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PERIODIC, FINITE-AMPLITUDE, AXISYMMETRIC GRAVITY WAVES

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CHAPTER I

INTRODUCTION

1. The Scope of the Investigation

This work is a study of finite-amplitude axisymmetric gravity waves in a circular basin of uniform depth. Only periodic, free oscillations of the fluid are considered. The analysis is carried out for a standing wave whose motion to the first approximation is that of the first mode. However, the same procedure may also be used for motion corresponding to another mode. The fluid is assumed to be a non-viscous incompressible liquid.

The relative depth of the liquid (that is, the ratio of the depth to the radius of the basin) is not limited *a priori* to either of the extreme cases of very large or very small values. Rather, the depth is allowed to be completely general. It is seen that at certain discrete values of the relative depth a coupled motion can occur in which a higher mode at a frequency equal to an integral multiple of the primary frequency is of the same order of magnitude as the primary mode. The motion for the depth equal to or very nearly equal to one of these particular depths is investigated by an appropriate modification of the general solution.

The main difficulty in obtaining the solution to the problem is the task of satisfying the two non-linear free-surface boundary conditions. These conditions, in addition to being non-linear, must be applied at a moving boundary whose position is itself an unknown to be determined. An iteration process is followed in satisfying these conditions.

This problem was selected for study primarily because it is fundamental in the fields of fluid mechanics and non-linear vibrations. It may, however, have practical application to the phenomenon of seiche or mass oscillations in harbors. It may also be useful in pointing the way toward the solution of other, not necessarily related, non-linear problems in cylindrical co-ordinates.

2. Historical Outline

In the past hundred fifty years many investigations have been devoted to gravity waves. A number of the more important works will be mentioned briefly here; others are listed in the bibliography to this dissertation and in the excellent bibliographical sections of the books by H. F. Thorade(59) [1] and J. J. Stoker(58). It is unfortunate that until very recently finite-amplitude waves in deep water and in shallow water have been treated as separate problems rather than as two aspects of the same problem.

The earliest analytical study of progressive gravity waves apparently is that for the case of infinite fluid depth given by F. J. von Gerstner(8) in 1802, and also independently at a later period by W. J. M. Rankine(18) in 1863, which presents a form of wave motion possessing vorticity. Virtually all subsequent writers, however, have rejected this form, arguing that the wave motion can be generated from rest and hence must be irrotational.

Progressive finite-amplitude waves in water of infinite depth and in water of large but finite depth were studied by G. G. Stokes(24) in 1847 and Lord Rayleigh(19) in 1876. The existence of such waves when the depth is infinite was proved in 1925 by T. Levi-Civita(12); the next year D. J. Struik(26) extended this proof to the case of finite depth.

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1 The numbers in raised parentheses refer to entries in the bibliography.

Periodic waves, progressing without change of form, in shallow water were first indicated by J. Boussinesq[6] in 1877. The name "cnoidal waves" was applied to these by D. J. Korteweg and G. de Vries[10] in 1895. Further study of cnoidal waves was made in 1940 by G. H. Keulegan and G. W. Patterson[41].

The solitary wave, which consists of a single intumescence, was first observed by J. Scott Russell[21][22][23] in 1838 and subsequently studied by H. Bazin[2] in 1865 and J. Boussinesq[4] in 1871. The solitary wave may be thought of as the limiting case of a wave of infinite wave length and hence belongs in the shallow water class. K. O. Friedrichs and D. H. Hyers[34] proved the existence of solitary waves in 1954.

By replacing the exact non-linear dynamic free-surface boundary condition by another non-linear condition approximating it,[2] T. V. Davies[32] in 1952 obtained a solution for progressive finite-amplitude gravity waves which is applicable over the entire range of depths. Provided his approximation is valid, both the solitary wave and waves in an infinitely deep fluid are special cases of Davies' solution.

Much less study has been devoted to standing waves. The motion of an infinitesimal-amplitude standing wave is rather easily obtained and is discussed by H. Lamb[11] for several geometrical configurations. The theoretical and experimental work of J. S. McNown[46] in 1953 should also be noted. To the author's knowledge the only theoretical study of finite-amplitude standing waves prior to the present work is that of W. G. Penney and A. I. Price[49] who in 1952 analyzed such waves in a rectangular co-ordinate system.

