P}(u)$ are not the same for K and K'. So we cannot replace K by its maximal separable subextension $K_s \subset K$ over $k(\!(u_0)\!)$, so the assumption that $K/k(\!(u_0)\!)$ is separable seems to give a genuine restriction.[8]

Our technique is analogous to Kisin's technique for studying the local structure of potentially semi-stable deformation rings [55, §3], with the difference that we work with weakly admissible Hodge-Pink structures while Kisin works with weakly admissible filtered (φ, N)-modules. This permits us to allow the case $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ too, as we shall do.

By Theorem 4.3.4, we have an equivalence of categories $\underline{\mathbb{H}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)[\frac{1}{\pi_0}] \xrightarrow{\sim} \mathcal{HP}_K^{wa,\geqslant 0}(\varphi)$ which restricts to $\underline{\mathbb{H}} : \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}[\frac{1}{\pi_0}] \xrightarrow{\sim} \mathcal{HP}_K^{wa,[0,h]}(\varphi)$, where the target category is the full subcategory of objects with all Hodge-Pink weights in $[0, h]$. We now generalize this to allow A-coefficients, where A is any finite F_0-algebra.

Let A be a finite F_0-algebra. We first make the following definition which is satisfied by objects of the form $(D_A, \Lambda_A) := \underline{\mathbb{H}}(\mathfrak{M}_A)$ for $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$:

Definition 11.3.1. An A-*isocrystal* is étale φ-module D_A which is free over $(\mathscr{K}_0)_A := \mathscr{K}_0 \otimes_{F_0} A$. For a free $(\mathscr{K}_0)_A$-module D_A, we set $(\mathcal{D}_A)^{\widehat{}}_{x_0} := \mathcal{O}_{\widehat{\boldsymbol{\Delta}},x_0} \otimes_{\mathscr{K}_0} D_A$. An

[8] The author does not have an example of a $\boldsymbol{\mathcal{G}}_K$-representation of $\mathcal{P}$-height $\leqslant h$ which is not $\mathcal{P}$-height $\leqslant h$ as a $\boldsymbol{\mathcal{G}}_{K_s}$-representation.

A-Hodge-Pink structure for a finite free $(\mathscr{K}_0)_A$-module D_A is a $(\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0})_A$-lattice $\Lambda_A \subset (\mathcal{D}_A)^{\wedge}_{x_0}[\frac{1}{\mathcal{P}(u)}]$, which is a *direct factor* as an A-module (i.e., for $h \gg 0$, the cokernel $\mathcal{P}(u)^{-h}(\mathcal{D}_A)^{\wedge}_{x_0}/\Lambda_A$ is a projective A-module[9]). The A-Hodge-Pink structure Λ_A is effective if Λ_A contains the standard lattice $(\mathcal{D}_A)^{\wedge}_{x_0}$. We define *Hodge-Pink weights* and *multiplicities* for A-Hodge-Pink structure Λ_A as Hodge-Pink weights and multiplicities for Λ_A as Hodge-Pink structure (via forgetting A-action).

We say that an A-isocrystal with A-Hodge-Pink structure is *weakly admissible* if it is weakly admissible as a F_0-isocrystal with F_0-Hodge-Pink structure (i.e., if it is weakly admissible after forgetting A-action). In other words, the weak admissibility is checked for all the subobjects or quotients which do not necessarily respect the A-module structure.

For any map $A \to B$ of finite F_0-algebras, we have a natural definition of "change of coefficients" for A-isocrystals with A-Hodge-Pink structure, namely $(D_A, \Lambda_A) \mapsto (D_A \otimes_A B, \Lambda_A \otimes_A B)$. Note that the natural map $\Lambda_A \otimes_A B \to (\mathcal{D}_B)^{\wedge}_{x_0}[\frac{1}{\mathcal{P}(u)}]$ is injective since the natural inclusion $\Lambda_A \hookrightarrow (\mathcal{D}_A)^{\wedge}_{x_0}[\frac{1}{\mathcal{P}(u)}]$ splits as an A-module by definition. The functor $\underline{\mathbb{H}}$ commutes with the change of coefficients.

We generalize Theorem 4.3.4 to allow A-coefficients as follows, where A is a finite F_0-algebra.

Lemma 11.3.2. *The functor $\underline{\mathbb{H}}$ induces an equivalence of categories from $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ onto the category of A-isocrystals with weakly admissible A-Hodge-Pink structure whose Hodge-Pink weights are in $[0,h]$.*

Proof. The full faithfulness follows from Theorem 4.3.4, and by Corollary 11.2.8 the essential surjectivity of $\underline{\mathbb{H}}$ follows if we show that $\underline{V}^*_{\mathcal{HP}}(D_A, \Lambda_A)$ is free over A, where $\underline{V}^*_{\mathcal{HP}}$ is defined in Corollary 5.2.4. We put $V_A := \underline{V}^*_{\mathcal{HP}}(D_A, \Lambda_A)$, and may

[9] For $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$, $(D_A, \Lambda_A) = \underline{\mathbb{H}}(\mathfrak{M}_A)$ satisfies this condition thanks to Proposition 8.2.3.

assume that A is local. First, we have $V_A \otimes_A A/\mathfrak{m}_A \cong \underline{V}^*_{\mathcal{HP}}(D_{A/\mathfrak{m}_A}, \Lambda_{A/\mathfrak{m}_A})$, so its $A/\mathfrak{m}_A$-dimension equals $\operatorname{rank}_{(\mathscr{K}_0)_A} D_A$. On the other hand, observe that $\dim_{F_0} V_A = \big(\operatorname{rank}_{(\mathscr{K}_0)_A} D_A\big)\cdot(\dim_{F_0} A)$, which forces V_A to be free with $\operatorname{rank}_A V_A = \operatorname{rank}_{(\mathscr{K}_0)_A} D_A$. (Indeed, by Nakayama's lemma there exists a A-linear surjection $A^{\oplus n} \twoheadrightarrow V_A$ with $n := \operatorname{rank}_{(\mathscr{K}_0)_A}(D_A)$ and both sides have the same F_0-dimension.) $\square$

Remark 11.3.3. In fact, Theorem 11.2.9 can be proved more easily using Lemma 11.3.2, namely from the fact that affine grassmannian is formally smooth and that weak admissibility lifts under the infinitesimal thickening of coefficient rings. (The last assertion follows from applying Proposition 2.3.8 to the short exact sequence (11.3.10.1) below.)

By Corollary 11.2.10, the noetherian rings $R^{\leqslant h}_{\rho_{\mathbb{F}}}[\frac{1}{\pi_0}]$ and $R^{\square,\leqslant h}_{\rho_{\mathbb{F}}}[\frac{1}{\pi_0}]$ are formally smooth over F (and in particular, geometrically regular). In order to compute their dimensions we introduce an invariant which picks out an equi-dimensional union of connected components, generalizing the Hodge-Pink type defined in §2.2.9. The dimension will be expressed in terms of the corresponding "Hodge-Pink type."

11.3.4 Hodge-Pink type with coefficients

We seek to define a "Hodge-Pink type" for $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$, with $A \in \mathfrak{AR}_E$ and E/F_0 finite (or rather, for the A-isocrystal $\underline{\mathbb{H}}(\mathfrak{M}_A)$ with A-Hodge-Pink type). Consider a finite free $(\mathscr{K}_0)_A$-module D_A (where $(\mathscr{K}_0)_A := \mathscr{K}_0 \otimes_{F_0} A$) and an A-Hodge-Pink structure Λ_A for D_A (Definition 11.3.1). Let $\widehat{\mathcal{D}}_{A,x_0} := \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0,A} \otimes_{(\mathscr{K}_0)_A} D_A$ (where $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0,A} := \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \otimes_{F_0} A$). Motivated by the discussion about Hodge-Pink types in §2.2.9 and §2.3.3, we make the following definition.

Definition 11.3.4.1. For a finite extension E/F_0, an *E-Hodge-Pink type* $\mathbf{v}$ is a pair $(n, \bar{\Lambda}^{\mathbf{v}}_E)$ where n is a positive integer and $\bar{\Lambda}^{\mathbf{v}}_E$ is a $\mathfrak{S}_E$-quotient of $(\mathfrak{S}_E/(\mathcal{P}(u)^h))^{\oplus n}$.

For a finite E-algebra A, an A-Hodge-Pink structure Λ_A for a finite free $(\mathscr{K}_0)_A$-module D_A is of *E-Hodge-Pink type* $\mathbf{v}$ (or simply, *Hodge-Pink type* $\mathbf{v}$) if D_A is of $(\mathscr{K}_0)_A$-rank n and there is an $\mathfrak{S}_A/(\mathcal{P}(u)^h)$-isomorphism $\bar{\Lambda}_E^{\mathbf{v}} \otimes_E A \cong \Lambda_A/(\mathcal{D})\widehat{_{x_0}}$. (Note that $\mathfrak{S}_A/(\mathcal{P}(u)^h) \cong \mathcal{O}_{\widehat{\boldsymbol{\Delta}},x_0,A}/(\mathcal{P}(u)^h)$.) For $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ with A a finite E-algebra, we say that $\mathfrak{M}_A$ is *of E-Hodge-Pink type* $\mathbf{v} = (n, \bar{\Lambda}_E^{\mathbf{v}})$ (or simply *of Hodge-Pink type* $\mathbf{v}$ if there is no risk of confusion), if $\underline{\mathbb{H}}(\mathfrak{M}_A)$ is of E-Hodge-Pink type $\mathbf{v}$; or equivalently by §3.2.6, if $\mathfrak{M}_A$ is of $\mathfrak{S}_A$-rank n and there is an $\mathfrak{S}_A/\mathcal{P}(u)^h$-isomorphism $\bar{\Lambda}_E^{\mathbf{v}} \otimes_E A \cong \operatorname{coker} \varphi_{\mathfrak{M}_A}$.

We define the Hodge-Pink type for objects in $\mathscr{D}_{\eta_E}^{\leqslant h}$, as follows: the Hodge-Pink type for $\xi_A \in \mathscr{D}_{\eta_E}^{\leqslant h}(A)$ is the Hodge-Pink type for the unique $\mathfrak{S}_A$-lattice $\mathfrak{M}_{\xi_A}$ of $\mathcal{P}$-height $\leqslant h$ for $M_\xi \otimes_R A$, where $M_\xi := \underline{D}_{\mathcal{E}}^{\leqslant h}(\xi)$. (The existence of $\mathfrak{M}_{\xi_A}$ is proved in Corollary 11.2.8.)

Hodge-Pink type with coefficients behaves well under change of coefficients in the following sense. Let $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ for a finite E-algebra A, and assume that $\mathfrak{M}_A$ is of E-Hodge-Pink type $\mathbf{v} = (n, \bar{\Lambda}_E^{\mathbf{v}})$. Then for any finite A-algebra A', $\mathfrak{M}_A \otimes_A A'$ is of E-Hodge-Pink type $\mathbf{v}$. Also for any finite extension E'/E and an E'-algebra A, $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ is of E-Hodge-Pink type $\mathbf{v} := (n, \bar{\Lambda}_E^{\mathbf{v}})$ if and only if $\mathfrak{M}_A$ is of E'-Hodge-Pink type $\mathbf{v}' := (n, \bar{\Lambda}_E^{\mathbf{v}} \otimes_E E')$. Using these, we will often replace E with a suitable finite extension of E in applications.

It is not *a priori* clear if any A-Hodge-Pink structure with $A \in \mathfrak{AR}_E$ has an E-Hodge-Pink type. But when $\mathscr{K}/\mathscr{K}_0$ is separable, the following equivalent definition of E-Hodge-Pink type can be used to show that any A-Hodge-Pink structure has an E-Hodge-Pink type.

Consider a finite free $(\mathscr{K}_0)_A$-module D_A and an A-Hodge-Pink structure Λ_A for D_A. If all the Hodge-Pink weights for Λ_A are in $[0,h]$ (i.e., $\widehat{\mathcal{D}}_{A,x_0} \subset \Lambda_A \subset \mathcal{P}(u)^{-h}\cdot$

$\widehat{\mathcal{D}}_{A,x_0}$) then by definition $\Lambda_A/\widehat{\mathcal{D}}_{A,x_0}$ and $(\mathcal{P}(u)^{-h}\cdot\widehat{\mathcal{D}}_{A,x_0})/\Lambda_A$ are finite projective A-modules. As in §2.3.3, we can associate decreasing separated exhaustive filtrations $\mathrm{Fil}^\bullet(\widehat{\mathcal{D}}_{A,x_0})$ of $\widehat{\mathcal{D}}_{A,x_0}$ by $(\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0})_A$-submodules, and $\mathrm{Fil}^\bullet(D_{A,\mathscr{K}})$ of $D_{A,\mathscr{K}}$ by $\mathscr{K}_A$-submodules, respectively, as follows:

$$\mathrm{Fil}^w(\widehat{\mathcal{D}}_{A,x_0}) := (\widehat{\mathcal{D}}_{A,x_0})\cap(\mathcal{P}(u)^w\cdot\Lambda_A)\subset\widehat{\mathcal{D}}_{A,x_0} \tag{11.3.4.2}$$

$$\mathrm{Fil}^w(D_{A,\mathscr{K}}) := \frac{\mathrm{Fil}^w(\widehat{\mathcal{D}}_{A,x_0})}{(\mathcal{P}(u)\cdot\widehat{\mathcal{D}}_{A,x_0})\cap\mathrm{Fil}^w(\widehat{\mathcal{D}}_{A,x_0})}\subset\frac{\widehat{\mathcal{D}}_{A,x_0}}{\mathcal{P}(u)\cdot\widehat{\mathcal{D}}_{A,x_0}}\cong D_{A,\mathscr{K}},\quad \text{for } w\in\mathbb{Z}, \tag{11.3.4.3}$$

where all the intersections are taken inside $\widehat{\mathcal{D}}_{A,x_0}[\frac{1}{\mathcal{P}(u)}]$. Note that if we forget the A-action and view $\mathrm{Fil}^\bullet(D_{A,\mathscr{K}})$ as a filtration by $\mathscr{K}$-subspaces, then $\mathrm{Fil}^\bullet(D_{A,\mathscr{K}})$ coincides with the filtration (2.4.3.1) or its analogue for the case $\mathfrak{o}_0=\mathbb{F}_q[[\pi_0]]$.

One can also construct $\mathrm{Fil}^w(D_{A,\mathscr{K}})$ from $\bar\Lambda_A:=\Lambda_A/\widehat{\mathcal{D}}_{A,x_0}$, as follows: since the submodule $\bar\Lambda_A[\mathcal{P}(u)^w]\subset\bar\Lambda_A$ of elements killed by $\mathcal{P}(u)^w$ is the image of $\mathcal{P}(u)^{-w}\cdot\mathrm{Fil}^w(\widehat{\mathcal{D}}_{A,x_0})$ under the natural projection, we have an $\mathscr{K}_A$-isomorphism

$$\mathrm{Fil}^w(D_{A,\mathscr{K}})\xleftarrow{\sim}\frac{\bar\Lambda_A[\mathcal{P}(u)^w]}{\bar\Lambda_A[\mathcal{P}(u)^{w-1}]} \tag{11.3.4.4}$$

for each w, where the isomorphism is induced from multiplication by $\mathcal{P}(u)^w$. Now, for any E-Hodge-Pink type $(n,\bar\Lambda^{\mathbf{v}}_E)$ we define $\mathrm{Fil}^w_{\mathbf{v}}:=\bar\Lambda^{\mathbf{v}}_E[\mathcal{P}(u)^w]/\bar\Lambda^{\mathbf{v}}_E[\mathcal{P}(u)^{w-1}]\subset D_{E,\mathscr{K}}$. It is clear from the isomorphism (11.3.4.4) that if an A-Hodge-Pink structure Λ_A is of E-Hodge-Pink type $\mathbf{v}$ then there exists a $\mathscr{K}_A$-isomorphism $\mathrm{Fil}^w(D_{A,\mathscr{K}})\cong\mathrm{Fil}^w_{\mathbf{v}}\otimes_E A$. We will show later in Lemma 11.3.5 that the converse is also true if $\mathscr{K}/\mathscr{K}_0$ is separable.

For $\mathfrak{M}_A\in(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$, the filtration $\mathrm{Fil}^\bullet(D_{A,\mathscr{K}})$ for $(D_A,\Lambda_A):=\underline{\mathbb{H}}(\mathfrak{M}_A)$ can be expressed in terms of $\mathfrak{M}_A$, as follows:

(11.3.4.5)

$$\mathrm{Fil}^w(D_{A,\mathscr{K}})\cong\frac{\mathrm{im}(\varphi_{\mathfrak{M}_A})\cap\mathcal{P}(u)^w\cdot\mathfrak{M}_A}{(\mathcal{P}(u)\cdot\mathrm{im}(\varphi_{\mathfrak{M}_A}))\cap\mathcal{P}(u)^w\cdot\mathfrak{M}_A}\subset\frac{\mathrm{im}(\varphi_{\mathfrak{M}_A})}{\mathcal{P}(u)\cdot\mathrm{im}(\varphi_{\mathfrak{M}_A})}\cong D_{A,\mathscr{K}},\quad\text{for } w\in\mathbb{Z}.$$

Lemma 11.3.5. *Assume that $\mathscr{K}/\mathscr{K}_0$ is separable.*

1. *For a finite E-algebra A, an A-Hodge-Pink structure Λ_A for $D_A := D_E \otimes_E A$ is of E-Hodge-Pink type $\mathbf{v}$ if and only if there is an $\mathscr{K}_A$-isomorphism* $\mathrm{Fil}^w(D_{A,\mathscr{K}}) \cong (\mathrm{Fil}^w_{\mathbf{v}}) \otimes_E A$ *for all w, where* $\mathrm{Fil}^w(D_{A,\mathscr{K}})$ *is as defined in* (11.3.4.3)*.*

2. *For a finite F_0-algebra A, the grading* $\mathrm{gr}^w(D_{A,\mathscr{K}}) := \frac{\mathrm{Fil}^w(D_{A,\mathscr{K}})}{\mathrm{Fil}^{w+1}(D_{A,\mathscr{K}})}$ *associated to an A-Hodge-Pink type Λ_A is a finite projective $\mathscr{K}_A$-module for any $w \in \mathbb{Z}$. In particular,* $\mathrm{Fil}^w(D_{A,\mathscr{K}})$ *is a finite projective $\mathscr{K}_A$-module for any $w \in \mathbb{Z}$.*

3. *If A is finite radicial E-algebra (e.g. $A \in \mathfrak{AR}_E$), then any A-Hodge-Pink type Λ_A has an E-Hodge-Pink type.*

When $\mathscr{K}/\mathscr{K}_0$ is separable, we often let $\mathbf{v}$ denote the corresponding filtration $\mathrm{Fil}^\bullet_{\mathbf{v}}$ which is equivalent information by Lemma 11.3.5(1).

Lemma 11.3.5(2) is false (even when A is a field) if $\mathscr{K}$ is *not* separable over $\mathscr{K}_0$. See Remark 11.3.6 for an example. But the A-Hodge-Pink structure in this counterexample *cannot* appear as a weakly admissible A-Hodge-Pink structure. The author does not know whether Lemma 11.3.5 holds for any weakly admissible E-Hodge-Pink structure without assuming that $\mathscr{K}/\mathscr{K}_0$ is separable.

Proof. The "only if" direction of (1) is already discussed; see the discussion below (11.3.4.4). To show the "if" direction of (1), we may assume that A is a finite local E-algebra. Let $E' \subset A$ be a subfield containing E which makes A a radicial E'-algebra. (For example, we may take E' to be the maximal separable subextension of $A/\mathfrak{m}_A$ over E.) To prove the lemma, we may replace E by E' and $\mathbf{v} := (n, \bar{\Lambda}^{\mathbf{v}}_E)$ by $\mathbf{v}' := (n, \bar{\Lambda}^{\mathbf{v}}_E)$, so we are reduced to the case when A is finite radicial E-algebra. (The point of this step is that for any finite extension E'' of E, $A \otimes_E E''$ is *local*.)

Since $\mathscr{K}/\mathscr{K}_0$ and $\mathscr{K}_0/F_0$ are separable it follows that $\mathscr{K}/F_0$ is separable, so we have an isomorphism $\mathscr{K}\otimes_{F_0} E \cong \bigoplus_i E_i$ for some finite separable extensions E_i/E equipped with a fixed F_0-embedding $\mathscr{K}\hookrightarrow E_i$. Also we have a unique $\mathscr{K}_0$-isomorphism $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \cong \mathscr{K}[[\mathcal{P}(u)]]$ (using separability of $\mathscr{K}/\mathscr{K}_0$), so we have an isomorphism $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0,E} \cong \bigoplus_i E_i[[\mathcal{P}(u)]]$.

Claim 11.3.5.1. *For a finite radicial E-algebra A, any $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0,A}/(\mathcal{P}(u)^h)$-quotient $\bar{\Lambda}_A$ of $\big(\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0,A}/(\mathcal{P}(u)^h)\big)^{\oplus n}$ which is* projective *as an A-module can be written as follows:*

$$\bar{\Lambda}_A \cong \bigoplus_i \bigoplus_{w=0,\cdots,h} \left(\frac{(E_i\otimes_E A)[[\mathcal{P}(u)]]}{(\mathcal{P}(u)^w)}\right)^{m_{w,i}}. \tag{11.3.5.2}$$

We choose $m_{0,i}\geq 0$ for each i so that we have $\sum_{w=0}^h m_{w,i} = n$.

To show Claim 11.3.5.1, it is enough to show that $\bar{\Lambda}_A$ is projective over $\mathscr{K}_A$ (which is E-isomorphic to $\bigoplus_i E_i\otimes_E A$). Since $\mathscr{K}_0/F_0$ is separable $\mathscr{K}_A$ is an étale A-algebra. So $\mathscr{K}_A\otimes_A (A/\mathfrak{m}_A)$ is a product of finite separable extensions of $A/\mathfrak{m}_A$, hence any $\mathscr{K}_A\otimes_A(A/\mathfrak{m}_A)$-module is projective. Now by local flatness criterion (especially, [62, Theorem 22.3(4)]), a finitely generated $\mathscr{K}_A$-module is $\mathscr{K}_A$-flat if and only if it is A-flat. But by assumption $\bar{\Lambda}_A$ is A-flat, so we proved Claim 11.3.5.1.

Now let us deduce (2), (3), and the "if" direction from Claim 11.3.5.1. First, observe that $\bar{\Lambda}_A$ in Claim 11.3.5.1 is isomorphic to $\bar{\Lambda}_E\otimes_E A$ where

$$\bar{\Lambda}_E \cong \bigoplus_i \bigoplus_{w=0,\cdots,h} \left(\frac{E_i[[\mathcal{P}(u)]]}{(\mathcal{P}(u)^w)}\right)^{m_{w,i}}.$$

This shows (3). To show (2), we may assume that A is local and radicial over some finite extension E/F_0 (e.g. by taking E to be the maximal separable subextension of $A/\mathfrak{m}_A$ over F_0). Then any A-Hodge-Pink type Λ_A, $\bar{\Lambda}_A := \Lambda_A/\widehat{\mathcal{D}}_{A,x_0}$ satisfies the assumption of Claim 11.3.5.1 by definition. Using the isomorphism (11.3.4.4), we obtain for any $w \in \mathbb{Z}$

$$\text{(11.3.5.3)} \qquad \mathrm{gr}^w(D_{A,\mathscr{K}}) \cong \bigoplus_i (E_i \otimes_E A)^{\oplus m_{w,i}},$$

which is visibly projective over $\mathscr{K}_A$. The "if" direction of (1) also follows because for any A-Hodge-Pink type Λ_A (with A a finite radicial E-algebra), $\Lambda_A/\widehat{\mathcal{D}}_{A,x_0}$ is uniquely determined by non-negative integers $\{m_{w,i}\}_{w,i}$ up to isomorphism, but $\{m_{w,i}\}_{w,i}$ is determined by $\mathrm{gr}^\bullet(D_{A,\mathscr{K}})$ as in (11.3.5.3). □

Remark 11.3.6. Consider $K := \mathbb{F}_q(\!(u)\!)$ with $u_0 = u^p$ (so $\mathcal{P}(u) = \pi_0 - u_0 = \pi_0 - u^p$). In particular, the image $\pi_{\mathscr{K}}$ of u in $\mathfrak{o}_{\mathscr{K}} = \mathbb{F}_q[[\pi_0, u]]/(\pi_0 - u^p)$ is a uniformizer satisfying $\pi_{\mathscr{K}}^p = \pi_0$. We take $E := F_0[\pi_E]/(\pi_0 - \pi_E^p)$ so we have $\mathscr{K}_E \cong \mathscr{K}[\pi_E]/(\pi_0 - \pi_E^p) \cong \mathscr{K}[\pi_E]/(\pi_{\mathscr{K}} - \pi_E)^p$.

Consider $D_E := (\mathscr{K}_0)_E \cdot \mathbf{e} \cong E \cdot \mathbf{e}$ and set $\widehat{\mathcal{D}}_{E,x_0} := (\widehat{\mathcal{O}_{\mathbf{\Delta},x_0}})_E \cdot \mathbf{e}$. (Note that $\mathscr{K}_0 = F_0$, so $(\mathscr{K}_0)_E \cong E$.) Consider the following E-Hodge-Pink type:

$$\Lambda_E := \sum_{w=0}^{p-1} (u - \pi_E)^w (\pi_0 - u^p)^{-w} \cdot \widehat{\mathcal{D}}_{E,x_0}.$$

Clearly Λ_E is of height $\leqslant p-1$, and one can that the associated filtration is $\mathrm{Fil}^w(\mathcal{D}_{E,\mathscr{K}}) = (\pi_{\mathscr{K}} - \pi_E)^w \cdot D_{E,\mathscr{K}}$ for $w \in [0, p-1]$ which is *not* free over $\mathscr{K}_E$.

It is impossible to give D_E an E-isocrystal structure which makes Λ_E weakly admissible; any E-isocrystal structure is pure of some slope w since D_E is of $(\mathscr{K}_0)_E$-rank 1, but this forces any weakly admissible Hodge-Pink structure to be of the from $\mathcal{P}(u)^{-w}\widehat{\mathcal{D}}_{E,x_0}$.

Proposition 11.3.7. *Assume that $\mathscr{K}$ is separable over $\mathscr{K}_0$. Let $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$ for some $R \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$. Then, for any E-Hodge-Pink type $\mathbf{v}$ with E a finite extension over $F = \mathrm{Frac}(\mathfrak{o})$,*[10] *there exists a (possibly empty) union of connected components*

[10]One can allow E/F_0 which does not necessarily contain F, as follows. Pick a finite extension E' of E which contains F, and replace $\mathbf{v}$ by $\mathbf{v}' := \{\mathrm{Fil}^\bullet_{\mathbf{v}} \otimes_E E'\}$.

$\mathscr{G}\mathscr{R}_\xi^{\mathbf{v}} \subset \mathscr{G}\mathscr{R}_\xi^{\leqslant h} \otimes_{\mathfrak{o}} E \cong \operatorname{Spec} R_E$ *(where $R_E := R \otimes_{\mathfrak{o}} E$), with the property that for any finite E-algebra A, an A-point $\zeta_A \in \mathscr{G}\mathscr{R}_\xi^{\leqslant h}(A)$ is of E-Hodge-Pink type $\mathbf{v}$ if and only if ζ_A is supported in $\mathscr{G}\mathscr{R}_\xi^{\mathbf{v}}$.*

Proof. Recall that we have a $\mathfrak{S}_{\mathscr{G}\mathscr{R}_\xi^{\leqslant h}}$-lattice $\underline{\mathfrak{M}}_\xi^{\leqslant h}$ of height $\leqslant h$ in $M_\xi \otimes_R \mathcal{O}_{\mathscr{G}\mathscr{R}_\xi^{\leqslant h}}$ which is "universal" in the sense of Corollary 11.1.11, where $\mathfrak{S}_{\mathscr{G}\mathscr{R}_\xi^{\leqslant h}} := \mathfrak{S} \otimes_{\mathfrak{o}_0} \mathcal{O}_{\mathscr{G}\mathscr{R}_\xi^{\leqslant h}}$. We view $\underline{\mathfrak{M}}_{\xi,E}^{\leqslant h} := \underline{\mathfrak{M}}_\xi^{\leqslant h} \otimes_{\mathfrak{o}} E$ as a $\mathfrak{S} \otimes_{\mathfrak{o}_0} R_E$-lattice of $\mathcal{P}$-height $\leqslant h$ in $M_\xi \otimes_{\mathfrak{o}} E$ via the structure morphism $\mathscr{G}\mathscr{R}_\xi^{\leqslant h} \otimes_{\mathfrak{o}} E \xrightarrow{\sim} \operatorname{Spec} R_E$. Now, set

(11.3.7.1)
$$\mathrm{Fil}^w_{\mathfrak{S},\xi,E} := \frac{\varphi(\sigma^*\underline{\mathfrak{M}}_{\xi,E}^{\leqslant h}) \cap \mathcal{P}(u)^w \cdot \mathfrak{M}_{\xi,E}^{\leqslant h}}{\mathcal{P}(u)\cdot\varphi(\sigma^*\underline{\mathfrak{M}}_{\xi,E}^{\leqslant h}) \cap \mathcal{P}(u)^w \cdot \mathfrak{M}_{\xi,E}^{\leqslant h}} \subset \frac{\underline{\mathfrak{M}}_{\xi,E}^{\leqslant h}}{\mathcal{P}(u)\cdot\underline{\mathfrak{M}}_{\xi,E}^{\leqslant h}}, \quad \text{for } w = 0, 1, \cdots, h.$$

(Compare the left side with (11.3.4.5).)

Let $\mathscr{G}\mathscr{R}_\xi^{\mathbf{v}} \subset \operatorname{Spec} R_E$ be a set of primes $\mathfrak{p} \subset R_E$ such that there exists an $\mathscr{K} \otimes_{F_0} (R_E)_{\mathfrak{p}}$-isomorphism

$$\text{(11.3.7.2)} \qquad \mathrm{Fil}^w_{\mathfrak{S},\xi,E} \otimes_{R_E} (R_E)_{\mathfrak{p}} \cong \mathrm{Fil}^w_{\mathbf{v}} \otimes_E (R_E)_{\mathfrak{p}}.$$

Clearly, $\mathscr{G}\mathscr{R}_\xi^{\mathbf{v}}$ is an open subspace of $\operatorname{Spec} R_E$. We will now show that $\mathscr{G}\mathscr{R}_\xi^{\mathbf{v}}$ is a union of connected components of $\operatorname{Spec} R_E$ and has the desired property for E-finite points.

We first show that $\mathrm{Fil}^w_{\mathfrak{S},\xi,E}$ is finite projective over $\mathscr{K} \otimes_{F_0} R_E$.[11] It suffices to show that $\mathrm{Fil}^w_{\mathfrak{S},\xi,E} \otimes_{R_E} (R_E)_{\mathfrak{p}}$ is finite projective over $\mathscr{K} \otimes_{F_0} (R_E)_{\mathfrak{p}}$ for all maximal ideals $\mathfrak{p} \subset R_E$. Let $E' := R_E/\mathfrak{p}$, and let $\mathbf{v}'$ be the E'-Hodge-Pink type for the $\mathfrak{S}_{E'}$-lattice $\underline{\mathfrak{M}}_{\xi,E}^{\leqslant h} \otimes_{R_E} R_E/\mathfrak{p}$ of $\mathcal{P}$-height $\leqslant h$ (which corresponds to the closed point $\mathfrak{p} \in \operatorname{Spec} R_E \xleftarrow{\sim} \mathscr{G}\mathscr{R}_\xi^{\leqslant h} \otimes_{\mathfrak{o}} E$). Choose a maximal ideal $\mathfrak{p}' \subset R_{E'} = R \otimes_{\mathfrak{o}} E'$ over $\mathfrak{p}$. By applying Lemma 11.3.5 to the $\mathfrak{S}_{R_{E'}/\mathfrak{p}'^n}$-lattice $\underline{\mathfrak{M}}_{\xi,E}^{\leqslant h} \otimes_{R_E} R_{E'}/\mathfrak{p}'^n$ of $\mathcal{P}$-height

[11] For a different proof, one can adopt the proof of Lemma 11.3.5 as in [55, Lemma 2.6.1].

$\leqslant h$, we obtain an $\mathscr{K} \otimes_{F_0} (R_{E'})^{\widehat{}}_{\mathfrak{p}'}$-isomorphism

$$\mathrm{Fil}^w_{\mathfrak{S},\xi,E} \otimes_{R_E} (R_{E'})^{\widehat{}}_{\mathfrak{p}'} \cong \mathrm{Fil}^w_{\mathbf{v}'} \otimes_{E'} (R_{E'})^{\widehat{}}_{\mathfrak{p}'}$$

for all w. Since $\mathrm{Fil}^w_{\mathfrak{S},\xi,E} \otimes_{R_E} (R_E)_{\mathfrak{p}}$ is finitely generated over $\mathscr{K} \otimes_{F_0} (R_E)_{\mathfrak{p}}$, it is finite projective by faithful flatness of $\mathscr{K} \otimes_{F_0} (R_{E'})^{\widehat{}}_{\mathfrak{p}'}$ over $\mathscr{K} \otimes_{F_0} (R_E)_{\mathfrak{p}}$.

Since both $\mathrm{Fil}^w_{\mathfrak{S},\xi,E}$ and $\mathrm{Fil}^w_{\mathbf{v}} \otimes_E R_E$ are projective $\mathscr{K} \otimes_{F_0} R_E$-modules, there exists a (possibly empty) union of connected components $U \subset \mathrm{Spec}(\mathscr{K} \otimes_{F_0} R_E)$ over which the ranks of both modules coincide (since the rank of a projective A-module is a locally constant function on $\mathrm{Spec}\, A$). Clearly, $\mathfrak{p}$ lies in $\mathscr{GR}^{\mathbf{v}}_\xi$ if and only if the fiber $\mathrm{Spec}(\mathscr{K} \otimes_{F_0} R_E/\mathfrak{p})$ over $\mathfrak{p}$ is contained in U. Thus, $\mathscr{GR}^{\mathbf{v}}_\xi$ is precisely the union of all the connected components whose preimages in $\mathrm{Spec}(\mathscr{K} \otimes_{F_0} R_E)$ lie in U.

It is clear from the definition of $\mathscr{GR}^{\mathbf{v}}_\xi$ that for any A-point $\zeta_A \in \mathscr{GR}^{\mathbf{v}}_\xi(A)$ with A finite over E, the $\mathfrak{S}_A$-lattice $\zeta_A^*(\underline{\mathfrak{M}}^{\leqslant h}_{\xi,E})$ of $\mathcal{P}$-height $\leqslant h$ in $M_\xi \otimes_{R,\zeta_A} A$ is of E-Hodge-Pink type $\mathbf{v}$. Now, let us show that for any $\zeta_A \in (\mathscr{GR}^{\leqslant h}_\xi \otimes_{\mathfrak{o}} E)(A)$ with A finite over E, if $\zeta_A^*(\underline{\mathfrak{M}}^{\leqslant h}_{\xi,E})$ is of E-Hodge-Pink type $\mathbf{v}$ then ζ_A factors through $\mathscr{GR}^{\mathbf{v}}_\xi$. We may assume that A is local, and let $\mathfrak{p}$ be the closed point of $\mathscr{GR}^{\leqslant h}_\xi \otimes_{\mathfrak{o}} E$ on which ζ_A is supported. By the assumption on ζ_A, the $\mathfrak{S}_{R_E/\mathfrak{p}}$-lattice $\underline{\mathfrak{M}}^{\leqslant h}_{\xi,E} \otimes_{R_E} R_E/\mathfrak{p}$ of $\mathcal{P}$-height $\leqslant h$ is of E-Hodge-Pink type $\mathbf{v}$, so $\mathrm{Fil}^w_{\mathfrak{S},\xi,E} \otimes_{R_E} (R_E)_{\mathfrak{p}}$ and $\mathrm{Fil}^w_{\mathbf{v}} \otimes_E (R_E)_{\mathfrak{p}}$ are projective $\mathscr{K} \otimes_{F_0} (R_E)_{\mathfrak{p}}$-modules with same (locally constant) $\mathscr{K} \otimes_{F_0} (R_E)_{\mathfrak{p}}$-ranks; i.e., they are isomorphic as $\mathscr{K} \otimes_{F_0} (R_E)_{\mathfrak{p}}$-modules. Thus, $\mathfrak{p} \in \mathscr{GR}^{\mathbf{v}}_\xi$. □

11.3.8

By Proposition 11.2.6, the structure morphism $\mathscr{GR}^{\leqslant h}_\xi \otimes_{\mathfrak{o}} F \to \mathrm{Spec}(R \otimes_{\mathfrak{o}} F)$ is an isomorphism, where ξ is over R. Let $R^{\mathbf{v}}$ be the direct factor of $R \otimes_{\mathfrak{o}} F$ such that the isomorphism induces $\mathscr{GR}^{\mathbf{v}}_\xi \xrightarrow{\sim} \mathrm{Spec}\, R^{\mathbf{v}}$. If $\mathrm{End}_{\mathcal{G}_K}(\rho_{\mathbb{F}}) = \mathbb{F}$ (so $\xi = \xi_{\mathrm{univ}}$ exists over $R := R^{\leqslant h}_{\rho_{\mathbb{F}}}$), then we write $R^{\mathbf{v}}_{\rho_{\mathbb{F}}}$ to denote $R^{\mathbf{v}}$. We similarly define $R^{\square,\mathbf{v}}_{\rho_{\mathbb{F}}}$

using $\xi = \xi^{\square}_{\mathrm{univ}}$. The rest of this section is devoted to computing the dimensions of F-algebras $R^{\mathbf{v}}_{\rho_{\mathbb{F}}}$ and $R^{\square,\mathbf{v}}_{\rho_{\mathbb{F}}}$ for fixed Hodge-Pink type $\mathbf{v}$. Since we already know that are geometrically regular F-algebras, it is enough to compute the dimension of the tangent space at each closed point, which can be done after increasing F so that the closed point becomes an F-rational point and passing to the completed local ring.

Fix an E-Hodge-Pink type $\mathbf{v}$ and a deformation $\eta \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(\mathfrak{o}_E)$ such that η_E is of E-Hodge-Pink type $\mathbf{v}$. We fix a framing $\beta_{\mathfrak{o}_E}$ for η to obtain a framed deformation $\eta^{\square} = (\eta, \beta_{\mathfrak{o}_E}) \in \mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}}}(\mathfrak{o}_E)$. As mentioned in Proposition 11.2.2, the tangent space $\left|\mathscr{D}^{\square,\leqslant h}_{\eta_E}\right|(E[\epsilon])$ is exactly the Zariski tangent space of $R^{\square,\mathbf{v}}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} E$ at the E-point $\eta^{\square}_E$, and similarly if $\mathrm{End}_{\boldsymbol{\mathcal{G}}_K}(\rho_{\mathbb{F}}) = \mathbb{F}$ then the tangent space $\left|\mathscr{D}^{\leqslant h}_{\eta_E}\right|(E[\epsilon])$ is exactly the Zariski tangent space of $R^{\mathbf{v}}_{\rho_{\mathbb{F}}} \otimes_{\mathfrak{o}} E$ at the E-point η_E. Note also that even if $\left|\mathscr{D}^{\leqslant h}_{\eta_E}\right|$ is not representable, the Zariski tangent space $\left|\mathscr{D}^{\leqslant h}_{\eta_E}\right|(E[\epsilon])$ makes sense as a finite-dimensional E-vector space since $\left|\mathscr{D}^{\leqslant h}_{\eta_E}\right|$ satisfies Schlessinger's criteria (H1)–(H3) by §11.7.1 and Theorem 11.7.2.

