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ON THE FUNCTIONING OF A FAMILIAR NONLINEAR THERMODYNAMIC OSCILLATOR

Abstract

The paper analyzes the mechanism of vibration of the child's toy known as the putt-putt boat. This system has many nonlinear features which lead to a variety of interesting wave forms. Self-excitation in this system is shown to depend principally on a condensation rate parameter. Analog tests, based on the analysis, are compared with tests on the physical system. General agreement is good although the system shows a number of features not predicted by the simple analysis.

Introduction

This paper will consider a nonlinear mechanical system in which stable self-induced oscillations are produced by the application of heat. The particular form of the system, which will be analyzed, is shown schematically in Fig. 1 on the left. A shallow circular chamber is covered by a thin diaphragm which is assumed to be attached without initial tension. From the base of the chamber a pipe or pipes lead to a vessel containing water. If the chamber is initially full or partly full of water and a heat source is applied to its base, the diaphragm and the water in the pipes begin to vibrate.

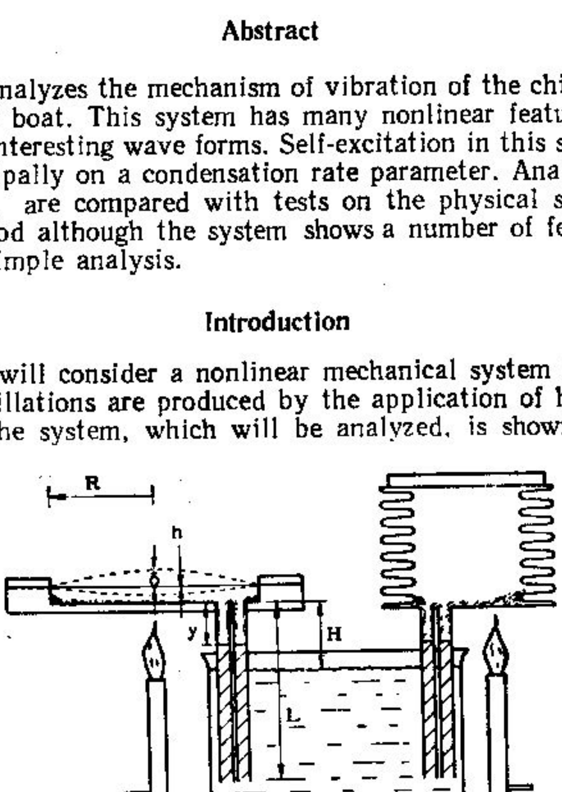

Fig. 1. Two systems which exhibit self-induced vibration.

Another arrangement which vibrates readily is shown in Fig. 1 at the right. If the top of the bellows is closed with glass, it is possible to observe its contents during vibration.

This type of vibration has been known for some time since it forms the basis of the child's toy known as the pop-pop or putt-putt boat. For the benefit of those who have not experimented with such a boat, it may be permissible to digress and describe its operation. Referring to Fig. 2 the chamber *A* is filled with water and a heat source *B* such as a candle or alcohol lamp is inserted. After a short time, the diaphragm *E* begins to vibrate and the boat moves forward making characteristic putt-putt noises. The only attempt which we know of to explain this behaviour was given by J. G. Baker [1] in 1933 in the course of a very interesting general article on self-induced vibration. The description is repeated here.

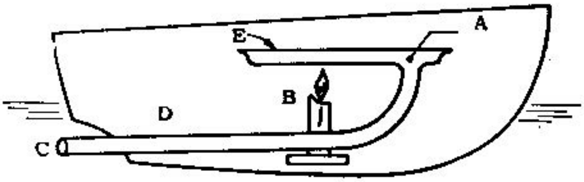

Fig. 2. Schematic drawing of putt-putt boat.

«The source of energy is the alcohol lamp *B*, which heats the chamber *A*; the latter is partly or wholly full of water. The top of *A* is a diaphragm constructed of bimetal in such a way that when hot it is bulging upwards and cold it is bulging downward. The chamber *A* is connected to water astern of the boat by the tube *D*. With the diaphragm down, *B* heats *A* and its contents, until the diaphragm snaps upward to its hot equilibrium position. This change in volume of *A* draws cool water in through *D* and cools *A*, so that the diaphragm snaps down again, forcing out the surplus water. With further heating, the cycle repeats several hundred times a minute». This explanation, based on a bimetallic diaphragm, appears plausible. However, it cannot apply to the boats presently on sale since they have solid brass diaphragms, nor can it be applied to the bellows design of Fig. 1. In addition, as equally satisfactory operation is obtained with a steel diaphragm and brass chamber, the previous explanation cannot be modified to predict vibration based on the differential thermal expansion of chamber and diaphragm.