2 This approximation is of the type, $\theta \approx \frac{1}{3} \sin 3\,\theta$, $\left(|\theta| \leq \frac{\pi}{6} \right)$.

CHAPTER II

THE PROBLEM AND ITS SOLUTION

1. The Governing Equations for Axisymmetric Standing Waves

The equation governing the irrotational axisymmetric motion of an incompressible non-viscous fluid is, in terms of the velocity potential $\bar{\phi}$,

$$\nabla_1^2 \bar{\phi} = 0 \quad , \tag{1}$$

in which

$$\nabla_1^2 \equiv \frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} + \frac{\partial^2}{\partial \bar{z}^2} \tag{2}$$

is the two-dimensional Laplacian operator. As shown in Figure 1 the origin of the co-ordinate system is on the axis of the cylinder a distance $\bar{H}$ above the bottom, where $\bar{H}$ is the mean depth of the fluid.

The requirement that the velocity normal to the solid boundaries must vanish is expressed by the conditions

$$\frac{\partial \bar{\phi}}{\partial \bar{z}} = 0 \qquad at \quad \bar{z} = -\bar{H} \tag{3}$$

$$\frac{\partial \bar{\phi}}{\partial \bar{r}} = 0 \qquad at \quad \bar{r} = R \, . \tag{4}$$

On the free surface, given by $\bar{z} = \bar{\eta}(\bar{r}, \bar{t})$, both kinematic and dynamic boundary conditions must be satisfied. The kinematic condition is

$$\frac{DF}{D\bar{t}} = 0 \qquad at \quad \bar{z} = \bar{\eta} \tag{5}$$

in which $\frac{D}{D\bar{t}}$ is the substantial derivative and F = 0 is the equation of the free surface. Setting $F = \bar{z} - \bar{\eta}$ and performing the indicated differentiation yields

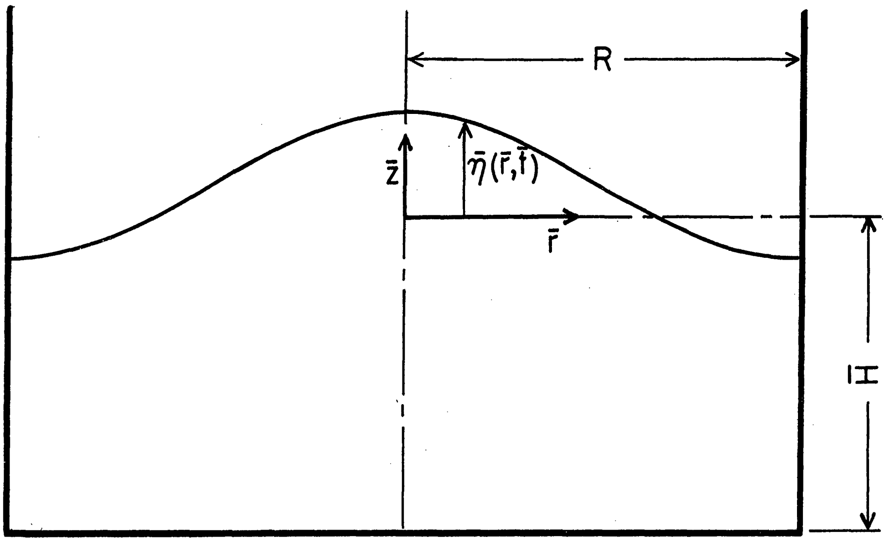

Figure 1. Co-ordinate System and Geometric Configuration

$$\frac{\partial \bar{\phi}}{\partial \bar{z}} - \frac{\partial \bar{\phi}}{\partial \bar{r}}\frac{\partial \bar{\eta}}{\partial \bar{r}} = -\frac{\partial \bar{\eta}}{\partial \bar{t}} \qquad \text{at} \quad \bar{z} = \bar{\eta} \tag{6}$$

as the kinematic surface boundary condition.