Let $\mathrm{Ad}(\eta_E)$ be the (natural) $\boldsymbol{\mathcal{G}}_K$-representation on $\mathrm{End}_E(V_\eta)$. In particular, we have $(\mathrm{Ad}(\eta_E))^{\boldsymbol{\mathcal{G}}_K} = \mathrm{End}_{\boldsymbol{\mathcal{G}}_K}(\eta_E)$. Then we can see that $\left|\mathscr{D}^{\square,\leqslant h}_{\eta_E}\right|(E[\epsilon])$ is a torsor over $\left|\mathscr{D}^{\leqslant h}_{\eta_E}\right|(E[\epsilon])$ under the natural tranitive action of $\mathrm{Ad}(\eta_E)/\left(\mathrm{Ad}(\eta_E)\right)^{\boldsymbol{\mathcal{G}}_K}$, which can be seen as follows: for a fixed deformation $\eta_{E[\epsilon]} \in \mathscr{D}^{\leqslant h}_{\eta_E}(E[\epsilon])$, any two lift of the framing (i.e., the ordered basis) for η_E are related by the action of $\mathrm{id} + \epsilon \cdot \mathrm{Ad}(\eta_E)$, and two lifts of the framing define isomorphic objects in $\mathscr{D}^{\square,\leqslant h}_{\eta_E}(E[\epsilon])$ if and only if they are related by the action of $\mathrm{id} + \epsilon \cdot (\mathrm{Ad}(\eta_E))^{\boldsymbol{\mathcal{G}}_K}$. So we obtain

(11.3.8.1)

$$\dim_E \left|\mathscr{D}^{\square,\leqslant h}_{\eta_E}\right|(E[\epsilon]) = \dim_E \left|\mathscr{D}^{\leqslant h}_{\eta_E}\right|(E[\epsilon]) + \dim_E \mathrm{Ad}(\eta_E) - \dim_E \left(\mathrm{Ad}(\eta_E)\right)^{\boldsymbol{\mathcal{G}}_K}.$$

Therefore, it is enough to compute the dimension of the tangent space $\left|\mathscr{D}^{\leqslant h}_{\eta_E}\right|(E[\epsilon])$. Thanks to Corollary 11.2.8 and Lemma 11.3.2, this can be done by studying (first-order) deformations of (weakly admissible) Hodge-Pink structures with coefficients.

11.3.9

The following discussion is an analogue of Kisin's technique [55, §3] for studying deformations of weakly admissible filtered isocrystals with coefficients. Let (D_E, Λ_E) be a weakly admissible E-Hodge-Pink structure of E-Hodge-Pink type $\mathbf{v}$. We write $(\mathrm{Ad}(D_E), \mathrm{Ad}(\Lambda_E)) := \mathrm{End}_E(D_E, \Lambda_E)$, where the right side is the internal hom of weakly admissible Hodge-Pink structures in the sense of §2.3.2. (If $V^*_{\mathcal{HP}}(D_E, \Lambda_E) \cong \eta_E$ then we have $V^*_{\mathcal{HP}}(\mathrm{Ad}(D_E), \mathrm{Ad}(\Lambda_E)) \cong \mathrm{Ad}(\eta_E)$.) The Hodge-Pink type $\mathrm{Ad}(\Lambda_E)$ is not effective if there are distinct Hodge-Pink weights for Λ_E.

Let $\mathrm{Ad}(\widehat{\mathcal{D}}_{E,x_0}) := \mathcal{O}_{\widehat{\mathbf{\Delta}},x_0} \otimes_{\mathscr{K}_0} \mathrm{Ad}(D_E)$ denote the standard lattice. Recall from §2.3.3 that we also have defined a filtration $\mathrm{Fil}^w_{\mathrm{Ad}(\Lambda_E)}$ on the standard lattice $\mathrm{Ad}(\widehat{\mathcal{D}}_{E,x_0})$. The zeroth filtration $\mathrm{Fil}^0_{\mathrm{Ad}(\Lambda_E)}$ is $\mathrm{Ad}(\Lambda_E) \cap \mathrm{Ad}(\widehat{\mathcal{D}}_{E,x_0})$, where the intersection is taken inside $\mathrm{Ad}(\widehat{\mathcal{D}}_{E,x_0})[\frac{1}{\mathcal{P}(u)}]$, and can be interpreted as a submodule of endomorphisms on $\widehat{\mathcal{D}}_{E,x_0}$ which take Λ_E into itself when extended to $\widehat{\mathcal{D}}_{E,x_0}[\frac{1}{\mathcal{P}(u)}]$. In particular, the image of an endomorphism $f \in \mathrm{Ad}(D_E)$ via the natural inclusion $\jmath : \mathrm{Ad}(D_E) \hookrightarrow \mathrm{Ad}(\widehat{\mathcal{D}}_{E,x_0})$ lies in $\mathrm{Fil}^0_{\mathrm{Ad}(\Lambda_E)}$ if and only if f respects E-Hodge-Pink structure Λ_E. Now we define the following 2-term complex:

$$\mathcal{C}^\bullet(D_E, \Lambda_E) := \left[\mathrm{Ad}(D_E) \overset{(\mathrm{id}-\varphi, \jmath)}{\longrightarrow} \mathrm{Ad}(D_E) \oplus \frac{\mathrm{Ad}(\widehat{\mathcal{D}}_{E,x_0})}{\mathrm{Fil}^0_{\mathrm{Ad}(\Lambda_E)}} \right].$$

We denote by $\mathcal{H}^i(D_E, \Lambda_E)$ the ith cohomology of the complex $\mathcal{C}^\bullet(D_E, \Lambda_E)$.

We discuss how this complex can be used to study the infinitesimal liftings of weakly admissible Hodge-Pink structures with coefficients. Let $A \in \mathfrak{AR}_E$, and let $I \subset A$ be an ideal with $\mathfrak{m}_A \cdot I = 0$. Put $\bar{A} := A/I \in \mathfrak{AR}_E$. We fix a weakly admissible $\bar{A}$-Hodge-Pink structure $(D_{\bar{A}}, \Lambda_{\bar{A}})$ which lifts (D_E, Λ_E). By Theorem 11.2.9, we already know that there exists a (weakly admissible) A-Hodge-Pink structures which lifts $(D_{\bar{A}}, \Lambda_{\bar{A}})$. So we would like to obtain the set of equivalence classes of such lifts, where two lifts (D_A, Λ_A) and (D'_A, Λ'_A) are equivalent if there is an isomorphism $(D_A, \Lambda_A) \to (D'_A, \Lambda'_A)$ which reduces to the identity map modulo I.

Proposition 11.3.10. *The set of equivalence classes of weakly admissible A-lifts of $(D_{\bar{A}}, \Lambda_{\bar{A}})$ is a principal homogeneous space under the action of $\mathcal{H}^1(D_E, \Lambda_E) \otimes_E I$. For any fixed such A-lift (D_A, Λ_A), the group of infinitesimal automorphisms is isomorphic to $\mathcal{H}^0(D_E, \Lambda_E) \otimes_E I$.*

It is natural to expect that there exists a functorial construction of the "obstruction class" in $\mathcal{H}^2(D_E, \Lambda_E) \otimes_E I$ for the liftability. But the second cohomology is trivial, which is consistent with Theorem 11.2.9.

Proof. The claim about the infinitesimal automorphisms is immediate, so we concentrate on the other claim.

Let (D_A, Λ_A) be an isocrystal with Hodge-Pink structure with A-coefficients such that $(D_A, \Lambda_A) \otimes_A \bar{A} \cong (D_{\bar{A}}, \Lambda_{\bar{A}})$. Then (D_A, Λ_A) is automatically weakly admissible since we have the following short exact sequence

$$0 \to (D_E, \Lambda_E) \otimes_E I \to (D_A, \Lambda_A) \to (D_{\bar{A}}, \Lambda_{\bar{A}}) \to 0, \tag{11.3.10.1}$$

where the flanking terms are weakly admissible. Being an extension of weakly admissible Hodge-Pink structures, (D_A, Λ_A) is weakly admissible, thanks to Proposition 2.3.8. Therefore, we are reduced to showing that the set of A-lifts (D_A, Λ_A) of $(D_{\bar{A}}, \Lambda_{\bar{A}})$ as isocrystals with Hodge-Pink structures with coefficients (without *a priori* imposing the weak admissibility) is a torsor under $\mathcal{H}^1(D_E, \Lambda_E) \otimes_E I$.

Let $\bar{\varphi} : \sigma^* D_{\bar{A}} \to D_{\bar{A}}$ be the Frobenius structure for the $\bar{A}$-isocrystal $(D_{\bar{A}}, \Lambda_{\bar{A}})$. Fixing the underlying $\mathscr{K}_0 \otimes_{F_0} A$-module for the A-isocrystal D_A that lifts $D_{\bar{A}}$, the set of A-lifts of $(D_{\bar{A}}, \bar{\varphi}, \Lambda_{\bar{A}})$ is a set of (φ, Λ_A) up to some equivalence relation, where

$\varphi : \sigma^* D_A \to D_A$ is an isomorphism which reduces to $\bar{\varphi}$ modulo I, and $\Lambda_A \subset (\mathcal{D}_A)\widehat{_{x_0}}$ is an an $(\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0})_A$-lattice which reduces to $\Lambda_{\bar{A}} \subset (\mathcal{D}_{\bar{A}})\widehat{_{x_0}}$ modulo I.

We fix an A-lift $(D_A, \varphi, \Lambda_A)$. For any other lift φ' of $\bar{\varphi}$, we can always find $\gamma_D \in \mathrm{Ad}(D_E) \otimes_E I$ such that $\varphi' = (\mathrm{id} + \gamma_D) \circ \varphi$, since $\varphi \otimes_A \bar{A} = \varphi' \otimes_A \bar{A} = \bar{\varphi}$. Conversely, given any $\gamma_D \in \mathrm{Ad}(D_E) \otimes_E I$, we obtain another lift $\varphi' := (\mathrm{id} + \gamma_D)\varphi$. For any other lift Λ'_A of $\Lambda_{\bar{A}}$, choose an automorphism of $(\mathcal{D}_A)\widehat{_{x_0}}[\frac{1}{\mathcal{P}(u)}]$ which takes Λ_A onto Λ'_A, and reduces to the identity modulo I. In fact, since (D_A, Λ_A) and (D_A, Λ'_A) should have the same E-Hodge-Pink type, it follows that this automorphism restricts to an automorphism $\mathrm{id} + \gamma_{\mathcal{HP}} : (\mathcal{D}_A)\widehat{_{x_0}} \to (\mathcal{D}_A)\widehat{_{x_0}}$, where $\gamma_{\mathcal{HP}} \in \mathrm{Ad}\left(\widehat{\mathcal{D}}_{E,x_0}\right) \otimes_E I$. Conversely, given any $\gamma_{\mathcal{HP}} \in \mathrm{Ad}\left(\widehat{\mathcal{D}}_{E,x_0}\right) \otimes_E I$, we can find $\Lambda'_A := (\mathrm{id} + \gamma_{\mathcal{HP}})(\Lambda_A)$, which clearly lifts $\Lambda_{\bar{A}} = \Lambda_A \otimes_A \bar{A}$. As remarked above, $\Lambda'_A = \Lambda_A$ if and only if $\gamma_{\mathcal{HP}} \in \mathrm{Fil}^0_{\mathrm{Ad}(\Lambda_A)}$.

To summarize, $\mathrm{Ad}(D_E) \oplus \left(\mathrm{Ad}(\widehat{\mathcal{D}}_{E,x_0}) / \mathrm{Fil}^0_{\mathrm{Ad}(\Lambda_E)}\right)$, which is a degree-1 term of $\mathcal{C}^\bullet(D_E, \Lambda_E)$, acts transitively on the set of equivalence classes of A-lifts. We now seek a condition for $(\gamma_D, \gamma_{\mathcal{HP}})$ for which the A-lifts $(D_A, \varphi, \Lambda_A)$ and $(D_A, \varphi', \Lambda'_A)$ are equivalent, where $\varphi' := (\mathrm{id} + \gamma_D)\varphi$ and $\Lambda'_A := (\mathrm{id} + \gamma_{\mathcal{HP}})(\Lambda_A)$. Assume that there exists $\beta \in \mathrm{Ad}(D_E) \otimes_E I$ such that the A-linear map $\mathrm{id} + \beta : D_A \xrightarrow{\sim} D_A$ which respects φ-structures (φ and φ') and A-Hodge-Pink structures (Λ_A and Λ'_A). In other words, we have

$$\varphi' = (\mathrm{id} + \beta) \circ \varphi \circ (\mathrm{id} - \sigma^* \beta) = \varphi + \left(\beta - \varphi \circ (\sigma^* \beta) \circ \varphi^{-1}\right) \circ \varphi = \varphi + (\beta - \varphi_{\mathrm{Ad}(D_E)}(\sigma^* \beta)) \circ \varphi,$$

(i.e., $\gamma_D = \beta - \varphi_{\mathrm{Ad}(D_E)}(\sigma^* \beta)$) and $\gamma_{\mathcal{HP}} = \jmath(\beta)$ (by considering the A-Hodge-Pink structure). In other words, $(D_A, \varphi, \Lambda_A)$ and $(D_A, \varphi', \Lambda'_A)$ are equivalent if and only if $(\gamma_D, \gamma_{\mathcal{HP}}) \in \mathrm{Im}(\mathrm{id} - \varphi_{\mathrm{Ad}(D_E)}, \jmath)$, which is the "coboundary condition." □

We apply the the previous proposition to the following special case. Let $A =$

$E[\epsilon]$ and $I = \epsilon\cdot A$, so necessarily we have $(D_{\bar{A}}, \Lambda_{\bar{A}}) = (D_E, \Lambda_E)$. By the previous proposition, the set of $E[\epsilon]$-deformations of (D_E, Λ_E), which has a natural E-vector space structure, is naturally E-isomorphic to $\mathcal{H}^1(D_E, \Lambda_E)$. (We can directly check that the $\mathcal{H}^1(D_E, \Lambda_E)$-action on the E-vector space of $E[\epsilon]$-deformations is E-linear.)

We use this result to compute the dimension of the tangent space $\left|\mathscr{D}_{\eta_E}^{\leqslant h}\right|(E[\epsilon])$. Choose (D_E, Λ_E) so that $\eta_E \cong V^*_{\mathcal{HP}}(D_E, \Lambda_E)$. By Corollary 11.2.8, Lemma 11.3.2 and the discussion immediately above, we have a natural E-linear isomorphism $\mathcal{H}^1(D_E, \Lambda_E) \cong \left|\mathscr{D}_{\eta_E}^{\leqslant h}\right|(E[\epsilon])$. One can compute the E-dimension of $\mathcal{H}^1(D_E, \Lambda_E)$, using the well-known trick that the "Euler characteristic" is equal to the alternating sum of dimensions of the terms of the complex:

$$\begin{aligned} \dim_E \mathcal{H}^1(D_E, \Lambda_E) &= \dim_E \mathcal{H}^0(D_E, \Lambda_E) + \dim_E \left(\frac{\operatorname{Ad}(\widehat{\mathcal{D}}_{E,x_0})}{\operatorname{Fil}^0_{\operatorname{Ad}(\Lambda_E)}}\right) \\ &= \dim_E \left(\operatorname{Ad}(\rho_{\eta_E})^{\mathcal{G}_K}\right) + \dim_E \left(\frac{\operatorname{Ad}(\widehat{\mathcal{D}}_{E,x_0})}{\operatorname{Fil}^0_{\operatorname{Ad}(\Lambda_E)}}\right), \end{aligned}$$

where the second equality follows from Corollary 5.2.4. Using the equation (11.3.8.1), we have the following corollary which is the main goal of this section.

Corollary 11.3.11. *There exists a natural E-linear isomorphism $\mathcal{H}^1(D_E, \Lambda_E) \xrightarrow{\sim} \left|\mathscr{D}_{\eta_E}^{\leqslant h}\right|(E[\epsilon])$. Any connected component of* $\operatorname{Spec} R_{\rho_{\mathbb{F}}}^{\square,\leqslant h}[\frac{1}{\pi_0}]$ *which contains a closed point corresponding to (D_E, Λ_E) of E-Hodge-Pink type $\mathbf{v}$ is of dimension*

$$\dim\left(R_{\rho_{\mathbb{F}}}^{\square,\mathbf{v}}\right) = d^2 + \dim_E \left(\frac{\operatorname{Ad}(\widehat{\mathcal{D}}_{E,x_0})}{\operatorname{Fil}^0_{\operatorname{Ad}(\Lambda_E)}}\right).$$

If $\operatorname{End}_{\mathcal{G}_K}(\rho_{\mathbb{F}}) = \mathbb{F}$, *then any connected component of* $\operatorname{Spec} R_{\rho_{\mathbb{F}}}^{\leqslant h}[\frac{1}{\pi_0}]$ *which contains a closed point corresponding to (D_E, Λ_E) of E-Hodge-Pink type $\mathbf{v}$ is of dimension*

$$\dim\left(R_{\rho_{\mathbb{F}}}^{\mathbf{v}}\right) = 1 + \dim_E \left(\frac{\operatorname{Ad}(\widehat{\mathcal{D}}_{E,x_0})}{\operatorname{Fil}^0_{\operatorname{Ad}(\Lambda_E)}}\right).$$

If furthermore $\mathscr{K}/\mathscr{K}_0$ is separable (e.g. when $\mathfrak{o}_0 = \mathbb{Z}_p$), then the formally smooth F-algebras $R_{\rho_{\mathbb{F}}}^{\square,\mathbf{v}}$ and $R_{\rho_{\mathbb{F}}}^{\mathbf{v}}$ (if it exists) are equi-dimensional.

Proof. It remains to show that equi-dimensionality assertion; i.e., the E-dimension of $\mathrm{Ad}(\widehat{\mathcal{D}}_{E,x_0})/\operatorname{Fil}^0_{\mathrm{Ad}(\Lambda_E)}$ only depends on the fixed E-Hodge-Pink type $\mathbf{v}$. Consider A-Hodge-Pink structures Λ_A and Λ'_A of $\mathcal{P}$-height $\leqslant h$ for a rank-d free $(\mathscr{K}_0)_A$-module D_A (with A finite over F_0) such that there exists an $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0,A}/(\mathcal{P}(u)^h)$-isomorphism $\Lambda_A/\widehat{\mathcal{D}}_{A,x_0} \cong \Lambda'_A/\widehat{\mathcal{D}}_{A,x_0}$. Then we can lift this ismorphism to an $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0,A}$-isomorphism $\Lambda_A \cong \Lambda'_A$ which maps $\widehat{\mathcal{D}}_{A,x_0} \subset \Lambda_A$ onto $\widehat{\mathcal{D}}_{A,x_0} \subset \Lambda'_A$. So we have $\operatorname{rank}_A\left(\mathrm{Ad}(\widehat{\mathcal{D}}_{A,x_0})/\operatorname{Fil}^0_{\mathrm{Ad}(\Lambda_A)}\right) = \operatorname{rank}_A\left(\mathrm{Ad}(\widehat{\mathcal{D}}_{A,x_0})/\operatorname{Fil}^0_{\mathrm{Ad}(\Lambda'_A)}\right)$. □

11.3.12 2-dimensional example

Let $\rho_{\mathbb{F}}$ be a 2-dimensional $\boldsymbol{\mathcal{G}}_K$-representation. Let us fix the following Hodge-Pink type (or rather F_0-Hodge-Pink type) $\mathbf{v} = (n=2, \bar{\Lambda}^{\mathbf{v}} := \mathfrak{S}_{F_0}/\mathcal{P}(u)^h)$. Choose $\mathfrak{M}_E \in \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}(E)$ with E a fixed finite extension $F = \operatorname{Frac}(\mathfrak{o})$ such that $\mathfrak{M}_E$ has Hodge-Pink type $\mathbf{v}$. Now, set $(D_E, \Lambda_E) := \underline{\mathbb{H}}(\mathfrak{M}_E)$, and choose a $(\mathscr{K}_0)_A$-basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for D_A, such that Λ_E is the $\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0}$-span of $\{\frac{1}{\mathcal{P}(u)^h}\mathbf{e}_1, \mathbf{e}_2\}$. Under the basis $\{\mathbf{e}_{ij} := \mathbf{e}_i^* \otimes \mathbf{e}_j\}_{i,j=1,2}$ for $\mathrm{Ad}\left(\widehat{\mathcal{D}}_{E,x_0}\right)$, the Hodge-Pink structure $\mathrm{Ad}(\Lambda_E)$ is the $(\mathcal{O}_{\widehat{\mathbf{\Delta}},x_0})_E$-span of $\left\{\mathbf{e}_{11},\, \frac{1}{\mathcal{P}(u)^h}\mathbf{e}_{12},\, \mathcal{P}(u)^h\mathbf{e}_{21},\, \mathbf{e}_{22}\right\}$. Therefore, $\{\mathbf{e}_{11},\, \mathbf{e}_{12},\, \mathcal{P}(u)^h\mathbf{e}_{21},\, \mathbf{e}_{22}\}$ spans $\operatorname{Fil}^0_{\mathrm{Ad}(\Lambda_E)}$, so we have

$$\frac{\mathrm{Ad}(\widehat{\mathcal{D}}_{E,x_0})}{\operatorname{Fil}^0_{\mathrm{Ad}(\Lambda_E)}} \cong \frac{\mathfrak{S}_E}{\mathcal{P}(u)^h}\mathbf{e}_{21}.$$

In particular, it follows from Corollary 11.3.11 that if $\mathscr{K}/\mathscr{K}_0$ is separable then $\dim\left(R^{\square,\mathbf{v}}_{\rho_{\mathbb{F}}}\right) = 4 + h\cdot[\mathscr{K} : F_0]$, and if furthermore $\operatorname{End}_{\boldsymbol{\mathcal{G}}_K}(\rho_{\mathbb{F}}) = \mathbb{F}$ then $\dim\left(R^{\mathbf{v}}_{\rho_{\mathbb{F}}}\right) = 1 + h\cdot[\mathscr{K} : F_0]$. Here, $\mathfrak{o}_{\mathscr{K}} := \mathfrak{S}/\mathcal{P}(u)$, viewed as an integral extension of $\mathfrak{o}_0$.

For $h=1$, we will determine the connected components of $\operatorname{Spec} R^{\square,\mathbf{v}}_{\rho_{\mathbb{F}}}$ and $\operatorname{Spec} R^{\mathbf{v}}_{\rho_{\mathbb{F}}}$ later in §11.4.14 and §11.5 when $\mathscr{K}/\mathscr{K}_0$ is separable.

11.3.13 Relation with crystalline and semi-stable deformation rings

This paragraph is a continuation of §11.2.11. We assume that $\mathfrak{o}_0 = \mathbb{Z}_p$. We first recall the definition of p-adic Hodge type for a weakly admissible filtered (φ, N)-module. Let $E/\mathbb{Q}_p$ be a finite extension and fix a decreasing separated and exhaustive filtration $\mathbf{v} := \{\mathrm{Fil}^w_{\mathbf{v}} \subset (\mathscr{K}_E)^{\oplus n}\}$ by $\mathscr{K}_E$-submodules such that the associated graded module is concentrated in degrees in $[0, h]$. For a finite E-algebra A, we say that a weakly admissible filtered (φ, N)-module with A-coefficients $(D_A, \varphi, N, \mathrm{Fil}^\bullet(D_{A,\mathscr{K}})$ is *of Hodge type* $\mathbf{v}$ if there exists a filtered $(\mathscr{K}_A)$-linear isomorphism $(\mathscr{K} \otimes_{\mathbb{Q}_p} A)^{\oplus n} \cong (D_A)_{\mathscr{K}}$ where the filtration on the left side is $\{(\mathrm{Fil}^w_{\mathbf{v}}) \otimes_E A\}$. For a semi-stable A-representation V_A of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ with Hodge-Tate weights in $[0, h]$, we say V_A is of p-adic Hodge type $\mathbf{v}$ if $\underline{D}_{\mathrm{st}}(V_A(-h))$ is of p-adic Hodge type $\mathbf{v}$, where $\underline{D}_{\mathrm{st}} : \mathrm{Rep}^{\mathrm{st}}_{\mathbb{Q}_p}(\boldsymbol{\mathcal{G}}_{\mathscr{K}}) \to \mathcal{MF}^{wa}_{\mathscr{K}}(\varphi, N)$ is the covariant equivalence of categories. By Lemma 11.3.5(1), one can also view $\mathbf{v} := \{\mathrm{Fil}^w_{\mathbf{v}}\}$ as an E-Hodge-Pink type.

Let A be a finite E-algebra. We defined a functor $\underline{\mathrm{res}} : \mathcal{MF}^{wa}_{\mathscr{K}}(\varphi, N) \to \mathcal{HP}^{wa}_K(\varphi)$ in (5.2.12.1), which takes a filtered (φ, N)-module with A-coefficients into Hodge-Pink structure with A-coefficients. For a fixed $\mathbf{v} := \{\mathrm{Fil}^w_{\mathbf{v}}\}$, we can show that the weakly admissible filtered (φ, N)-module $D_A := (D_A, \varphi, N, \mathrm{Fil}^\bullet(D_A)_{\mathscr{K}})$ with A-coefficients is of p-adic Hodge type $\mathbf{v}$ if and only if $\underline{\mathrm{res}}(D_A)$ is of Hodge-Pink type $\mathbf{v}$. This claim essentially follows from [52, Lemma 1.2.1].

As in §11.2.11, fix a mod p representation $\bar{\rho}$ of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$. Let $R^{\square,\leqslant h}_{\mathrm{st}}$ and $R^{\square,\leqslant h}_{\mathrm{cris}}$ denote semi-stable and crystalline framed deformation ring for $\bar{\rho}$ in the sense of [59], respectively. (For what follows, the same discussion works if the framed deformation rings are replaced by deformation rings, provided $\mathrm{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}(\bar{\rho}_\infty) = \mathbb{F}$.) By [55, (2.6)], fixing the Hodge type $\mathbf{v}$ cuts out unions of connected components $\mathrm{Spec}\, R^{\square,\mathbf{v}}_{\mathrm{st}} \subset \mathrm{Spec}\, R^{\square,\leqslant h}_{\mathrm{st}}[\frac{1}{p}]$ and $\mathrm{Spec}\, R^{\square,\mathbf{v}}_{\mathrm{cris}} \subset \mathrm{Spec}\, R^{\square,\leqslant h}_{\mathrm{cris}}[\frac{1}{p}]$, respectively. Moreover, the map $\underline{\text{res}}^{\text{st}} : \operatorname{Spec} R_{\text{st}}^{\square,\leqslant h} \to \operatorname{Spec} R_{\infty}^{\square,\leqslant h}$ defined by "restricting to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$" restrict to $\operatorname{Spec} R_{\text{st}}^{\square,\mathbf{v}} \to \operatorname{Spec} R_{\infty}^{\square,\mathbf{v}}$, where $\mathbf{v}$ and $\mathbf{v}$ are chosen as above. Similarly, $\underline{\text{res}}^{\text{cris}}$ restricts to $\operatorname{Spec} R_{\text{cris}}^{\square,\mathbf{v}} \to \operatorname{Spec} R_{\infty}^{\square,\mathbf{v}}$.

The local structure of $R_{\text{cris}}^{\square,\mathbf{v}}$ and $R_{\text{st}}^{\square,\mathbf{v}}$ is studied in [55, §3] (including the case $p = 2$). For example, $R_{\text{cris}}^{\square,\mathbf{v}}$ is equi-dimensional and formally smooth, and $R_{\text{st}}^{\square,\mathbf{v}}$ is equi-dimensional and admits a dense open subscheme which is formally smooth. The dimensions can be computed. In particular, by comparing the dimension formulae for $R_{\text{cris}}^{\square,\mathbf{v}}$ and for $R_{\infty}^{\square,\mathbf{v}}$, we see that they have the *same* dimension if (and only if) $h = 1$.

We give an example in case $\bar{\rho}$ is a 2-dimensional $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representation. Let $\mathbf{v}$ be the filtration on $\mathscr{K}^{\oplus 2}$ such that $\dim_{\mathscr{K}} \operatorname{Fil}_{\mathbf{v}}^{w} = 2$ for $w \leq 0$, $\dim_{\mathscr{K}} \operatorname{Fil}_{\mathbf{v}}^{w} = 1$ for $1 \leq w \leq h$, and $\dim_{\mathscr{K}} \operatorname{Fil}_{\mathbf{v}}^{w} = 0$ for $w > h$. We obtain natural maps $\operatorname{Spec} R_{\text{st}}^{\square,\mathbf{v}} \to \operatorname{Spec} R_{\infty}^{\square,\mathbf{v}}$ and $\operatorname{Spec} R_{\text{cris}}^{\square,\mathbf{v}} \to \operatorname{Spec} R_{\infty}^{\square,\mathbf{v}}$, and similarly for the deformation rings. The first two equations of the following are from Kisin [55, (3.3)] and the rest from §11.3.12 above.

$$\begin{aligned}
\dim(R_{\text{cris}}^{\square,\mathbf{v}}) = \dim(R_{\text{st}}^{\square,\mathbf{v}}) &= 4 + [\mathscr{K} : \mathbb{Q}_p], \\
\dim(R_{\text{cris}}^{\mathbf{v}}) = \dim(R_{\text{st}}^{\mathbf{v}}) &= 1 + [\mathscr{K} : \mathbb{Q}_p], \quad \text{if } \operatorname{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}}}(\bar{\rho}) \cong \mathbb{F}. \\
\dim(R_{\infty}^{\square,\mathbf{v}}) &= 4 + h[\mathscr{K} : \mathbb{Q}_p] \\
\dim(R_{\infty}^{\mathbf{v}}) &= 1 + h[\mathscr{K} : \mathbb{Q}_p], \quad \text{if } \operatorname{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}(\bar{\rho}_\infty) \cong \mathbb{F}
\end{aligned}$$

For $h > 1$, this difference of dimensions reflects the "gap" between $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representations of $\mathcal{P}$-height $\leqslant h$ and crystalline or semi-stable $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations with Hodge-Tate weights in $[0, h]$.

11.4 "Ordinary" and "formal" components

We again allow $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$. In addition to Hodge-Pink types, we discuss two more conditions on $R_{\rho_{\mathbb{F}}}^{\square,\leqslant h}[\frac{1}{\pi_0}]$ (or rather, on $\mathscr{G}\mathscr{R}^{\square,\leqslant h} \otimes_{\mathfrak{o}} F$) which cut out unions of connected components: more precisely, we show that the "ordinary" and "formal" deformations, which will be defined below in §11.4.3 and §11.4.6, form unions of connected components in the generic fiber of the framed deformation ring of $\mathcal{P}$-height $\leqslant h$. Exactly the same results will hold for $R^{\leqslant h}_{\rho_{\mathbb{F}}}[\frac{1}{\pi_0}]$ whenever it exists.

Using these finer conditions, we work out a complete description of ordinary components of a 2-dimensional (framed) deformation ring of $\mathcal{P}$-height $\leqslant h$ with a certain fixed Hodge-Pink type (see Proposition 11.4.15)[12]. We end this discussion with an application to crystalline and semi-stable (framed) deformation rings in §11.4.17.

11.4.1

For the proof of Proposition 11.4.2 we need to extend Corollary 8.1.11 to allow coefficients in $(B,J) \in \mathfrak{Aug}_{\mathfrak{o}}$. First, recall that for $T_B \in \mathrm{Rep}^{\mathrm{free}}_B(\boldsymbol{\mathcal{G}}_K)$ with B an $\mathfrak{o}_0$-algebra where π_0 is nilpotent, we defined in (11.1.4.1) an étale $(\varphi, \mathfrak{o}_{\mathcal{E},B})$-module $\underline{D}^{\leqslant h}_{\mathcal{E}}(T_B)$ free with $\mathfrak{o}_{\mathcal{E},B}$-rank equal to $\mathrm{rank}_B(T_B)$. We also showed that $\underline{D}^{\leqslant h}_{\mathcal{E}}$ is exact and commutes with $\otimes$-products, internal homs, duality, and change of coefficients.

If, furthermore, T_B is unramified (i.e., I_K acts trivially on T_B), then we define

$$\underline{U}(T_B) := (W^{sh} \otimes_{\mathfrak{o}_0} T_B)^{\boldsymbol{\mathcal{G}}_K/I_K}, \qquad (11.4.1.1)$$

where W is as in §1.3.3 and W^{sh} denotes the strict henselization of W. We can show the following without difficulty:

1. For $T_B \in \mathrm{Rep}^{\mathrm{free}}_B(\boldsymbol{\mathcal{G}}_K/I_K)$, $\underline{U}(T_B)$ is an finite free étale (φ, W_B)-module with W_B-rank equal to $\mathrm{rank}_B(T_B)$. Here, $W_B := W \otimes_{\mathfrak{o}_0} B$.
2. A sequence $(\dagger):\ 0 \to T'_B \to T_B \to T''_B \to 0$ in $\mathrm{Rep}^{\mathrm{free}}_B(\boldsymbol{\mathcal{G}}_K/I_K)$ is short exact if and only if $\underline{U}(\dagger)$ is short exact.

[12] For this result, we do not require $\mathscr{K}/\mathscr{K}_0$ to be separable.

3. The formation of $\underline{U}$ commutes with $\otimes$-products, internal homs, duality, and change of coefficients.

4. There is a natural isomorphism $\mathfrak{o}_{\mathcal{E},B} \otimes_{W_B} \underline{U}(T_B) \cong \underline{D}_{\mathcal{E}}^{\leqslant h}(T_B(h))$ of étale $(\varphi, \mathfrak{o}_{\mathcal{E},B})$-module, where $T_B \in \mathrm{Rep}_B^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K/I_K)$.

5. If $\mathfrak{M}_B \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ is étale, then $T_B := \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_B)$ is of Lubin-Tate type of $\mathcal{P}$-height h (i.e., $T_B(-h)$ is unramified) and we have a natural φ-compatible isomorphism $\mathfrak{M}_B \cong \mathfrak{S}_B \otimes_{W_B} \underline{U}(T_B(-h))$.

If $\#(B) < \infty$ then (1)–(4) can be proved using Corollary 8.1.11, following the argument in §8.2.4. The general case of (1)–(4) follows from this case because T_B descends to $T_{B'} \in \mathrm{Rep}_{B'}^{\mathrm{free}}(\boldsymbol{\mathcal{G}}_K/I_K)$ with $B' \subset B$ some finite $\mathfrak{o}_0$-subalgebra, and we have a φ-compatible isomorphism $\underline{U}(T_B) \cong \underline{U}(T_{B'}) \otimes_{B'} B$. To show (5), first observe that $T_B(-h) \cong \underline{T}_{\mathfrak{S}}^{\leqslant 0}(\mathfrak{M}_B)$, where the right side makes sense since $\mathfrak{M}_B$ is étale. Now T_B is of $\mathcal{P}$-height $\leqslant 0$ by Proposition 11.1.6, so T_B is unramified by Proposition 8.1.10. The second part of (5) is reduced to the case when $\#(B) < \infty$ by a similar argument as previously, and then we apply Corollary 8.1.11.

We now state the following proposition which shows the existence of the connected-étale sequence for $\mathfrak{M}_A \in \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h}(A, I)$. This generalizes Proposition 8.2.7.

Proposition 11.4.2. *Consider* $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$*, and assume that* $\mathrm{Spec}\, A$ *is connected. Any* $\mathfrak{M}_A \in \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h}(A, I)$ *has a "maximal" étale submodule* $\mathfrak{M}_A^{\text{ét}} \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ *and a "maximal" Lubin-Tate type quotient* $\mathfrak{M}_A^{\mathcal{LT}} \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ *with the following properties.*

1. *Both the quotient* $\mathfrak{M}_A/\mathfrak{M}_A^{\text{ét}}$ *and the kernel of* $\mathfrak{M}_A \twoheadrightarrow \mathfrak{M}_A^{\mathcal{LT}}$ *are finite locally free over* $\mathfrak{S}_A$*; i.e. they are objects in* $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$.

2. *For any morphism $(A,I) \to (B,J)$ in $\mathfrak{Aug}_{\mathfrak{o}}$, the natural morphisms*

$$\mathfrak{M}_A^{\text{ét}} \otimes_A B \to (\mathfrak{M}_A \otimes_A B)^{\text{ét}}, \quad (\mathfrak{M}_A \otimes_A B)^{\mathcal{LT}} \to \mathfrak{M}_A^{\mathcal{LT}} \otimes_A B$$

are isomorphisms.

3. *The natural morphisms $(\mathfrak{M}_A^{\mathcal{LT}})^\vee \to (\mathfrak{M}_A^\vee)^{\text{ét}}$ and $(\mathfrak{M}_A^\vee)^{\mathcal{LT}} \to (\mathfrak{M}_A^{\text{ét}})^\vee$ are isomorphisms.*

4. *The formation of $\mathfrak{M}_A^{\text{ét}}$ and $\mathfrak{M}_A^{\mathcal{LT}}$ is "functorial," in the following sense: any φ-compatible map $\mathfrak{M}_A \to \mathfrak{M}'_A$ in $(\text{ModFI}/\mathfrak{S})_A^{\leqslant h}$ takes $\mathfrak{M}_A^{\text{ét}}$ into $(\mathfrak{M}'_A)^{\text{ét}}$ and induces a map $\mathfrak{M}_A^{\mathcal{LT}} \to (\mathfrak{M}'_A)^{\mathcal{LT}}$.*

Proof. The proof is essentially identical to [51, Proposition 2.4.14]. The existence and properties of $\mathfrak{M}_A^{\mathcal{LT}}$ can be reduced to the corresponding claims on $\mathfrak{M}_A^{\text{ét}}$ by duality of $\mathcal{P}$-height h, so it suffices to handle the claims on $\mathfrak{M}_A^{\text{ét}}$. We may assume that A is finitely generated over $\mathfrak{o}$. Then A is Jacobson since A_{red} is finitely generated over $\mathbb{F}_q$. The key step is to show that for any closed point $x \in \operatorname{Spec} A$, $d(x) := \operatorname{rank}_{\mathfrak{S}_{\kappa(x)}}(\mathfrak{M}_A \otimes_A \kappa(x))^{\text{ét}}$ is locally constant in $\operatorname{MaxSpec} A$ (hence in $\operatorname{Spec} A$), where $\kappa(x)$ denotes the residue field at $x \in \operatorname{Spec} A$.

We first show that $d(x)$ is lower semi-continuous (i.e., $d(x)$ goes down along a closed subset), as follows. Consider a φ-module $\mathfrak{M}_A/u\mathfrak{M}_A$ and choose a basis. Let $P(T)$ be the characteristic polynomial for the matrix representation of $\bar{\varphi} : \sigma^*(\mathfrak{M}_A/u\mathfrak{M}_A) \to \mathfrak{M}_A/u\mathfrak{M}_A$ with respect to the chosen basis. Then $d(x)$ equals the largest integer d such that the coefficient of T^{n-d} in $P(T)$ does not vanish at x.

In order to show that $d(x)$ is upper semi-continuous, we will define for each $d \in [0,n]$ a projective A-scheme $X^d_{\mathfrak{M}_A}$, such that $d(x) \geq d$ if and only if x is in the image of $X^d_{\mathfrak{M}_A}$. To construct $X^d_{\mathfrak{M}_A}$, consider $\mathfrak{M}_A \in \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}(A,I)$ with $(A,I) \in \mathfrak{Aug}_{\mathfrak{o}}$, and let $T_A := \underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}_A)$ as defined in (the proof of) Proposition 11.1.6. For any A-algebra

B, we define $X^d_{\mathfrak{M}_A}(B)$ to be the set of $\boldsymbol{\mathcal{G}}_K$-stable B-submodules $L_B \subset T_A \otimes_A B$ with the following properties.