Before turning to an alternative explanation of the self-excitation a photograph of two boats now on the market may be of interest. In Fig. 3, the larger boat is made in Mexico and the smaller in Japan. At the risk of being thought commercial we mention that the most unusual feature of the smaller boat is not the mechanisms of vibration or propulsion but the retail price of 10 c.

In what follows we will attempt to derive equations relating the motions of the diaphragm and water in the pipes to the other variables. Analog computer solution of the equations indicates the general types of behaviour to be expected. This will be compared with the behaviour of the physical system. Finally a few comments will be made about the method of propulsion.

Analysis of Vibration

For simplicity, the arrangement shown on the left side of Fig. 1 is considered. By making the entire unit, except for the diaphragm, out of glass, it is possible to observe its contents during vibration. In general, the chamber contains steam and water. This agrees with the observations that the heat source must be applied for some time before vibration commences and that vibration usually ceases if too much heat is applied. The mean water level in the pipes is usually below the base of the chamber. The mean level is normally lowered by increased heating and raised, closer to the chamber, if the tubes are cooled. During vibration of the water column there is often a «sloshing» of water back into the chamber. Under this condition, continuous oscillation of reasonably constant amplitude and frequency has been observed for periods of over one hour. However, the return of water to the chamber does not appear to be essential since vibration may occur for extended periods in its absence.

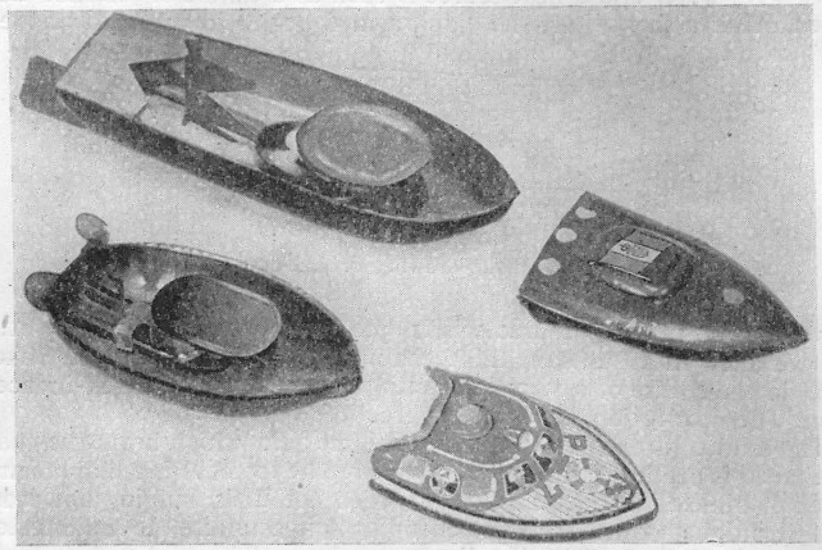

Fig. 3. Two boats with top decks removed.

From the preceding observations, a reasonable assumption is that steam is continuously generated in the chamber and condensed in the pipes. As heat is supplied from a source of high temperature, the rate of steam generation will be taken as a constant and the rate of steam condensation will be taken as proportional to the area of the available condensing surface. On this basis the net rate of steam generation is written as

$$\dot{V} = k_1 - k_2 y$$

where the symbols are defined under Nomenclature. This expression gives a highly simplified picture of what is probably a very complicated process of time-varying condensation. In addition, it assumes that the effect of pressure changes on steam generation and condensation rates may be neglected.

The total volume of steam present in the system may be written as

$$V = V_0 + 2ay + (e\pi R^2 h)z$$

if the small volume of water present in the chamber is neglected. $e = \frac{1}{3}$ for small z and increases to $^1/_2$ with increasing z. Combining the two preceding equations leads to

$$b\dot{z} = K_1 - K_2 y - \dot{y} \tag{1}$$

where b is approximately constant for a given system. K_1 may be interpreted as the rate at which the water level would drop in the pipes due to a given heat input if the diaphragm were held fixed. K_2 may be interpreted similarly as the rate at which the water level would rise due to condensation in a unit length of each pipe.

As the diaphragm is very thin, its mass is small compared to that of the water in the pipes. If the mass of the diaphragm is neglected, its motion may be related to the chamber pressure changes by static deflection considerations. Following Nadai [2, 3], the elastic deflection of a clamped circular diaphragm loaded by pressure differential $p - p_0$ may be written as

$$z + 0.583z^3 = 0{,}171\,\frac{(p - p_0)}{E}\left(\frac{R}{h}\right)^4.$$

Unfortunately, this expression is in serious disagreement with static deflection tests on diaphragms for a number of models built as shown in the left side of Fig. 1. The measured deflection is often considerably greater than predicted and varies from one model to another. This is due apparently to the inherent geometrical imperfections in the thin diaphragms. These nonuniformities also provide an explanation for the noise generated since they allow a coupling of the basic motion to higher modes of vibration which produce the characteristic noise. Despite its disagreement with experiment, the preceding expression will be carried through in the analysis since it represents the ideal case.