In the present co-ordinate system the Bernoulli equation takes the form

$$\frac{\partial \bar{\phi}}{\partial \bar{t}} + F(\bar{t}) + \bar{C}_1 - \frac{1}{2}\left(\frac{\partial \bar{\phi}}{\partial \bar{r}}\right)^2 - \frac{1}{2}\left(\frac{\partial \bar{\phi}}{\partial \bar{z}}\right)^2 = \bar{g}\bar{z} + \frac{\bar{p} - \bar{p}_0}{\rho} \tag{7}$$

in which the reference pressure $\bar{p}_0$ is chosen to be the atmospheric pressure acting upon the surface of the liquid For convenience later on the time function of integration $F(\bar{t})$ is merged in $\bar{\phi}$. When a specific form of $\bar{\phi}$ is assumed in the next section, it will be pointed out which parts of $\bar{\phi}$ correspond to the $F(\bar{t})$ of equation (7). At the surface $\bar{p}$ equals $\bar{p}_0$ and the dynamic surface boundary condition is

$$\frac{\partial \bar{\phi}}{\partial \bar{t}} + \bar{C}_1 - \frac{1}{2}\left(\frac{\partial \bar{\phi}}{\partial \bar{r}}\right)^2 - \frac{1}{2}\left(\frac{\partial \bar{\phi}}{\partial \bar{z}}\right)^2 = \bar{g}\bar{\eta} \qquad \text{at} \quad \bar{z} = \bar{\eta}\,. \tag{8}$$

Since the volume of the fluid remains constant,

$$\iint_A \bar{\eta}\; dA = 0\,, \tag{9}$$

where dA is an element of area in a plane normal to the $\bar{z}$-axis. Setting $dA = \bar{r}\, d\bar{r}\, d\theta$ and performing the integration with respect to θ, we obtain

$$\int_0^R \bar{r}\,\bar{\eta}\; d\bar{r} = 0\,. \tag{10}$$

Equation (1) together with the boundary conditions (3), (4), (6) and (8) and the statement (10) constitute the governing equations of the system. The barred quantities appearing in these equations are dimensional.

Let us now introduce dimensionless independent and dependent variables as follows:

$$r = \frac{\bar{r}}{R}\ , \qquad z = \frac{\bar{z}}{R}\ , \qquad t = \omega \bar{t} \tag{11}$$

$$\eta = \frac{\bar{\eta}}{R}\ , \quad \phi = \frac{\bar{\phi}}{\omega R^2}\ , \quad G = \frac{\bar{g}}{\omega^2 R}\ , \quad p = \frac{\bar{p}}{\rho \omega^2 R^2}\ . \tag{12}$$

ω (dimensional) is the frequency of the oscillation. The depth is made dimensionless by

$$H = \frac{\bar{H}}{R}\ . \tag{13}$$

H is thus the relative depth parameter (hereafter called merely "depth"). An upper case G is written for the non-dimensional gravitational acceleration to emphasize the fact that, although it takes the place of $\bar{g}$ in (8), it is a dependent variable which determines the frequency.

The dimensionless equations governing the system are thus

$$\nabla_1^2 \phi = 0 \tag{14}$$

$$\frac{\partial \phi}{\partial z} = 0 \qquad \text{at} \quad z = -H \tag{15}$$

$$\frac{\partial \phi}{\partial r} = 0 \qquad \text{at} \quad r = 1 \tag{16}$$

$$\frac{\partial \phi}{\partial z} - \frac{\partial \phi}{\partial r}\frac{\partial \eta}{\partial r} = -\frac{\partial \eta}{\partial t} \qquad \text{at} \quad z = \eta \tag{17}$$

$$\frac{\partial \phi}{\partial t} + C_1 - \frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial z}\right)^2 = G\eta \qquad \text{at} \quad z = \eta \tag{18}$$

$$\int_0^1 r\, \eta\, dr = 0\ . \tag{19}$$

2. The Solution at General Depth

Through the use of the method of separation of variables it is seen that the solution of equation (14) which is finite throughout the fluid region and which satisfies the linear boundary conditions (15) and (16) is

$$\phi = A \cosh K(z+H)\; J_0(Kr)\; T(t), \tag{20}$$

provided that K assumes the discrete positive eigenvalues K_n for which

$$J_1(K_n) = 0 \qquad (n = 1, 2, \cdots). \tag{21}$$

$J_0(Kr)$ is a Bessel function of the first kind. The first five eigenvalues are given in Table 1.[3]

TABLE 1

THE EIGENVALUES K_n

n	K_n
1	3.83170 59702
2	7.01558 66698
3	10.17346 81351
4	13.32369 19363
5	16.47063 00509

Equations (14), (15), and (16) are linear; hence any linear combination of the eigenfunctions given in (20) will also satisfy (14), (15), and (16). Restricting our attention to periodic solutions, we accordingly choose the following form for ϕ:

$$\phi = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} A_{mn} \frac{\cosh K_n(z+H)}{\cosh K_n H} J_0(K_n r) \sin mt + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} B_{mn} \frac{\cosh K_n(z+H)}{\cosh K_n H} J_0(K_n r) \cos mt. \tag{22}$$

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3 The first 150 values of K_n are given to ten decimal places (eleven to thirteen significant figures) in the British Association Tables[60].