1. The submodule L_B is locally free of B-rank d and the quotient $(T_A \otimes_A B)/L_B$ is locally free over B.

2. The Tate twist $L_B(-h)$ is unramified.

3. We identify $\mathfrak{M}_A[\frac{1}{u}]$ with $\underline{D}^{\leqslant h}_{\mathcal{E},A}(T_A)$ using Proposition 11.1.6. The φ-stable W_B-submodule $\underline{U}(L_B(-h)) \subset \underline{D}^{\leqslant h}_{\mathcal{E},A}(T_A)$ is contained in $\mathfrak{M}_A \otimes_A B$.

We now show that the functor $X^d_{\mathfrak{M}_A}$ is representable by a projective A-scheme equipped with a universal rank-d $\boldsymbol{\mathcal{G}}_K$-stable subbundle $L_{X^d_{\mathfrak{M}_A}} \subset T_A \otimes_A \mathcal{O}_{X^d_{\mathfrak{M}_A}}$ of Lubin-Tate type of $\mathcal{P}$-height h. It is clear from the definition that the formation of $X^d_{\mathfrak{M}_A}$ and $L_{X^d_{\mathfrak{M}_A}}$, if they exist, commutes with arbitrary scalar extension for $A \to B$. For the proof of representability, first observe that the conditions (1) and (2) obviously define a closed subscheme (which we denote by $Y^d_{\mathfrak{M}_A}$) of the grassmannian of rank-d subspaces of T_A. We now show that the third condition is closed in $Y^d_{\mathfrak{M}_A}$, as follows. It is enough to show that for any A-algebra B and any B-point $L_B \in Y^d_{\mathfrak{M}_A}(B)$, there exists an ideal $J \in B$ with the property that $L_B \otimes_B C \subset T_A \otimes_A C$ satisfies (3) above for a B-algebra C if and only if $JC = 0$. It is clear that the construction of $J \subset B$ is compatible with scalar extension, so we obtain the universal closed subscheme of the grassmannian with conditions (1)–(3) by gluing such ideals $J_\alpha \subset B_\alpha$ for some open affine covering $\{\operatorname{Spec} B_\alpha\}_\alpha$ of $Y^d_{\mathfrak{M}_A}$.

To construct the ideal $J \subset B$ as above, put $\mathfrak{M}_B := \mathfrak{M}_A \otimes_A B$ and $T_B := T_A \otimes_A B$. Since $\mathfrak{S}_B[\frac{1}{u}]/\mathfrak{S}_B$ is free over B, $\mathfrak{M}_B[\frac{1}{u}]/\mathfrak{M}_B$ is also free over B. Choose an B-basis $\{\mathbf{e}_j\}_j$ for $\mathfrak{M}_B[\frac{1}{u}]/\mathfrak{M}_B$. Now consider the following composite of B-linear maps

$$r_B : \underline{U}(L_B(-h)) \hookrightarrow \underline{D}^{\leqslant h}_{\mathcal{E}}(T_B) \twoheadrightarrow \underline{D}^{\leqslant h}_{\mathcal{E}}(T_B)/\mathfrak{M}_B \xrightarrow{\sim} \mathfrak{M}_B[1/u]/\mathfrak{M}_B \xrightarrow{\sim} \bigoplus_j B\mathbf{e}_j.$$

Note that $\underline{U}(L_B(-h))$ is finite free over B since W_B is so. Choose a B-basis $\{u_i\}$ for $\underline{U}(L_B(-h))$, and let $J \subset B$ be the ideal generated by $r_B(u_i)$. Since the formation of $\underline{U}$ and the B-linear map r_B commutes with change of coefficients, the ideal J has the required property.

Now, let us show that for a closed point $x \in \operatorname{Spec}(A)$, we have $d(x) \geq d$ if and only if x is in the image of $X^d_{\mathfrak{M}_A}$. (This shows that $d(x)$ is locally constant on $\operatorname{Spec} A$, so it is constant if $\operatorname{Spec} A$ is connected.) First, it directly follows from the definition of $X^d_{\mathfrak{M}_A}$ that for any map $(A, I) \to (B, J)$ in $\mathfrak{Aug}_{\mathfrak{o}}$ we have a natural isomorphism $X^d_{\mathfrak{M}_A} \otimes_A B \cong X^d_{\mathfrak{M}_B}$ where $\mathfrak{M}_B := \mathfrak{M}_A \otimes_A B$. By taking $(B, J) = (\kappa(x), (0))$ where $\kappa(x)$ is the residue field at x, we are reduced to showing that the $X^d_{\mathfrak{M}_{\kappa(x)}}$ is non-empty if and only if $\operatorname{rank}_{\mathfrak{S}_{\kappa(x)}}(\mathfrak{M}_{\kappa(x)})^{\text{ét}} \geq d$, where $\mathfrak{M}_{\kappa(x)} := \mathfrak{M}_A \otimes_A \kappa(x)$. If we have the inequality $\operatorname{rank}_{\mathfrak{S}_{\kappa(x)}} \left((\mathfrak{M}_{\kappa(x)})^{\text{ét}}\right) \geq d$, then any d-dimensional $\boldsymbol{\mathcal{G}}_K$-stable subspace $L_{\kappa(x)}$ of $\underline{T}^{\leqslant h}_{\mathfrak{S}}\left((\mathfrak{M}_{\kappa(x)})^{\text{ét}}\right)$ defines a $\kappa(x)$-point of $X^d_{\mathfrak{M}_{\kappa(x)}}$. (That $L_{\kappa(x)}$ satisfies condition (3) of the definition of $X^d_{\mathfrak{M}_A}$ follows Corollary 8.1.11, especially the special case of (2) and (5) in §11.4.1.) Conversely, if $X^d_{\mathfrak{M}_{\kappa(x)}}$ is non-empty then there exists a κ-point $L_\kappa \in X^d_{\mathfrak{M}_{\kappa(x)}}(\kappa)$ for some finite extension $\kappa/\kappa(x)$. By definition of $X^d_{\mathfrak{M}_{\kappa(x)}}$, especially by condition (3), we have $\mathfrak{S}_\kappa \otimes_{W_\kappa} \underline{U}(L_\kappa(-h)) \subset (\mathfrak{M}_{\kappa(x)} \otimes_{\kappa(x)} \kappa)^{\text{ét}} \cong (\mathfrak{M}_{\kappa(x)})^{\text{ét}} \otimes_{\kappa(x)} \kappa$ (where the isomorphism is obtained from Proposition 8.2.7), so we have the desired inequality $\operatorname{rank}_{\mathfrak{S}_{\kappa(x)}}(\mathfrak{M}_{\kappa(x)})^{\text{ét}} \geq d$.

This shows that $d(x)$ is locally constant on $\operatorname{Spec} A$. We can furthermore show that the structure morphism $X^d_{\mathfrak{M}_A} \to \operatorname{Spec} A$ induces an isomorphism over the (possibly empty) union of connected components on which $d(x) = d$. Assume that $\operatorname{Spec} A$ is connected and $d(x) = d$ for all closed point $x \in \operatorname{Spec} A$. Since $X^d_{\mathfrak{M}_A}$ is proper over A, it is enough to show that if the formal completion at each closed point of $\operatorname{Spec} A$ is an isomorphism. Since the formation of $X^d_{\mathfrak{M}_A}$ commutes with scalar extension

$A \to A/\mathfrak{m}_x^n$ (where $\mathfrak{m}_x$ is the maximal ideal corresponding to x), it suffices to show that if A is local with $\#(A) < \infty$ then we have $\mathrm{rank}_{\mathfrak{S}_A}(\mathfrak{M}_A^{\text{ét}}) = d$ and $X^d_{\mathfrak{M}_A}(B) \to (\mathrm{Spec}\, A)(B)$ is a bijection for any finite A-algebra B. To prove this claim, observe that for any B-point $L_B \in X^d_{\mathfrak{M}_A}(B)$ we have $\underline{U}(L_B) \subset (\mathfrak{M}_A \otimes_A B)^{\text{ét}} \cong \mathfrak{M}_A^{\text{ét}} \otimes_A B$ by definition of $X^d_{\mathfrak{M}_A}$ and Proposition 8.2.7, so (essentially by Corollary 8.1.11) we have an inclusion $L_B \subset \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A^{\text{ét}}) \otimes_A B$ of rank-d free B-modules which are direct factors in T_B (as abstract B-modules). Thus, we have an equality $L_B = \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A^{\text{ét}}) \otimes_A B$, which proves the claim.

Now, assume that A is connected and finitely generated over $\mathfrak{o}_0$ with $d(x) = d$ for all closed point $x \in \mathrm{Spec}\, A$. Since the structure morphism $X^d_{\mathfrak{M}_A} \to \mathrm{Spec}\, A$ is an isomorphism, we obtain the universal $\boldsymbol{\mathcal{G}}_K$-stable submodule $L_A \subset T_A$ of Lubin-Tate type of $\mathcal{P}$-height h. When $\#(A) < \infty$, it follows from Corollary 8.1.11 and the discussion of the paragraph immediately above that $\mathfrak{M}_A^{\text{ét}} = \mathfrak{S}_A \otimes_{W_A} \underline{U}(L_A)$ as submodules of $\mathfrak{M}_A$. In general, we put $\mathfrak{M}_A^{\text{ét}} := \mathfrak{S}_A \otimes_{W_A} \underline{U}(L_A)$. Since the formations of L_A and $\underline{U}$ commute with any change of coefficients for $A \to B$, we obtain the equality $\mathfrak{M}_A^{\text{ét}} \otimes_A B = (\mathfrak{M}_A \otimes_A B)^{\text{ét}}$ of submodules of $\mathfrak{M}_A \otimes_A B$ for any $A \to B$, and if $\#(B) < \infty$ then this is known to be a maximal étale submodule of $\mathfrak{M}_A \otimes_A B$. In particular, $\mathfrak{M}_A^{\text{ét}} \otimes_A B$ contains the image of $\bigcap_{i=1}^{\infty} \varphi^r(\sigma^{r*}\mathfrak{M}_A)$ in $\mathfrak{M}_A \otimes B$. So for any maximal ideal $\mathfrak{m} \subset A$ and any positive integer i, we have

$$\bigcap_{i=1}^{\infty} \varphi^r(\sigma^{r*}\mathfrak{M}_A)/\mathfrak{M}_A^{\text{ét}} \subset \mathfrak{m}^i(\mathfrak{M}_A/\mathfrak{M}_A^{\text{ét}}),$$

thus, we have $\mathfrak{M}_A^{\text{ét}} = \bigcap_{i=1}^{\infty} \varphi^r(\sigma^{r*}\mathfrak{M}_A)$. This shows that $\mathfrak{M}_A^{\text{ét}}$ is a maximal étale submodule of $\mathfrak{M}_A$. To see $\mathfrak{M}_A/\mathfrak{M}_A^{\text{ét}}$ is a finite locally free $\mathfrak{S}_A$-module, note that $(\mathfrak{M}_A/\mathfrak{M}_A^{\text{ét}}) \otimes_{\mathfrak{S}_A} \mathfrak{S}_{\widehat{A}_{\mathfrak{m}}}$ is finite locally free over $\mathfrak{S}_{\widehat{A}_{\mathfrak{m}}}$ for any maximal ideal $\mathfrak{m} \subset A$ because the formation of $\mathfrak{M}_A^{\text{ét}}$ commutes with change of coefficients and $\mathfrak{M}_A/\mathfrak{M}_A^{\text{ét}}$ is finite locally free over $\mathfrak{S}_A$ when $\#(A) < \infty$ by Proposition 8.2.7. The functoriality assertion is clear. □

11.4.3 Definition: formal $\mathcal{G}_K$-representations

A torsion $\mathfrak{o}_0$-representation $\bar{T}$ is said to be *formal* if there exists a a unipotent[13] torsion φ-module $\overline{\mathfrak{M}} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ such that $\bar{T} \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\overline{\mathfrak{M}}) = \underline{T}_{\mathfrak{S}}^{*}(\overline{\mathfrak{M}}^{\vee})$. A lattice $\mathfrak{o}_0$-representation T is said to be *formal* if there exists a unipotent φ-module $\mathfrak{M} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ such that $T \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}) = \underline{T}_{\mathfrak{S}}^{*}(\mathfrak{M}^{\vee})$. It follows from Proposition 9.2.6 and the existence of (the dual of) connected-étale sequence that a lattice $\mathfrak{o}_0$-representation T is formal if and only if $T/\pi_0^n T$ is formal as a torsion representation for each n.

Let A^o be a complete local noetherian $\mathfrak{o}_0$-algebra with finite residue field. We say an A^o-representation T_{A^o} is *formal* if $T_{A^o} \otimes_{A^o} (A^o/\mathfrak{m}_{A^o}^n)$ for each n is formal as a torsion $\mathfrak{o}_0$-representation. Observe that if $A^o = \mathfrak{o}_0$ then this definition recovers the definition of formal $\mathfrak{o}_0$-lattice representations by the above application of Proposition 9.2.6.

An F_0-representation V is said to be *formal* if there exists a $\mathcal{G}_K$-stable $\mathfrak{o}_0$-lattice $T \subset V$ which is formal as a lattice $\mathfrak{o}_0$-representation. In fact, for an F_0-representation V, if a $\mathcal{G}_K$-stable lattice $T \subset V$ is formal then any other $\mathcal{G}_K$-stable lattice is formal. (Proof: if $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$ is φ-nilpotent then any $\mathfrak{M}' \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$ which is isogenous to $\mathfrak{M}$ is φ-nilpotent. Now apply Proposition 5.2.9.) For a finite F_0-algebra A, we say an A-representation V_A is *formal* if it is formal as an F_0-representation.

We record some special cases of this, which justifies the name "formal" $\mathcal{G}_K$-representation

If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$ and A is finite flat over $\mathbb{F}_q[[\pi_0]]$, then formal $\mathcal{G}_K$-representations (over A) of $\mathcal{P}$-height $\leqslant h$ are exactly those that come from formal (i.e., connected)

[13] Recall from §8.3.6 that $\overline{\mathfrak{M}}$ is unipotent if and only if $\overline{\mathfrak{M}}^{\vee}$ is φ-nilpotent.

π_0-divisible groups of $\mathcal{P}$-height $\leqslant h$ (with A-action). Next, suppose $\mathfrak{o}_0 = \mathbb{Z}_p$ and $h = 1$. We shall use the notations from §1.3.1.2. Assume that $p > 2$ so that we can use the Breuil-Kisin classification of Barsotti-Tate groups and finite flat group schemes. Let A be a finite flat $\mathbb{Z}_p$-algebra, and consider a Barsotti-Tate group G over $\mathfrak{o}_{\mathscr{K}}$ with an action of A. Then the $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-restriction of the Tate module $T_p(G)$ is formal if and only if G is a formal (i.e., connected) Barsotti-Tate group. Similarly, let A be a finite $\mathbb{Z}_p$-algebra of finite length and let G be a finite flat group scheme over $\mathfrak{o}_{\mathscr{K}}$ with an action of A. Then the $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-restriction of the torsion $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representation $G(\overline{\mathscr{K}})$ is formal if and only if G is connected.[14]

Now let us define the full subgroupoids $\mathscr{D}^{u,\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}} \subset \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}$ whose objects are *unipotent of $\mathcal{P}$-height $\leqslant h$*, and $\mathscr{D}^{f,\leqslant h}_{\rho_{\mathbb{F}}} \subset \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ whose objects are formal deformations. That they are sub*groupoids* follows from the fact that if $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})^{\leqslant h}_A$ is φ-nilpotent then the change of coefficients $\mathfrak{M}_A \otimes_A A'$ is also φ-nilpotent (so we have enough co-cartesian lifts). The composition of 1-morphisms $\mathscr{D}^{u,\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}} \hookrightarrow \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}} \xrightarrow{T^{\leqslant h}_{\mathfrak{S}}} \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ factors through $\mathscr{D}^{f,\leqslant h}_{\rho_{\mathbb{F}}}$.

Proposition 11.4.4. *Let $\mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ be the category of finitely generated $\mathfrak{o}_0$-modules with a continuous $\boldsymbol{\mathcal{G}}_K$-action, and $\mathrm{Rep}_{F_0}(\boldsymbol{\mathcal{G}}_K)$ the category of F_0-representations of $\boldsymbol{\mathcal{G}}_K$. The full subcategories of $\mathrm{Rep}_{\mathfrak{o}_0}(\boldsymbol{\mathcal{G}}_K)$ and $\mathrm{Rep}_{F_0}(\boldsymbol{\mathcal{G}}_K)$ whose objects are formal of $\mathcal{P}$-height $\leqslant h$ are closed under subobjects, quotients and direct sums. Therefore, the natural inclusion $\mathscr{D}^{f,\leqslant h}_{\rho_{\mathbb{F}}} \subset \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ of $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$-groupoids is relatively representable by surjections of rings.*

For $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$ and $R \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$, let R^f be the universal quotient of R which represents the inclusion $\mathscr{D}^{f,\leqslant h}_{\rho_{\mathbb{F}}} \subset \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$. Then the subscheme $\mathrm{Spec}(R^f \otimes_{\mathfrak{o}} F) \subset \mathrm{Spec}(R \otimes_{\mathfrak{o}} F)$ is open and closed.

[14]The "if" direction is still true when $p = 2$ by [53].

Assume that $\mathscr{K}/\mathscr{K}_0$ is separable (which is automatic if $\mathfrak{o}_0 = \mathbb{Z}_p$) and let $\mathbf{v}$ be any E-Hodge-Pink type for some finite extension E/F. Let $\operatorname{Spec} R^{\mathbf{v}}$ denote the union of connected components of $\operatorname{Spec}(R \otimes_{\mathfrak{o}} E)$ whose closed points have Hodge-Pink type $\mathbf{v}$. (Such a quotient $R_{\mathbf{v}}$ exists by Proposition 11.3.7.) It follows from the proposition above that there exists an open and closed subscheme $\operatorname{Spec} R^{f,\mathbf{v}} \subset \operatorname{Spec} R^{\mathbf{v}}$ whose closed points corresponds to formal $\boldsymbol{\mathcal{G}}_K$-representation of Hodge-Pink type $\mathbf{v}$.

Proof. The first claim is reduced to the fact that φ-nilpotentness of φ-modules is closed under subobjects, quotients, and direct sums, by a schematic closure argument similar to Proposition 9.2.2. (The claims for formal F_0-representations are reduced to the claims for formal lattice $\mathfrak{o}_0$-representations.) Applying Ramakrishna's relative representability criterion [68], it follows that the natural inclusion $\mathscr{D}^{f,\leqslant h}_{\rho_{\mathbb{F}}} \subset \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ is relatively representable by surjections of rings.

It is left to show that the map $\operatorname{Spec}(R^f \otimes_{\mathfrak{o}} F) \hookrightarrow \operatorname{Spec}(R \otimes_{\mathfrak{o}} F)$ is formally étale at each closed point (since $R \otimes_{\mathfrak{o}} F$ is Jacobson). Put $V_\xi := T_\xi[\frac{1}{\pi_0}]$ where T_ξ is the representation space which corresponds to ξ. Let A be a finite artin local F-algebra, let $I \subset A$ be a square-zero ideal, and put $\bar{A} := A/I$. Let us fix an A-point $x : R \to A$ such that $\bar{x} : R \xrightarrow{x} A \twoheadrightarrow \bar{A}$ factors through R^f. Set $V_x := V_\xi \otimes_{R,x} A$ and $V_{\bar{x}} := V_\xi \otimes_{R,\bar{x}} \bar{A}$. Then we have a short exact sequence $0 \to V_{\bar{x}} \otimes_{\bar{A}} I \to V_x \to V_{\bar{x}} \to 0$, Now it follows that V_x is formal, being an extension of formal $\boldsymbol{\mathcal{G}}_K$-representations. In other words, x factors through R^f. □

Proposition 11.4.5. *The natural inclusion* $\mathscr{D}^{u,\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}} \hookrightarrow \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}$ *is open and closed. In particular, for any* $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$ *and* $R \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$, *there exists a universal open and closed immersion* $\mathscr{GR}^{u,\leqslant h}_{\xi} \hookrightarrow \mathscr{GR}^{\leqslant h}_{\xi}$ *of* R*-schemes which represents the fully faithful inclusion* $\mathscr{D}^{u,\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi} \hookrightarrow \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}$, *where* $\mathscr{D}^{u,\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi} := (\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \times_{\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}} \mathscr{D}^{u,\leqslant h}_{\rho_{\mathbb{F}}}$. *Furthermore* *the composition* $\mathscr{GR}_\xi^{u,\leqslant h} \hookrightarrow \mathscr{GR}_\xi^{\leqslant h} \to \operatorname{Spec} R$ *factors through* $\operatorname{Spec} R^f$ *and induces an isomorphism* $\mathscr{GR}_\xi^{u,\leqslant h} \otimes_{\mathfrak{o}} F \to \operatorname{Spec}(R^f \otimes_{\mathfrak{o}} F)$*.*

From the proposition above, one can immediately deduce that an A-point $\mathfrak{M}_A \in \mathscr{GR}_\xi^{\leqslant h}(A)$ (with A finite over F) is supported in $\mathscr{GR}_\xi^{u,\leqslant h}$ if and only if $\mathfrak{M}_A$ is unipotent of $\mathcal{P}$-height $\leqslant h$ (i.e., $\mathfrak{M}_A$ allows a $\mathfrak{S}_{A^o}$-lattice $\mathfrak{M}_{A^o} \subset \mathfrak{M}_A$ which is "unipotent" of $\mathcal{P}$-height $\leqslant h$, where $A^o \subset A$ is a finite flat $\mathfrak{o}$-subalgebra with $A^o[\frac{1}{\pi_0}] = A$).

Proof. Let us first show that $\mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{u,\leqslant h} \hookrightarrow \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h}$ is open and closed. Consider $\mathfrak{M}_A \in \mathscr{D}_{\mathfrak{S},M_{\mathbb{F}}}^{\leqslant h}(A,I)$ for $(A,I) \in \mathfrak{Aug}_{\mathfrak{o}}$. Let $\operatorname{Spec} A^f \subset \operatorname{Spec} A$ be the locus where the rank of maximal Lubin-Tate quotient $d_{\mathcal{LT}}$ is zero, which is open and closed by Proposition 11.4.2 applied to connected components of $\operatorname{Spec} A$. Thus, there is a unique union of connected components $\widehat{\mathscr{GR}_\xi}^{u,\leqslant h} \subset \widehat{\mathscr{GR}_\xi}^{\leqslant h}$ such that its functorial points are exactly the "unipotent points" of $\mathscr{GR}_\xi^{\leqslant h}$. Now clearly $\widehat{\mathscr{GR}_\xi}^{u,\leqslant h}$ is obtained from $\mathfrak{m}_R$-adic completion of an open and closed subscheme $\mathscr{GR}_\xi^{u,\leqslant h} \subset \mathscr{GR}_\xi^{\leqslant h}$. The last claim in the proposition readily follows from the definition of formal $\boldsymbol{\mathcal{G}}_K$-representations of $\mathcal{P}$-height $\leqslant h$ and the structure morphism $\mathscr{GR}_\xi^{\leqslant h} \otimes_{\mathfrak{o}} F \to \operatorname{Spec}(R \otimes_{\mathfrak{o}} F)$ being isomorphic. $\square$

11.4.6 Definitions: ordinary $\boldsymbol{\mathcal{G}}_K$-representations

Let A be one of the following: $A \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$, $(A,I) \in \mathfrak{Aug}_{\mathfrak{o}}$, or a finite F-algebra A. A (continuous) A-representation V_A of $\boldsymbol{\mathcal{G}}_K$ of $\mathcal{P}$-height $\leqslant h$ is said to be *ordinary* if there exists a $\boldsymbol{\mathcal{G}}_K$-stable A-submodule $L_A \subset V_A$ such that V_A/L_A is a projective A-module (of constant rank) and both V_A/L_A and $L_A(-h)$ are unramified representations[15]. In other words, V_A is an extension of an unramified representation by a Lubin-Tate type representation of $\mathcal{P}$-height h. For a complete local noetherian $\mathfrak{o}$-algebra A,

[15] If $\mathfrak{o}_0 = \mathbb{Z}_p$, then $\boldsymbol{\mathcal{G}}_K \cong \boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ acts on $\mathbb{Z}_p(1)$ via the restriction of the p-adic cyclotomic character; and if $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then $\boldsymbol{\mathcal{G}}_K$ acts on $\mathbb{F}_q[[\pi_0]](1)$ via the Lubin-Tate character.

the ordinary-ness for V_A is equivalent to the ordinary-ness for each $V_A \otimes_A (A/\mathfrak{m}_A^n)$. If A is a finite F_0-algebra, then the ordinary-ness for V_A is equivalent to requiring the existence of an ordinary "$\boldsymbol{\mathcal{G}}_K$-stable A^o-lattice" for some finite flat $\mathfrak{o}_0$-subalgebra $A^o \subset A$ by Lemma 11.4.7 below.

Let A be either an object in $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ or $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$ for some ideal I with $\operatorname{Spec} A$ connected. By Proposition 11.4.2 any $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ admits the maximal étale subobject $\mathfrak{M}_A^{\text{ét}} \subset \mathfrak{M}_A$ and the maximal φ-nilpotent quotient $\mathfrak{M}_A/\mathfrak{M}_A^{\text{ét}}$, which are both objects in $(\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$. In other words, there exists a "connected-étale sequence" for any $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$. We say that $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ is *ordinary* if the maximal φ-nilpotent quotient $\mathfrak{M}_A/\mathfrak{M}_A^{\text{ét}}$ is of Lubin-Tate type of $\mathcal{P}$-height h. When $A \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$, the ordinary-ness of $\mathfrak{M}_A$ is equivalent to the ordinary-ness of $\mathfrak{M}_A \otimes_A (A/\mathfrak{m}_A^n)$ for each n. For a finite F-algebra A, we say $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$ is *ordinary* if there exists a finite flat $\mathfrak{o}$-subalgebra $A^o \subset A$ and $\mathfrak{M}_{A^o} \in (\mathrm{ModFI}\,/\mathfrak{S})_{A^o}^{\leqslant h}$ such that $\mathfrak{M}_A = \mathfrak{M}_{A^o}[\frac{1}{\pi_0}]$ and $\mathfrak{M}_{A^o}$ is ordinary. The ordinary-ness is stable under the duality of $\mathcal{P}$-height h.

Let A be either a complete local noetherian $\mathfrak{o}_0$-algebra or a finite F_0-algebra, and consider $\mathfrak{M}_A \in (\mathrm{ModFI}\,/\mathfrak{S})_A^{\leqslant h}$. Then we can see that $\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A)$ is ordinary as a $\boldsymbol{\mathcal{G}}_K$-representation if and only $\mathfrak{M}_A$ is ordinary. (The 'if" direction of the case when A is F_0-finite uses Lemma 11.4.7 below and Theorem 5.2.3.)

Lemma 11.4.7. *Let A be a finite F_0-algebra. Consider a short exact sequence $V^\bullet$: $0 \to V'_A \to V_A \to V''_A \to 0$ of finite free A-modules with continuous $\boldsymbol{\mathcal{G}}_K$-action and assume that all the maps are $\boldsymbol{\mathcal{G}}_K$-equivariant. Then there exists a finite flat $\mathfrak{o}$-subalgebra $A^o \subset A$ with $A^o[\frac{1}{\pi_0}] = A$, and $\boldsymbol{\mathcal{G}}_K$-stable A^o-lattices $T'_{A^o} \subset V'_A$, $T_{A^o} \subset V_A$, and $T''_{A^o} \subset V''_A$, such that the short exact sequence $V^\bullet$ restricts to a short exact sequence $0 \to T'_{A^o} \to T_{A^o} \to T''_{A^o} \to 0$.*

Proof. We modify the argument in [50, Proposition 9.5], at the bottom of page 433. We may assume that A is local, and let E be its residue field. Let A^+ be the preimage of $\mathfrak{o}_E$ under the natural projection $A \twoheadrightarrow E$. Note that A^+ is a rising union of finite flat $\mathfrak{o}_0$-subalgebras $A^o \subset A$. Since the claim is clear when $A = E$ (by taking $A^o := \mathfrak{o}_E$), we may choose an A-basis $\{\mathbf{e}_1, \cdots, \mathbf{e}_{r'}, \mathbf{e}_{r'+1}, \cdots, \mathbf{e}_{r'+r''}\}$ of V_A such that $\{\mathbf{e}_1, \cdots, \mathbf{e}_{r'}\}$ is an A-basis for the image of V'_A in V_A, $\{\mathbf{e}_{r'+1}, \cdots, \mathbf{e}_{r'+r''}\}$ reduces to an A-basis for V''_A, and the image of $\{\mathbf{e}_1, \cdots, \mathbf{e}_{r'+r''}\}$ in $V \otimes_A E$ generates a $\boldsymbol{\mathcal{G}}_K$-stable $\mathfrak{o}_E$-lattice (i.e., the A^+-span of $\{\mathbf{e}_1, \cdots, \mathbf{e}_{r'+r''}\}$ is $\boldsymbol{\mathcal{G}}_K$-stable). By compactness of $\boldsymbol{\mathcal{G}}_K$, the image of $\boldsymbol{\mathcal{G}}_K$ in $\mathrm{GL}_{r'+r''}(A^+)$ has to lie in $\mathrm{GL}_{r'+r''}(A^o)$ for some finite flat $\mathfrak{o}_0$-subalgebra $A^o \subset A^+$. Now, we put $T_{A^o} \subset V_A$ be the A^o-span of $\{\mathbf{e}_1, \cdots, \mathbf{e}_{r'+r''}\}$, $T'_{A^o} \subset V'_A$ the A^o-span of $\{\mathbf{e}_1, \cdots, \mathbf{e}_{r'}\}$, and $T''_{A^o} \subset V''_A$ the A^o-span of $\{\mathbf{e}_{r'+1}, \cdots, \mathbf{e}_{r'+r''}\}$. □

11.4.8

Let $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$ for some $R \in \widehat{\mathfrak{AR}_{\mathfrak{o}}}$. We now show that the "ordinary-ness condition" cuts out a union of connected components in $\mathscr{GR}^{\leqslant h,d}_{\xi}$. Choose non-negative integers $\mathbf{d} := \{d_{\text{ét}}, d_{\mathcal{LT}}\}$ such that $d_{\text{ét}} + d_{\mathcal{LT}} \leq n := \dim_{\mathbb{F}}(\rho_{\mathbb{F}})$. We define a full $\mathfrak{Aug}_{\mathfrak{o}}$-subgroupoid $\mathscr{D}^{\leqslant h,\mathbf{d}}_{\mathfrak{S},M_{\mathbb{F}}} \subset \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}$ such that $\mathfrak{M}_A \in \mathscr{D}^{\leqslant h,\mathbf{d}}_{\mathfrak{S},M_{\mathbb{F}}}(A, I)$ for $(A, I) \in \mathfrak{Aug}_{\mathfrak{o}}$ if and only if $\mathfrak{M}^{\text{ét}}_A$ is of $\mathfrak{S}_A$-rank $d_{\text{ét}}$ and $\mathfrak{M}^{\mathcal{LT}}_A$ is of $\mathfrak{S}_A$-rank $d_{\mathcal{LT}}$. This *is* a $\mathfrak{Aug}_{\mathfrak{o}}$-subgroupoid by Proposition 11.4.2, especially by (2). If $d = d_{\text{ét}}$ and $d_{\text{ét}} + d_{\mathcal{LT}} = n$, then we put $\mathscr{D}^{\leqslant h,d,\text{ord}}_{\mathfrak{S},M_{\mathbb{F}}} := \mathscr{D}^{\leqslant h,\mathbf{d}}_{\mathfrak{S},M_{\mathbb{F}}}$; i.e., $\mathfrak{M}_A \in \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}$ is an object in $\mathscr{D}^{\leqslant h,d,\text{ord}}_{\mathfrak{S},M_{\mathbb{F}}}(A, I)$ if and only if it is an extension of a Lubin-Tate type object of rank $n - d$ by an étale object of rank d.

For any $R \in \widehat{\mathfrak{AR}_{\mathfrak{o}}}$ and $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$, the 2-fiber product $\mathscr{D}^{\leqslant h,\mathbf{d}}_{\mathfrak{S},M_{\mathbb{F}},\xi} := (\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \times_{\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}} \mathscr{D}^{\leqslant h,\mathbf{d}}_{\mathfrak{S},M_{\mathbb{F}}}$ is a full $\mathfrak{Aug}_R$-subgroupoid of $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}$. By Corollary 11.1.11, $\mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}},\xi}$ can be represented by a projective R-scheme $\mathscr{GR}^{\leqslant h}_{\xi}$.

Proposition 11.4.9. *The full* $\mathfrak{Aug}_R$*-subgroupoid* $\mathscr{D}^{\leqslant h,\mathbf{d}}_{\mathfrak{S},M_{\mathbb{F}},\xi}$ *is representable by an open and closed* R*-subscheme* $\mathscr{GR}^{\leqslant h,\mathbf{d}}_{\xi}$ *of* $\mathscr{GR}^{\leqslant h}_{\xi}$*. As a special case, the full* $\mathfrak{Aug}_R$*-subgroupoid* $\mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{S},M_{\mathbb{F}}}$ *can be represented by an open and closed* R*-subscheme* $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}_{\xi}$ *of* $\mathscr{GR}^{\leqslant h}_{\xi}$*.*

Proof. The proof is essentially identical to the proof of Proposition 11.4.5. Consider $\mathfrak{M}_A \in \mathscr{D}^{\leqslant h}_{\mathfrak{S},M_{\mathbb{F}}}(A,I)$ for $(A,I) \in \mathfrak{Aug}_{\mathfrak{o}}$. Let $\operatorname{Spec} A^{\mathbf{d}} \subset \operatorname{Spec} A$ be the locus where the rank of $\mathfrak{M}_A^{\mathcal{LT}}$ is $d_{\mathcal{LT}}$ and the rank of $\mathfrak{M}_A^{\text{ét}}$ is $d_{\text{ét}}$, which is open and closed by Proposition 11.4.2 applied to connected components of $\operatorname{Spec} A$. Thus, there is a unique union of connected components $\widehat{\mathscr{GR}}^{\leqslant h,\mathbf{d}}_{\xi} \subset \widehat{\mathscr{GR}}^{\leqslant h}_{\xi}$ such that its functorial points are exactly the points of $\mathscr{GR}^{\leqslant h}_{\xi}$ with the "condition $\mathbf{d}$" on the ranks of a maximal étale subobject and a maximal Lubin-Tate type quotient. Now clearly $\widehat{\mathscr{GR}}^{\leqslant h,\mathbf{d}}_{\xi}$ is obtained from $\mathfrak{m}_R$-adic completion of an open and closed subscheme $\mathscr{GR}^{\leqslant h,\mathbf{d}}_{\xi} \subset \mathscr{GR}^{\leqslant h}_{\xi}$. □

Let $\operatorname{Spec}(R[\frac{1}{\pi_0}])^{\mathbf{d}} \subset \operatorname{Spec} R[\frac{1}{\pi_0}]$ be the union of connected components which is the image of $\mathscr{GR}^{\leqslant h,\mathbf{d}}_{\xi}$ under the structure morphism $\mathscr{GR}^{\leqslant h}_{\xi} \otimes_{\mathfrak{o}} F \xrightarrow{\sim} \operatorname{Spec} R[\frac{1}{\pi_0}]$. We put $(R[\frac{1}{\pi_0}])^{d,\mathrm{ord}} := (R[\frac{1}{\pi_0}])^{\mathbf{d}}$ with $d = d_{\text{ét}}$ and $d_{\mathcal{LT}} = n - d$ (where $n = \dim_{\mathbb{F}} \rho_{\mathbb{F}}$). From the discussion in §11.4.6 and the proposition above, we can easily deduce that an $\mathfrak{o}$-map $R[\frac{1}{\pi_0}] \to A$ (with A finite over F) factors through $(R[\frac{1}{\pi_0}])^{d,\mathrm{ord}}$ if and only if $\xi \otimes_R A$ is ordinary with maximal Lubin-Tate A-subrepresentation of rank d. One can deduce a similar assertion for A-points of $\mathscr{GR}^{\leqslant h,\mathbf{d}}_{\xi}$, $\operatorname{Spec}(R[\frac{1}{\pi_0}])^{\mathbf{d}}$, and $\mathscr{GR}^{u,\leqslant h}_{\xi}$ (with A finite over F).

We will often apply this discussion to $R = R^{\square,\leqslant h}_{\rho_{\mathbb{F}}}$ and $R = R^{\leqslant h}_{\rho_{\mathbb{F}}}$, in which case we respectively write $R^{\square,\leqslant h,d,\mathrm{ord}}_{\rho_{\mathbb{F}}}$ and $R^{\leqslant h,d,\mathrm{ord}}_{\rho_{\mathbb{F}}}$ for $R^{d,\mathrm{ord}}$.

Remark 11.4.10. Consider an F_0-Hodge-Pink type $\mathbf{v}_d := \big(n, (\mathfrak{S}_F/\mathcal{P}(u))^{\oplus d}\big)$. (If $\mathscr{K}/\mathscr{K}_0$ is separable then fixing the Hodge-Pink type $\mathbf{v}_d$ is equivalent by Lemma 11.3.5(1) to fixing the filtration $\mathrm{Fil}^{\bullet}_{\mathbf{v}_d}$ of $\mathscr{K}^{\oplus n}$ given as follows: $\dim_{\mathscr{K}}(\mathrm{gr}^0_{\mathbf{v}_d}) = n - d$,

$\dim_{\mathscr{K}}(\mathrm{gr}^h_{\mathbf{v}_d}) = d$, and $\dim_{\mathscr{K}}(\mathrm{gr}^w_{\mathbf{v}_d}) = 0$ if $w \neq 0, h$.) We claim that for a finite F-algebra A, an A-point $\zeta_A \in \mathscr{GR}^{\mathbf{v}_d}_{\xi}(A)$ factors through $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}_{\xi} \otimes_{\mathfrak{o}} F$ if and only if the corresponding $\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$ is ordinary with F_0-Hodge-Pink type $\mathbf{v}_d$. In particular, if $\mathscr{K}/\mathscr{K}_0$ is separable then $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}_{\xi} \otimes_{\mathfrak{o}} F$ is contained in $\mathscr{GR}^{\mathbf{v}_d}_{\xi}$ (as a union of connected components).

In fact, we have the following general claim. Let A be one of the following: complete local noetherian $\mathfrak{o}_0$-algebra with finite residue field, an $\mathfrak{o}_0$-algebra with $\pi_0^N \cdot A = 0$, and a finite F_0-algebra. For any $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ of *Lubin-Tate type of $\mathcal{P}$-height h* (as defined in §8.3.5), the image of $\varphi_{\mathfrak{M}_A}$ is precisely $\mathcal{P}(u)^h\mathfrak{M}_A$. If $\mathfrak{M}_A$ fits in the following short exact sequence $0 \to \mathfrak{M}_A^{\text{ét}} \to \mathfrak{M}_A \to \mathfrak{M}_A^{\mathcal{LT}} \to 0$ where $\mathfrak{M}_A^{\text{ét}}$ is an finite free étale $(\varphi, \mathfrak{S}_A)$-module and $\mathfrak{M}_A^{\mathcal{LT}} \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ is of Lubin-Tate type of $\mathcal{P}$-height h, then we have an $\mathfrak{S}_A/(\mathcal{P}(u)^h)$-isomorphism $\mathrm{coker}(\varphi_{\mathfrak{M}_A}) \cong \mathfrak{M}_A^{\mathcal{LT}}/\mathcal{P}(u)^h\mathfrak{M}_A^{\mathcal{LT}} \cong (\mathfrak{S}_A/(\mathcal{P}(u)^h))^{\oplus d_{\mathcal{LT}}}$ where the second isomorphism is obtained by choosing $\mathfrak{S}_A$-basis for $\mathfrak{M}_A^{\mathcal{LT}}$ (where $d_{\mathcal{LT}}$ is the $\mathfrak{S}_A$-rank of $\mathfrak{M}_A^{\mathcal{LT}}$).