An expression may be obtained relating the pressure changes in the chamber to the motion of water in the pipes. If the motion of water outside the pipes is neglected, but with allowance for the changing mass in the tubes, then

$$2aL\varrho\left(1 - \frac{y}{L}\right)\ddot{y} + 2aF(\dot{y}) + 2a\varrho g(y - H) = 2a(p - p_0).$$

Eliminating $p - p_0$ from the two preceding equations leads to

$$\ddot{y}\left(1 - \frac{y}{L}\right) + F(\dot{y})/\varrho L + g(y - H)/L = \alpha(z + 0{,}583z^3). \tag{2}$$

The damping term $F(\dot{y})$, as in many vibration problems, presents some uncertainty since it involves the resistance to oscillating flow in the pipes. However, there is another important source of damping in this system due to the loss at entry to the pipes. It will be pointed out later that this contributes a term $-\rho\dot{y}^2$ to $F(\dot{y})$ only for $\dot{y} < 0$. In the case of smooth pipes with not too large a L/d ratio this term may be the principal source of damping. For the general case

$$F(\dot{y}) = C_1\dot{y} + C_2\dot{y}/\dot{y}/ + \frac{\varrho}{4}\{\dot{y} - |\dot{y}|\}^2.$$

To determine whether or not vibration will start, z^3, $\dot{y}^2$ and $\ddot{y}y$ terms are ignored to obtain

$$\dddot{y} + a_1C_1\ddot{y} + a_2\dot{y} + a_3K_2y + a_4 = 0$$

where the a's are positive constants for a given configuration.

Routh's rule states that oscillations will grow if

$$K_2 = C_1a_1a_2/a_3.$$

That is, the minimum coefficient of condensation in the pipes K_2 for which vibration will commence is some multiple of the resistance to laminar flow in the pipes. In the system studied, there was no difficulty in initiating vibration and this aspect of the problem was not pursued.

Analog Tests

The equations were arranged for analog computation so that the damping terms, the condensing term and the heat input term could be adjusted during operation. The procedure usually followed was to choose the damping terms C_1 and C_2 such that a simulated damped free vibration curve on the analog would be similar to that in a physical system. Then for arbitrary values of the condensing term the system behaviour was studied for varying amounts of heat input. For simplicity, and for ease of comparison with the physical system, only the cases in which the condensing term K_2y in Eq. (1) is small or large relative to $\dot{y}$ will be considered.

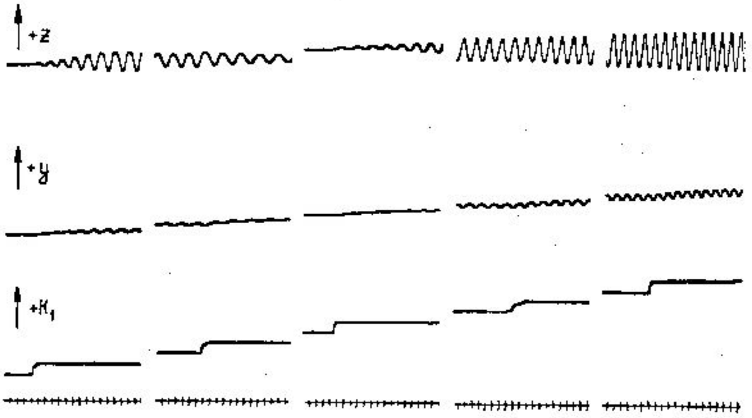

Fig. 4. Analog simulation for system with 2 in. dia.×0.002 in. diaphragm and 6 in. long ×0.18 in. dia. pipes. Water level outside pipes is 1 inch below base of chamber. Damping taken as the same in both directions. Records show z, y and K_1. Timing marks every 0.1 second.

When K_2 is small it follows from Eq. (1) that $z \sim -y$, i. e., the diaphragm and water column move down almost in phase. Fig. 4 shows z, y and K_1 for this type of behaviour. In this case the water level is initially one inch below the diaphragm. Damping was assumed to be the same for flow in both directions. Increasing heat input drives the diaphragm upwards. Its amplitude and frequency of vibration at first decrease, then increase with increasing heat input. Since the condensing term is low, there is, as might be expected, a large change in the mean level of the water when the heat input is increased.

Fig. 5 shows also a case in which the condensing term is small. Again the diaphragm and water deflections are nearly 180° out of phase. In this case the water outside of the tubes was level with the diaphragm. In this case the velocity squared damping was assumed to act only on inflow. This produces an interesting change in the wave form at higher amplitudes.