The n = 0 terms, in which K_0 is defined to be zero, are independent of the space variables and thus correspond to the F(t) which was merged with ϕ in writing the Bernoulli equation in the form (18). These terms make no contribution to the velocities; they are included with ϕ solely for convenience.

The surface elevation $\eta(r,t)$ is represented in the form

$$\eta(r,t) = \sum_{m=0}^{\infty} \mathcal{J}_m(r) \cos mt + \sum_{m=1}^{\infty} \mathcal{J}_{-m}(r) \sin mt \quad . \tag{23}$$

Because of the long and complicated nature of the expressions obtained for A_{mn}, B_{mn}, $\mathcal{J}_{\pm m}$, and G, several shorthand notations are defined below and used hereafter:

$$\left.\begin{aligned} J_{ij} &\equiv J_i(K_j r), \\ ch_j \equiv \cosh K_j H, \quad & sh_j \equiv \sinh K_j H, \quad th_j \equiv \tanh K_j H \end{aligned}\right\} . \tag{24}$$

The subscript i appears in this work solely as 0 and 1.

The frequency of an oscillation whose motion is to the first (that is, linear) approximation that of the first axisymmetric mode will be denoted by ω (dimensional). Then the A_{11} and B_{11} terms in (22) correspond to this motion. Since this is a steady-state oscillation, there is no natural time origin in contrast, for example, to the case of release from rest at some initial configuration. Therefore, the time origin can be chosen arbitrarily. In particular, if the time origin is chosen such that B_{11} is identically zero, A_{11} is the parameter which determines the amplitude of the motion. Its magnitude is allowed to be arbitrary within a certain upper limit. This upper limit, corresponding to a breaking wave, will be discussed in detail in Section 4 of Chapter III. It is assumed that all other A_{mn} and B_{mn} are of order

$\mathcal{O}(A_{11}^2)$ or higher. The order-of-magnitude assumptions may be summarized as

$$\left.\begin{array}{l} A_{11} = \mathcal{O}(A_{11}) \\ B_{11} \equiv 0 \\ \text{All other } A_{mn} \text{ and } B_{mn} = \mathcal{O}(A_{11}^2) \text{ or higher.} \end{array}\right\} \qquad (25)$$

It is further assumed that in satisfying the free-surface boundary conditions (17) and (18) terms of order $\mathcal{O}(A_{11}^2)$ may be neglected in comparison with terms of order $\mathcal{O}(A_{11})$ in the first approximation. Similarly, in the second approximation terms of orders $\mathcal{O}(A_{11}^2)$ and $\mathcal{O}(A_{11})$ are retained while terms of order $\mathcal{O}(A_{11}^3)$ or higher are neglected, et cetera.

The following procedure is used for satisfying the surface boundary conditions. The approximations are made in order, first, second, third, etc. The assumed forms of ϕ and η in (22) and (23) are substituted into the kinematic surface condition (17). In evaluating the derivatives of ϕ at $z = \eta$ the functions sinh $K_n(\eta + H)$ and cosh $K_n(\eta + H)$ are expanded as

$$\begin{aligned} \sinh K_m(\eta+H) &= \left[1 + \frac{(K_m\eta)^2}{2!} + \cdots\right] sh_m + \left[K_m\eta + \frac{(K_m\eta)^3}{3!} + \cdots\right] ch_m \\ \cosh K_m(\eta+H) &= \left[1 + \frac{(K_m\eta)^2}{2!} + \cdots\right] ch_m + \left[K_m\eta + \frac{(K_m\eta)^3}{3!} + \cdots\right] sh_m \, . \end{aligned} \qquad (26)$$

Each non-linear term is of the form XY where both X and Y contain one or more terms of order $\mathcal{O}(A_{11})$ as well as higher order terms. In order to retain all terms of order $\mathcal{O}(A_{11}^j)$ in the product XY it is necessary to use only those terms of order $\mathcal{O}(A_{11}^{j-1})$ or less in X and in Y. Hence the use of the results of the (j-1)st approximation in the non-linear terms of the j th approximation is permissible.