11.4.11

Let $n := \dim_{\mathbb{F}}(\rho_{\mathbb{F}})$. We choose $\mathfrak{M}_{\mathbb{F}} \in \mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{S},M_{\mathbb{F}}}(\mathbb{F})$; i.e., $\mathfrak{M}_{\mathbb{F}}$ is a $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$ for $\underline{D}^{\leqslant h}_{\mathcal{E}}(\rho_{\mathbb{F}})$ such that $\mathfrak{M}_{\mathbb{F}}^{\text{ét}}$ is of rank d and $\mathfrak{M}_{\mathbb{F}}/\mathfrak{M}_{\mathbb{F}}^{\text{ét}}$ is of Lubin-Tate type of $\mathcal{P}$-height h. We consider an $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}}$ whose objects over A are $(\mathfrak{M}_A, \iota_A)$, where $\mathfrak{M}_A \in \mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{S},M_{\mathbb{F}}}(A)$ and $\iota_A : \mathfrak{M}_A \otimes_A A/\mathfrak{m}_A \xrightarrow{\sim} \mathfrak{M}_{\mathbb{F}}$. There is a natural 1-morphism $\mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}} \to \mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{S},M_{\mathbb{F}}}$, defined by forgetting ι_A. If $\mathrm{End}_{\mathcal{G}_K}(\rho_{\mathbb{F}}) \cong \mathbb{F}$, then $\mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}}$ is pro-representable by the completed local ring of $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}$ at the closed point corresponding to $\mathfrak{M}_{\mathbb{F}}$. In general, the 2-fiber product $\mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}},\xi} := (\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \times_{\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}} \mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}}$ is pro-representable by the completed local ring of $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}_{\xi}$ at the closed point corresponding to $\mathfrak{M}_{\mathbb{F}}$. By increasing $\mathbb{F}$, we obtain the completed local ring of $\mathscr{GR}^{\leqslant h,d}_{\xi}$ at any closed point.

There is an $\mathfrak{S}_{\mathbb{F}}/(\mathcal{P}(u)^h)$-isomorphism $\beta_{\mathbb{F}} : (\mathfrak{S}_{\mathbb{F}}/\mathcal{P}(u)^h)^{\oplus n} \xrightarrow{\sim} \mathfrak{M}_{\mathbb{F}}/\mathcal{P}(u)^h\mathfrak{M}_{\mathbb{F}}$ by Remark 11.4.10, and we choose one. We define an $\mathfrak{AR}_{\mathfrak{o}}$-groupoid[16] $\widetilde{\mathscr{D}}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}}$, where an object over A is $\mathfrak{M}_A \in \mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}}(A)$ together with an $\mathfrak{S}_A/\mathcal{P}(u)^h$-isomorphism $\beta_A : (\mathfrak{S}_A/\mathcal{P}(u)^h)^{\oplus n} \xrightarrow{\sim} \mathfrak{M}_A/\mathcal{P}(u)^h\mathfrak{M}_A$ which lifts $\beta_{\mathbb{F}}$. (Note that such an isomorphism exists by Remark 11.4.10.) By forgetting this isomorphism, we obtain a 1-morphism $\widetilde{\mathscr{D}}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}} \to \mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}}$, which makes the former into a torsor under the formal completion of the Weil restriction $\mathrm{Res}^{\mathfrak{S}_{\mathfrak{o}}/\mathcal{P}(u)^h}_{\mathfrak{o}} \mathrm{GL}_n$ at the identity section. In particular, this 1-morphism is formally smooth.

Now, we define another $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\mathrm{Gr}^{\leqslant h}(n,d)$ whose objects are quotients of $(\mathfrak{S}_A/\mathcal{P}(u)^h)^{\oplus n}$ which are free of $\mathfrak{S}_A/\mathcal{P}(u)^h$-rank d. This groupoid is representable by (the π_0-adic completion of) a grassmannian for $\mathrm{Res}^{\mathfrak{S}_{\mathfrak{o}}/\mathcal{P}(u)^h}_{\mathfrak{o}} \mathrm{GL}_n$, which is a smooth formal $\mathfrak{o}$-scheme. We have a 1-morphism $\widetilde{\mathscr{D}}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}} \to \mathrm{Gr}^{\leqslant h}(n,d)$ by sending $(\mathfrak{M}_A, \beta_A)$ to the composite $(\mathfrak{S}_A/\mathcal{P}(u)^h)^{\oplus n} \underset{\beta_A}{\xrightarrow{\sim}} \mathfrak{M}_A/\mathcal{P}(u)^h\mathfrak{M}_A \xrightarrow{\varphi} \operatorname{coker}\varphi$. We now show that this 1-morphism is formally smooth, as follows. Let $A \in \mathfrak{AR}_{\mathfrak{o}}$ and let $I \subset A$ be a square-zero ideal. Put $\bar{A} := A/I$. Let $(\mathfrak{M}_{\bar{A}}, \beta_{\bar{A}}) \in \widetilde{\mathscr{D}}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}}(\bar{A})$, and we put $(\mathfrak{S}_{\bar{A}}/\mathcal{P}(u)^h)^{\oplus n} \twoheadrightarrow \bar{\Lambda}^d_{\bar{A}}$ be the corresponding point in $\mathrm{Gr}^{\leqslant h}(n,d)(\bar{A})$ and fix a lift $(\mathfrak{S}_A/\mathcal{P}(u)^h)^{\oplus n} \twoheadrightarrow \bar{\Lambda}^d_A$. We put $\mathfrak{M}_A := \mathfrak{S}_A^{\oplus n}$ and choose $\mathfrak{M}_A \twoheadrightarrow \mathfrak{M}_{\bar{A}}$ which lifts $\beta_{\bar{A}}$. Now, we can give φ on $\mathfrak{M}_A$ by choosing a lift of $\bar{\varphi} := \varphi_{\mathfrak{M}_{\bar{A}}}$ in the commutative diagram below with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \sigma^*\mathfrak{M}_A & \overset{\varphi}{\dashrightarrow} & \mathfrak{M}_A & \longrightarrow & \bar{\Lambda}^d_A & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \sigma^*\mathfrak{M}_{\bar{A}} & \xrightarrow{\bar{\varphi}} & \mathfrak{M}_{\bar{A}} & \longrightarrow & \bar{\Lambda}^d_{\bar{A}} & \longrightarrow & 0. \end{array}$$

Now, let $\mathfrak{M}^{\text{ét}}_A$ and $\mathfrak{M}^{\mathcal{LT}}_A$ be a maximal étale subobject and a maximal Lubin-Tate type quotient of $\mathfrak{M}_A$, respectively. Since the formation of $\mathfrak{M}^{\text{ét}}_A$ and $\mathfrak{M}^{\mathcal{LT}}_A$ commutes with change of coefficients by Proposition 8.2.7, we have natural isomorphisms $\mathfrak{M}^{\text{ét}}_A \otimes_A \bar{A} \cong$

[16]Here, the tilde in the notation does *not* mean the extension by 2-direct limits, which is defined in §10.4.4.

$\mathfrak{M}^{\text{ét}}_{\bar{A}}$ and $\mathfrak{M}^{\mathcal{LT}}_{A} \otimes_A \bar{A} \cong \mathfrak{M}^{\mathcal{LT}}_{\bar{A}}$. This shows that $\mathfrak{M}_A$ together with the obvious choice of β_A defines an object in $\widetilde{\mathscr{D}}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}}(A)$.

Now, we are ready to prove the following

Proposition 11.4.12. *Assume that* $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ *is formally smooth. Then* $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}_{\xi}$ *is formally smooth over* $\mathfrak{o}$*. In particular,* $\mathscr{GR}^{\square,\leqslant h,d,\mathrm{ord}}$ *is formally smooth over* $\mathfrak{o}$ *and* $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}$ *is formally smooth over* $\mathfrak{o}$ *if* $\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ *is representable.*

Proof. It is enough to show the completed local ring of $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}_{\xi}$ at each closed point is a formally smooth $\mathfrak{o}$-algebra. Now consider the following diagrams where all the arrows are formally smooth.

$$\begin{array}{ccccc} \widetilde{\mathscr{D}}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}},\xi} & \longrightarrow & \widetilde{\mathscr{D}}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}} & \longrightarrow & \mathrm{Gr}^{\leqslant h}(n,d) \\ \downarrow & & \downarrow & & \\ \mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}},\xi} & \longrightarrow & \mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}}} & & \end{array}$$

The two horizontal arrows in the square are formally smooth since they are 2-pull back of the formally smooth 1-morphism $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ and the formal smoothness pulls back under 2-base changes (Proposition 10.2.8). Since the square is 2-cartesian and the right vertical arrow is formally smooth, the left vertical arrow is because the formal smoothness pulls back under 2-base changes. Finally, we have seen that $\mathrm{Gr}^{\leqslant h}(n,d)$ is a smooth formal $\mathfrak{o}$-scheme, and $\mathscr{D}^{\leqslant h,d,\mathrm{ord}}_{\mathfrak{M}_{\mathbb{F}},\xi}$ is prorepresentable by the completed local ring of $\mathscr{GR}^{\leqslant h,d}_{\xi}$ at the closed point which corresponds to $\mathfrak{M}_{\mathbb{F}}$. $\square$

Let $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}_{\xi,0}$ be the fiber over the closed point of $\operatorname{Spec} R$ under $\underline{T}^{\leqslant h}_{\mathfrak{S}} : \mathscr{GR}^{\leqslant h,d,\mathrm{ord}}_{\xi} \to \operatorname{Spec} R$. The following corollary shows that distinct connected components of the generic fiber $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}_{\xi} \otimes_{\mathfrak{o}} F$ "reduce" to distinct connected components of $\mathscr{GR}^{\leqslant h,d,\mathrm{ord}}_{\xi,0}$. We let $H_0(X)$ denote the set of connected components of X.

Corollary 11.4.13. *We keep the assumption that* $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ *is formally smooth. Then the natural maps below*

$$H_0(\mathscr{G}\mathscr{R}^{\leqslant h,d,\mathrm{ord}}_{\xi} \otimes_{\mathfrak{o}} F) \to H_0(\mathscr{G}\mathscr{R}^{\leqslant h,d,\mathrm{ord}}_{\xi}) \leftarrow H_0(\mathscr{G}\mathscr{R}^{\leqslant h,d,\mathrm{ord}}_{\xi,0})$$

are bijective.

Proof. The first bijection is clear from the formal smoothness of $\mathscr{G}\mathscr{R}^{\leqslant h,d,\mathrm{ord}}_{\xi}$. By the theorem on formal functions, the natural map $H_0(\widehat{\mathscr{G}\mathscr{R}}^{\leqslant h,d,\mathrm{ord}}_{\xi}) \to H_0(\mathscr{G}\mathscr{R}^{\leqslant h,d,\mathrm{ord}}_{\xi})$ is a bijection, where $\widehat{\mathscr{G}\mathscr{R}}^{\leqslant h,d,\mathrm{ord}}_{\xi}$ is the $\mathfrak{m}_R$-adic completion. Since $\widehat{\mathscr{G}\mathscr{R}}^{\leqslant h,d,\mathrm{ord}}_{\xi}$ and $\mathscr{G}\mathscr{R}^{\leqslant h,d,\mathrm{ord}}_{\xi,0}$ have the same underlying topological space, we have the second bijection. □

11.4.14 Rank-2 example

We assume that $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ is formally smooth, so $\mathscr{G}\mathscr{R}^{\leqslant h,d,\mathrm{ord}}_{\xi}$ is formally smooth over $\mathfrak{o}$. Thus, to compute the connected components of $\mathscr{G}\mathscr{R}^{\leqslant h,d,\mathrm{ord}}_{\xi} \otimes_{\mathfrak{o}} F$ it is enough to compute the connected component of the fiber $\mathscr{G}\mathscr{R}^{\leqslant h,d,\mathrm{ord}}_{\xi,0}$ over the closed point of $\operatorname{Spec} R$ (by the theorem on formal functions). We now do this computation for the case when that $\dim_{\mathbb{F}}(\rho_{\mathbb{F}}) = 2$ and $d := d_{\text{ét}} = 1$. We let $\mathscr{G}\mathscr{R}^{\leqslant h,\mathrm{ord}}_{\xi}$ denote the ordinary locus of $\mathscr{G}\mathscr{R}^{\leqslant h}_{\xi}$ with $d := d_{\text{ét}} = 1$. When $\mathscr{K}/\mathscr{K}_0$ is separable, $\mathscr{G}\mathscr{R}^{\leqslant h,\mathrm{ord}}_{\xi} \otimes_{\mathfrak{o}} F$ is a union of connected components of the $\mathscr{G}\mathscr{R}^{\mathbf{v}}_{\xi}$'s where $\mathbf{v} := (n = 2, \bar{\Lambda}^{\mathbf{v}} := \mathfrak{S}_{F_0}/\mathcal{P}(u)^h)$ (by Remark 11.4.10), and its complement is precisely the "unipotent locus" in $\mathscr{G}\mathscr{R}^{\mathbf{v}}_{\xi}$ (which is open and closed, by Proposition 11.4.5). But even when $\mathscr{K}/\mathscr{K}_0$ is not separable, any A-point of $\mathscr{G}\mathscr{R}^{\leqslant h,\mathrm{ord}}_{\xi} \otimes_{\mathfrak{o}} F$ has F_0-Hodge-Pink type $\mathbf{v} := (n = 2, \bar{\Lambda}^{\mathbf{v}} := \mathfrak{S}_{F_0}/\mathcal{P}(u)^h)$ by Remark 11.4.10. In this sense, the following discussion is the continuation of §11.3.12.

We now set up some notations. Let $\chi_{\mathcal{LT}} : \boldsymbol{\mathcal{G}}_K \to \mathbb{F}_q^{\times}$ be the character by which $\boldsymbol{\mathcal{G}}_K$ acts on $\mathbb{F}_q(1) := \underline{T}^*_{\mathfrak{S}}(\mathfrak{S}(1)) \otimes_{\mathfrak{o}_0} \mathbb{F}_q$. If $\mathfrak{o}_0 = \mathbb{Z}_p$, then $\chi_{\mathcal{LT}}$ is the restriction of the p-adic cyclotomic character to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty} \cong \boldsymbol{\mathcal{G}}_K$. (Note that $q = p$ in this case.) If $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, then $\chi_{\mathcal{LT}}$ is obtained from the π_0-torsion points of the Lubin-Tate formal group. For an unramified character ψ, let $\mathfrak{M}_\psi$ denote the unique Lubin-Tate type φ-module over $\mathfrak{S}_{\mathbb{F}}$ of $\mathcal{P}$-height h such that $\underline{T}^{\leqslant h}(\mathfrak{M}_\psi) \cong \psi$. (So the Tate twist $\mathfrak{M}_\psi(-h)$ is an étale $(\varphi, \mathfrak{S}_{\mathbb{F}})$-module such that $\boldsymbol{\mathcal{G}}_K$ acts on $\underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}_\psi(-h))$ via $\psi\chi^h_{\mathcal{LT}}$.)

Proposition 11.4.15. *Assume that $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ is formally smooth. If $\mathscr{GR}^{\leqslant h,\mathrm{ord}}_{\xi,0}$ is non-empty then it consists of a single point, unless $\rho_{\mathbb{F}} \cong \left(\begin{smallmatrix}\psi_1 & 0\\ 0 & \psi_2\end{smallmatrix}\right)$ where both ψ_1 and ψ_2 are unramified (so necessarily $\chi_{\mathcal{LT}}$ is unramified). In the latter case, we have two possibilities:*

1. *If $\psi_1 \neq \psi_2$ then $\mathscr{GR}^{\leqslant h,\mathrm{ord}}_{\xi,0}$ consists of two (reduced) points which correspond to*

$$\left(\mathfrak{M}_{\psi_1\chi^{-h}_{\mathcal{LT}}}\right)(-h)\oplus\mathfrak{M}_{\psi_2}, \quad \text{and} \quad \left(\mathfrak{M}_{\psi_2\chi^{-h}_{\mathcal{LT}}}\right)(-h)\oplus\mathfrak{M}_{\psi_1}, \quad \text{respectively.} \tag{11.4.15.1}$$

2. *If $\psi = \psi_1 = \psi_2$ then any $\mathfrak{M}_{\mathbb{F}} \in \mathscr{GR}^{\leqslant h,\mathrm{ord}}_{\xi,0}(\mathbb{F})$ is of the following form:*

$$\mathfrak{M}_{\mathbb{F}} \cong \left(\mathfrak{M}_{\psi\chi^{-h}_{\mathcal{LT}}}\right)(-h) \oplus \mathfrak{M}_\psi. \tag{11.4.15.2}$$

   Furthermore, we have a natural isomorphism $\mathscr{GR}^{\leqslant h,\mathrm{ord}}_{\xi,0} \xrightarrow{\sim} \mathbb{P}^1_{\mathbb{F}}$ of $\mathbb{F}$-schemes, and this sends $\mathfrak{M}_{\mathbb{F}} \in \mathscr{GR}^{\leqslant h,\mathrm{ord}}_{\xi,0}(\mathbb{F})$ to $L_{\mathbb{F}} := \underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}^{\text{ét}}_{\mathbb{F}}) \subset T_{\mathbb{F}}$ (which defines an $\mathbb{F}$-point of $\mathbb{P}^1_{\mathbb{F}}$), where $T_{\mathbb{F}} := \underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}_{\mathbb{F}})$ is the representation space for $\rho_{\mathbb{F}}$. Note that under the isomorphism (11.4.15.2)*, we have $\mathfrak{M}^{\text{ét}}_{\mathbb{F}} = \left(\mathfrak{M}_{\psi\chi^{-h}_{\mathcal{LT}}}\right)(-h)$.*

Proof. Let A be a finite artin local $\mathbb{F}$-algebra, and put $T_A := T_{\mathbb{F}} \otimes_{\mathbb{F}} A$, $M_A := \underline{D}^{\leqslant h}_{\mathcal{E}}(T_A)$. We first consider a point $\mathfrak{M}_A \in \mathscr{GR}^{\leqslant h,\mathrm{ord}}_{\xi,0}(A)$, which is a $\mathfrak{S}_A$-lattice of $\mathcal{P}$-height $\leqslant h$ for M_A. Then we have a short exact sequence

$$0 \to \mathfrak{M}^{\text{ét}}_A \to \mathfrak{M}_A \to \mathfrak{M}^{\mathcal{LT}}_A \to 0.$$

We put $L_A := \underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}^{\text{ét}}_A)$, so the locally free quotient $T_A/L_A \cong \underline{T}^{\leqslant h}_{\mathfrak{S}}(\mathfrak{M}^{\mathcal{LT}}_A)$ is unramified.

Claim. *Let T_A be an A-representation of $\boldsymbol{\mathcal{G}}_K$ of $\mathcal{P}$-height $\leqslant h$ (as an $\mathfrak{o}_0$-torsion $\boldsymbol{\mathcal{G}}_K$-representation in the sense of Definition 8.1.7*[17]*) and we put $M_A := \underline{D}_{\mathcal{E}}^{\leqslant h}(T_A)$. For any $\boldsymbol{\mathcal{G}}_K$-stable A-line $L_A \subset T_A$ (i.e., T_A/L_A is A-projective of constant rank) such that $L_A(-h)$ and T_A/L_A are unramified, there exists a unique $\mathfrak{S}_A$-lattice $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ in M_A such that $\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A^{\text{ét}}) = L_A$ and $\underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A^{\mathcal{LT}}) = T_A/L_A$.*

We first grant this claim and deduce the proposition. The proposition follows straightforwardly from the claim except when $\rho_{\mathbb{F}} \cong \left(\begin{smallmatrix} \psi & 0 \\ 0 & \psi \end{smallmatrix}\right)$, in which case we only have a functorial isomorphism $\mathscr{GR}_{\xi,0}^{\leqslant h,\mathrm{ord}}(A) \cong \mathbb{P}^1_{\mathbb{F}}(A)$ for finite artinian $\mathbb{F}$-algebras A. However, this implies that the smooth proper $\mathbb{F}$-scheme $\mathscr{GR}_{\xi,0}^{\leqslant h,\mathrm{ord}}$ has zeta function coinciding with that of $\mathbb{P}^1_{\mathbb{F}}$, which forces $\mathscr{GR}_{\xi,0}^{\leqslant h,\mathrm{ord}}$ to be a smooth curve of genus 0. Since $\mathbb{F}$ is finite, we have $\mathscr{GR}_{\xi,0}^{\leqslant h,\mathrm{ord}} \cong \mathbb{P}^1_{\mathbb{F}}$.

It remains to show the claim. Consider the étale $\mathfrak{S}$-lattice $\mathfrak{M}_{L_A}^{\text{ét}} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ in $\underline{D}_{\mathcal{E}}^{\leqslant h}(L_A)$, and the Lubin-Tate type $\mathfrak{S}$-lattice $\mathfrak{M}_{T_A/L_A}^{\mathcal{LT}} \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ of $\mathcal{P}$-height h in $\underline{D}_{\mathcal{E}}^{\leqslant h}(T_A/L_A)$. Note that the A-action on L_A induces a φ-compatible A-action on $\mathfrak{M}_{L_A}^{\text{ét}}$ (by functoriality of a maximal $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$), and one can show that this makes $\mathfrak{M}_{L_A}^{\text{ét}}$ a finite free $\mathfrak{S}_A$-module by an argument as sketched in §8.2.4[18]; i.e., $\mathfrak{M}_{L_A}^{\text{ét}} \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$. By duality of $\mathcal{P}$-height h, we also have $\mathfrak{M}_{T_A/L_A}^{\mathcal{LT}} \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$.

Now, we can rephrase the claim that there exists a unique $\mathfrak{S}_A$-lattice $\mathfrak{M}_A$ of $\mathcal{P}$-height $\leqslant h$ in $M_{\mathbb{F}} \otimes_{\mathbb{F}} A$ which is an extension of $\mathfrak{M}_{T_A/L_A}^{\mathcal{LT}}$ by $\mathfrak{M}_{L_A}^{\text{ét}}$ in $(\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$. To show the existence of $\mathfrak{M}_A$ with the required property, consider a maximal $\mathfrak{S}_A$-lattice $\mathfrak{M}_A^+$ of $\mathcal{P}$-height $\leqslant h$ in $M_{\mathbb{F}} \otimes_{\mathbb{F}} A$, so the inclusion $L_A \hookrightarrow T_A$ induces $\mathfrak{M}_{L_A}^{\text{ét}} \hookrightarrow \mathfrak{M}_A^+$.

[17] *A priori*, we do not necessarily have $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})_A^{\leqslant h}$ such that $T_A \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A)$; the definition only guarantees the existence of $\mathfrak{M}_A \in (\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ which may not be a $(\varphi, \mathfrak{S}_A)$-module, such that $T_A \cong \underline{T}_{\mathfrak{S}}^{\leqslant h}(\mathfrak{M}_A)$ as $\mathfrak{o}_0$-torsion $\boldsymbol{\mathcal{G}}_K$-representations.

[18] We briefly recall the argument. Essentially by Corollary 8.1.11, we have $\mathfrak{M}_{L_A}^{\text{ét}} \cong \mathfrak{S} \otimes_W \underline{U}(L_A)$ in $(\mathrm{Mod}/\mathfrak{S})^{\leqslant h}$ which respects the natural A-actions on both sides. (See the proof of Proposition 8.1.10 for the definition of $\underline{U}$.) Now, we repeat the argument in §8.2.4 to show that $\underline{U}(L_A)$ is W_A-free with rank equal to $\mathrm{rank}_A(L_A)$.

Since $\mathfrak{M}^{\text{ét}}_{L_A}$ is a maximal $\mathfrak{S}_A$-lattice in $\underline{D}^{\leqslant h}_{\mathcal{E}}(L_A)$, it follows that $\mathfrak{M}^+_A/\mathfrak{M}^{\text{ét}}_{L_A}$ has no non-zero u-torsion so it is a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ in $\underline{D}^{\leqslant h}_{\mathcal{E}}(T_A/L_A)$. Now, $\mathfrak{M}^+_A/\mathfrak{M}^{\text{ét}}_{L_A}$ contains $\mathfrak{M}^{\mathcal{LT}}_{T_A/L_A}$ because the latter is a minimal $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$, and let $\mathfrak{M}_A$ be the preimage of $\mathfrak{M}^{\mathcal{LT}}_{T_A/L_A}$ under the natural projection $\mathfrak{M}^+_A \twoheadrightarrow \mathfrak{M}^+_A/\mathfrak{M}^{\text{ét}}_{L_A}$. By construction $\mathfrak{M}_A$ has a natural φ-compatible A-action, so we have a short exact sequence $0 \to \mathfrak{M}^{\text{ét}}_{L_A} \to \mathfrak{M}_A \to \mathfrak{M}^{\mathcal{LT}}_{T_A/L_A} \to 0$ of $(\varphi, \mathfrak{S}_A)$-modules. Since both flanking terms are finite free over $\mathfrak{S}_A$, so is the middle term $\mathfrak{M}_A$; i.e., $\mathfrak{M}_A \in (\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$.

To show the uniqueness, observe that if $\mathfrak{M}_A$ and $\mathfrak{M}'_A$ are extensions of $\mathfrak{M}^{\mathcal{LT}}_{T_A/L_A}$ by $\mathfrak{M}^{\text{ét}}_{L_A}$ in $(\mathrm{ModFI}/\mathfrak{S})^{\leqslant h}_A$ then so is $\mathfrak{M}_A + \mathfrak{M}'_A$ where the sum is taken inside $M_{\mathbb{F}} \otimes_{\mathbb{F}} A$; clearly $\mathfrak{M}_A + \mathfrak{M}'_A$ is an extension of $\mathfrak{M}^{\mathcal{LT}}_{T_A/L_A}$ by $\mathfrak{M}^{\text{ét}}_{L_A}$ as a $(\varphi, \mathfrak{S}_A)$-module, so $\mathfrak{S}_A$-freeness follows. By Lemma 9.2.4 there exist maximal and minimal $\mathfrak{S}_A$-lattices $\mathfrak{M}^{(+)}_A, \mathfrak{M}^{(-)}_A \subset M_{\mathbb{F}} \otimes_{\mathbb{F}} A$ of $\mathcal{P}$-height $\leqslant h$ among the extensions of $\mathfrak{M}^{\mathcal{LT}}_{T_A/L_A}$ by $\mathfrak{M}^{\text{ét}}_{L_A}$. Now, we have the following commutative diagram with short exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{M}^{\text{ét}}_{L_A} & \longrightarrow & \mathfrak{M}^{(-)}_A & \longrightarrow & \mathfrak{M}^{\mathcal{LT}}_{T_A/L_A} & \longrightarrow & 0\,, \\
 & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & \mathfrak{M}^{\text{ét}}_{L_A} & \longrightarrow & \mathfrak{M}^{(+)}_A & \longrightarrow & \mathfrak{M}^{\mathcal{LT}}_{T_A/L_A} & \longrightarrow & 0
\end{array}$$

where the vertical map in the middle is the natural inclusion. By 5-lemma, the vertical map in the middle is an isomorphism, which shows the uniqueness. $\square$

Corollary 11.4.16. *Assume that* $\dim_{\mathbb{F}}(\rho_{\mathbb{F}}) = 2$ *and* $\mathscr{K}/\mathscr{K}_0$ *is separable (so that we can apply Proposition 11.3.7). Let* R *be* $R^{\square, \leqslant h}_{\rho_{\mathbb{F}}}$, *or* $R^{\leqslant h}_{\rho_{\mathbb{F}}}$ *if it exists. Let* E/F *be a finite extension, and let* $x_1, x_2 \in (\mathrm{Spec}\, R^{\mathbf{v}})(E)$, *where* $\mathbf{v}$ *is the* F_0*-Hodge-Pink type* $(n = 2, \mathfrak{S}_F/\mathcal{P}(u)^h)$. *If* x_1 *and* x_2 *lie in the same connected component of* $\mathrm{Spec}\, R^{\mathbf{v}}$ *then* V_{x_1} *and* V_{x_2} *are either both ordinary or both non-ordinary.*

If both V_{x_1} *and* V_{x_2} *are ordinary, then* x_1 *and* x_2 *are in the same connected component if for the unique* E*-line* $L_i \subset V_{x_i}$ *on which* I_K *acts via* $\chi^h_{\mathcal{LT}}$, *the Galois group*

$\mathcal{G}_K$ acts on L_1 and L_2 via $\mathfrak{o}_E^\times$-valued characters with the same reduction modulo $\mathfrak{m}_E$.

Assuming that $\mathscr{K}/\mathscr{K}_0$ is separable, the natural question that arises is to compute the non-ordinary connected components of $\operatorname{Spec} R^{\mathbf{v}}$, where $\mathbf{v} := (n = 2, \mathfrak{S}_F/\mathcal{P}(u)^h)$ is as in the statement of the corollary. If $h = 1$, then we can show that the non-ordinary locus in $\operatorname{Spec} R^{\mathbf{v}}$ is connected, which will be seen in the next section. On the other hand, this question for $h > 1$ seems to require a new idea.

11.4.17 Application to crystalline and semi-stable deformation rings

Assume $\mathfrak{o}_0 = \mathbb{Z}_p$, and use the same notations as in §11.2.11. Let V be a p-adic $\mathcal{G}_{\mathscr{K}}$-representation which is semi-stable with Hodge-Tate weights in $[0, h]$. We say that V is *ordinary* if there exists a $\mathcal{G}_{\mathscr{K}}$-stable subspace $L \subset V$ such that both $L(-h)$ and V/L are unramified. Equivalently, one can require that $D^*_{\mathrm{st}}(V)$, or equivalently $D_{\mathrm{st}}(V)(h)$, is an extension of a weakly admissible filtered φ-module pure of slope h by an étale filtered φ-module. We say that V is *formal* if V admits no non-trivial unramified quotient. Equivalently, one can require that $D^*_{\mathrm{st}}(V)$ has no non-trivial étale subobject, or equivalently that $D_{\mathrm{st}}(V)(h)$ admits no weakly admissible quotient which is pure of slope h. We can naturally extend these definitions to semi-stable A-representations V_A of $\mathcal{G}_{\mathscr{K}}$ where A is a finite $\mathbb{Q}_p$-algebra, as follows: we say that V_A is *ordinary* if it is ordinary as a p-adic representation and the maximal unramified quotient V_A/L_A is projective as an A-module; we say that V_A is *formal* if it is formal as a p-adic representation. (Note that the maximal unramified $\mathbb{Q}_p$-linear quotient $V_A^{\text{ét}}$ of V_A is automatically an A-linear quotient.)

As before, we fix a mod p representation $\bar{\rho}$ of $\mathcal{G}_{\mathscr{K}}$. Let $R^{\leqslant h}_{\mathrm{st}}$ and $R^{\leqslant h}_{\mathrm{cris}}$ respectively denote the semi-stable and crystalline deformation ring or framed deformation ring of $\bar{\rho}$ in the sense of [59]. We will use Propositions 11.4.4 and 11.4.9, and the maps

$\underline{\mathrm{res}}^{\mathrm{cris}}$ and $\underline{\mathrm{res}}^{\mathrm{st}}$ to prove:

Proposition 11.4.18. *Let $R \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$, and consider a semi-stable deformation ρ_R of $\bar{\rho}$ with Hodge-Tate weights in $[0,h]$; i.e., each artinian quotient $\rho_R \otimes_R (R/\mathfrak{m}_R^n)$ is torsion semi-stable with Hodge-Tate weights in $[0,h]$.*

1. *There exists a unique open and closed subscheme $\mathrm{Spec}(R^f \otimes_{\mathfrak{o}} F)$ of $\mathrm{Spec}(R \otimes_{\mathfrak{o}} F)$ with the following property: for any finite $\mathbb{Q}_p$-algebra A, a map $x : R \to A$ factors through R^f if and only if the corresponding representation $\rho_R \otimes_{R,x} A$ is formal.*

2. *There exists a unique open and closed subscheme $\mathrm{Spec}\, R^{d,\mathrm{ord}}$ of $\mathrm{Spec}(R \otimes_{\mathfrak{o}} F)$ with the following property: for any finite $\mathbb{Q}_p$-algebra A, a map $x : R \otimes_{\mathfrak{o}} F \to A$ factors through $R^{d,\mathrm{ord}}$ if and only if the corresponding representation $\rho_R \otimes_{R,x} A$ is ordinary and its maximal unramified quotient is of A-rank $(n-d)$.*

Proof. The uniqueness is clear, so we just have to prove existence. Let $\mathrm{Spec}\, R^f \subset \mathrm{Spec}\, R$ be the maximal closed subscheme (which is also open in the $\mathbb{Q}_p$-fiber) such that $(\rho_R \otimes_R R^f)|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}$ is formal as a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representation, and let $\mathrm{Spec}\, R^{d,\mathrm{ord}} \subset \mathrm{Spec}(R \otimes_{\mathfrak{o}} F)$ be the maximal open and closed subscheme such that $(\rho_R \otimes_R R^{d,\mathrm{ord}})|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}$ is ordinary and its maximal unramified quotient is of rank $(n-d)$. The existence of R^f and $R^{d,\mathrm{ord}}$ is proved in Propositions 11.4.4 and 11.4.9, respectively. It follows from the lemma below that R^f and $R^{d,\mathrm{ord}}$ satisfy the desired properties. □

Lemma 11.4.19. *Let A be a finite $\mathbb{Q}_p$-algebra and let V_A be a rank-n semi-stable A-representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ with Hodge-Tate weights in $[0,h]$. Let $V_A^{\text{ét}}$ be the maximal unramified A-quotient of $V_A|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}$ as a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representation, which exists and is a projective A-module by Proposition 8.2.7. Then $V_A^{\text{ét}}$ is the maximal unramified A-quotient of V_A as a $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representation; i.e., the kernel of the natural projection $V_A \twoheadrightarrow$*

$V_A^{\text{ét}}$ *is* $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$*-stable and has no non-trivial unramified quotient as a* $\mathbb{Q}_p$*-representation space.*

As special cases, we have the following:

1. *The* $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$*-representation* $V_A|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}$ *is formal if and only if* V_A *is formal as a* $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$*-representation.*

2. *The* $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$*-representation* $V_A|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}$ *is ordinary of* $\mathcal{P}$*-height* $\leqslant h$ *with maximal unramified* $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$*-quotient of* A*-rank* d *if and only if* V_A *is ordinary as a* $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$*-representation with maximal unramified* $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$*-quotient of* A*-rank* d*.*

Proof. Let $D_A = (D_A, \varphi, N, \mathrm{Fil}^\bullet(D_A)_{\mathscr{K}}) := \underline{D}^*_{\mathrm{st}}(V_A)$ be the weakly admissible filtered (φ, N)-module which correspond to V_A. Let $\underline{\mathrm{res}}(D_A)$ be the weakly admissible Hodge-Pink structure corresponding to $V_A|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}$. (The functor $\underline{\mathrm{res}}$ is defined in §5.2.12.)

We may assume that the residue field k of $\mathfrak{o}_{\mathscr{K}}$ is algebraically closed, and that A is local. Let $V_A^{\text{ét}}$ be the maximal unramified quotient of $V_A|_{\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}}$ as a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representation, and set $d := \mathrm{rank}_A(V_A^{\text{ét}})$. Let $D_A^{\text{ét}}$ be the maximal étale subobject of $\underline{\mathrm{res}}(D_A)$ as an isocrystal with Hodge-Pink structure, so $D_A^{\text{ét}}$ is of $\mathscr{K}_{0,A}$-rank d. Since $k = \bar{k}$, by the Dieudonné-Manin decomposition $D_A \cong D_A^{\text{ét}} \oplus D'_A$, where any subquotient of D'_A has positive slopes. Thus, the relation $N\varphi = p\varphi N$ implies that $N|_{D_A^{\text{ét}}} = 0$. So by the weak admissibility, we see that $(D_A^{\text{ét}})_{\mathscr{K}} \cap \mathrm{Fil}^w(D_A)_{\mathscr{K}} = 0$ for all $w > 0$. Thus, $D_A^{\text{ét}}$ defines a weakly admissible subobject of D_A. Clearly, $V^*_{\mathrm{st}}(D_A^{\text{ét}})$ is the maximal unramified quotient of V_A as a $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representation over $\mathbb{Q}_p$, and is an A-quotient of $V_A^{\text{ét}}$. Now, we claim that $V_A^{\text{ét}} = V^*_{\mathrm{st}}(D_A^{\text{ét}})$ as A-quotients of V_A. For this, it suffices to show the inequality $\mathrm{rank}_{\mathfrak{S}_A}(\mathfrak{M}_A^{\text{ét}}) \leqslant \mathrm{rank}_{(\mathscr{K}_0)_A}(D_A^{\text{ét}})$. First, observe that $\mathfrak{M}_A^{\text{ét}}/u\mathfrak{M}_A^{\text{ét}}$ is pure of slope 0, because by definition there exists a finite flat $\mathbb{Z}_p$-subalgebra $A^o \subset A$ and a finite free étale $(\varphi, \mathfrak{S}_{A^o})$-module $\mathfrak{M}_{A^o}$ with $\mathfrak{M}_{A^o}[\frac{1}{p}] = \mathfrak{M}_A$.

But since $D_A \cong \mathfrak{M}_A/u\mathfrak{M}_A$, we have $\mathfrak{M}_A^{\text{ét}}/u\mathfrak{M}_A^{\text{ét}} \subset D_A^{\text{ét}}$, thus we obtain the desired inequality. □

Consider a filtration $\mathrm{Fil}^{\bullet}_{\mathbf{v}}$ of $\mathscr{K}^{\oplus n}$ (i.e., a p-adic Hodge type $\mathbf{v}$). Proposition 11.4.18 provides a "universal" open and closed subscheme $\operatorname{Spec} R^{f,\mathbf{v}} \subset \operatorname{Spec} R^{\mathbf{v}}$, where $\operatorname{Spec} R^{\mathbf{v}}$ is the open and closed subscheme of $\operatorname{Spec}(R \otimes_{\mathfrak{o}} F)$ corresponding to the Hodge type $\mathbf{v}$. Let $\mathbf{v}_d$ be a filtration of $\mathscr{K}^{\oplus n}$ such that $\dim_{\mathscr{K}} \mathrm{gr}^0_{\mathbf{v}_d} = n-d$ for $w \leq 0$, $\dim_{\mathscr{K}} \mathrm{gr}^h_{\mathbf{v}_d} = d$, and $\dim_{\mathscr{K}} \mathrm{gr}^w_{\mathbf{v}_d} = 0$ for $w \neq 0, h$. It follows from Proposition 11.4.18 that the natural open and closed inclusions $\operatorname{Spec} R^{d,\mathrm{ord}} \hookrightarrow \operatorname{Spec}(R \otimes_{\mathfrak{o}} F)$ factors through $\operatorname{Spec} R^{\mathbf{v}_d}$.