In general, whether the condensing term is large or small, the introduction of additional damping on only half the cycle tends to some very peculiar waveforms. These appear to correspond to the introduction of a component at twice the basic frequency. In some cases this dominates and the frequency of the record suddenly doubles.

For the analog runs with large condensing term it was found as expected that $\dot{y} \backsim z$ and the mean water level, for a given heat input, moved closer to the top of the tubes.

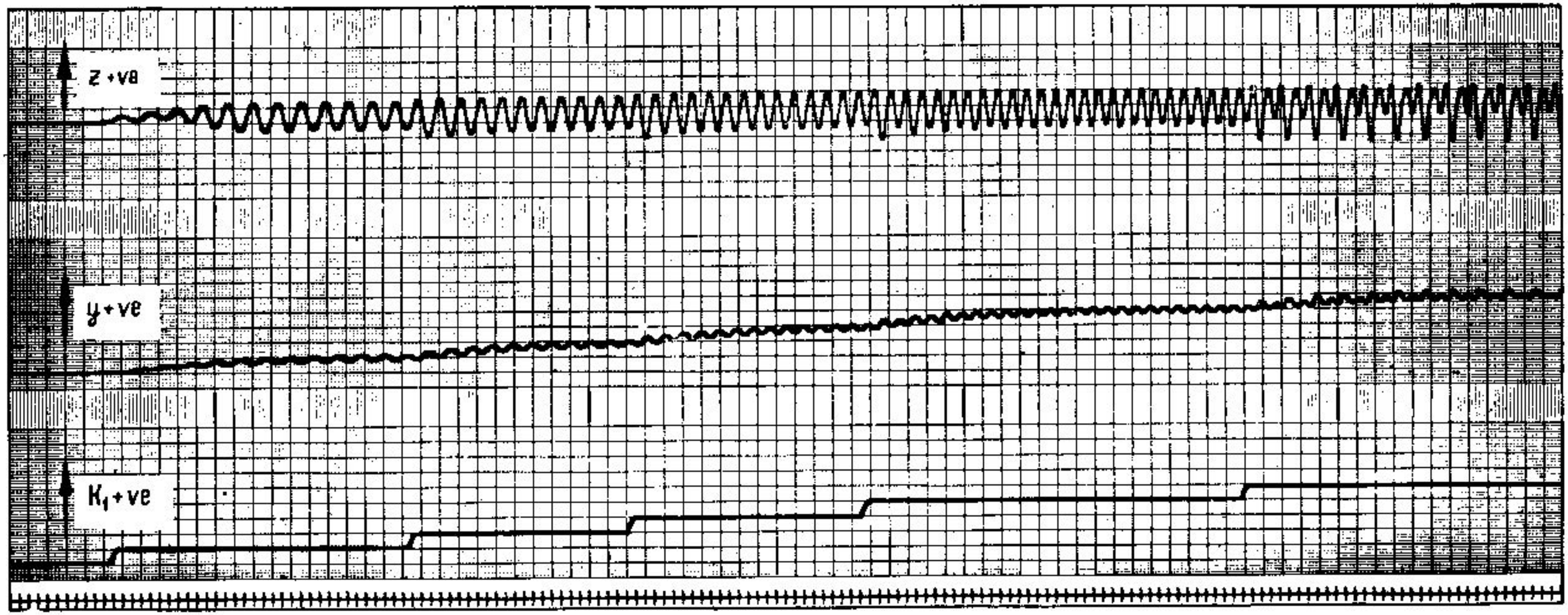

Fig. 5. Analog simulation for system with 2 in. dia.×0.002 in. diaphragm and 6 in. long×0.18 in. dia. pipes. Water outside pipes level with base of chamber. Damping proportional to (velocity)^2 is taken to act only on inlet to pipe. Records show z, y, and K_1. Timing marks every 0.1 second.

Tests on the Physical System

The models tested were of the form shown in Fig. 1 and all vibrated readily. Various methods of heating were employed but eventually we chose a cigar lighter element connected to a battery charger as the most easily controlled heat source. The main problem in comparing experiment with prediction is that of choosing a coefficient K_2 to characterize condensation rates in the model. Estimates of rates of condensing of pure vapors on the tube wall indicate that K_2 is very large.

This is in agreement with observations on the models, Fig. 6, which show that the diaphragm deflection z is almost in phase with the velocity $\dot{y}$ of the water column. This figure also demonstrates the frequency and amplitude modulation which is sometimes, but not always, observed. Observations on a model with a glass base show that this behaviour is connected with the return of water to the chamber. This feature was not incorporated in the derivation of the equations.