Equation (17) after the substitution of (22) and (23) may be written in the form

$$\sum_{m=0}^{\infty} \beta_{-m} \cos mt + \sum_{m=1}^{\infty} \beta_m \sin mt = 0 \tag{27}$$

in which each $\beta_{\pm m}$ is of the form

$$\beta_{\pm m} = m J_{\pm m}(r) + f_{\pm m}(A_{11}, A_{mn}, B_{mn}, H, r) \qquad (m \geq 0) \tag{28}$$

where each $f_{\pm m}$ is known. Because of the orthogonality of the trigonometric functions each $\beta_{\pm m}$ must vanish independently.

$$\beta_{\pm m} = 0 \qquad (m \geq 0) . \tag{29}$$

For all $m \geq 1$, (29) becomes

$$J_{\pm m}(r) = -\frac{1}{m} f_{\pm m}(A_{11}, A_{mn}, B_{mn}, H, r) \qquad (m \geq 1) \tag{30}$$

and the functions $J_{\pm m}(r)$ are determined in terms of the A_{mn} and B_{mn}. For $m = 0$

$$f_0 (A_{11}, B_{0n}, H, r) = 0 . \tag{31}$$

Each function of r occurring in the expression (31) is expanded in a Dini series of Bessel functions. That is,

$$F(r) = \sum_{n=0}^{\infty} \alpha_n(F) J_{0n} \tag{32}$$

where the constants α_n are given by

$$\alpha_n(F) = \frac{\int_0^1 r F(r) J_{0n} dr}{\frac{1}{2} J_0^2(K_n)} . \tag{33}$$

The Dini expansions are explained more fully and the α_n's for the particular functions F(r) of interest here are tabulated in Appendix I. Multiplying equation (31) by $r \, J_{0j} \, dr$, integrating with respect to r from 0 to 1, and noting the orthogonality relations (II-1 and II-3) for Bessel functions, we obtain explicit expressions for the B_{0n}, $n \geq 1$, in terms of the parameters A_{11} and H.

Now substituting (22), (23), and (30) into the dynamic surface boundary condition (18), using the value of G from the previous approximation in all terms of order $\mathcal{O}(A_{11}^2)$ or higher on the right side of (18), and applying the same techniques as described above for the kinematic surface condition, we obtain solutions for A_{mn}, B_{mn}, and G solely in terms of the parameters A_{11} and H. Substitution of A_{mn} and B_{mn} into the equations (30) gives $\mathcal{J}_{\pm m}(r)$ in terms of A_{11} and H. From (18) a solution for $\mathcal{J}_0(r)$ in terms of A_{11}, H, and C_1, the constant of equation (18), is also obtained. Equation (19) is now applied to determine C_1 and hence $\mathcal{J}_0(r)$ in terms solely of the parameters A_{11} and H. To show that the time-dependent terms of η vanish when integrated in the manner of (19) it is necessary to prove certain indentities. Those identities needed through the third approximation are proved in Appendix II as equations (II-11), (II-12), and (II-13).

The procedure which has been outlined for satisfying the non-linear free-surface boundary conditions has been carried out through the third approximation. Further approximations do not seem practicable for two reasons. First, the expressions obtained are quite complicated. Secondly, for higher approximations the coefficients α_n in the Dini expansions of many more functions F(r) must be computed. Since these coefficients are evaluated by numerical integration, the labor involved is considerable.

The solution to the first approximation is

$$
\begin{aligned}
A_{11} &= \mathcal{A}_{11} \,, \qquad B_{11} \equiv 0 \\
\mathcal{J}_1(r) &= A_{11} K_1 \, \mathrm{th}_1 \, J_{01} \\
\frac{1}{G} &= K_1 \, \mathrm{th}_1 \qquad .
\end{aligned}
\tag{34}
$$

C_1 and all the other A_{mn}, B_{mn}, and $\mathcal{J}_{\pm m}$ are either zero or of higher order than $\mathcal{O}(A_{11})$. Equations (34) are the linear solution for infinitesimal-amplitude waves as given in Lamb[11] and due to Rayleigh[19][4].