We will often apply Proposition 11.4.18 to the following cases. We let R denote one of the following: $R^{\square,\leqslant h}_{\mathrm{cris}}$, $R^{\square,\leqslant h}_{\mathrm{st}}$, $R^{\leqslant h}_{\mathrm{cris}}$, and $R^{\leqslant h}_{\mathrm{st}}$. With these choices of R, the ordinary and formal loci $R^{d,\mathrm{ord}}$ and $R^f \otimes_{\mathfrak{o}} F$ in $R[\frac{1}{p}]$ have an obvious "mapping property." For example, for any finite F-algebra A, an A-point of $R^{\square,\leqslant h}_{\mathrm{cris}}[\frac{1}{p}]$ factors through $R^{\square,\leqslant h,d,\mathrm{ord}}_{\mathrm{cris}}$ if and only if the corresponding framed A-deformation is ordinary such that the maximal étale quotient is of A-rank $n-d$. With this said, the proposition can be rephrased as follows. *Let R denote one of the following: $R^{\square,\leqslant h}_{\mathrm{cris}}$, $R^{\square,\leqslant h}_{\mathrm{st}}$, $R^{\leqslant h}_{\mathrm{cris}}$, and $R^{\leqslant h}_{\mathrm{st}}$. Let x_1, x_2 be closed points of* $\operatorname{Spec}(R \otimes_{\mathfrak{o}} F)$*, and V_{x_1}, V_{x_2} be corresponding $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations. If x_1 and x_2 lie in the same component then either both V_{x_1} and V_{x_2} are ordinary or both are non-ordinary. Similarly, if x_1 and x_2 lie in the same component then either both V_{x_1} and V_{x_2} are formal or both are non-formal.*

11.5 Connected components: $h = 1$ Case

Now we restrict ourselves to the case when $h = 1$ and $\mathscr{K}/\mathscr{K}_0$ is separable. We assume that $\dim_{\mathbb{F}}(\rho_{\mathbb{F}}) = 2$, and choose $\xi \in \mathscr{D}^{\leqslant 1}_{\rho_{\mathbb{F}}}(R)$ for some $R \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$ such that $(\mathscr{D}^{\leqslant 1}_{\rho_{\mathbb{F}}}/\xi) \to \mathscr{D}^{\leqslant 1}_{\rho_{\mathbb{F}}}$ is *formally smooth*. Important examples are $\xi := \xi_{\mathrm{univ}}$ if

$\mathrm{End}_{\mathcal{G}_K}(\rho_{\mathbb{F}}) \cong \mathbb{F}$, and $\xi := \xi^{\square}_{\mathrm{univ}}$.

We fix a Hodge-Pink type $\mathbf{v} := (n = 2, \bar{\Lambda}^{\mathbf{v}} := \mathfrak{S}_{F_0}/\mathcal{P}(u))$ as in §11.3.12. We already described connected components of $\mathscr{G}\mathscr{R}^{\mathbf{v}}_{\xi}$ which correspond to ordinary lifts in Proposition 11.4.15 and Corollary 11.4.16. In this section, we show that the non-ordinary locus in $\mathscr{G}\mathscr{R}^{\mathbf{v}}_{\xi}$ is connected, which completes the description of the connected components of $\mathscr{G}\mathscr{R}^{\mathbf{v}}_{\xi}$. Actually, we will content ourselves with reducing the proof to the affine grassmannian computation which is done in [51, §2.5] and [43].

We briefly explain the idea and indicate where we need the assumption $h = 1$. We start with defining a closed subscheme $\mathscr{G}\mathscr{R}^{\mathbf{v},\mathrm{int}}_{\xi} \subset \mathscr{G}\mathscr{R}^{\leqslant 1}_{\xi}$ such that $\mathscr{G}\mathscr{R}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} F = \mathscr{G}\mathscr{R}^{\mathbf{v}}_{\xi}$, and $\mathscr{G}\mathscr{R}^{\mathbf{v},\mathrm{int}}_{\xi}$ is "reasonably nice" as a scheme so that each connected component of $\mathscr{G}\mathscr{R}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} F$ "uniquely reduces" to a connected component of $\mathscr{G}\mathscr{R}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_R R/\mathfrak{m}_R$. The author does not know any analogue of $\mathscr{G}\mathscr{R}^{\mathbf{v},\mathrm{int}}_{\xi}$ for $h > 1$. Once we show this, then the affine grassmannian computation in *loc.cit.* gives the connectedness result we want.

11.5.1

For an $\mathfrak{o}_0$-algebra A, let $\overline{\mathcal{D}}_A$ be a free $\mathfrak{S}_A/\mathcal{P}(u)$-module of rank 2. We say a $\mathfrak{S}_A/\mathcal{P}(u)$-submodule $\mathcal{L}_A \subset \overline{\mathcal{D}}_A$ is *Lagrangian* if it is a direct factor as an A-module and the submodule $\mathcal{L}_A$ is its own annihilator under a $\mathfrak{S}_A/\mathcal{P}(u)$-bilinear symplectic pairing on $\overline{\mathcal{D}}_A$ (which is unique up to unit multiple).

If $\mathscr{K}/\mathscr{K}_0$ is separable and A is a finite F_0-algebra, then $\varphi(\sigma^*\mathfrak{M}_A)/\mathcal{P}(u)\mathfrak{M}_A$ is necessarily projective $\mathfrak{S}_A/\mathcal{P}(u)$-module of rank 1; since $\mathfrak{S}_A/\mathcal{P}(u)$ is a finite étale A-algebra, A-flatness of $\overline{\mathcal{D}}_A/\mathcal{L}_A$ implies $\mathfrak{S}_A/\mathcal{P}(u)$-flatness by local flatness criterion. But in general, a Lagrangian is not necessarily projective over $\mathfrak{S}_A/\mathcal{P}(u)$. If $\pi_0 \cdot A = 0$ then the $\mathfrak{S}_A/\mathcal{P}(u)$-span of $\{u^i\mathbf{e}_1, u^{e-i}\mathbf{e}_2\}$ for $i \in [0, e]$ is a Lagrangian in $\bigoplus_{i=1,2}\mathfrak{S}_A/\mathcal{P}(u)\mathbf{e}_i$. If $\mathscr{K}/\mathscr{K}_0$ is not separable then one can construct Lagragians in

$\overline{\mathcal{D}}_A$ which is not projective over $\mathfrak{S}_A/\mathcal{P}(u)$ even when $A = E$ is a finite extension of F_0, using an idea similar to Remark 11.3.6.

We now define a full subgroupoid $\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{S},M_{\mathbb{F}}} \subset \mathscr{D}^{\leqslant 1}_{\mathfrak{S},M_{\mathbb{F}}}$ over $\mathfrak{Aug}_{\mathfrak{o}}$ and $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ whose objects $\mathfrak{M}_A$ over A are those satisfying that the $\mathfrak{S}_A/\mathcal{P}(u)$-submodule $\varphi(\sigma^*\mathfrak{M}_A)/\mathcal{P}(u)\mathfrak{M}_A \subset \mathfrak{M}_A/\mathcal{P}(u)\mathfrak{M}_A$ is a Lagrangian. Note that this submodule is a direct factor as an A-module, by Proposition 8.2.3.

Proposition 11.5.2. *The natural inclusion of $\mathfrak{Aug}_{\mathfrak{o}}$-groupoids $\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{S},M_{\mathbb{F}}} \hookrightarrow \mathscr{D}^{\leqslant 1}_{\mathfrak{S},M_{\mathbb{F}}}$ is relatively representable by closed immersions; i.e., for any $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$ with $R \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$, the $\mathfrak{Aug}_R$-groupoid $\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{S},M_{\mathbb{F}},\xi} := (\mathscr{D}^{\leqslant 1}_{\rho_{\mathbb{F}}}/\xi) \times_{\mathscr{D}^{\leqslant 1}_{\rho_{\mathbb{F}}}} \mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{S},M_{\mathbb{F}}}$ is representable by a closed subscheme $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \subset \mathscr{GR}^{\leqslant 1}_{\xi}$. If $\mathscr{K}/\mathscr{K}_0$ is separable, then $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} F \subset \mathscr{GR}^{\leqslant 1}_{\xi} \otimes_{\mathfrak{o}} F$ is precisely $\mathscr{GR}^{\mathbf{v}}_{\xi}$ with $\mathbf{v} := (n = 2, \bar{\Lambda}^{\mathbf{v}} := \mathfrak{S}_{F_0}/\mathcal{P}(u))$. (In particular, $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} F$ is a union of connected components of $\mathscr{GR}^{\leqslant 1}_{\xi} \otimes_{\mathfrak{o}} F$.)*

Even though $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} F$ makes sense without the separability assumption on $\mathscr{K}/\mathscr{K}_0$, the author does not know whether all closed points of $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} F$ have Hodge-Pink type $\mathbf{v} := (n = 2, \bar{\Lambda}^{\mathbf{v}} := \mathfrak{S}_{F_0}/\mathcal{P}(u))$, nor whether $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} F$ is a union of connected components of $\mathscr{GR}^{\leqslant 1}_{\xi}$.

If $\mathscr{K}/\mathscr{K}_0$ is separable, one can adapt the discussions in [51, (2.2)] to define a closed subscheme $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi}$ of $\mathscr{GR}_{\xi} \otimes_{\mathfrak{o}} \mathfrak{o}_E$ for any E-Hodge-Pink type $\mathbf{v}$ of $\mathcal{P}$-height $\leqslant 1$ (with E/F a finite extension) such that $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}_E} E = \mathscr{GR}^{\mathbf{v}}_{\xi}$.

Proof. We construct $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi}$ as follows. Put $\mathfrak{S}_{\mathscr{GR}^{\leqslant 1}_{\xi}} := \mathfrak{S} \otimes_{\mathfrak{o}_0} \mathcal{O}_{\mathscr{GR}^{\leqslant 1}_{\xi}}$ and consider the universal $\mathfrak{S}_{\mathscr{GR}^{\leqslant 1}_{\xi}}$-lattice $\underline{\mathfrak{M}}^{\leqslant 1}_{\xi}$ of $\mathcal{P}$-height $\leqslant 1$ in $M_{\xi} \otimes_R \mathcal{O}_{\mathscr{GR}^{\leqslant 1}_{\xi}}$. Let φ_{ξ} denote the universal φ-structure on $\underline{\mathfrak{M}}^{\leqslant 1}_{\xi}$. By Proposition 8.2.3, $\mathrm{im}(\varphi_{\xi})/\mathcal{P}(u)\underline{\mathfrak{M}}^{\leqslant 1}_{\xi} \subset \underline{\mathfrak{M}}^{\leqslant 1}_{\xi}/\mathcal{P}(u)\underline{\mathfrak{M}}^{\leqslant 1}_{\xi}$ is a direct factor as a vector bundle over $\mathscr{GR}^{\leqslant 1}_{\xi}$. Now, choose a $\mathfrak{S}_{\mathscr{GR}^{\leqslant 1}_{\xi}}/\mathcal{P}(u)$-basis for $\underline{\mathfrak{M}}^{\leqslant 1}_{\xi}/\mathcal{P}(u)\underline{\mathfrak{M}}^{\leqslant 1}_{\xi}$ and let $\langle , \rangle$ denote the standard symplectic pairing on $\underline{\mathfrak{M}}_\xi^{\leqslant 1}/\mathcal{P}(u)\underline{\mathfrak{M}}_\xi^{\leqslant 1}$ with respect to the fixed basis. Choose an open (affine) covering $\{U_\alpha\}$ of $\mathscr{GR}_\xi^{\leqslant 1}$ which trivializes $\mathrm{im}(\varphi_\xi)/\mathcal{P}(u)\underline{\mathfrak{M}}_\xi^{\leqslant 1}$, and choose an $\mathcal{O}_{U_\alpha}$-basis $\{\mathbf{e}_{1,\alpha},\cdots,\mathbf{e}_{r_\alpha,\alpha}\}$ of $(\mathrm{im}(\varphi_\xi)/\mathcal{P}(u)\underline{\mathfrak{M}}_\xi^{\leqslant 1})|_{U_\alpha}$. Now, let $\mathscr{GR}_\xi^{\mathbf{v},\mathrm{int}}$ be the closed subscheme of $\mathscr{GR}_\xi^{\leqslant 1}$ cut out by a coherent ideal $\mathscr{I}$, where $\mathscr{I}|_{U_\alpha}$ is generated by $\{\langle \mathbf{e}_{i,\alpha},\mathbf{e}_{j,\alpha}\rangle\}_{i,j=1,\cdots,r_\alpha}$, viewing $\langle,\rangle$ as an $\mathcal{O}_{\mathscr{GR}_\xi^{\leqslant 1}}$-bilinear pairing. Clearly $\mathscr{GR}_\xi^{\mathbf{v},\mathrm{int}}$ represents the groupoid $\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{S},M_{\mathbb{F}},\xi}$. If $\mathscr{K}/\mathscr{K}_0$ is separable then any A-point $\mathfrak{M}_A\in\mathscr{GR}_\xi^{\leqslant 1}(A)$ with A finite over F is supported in $\mathscr{GR}_\xi^{\mathbf{v},\mathrm{int}}\otimes_{\mathfrak{o}}F$ if and only if $\mathfrak{M}_A$ is of Hodge-Pink type $\mathbf{v} := (n=2,\bar{\Lambda}^{\mathbf{v}} := \mathfrak{S}_{F_0}/\mathcal{P}(u))$ since any Lagrangian in $(\mathfrak{S}_A/\mathcal{P}(u))^{\oplus 2}$ is free of rank 1 over $\mathfrak{S}_A/\mathcal{P}(u)$. Since both $\mathscr{GR}_\xi^{\mathbf{v},\mathrm{int}}\otimes_{\mathfrak{o}}F$ and $\mathscr{GR}_\xi^{\leqslant 1}\otimes_{\mathfrak{o}}F$ are Jacobson, this implies $\mathscr{GR}_\xi^{\mathbf{v},\mathrm{int}}\otimes_{\mathfrak{o}}F=\mathscr{GR}_\xi^{\mathbf{v}}$ as a subscheme of $\mathscr{GR}_\xi^{\leqslant 1}\otimes_{\mathfrak{o}}F$. □

11.5.3

As in §11.4.11, we construct a common "formally smooth covering" of the completed local rings of $\mathscr{GR}_\xi^{\mathbf{v},\mathrm{int}}$ and a "model space" whose local structure can be understood. Using this technique, we will show that $\mathscr{GR}_\xi^{\mathbf{v},\mathrm{int}}$ is $\mathfrak{o}$-flat and a relative complete intersection, and that $\mathscr{GR}_\xi^{\mathbf{v},\mathrm{int}}\otimes_{\mathfrak{o}}\mathfrak{o}/\mathfrak{m}_{\mathfrak{o}}$ is reduced. If $\mathfrak{o}_0=\mathbb{Z}_p$ then the model space will coincide with the Deligne-Pappas étale-local model for Hilbert-Brumenthal modular surfaces [22], as one expects from Kisin's work [51]. (For more general Hodge-Pink types $\mathbf{v}$, the model space that appears is the Pappas-Rapoport étale-local model for a certain type of Shimura varieties [66] in the case $\mathfrak{o}_0=\mathbb{Z}_p$.)

We fix $\mathfrak{M}_{\mathbb{F}}\in\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{S},M_{\mathbb{F}}}(\mathbb{F})$. We consider an $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}}$ whose objects over A are $(\mathfrak{M}_A,\iota_A)$, where $\mathfrak{M}_A\in\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{S},M_{\mathbb{F}}}(A)$ and $\iota_A:\mathfrak{M}_A\otimes_A A/\mathfrak{m}_A\xrightarrow{\sim}\mathfrak{M}_{\mathbb{F}}$. There is a natural 1-morphism $\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}}\to\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{S},M_{\mathbb{F}}}$, defined by replacing ι_A with $\iota_A[\frac{1}{\pi_0}]:(\mathfrak{M}_A\otimes_A A/\mathfrak{m}_A)[\frac{1}{\pi_0}]\xrightarrow{\sim}M_{\mathbb{F}}$. If $\mathrm{End}_{\mathcal{G}_K}(\rho_{\mathbb{F}})\cong\mathbb{F}$, then $\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}}$ is pro-representable by the completed local ring of $\mathscr{GR}^{\mathbf{v},\mathrm{int}}$ at the closed point corresponding to $\mathfrak{M}_{\mathbb{F}}$. In general, the 2-fiber product $\mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}},\xi} := (\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \times_{\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}} \mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}}$ is pro-representable by the completed local ring of $\mathscr{G}\mathscr{R}^{\mathbf{v},\mathrm{int}}_{\xi}$ at the closed point corresponding to $\mathfrak{M}_{\mathbb{F}}$. By extending $\mathbb{F}$, we obtain all completed local rings of $\mathscr{G}\mathscr{R}^{\mathbf{v},\mathrm{int}}_{\xi}$ at closed points.

We fix a $\mathfrak{S}_{\mathbb{F}}/\mathcal{P}(u)$-isomorphism $\beta_{\mathbb{F}} : (\mathfrak{S}_{\mathbb{F}}/\mathcal{P}(u))^{\oplus 2} \xrightarrow{\sim} \mathfrak{M}_{\mathbb{F}}/\mathcal{P}(u)\mathfrak{M}_{\mathbb{F}}$. We define an $\mathfrak{AR}_{\mathfrak{o}}$-groupoid[19] $\widetilde{\mathscr{D}}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}}$, where an object over A is $\mathfrak{M}_A \in \mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}}(A)$ together with a $\mathfrak{S}_A/\mathcal{P}(u)$-linear isomorphism $\beta_A : (\mathfrak{S}_A/\mathcal{P}(u))^{\oplus 2} \xrightarrow{\sim} \mathfrak{M}_A/\mathcal{P}(u)\mathfrak{M}_A$ which lifts $\beta_{\mathbb{F}}$. By forgetting this isomorphism, we obtain a 1-morphism $\widetilde{\mathscr{D}}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}} \rightarrow \mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}}$, which makes the former into a torsor under the formal completion of the Weil restriction $\mathrm{Res}^{\mathfrak{S}_{\mathfrak{o}}/\mathcal{P}(u)}_{\mathfrak{o}} \mathrm{GL}_n$ at the identity section. In particular, this 1-morphism is formally smooth.

Now, we define another $\mathfrak{AR}_{\mathfrak{o}}$-groupoid $\underline{M}_{\mathbf{v}}$ whose objects are Lagrangians of $(\mathfrak{S}_A/\mathcal{P}(u))^{\oplus 2}$ under the standard symplectic form (in the sense of §11.5.1). This groupoid is representable by (the π_0-adic completion of) a closed subscheme of a grassmannian. We let the same notation $\underline{M}_{\mathbf{v}}$ denote the representing projective $\mathfrak{o}$-scheme. The argument given in [22, §4], which also works in the case of $\mathfrak{o}_0 = \mathbb{F}_q[[\pi_0]]$, shows that $\underline{M}_{\mathbf{v}}$ is $\mathfrak{o}$-flat and a relative complete intersectionT and that $\underline{M}_{\mathbf{v}} \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{m}_{\mathfrak{o}}$ is reduced. (For the $\mathfrak{o}$-flatness, see [27, IV_2, 3.4.6.1].) We have a 1-morphism $\widetilde{\mathscr{D}}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}} \rightarrow \underline{M}_{\mathbf{v}}$ by sending $(\mathfrak{M}_A, \beta_A)$ to the kernel of $(\mathfrak{S}_A/\mathcal{P}(u))^{\oplus 2} \xrightarrow[\beta_A]{\sim} \mathfrak{M}_A/\mathcal{P}(u)\mathfrak{M}_A \xrightarrow{\varphi} \operatorname{coker}\varphi$, which is seen to be formally smooth by an argument similar to §11.4.11.

Now, we are ready to prove the following

Proposition 11.5.4. *Assume that* $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \rightarrow \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ *is formally smooth. Then* $\mathscr{G}\mathscr{R}^{\mathbf{v},\mathrm{int}}_{\xi}$ *is* $\mathfrak{o}$*-flat and a relative complete intersection, and* $\mathscr{G}\mathscr{R}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{m}_{\mathfrak{o}}$ *is reduced.*

Proof. The proof is similar to that of Proposition 11.4.12. Consider the following

[19] Again, the tilde in the notation does *not* mean the extension by 2-direct limit, which is defined in §10.4.4.

diagrams where all the arrows are formally smooth.

$$\begin{array}{ccccc} \widetilde{\mathscr{D}}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}},\xi} & \longrightarrow & \widetilde{\mathscr{D}}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}} & \longrightarrow & \underline{M}_{\mathbf{v}} \\ \downarrow & & \downarrow & & \\ \mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}},\xi} & \longrightarrow & \mathscr{D}^{\mathbf{v},\mathrm{int}}_{\mathfrak{M}_{\mathbb{F}}} & & \end{array}$$

Since $\underline{M}_{\mathbf{v}}$ has the desired properties, we conclude that $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi}$ has the desired properties. □

Let $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi,0}$ be the fiber over the closed point of $\operatorname{Spec} R$ under $\underline{T}^{\leqslant h}_{\mathfrak{S}} : \mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \to \operatorname{Spec} R$. The following corollary shows that distinct connected components of the generic fiber $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} F$ "reduce" to distinct connected components of $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi,0}$. We let $H_0(X)$ denote the set of connected components of X.

Corollary 11.5.5. *We keep the assumption that* $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ *is formally smooth. Then the natural maps below*

$$H_0(\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} F) \to H_0(\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi}) \leftarrow H_0(\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi,0})$$

are bijective.

Proof. The second bijection follows from the theorem on formal functions. The first bijection follows from an argument similar to the proof of [51, Corollary 2.4.10] using $\mathfrak{o}$-flatness and the reducedness of $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi} \otimes_{\mathfrak{o}} \mathfrak{o}/\mathfrak{m}_{\mathfrak{o}}$. □

We now state the following theorem.

Theorem 11.5.6. *Assume that* $\mathscr{K}/\mathscr{K}_0$ *is separable and* $\xi \in \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}(R)$ *is such that* $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}/\xi) \to \mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}$ *is formally smooth. Let* $R^{\mathbf{v}}$ *be the universal quotient of* $R[\frac{1}{\pi_0}]$ *whose points are of Hodge-Pink type* $\mathbf{v} := (n = 2, \bar{\Lambda}^{\mathbf{v}} = \mathfrak{S}_{F_0}/\mathcal{P}(u))$*. Then the non-ordinary locus of* $\operatorname{Spec} R^{\mathbf{v}}$ *is connected.*

Proof. By Corollary 11.5.5, the problem is reduced to showing the connectedness of the non-ordinary locus in $\mathscr{GR}^{\mathbf{v},\mathrm{int}}_{\xi,0}$. This follows from the affine grassmannian computation in characteristic p by Kisin [51, (2.5)] in the case of $k = \mathbb{F}_q$ (where $q = p$ if $\mathfrak{o}_0 = \mathbb{Z}_p$), and by Imai [43] in the general case[20]. □

11.6 Application to flat deformation rings

Throughout this section, we assume that $\mathfrak{o}_0 = \mathbb{Z}_p$ and let $\bar{\rho}$ be a 2-dimensional "finite flat" $\mathbb{F}$-representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$; i.e., we assume that $\bar{\rho}$ comes from the generic fiber of a finite flat group scheme over $\mathfrak{o}_{\mathscr{K}}$. Let $R^{\square,\leqslant 1}_{\mathrm{cris}}$, $R^{\leqslant 1}_{\mathrm{cris}}$, $R^{\square,\leqslant 1}_{\infty}$ and $R^{\leqslant 1}_{\infty}$ be as in §11.2.11. By Kisin's theorem[21] (also stated in Theorem 2.4.11(2)), any crystalline $\mathbb{Q}_p$-representation with Hodge-Tate weights in $[0,1]$ comes from the p-adic Tate module of a Barsotti-Tate group over $\mathfrak{o}_{\mathscr{K}}$. Therefore, the crystalline deformation rings $R^{\square,\leqslant 1}_{\mathrm{cris}}$ and $R^{\leqslant 1}_{\mathrm{cris}}$ coincide with the flat deformation rings $R^{\square}_{\mathrm{fl}}$ and R_{fl}, respectively.

The goal of this section is to prove the following theorem, which was originally proved by Kisin [51, Corollary 2.5.16] under the assumption that $p > 2$, and [53, §2] for any p (especially, $p = 2$). Note that this theorem plays a crucial role in Kisin's modularity lifting theorem for potentially Barsotti-Tate representations.

Theorem 11.6.1 (Kisin)**.** *Assume that $\bar{\rho}$ is finite flat.*[22] *Let $\mathbf{v}$ be the p-adic Hodge type such that $\dim_{\mathscr{K}} \mathrm{gr}^w_{\mathbf{v}} = 1$ for $w = 0$ or 1, and $\dim_{\mathscr{K}} \mathrm{gr}^w_{\mathbf{v}} = 0$ for $w \neq 0, 1$.*

1. *There is at most one non-ordinary connected component in* $\operatorname{Spec} R^{\square,\mathbf{v}}_{\mathrm{cris}}$.
2. *There exists at most one ordinary connected component in* $\operatorname{Spec} R^{\square,\mathbf{v}}_{\mathrm{cris}}$ *if and only if $\bar{\rho} \not\cong \left(\begin{smallmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{smallmatrix}\right)$ where both χ_1 and χ_2 are distinct unramified characters.*

[20] The author believes, but has not carefully checked, that Imai's computation works in the case $p = 2$.

[21] Breuil [14] gave the first proof of this theorem for the case $p > 2$, and Kisin reproved the theorem without assuming $p > 2$.

[22] It follows from [52, Corollary 2.2.6] that any torsion crystalline $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representation comes from the generic fiber of a finite flat group scheme over $\mathfrak{o}_{\mathscr{K}}$ (even when $p = 2$), so this assumption can be removed.

3. *If* $\bar\rho \cong \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix}$ *where* $\chi_1 \neq \chi_2$ *are both unramified, then there exist exactly two ordinary connected components in* $\operatorname{Spec} R^{\square,\mathbf{v}}_{\mathrm{fl}}$*. For a finite extension* E/F*, let* x_1 *and* x_2 *be* E*-points of* $\operatorname{Spec} R^{\square,\mathbf{v}}_{\mathrm{cris}}$ *such that the corresponding* $\mathcal{G}_{\mathscr{K}}$*-representations* V_{x_1} *and* V_{x_2} *are ordinary. Then* x_1 *and* x_2 *are in the same connected component if and only if for the unique* E*-line* $L_i \subset V_{x_i}$ *on which* I_K *acts via* $\chi^h_{\mathcal{LT}}$*, the Galois group* $\mathcal{G}_K$ *acts on* L_1 *and* L_2 *via* $\mathfrak{o}_E^\times$*-valued characters with the same reduction modulo* $\mathfrak{m}_E$*.*

The same holds for $\operatorname{Spec} R^{\mathbf{v}}_{\mathrm{cris}}$ *if* $\operatorname{End}_{\mathcal{G}_{\mathscr{K}}}(\bar\rho) \cong \mathbb{F}$*.*

11.6.2 Preliminary reduction: the case $p > 2$

Let $\mathbf{v}$ be as in the statement of Theorem 11.6.1 and set $\mathbf{v} := (n = 2, \bar\Lambda^{\mathbf{v}} = \mathfrak{S}_{F_0}/\mathcal{P}(u))$. Recall from §11.3.13 that a semi-stable $\mathbb{Q}_p$-representation is of p-adic Hodge type $\mathbf{v}$ if and only if its restriction to $\mathcal{G}_{\mathscr{K}_\infty}$ is of Hodge-Pink type $\mathbf{v}$. In particular, the map $\underline{\mathrm{res}}^{\mathrm{cris}} : \operatorname{Spec} R^{\square,\leqslant 1}_{\mathrm{cris}}[\frac{1}{p}] \to \operatorname{Spec} R^{\square,\leqslant 1}_{\infty}[\frac{1}{p}]$ restricts to $\operatorname{Spec} R^{\square,\mathbf{v}}_{\mathrm{cris}} \to \operatorname{Spec} R^{\square,\mathbf{v}}_{\infty}$. Now assume that $\bar\rho$ comes from a finite flat group scheme and $\operatorname{End}_{\mathcal{G}_{\mathscr{K}}}(\bar\rho) \cong \mathbb{F}$. Then we will show later in Lemma 11.6.12 that $\operatorname{End}_{\mathcal{G}_{\mathscr{K}_\infty}}(\bar\rho_\infty) \cong \mathbb{F}$, so we get $\operatorname{Spec} R^{\mathbf{v}}_{\mathrm{cris}} \to \operatorname{Spec} R^{\mathbf{v}}_{\infty}$.

On the other hand, we have obtained the complete description of the connected components of $\operatorname{Spec} R^{\square,\mathbf{v}}_{\infty}$, which is very similar to the statement of Theorem 11.6.1. See §11.4-§11.5, especially Proposition 11.4.9 and Theorem 11.5.6. So in order to obtain Theorem 11.6.1 from this, we need more information about the map $\operatorname{Spec} R^{\square,\mathbf{v}}_{\mathrm{cris}} \to \operatorname{Spec} R^{\square,\mathbf{v}}_{\infty}$, and the map $\operatorname{Spec} R^{\mathbf{v}}_{\mathrm{cris}} \to \operatorname{Spec} R^{\mathbf{v}}_{\infty}$ if $\operatorname{End}_{\mathcal{G}_{\mathscr{K}}}(\bar\rho) \cong \mathbb{F}$.

Proposition 11.6.3. *Assume that* $p > 2$ *and* $\bar\rho$ *is finite flat. The natural map* $\underline{\mathrm{res}}^{\mathrm{cris}}$: $\operatorname{Spec} R^{\square,\leqslant 1}_{\mathrm{cris}}[\frac{1}{p}] \to \operatorname{Spec} R^{\square,\leqslant 1}_{\infty}[\frac{1}{p}]$ *defined by the restriction to* $\mathcal{G}_{\mathscr{K}_\infty}$ *is an isomorphism. If furthermore* $\operatorname{End}_{\mathcal{G}_{\mathscr{K}_\infty}}(\bar\rho|_{\mathcal{G}_{\mathscr{K}_\infty}}) \cong \mathbb{F}$*, then the natural map* $\underline{\mathrm{res}}^{\mathrm{cris}} : \operatorname{Spec} R^{\leqslant 1}_{\mathrm{cris}}[\frac{1}{p}] \to$

$\operatorname{Spec} R_\infty^{\leqslant 1}[\frac{1}{p}]$ *is an isomorphism.*

11.6.4 Preliminary reduction: the case $p = 2$

It is conjectured that we can remove the hypothesis $p > 2$ from the statement of Proposition 11.6.3.[23] On the other hand, the hard part in proving Theorem 11.6.1 is to show the connectedness of the non-ordinary locus in $\operatorname{Spec} R_{\mathrm{cris}}^{\mathbf{v}}$ (i.e., the "formal" locus in the sense of §11.4.17); the ordinary connected components can be analyzed using Kummer theory and some Galois cohomology considerations.[24] (See [53, §2.4] for an argument.)

This leads us to consider the following setting. By Proposition 11.4.4, there exists the universal closed subscheme $\operatorname{Spec} R_{\mathrm{cris}}^{\Box,\leqslant 1,f}$ of $\operatorname{Spec} R_{\mathrm{cris}}^{\Box,\leqslant 1}$ whose points correspond to deformations which restrict to a *formal* $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representation (in the "torsion" sense). Recall that on the $\mathbb{Q}_p$-fiber, the subscheme $\operatorname{Spec}(R_{\mathrm{cris}}^{\Box,\leqslant 1,f}[\frac{1}{p}]) \subset \operatorname{Spec}(R_{\mathrm{cris}}^{\Box,\leqslant 1}[\frac{1}{p}])$ is open and closed, with finite artinian points corresponding to "formal" lifts of $\bar\rho$ in the sense of §11.4.17. (See Proposition 11.4.5 and Lemma 11.4.19 for more details.)

By definition, the natural map $\underline{\mathrm{res}}^{\mathrm{cris}} : \operatorname{Spec} R_{\mathrm{cris}}^{\Box,\leqslant 1} \to \operatorname{Spec} R_\infty^{\Box,\leqslant 1}$ restricts to $\underline{\mathrm{res}}^{\mathrm{cris}} : \operatorname{Spec} R_{\mathrm{cris}}^{\Box,\leqslant 1,f} \to \operatorname{Spec} R_\infty^{\Box,\leqslant 1,f}$, where $R_\infty^{\Box,\leqslant 1,f}$ is a universal quotient of $R_\infty^{\Box,\leqslant 1}$ classifying "formal" framed deformations. If $\operatorname{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}}}(\bar\rho) \cong \mathbb{F}$, then we can apply the same discussion to "unframed" deformation rings. Now, we are ready to state the following modification of Proposition 11.6.3 which we prove with no assumption on p.

Proposition 11.6.5. *Assume that $\bar\rho$ is finite flat. The natural map*

$$\underline{\mathrm{res}}^{\mathrm{cris}} : \operatorname{Spec} R_{\mathrm{cris}}^{\Box,\leqslant 1,f}[\frac{1}{p}] \to \operatorname{Spec} R_\infty^{\Box,\leqslant 1,f}[\frac{1}{p}]$$

[23] If the Breuil-Kisin classification of finite flat group schemes work in the case $p = 2$, which is conjectured in [11], then the proposition 11.6.3 for $p = 2$ follows.

[24] In particular, the argument does not use the Breuil-Kisin classification of finite flat group schemes over $\mathfrak{o}_{\mathscr{K}}$.

defined by the restriction to $\mathcal{G}_{\mathscr{K}_\infty}$ is an isomorphism. Furthermore, if we have $\mathrm{End}_{\mathcal{G}_{\mathscr{K}_\infty}}(\bar\rho|_{\mathcal{G}_{\mathscr{K}_\infty}}) \cong \mathbb{F}$ *then the natural map* $\underline{\mathrm{res}}^{\mathrm{cris}} : \mathrm{Spec}\, R^{\leqslant 1, f}_{\mathrm{cris}}[\frac{1}{p}] \to \mathrm{Spec}\, R^{\leqslant 1, f}_{\infty}[\frac{1}{p}]$ *is an isomorphism.*

Combining the proposition above with Theorem 11.5.6, one obtains the connectedness of $\mathrm{Spec}\, R^{\square, \mathbf{v}, f}_{\mathrm{cris}}$, and so completes the proof of Theorem 11.6.1.

11.6.6

Kisin's original proof of Theorem 11.6.1, or rather Propositions 11.6.3 and 11.6.5, can be rephrased as follows (using our deformation rings $R^{\square, \leqslant 1}_\infty, R^{\leqslant 1}_\infty$ for $\mathcal{G}_{\mathscr{K}_\infty}$ that were not considered in [51, 53]). If $p > 2$, then we can use the *Breuil-Kisin classification of finite flat group schemes*[25] over $\mathfrak{o}_{\mathscr{K}}$ to show that the restriction to $\mathcal{G}_{\mathscr{K}_\infty}$ induces an equivalence of categories $\mathrm{Rep}^{\mathrm{tor,cris},[0,1]}_{\mathbb{Z}_p}(\mathcal{G}_{\mathscr{K}}) \to \mathrm{Rep}^{\mathrm{tor},[0,1]}_{\mathbb{Z}_p}(\mathcal{G}_{\mathscr{K}_\infty})$. (See [15, Theorem 3.4.3] for a proof.) In particular, the natural maps $\underline{\mathrm{res}}^{\mathrm{cris}} : R^{\square, \leqslant 1}_\infty \to R^{\square}_{\mathrm{fl}}$ and $\underline{\mathrm{res}}^{\mathrm{cris}} : R^{\leqslant 1}_\infty \to R_{\mathrm{fl}}$ are isomorphisms[26]. This proves Proposition 11.6.3.

For the case $p = 2$, Kisin [53, §1] extended the classification theorem to *connected* finite flat group schemes over $\mathfrak{o}_{\mathscr{K}}$. Now repeating the same argument, one obtains Proposition 11.6.5 (in a stronger form, without inverting p). We note that Kisin's work in [53, §1] uses Zink's theory of windows and displays. We re-emphasize that the modularity lifting theorem for 2-adic Barsotti-Tate representations has an important consequence, namely the even conductor case of Serre's modularity conjecture.

We now present a different proof of Propositions 11.6.3 and 11.6.5 (hence, of Theorem 11.6.1), which avoids the Breuil-Kisin classification of finite flat group schemes (so in turn, it eliminates the use of Zink's theory of windows and displays).

Let us discuss the proof of Proposition 11.6.3. By avoiding the classification of

[25] See [52, Theorem 2.3.5] for the precise statement.

[26] By the full faithfulness, $\mathrm{End}_{\mathcal{G}_{\mathscr{K}}}(\bar\rho) \cong \mathbb{F}$ implies $\mathrm{End}_{\mathcal{G}_{\mathscr{K}_\infty}}(\bar\rho|_{\mathcal{G}_{\mathscr{K}_\infty}}) \cong \mathbb{F}$, in which case $R^{\leqslant 1}_\infty$ exists.

finite flat group schemes, we lose our grip on the torsion theory to some extent. So instead of trying to study the artinian points of the deformation ring (concentrated on the closed point), we use the following theorem of Gabber to study the $\mathbb{Q}_p$-fiber of the deformation ring more directly. We state Gabber's theorem in the form that we will use in our situation, but the original statement in [54, Appendix] is more general.

Theorem 11.6.7 (Gabber)**.** *Let R and R' be complete local noetherian $\mathfrak{o}$-algebras with residue fields finite over $\mathfrak{o}/\mathfrak{m}_{\mathfrak{o}}$. Assume that both R and R' are p-torsion free, R is reduced, and R' is normal. Let $f : R' \to R$ be a $\mathfrak{o}$-algebra map such that* $\mathrm{Spec}(f[\frac{1}{p}]) : \mathrm{Spec}\, R[\frac{1}{p}] \to \mathrm{Spec}\, R'[\frac{1}{p}]$ *induces a bijection between the set of closed points and isomorphisms on the residue fields at each closed point. Then f is an isomorphism.*

This theorem is certainly very delicate – it is not even obvious that the assumption implies that f is of finite type. The proof uses the flattening technique of Raynaud-Gruson (and very ingenious commutative algebra). See Gabber's appendix in [54] for more details.

11.6.8

We outline how to use Gabber's theorem to prove Proposition 11.6.3. Fix an $\mathbb{F}$-representation $\bar{\rho}$ of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ (of arbitrary dimension) and let $\bar{\rho}_\infty$ denote the restriction of $\bar{\rho}$ to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$. Let R and R_∞ be one of the following:

1. Assuming $p > 2$, we set $R := R_{\mathrm{cris}}^{\square,\leqslant 1}$ and $R_\infty := R_\infty^{\square,\leqslant 1}$.

2. Assuming $p > 2$ and $\mathrm{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}}}(\bar{\rho}) \cong \mathbb{F}$, we set $R := R_{\mathrm{cris}}^{\leqslant 1}$ and $R_\infty := R_\infty^{\leqslant 1}$.

3. Under no assumption on p, we set $R := R_{\mathrm{cris}}^{\square,\leqslant 1,f}$ and $R_\infty := R_\infty^{\square,\leqslant 1,f}$.

4. Assuming $\mathrm{End}_{\boldsymbol{\mathcal{G}}_{\mathscr{K}}}(\bar{\rho}) \cong \mathbb{F}$ and under no assumption on p, we set $R := R_{\mathrm{cris}}^{\leqslant 1,f}$

and $R_\infty := R_\infty^{\leqslant 1, f}$.

In all the cases above, the restriction to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ induces a natural map $\underline{\mathrm{res}} : R_\infty \to R$. Although both the source and the target of $\underline{\mathrm{res}}$ are each finite over some formal power series ring (being complete local noetherian $\mathfrak{o}$-algebras with the same residue field as $\mathfrak{o}$), they may not be normal nor reduced. But we know that both $R[\frac{1}{p}]$ and $R_\infty[\frac{1}{p}]$ are formally smooth over $\mathbb{Q}_p$. We fix this situation by applying normalization, as follows.

We let $\widetilde{R}_\infty$ be the normalization of the image of R_∞ in $R_\infty[\frac{1}{p}]$. (More naturally speaking, $\widetilde{R}_\infty$ is the normalization of $(R_\infty)_{\mathrm{red}}/(R_\infty)_{\mathrm{red}}[p^\infty]$.) Note that $\widetilde{R}_\infty$ is finite over R_∞ since every complete local noetherian ring is excellent [27, IV_2, (7.8.3)(iii)], so $\widetilde{R}_\infty$ satisfies the assumptions on R' in the statement of Gabber's theorem. By the property of normalization, we have a natural map $\widetilde{R}_\infty \to R_\infty[\frac{1}{p}]$ which induces an isomorphism $\widetilde{R}_\infty[\frac{1}{p}] \xrightarrow{\sim} R_\infty[\frac{1}{p}]$. We identify $\widetilde{R}_\infty$ with its image in $R_\infty[\frac{1}{p}]$.