A basic check on the explanation of self-excitation can be obtained by running the physical system with lower values of K_2, the condensation parameter. Perhaps the easiest way to do this is to introduce air into the chamber since the presence of non-condensible gases will greatly slow down the rate of condensation of steam. When enough air is added, the model is seen to vibrate with $z \sim -\dot{y}$ as would be expected from the analysis if the condensation rate parameter K_2 is low. Adding air during operation also lowers the mean water level, as predicted. The general behaviour for the case of low condensation rate appears to be as anticipated, although there are certain effects which occur occasionally and have not been explained. Fig. 7 shows one of these. In this case, vibration was occurring with the water level in the tubes well away from the chamber. Note that y and z are almost 180° out of phase in these records.

The preceding two figures apply to model tests run with pipes whose L/d ratio was such that the inlet damping term should be small compared to the pipe friction terms. To study the influence of the unidirectional damping, a few tests were made on a unit with pipes of smaller L/d ratio. Fig. 8 shows a reasonably typical result. From the analog simulation it appears that the peculiar waveform is due to the unidirectional damping.

Mechanism of Propulsion

Another interesting aspect of the boat is the forward propulsion which results from the alternating flow. Baker gives the following discussion of this point.

«During the time water is flowing out, it forms a «jet», issuing from C which pushes the boat forward by its reaction. When the water is going in, however, the point C is a «sink»; the water is attracted to it from all sides, so that no retarding action is felt by the boat. In total, therefore, there is a driving force during a full cycle». Baker continues in a footnote: «It is an experimental fact that the pressure of water issuing from a tube under water is the static pressure of the water at the level of the tube and that the pressure of water being sucked in under similar conditions is less than the static pressure by an amount corresponding to the velocity with which it is sucked in. The difference is known to be due to the effect of the viscosity of the water at the surfaces of the tube».