The results of the second approximation, in which the first effects due to the finiteness of the amplitude appear are

$$A_{11} = \mathcal{A}_{11} \,, \qquad B_{11} \equiv 0 \tag{35}$$

$$A_{2n} = -\frac{A_{11}^2 K_1^2}{8} \Gamma_n \qquad (n \geq 0) \tag{36}$$

$$C_1 = \frac{A_{11}^2 K_1^2}{4} \left(1 - \mathrm{th}_1^2\right) J_0^2(K_1) \tag{37}$$

$$\mathcal{J}_0(r) = \frac{A_{11}^2 K_1^3 \mathrm{th}_1}{4} \left[\mathrm{th}_1^2 J_{01}^2 - J_{11}^2 + \left(1 - \mathrm{th}_1^2\right) J_0^2(K_1) \right] \tag{38}$$

$$\mathcal{J}_1(r) = A_{11} K_1 \, \mathrm{th}_1 \, J_{01} \tag{39}$$

$$\mathcal{J}_2(r) = \frac{A_{11}^2 K_1^3 \mathrm{th}_1}{4} \left[J_{01}^2 - J_{11}^2 - \frac{1}{4} \sum_{n=1}^{\infty} \frac{K_n \mathrm{th}_n}{K_1 \mathrm{th}_1} \Gamma_n J_{0n} \right] \tag{40}$$

$$\frac{1}{G} = K_1 \, \mathrm{th}_1 \qquad . \tag{41}$$

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[4] This problem was considered as early as 1828 by Poisson[17]. However, because the theory of Bessel functions had not yet been worked out, his results were not interpreted.

All other A_{mn}, B_{mn}, and $\mathcal{J}_{\pm m}$ are either zero or of order higher than $\mathcal{O}(A_{11}^2)$. The quantity Γ_m is a function of H only and is defined as

$$\Gamma_m \equiv \frac{(3\,th_1^2-1)\,\alpha_m(J_{01}^2) + 2\,\alpha_m(J_{11}^2)}{1 - \dfrac{K_m\, th_m}{4\,K_1\, th_1}} \quad . \tag{42}$$

The solution to the third approximation, upon which much of the discussion of Chapter III will be based, is presented in Appendix III. In carrying out the computations it is found that all B_{mn} and $\mathcal{J}^{(n)}_{-m}$, $(m \geq 1)$, are proportional to B_{11} and hence, because the time origin was chosen such that $B_{11} \equiv 0$, are identically zero. It is also found that B_{0m} are zero, at least to order $\mathcal{O}(A_{11}^3)$, independently of the assumption that B_{11} is zero.

The existence of finite-amplitude axisymmetric standing waves and the convergence of the iterative process used for obtaining ϕ and η are not proved. The existence and convergence theorems for progressive waves (16)(12)(26)(34) all depend essentially on the possibility of transforming the problem to one of steady flow by adopting co-ordinates travelling with the wave. This simplification is in the present case unavailable. The author is not aware of any existence or convergence theorems for finite-amplitude standing waves; Penney and Price[(49, p. 268)] specifically state, "There seems little likelihood that a proof of the existence of the stationary waves will ever be given." One observation can be made concerning the convergence of the present solution. The factor $\frac{th_n}{th_1}$, which is present in a number of the summations of the third approximation, behaves as unity for H very large and as $\frac{K_n}{K_1}$ for H very small; thus for small depths this factor will give a greater amplification to the terms for $n \geq 2$ of these summations than it will for large depths. It can therefore be anticipated that the convergence of the present solution will be more rapid for large depths than for small.

3. The Solution at the Critical Depths

Inspection of the second approximation to the solution at general depth, in particular equations (36) and (40) and the definition (42) of Γ_m, reveals the presence of the factor $1 - \frac{K_m \, th_m}{4 K_1 \, th_1}$ in the denominator. Likewise the third approximation contains the factor $1 - \frac{K_m \, th_m}{9 K_1 \, th_1}$ in the denominator of certain terms in equations (III-4) and (III-9). These factors are both of the type

$$1 - \frac{K_\ell \, th_\ell}{q^2 K_1 \, th_1} \qquad \begin{matrix} (q = 1, 2, 3 \ldots\ldots) \\ (\ell = 0, 1, 2 \ldots\ldots) \end{matrix} \quad . \tag{43}$$

In general the j th approximation will introduce the factor (43) with q = j into the denominator of certain terms which were expected to be of order $\mathcal{O}(A_{11}^j)$. If there are any values of ℓ, q, and H for which this factor equals zero, then particular terms in both ϕ and η become infinite and the general-depth solution must be rejected at those depths. Accordingly, it is desirable to look for the roots of the equation

$$1 - \frac{K_\ell \, th_\ell}{q^2 K_1 \, th_1} = 0 \; . \tag{44}$$

When q = 1, ℓ = 1 is a root for all values of H. This root causes no trouble, however, since the factor (43) is identically zero when q = ℓ = 1 and hence we do not divide by it assuming it to be non-zero. Therefore attention can be restricted to q $\geq$ 2, for which it is evident that (44) can have no roots when ℓ = 0 or 1.