We also define $\widetilde{R}$ to be the normalization of the image of R in $R[\frac{1}{p}]$. We view $\widetilde{R}$ as an $\mathfrak{o}$-subalgebra of $R[\frac{1}{p}]$ via the natural isomorphism $\widetilde{R}[\frac{1}{p}] \xrightarrow{\sim} R[\frac{1}{p}]$. The normalization $\widetilde{R}$ is finite over R, reduced and p-torsion free. (Thus, $\widetilde{R}$ satisfies the assumptions on R in the statement of Gabber's theorem.) Furthermore the map $\underline{\mathrm{res}}^{\mathrm{cris}}[\frac{1}{p}] : R_\infty[\frac{1}{p}] \to R[\frac{1}{p}]$ restricts to $\underline{\widetilde{\mathrm{res}}} : \widetilde{R}_\infty \to \widetilde{R}$.

Now, we prove the following proposition.

Proposition 11.6.9. *Let R and R_∞ be as above §11.6.8. Then the F-morphism $\underline{\mathrm{res}}[\frac{1}{p}] : \operatorname{Spec} R[\frac{1}{p}] \to \operatorname{Spec} R_\infty[\frac{1}{p}]$ induces a bijection between the sets of closed points and trivial residue field extension at each closed point.*

By Gabber's theorem (Theorem 11.6.7), the proposition implies that the map $\underline{\widetilde{\mathrm{res}}} : \widetilde{R}_\infty \to \widetilde{R}$ induced on the normalized deformation rings is an isomorphism, so in particular, $\underline{\mathrm{res}}[\frac{1}{p}] : R_\infty[\frac{1}{p}] \to R[\frac{1}{p}]$ is an isomorphism.

Remark 11.6.10. A similar situation to Proposition 11.6.9 came up in Kisin's work [54, Proposition 3.13], where he analyzed a certain crystalline deformation ring of "intermediate" Hodge-Tate weights (with $\mathscr{K} = \mathbb{Q}_p$). Kisin constructed a concrete ring which maps into the normalized crystalline deformation ring, and after inverting p induces a bijection on the set of closed points and induces trivial residue field extensions at such points. Kisin uses Gabber's theorem and obtained the connectedness result of the crystalline deformation ring which is strong enough to prove the modularity lifting theorem in his setup.

11.6.11

We outline the proof of Proposition 11.6.9. Let R and R_∞ be as in §11.6.8. By the full faithfulness of the restriction to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$ on crystalline $\mathbb{Q}_p$-representations (as stated in Theorem 2.4.10), we see that $\underline{\mathrm{res}}[\frac{1}{p}] : \operatorname{Spec} R[\frac{1}{p}] \to \operatorname{Spec} R_\infty[\frac{1}{p}]$ induces an injective map on the sets of closed points. In order to show the surjectivity and triviality of residue field extensions at closed points, it suffices to show that for any finite extension E/F, the map $(\operatorname{Spec} R)(E) \to (\operatorname{Spec} R_\infty)(E)$ induced by $\underline{\mathrm{res}}$ is surjective. Let $x \in (\operatorname{Spec} R_\infty)(E)$ and let V_x be the corresponding E-representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$. Since the $\mathfrak{o}$-algebra map $x : R_\infty \to E$ factors through $\mathfrak{o}_E$, we also obtain a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable $\mathfrak{o}_E$-lattice $T_x \subset V_x$, such that $T_x \otimes_{\mathfrak{o}_E} \mathfrak{o}_E/\mathfrak{m}_E \cong \bar{\rho}_\infty \otimes_{\mathbb{F}} \mathfrak{o}_E/\mathfrak{m}_E$ as a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representation. We now proceed by showing the following.

Step(1) The $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representation V_x (uniquely) extends to a crystalline representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ with Hodge-Tate weights in $[0, 1]$.

Step(2) The $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable $\mathfrak{o}_E$-lattice $T_x \subset V_x$ is $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable.

Step(3) We have a $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-equivariant isomorphism $T_x \otimes_{\mathfrak{o}_E} \mathfrak{o}_E/\mathfrak{m}_E \cong \bar{\rho} \otimes_{\mathbb{F}} \mathfrak{o}_E/\mathfrak{m}_E$ extending the initial such $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-isomorphism.

The above claims imply that T_x defines an $\mathfrak{o}_E$-point of $\operatorname{Spec} R$ which maps to $x \in (\operatorname{Spec} R_\infty)(E)$ by $\underline{\text{res}}$, hence we obtain the desired surjectivity.

Step (1) is an immediate consequence of [52, Lemma 2.2.2], which is also stated as Corollary 2.4.7 in this paper. The following lemma takes care of **Step (3)**.

Lemma 11.6.12. *The functor from the category of mod p finite flat representations of $\mathcal{G}_{\mathscr{K}}$ to the category of representations of $\mathcal{G}_{\mathscr{K}_\infty}$ over $\mathbb{F}_p$ defined by "restricting to $\mathcal{G}_{\mathscr{K}_\infty}$" is* fully faithful.

Proof. The main idea is to use Fontaine's ramification estimates for mod p finite flat $\mathcal{G}_{\mathscr{K}}$-representations [30], which turn out to be "very sharp."[27] Fontaine's ramification estimate asserts that the higher ramification group $I_{\mathscr{K}}^{e^*}$ is in the kernel of any mod p finite flat $\mathcal{G}_{\mathscr{K}}$-representation, where $e^* = \frac{ep}{p-1}$. (We follow Serre's upper indexing [72, Ch.IV] while the convention Fontaine used in [30] differs from Serre's by a shift by 1.) The idea is to show that the natural inclusion induces an isomorphism

$$\mathcal{G}_{\mathscr{K}_\infty}/(I_{\mathscr{K}}^{e^*} \cap \mathcal{G}_{\mathscr{K}_\infty}) \xrightarrow{\sim} \mathcal{G}_{\mathscr{K}}/I_{\mathscr{K}}^{e^*}. \tag{11.6.12.1}$$

The lemma follows from this isomorphism, since any mod p finite flat representation $\bar{\rho} : \mathcal{G}_{\mathscr{K}} \to \mathrm{GL}(n, \overline{\mathbb{F}}_p)$ factors through $\mathcal{G}_{\mathscr{K}}/I_{\mathscr{K}}^{e^*}$ so the isomorphism (11.6.12.1) shows the equality $\bar{\rho}(\mathcal{G}_{\mathscr{K}_\infty}) = \bar{\rho}(\mathcal{G}_{\mathscr{K}})$ of the images (i.e., $\bar{\rho}$ and $\bar{\rho}|_{\mathcal{G}_{\mathscr{K}_\infty}}$ are essentially the same representation.)

To rephrase the isomorphism (11.6.12.1), we want to show that the open subgroup $I_{\mathscr{K}}^{e^*} \cdot \mathcal{G}_{\mathscr{K}_\infty} \subset \mathcal{G}_{\mathscr{K}}$ fills up the full Galois group $\mathcal{G}_{\mathscr{K}}$; i.e., there is no non-trivial subextension of $\mathscr{K}_\infty/\mathscr{K}$ fixed by $I_{\mathscr{K}}^{e^*}$. This follows from the claim below, which is a nice exercise with higher ramification groups.

Claim. *Let $\mathscr{K}_1 := \mathscr{K}(\pi^{(1)})$ for $\pi^{(1)} \in \mathscr{K}_\infty$ such that $(\pi^{(1)})^p = \pi$. Then $I_{\mathscr{K}}^{e^*}$ does*

[27] The author learned this idea from [2, Proposition 8.5.1]

not *fix* $\mathscr{K}_1$.

Put $\mathscr{K}' := \mathscr{K}(\zeta_p)$ where $\zeta_p \in \overline{\mathscr{K}}$ is a primitive p-th root of unity, and consider $\mathscr{K}'_1 := \mathscr{K}'(\pi^{(1)})$, which is a Galois closure of $\mathscr{K}_1/\mathscr{K}$. We put $\boldsymbol{\mathcal{G}} := \mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}) \cong \mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}') \rtimes \mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}_1)$. Here, $\mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}') \cong \mathbb{Z}/p\mathbb{Z}$ is the wild inertia subgroup of $\boldsymbol{\mathcal{G}}$, and $\mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}_1) \subset (\mathbb{Z}/p\mathbb{Z})^\times$ acts on $\mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}')$ by Kummer theory.

Since the upper indexing is well-behaved under passing to quotients [72, IV.§3, Proposotion 14], it is enough to show that $\boldsymbol{\mathcal{G}}^{e^*}$ does not fix $\mathscr{K}_1$. Indeed, we show that $\boldsymbol{\mathcal{G}}^{e^*} = \mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}')$ by computing the higher ramification subgroups $\boldsymbol{\mathcal{G}}_i$ in the lower numbering and using the Herbrand function, exploiting the explicitness of the situation.

Clearly, $\boldsymbol{\mathcal{G}}_1 = \mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}')$, and $\boldsymbol{\mathcal{G}}_i$ is a subgroup of $\mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}')$ for all $i > 0$. Let $c := \left|I_{\mathscr{K}'_1/\mathscr{K}_1}\right|$ denote the ramification index of $\mathscr{K}'_1/\mathscr{K}_1$, so the absolute ramification index of $\mathscr{K}'_1$ is epc. (Also note that $[\boldsymbol{\mathcal{G}}_0 : \boldsymbol{\mathcal{G}}_1] = c$.) Since lower indexing is well-behaved under passing to subgroups, we may replace $\mathscr{K}$ by $\mathscr{K}'$ and assume $\boldsymbol{\mathcal{G}} = \mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}')$ in order to compute $\boldsymbol{\mathcal{G}}_i$. For any $\gamma \in \boldsymbol{\mathcal{G}}$, we have $\gamma(\pi^{(1)}) = \zeta_p^{\varepsilon(\gamma)}\pi^{(1)}$ for the cocycle ε given by Kummer theory, so

$$v_{\mathscr{K}'_1}\left(\gamma(\pi^{(1)}) - \pi^{(1)}\right) - 1 \ = v_{\mathscr{K}'_1}\left(\zeta_p^{\varepsilon(\gamma)} - 1\right) + v_{\mathscr{K}'_1}\left(\pi^{(1)}\right) - 1 \ = e^*c + c - 1.$$

This shows that $\boldsymbol{\mathcal{G}}_i = \mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}')$ for $0 < i \leq e^*c + c - 1$, and $\boldsymbol{\mathcal{G}}_i = \{\mathrm{id}\}$ for $i > e^*c+c-1$ (without assuming $\mathscr{K} = \mathscr{K}'$). Since $[\boldsymbol{\mathcal{G}}_0 : \boldsymbol{\mathcal{G}}_i] = c$ for $0 < i \leq e^*c+c-1$, we obtain

$$\begin{aligned} \boldsymbol{\mathcal{G}}^r &= \mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}'), && \text{for } 0 < r \leq e^* + \tfrac{c-1}{c} \\ \boldsymbol{\mathcal{G}}^r &= \{\mathrm{id}\}, && \text{for } r > e^* + \tfrac{c-1}{c}. \end{aligned}$$

In particular, $\boldsymbol{\mathcal{G}}^{e^*} = \mathrm{Gal}(\mathscr{K}'_1/\mathscr{K}')$ does not fix $\mathscr{K}_1$. □

For **Step (2)**, we use Breuil's theory of strongly divisible lattices (of weight $\leqslant 1$). We only use the fact that one can naturally associate a $\mathcal{G}_{\mathscr{K}}$-stable $\mathbb{Z}_p$-lattice of V_x to a strongly divisible "lattice" by purely semilinear algebra means (i.e., without relating the strongly divisible modules of weight $\leqslant 1$ with Barsotti-Tate groups over $\mathfrak{o}_{\mathscr{K}}$). Since it takes a significant digression to introduce the relevant definitions, we carry out these steps in a separate chapter §XII.

11.7 Representability

In this section, we prove Theorem 11.1.2. In fact, the proof is via Ramakrishna's theory [68, Theorem 1.1], which is built upon Schlessinger's criterion [71, Theorem 2.11]. This is familiar from the flat deformation problem for the Galois group of a finite extension of $\mathbb{Q}_p$. But the crucial difference is that the tangent spaces of $|\mathscr{D}_{\rho_{\mathbb{F}}}|$ and $\left|\mathscr{D}^{\square}_{\rho_{\mathbb{F}}}\right|$ are *not* finite, even when the residue field k of K is finite, so we additionally have to show that the finiteness of the tangent spaces of $\left|\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}\right|$ and $\left|\mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}}}\right|$ when k is finite.

11.7.1 Resumé of Mazur's and Ramakrishna's theory

Given a functor $F : \mathfrak{AR}_{\mathfrak{o}} \to (\mathbf{Sets})$, Schlessinger found three conditions (H1)-(H3) which are equivalent for F to have a hull. He also showed that F is pro-representable if and only if F satisfies an additional condition (H4). For the statement and a proof, see [71, Thm 2.11].

Mazur [63, §1.2] showed that for a profinite group Γ and a continuous $\mathbb{F}$-linear Γ-representation $\rho_{\mathbb{F}}$, the deformation functor $|\mathscr{D}_{\rho_{\mathbb{F}}}|$ always satisfies (H1)-(H2), and satisfies (H4) if $\rho_{\mathbb{F}}$ is absolutely irreducible. In fact, the argument can be modified to show (H4) if $\mathrm{End}_{\mathcal{G}_K}(\rho_{\mathbb{F}}) \cong \mathbb{F}$. Furthermore, Mazur showed that $\left|\mathscr{D}^{\square}_{\rho_{\mathbb{F}}}\right|$ always satisfies (H1), (H2) and (H4) with no assumption on $\rho_{\mathbb{F}}$.

On the other hand, in order to show that the deformation functor and the framed deformation functor satisfy (H3) (i.e., the tangent space is a finite-dimensional $\mathbb{F}$-vector space) we need a p-finiteness assumption on Γ [63, §1.1], which is satisfied by an absolute Galois group for a finite extension of $\mathbb{Q}_p$ (and certain quotients of the absolute Galois group of any finite extension of $\mathbb{Q}$). Unfortunately, $\boldsymbol{\mathcal{G}}_K$ does not satisfy the p-finiteness even when the residue field k of K is finite. In fact, (H3) fails even when $\rho_{\mathbb{F}}$ is 1-dimensional. To see this, consider the cohomological interpretation of the tangent space; i.e., $|\mathscr{D}_{\rho_{\mathbb{F}}}|(\mathbb{F}[\epsilon]) \cong \mathrm{H}^1(K, \mathrm{Ad}(\rho_{\mathbb{F}}))$, where $\mathrm{Ad}(\rho_{\mathbb{F}})$ is $\mathrm{End}_{\mathbb{F}}(T_{\mathbb{F}})$ with the natural $\boldsymbol{\mathcal{G}}_K$-action. If $\rho_{\mathbb{F}}$ is 1-dimensional, then $\mathrm{Ad}(\rho_{\mathbb{F}})$ is the trivial 1-dimensional $\boldsymbol{\mathcal{G}}_K$-representation, so $\mathrm{H}^1(K, \mathrm{Ad}(\rho_{\mathbb{F}})) \cong \mathrm{Hom}_{\mathrm{cont}}(\boldsymbol{\mathcal{G}}_K, \mathbb{F})$, which is always infinite since we have infinitely many Artin-Schreier cyclic p-extensions (via the theory of norm fields and local class field theory in characteristic $p > 0$). This also shows that $|\mathscr{D}_{\rho_{\mathbb{F}}}|$ and $\left|\mathscr{D}^{\square}_{\rho_{\mathbb{F}}}\right|$ never satisfies (H3) for any finite-dimensional $\rho_{\mathbb{F}}$, since we have a surjective map $|\mathscr{D}_{\rho_{\mathbb{F}}}|(\mathbb{F}[\epsilon]) \twoheadrightarrow \left|\mathscr{D}_{\det(\rho_{\mathbb{F}})}\right|(\mathbb{F}[\epsilon])$ induced by taking determinant[28], and in particular these 'unrestricted' deformation functors are never represented by a complete local noetherian ring.

Now, let us look at the subfunctors $\left|\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}}\right| \subset |\mathscr{D}_{\rho_{\mathbb{F}}}|$ and $\left|\mathscr{D}^{\square,\leqslant h}_{\rho_{\mathbb{F}}}\right| \subset \left|\mathscr{D}^{\square}_{\rho_{\mathbb{F}}}\right|$ which consist of deformations with extra properties of interest. We have seen, in Proposition 9.2.2, that these subfunctors are closed under subobjects, quotients, and direct sums. Under this setup, Ramakrishna [68, proof of Theorem 1.1] proved that if the ambient functor satisfies (Hi) for some i =1, 2, 3 or 4, then so does the subfunctor. (For this result, see also §25 and §23 of [64].)

Applying this to our setup, we obtain the following results.

[28] For any deformation $\det(\rho_{\mathbb{F}}) + \epsilon \cdot c \in \left|\mathscr{D}_{\det(\rho_{\mathbb{F}})}\right|(\mathbb{F}[\epsilon])$ (where $c : \boldsymbol{\mathcal{G}}_K \to F$ is a cocycle), consider $\rho_{\mathbb{F}} + \epsilon \cdot \tilde{c}$ with $\tilde{c} := \begin{pmatrix} c & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{pmatrix}$ has determinant $\det(\rho_{\mathbb{F}}) + \epsilon \cdot c$.

1. The functor $\left|\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}\right|$ always satisfies (H1)-(H2), and satisfies (H4) if $\mathrm{End}_{\mathcal{G}_K}(\rho_{\mathbb{F}}) \cong \mathbb{F}$.

2. The functor $\left|\mathscr{D}_{\rho_{\mathbb{F}}}^{\square,\leqslant h}\right|$ always satisfies (H1), (H2), and (H4) with no assumptions on $\rho_{\mathbb{F}}$.

Recall that the natural 1-morphism $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h} \to \left|\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}\right|$ is a 1-isomorphism if $\mathrm{End}_{\mathcal{G}_K}(\rho_{\mathbb{F}}) \cong \mathbb{F}$. Therefore, the representability assertion of Theorem 11.1.2 reduces to the following theorem.

Theorem 11.7.2. *Assume that the residue field k of $\mathfrak{o}_K$ is finite. Then the tangent spaces $\left|\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}\right|(\mathbb{F}[\epsilon])$ and $\left|\mathscr{D}_{\rho_{\mathbb{F}}}^{\square,\leqslant h}\right|(\mathbb{F}[\epsilon])$ are finite-dimensional $\mathbb{F}$-vector spaces.*

Proof. Since $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square,\leqslant h}$ is a $\widehat{\underline{\mathrm{PGL}}}(n)$-torsor over $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}$, it is enough to show that the set $\left|\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}\right|(\mathbb{F}[\epsilon])$ is finite. We proceed in the following steps.

11.7.2.1 Setup

Let $[(\rho_{\mathbb{F}[\epsilon]}, T_{\mathbb{F}[\epsilon]}, \iota)] \in \left|\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h}\right|(\mathbb{F}[\epsilon])$. Set $M_{\mathbb{F}} := \underline{D}_{\mathcal{E}}^{\leqslant h}(T_{\mathbb{F}})$ and $M_{\mathbb{F}[\epsilon]} := \underline{D}_{\mathcal{E}}^{\leqslant h}(T_{\mathbb{F}[\epsilon]})$. See §11.1.4 for the definition of $\underline{D}_{\mathcal{E}}^{\leqslant h}$. Viewing $M_{\mathbb{F}[\epsilon]}$ as a $\mathfrak{o}_{\mathcal{E},\mathbb{F}}$-module, there is a $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M}_{\mathbb{F}[\epsilon]} \subset M_{\mathbb{F}[\epsilon]}$ of $\mathcal{P}$-height $\leqslant h$ for $M_{\mathbb{F}[\epsilon]}$. In general, there may be no $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-lattice of $\mathcal{P}$-height $\leqslant h$ for $M_{\mathbb{F}[\epsilon]}$, as we saw in Remark 11.1.7

11.7.2.2 Strategy and Outline

Using the 1-isomorphism $\underline{D}_{\mathcal{E}}^{\leqslant h}$, we rephrase our goal. We need to show that there exist only finitely many equivalence classes of étale φ-modules $M_{\mathbb{F}[\epsilon]}$ which are free over $\mathfrak{o}_{\mathcal{E},\mathbb{F}[\epsilon]}$ and equipped with an isomorphism $\iota : M_{\mathbb{F}} \xrightarrow{\sim} M_{\mathbb{F}[\epsilon]} \otimes_{\mathbb{F}[\epsilon]} \mathbb{F}$, where two such lifts $(M_{\mathbb{F}[\epsilon]}, \iota)$ and $(M'_{\mathbb{F}[\epsilon]}, \iota')$ are *equivalent* if there exists an isomorphism $M_{\mathbb{F}[\epsilon]} \xrightarrow{\sim} M'_{\mathbb{F}[\epsilon]}$ which respects ι and ι'.

One possible approach is to fix a $\mathfrak{o}_{\mathcal{E},\mathbb{F}}$-basis for $M_{\mathbb{F}}$ and a lift to a $\mathfrak{o}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis for each deformation $M_{\mathbb{F}[\epsilon]}$ once and for all, and identify $M_{\mathbb{F}[\epsilon]}$ with the "φ-matrix" with respect to the fixed basis and interpret the equivalence relations in terms of the "φ-matrix." Then the problem turns into showing the finiteness of equivalence classes of matrices with some constraints – namely, having some "integral structure" or more precisely, having a $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$. So the fixed basis has to "reflect" the integral structure.

This approach faces the following obstacles. First, the deformations $M_{\mathbb{F}[\epsilon]}$ we consider do not necessarily allow any $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-lattice of $\mathcal{P}$-height $\leqslant h$. In other words, we cannot expect, in general, to find a $\mathfrak{o}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\{e_i\}$ for $M_{\mathbb{F}[\epsilon]}$ in such a way that $\{e_i, \epsilon e_i\}$ generates a $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$. In §11.7.2.3–§11.7.2.5 we show that a slightly weaker statement is true. Roughly speaking, we show that there is an $\mathfrak{o}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis $\{\mathbf{e}_i\}$ for $M_{\mathbb{F}[\epsilon]}$ so that there exists an $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$ with a $\mathfrak{S}_{\mathbb{F}}$-basis only involving "uniformly" u-adically bounded denominators as coefficients relative to the $\mathfrak{o}_{\mathcal{E},\mathbb{F}}$-basis $\{\mathbf{e}_i, \epsilon \cdot \mathbf{e}_i\}$ of $M_{\mathbb{F}[\epsilon]}$.

Second, we may have more than one $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$ for $M_{\mathbb{F}}$ or for $M_{\mathbb{F}[\epsilon]}$, especially when he is large. In particular, a fixed $\mathfrak{S}_{\mathbb{F}}$-lattice for $M_{\mathbb{F}}$ may not "lift" to any $\mathfrak{S}_{\mathbb{F}}$-lattice for some deformation $M_{\mathbb{F}[\epsilon]}$ of $\mathcal{P}$-height $\leqslant h$. We get around this issue by varying the basis for $M_{\mathbb{F}}$ among finitely many choices. This step is carried out in §11.7.2.6. In fact, we only need finitely many choices of bases because there are only finitely many $\mathfrak{S}_{\mathbb{F}}$-lattices of $\mathcal{P}$-height $\leqslant h$ for a fixed $M_{\mathbb{F}}$, thanks to Lemma 9.2.4.

Once we get around these technical problems, we show the finiteness by a σ-conjugacy computation of matrices. This is the key technical step and crucially uses the assumption that the $\mathbb{F}[\epsilon]$-deformations we consider (or rather, the corresponding étale φ-module $M_{\mathbb{F}[\epsilon]}$) admits a $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$ (in $M_{\mathbb{F}[\epsilon]}$). See Claim 11.7.2.1 for more details.

11.7.2.3

Let $M_{\mathbb{F}[\epsilon]}$ correspond to some deformation of $\mathcal{P}$-height $\leqslant h$. Even though there may not exist any $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-lattice of $\mathcal{P}$-height $\leqslant h$ for $M_{\mathbb{F}[\epsilon]}$, we can find a $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M}_{\mathbb{F}[\epsilon]}$ with $\mathcal{P}$-height $\leqslant h$ such that $\mathfrak{M}_{\mathbb{F}[\epsilon]}$ is stable under multiplication by ϵ.[29] In fact, the maximal $\mathfrak{S}$-lattice $\mathfrak{M}^+_{\mathbb{F}[\epsilon]}$ of $\mathcal{P}$-height $\leqslant h$ does the job. (More generally, a maximal $\mathfrak{S}$-lattice $\mathfrak{M}^+$ of $\mathcal{P}$-height $\leqslant h$ in a torsion étale φ-module M is easily seen to be functorial in M.)

11.7.2.4

For a $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M}_{\mathbb{F}[\epsilon]} \subset M_{\mathbb{F}[\epsilon]}$ of $\mathcal{P}$-height $\leqslant h$ which is stable under the ϵ-multiplication, we can find a $\mathfrak{S}_{\mathbb{F}}$-basis which can be "nicely" written in terms of some $\mathfrak{o}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis of $M_{\mathbb{F}[\epsilon]}$, as follows. Let $\mathfrak{M}_{\mathbb{F}}$ be the image of $\mathfrak{M}_{\mathbb{F}[\epsilon]} \to M_{\mathbb{F}}$ induced by the natural projection $M_{\mathbb{F}[\epsilon]} \to M_{\mathbb{F}}$, which is a $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$ in $M_{\mathbb{F}}$. Now, consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{N} & \longrightarrow & \mathfrak{M}_{\mathbb{F}[\epsilon]} & \longrightarrow & \mathfrak{M}_{\mathbb{F}} & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & \epsilon\cdot M_{\mathbb{F}[\epsilon]} & \longrightarrow & M_{\mathbb{F}[\epsilon]} & \longrightarrow & M_{\mathbb{F}} & \longrightarrow & 0, \end{array}$$

where $\mathfrak{N} := \mathrm{Ker}[\mathfrak{M}_{\mathbb{F}[\epsilon]} \twoheadrightarrow \mathfrak{M}_{\mathbb{F}}]$ is a $\mathfrak{S}_{\mathbb{F}}$-lattice of $\mathcal{P}$-height $\leqslant h$ in $M_{\mathbb{F}[\epsilon]}$. We choose a $\mathfrak{S}_{\mathbb{F}}$-basis $\{e_1, \cdots, e_n\}$ of $\mathfrak{M}_{\mathbb{F}}$. Viewing them as a $\mathfrak{o}_{\mathcal{E},\mathbb{F}}$-basis of $M_{\mathbb{F}}$, we lift $\{e_i\}$ to an $\mathfrak{o}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis of $M_{\mathbb{F}[\epsilon]}$ (again denoted $\{e_i\}$). By assumption from the previous step, we have $\bigoplus_{i=1}^n \mathfrak{S}_{\mathbb{F}}\cdot(\epsilon e_i) \subset \mathfrak{N}$, where both are $\mathfrak{S}_{\mathbb{F}}$-lattices of $\mathcal{P}$-height $\leqslant h$ for $\epsilon\cdot M_{\mathbb{F}[\epsilon]}$. It follows that $(\frac{1}{u^{r_i}}\epsilon)e_i$ form a $\mathfrak{S}_{\mathbb{F}}$-basis of $\mathfrak{N}$, for some non-negative integers r_i. Therefore, $\{e_i, (\frac{1}{u^{r_i}}\epsilon)e_i\}$ is a $\mathfrak{S}_{\mathbb{F}}$-basis of $\mathfrak{M}_{\mathbb{F}[\epsilon]}$. We also use the isomorphism $M_{\mathbb{F}} \xrightarrow{\sim} \epsilon\cdot M_{\mathbb{F}[\epsilon]}$ via multiplication by ϵ.

[29]This means that $\mathfrak{M}_{\mathbb{F}[\epsilon]}$ is a φ-module over $\mathfrak{S}_{\mathbb{F}[\epsilon]}$, but does not force $\mathfrak{M}_{\mathbb{F}[\epsilon]}$ to be a projective $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-module. Hence, such $\mathfrak{M}_{\mathbb{F}[\epsilon]}$ may not be a $\mathfrak{S}_{\mathbb{F}[\epsilon]}$-module of $\mathcal{P}$-height $\leqslant h$. The example $\mathfrak{M}_{\mathbb{F}[\epsilon]} \cong \mathfrak{S}_{\mathbb{F}}\cdot\mathbf{e} \oplus \mathfrak{S}_{\mathbb{F}}\cdot(\frac{1}{u}\epsilon\mathbf{e})$ discussed in Remark 11.1.7 is such an example.

11.7.2.5

In this step, we find an upper bound for the non-negative integers r_i only depending on $\mathfrak{M}_{\mathbb{F}}$ and the choice of $\mathfrak{S}_{\mathbb{F}}$-basis of $\mathfrak{M}_{\mathbb{F}}$. Since $\mathfrak{N}$ is a φ-stable submodule, it contains

$$(\dagger) \qquad \varphi_{M_{\mathbb{F}[\epsilon]}}\left(\sigma^*\left(\frac{1}{u^{r_i}}\epsilon e_i\right)\right) = \left(\frac{1}{u^{qr_i}}\epsilon\right)\cdot\varphi_{\mathfrak{M}_{\mathbb{F}}}(\sigma^* e_i) = \frac{1}{u^{qr_i}}\epsilon\cdot\sum_{j=1}^{n}\alpha_{ij}e_j,$$

where $\alpha_{ij} \in \mathfrak{S}_{\mathbb{F}}$ satisfy $\varphi_{\mathfrak{M}_{\mathbb{F}}}(\sigma^* e_i) = \sum_{j=1}^{n}\alpha_{ij}e_j$. Note that we obtain the first identity because $\varphi_{M_{\mathbb{F}[\epsilon]}}(\sigma^* e_i)$ lifts $\varphi_{\mathfrak{M}_{\mathbb{F}}}(\sigma^* e_i)$ and the ϵ-multiple ambiguity in the lift disappears when we multiply against ϵ. Since any element of $\mathfrak{N}$ is a $\mathfrak{S}_{\mathbb{F}}$-linear combination of $(\frac{1}{u^{r_i}}\epsilon)e_i$, we obtain inequalities $\mathrm{ord}_u(\alpha_{ij}) - qr_i \geq -r_j$ for all i,j from the above equation ($\dagger$). Let $r := \max_j\{r_j\}$ and we obtain $qr_i \leq r + \min_j\{\mathrm{ord}_u(\alpha_{ij})\}$ for all i. (Note that the right side of the inequality is always finite.) Now, by taking the maximum among all i, we obtain

$$r \leq \frac{1}{q-1}\max_i\left\{\min_j\{\mathrm{ord}_u(\alpha_{ij})\}\right\} < \infty$$

This shows that the non-negative integers r_i has an upper bound which only depends on the matrices entries for $\varphi_{\mathfrak{M}_{\mathbb{F}}}$ with respect to the $\mathfrak{S}_{\mathbb{F}}$ basis of $\mathfrak{M}_{\mathbb{F}}$.

11.7.2.6 Recapitulation

For each $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M}_{\mathbb{F}}^{(a)}$ of $\mathcal{P}$-height $\leqslant h$ for $M_{\mathbb{F}}$, we fix a $\mathfrak{S}_{\mathbb{F}}$-basis $\{e_i^{(a)}\}$ and let $\alpha^{(a)} = (\alpha_{ij}^{(a)}) \in \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$ be the "φ-matrix" with respect to $\{e_i^{(a)}\}$. In other words, $\varphi_{\mathfrak{M}_{\mathbb{F}}^{(a)}}(\sigma^* e_i^{(a)}) = \sum_{i=1}^{n}\alpha_{ij}^{(a)}e_j^{(a)}$. We also view $\{e_i^{(a)}\}$ as a $\mathfrak{o}_{\mathcal{E},\mathbb{F}}$-basis for $M_{\mathbb{F}}$ and $(\alpha_{ij}^{(a)})$ is the matrix for $\varphi_{M_{\mathbb{F}}}$ with respect to $\{e_i^{(a)}\}$. Note that $(\alpha_{ij}^{(a)})$ is invertible over $\mathfrak{o}_{\mathcal{E},\mathbb{F}}$ since $M_{\mathbb{F}} = \mathfrak{M}_{\mathbb{F}}^{(a)}[\frac{1}{u}]$ is an étale $(\varphi, \mathfrak{o}_{\mathcal{E},\mathbb{F}}$-module. We pick an integer $r^{(a)} \geq \frac{1}{q-1}\max_i\left\{\min_j\{\mathrm{ord}_u(\alpha_{ij})\}\right\} < \infty$, for each index a. As remarked earlier in §11.7.2.2, there exist only finitely many $\mathfrak{S}_{\mathbb{F}}$-lattices of $\mathcal{P}$-height $\leqslant h$ for $M_{\mathbb{F}}$, thanks to Lemma 9.2.4, so the index a runs through a finite set.

For any $M_{\mathbb{F}[\epsilon]}$ which corresponds to a deformation of $\mathcal{P}$-height $\leqslant h$, we may find a $\mathfrak{S}_{\mathbb{F}}$-lattice $\mathfrak{M}_{\mathbb{F}[\epsilon]} \subset M_{\mathbb{F}[\epsilon]}$ of $\mathcal{P}$-height $\leqslant h$ which is stable under ϵ-multiplication §11.7.2.3. The image of $\mathfrak{M}_{\mathbb{F}[\epsilon]}$ inside $M_{\mathbb{F}}$ is equal to some $\mathfrak{M}_{\mathbb{F}}^{(a)}$. Lift the chosen basis $\{e_i^{(a)}\}$ to an $\mathfrak{o}_{\mathcal{E},\mathbb{F}[\epsilon]}$-basis for $M_{\mathbb{F}[\epsilon]}$. Then $\mathfrak{M}_{\mathbb{F}[\epsilon]}$ admits a $\mathfrak{S}_{\mathbb{F}}$-basis of form $\{e_i^{(a)}, (\frac{1}{u^{r_i}}\epsilon)e_i^{(a)}\}$ for some integers $r_i \leq r^{(a)}$ (§11.7.2.4–§11.7.2.5).

Let us consider the matrix representation of $\varphi_{M_{\mathbb{F}[\epsilon]}}$ with respect to the basis $\{e_i^{(a)}\}$. We have $\varphi_{M_{\mathbb{F}[\epsilon]}}(e_i^{(a)}) = \sum_i(\alpha_{ij}^{(a)} + \epsilon\beta_{ij}^{(a)})e_j^{(a)}$ for some $\beta^{(a)} = (\beta_{ij}^{(a)}) \in \mathrm{Mat}_n(\mathfrak{o}_{\mathcal{E},\mathbb{F}})$ because $\varphi_{M_{\mathbb{F}[\epsilon]}}$ lifts $\varphi_{M_{\mathbb{F}}}$. In fact, since $\mathfrak{M}_{\mathbb{F}[\epsilon]}$ is φ-stable, it follows that $\beta \in \frac{1}{u^{r^{(a)}}} \cdot \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$. We say two such matrices β and β' are *equivalent* if there exists a matrix $X \in \mathrm{Mat}_n(\mathfrak{o}_{\mathcal{E},\mathbb{F}})$ such that $\beta' = \beta + (\alpha^{(a)}\cdot\sigma(X) - X\cdot\alpha^{(a)})$. This equation is obtained from the following:

$$(\alpha^{(a)} + \epsilon\beta') = (\mathrm{Id}_n + \epsilon X)^{-1}\cdot(\alpha^{(a)} + \epsilon\beta)\cdot\sigma(\mathrm{Id}_n + \epsilon X),$$

which defines the equivalence of two étale φ-modules whose φ-structures are given by $(\alpha^{(a)} + \epsilon\beta)$ and $(\alpha^{(a)} + \epsilon\beta')$, respectively.

Now, the theorem is reduced to prove the following claim: *for each a, there exist only finitely many equivalence classes of matrices $\beta \in \frac{1}{u^{r^{(a)}}}\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$.* Indeed, by varying both a and the equivalence classes of β, we cover all the possible deformations $M_{\mathbb{F}[\epsilon]}$ of "$\mathcal{P}$-height $\leqslant h$" up to isomorphism, hence the theorem is proved.

From now on, we fix a and suppress the superscript $^{(a)}$ everywhere. For example, $r := r^{(a)}$ and $\alpha := \alpha^{(a)}$. Proving the following claim is the last step of the proof.

Claim 11.7.2.1. *For any $X \in u^c \,\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$ with $c > 2he$, the matrices β and $\beta + X$ are equivalent.*[30]

(Granting this claim, we have a surjective map from $\left(\frac{1}{u^r}\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})\right) / \left(u^c\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})\right)$

[30] The inequality $c > 2he$ is used to ensure $q(c - he) > c$. Therefore, if $q \neq 2$ then $c = 2he$ also works.

onto the set of equivalence classes of β's, and the former is a finite set[31], as desired.)

We prove the claim by "successive approximation." Let $\gamma = u^{he}\cdot\alpha^{-1}$. Since $\mathfrak{M}_{\mathbb{F}} := \mathfrak{M}_{\mathbb{F}}^{(a)}$ is of $\mathcal{P}$-height $\leqslant h$ and $\mathcal{P}(u)$ has image in $\mathfrak{S}_{\mathbb{F}} \cong (k\otimes_{Fq}\mathbb{F})[[u]]$ with u-order e, we know that $\gamma \in \mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$. We set $Y^{(1)} := \frac{1}{u^{he}}\cdot(X\gamma)$, which is in $u^{c-he}\,\mathrm{Mat}_n(\mathfrak{S}_F)$ by the assumption on X. Then $\beta+X$ is equivalent to

$$(\beta+X)+(\alpha\cdot\sigma(Y^{(1)})-Y^{(1)}\alpha)=\beta+\alpha\cdot\sigma(Y^{(1)}) =: \beta+X^{(1)}$$

with $X^{(1)}\in u^{c^{(1)}}\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$, where $c^{(1)} := q(c-he) > c$. Now for any positive integer i, we recursively define the following

$$Y^{(i)} := \frac{1}{u^{he}}\cdot(X^{(i-1)}\gamma), \qquad X^{(i)} := \alpha\cdot\sigma(Y^{(i)}), \qquad c^{(i)} := q(c^{(i-1)}-he).$$

One can check that $c^{(i)} > c(i-1)(> 2he)$, $X^{(i)} \in u^{c^{(i)}}\cdot\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$, and $Y^{(i)} \in u^{c^{(i-1)}-he}\,\mathrm{Mat}_n(\mathfrak{S}_{\mathbb{F}})$. Also, $\beta+X$ is equivalent to

$$(\beta+X)+\big(\alpha\cdot\sigma(Y^{(1)}+\cdots+Y^{(i)})-(Y^{(1)}+\cdots+Y^{(i)})\alpha\big)=\beta+X^{(i)}.$$

From the inequality $c^{(i)} > c^{(i-1)}$, it follows that the infinite sum $Y := \sum_{i=1}^{\infty} Y^{(i)}$ converges and $X^{(i)}\to 0$ as $i\to\infty$. Therefore we see that $\beta+X$ is equivalent to

$$\begin{aligned}(\beta+X)+(\alpha\cdot\sigma(Y)-y\cdot\alpha) &= (\beta+X)+\left(\alpha\cdot\sigma\big(\sum_{i=1}^{\infty}Y^{(i)}\big)-\big(\sum_{i=1}^{\infty}Y^{(i)}\big)\cdot\alpha\right)\\ &= \lim_{i\to\infty}(\beta+X^{(i)})=\beta,\end{aligned}$$

so we are done. □

To complete the proof of Theorem 11.1.2, it remains to show the following relative representability result, which is, again, "essentially" a consequence of Ramakrishna's theory [68, Theorem 1.1].

[31]We crucially used the fact that we can bound the denominator.