This explanation would appear to be essentially correct and may be con-

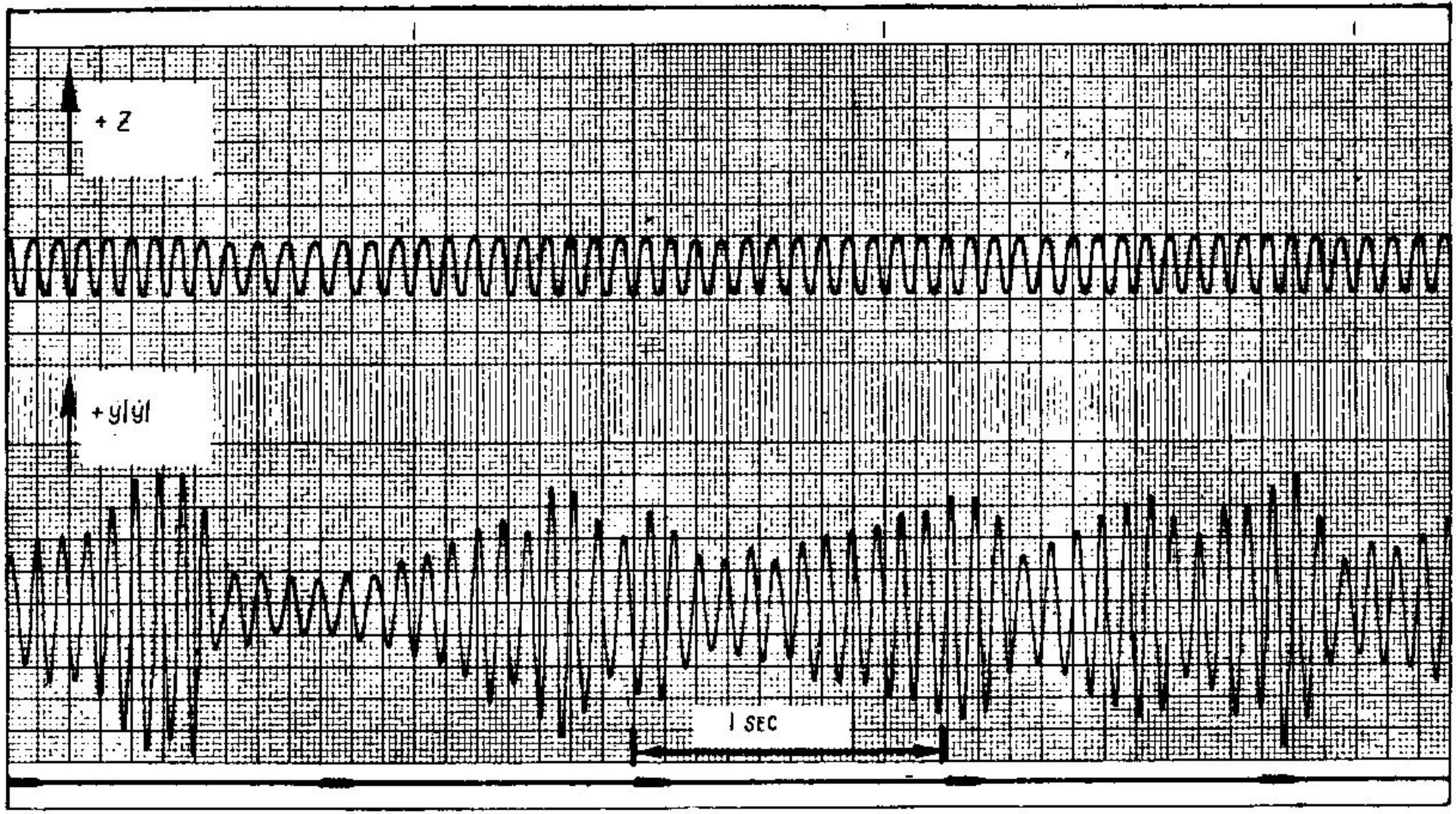

Fig. 6. Observations on a model with 2 in. dia×0.002 in. diaphragm and 6 in. long×0.18 in. diameter pipes. Water level outside pipes was 1 inch below base of chamber. Heat input 60 watts. Top record shows frequency and phase of z but waveform is distorted by measuring device. Bottom record shows $y(\dot{y})$ as measured by Pitot tube.

firmed by a simple experiment with an immersed rotary type lawn sprinkler. The average thrust is thus the average value of the momentum flux from the tubes, $\frac{\varrho a}{2}(\dot{y}+|\dot{y}|)^2$ over the cycle. Since during inflow the pressure within the pipes, after parallel though turbulent flow has been established, is less than the static pressure in the surroundings by $\frac{\varrho a}{2}(\dot{y}-|\dot{y}|)^2$ this term must appear as a damping term in the equation of motion. There is thus inherent damping in this system in the form of an energy loss at inlet, quite apart from the friction losses due to pipe flow. The net contribution to the velocity of the boat due to acceleration of water in the tubes cancels in alternate half cycles if the boat velocity is proportional to thrust, as is probably the case.

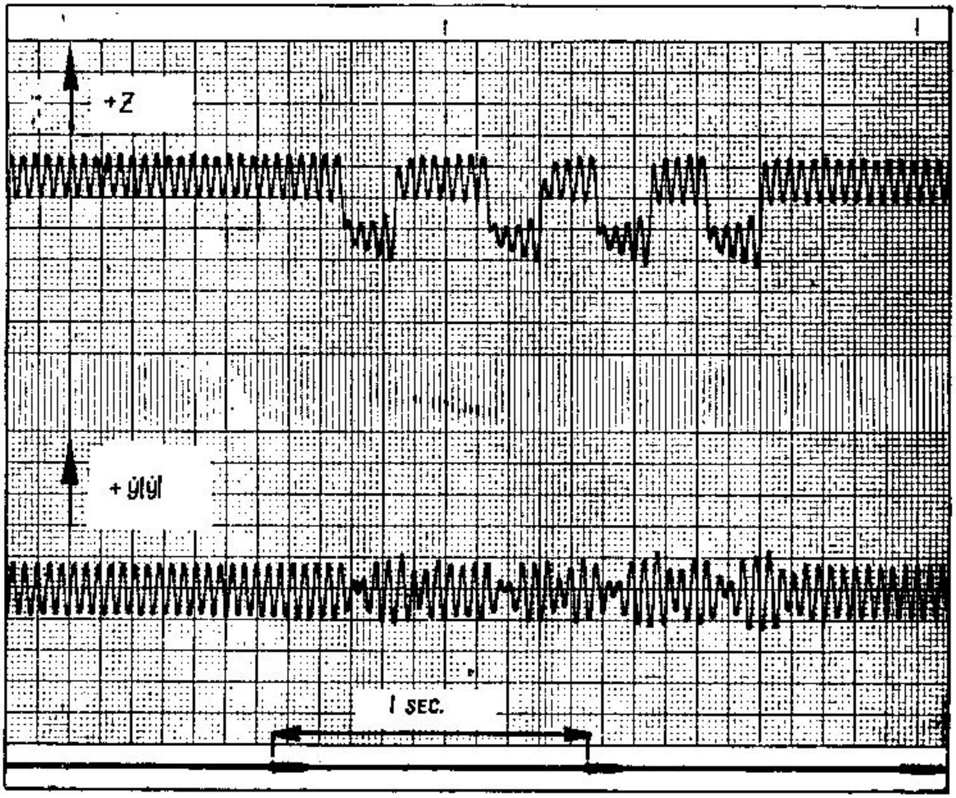

Fig. 7. Observations on a glass model with 1—½ in. dia. ×0.002 in. diaphragm and 6 in. long 0.18 in. dia. pipes. A considerable amount of air was introduced into the chamber. Comments on Figure 6 on z and $\dot{y}$ measurements apply here also.

Assuming simple harmonic motion of the water column, $y = y_0 \cos \omega t$, and constant velocity V of the boat, the ratio of useful propulsion work to work dissipated as the inlet loss may be shown to be equal to $(3\pi/8)(V/\omega y_0)$. For a typical boat this has a value of about 0.1. To increase this ratio requires a low jet velocity relative to the boat velocity. Since this also implies low thrust per unit cross-section area of pipe, this method of propulsion cannot be considered attractive for other than very specialized applications.

The maximum possible *thermodynamic efficiency* of the vibrating mechanism is also very low since steam is generated and condensed at nearly the

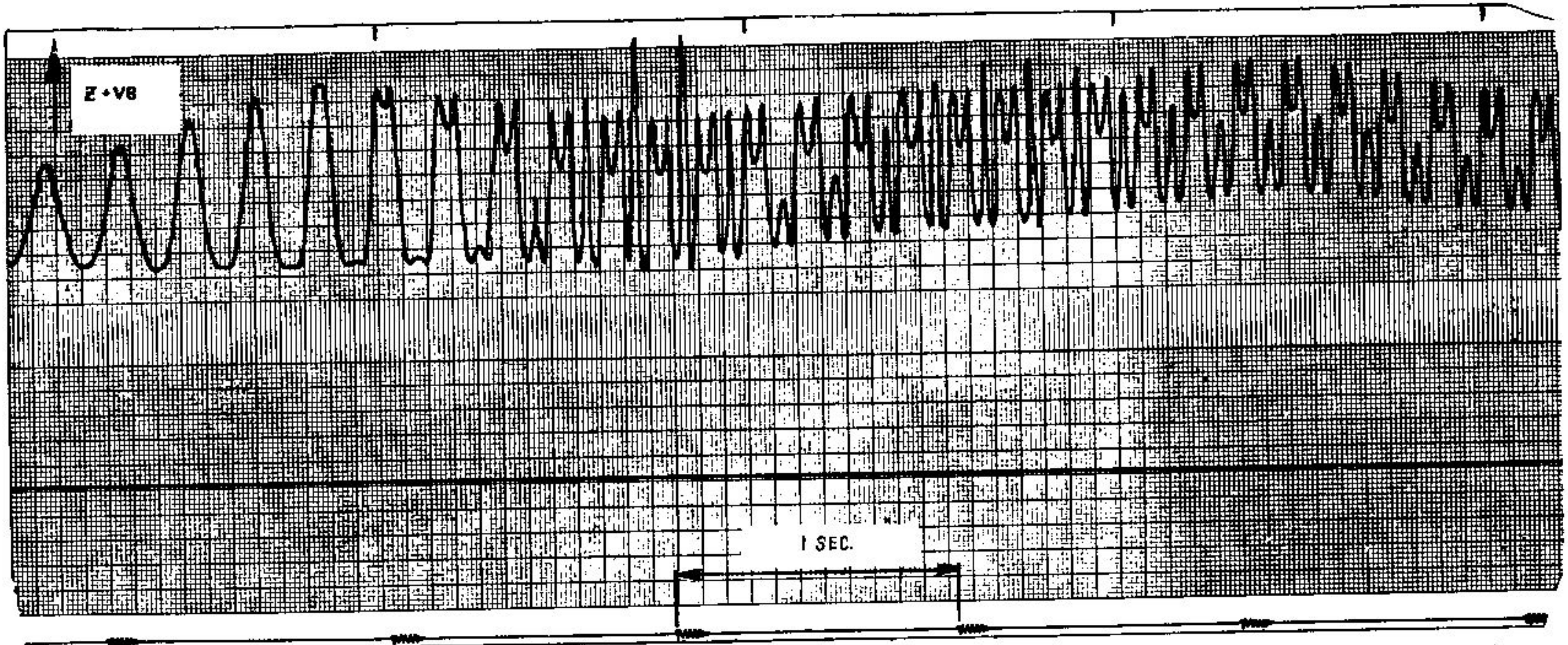

Fig. 8. Observations on a model with 6 in. dia. ×0.007 in. diaphragm and 5.75 in. long ×0.43 in. dia. pipes. Deflection measured with wire resistance strain gage attached to diaphragm. Record has been inked over for clarity.

same temperature. However, the device shown on the right side of Fig. 1 will generate considerable force and could yield useful work if the bellows were coupled to a suitable mechanism. Such a device would be a self-excited valveless) version of the early Watt (1769) steam engine.