Proposition 11.7.3. *The natural inclusions* $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h} \hookrightarrow \mathscr{D}_{\rho_{\mathbb{F}}}$ *and* $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square,\leqslant h} \hookrightarrow \mathscr{D}_{\rho_{\mathbb{F}}}^{\square}$ *of* $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$*-groupoids are relatively representable by surjective maps in* $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ *and its formation commutes with 2-projective limits in the sense of Definition 10.4.8. In other words, for any given deformation or framed deformation over* $A \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$*, there exists a universal quotient* $A^{\leqslant h}$ *of* A *over which the deformation or framed deformation is of* $\mathcal{P}$*-height* $\leqslant h$*.*

Recall that the formation of the natural inclusions $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h} \hookrightarrow \mathscr{D}_{\rho_{\mathbb{F}}}$ and $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square,\leqslant h} \hookrightarrow \mathscr{D}_{\rho_{\mathbb{F}}}^{\square}$ commutes with 2-projective limit (as observed below Definition 10.4.2). From this we obtain a natural isomorphism $A^{\leqslant h} \cong \varprojlim_n (A/\mathfrak{m}_A^n)^{\leqslant h}$ for any $A \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$.

Proof. It is enough to show that $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h} \hookrightarrow \mathscr{D}_{\rho_{\mathbb{F}}}$ is relatively representable by surjective maps, since the other inclusion $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square,\leqslant h} \hookrightarrow \mathscr{D}_{\rho_{\mathbb{F}}}^{\square}$ is a "2-base change" of $\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h} \hookrightarrow \mathscr{D}_{\rho_{\mathbb{F}}}$ under $\mathscr{D}_{\rho_{\mathbb{F}}}^{\square} \to \mathscr{D}_{\rho_{\mathbb{F}}}$, and relative representability pulls back.

Consider $\xi \in \mathscr{D}_{\rho_{\mathbb{F}}}(A)$ for $A \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$. The natural projection $\mathrm{pr}_1 : (\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h})_\xi = (\mathscr{D}_{\rho_{\mathbb{F}}}/\xi) \times_{\mathscr{D}_{\rho_{\mathbb{F}}}} \mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h} \to (\mathscr{D}_{\rho_{\mathbb{F}}}/\xi)$ is fully faithful, so we regard the left side as a full subcategory of the right side via pr_1. And since $(\mathscr{D}_{\rho_{\mathbb{F}}}/\xi)$ is co-fibered in equivalence relations, so is its full subcategory $(\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h})_\xi$. Therefore, it is enough to show the natural monomorphism of functors $|\mathrm{pr}_1| : \left|(\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h})_\xi\right| \hookrightarrow |(\mathscr{D}_{\rho_{\mathbb{F}}}/\xi)|$ is relatively representable by surjective maps of rings.

The objects of $|(\mathscr{D}_{\rho_{\mathbb{F}}}/\xi)|$ are the isomorphism classes $[\xi \to \eta]$ of morphisms in $\mathscr{D}_{\rho_{\mathbb{F}}}$, so we have a natural notion of direct sums, sub-objects and quotients using the corresponding notion for η. By Proposition 9.2.2, the subfunctor $\left|(\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h})_\xi\right|$ is closed under these operations. We can therefore repeat the proof of [68, Theorem 1.1] for our setup to show that $\left|(\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h})_\xi\right|$ is representable.

Now let $A' \in \widehat{\mathfrak{AR}}_{\mathfrak{o}}$ be the object which represents $(\mathscr{D}_{\rho_{\mathbb{F}}}^{\leqslant h})_\xi$, and let $A \to A'$ be the morphism in $\widehat{\mathfrak{AR}}_{\mathfrak{o}}$ which represents the 1-morphism $(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}})_\xi \hookrightarrow (\mathscr{D}_{\rho_{\mathbb{F}}}/\xi)$. It is left to show that $A \to A'$ is surjective. Since both rings are complete local noetherian with the same residue field, it is enough to show that the morphism induces a surjective map on "reduced" Zariski cotangent spaces $\mathfrak{m}_A/(\mathfrak{m}_{\mathfrak{o}} + \mathfrak{m}_A^2) \to \mathfrak{m}_{A'}/(\mathfrak{m}_{\mathfrak{o}} + \mathfrak{m}_{A'}^2)$, which in turn is equivalent to the injectivity of $\left|(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}})_\xi\right|(\mathbb{F}[\epsilon]) \to |(\mathscr{D}_{\rho_{\mathbb{F}}}/\xi)|(\mathbb{F}[\epsilon])$ by the duality of finite-dimensional $\mathbb{F}$-vector spaces. But the morphism of functors $\left|(\mathscr{D}^{\leqslant h}_{\rho_{\mathbb{F}}})_\xi\right| \to |(\mathscr{D}_{\rho_{\mathbb{F}}}/\xi)|$ is a monomorphism, by assumption. □

CHAPTER XII

Integral p-adic Hodge theory

Assume $\mathfrak{o}_0 = \mathbb{Z}_p$. We introduce new semilinear algebra objects which give rise to lattice semi-stable $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations of low Hodge-Tate weights, initiated by Breuil. Using these, we complete **Steps (2)** in §11.6.11, hence the proof of Theorem 11.6.1. Even though we will apply the results only for crystalline representations with Hodge-Tate weights in $[0,1]$, we present the theory in more generality than we need.

Since we only need classical, if not basic, results in this subject, we direct interested readers to [15] for an overview of the theory. See [58] for more recent developments in this subject.

12.1 Definitions

12.1.1 Basic assumption

Let V be a p-adic $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representation with Hodge-Tate weights in $[0,h]$. *Throughout this chapter we assume that* $0 < h \leqslant p-1$*. If* $h = p-1$*, then we additionally require* V *to be "formal" in the sense of* §*11.4.17.* For example, if $p = 2$ (so $h = 1$) we only consider "formal" representations.

Let $D := \underline{D}_{\mathrm{st}}(V(-h))$ be the weakly admissible filtered (φ, N)-module[1] covari-

[1]Following the usual convention, φ is a σ-semilinear endomorphism throughout this chapter.

antly associated to V with filtration jumps in $[0,h]$. Then V is "formal" in the sense of §11.4.17 if and only if D is "unipotent" in the sense of [15, Definition 2.1.1]; i.e., D does not admit any weakly admissible quotient pure of slope h. So we assume throughout this chapter that *if $h = p-1$ then we only consider weakly admissible filtered (φ, N)-modules that are unipotent.*

Although the key lemma below (Lemma 12.2.4) requires this basic assumption (not to mention the full force of Breuil's theory of strongly divisible modules requires this assumption), a lot of the results proven in this chapter do not require this assumption. So we will indicate whenever we actually need this assumption.

12.1.2 Breuil's theory of "filtered modules"

Let S be the p-adic completion of the divided power envelop of $W(k)[u]$ with respect to the ideal generated by $\mathcal{P}(u)$. It can be shown, with some work, that S can be viewed as a subring of $\mathscr{K}_0[[u]]$ whose elements are precisely those of the form $\sum_{i\geq 0} a_i \frac{u^i}{q(i)}$, where $q(i) := \lfloor \frac{i}{e} \rfloor$ with $e := \deg \mathcal{P}(u)$, and $a_i \in W(k)$ converge to 0 as $i \to \infty$. We define a differential operator $N := -u\frac{d}{du}$ on S. We define $\sigma : S \to S$ via extending the Witt vector Frobenius on the coefficients by $\sigma(u) = u^p$. We let $\operatorname{Fil}^h S \subset S$ denote the ideal topologically generated by $\mathcal{P}(u)^i/i!$ for $i \geqslant h$. If $h \leqslant p-1$ then $\sigma_h := \frac{\sigma}{p^h} : \operatorname{Fil}^h S \to S$ is well-defined. On the other hand, if $h > p-1$ then the image of $\operatorname{Fil}^h S$ under σ is *not* divisible by p^h. (Idea of proof: consider $\sigma(\frac{u^{ei}}{i!}) = \frac{u^{eip}}{i!} = \frac{(ip)!}{i!} \cdot \frac{u^{eip}}{(ip)!}$ and compute $\operatorname{ord}_p(\frac{(ip)!}{i!})$.) As will be clear later in this chapter, this is one of the main reasons why we work under the running assumptions in §12.1.1.

12.1.2.1 $\mathbb{Q}_p$-theory

Let D be a weakly admissible filtered (φ, N)-module over $\mathscr{K}$ with non-negative Hodge-Tate weights. We consider the finite free $S[\frac{1}{p}]$-module $\widehat{\mathcal{D}} := S \otimes_{W(k)} D$ equipped with the σ-linear endomorphism $\varphi_{\widehat{\mathcal{D}}} := \sigma_S \otimes \varphi_D$, the differential operator $N_{\widehat{\mathcal{D}}} := N \otimes \mathrm{id} + \mathrm{id} \otimes N$ over $N : S \to S$, and the decreasing filtration $\mathrm{Fil}^\bullet \widehat{\mathcal{D}}$ which is defined as follows: set $\mathrm{Fil}^0 \widehat{\mathcal{D}} := \widehat{\mathcal{D}}$, and for any $i \geq 0$ we set

$$\mathrm{Fil}^{i+1} \widehat{\mathcal{D}} := \{x \in \widehat{\mathcal{D}} |\ N_{\widehat{\mathcal{D}}}(x) \in \mathrm{Fil}^i \widehat{\mathcal{D}},\ \mathrm{pr}_\pi(x) \in \mathrm{Fil}^{i+1} D_{\mathscr{K}}\},$$

where $\mathrm{pr}_\pi : \widehat{\mathcal{D}} \twoheadrightarrow D_{\mathscr{K}}$ is induced from $S \twoheadrightarrow S/\mathcal{P}(u) \cong \mathfrak{o}_{\mathscr{K}}$ where $u \mapsto \pi$. If all the Hodge-Tate weights of D are in $[0, h]$, then the associated grading to $\mathrm{Fil}^\bullet \widehat{\mathcal{D}}$ is concentrated in degrees $[0, h]$.

Let us record the following observations:

1. There exists a unique section $D \hookrightarrow \widehat{\mathcal{D}}$ to the projection $\widehat{\mathcal{D}} \twoheadrightarrow \widehat{\mathcal{D}}/u\widehat{\mathcal{D}} \cong D$ which is compatible with φ and N. This identifies D with the $\mathscr{K}_0$-subspace of $\widehat{\mathcal{D}}$ which consists of elements killed by some power of $N_{\widehat{\mathcal{D}}}$. The filtration on $D_{\mathscr{K}}$ coincides with the image of $\mathrm{Fil}^\bullet \widehat{\mathcal{D}}$ by under $\mathrm{pr}_\pi : \widehat{\mathcal{D}} \twoheadrightarrow \widehat{\mathcal{D}}/\mathcal{P}(u)\widehat{\mathcal{D}} \cong D_{\mathscr{K}}$. In particular, the construction $D \mapsto \widehat{\mathcal{D}}$ defines a fully faithful functor into a suitable target category. See [10, §6] for the proofs and more details.

2. Assume that all the Hodge-Tate weights of D are in $[0, h]$. Then the filtration $\mathrm{Fil}^\bullet \widehat{\mathcal{D}}$ can be (uniquely) recovered from $\mathrm{Fil}^h \widehat{\mathcal{D}}$ as follows:

   $$\mathrm{Fil}^i \widehat{\mathcal{D}} = \{x \in \widehat{\mathcal{D}} :\ \mathcal{P}(u)^{h-1} x \in \mathrm{Fil}^h \widehat{\mathcal{D}}\}.$$

   Also $\varphi_{\widehat{\mathcal{D}}}$ can be recovered from $\varphi_h := \frac{\varphi}{p^h} : \mathrm{Fil}^h \widehat{\mathcal{D}} \to \widehat{\mathcal{D}}$. (*c.f.* See the definition of strongly divisible lattices in §12.1.2.2.)

3. The monodromy operator $N : D \to D$ is the zero map if and only if $N_{\widehat{\mathcal{D}}} \equiv 0 \bmod u\widehat{\mathcal{D}}$.

Later, we will associate to $\widehat{\mathcal{D}}$ a $\mathbb{Q}_p$-representation $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$ of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ which is naturally isomorphic to $\underline{V}^*_{\mathrm{st}}(D)$ (so $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$ is semi-stable with Hodge-Tate weights in $[0, h]$). We will define $\widehat{\underline{V}^*}_{\mathrm{st}}$ later in §12.2.1, for which we need to define $\widehat{A}_{\mathrm{st}}$, an S-algebra where the "integral structure" of periods lie in. See §12.1.3 for the definition of $\widehat{A}_{\mathrm{st}}$, and see [10, §6] or [58, §2.2] for more details.

The construction $D \mapsto \widehat{\mathcal{D}}$ makes sense without any assumptions on h. But for the case $h \leqslant p - 1$, one can give an intrinsic characterization[2] of the "filtered $S[\frac{1}{p}]$-modules" $\widehat{\mathcal{D}}$ which can be obtained as $\widehat{\mathcal{D}} := S \otimes_{W(k)} D$ for a weakly admissible filtered (φ, N)-module D over $\mathscr{K}$ with Hodge-Tate weights in $[0, h]$. This is done in [10, 12] under the assumption $h < p - 1$, but it is claimed in [15, Theorem 2.2.3] that this can be done when $h = p - 1$. In the intended application in this paper, any "filtered $S[\frac{1}{p}]$-module" we study are known to come from a weakly admissible filtered (φ, N)-module, so we do not need the intrinsic characterization of the essential image of the functor $D \mapsto \widehat{\mathcal{D}}$.

12.1.2.2 $\mathbb{Z}_p$-theory

Let D be a weakly admissible filtered (φ, N)-module with Hodge-Tate weights in $[0, h]$. We impose our running assumptions; i.e., $h \leqslant p-1$, and that D is "unipotent" if $h = p - 1$. Let $\widehat{\mathcal{D}} := S \otimes_{W(k)} D$ be the "filtered $S[\frac{1}{p}]$-module." We say that an S-lattice $\boldsymbol{\mathcal{M}} \subset \widehat{\mathcal{D}}$ is a *strongly divisible lattice (of weight $\leqslant h$)* if $\boldsymbol{\mathcal{M}}$ satisfies the following properties.

(SD1) $\boldsymbol{\mathcal{M}}$ is a finite free S-submodule of $\widehat{\mathcal{D}}$ which is stable under $\varphi : \widehat{\mathcal{D}} \to \widehat{\mathcal{D}}$ and

[2] i.e., a description purely in terms of φ, N, and the filtration on $\widehat{\mathcal{D}}$, without mentioning the weakly admissible filtered (φ, N)-module D from which $\widehat{\mathcal{D}}$ was constructed. For this statement, we do not need to assume that all the weakly admissible filtered (φ, N)-modules are unipotent when $h = p - 1$.

such that $\boldsymbol{\mathcal{M}}[\frac{1}{p}] = \widehat{\mathcal{D}}$.

(SD2) Set $\mathrm{Fil}^h \boldsymbol{\mathcal{M}} := \boldsymbol{\mathcal{M}} \cap \mathrm{Fil}^h \widehat{\mathcal{D}}$, then we have $\varphi(\mathrm{Fil}^h \boldsymbol{\mathcal{M}}) \subset p^h \boldsymbol{\mathcal{M}}$. We set $\varphi_h := \frac{1}{p^h}\varphi : \mathrm{Fil}^h \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}}$. (In fact, this axiom implies the seemingly stronger axiom, namely that $\varphi_h(\mathrm{Fil}^h \boldsymbol{\mathcal{M}})$ generates $\boldsymbol{\mathcal{M}}$.)

(SD3) $\boldsymbol{\mathcal{M}}$ is stable under $N : \widehat{\mathcal{D}} \to \widehat{\mathcal{D}}$; i.e., $N(\boldsymbol{\mathcal{M}}) \subset \boldsymbol{\mathcal{M}}$.

Any strongly divisible lattice $\boldsymbol{\mathcal{M}}$ in $\mathcal{D}$ is equipped with a S-submodule $\mathrm{Fil}^h \boldsymbol{\mathcal{M}} \subset \boldsymbol{\mathcal{M}}$, σ-linear map $\varphi_h : \mathrm{Fil}^h \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}}$ whose image generates $\boldsymbol{\mathcal{M}}$, and a differential operator $N_{\boldsymbol{\mathcal{M}}} : \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}}$ over $N : S \to S$. As previously, the datum $(\boldsymbol{\mathcal{M}}, \mathrm{Fil}^h \boldsymbol{\mathcal{M}}, \varphi_h, N_{\boldsymbol{\mathcal{M}}})$ which is obtained as a strongly divisible lattice in some $\widehat{\mathcal{D}} := S \otimes_{W(k)} D$ can be characterized purely in terms of $\mathrm{Fil}^h \boldsymbol{\mathcal{M}}$, φ_h, and $N_{\boldsymbol{\mathcal{M}}}$. See [15, Theorem 2.2.3] for the statement. We will later construct a $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable lattice $\underline{T}^*_{\mathrm{st}}(\boldsymbol{\mathcal{M}})$ in $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$, which is semi-stable with Hodge-Tate weights in $[0, h]$.

We say that an S-lattice $\boldsymbol{\mathcal{M}} \subset \widehat{\mathcal{D}}$ is a *quasi-strongly divisible lattice*[3] *(of weight* $\leqslant h$*)* if $\boldsymbol{\mathcal{M}}$ only satisfies (SD1) and (SD2). Such $\boldsymbol{\mathcal{M}}$ is equipped with $\mathrm{Fil}^h \boldsymbol{\mathcal{M}} := \boldsymbol{\mathcal{M}} \cap \mathrm{Fil}^h \widehat{\mathcal{D}}$ and $\varphi_h : \mathrm{Fil}^h \boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}}$, but has no differential operator $N_{\boldsymbol{\mathcal{M}}}$. For such an object, we can only associate a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable $\mathbb{Z}_p$-lattice $\underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})$ in $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$, but not necessarily $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable.

Tong Liu [58] showed that any $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable $\mathbb{Z}_p$-lattice of $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}}) \cong \underline{V}^*_{\mathrm{st}}(D)$ comes from a strongly divisible lattice in $\widehat{\mathcal{D}}$. Note that it is *not* obvious (and was not fully known before Tong Liu's theorem) that for any weakly admissible filtered (φ, N)-module D over $\mathscr{K}$ with Hodge-Tate weights in $[0, h]$, $\widehat{\mathcal{D}} := S \otimes_{W(k)} D$ admits a strongly divisible lattice $\boldsymbol{\mathcal{M}} \in \widehat{\mathcal{D}}$. These seemingly more complicated objects $\widehat{\mathcal{D}}$ are introduced and studied because one can obtain "integral" p-adic Hodge theory. On the other hand, in our intended application we will be given $\widehat{\mathcal{D}}$ together with a

[3] This terminology is introduced by Tong Liu [60].

strongly divisible lattice $\boldsymbol{\mathcal{M}}$ from the outset. All we need for the application would be that $\boldsymbol{\mathcal{M}}$ gives rise to a $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable $\mathbb{Z}_p$-lattice of $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$.

12.1.2.3 Coefficients

We can extend the definitions to allow various coefficients by requiring that all the structures are linear over the coefficient ring. We give an example which will be used later. Let $E/\mathbb{Q}_p$ be a finite extension and $\mathfrak{o}_E$ its valuation ring. We put $S_E := S \otimes_{\mathbb{Z}_p} E$ and $S_{\mathfrak{o}_E} := S \otimes_{\mathbb{Z}_p} \mathfrak{o}_E$. Now for a weakly admissible filtered (φ, N)-module D_E with E-coefficients, we see that $\widehat{\mathcal{D}}_E := S \otimes_{W(k)} D_E$ is finite free over S_E. We consider strongly divisible lattices in $\widehat{\mathcal{D}}_E$ which are $S_{\mathfrak{o}_E}$-free.

12.1.3 More "period S-algebras"

In order to define the functors into the categories of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$- and $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$- representations, we need to introduce some S-algebras where the "integral structure" of p-adic periods lie. First of all, we put $\mathfrak{R} := \varprojlim_{x^p \leftarrow x} \mathfrak{o}_{\overline{\mathscr{K}}}/(p)$. It is well-known that the $\bar{k}$-algebra $\mathfrak{R}$ is complete with respect to a naturally given valuation and $\mathrm{Frac}(\mathfrak{R})$ is algebraically closed. See [32, §1] for basic properties of $\mathfrak{R}$. As in §1.3.1.2, we fix a uniformizer $\pi \in \mathfrak{o}_{\mathscr{K}}$ such that $\mathcal{P}(\pi) = 0$ and we choose successive p-power roots $\pi^{(n)}$; i.e., $\pi^{(0)} = \pi$ and $(\pi^{(n+1)})^p = \pi^{(n)}$. The sequence $\underline{\pi} := \{\pi^{(n)}\}$ is an element of $\mathfrak{R}$, and we embed $\mathfrak{o}_K := k[[u]] \hookrightarrow \mathfrak{R}$ over k via $u \mapsto \underline{\pi}$.

Take the "canonical lift" $\theta : W(\mathfrak{R}) \twoheadrightarrow \mathfrak{o}_{\mathbb{C}_{\mathscr{K}}}$ of the first projection $\mathfrak{R} \twoheadrightarrow \mathfrak{o}_{\overline{\mathscr{K}}}/(p)$. This map is $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-equivariant for the natural actions on both sides and is a topological quotient map (for the "product topology" on the source and the natural p-adic topology on the target). We define A_{cris} as the p-adic completion of the divided power envelop of $W(\mathfrak{R})$ with respect to $\ker(\theta)$. The Witt vector Frobenius map and the $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-action on $W(\mathfrak{R})$ extend to A_{cris}. By construction, the inclusion $W(k)[u] \hookrightarrow W(\mathfrak{R})$, which satisfies $u \mapsto [\underline{\pi}]$ (where $[\underline{\pi}]$ is the Teichmüller lift of $\underline{\pi}$), uniquely extends to $S \hookrightarrow A_{\mathrm{cris}}$ which makes A_{cris} an S-algebra. (This map is well-defined since $\mathcal{P}([\underline{\pi}]) \in \ker(\theta)$.) This inclusion respects the Frobenius structures and is $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable, but *not* $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable. We let $\mathrm{Fil}^h A_{\mathrm{cris}}$ be the ideal topologically generated by $\frac{1}{i!}(\ker\theta)^i$ for $i \geqslant h$. We have $(\mathrm{Fil}^h S)A_{\mathrm{cris}} \subset \mathrm{Fil}^h A_{\mathrm{cris}}$, and from the running assumption $h \leqslant p-1$ we have $\sigma(\mathrm{Fil}^h A_{\mathrm{cris}})A_{\mathrm{cris}} \subset p^h A_{\mathrm{cris}}$, so we can define $\sigma_h := \frac{\sigma}{p^h} : \mathrm{Fil}^h A_{\mathrm{cris}} \to A_{\mathrm{cris}}$.

As observed above, A_{cris} cannot produce $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations from Breuil's divisible S-modules because the map $S \hookrightarrow A_{\mathrm{cris}}$ is not $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable. Also, A_{cris} does not have a "monodromy operator." For these reasons, we introduce a "bigger ring" with more structures. Let $\widehat{A}_{\mathrm{st}}$ be the p-adic completion of the divided power "polynomial ring" $A_{\mathrm{cris}}[X, \frac{X^i}{i!}]_{i\geq 1}$. We first define the embedding $S \hookrightarrow \widehat{A}_{\mathrm{st}}$ by $u \mapsto \frac{[\underline{\pi}]}{1+X}$ and then define the structures on $\widehat{A}_{\mathrm{st}}$ in such a way that this embedding respects all the structures: define a Frobenius map $\sigma : \widehat{A}_{\mathrm{st}} \to \widehat{A}_{\mathrm{st}}$ using $\sigma : A_{\mathrm{cris}} \to A_{\mathrm{cris}}$ on the coefficients and $\sigma(1+X) = (1+X)^p$, so $\sigma(\frac{[\underline{\pi}]}{1+X}) = \left(\frac{[\underline{\pi}]}{1+X}\right)^p$. We define the ideal

$$\mathrm{Fil}^h \widehat{A}_{\mathrm{st}} := \{\sum_{i\geq 0} a_i \frac{X^i}{i!} \in \widehat{A}_{\mathrm{st}} |\ a_i \in \mathrm{Fil}^{i-h} A_{\mathrm{cris}},\ \lim_{i\to\infty} a_i = 0\},$$

where we set $\mathrm{Fil}^w A_{\mathrm{cris}} := A_{\mathrm{cris}}$ for $w \leq 0$. Then $(\mathrm{Fil}^h S)\widehat{A}_{\mathrm{st}} \subset \mathrm{Fil}^h \widehat{A}_{\mathrm{st}}$ and the map $\sigma_h := \frac{\sigma}{p^h} : \mathrm{Fil}^h \widehat{A}_{\mathrm{st}} \to \widehat{A}_{\mathrm{st}}$ is well-defined. Let $N : \widehat{A}_{\mathrm{st}} \to \widehat{A}_{\mathrm{st}}$ be the A_{cris}-derivation $(1+X)\frac{d}{dX}$, so that $N(\frac{[\underline{\pi}]}{1+X}) = -\frac{[\underline{\pi}]}{1+X}$. For any $\gamma \in \boldsymbol{\mathcal{G}}_{\mathscr{K}}$, we let $\epsilon(\gamma) := \frac{\gamma[\underline{\pi}]}{[\underline{\pi}]} \in A_{\mathrm{cris}}$, and $\gamma \mapsto \epsilon(\gamma)$ is a continuous cocycle. We define $\gamma(1+X) := \epsilon(\gamma)(1+X)$, so $\gamma(\frac{[\underline{\pi}]}{1+X}) = \frac{[\underline{\pi}]}{1+X}$. In particular, the embedding $S \hookrightarrow \widehat{A}_{\mathrm{st}}$ is $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable! (Actually, we even have $S \xrightarrow{\sim} (\widehat{A}_{\mathrm{st}})^{\boldsymbol{\mathcal{G}}_{\mathscr{K}}}$. See [10, §4] for the proof.) The choice of the coordinate X depends on the choice of $\underline{\pi} := \{\pi^{(n)}\}$, but if we replacing $\underline{\pi}$ with $\underline{\pi}' := \underline{\varepsilon}\cdot\underline{\pi}$ where $\underline{\epsilon} = \{\epsilon^{(i)}\}_{i\geqslant 0} \in \mathfrak{R}$ with $\epsilon^{(0)} = 1$, then X gets replaced by $X' = [\underline{\epsilon}]X + ([\underline{\epsilon}] - 1) \in \mathrm{Fil}^1 \widehat{A}_{\mathrm{st}} \setminus \mathrm{Fil}^2 \widehat{A}_{\mathrm{st}}$ (obtained by setting $\frac{[\underline{\pi}]}{1+X} = \frac{[\underline{\epsilon}][\underline{\pi}]}{1+X'}$). One can directly check that this change of coordinates does not modify the embedding $S \hookrightarrow \widehat{A}_{\mathrm{st}}$ defined by $u \mapsto \frac{[\pi]}{1+X} = \frac{[\epsilon][\pi]}{1+X'}$, and it respects σ, N, and the filtration. So $\widehat{A}_{\mathrm{st}}$ only depends on the choice of $\pi = \pi^{(0)}$. (In fact, $\widehat{A}_{\mathrm{st}}$ has a coordinate-free description in terms of log-crystalline cohomology which only depends on the choice of $\pi = \pi^{(0)}$.)

We also record that the map $\widehat{A}_{\mathrm{st}} \to A_{\mathrm{cris}}$ defined by $X \mapsto 0$ is a map of S-algebras which respects the Frobenius structures and $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-actions on both sides. We emphasize that even though we are only interested in crystalline representations, we still need to work with $\widehat{A}_{\mathrm{st}}$ to obtain functors into $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations because the embedding $S \hookrightarrow A_{\mathrm{cris}}$ is only $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable. (Readers are advised not to be tricked by the notations A_{cris} and $\widehat{A}_{\mathrm{st}}$.)

We discuss the relation of $\widehat{A}_{\mathrm{st}}$ with Fontaine's period rings. Let B_{dR}^{+} be the completion of $W(\mathfrak{R})[\frac{1}{p}]$ with respect to the kernel of $\theta[\frac{1}{p}]$, and let $B_{\mathrm{dR}} := B_{\mathrm{dR}}^{+}[\frac{1}{t}]$, where t is Fontaine's p-adic analogue of $2\pi i$. Recall that A_{cris} naturally embeds into B_{dR} (and in fact $\mathscr{K} \otimes_{\mathscr{K}_0} A_{\mathrm{cris}}$ embeds into B_{dR} by [32, Théorème 4.2.4]), and we define an embedding $\widehat{A}_{\mathrm{st}} \hookrightarrow B_{\mathrm{dR}}$ over A_{cris} by $X \mapsto \frac{[\pi]}{\pi} - 1$. This embedding respects the natural $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-actions and the filtrations on both sides.

From A_{cris} we obtain Fontaine's crystalline period ring $B_{\mathrm{cris}} := A_{\mathrm{cris}}[\frac{1}{p}, \frac{1}{t}] = A_{\mathrm{cris}}[\frac{1}{t}]$. On the other hand, $\widehat{B_{\mathrm{st}}} := \widehat{A}_{\mathrm{st}}[\frac{1}{p}, \frac{1}{t}] = \widehat{A}_{\mathrm{st}}[\frac{1}{t}]$ is strictly larger than Fontaine's semi-stable period ring B_{st}. In fact, B_{st} has an embedding into $\widehat{B_{\mathrm{st}}}$ which respects all the structures, and the image is the B_{cris}-subalgebra $B_{\mathrm{cris}}[\log(1+X)] \subset \widehat{B_{\mathrm{st}}}$, or equivalently the subring of elements on which N is nilpotent. See [10, Lemma 7.1] for the proof.

Now we are ready to functorially associate to a strongly divisible S-lattice in $\widehat{\mathcal{D}} := S \otimes_{W(k)} D$ a $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable $\mathbb{Z}_p$-lattice of $\underline{V}_{\mathrm{st}}^{*}(D)$. We also define functors from other semi-linear algebra categories to the category of strongly divisible S-modules, and compare the associated Galois representations.

12.2 Galois representations

12.2.1 Construction of $\mathcal{G}_{\mathscr{K}}$-stable lattices of a semi-stable representation

Let D be the weakly admissible filtered (φ, N)-module with Hodge-Tate weights in $[0, h]$, and consider $\widehat{\mathcal{D}} := S \otimes_{W(k)} D$ with the structure of φ, N, and filtration as discussed in §12.1.2.1. Let $\mathcal{M} \subset \widehat{\mathcal{D}}$ be a strongly divisible lattice. Now, we define a $\mathbb{Q}_p[\mathcal{G}_{\mathscr{K}}]$-module $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$ and its $\mathcal{G}_{\mathscr{K}}$-stable $\mathbb{Z}_p$-submodule $\underline{T}^*_{\mathrm{st}}(\mathcal{M})$, as follows.

$$\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}}) := \mathrm{Hom}_{S[\frac{1}{p}],\varphi_h,N,\mathrm{Fil}^\bullet}(\widehat{\mathcal{D}}, \widehat{B_{\mathrm{st}}}) \tag{12.2.1.1}$$

$$\underline{T}^*_{\mathrm{st}}(\mathcal{M}) := \mathrm{Hom}_{S,\varphi_h,N,\mathrm{Fil}^h}(\mathcal{M}, \widehat{A}_{\mathrm{st}}), \tag{12.2.1.2}$$

where $\mathcal{G}_{\mathscr{K}}$ acts through $\widehat{B_{\mathrm{st}}} := \widehat{A}_{\mathrm{st}}[\frac{1}{t}]$ and $\widehat{A}_{\mathrm{st}}$ respectively. The natural inclusion $\underline{T}^*_{\mathrm{st}}(\mathcal{M}) \hookrightarrow \widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$ induces an isomorphism $\underline{T}^*_{\mathrm{st}}(\mathcal{M})[\frac{1}{p}] \xrightarrow{\sim} \widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$, which follows from the fact that an $S[\frac{1}{p}]$-map $\widehat{\mathcal{D}} \to \widehat{B_{\mathrm{st}}}$ respects the filtrations on both sides if and only if it respects Fil^h on both sides (by §12.1.2.1). Since $\underline{T}^*_{\mathrm{st}}(\mathcal{M})$ is clearly p-adically separated and complete, $\underline{T}^*_{\mathrm{st}}(\mathcal{M})$ is finite free over $\mathbb{Z}_p$ so it can naturally be viewed as a $\mathcal{G}_{\mathscr{K}}$-stable lattice of $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$.

Theorem 12.2.1.3 (Breuil)**.** *Let D be the weakly admissible filtered (φ, N)-module (with no assumptions on Hodge-Tate weights), and $\widehat{\mathcal{D}} := S \otimes_{W(k)} D$. Then there exists a natural $\mathcal{G}_{\mathscr{K}}$-isomorphism*

$$\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}}) \cong \underline{V}^*_{\mathrm{st}}(D) := \mathrm{Hom}_{\mathscr{K}_0,\varphi,N,\mathrm{Fil}^\bullet}(\widehat{\mathcal{D}}, B_{\mathrm{st}}),$$

Assume that all the Hodge-Tate weights of D are non-negative. Then under the above identification, we view $\underline{T}^*_{\mathrm{st}}(\mathcal{M})$ as a $\mathcal{G}_{\mathscr{K}}$-stable $\mathbb{Z}_p$-lattice in the semi-stable representation $\underline{V}^*_{\mathrm{st}}(D)$.

Proof. The proof of this theorem is sketched in [15, Proposition 2.2.5]. We can reduce to the case when all the Hodge-Tate weights of D are non-negative, which will be assumed from now on. We embed all the period rings and "period S-algebras" into B_{dR} in a compatible way.

We first define a map $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}}) \to \underline{V}^*_{\mathrm{st}}(D)$, and then we show this is an isomorphism. As discussed in §12.1.2.1, D can be identified with the $\mathscr{K}_0$-subspace of $\widehat{\mathcal{D}}$ whose elements are killed by some power of $N_{\widehat{\mathcal{D}}}$. For any $f \in \widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$, one can show that the image of D under f is contained in B^+_{st} since B_{st} has an embedding into $\widehat{B_{\mathrm{st}}}$ (which respects all the structures) and the image is precisely the subring of elements killed by some power of N. In order to show that $f|_D \in \underline{V}^*_{\mathrm{st}}(D)$, the only non-trivial part is to show that $f|_D$ respects the filtrations. For this, we use that $\mathrm{Fil}^\bullet D_{\mathscr{K}} = \mathrm{pr}_\pi(\mathrm{Fil}^\bullet \widehat{\mathcal{D}})$ where $\mathrm{pr}_\pi : \widehat{\mathcal{D}} \twoheadrightarrow \widehat{\mathcal{D}}/\mathcal{P}(u)\widehat{\mathcal{D}} \xrightarrow{\sim} D_{\mathscr{K}}$, and that the following diagram commutes:

(12.2.1.4)
$$\begin{array}{ccc} \widehat{\mathcal{D}} & \xrightarrow{\quad\quad f \quad\quad} & \widehat{B_{\mathrm{st}}}^{+} \\ {\scriptstyle \mathrm{pr}_\pi}\downarrow & & \downarrow \\ D_{\mathscr{K}} \xrightarrow[1\otimes f|_D]{} \mathscr{K} \otimes_{\mathscr{K}_0} B^+_{\mathrm{st}} & \hookrightarrow & B_{\mathrm{dR}}. \end{array}$$

So $f \mapsto f|_D$ defines an injective $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-equivariant map $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}}) \to \underline{V}^*_{\mathrm{st}}(D)$.

Now, we define its inverse as follows. For any $g \in \underline{V}^*_{\mathrm{st}}(D)$, consider $D \xrightarrow{g} B^+_{\mathrm{st}} \hookrightarrow \widehat{B_{\mathrm{st}}}^{+} := \widehat{A}_{\mathrm{st}}[\frac{1}{p}]$ and $S[\frac{1}{p}]$-linearly extend it to $\tilde{g} : \widehat{\mathcal{D}} \to \widehat{B_{\mathrm{st}}}^{+}$. Once we show that $\tilde{g} \in \widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$ then one can check that $g \mapsto \tilde{g}$ defines the inverse of the map $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}}) \to \underline{V}^*_{\mathrm{st}}(D)$ defined by $f \mapsto f|_D$.

Clearly, $\tilde{g}$ respects φ and N. So it is left to check that $\tilde{g}$ takes $\mathrm{Fil}^i \widehat{\mathcal{D}}$ to $\mathrm{Fil}^i \widehat{B_{\mathrm{st}}}^{+} := (\mathrm{Fil}^i \widehat{A}_{\mathrm{st}})[\frac{1}{p}]$ for all i. For this, we use the induction on i. The case $i = 0$ is trivial. Now, we assume that $\tilde{g}$ takes $\mathrm{Fil}^{i-1} \widehat{\mathcal{D}}$ to $\mathrm{Fil}^{i-1} \widehat{B_{\mathrm{st}}}^{+}$.

For any $x \in \operatorname{Fil}^i \widehat{\mathcal{D}}$, we have

(1) $\tilde{g}(x) = g(\mathrm{pr}_\pi(x)) \in \operatorname{Fil}^i(B^+_{\mathrm{st}} \otimes_{\mathscr{K}_0} \mathscr{K}) = \operatorname{Fil}^i B^+_{\mathrm{dR}} \cap (\mathscr{K} \otimes_{\mathscr{K}_0} B^+_{\mathrm{st}}),$ by (12.2.1.4),

(2) $N(\tilde{g}(x)) = \tilde{g}(N(x)) \in \operatorname{Fil}^{i-1} \widehat{B_{\mathrm{st}}}^+$ (by the induction hypothesis).

Write $\tilde{g}(x) = \sum_{n\geq 0} a_n \frac{X^n}{n!}$ where $a_n \in B^+_{\mathrm{cris}}$ such that $a_n \to 0$ p-adically. Then $\operatorname{Fil}^{i-1} \widehat{B_{\mathrm{st}}}^+$ contains

$$N(\tilde{g}(x)) = (1+X)\sum_{n\geq 0} a_{n+1}\frac{X^n}{n!} = a_1 + \sum_{n\geq 1}(a_n + a_{n+1})\frac{X^n}{n!}.$$

By the definition of $\operatorname{Fil}^{i-1} \widehat{B_{\mathrm{st}}}^+$, we have $a_1 \in \operatorname{Fil}^{i-1} B^+_{\mathrm{cris}}$ and $a_1 + a_2 \in \operatorname{Fil}^{i-2} B^+_{\mathrm{cris}}$, so we obtain $a_2 \in \operatorname{Fil}^{i-2} B^+_{\mathrm{cris}}$. By repeating this process, we get $a_n \in \operatorname{Fil}^{i-n} B^+_{\mathrm{cris}}$ (where $\operatorname{Fil}^w B^+_{\mathrm{cris}} := B_{\mathrm{cris}}$ for $w \leq 0$. This shows $\sum_{n\geq 1} a_n \frac{X^n}{n!} \in \operatorname{Fil}^i \widehat{B_{\mathrm{st}}}^+$. But (1) implies that $a_0 \in \operatorname{Fil}^i B^+_{\mathrm{cris}}$, which shows that $\tilde{g}(x) \in \operatorname{Fil}^i \widehat{B_{\mathrm{st}}}^+$. $\square$

12.2.2 $\mathcal{G}_{\mathscr{K}_\infty}$-representations

Let $\widehat{\mathcal{D}}$ be as in §12.2.1, and let $\mathcal{M} \subset \widehat{\mathcal{D}}$ be a quasi-strongly divisible lattice (for example, a strongly divisible lattice $\mathcal{M}$ by "ignoring" the differential operator $N : \mathcal{M} \to \mathcal{M}$). Now, we define a $\mathbb{Q}_p[\mathcal{G}_{\mathscr{K}_\infty}]$-module $\widehat{\underline{V}^*}_{\mathrm{qst}}(\widehat{\mathcal{D}})$ and its $\mathcal{G}_{\mathscr{K}_\infty}$-stable $\mathbb{Z}_p$-submodule $\underline{T}^*_{\mathrm{qst}}(\mathcal{M})$, as follows.

$$\widehat{\underline{V}^*}_{\mathrm{qst}}(\widehat{\mathcal{D}}) := \operatorname{Hom}_{S[\frac{1}{p}],\varphi_h,\operatorname{Fil}^h}(\widehat{\mathcal{D}}, A_{\mathrm{cris}}[1/p]) \tag{12.2.2.1}$$

$$\underline{T}^*_{\mathrm{qst}}(\mathcal{M}) := \operatorname{Hom}_{S,\varphi_h,\operatorname{Fil}^h}(\mathcal{M}, A_{\mathrm{cris}}), \tag{12.2.2.2}$$

where $\mathcal{G}_{\mathscr{K}_\infty}$ acts through $A_{\mathrm{cris}}[\frac{1}{p}]$ and A_{cris}, respectively. Clearly, $\underline{T}^*_{\mathrm{qst}}(\mathcal{M})$ is p-adically separated and complete, so the following lemma shows that $\underline{T}^*_{\mathrm{qst}}(\mathcal{M})$ is finite free over $\mathbb{Z}_p$. The following lemma is taken from [60, Lemma 3.4.3].

Lemma 12.2.2.3. *Let D be the weakly admissible filtered (φ, N)-module (with no assumptions on Hodge-Tate weights), and $\widehat{\mathcal{D}} := S \otimes_{W(k)} D$. Then the natural map*

(12.2.2.4)

$$\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}}) = \mathrm{Hom}_{S[\frac{1}{p}],\varphi_h,N,\mathrm{Fil}^h}(\widehat{\mathcal{D}}, \widehat{B_{\mathrm{st}}}^+) \to \mathrm{Hom}_{S[\frac{1}{p}],\varphi_h,\mathrm{Fil}^h}(\widehat{\mathcal{D}}, B^+_{\mathrm{cris}}) = \widehat{\underline{V}^*}_{\mathrm{qst}}(\mathcal{D}),$$

induced by the map $\widehat{B_{\mathrm{st}}}^+ \to B^+_{\mathrm{cris}}$ defined by $X \mapsto 0$ is a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-isomorphism.

Assume that all the Hodge-Tate weights of D are non-negative. For a strongly divisible lattice $\boldsymbol{\mathcal{M}}$ in $\widehat{\mathcal{D}}$, the above isomorphism $\widehat{\underline{V}^*}_{\mathrm{st}}(\mathcal{D}) \xrightarrow{\sim} \widehat{\underline{V}^*}_{\mathrm{qst}}(\mathcal{D})$ restricts to the isomorphism of $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable lattices $\underline{T}^*_{\mathrm{st}}(\boldsymbol{\mathcal{M}}) \xrightarrow{\sim} \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})$.

Using this lemma, we identify the representation spaces of $\underline{T}^*_{\mathrm{st}}(\boldsymbol{\mathcal{M}})$ and $\underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})$ so we regard $\underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})$ as the restriction of the $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-action on $\underline{T}^*_{\mathrm{st}}(\boldsymbol{\mathcal{M}})$ to $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$.

Proof. The second claim follows from the first. Let us show that the natural $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-equivariant map (12.2.2.4) is an isomorphism.

Let $f \in \widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}})$ and let $\bar{f} \in \underline{V}^*_{\mathrm{qst}}(\widehat{\mathcal{D}})$ denote the image of f. We identify D with the $\mathscr{K}_0$-subspace of $\widehat{\mathcal{D}}$ whose elements are killed by some power of N. Since the image of B_{st} in $\widehat{B_{\mathrm{st}}}^+$ is $B_{\mathrm{cris}}[\log(1+X)]$ and the map $\widehat{B_{\mathrm{st}}}^+ \twoheadrightarrow B^+_{\mathrm{cris}}$ defined by $X \mapsto 0$ maps $\frac{[\underline{\pi}]}{1+X}$ to $[\underline{\pi}] = \frac{[\underline{\pi}]}{1+X} \cdot \sum_{i \geqslant 0} \gamma^i(\log(1+X))$, it follows that for any $x \in D$ we have

$$f(x) = \sum_{i \geqslant 0} \bar{f}(N^i x)\gamma^i(\log(1+X)),$$

where γ^i is the standard ith divided power. In particular, if $\bar{f} = 0$ then $f = 0$. This shows that the natural map (12.2.2.4) is injective.

We now show the surjectivity. For any $\bar{f} \in \widehat{\underline{V}^*}_{\mathrm{qst}}(\widehat{\mathcal{D}})$, we consider the following "formal expression:"

$$f(x) = \sum_{i \geq 0} \bar{f}(N^i x)\gamma^i(\log(1+X)) \in B^+_{\mathrm{cris}}[[X]], \quad \text{for any } x \in \widehat{\mathcal{D}}.$$

If $x \in D$ (i.e., if $N^i(x) = 0$ for some i), then $f(x)$ converges in $\widehat{B_{\rm st}}^+$. On the other hand, f turns out to be S-linear, hence f defines a map into $\widehat{B_{\rm st}}^+$. Instead of proving the S-linearity, we give the following "heuristics" which can be turned into a proof. Recall that we embed $S \to A_{\rm cris}$ via $u \mapsto [\underline{\pi}]$ and $S \to \widehat{A}_{\rm st}$ via $u \mapsto \frac{[\underline{\pi}]}{1+X}$. We want to show $f(\gamma^n(u)x) = \gamma^n(\frac{[\underline{\pi}]}{1+X})\cdot f(x)$ for all $x \in \widehat{\mathcal{D}}$, where γ^n is the nth standard divided power. The following equation is a formal consequence of the Leibnitz rule:

$$N^i(\gamma^n(u)x) = \sum_{j=0}^{i} \binom{i}{j} N^j(\gamma^n(u))N^j(x) = \sum_{j=0}^{i} \binom{i}{j} (-n)^{i-j}\gamma^n(u)N^j(x).$$

By "reordering" the sum, we obtain

$$\begin{aligned} f(\gamma^n(u)x) \quad &= \quad \gamma^n([\underline{\pi}]) \sum_{i\geq 0} \sum_{j=0}^{i} \bar{f}(N^j(x))(-n)^{i-j} \binom{i}{j} \gamma^i(\log(1+X)) \\ &= \quad \gamma^n([\underline{\pi}]) \sum_{i\geq 0} \sum_{j=0}^{i} \bar{f}(N^j(x))\Big((-n)^{i-j}\gamma^{i-j}(\log(1+X))\Big)\gamma^j(\log(1+X)) \\ \text{“}=\text{”} \quad & \quad \gamma^n([\underline{\pi}]) \exp(-n\log(1+X)) \sum_{j\geq 0} \bar{f}(N^j(x))\gamma^j(\log(1+X)) \\ &= \quad \gamma^n(\frac{[\underline{\pi}]}{1+X})\cdot f(x). \end{aligned}$$

The step of "reordering the sum" (i.e., the equality in quotes) can be made precise by truncating both sides and estimating the error terms.

Now, it remains to check that:

1. f respects φ: for any $x \in \widehat{\mathcal{D}}$,

$$\begin{aligned} \sigma(f(x)) &= \sum_{i\geq 0} \sigma(\bar{f}(N^i x))\cdot\gamma^i(\sigma(\log(1+X))) \\ &= \sum_{i\geq 0} \bar{f}(p^{-i}N^i\varphi(x)))\cdot(\gamma^i(p\log(1+X))) \\ &= f(\varphi(x)). \end{aligned}$$

2. f respects N: for any $x \in \widehat{\mathcal{D}}$, we have

$$\begin{aligned} N(f(x)) &= \sum_{i\geq 0} \bar{f}(N^i x)\cdot N(\gamma^i(\log(1+X))) \\ &= \sum_{i\geq 1} \bar{f}(N^i x)\cdot \gamma^{i-1}(\log(1+X)) \\ &= f(N(x)). \end{aligned}$$

   The first equality follows since N is a derivation over A_{cris} and $\bar{f}(N^i x) \in A_{\text{cris}}$, and the second equality follows since $N(\log(1+X)) = 1$.

3. f respects the filtrations: if $x \in \mathrm{Fil}^w \widehat{\mathcal{D}}$, then we claim $f(x) \in \mathrm{Fil}^w \widehat{B_{\text{st}}}^+$. But we have $\gamma^i(\log(1+X)) \in \mathrm{Fil}^i \widehat{B_{\text{st}}}^+$, and since $\bar{f} : \widehat{\mathcal{D}} \to B^+_{\text{cris}}$ respects the filtrations we have $\bar{f}(N^i x) \in \mathrm{Fil}^{w-i} B^+_{\text{cris}}$, where $\mathrm{Fil}^w B^+_{\text{cris}} := B_{\text{cris}}$ for $w \leq 0$.

This shows that $f \in \widehat{\underline{V}^*}_{\text{st}}(\widehat{\mathcal{D}})$, which completes the proof. □

Kisin's theory [52] provides another category of φ-modules which classify $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable $\mathbb{Z}_p$-lattices in semi-stable $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representations, namely $\mathfrak{S}$-lattices of $\mathcal{P}$-height $\leqslant h$ in the (φ, N_∇)-vector bundle $\underline{\mathcal{M}}^{\mathcal{MF}}(D)$ over $\boldsymbol{\Delta}$. (See Theorem 2.4.5 for notations.) The next subsection associates a quasi-strongly divisible lattice $\boldsymbol{\mathcal{M}}$ in $\widehat{\mathcal{D}}$ to a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$ in $\underline{\mathcal{M}}^{\mathcal{MF}}(D)$ compatibly with the functors into the category of lattice $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representations.

12.2.3 Relation with Kisin's theory

Let D be a weakly admissible filtered (φ, N)-module over $\mathscr{K}$ with Hodge-Tate weights in $[0, h]$, where $h \leqslant p-1$. Kisin [52, §1] constructed a (φ, N_∇)-module $\mathcal{M} := \underline{\mathcal{M}}^{\mathcal{MF}}(D)$ over $\mathcal{O}_{\boldsymbol{\Delta}}$ from a filtered (φ, N)-module D and showed that D is weakly admissible if and only if $\mathcal{M}$ is pure of slope 0 in the sense of Kedlaya. Let $\mathfrak{M} \subset \mathcal{M}$ be a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$. We set $\boldsymbol{\mathcal{M}} := S \otimes_{\sigma,\mathfrak{S}} \mathfrak{M} \cong S \otimes_{\mathfrak{S}} (\sigma^*\mathfrak{M})$.

We have an S-linear map $\mathrm{id}\otimes\varphi_{\mathfrak{M}} : \boldsymbol{\mathcal{M}} \cong S\otimes_{\mathfrak{S}}(\sigma^*\mathfrak{M}) \to S\otimes_{\mathfrak{S}}\mathfrak{M}$. Using this, we define a S-submodule $\mathrm{Fil}^h\boldsymbol{\mathcal{M}} \subset \boldsymbol{\mathcal{M}}$ and $\varphi_h : \mathrm{Fil}^h\boldsymbol{\mathcal{M}} \to \boldsymbol{\mathcal{M}}$ as follows.

(12.2.3.1) $\mathrm{Fil}^h\boldsymbol{\mathcal{M}} := \{x\in\boldsymbol{\mathcal{M}}|\ \mathrm{id}\otimes\varphi_{\mathfrak{M}}(x)\in\mathrm{Fil}^h S\otimes_{\mathfrak{S}}\mathfrak{M}\subset S\otimes_{\mathfrak{S}}\mathfrak{M}\}$

(12.2.3.2) $\quad\varphi_h : \mathrm{Fil}^h\boldsymbol{\mathcal{M}} \xrightarrow{\mathrm{id}\otimes\varphi_{\mathfrak{M}}} \mathrm{Fil}^h S\otimes_{\mathfrak{S}}\mathfrak{M} \xrightarrow{\sigma_h\otimes\mathrm{id}} S\otimes_{\sigma,\mathfrak{S}}\mathfrak{M} = \boldsymbol{\mathcal{M}}$

The following lemma directly follows from [58, Corollary 3.2.3].

Lemma 12.2.3.3. *With the same notations as above, $\boldsymbol{\mathcal{M}} := S\otimes_{\sigma,\mathfrak{S}}\mathfrak{M}$ has a natural structure of quasi-strongly divisible lattice in $\widehat{\mathcal{D}} := S\otimes_{W(k)} D$.*

In fact, the above construction $\mathfrak{M} \rightsquigarrow S\otimes_{\sigma,\mathfrak{S}}\mathfrak{M}$ induces an equivalence of categories between $\underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$ and the category of quasi-strongly divisible lattices of weight $\leqslant h$ [16, Theorem 2.2.1]. We will not use this fact.

Sketch of Proof. Let $\iota : \mathcal{O}_{\boldsymbol{\Delta}} \hookrightarrow S[\frac{1}{p}]$ denote the embedding defined by $u\mapsto u$, which is well-defined as can be directly checked. Put $\sigma : \mathcal{O}_{\boldsymbol{\Delta}} \xrightarrow{\sigma_{\boldsymbol{\Delta}}} \mathcal{O}_{\boldsymbol{\Delta}} \overset{\iota}{\hookrightarrow} S[\frac{1}{p}]$. For a (φ, N_∇)-module $\mathcal{M}$ pure of slope 0, consider $\widehat{\mathcal{D}}_{\mathcal{M}} := S[\frac{1}{p}]\otimes_{\sigma,\mathcal{O}_{\boldsymbol{\Delta}}}\mathcal{M}$. We define $\mathrm{Fil}^h\widehat{\mathcal{D}}$ and φ_h in the same manner to (12.2.3.1) and (12.2.3.2). We put $N_{\widehat{\mathcal{D}}_{\mathcal{M}}} := N\otimes 1 + \frac{p}{\sigma(\lambda)}(1\otimes N_\nabla)$. (For any $f\in\mathcal{O}_{\boldsymbol{\Delta}} \overset{\iota}{\hookrightarrow} S$, we have $N(\iota(f)) = \iota(\frac{p}{\sigma(\lambda)}N_\nabla(f))$.) By direct computations, one show that $\widehat{\mathcal{D}}_{\mathcal{M}}$ satisfies the "intrinsic characterization" for filtered $S[\frac{1}{p}]$-modules which come from weakly admissible filtered (φ, N)-modules. See [60, Proposition 3.2.1] for the proof.

Let $\mathcal{M} := \underline{\mathcal{M}}^{\mathcal{MF}}(D)$ and let $\mathfrak{M}\subset\mathcal{M}$ be a $\mathfrak{S}$-lattice of $\mathcal{P}$-height $\leqslant h$. Then clearly, $\boldsymbol{\mathcal{M}} := S\otimes_{\sigma,\mathfrak{S}}\mathfrak{M}$ is a quasi-strongly divisible lattice in $\widehat{\mathcal{D}}_{\mathcal{M}} := S[\frac{1}{p}]\otimes_{\mathcal{O}_{\boldsymbol{\Delta}}} \mathcal{M}$. So the lemma will follow if we show that naturally $\widehat{\mathcal{D}}_{\mathcal{M}} \cong \widehat{\mathcal{D}}_D := S\otimes_{W(k)} D$ as a filtered $S[\frac{1}{p}]$-module. Note that the functor $D\rightsquigarrow\widehat{\mathcal{D}}_D$ from the category of weakly admissible filtered (φ, N)-modules to the category of filtered $S[\frac{1}{p}]$-modules is fully faithful; that we can recover the underlying (φ, N)-module D as $\widehat{\mathcal{D}}_D/u\widehat{\mathcal{D}}_D$, and the filtration $\mathrm{Fil}^\bullet D_{\mathscr{K}}$ is $\mathrm{pr}_\pi(\mathrm{Fil}^\bullet \widehat{\mathcal{D}}_D)$ where $\mathrm{pr}_\pi : \widehat{\mathcal{D}}_D \twoheadrightarrow \widehat{\mathcal{D}}_D/\mathcal{P}(u)\widehat{\mathcal{D}}_D \cong D_{\mathscr{K}}$ is the natural map. It can be directly seen that this recipe, when applied to $\widehat{\mathcal{D}}_{\mathcal{M}}$, precisely gives $\underline{D}^{\mathcal{MF}}(\mathcal{M})$ which is naturally isomorphic to D by Kisin's result (stated in Theorem 2.4.5 of this paper). This verification uses the construction of the functor $\underline{D}^{\mathcal{MF}}$. See [58, Corollary 3.2.3] for the complete proof. $\square$

Let $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$. Recall that we have a contravariant functor $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) = \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \mathfrak{o}_{\widehat{\mathcal{E}^{\mathrm{ur}}}}) \xleftarrow{\sim} \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}}^{\mathrm{ur}})$ if $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$. (That the arrows are isomorphisms follows from [31, §B Proposition 1.8.3], which is also stated as Lemma 8.1.6 in this paper.) Therefore, we obtain a natural $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-equivariant morphism:

$$\mathfrak{T}_{\mathfrak{M}} : \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \cong \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}}^{\mathrm{ur}}) \to \mathrm{Hom}_{S,\varphi_h,\mathrm{Fil}^h}(\boldsymbol{\mathcal{M}}, A_{\mathrm{cris}}) = \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}}) \tag{12.2.3.4}$$

where the arrow in the middle is defined as follows: for a $\mathfrak{S}$-linear map $f : \mathfrak{M} \to \widehat{\mathfrak{S}}^{\mathrm{ur}}$, we consider $\widetilde{f} : \boldsymbol{\mathcal{M}} = S \otimes_{\sigma,\mathfrak{S}} \mathfrak{M} \to A_{\mathrm{cris}}$ obtained by S-linearly extending $\mathfrak{M} \xrightarrow{f} \widehat{\mathfrak{S}}^{\mathrm{ur}} \xrightarrow{\sigma} A_{\mathrm{cris}}$, where we view S as a $\mathfrak{S}$-algebra via $\sigma : \mathfrak{S} \to S$. One can check that if f respects φ, then $\widetilde{f}$ respects φ_h and takes $\mathrm{Fil}^h \boldsymbol{\mathcal{M}}$ to $\mathrm{Fil}^h A_{\mathrm{cris}}$, so $f \mapsto \widetilde{f}$ defines a map $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \to \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})$. This map is furthermore $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-equivariant since $\sigma : \widehat{\mathfrak{S}}^{\mathrm{ur}} \hookrightarrow A_{\mathrm{cris}}$ is $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-equivariant.

Let us state the key lemma, which crucially uses all of the basic assumptions in §12.1.1. We postpone the proof to §12.3.

Lemma 12.2.4. *Let $h \leqslant p-1$ and $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$. Assume that $\mathfrak{M}$ is unipotent in the sense of §8.3.6 if $h = p-1$. Then the natural map $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \to T^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})$ in* (12.2.3.4) *is an isomorphism as a $\mathbb{Z}_p$-lattice $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-representation.*

The following lemma is very specific to the case of $h = 1$, which is proved in [14, Proposition 5.1.3] under the assumption that $p > 2$. The identical proof works when

$p = 2$ (even without assuming that D is "unipotent").

Lemma 12.2.5. *Let D be a weakly admissible filtered isocrystal with Hodge-Tate weights in $[0,1]$. (So the monodromy operator $N : D \to D$ is a zero map.) Let $\widehat{\mathcal{D}} := S[\frac{1}{p}] \otimes_{W(k)} D$ as before. Then any quasi-strongly divisible lattice $\boldsymbol{\mathcal{M}} \in \widehat{\mathcal{D}}$ is stable under $N_{\widehat{\mathcal{D}}} : \widehat{\mathcal{D}} \to \widehat{\mathcal{D}}$, hence $\boldsymbol{\mathcal{M}}$ is a strongly divisible lattice in $\widehat{\mathcal{D}}$.*

Corollary 12.2.6. *Let V be a crystalline $\mathbb{Q}_p$-representation of $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$ with Hodge-Tate weights in $[0,1]$, and let $T \subset V$ be a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable $\mathbb{Z}_p$-lattice. If $p = 2$ then assume that either V has no nontrivial unramified quotient, or V has no non-trivial $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-subrepresentation on which the inertia group $I_{\mathscr{K}}$ acts via the p-adic cyclotomic character. Then T is necessarily $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable.*

Proof. Let $D := \underline{D}_{\mathrm{cris}}(V(-1))$ be the corresponding weakly admissible filtered isocrystal, so D is unipotent if and only if V has no nontrivial unramified quotient. If V has no non-trivial $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-subrepresentation on which the inertia group $I_{\mathscr{K}}$ acts via the p-adic cyclotomic character, then we replace V with $V^*(1)$.

Let $\widehat{\mathcal{D}} := S \otimes_{W(k)} D$ be the corresponding filtered (φ, N)-module over S. We identify V with $\underline{V}^*_{\mathrm{cris}}(D^*(1))$ and $\widehat{\underline{V}^*}_{\mathrm{st}}(\widehat{\mathcal{D}}^*(1))$ via natural isomorphisms, where $\widehat{\mathcal{D}}^*(1)$ is the filtered S-module corresponding to $D^*(1)$. Kisin's theory produces a $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant 1}$ equipped with a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-equivariant isomorphism $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \cong T$. (See the comment below Theorem 2.4.10 or see [52, Proposition 2.1.15].) By Lemma 12.2.3.3, $\boldsymbol{\mathcal{M}} := S \otimes_{\sigma,\mathfrak{S}} \mathfrak{M}$ can be naturally viewed as a quasi-strongly divisible lattice in $\widehat{\mathcal{D}}$ such that $\underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}}) = T$ as a $\mathbb{Z}_p$-lattice in V. Lemma 12.2.5 asserts that $\boldsymbol{\mathcal{M}}$ is a strongly divisible lattice, hence $\underline{T}^*_{\mathrm{st}}(\boldsymbol{\mathcal{M}}) = \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}}) = T$ by Lemmas 12.2.2.3 and 12.2.4. In particular, T is a $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-stable lattice. ∎

12.3 Proof of Lemma 12.2.4

Let us recall the statement of of Lemma 12.2.4. Let $h \leqslant p-1$ and $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$. Assume that $\mathfrak{M}$ is unipotent in the sense of §8.3.6 if $h = p-1$. Consider the following natural $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-equivariant map

$$\mathfrak{T}_{\mathfrak{M}} : \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \cong \mathrm{Hom}_{\mathfrak{S},\varphi}(\mathfrak{M}, \widehat{\mathfrak{S}}^{\mathrm{ur}}) \to \mathrm{Hom}_{S,\varphi_h,\mathrm{Fil}^h}(\boldsymbol{\mathcal{M}}, A_{\mathrm{cris}}) = \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})$$

induced by $\sigma : \widehat{\mathfrak{S}}^{\mathrm{ur}} \to A_{\mathrm{cris}}$. See the discussion following (12.2.3.4) for the construction of this map.

Clearly, $\mathfrak{T}_{\mathfrak{M}}$ is injective since $\sigma : \widehat{\mathfrak{S}}^{\mathrm{ur}} \to A_{\mathrm{cris}}$ is injective. So in order to prove Lemma 12.2.4, it is enough to show the surjectivity of $\mathfrak{T}_{\mathfrak{M}}$. Since $\underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})$ is p-adically separated and complete (which is immediate from the definition), we may apply successive approximation[4] to reduce to showing the surjectivity of $\mathfrak{T}_{\mathfrak{M}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p : \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \to \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$.

Let us consider the "mod p reduction" $\overline{\mathfrak{M}} := \mathfrak{S}/p\mathfrak{S} \otimes_{\mathfrak{S}} \mathfrak{M}$, and $\overline{\boldsymbol{\mathcal{M}}} := S/pS \otimes_S \boldsymbol{\mathcal{M}}$ equipped with $\mathrm{Fil}^h \overline{\boldsymbol{\mathcal{M}}} := S/pS \otimes_S \mathrm{Fil}^h \boldsymbol{\mathcal{M}}$ and $\bar{\varphi}_h : \mathrm{Fil}^h \overline{\boldsymbol{\mathcal{M}}} \to \overline{\boldsymbol{\mathcal{M}}}$. By Lemma 5.1.9, the $\mathbb{F}_p[\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}]$-module $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is naturally isomorphic to $\underline{T}^*_{\mathfrak{S}}(\overline{\mathfrak{M}})$. Now, by Fontaine's lemma [31, §B, Proposition 1.8.3] (which is Lemma 8.1.6), we have the natural isomorphism

$$\mathrm{Hom}_{\mathfrak{S}/p\mathfrak{S},\varphi}(\overline{\mathfrak{M}}, \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{S}/p\mathfrak{S},\varphi}(\overline{\mathfrak{M}}, \mathfrak{o}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}/p\mathfrak{o}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}) =: \underline{T}^*_{\mathfrak{S}}(\overline{\mathfrak{M}}),$$

which is induced from the natural inclusion $\widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}} \hookrightarrow \mathfrak{o}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}/p\mathfrak{o}_{\widehat{\mathcal{E}}^{\mathrm{ur}}}$.

From the natural projection $A_{\mathrm{cris}} \twoheadrightarrow A_{\mathrm{cris}}/pA_{\mathrm{cris}}$, we obtain the following natural

[4] More precisely, we are applying Nakayama's lemma to $\mathfrak{T}_{\mathfrak{M}} \otimes_{\mathbb{Z}_p} \mathbb{Z}/(p^n)$, for which we don't *a priori* have to know the target is finitely generated. If we assume that $\mathfrak{M}$ corresponds to a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable $\mathbb{Z}_p$-lattice of some semi-stable representation with non-negative Hodge-Tate weights (which will be the case in the intended application) then $\mathfrak{T}_{\mathfrak{M}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is an isomorphism by Lemma 12.2.2.3 and Theorem 2.4.10, so we see that the target $\underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})$ of $\mathfrak{T}_{\mathfrak{M}}$ is a finitely generated $\mathbb{Z}_p$-module.

injective map:

$$\mathrm{Hom}_{S,\varphi_h,\mathrm{Fil}^h}(\boldsymbol{\mathcal{M}}, A_{\mathrm{cris}}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \to \mathrm{Hom}_{S/pS,\varphi_h,\mathrm{Fil}^h}(\overline{\boldsymbol{\mathcal{M}}}, A_{\mathrm{cris}}/pA_{\mathrm{cris}}), \tag{12.3.0.1}$$

where the left side is $\underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ and the right side is $\underline{T}^*_{\mathrm{qst}}(\overline{\boldsymbol{\mathcal{M}}})$. Therefore we obtain the natural map

$$\mathfrak{T}_{\overline{\mathfrak{M}}} : \underline{T}^*_{\mathfrak{S}}(\overline{\mathfrak{M}}) \overset{\mathfrak{T}_{\mathfrak{M}}\otimes_{\mathbb{Z}_p}\mathbb{F}_p}{\longrightarrow} \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \overset{(12.3.0.1)}{\longrightarrow} \underline{T}^*_{\mathrm{qst}}(\overline{\boldsymbol{\mathcal{M}}}). \tag{12.3.0.2}$$

We obtain the same map $\mathfrak{T}_{\overline{\mathfrak{M}}}$ using the map on the second arguments $\sigma : \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}} \to A_{\mathrm{cris}}/pA_{\mathrm{cris}}$, by the construction similar to (12.2.3.4). The following lemma is the main step of the proof.

Lemma 12.3.1. *Let $h \leqslant p-1$ and $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant h}$. Assume that $\mathfrak{M}$ is unipotent in the sense of §8.3.6 if $h = p-1$. Then, the natural map $\mathfrak{T}_{\overline{\mathfrak{M}}} : \underline{T}^*_{\mathfrak{S}}(\overline{\mathfrak{M}}) \to \underline{T}^*_{\mathrm{qst}}(\overline{\boldsymbol{\mathcal{M}}})$ is injective.*

12.3.2 Deducing Lemma 12.2.4 from Lemma 12.3.1

We now show that Lemma 12.3.1 implies that $\mathfrak{T}_{\mathfrak{M}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p : \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \to \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is an isomorphism, so Lemma 12.2.4 follows from successive approximation as we noted at the beginning of §12.3.

Since $\mathfrak{T}_{\overline{\mathfrak{M}}} : \underline{T}^*_{\mathfrak{S}}(\overline{\mathfrak{M}}) \to \underline{T}^*_{\mathrm{qst}}(\overline{\boldsymbol{\mathcal{M}}})$ is injective by Lemma 12.3.1, it follows the construction of $\mathfrak{T}_{\overline{\mathfrak{M}}}$ that $\mathfrak{T}_{\mathfrak{M}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p : \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \to \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is injective. So it is enough to show that the $\mathbb{F}_p$-dimensions of the source and the target are equal. We have $\dim_{\mathbb{F}_p} \underline{T}^*_{\mathrm{qst}}(\overline{\boldsymbol{\mathcal{M}}}) = \mathrm{rank}_{S/pS}\, \overline{\boldsymbol{\mathcal{M}}} = \mathrm{rank}_{\mathfrak{S}/p\mathfrak{S}}\, \overline{\mathfrak{M}}$ by [12, Lemme 2.3.1.2], which forces $\mathfrak{T}_{\overline{\mathfrak{M}}}$ and (12.3.0.1) to be isomorphisms.

The following special case of Lemma 12.2.4 which suffices for proving Corollary 12.2.6 can be deduced from Lemma 12.3.1 without invoking [12, Lemme 2.3.1.2]: namely when $\overline{\boldsymbol{\mathcal{M}}}$ comes from the mod p reduction of $\mathfrak{M}$ which comes from a $\boldsymbol{\mathcal{G}}_{\mathscr{K}_\infty}$-stable lattice of a semi-stable $\boldsymbol{\mathcal{G}}_{\mathscr{K}}$-representation. In this case, we know by Lemma

12.2.2.3 and Theorem 2.4.10 that $\mathfrak{T}_{\mathfrak{M}}[\frac{1}{p}] : \underline{T}^*_{\mathfrak{S}}(\mathfrak{M})[\frac{1}{p}] \to \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})[\frac{1}{p}]$ is an isomorphism, so in particular the $\mathbb{Z}_p$-ranks of $\underline{T}^*_{\mathfrak{S}}(\mathfrak{M})$ and $\underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}})$ are the same. Thus, injectivity of $\mathfrak{T}_{\mathfrak{M}}$ implies surjectivity.

The rest of the section is devoted to proving Lemma 12.3.1.

12.3.3 Proof of Lemma 12.3.1: the case $h \leqslant p-2$

We give a proof for the case $h \leqslant p-2$ following Breuil [13, Proposition 4.2.1][5]. (This automatically rules out the case when $p = 2$ and $h = 1$.)

We recall the notation from §12.1.3. Let $\mathfrak{R} := \varprojlim_{x^p \leftarrow x} \mathfrak{o}_{\overline{\mathscr{K}}}/(p)$ and consider the canonical lift $\theta : W(\mathfrak{R}) \twoheadrightarrow \mathfrak{o}_{\mathbb{C}_{\mathscr{K}}}$ of the "first projection" $\mathfrak{R} \twoheadrightarrow \mathfrak{o}_{\overline{\mathscr{K}}}/(p)$. We set $\mathrm{Fil}^1 W(\mathfrak{R}) := \ker\theta$ and $\mathrm{Fil}^1 \mathfrak{R} := \ker[\theta \otimes_{\mathbb{Z}_p} \mathbb{F}_p : \mathfrak{R} \twoheadrightarrow \mathfrak{o}_{\overline{\mathscr{K}}}/(p)]$. Recall that A_{cris} is the p-adic completion of the divided power envelop of $W(\mathfrak{R})$ with respect to $\mathrm{Fil}^1 W(\mathfrak{R})$. It can be checked that $A_{\mathrm{cris}}/pA_{\mathrm{cris}}$ is precisely the divided power envelop of $\mathfrak{R}$ with respect to $\mathrm{Fil}^1 \mathfrak{R}$. (See [7, Remark 3.20(8)] for the proof.)

Recall that the map

$$\mathfrak{T}_{\overline{\mathfrak{M}}} : \mathrm{Hom}_{\mathfrak{S}/p\mathfrak{S},\varphi}(\overline{\mathfrak{M}}, \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}}) \to \mathrm{Hom}_{S/pS,\varphi_h,\mathrm{Fil}^h}(\overline{\boldsymbol{\mathcal{M}}}, A_{\mathrm{cris}}/pA_{\mathrm{cris}})$$

is induced from the map $\sigma : \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}} \to A_{\mathrm{cris}}/pA_{\mathrm{cris}}$. (See the comment below (12.3.0.2).) One can check, by hand, that the kernel of $\sigma : \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}} \to A_{\mathrm{cris}}/pA_{\mathrm{cris}}$ is principally generated by u^e, where e is the ramification index of $\mathscr{K}$.

Assume that $f \in \underline{T}^*_{\mathfrak{S}}(\overline{\mathfrak{M}}) = \mathrm{Hom}_{\mathfrak{S}/p\mathfrak{S},\varphi}(\overline{\mathfrak{M}}, \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}})$ is in the kernel of $\mathfrak{T}_{\overline{\mathfrak{M}}}$. Then, for any $x \in \overline{\mathfrak{M}}$, we have $f(x) \in u^e(\widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}})$. Suppose $f(x) \in u^{e'}(\widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}})$ for some $e' \geq e$. Since $\overline{\mathfrak{M}}$ is of $\mathcal{P}$-height $\leqslant h$, there exists $y \in \overline{\mathfrak{M}}$ such that $\varphi_{\overline{\mathfrak{M}}}(\sigma^* y) = u^{eh}x$. Since $\mathcal{P}(u)$ mod p is (a unit multiple of) u^e, we have

$$f(x) = u^{-eh} f\left(\varphi_{\overline{\mathfrak{M}}}(\sigma^* y)\right) = u^{-eh}\sigma\left(f(y)\right) \in u^{e'p-eh}(\widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}}) \subset u^{2e'}(\widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}}),$$

[5] Although [13, Proposition 4.2.1] is only worked out when $e = 1$, the argument works with little modification for any e.

since we assumed that $h \leqslant p-2$. By iterating this process, we conclude that $f = 0$.

12.3.4 Non-example: the case $h = p - 1$

Before we present the proof of the case $h = p - 1$, we give an example of *non-unipotent* $\mathfrak{M} \in \underline{\mathrm{Mod}}_{\mathfrak{S}}(\varphi)^{\leqslant p-1}$ where the lemma fails to hold. Take $\mathfrak{M} := \mathfrak{S}(p-1)$; i.e., $\mathfrak{M} \cong \mathfrak{S}\mathbf{e}$ with $\varphi_{\mathfrak{M}}(\sigma^*\mathbf{e}) = \mathcal{P}(u)^{p-1}\mathbf{e}$. Let $\overline{\mathfrak{M}} := \mathfrak{M} \otimes_{\mathfrak{S}} \mathfrak{S}/p\mathfrak{S}$ and $\overline{\boldsymbol{\mathcal{M}}} := S/pS \otimes_{\sigma,\mathfrak{S}} \mathfrak{M}$. We show that $\mathfrak{T}_{\overline{\mathfrak{M}}} : \underline{T}^*_{\mathfrak{S}}(\overline{\mathfrak{M}}) \to \underline{T}^*_{\mathrm{qst}}(\overline{\boldsymbol{\mathcal{M}}})$ is the zero map, which in turn implies that $\mathfrak{T}_{\mathfrak{M}} \otimes_{\mathbb{Z}_p} \mathbb{F}_p : \underline{T}^*_{\mathfrak{S}}(\mathfrak{M}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p \to \underline{T}^*_{\mathrm{qst}}(\boldsymbol{\mathcal{M}}) \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ is the zero map. In particular, $\mathfrak{T}_{\overline{\mathfrak{M}}}$ cannot be injective (so $\mathfrak{T}_{\mathfrak{M}}$ cannot be an isomorphism).

Let $f \in \underline{T}^*_{\mathfrak{S}}(\overline{\mathfrak{M}}) = \mathrm{Hom}_{\mathfrak{S}/p\mathfrak{S},\varphi}(\overline{\mathfrak{M}}, \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}})$ be any element. Then we have

$$(f(\mathbf{e}))^p = \sigma\left(f(\mathbf{e})\right) = f\left(\varphi_{\overline{\mathfrak{M}}}(\sigma^*\mathbf{e})\right) = u^{(p-1)e} \cdot f(\mathbf{e}).$$

If f is non-zero then we have $(f(\mathbf{e}))^{p-1} = u^{(p-1)e}$, so $f(\mathbf{e}) = \alpha \cdot u^e$ where $\alpha \in \mathbb{F}_p^\times$. On the other hand, $\sigma : \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}} \to A_{\mathrm{cris}}/pA_{\mathrm{cris}}$ maps any multiple of u^e to 0. This proves that $\mathfrak{T}_{\overline{\mathfrak{M}}}$ is the zero map.

12.3.5 Proof of Lemma 12.3.1: the case $h = p - 1$

Now, we handle the remaining case.[6] Put $h = p - 1$, and assume that $\overline{\mathfrak{M}}$ is unipotent of $\mathcal{P}$-height $\leqslant h$ (or equivalently, $\mathfrak{M}$ is). Let $f \in \underline{T}^*_{\mathfrak{S}}(\overline{\mathfrak{M}})$ be in the kernel of $\mathfrak{T}_{\overline{\mathfrak{M}}}$. We set $\overline{\mathfrak{N}} := f(\overline{\mathfrak{M}}) \subset \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}}$, which is a $\mathfrak{S}/p\mathfrak{S}$-submodule stable under the pth power map $\sigma : \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}} \to \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}}$. This makes $\overline{\mathfrak{N}}$ into a $(\varphi, \mathfrak{S}/p\mathfrak{S})$-module. Since we have the φ-compatible surjection $f : \overline{\mathfrak{M}} \twoheadrightarrow \overline{\mathfrak{N}}$, it follows that $\overline{\mathfrak{N}}$ is of $\mathcal{P}$-height $\leqslant p-1$; i.e., $u^{(p-1)e}\overline{\mathfrak{N}} \subset \varphi_{\overline{\mathfrak{N}}}(\sigma^*\overline{\mathfrak{N}})$.

Since f is in the kernel of $\mathfrak{T}_{\overline{\mathfrak{M}}}$, the same argument as §12.3.3 implies that $\overline{\mathfrak{N}} \subset u^e(\widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}})$. Using that $\varphi_{\overline{\mathfrak{N}}}$ is induced from the pth power map $\sigma : \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}} \to \widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}}$, we have $\varphi_{\overline{\mathfrak{N}}}(\sigma^*\overline{\mathfrak{N}}) \subset u^{pe}(\widehat{\mathfrak{S}}^{\mathrm{ur}}/p\widehat{\mathfrak{S}}^{\mathrm{ur}})$, so $u^{(p-1)e}\overline{\mathfrak{N}} \supset \varphi(\sigma^*\overline{\mathfrak{N}})$. Since $\overline{\mathfrak{N}}$ is of

[6] The author thanks Tong Liu for providing his idea.

$\mathcal{P}$-height $\leqslant p-1$, we obtain $\varphi_{\overline{\mathfrak{N}}}(\sigma^*\overline{\mathfrak{N}}) = u^{(p-1)e}\overline{\mathfrak{N}}$; i.e., $\overline{\mathfrak{N}}$ is of Lubin-Tate type of $\mathcal{P}$-height $p-1$. But by the definition of unipotent-ness of $\mathcal{P}$-height $\leqslant p-1$, $\overline{\mathfrak{M}}$ does not admit any non-zero quotient of Lubin-Tate type of $\mathcal{P}$-height $p-1$. Therefore $\overline{\mathfrak{N}} = 0$, so $f = 0$.

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