NOMENCLATURE

a — Inside area of one tube, i. e., $\pi d^2/4$
a_1, a_2, a_3, a_4 — Constants for a given configuration
b — Approximately a constant for a given configuration. Equal to $e\pi R^2h/2a$
d — Inside diameter of one tube
e — Value lying between $^1/_3$ and $^1/_2$
g — Gravitational constant
h — Diaphragm thickness
k_1 — Rate of steam generation in chamber
k_2 — Rate of steam condensation in unit length of tubes
p — Chamber pressure
p_0 — Atmospheric pressure
t — Time
y — Distance from base of chamber to water level inside tubes
y_0 — Constant
z — Diaphragm deflection: diaphragm thickness. Positive for upward deflection
C_1C_2 — Constants
E — Elastic modulus of diaphragm material; also used as symbol in Fig. 2.
$F(\dot{y})$ — Function of $\dot{y}$
H — Distance from base of chamber to water level outside tubes
K_1 — Steam generation parameter equal to $k_1/2a$
K_2 — Steam condensation parameter equal to $k_2/2a$
L — Length of tubes
R — Radius of diaphragm
V — Boat velocity or steam volume
α — Constant for a given system. Equal to

$$\frac{E}{0.171\varrho L}\left(\frac{h}{R}\right)^4$$

δ — Diaphragm deflection
ϱ — Mass density of water
ω — Angular frequency

Reference

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2. A. Nadai, Elastische Platten, 1925, 288.
3. S. Timoshenko, Theory of Plates and Shells, McGraw-Hill Book Company, Inc., New York, 1940, 336.

И. ФИННИ, Р. Л. КЕРЛ

США

О РАБОТЕ ОБЫЧНОГО НЕЛИНЕЙНОГО ТЕРМОДИНАМИЧЕСКОГО ОСЦИЛЛЯТОРА

Резюме

Рассматриваемая система представляет собой детскую игрушку, известную под названием «пат-пат»-лодки. В лодке имеется полая камера, покрытая тонкой пленкой. От основания камеры к корме в воду проведены трубки. Если камера наполнена целиком или частично водой и нагревается

с основания, то пленка и вода в трубках начинают вибрировать и лодка движется вперед, производя характерный шум «пат-пат».

Если такую лодку сделать из стекла, то можно увидеть, что пар образуется в камере и конденсируется в трубках. Исходя из этого, выведены уравнения, связывающие движение пленки с движением воды в трубках. Показано, что самовозбуждение колебаний зависит главным образом от скорости конденсаций. Уравнения нелинейны не только вследствие того, что жесткость пленки и масса воды в трубках переменны, но и вследствие наличия нелинейного торможения, действующего только в фазе поступления воды в трубки. По-видимому, это нелинейное торможение обусловливает своеобразную форму волны, замеченную при решении уравнений задачи на аналоговом устройстве и при измерениях в натуре.

Известно, что в присутствии неконденсируемого газа, скажем воздуха, уменьшается скорость конденсации пара. Оказалось, что введение в камеру воздуха меняет характер волны так, как это следует из выведенных уравнений при изменении параметра — скорости конденсации.

Тонкие пленки не вполне точно описываются уравнениями теории упругости, что следует отнести за счет их неизбежных геометрических несовершенств. Этими несовершенствами можно объяснить возникновение шума, так как из-за них на основную форму движения накладываются высшие гармоники.

В заключение рассматривается механизм поступательного движения. Отмечено, что объяснение, данное Дж. Г. Бэкером в 1933 г., в основном верно. Показано, что коэффициент полезного действия при таком поступательном движении составляет только ≈ 0.1. Отсюда следует заключить, что этот метод получения движения не перспективен с практической точки зрения, хотя в связи с рассмотренной игрушкой возникает ряд весьма интересных проблем.

DISCUSSION

Questions

И. И. Блехман (СССР). Удалось ли проследить на машине причину прекращения колебаний при большом подводе тепла?

I. Finnie. When the quantity of heat increases the level of water draws down and the oscillations stop. I did not investigate it mathematically.

Speeches

И. И. Блехман (СССР). Заслушанный доклад представляет большой интерес. Данное исследование является первой попыткой разобраться в сложной, но весьма интересной термодинамической автоколебательной системе. Разумеется, разультаты этой работы приложимы не только к теории рассмотренной игрушки; не вызывает сомнения, что они будут использованы и развиты в дальнейшем.

Автоколебания в системах с термодинамическими элементами, представляя значительный научный интерес, всегда поражают воображение. Недавно нам довелось наблюдать самовозбуждающиеся колебания наполненного водой чайника с выпуклым дном, стоящего на гладкой стальной пластине, подогреваемой снизу газовой горелкой. При прекращении нагрева колебания прекращались.

Таким образом, в данном случае тепловые явления также играют первостепенную роль в механизме возбуждения автоколебаний. Однако построение адэкватной математической модели системы представляется весьма трудным.