For $\ell \geq 2$, $\frac{th_\ell}{th_1}$ ~~decreases~~ monotonically from $\frac{K_\ell}{K_1}$ to 1 as H increases from 0 to ∞. Thus only for those ℓ such that $q \leq \frac{K_\ell}{K_1} \leq q^2$ will there be roots to equation (44). Since the eigenvalues K_n tend ultimately to the form $K_n \approx \pi(n + \frac{1}{4})$, there will be for each integer q approximately $\frac{K_1}{\pi}(q^2 - q)$ discrete values of H, each associated with a particular ℓ , for which (44) will be satisfied. The roots of (44) for q = 2,3 are listed in Table 2. In addition the minimum and maximum roots for q = 10 have been computed and are also shown. For future reference we denote the values of H at which there are roots of (44) with q = 2 as the second-order critical depths and those values of H at which there are roots of (44) with q = 3 as the third-order critical depths, etc.

TABLE 2

ROOTS OF THE EQUATION,

$$1 - \frac{K_\ell \, th_\ell}{q^2 K_1 \, th_1} = 0$$

q	ℓ	H
2	3	0.19811
2	4	0.34698
3	4	0.084
3	5	0.132
3	6	0.168
3	7	0.207
3	8	0.255
3	9	0.321
3	10	0.440
......	......	......
10	12	0.004
......	......	......
10	121	0.761

The physical meaning attached to these critical depths may be seen rather easily. It has been assumed that there is a first mode of order $\mathcal{O}(A_{11})$ oscillating at frequency ω and that all other modes and harmonics (that is, all other A_{mn}, B_{mn}) are of order $\mathcal{O}(A_{11}^2)$ or higher. When the depth equals one of the critical depths, the assumption that all other A_{mn}, B_{mn} are of higher order is not valid. In particular, $A_{q\ell}$ and/or $B_{q\ell}$ will be of order $\mathcal{O}(A_{11})$, where q and ℓ are the q and ℓ associated with the particular critical depth. In fact the condition that A_{11}, representing a first mode at frequency ω, and $A_{q\ell}$ and/or $B_{q\ell}$, representing an ℓ th mode at frequency $q\omega$, both be of order $\mathcal{O}(A_{11})$ is, to the first approximation, that equation (44) be satisfied.

Since $A_{q\ell}$ and/or $B_{q\ell}$ are of order $\mathcal{O}(A_{11})$ when $H = H_c$, where H_c is a critical depth and q and ℓ have those values associated with H_c, and are of order $\mathcal{O}(A_{11}^2)$ when H is appreciably different from H_c, it is logical to assume that there is some transition range $H_c - \epsilon < H < H_c + \epsilon$, where ϵ is some small number dependent on A_{11}, in which $A_{q\ell}$ and/or $B_{q\ell}$ are between order $\mathcal{O}(A_{11}^2)$ and order $\mathcal{O}(A_{11})$.[5] Thus in order to obtain a solution of the system (14),(15),(16),(17),(18), and (19) which is valid when H is equal to or is very nearly equal to H_c, we revise the general-depth assumptions (25) to the following:

$$\left.\begin{aligned} &A_{11} = \mathcal{O}(A_{11}) \\ &B_{11} \equiv 0 \\ &A_{q\ell} \text{ and/or } B_{q\ell} = \mathcal{O}(A_{11}) \\ &\text{All other } A_{mn} \text{ and } B_{mn} = \mathcal{O}(A_{11}^2) \text{ or higher.} \end{aligned}\right\} \qquad (45)$$

To observe how the solution at a critical depth differs in form from the solution when H is not critical the solution can be carried out by

5 The alternative is to assume that $|A_{q\ell}(H)|$ and/or $|B_{q\ell}(H)|$ has a jump discontinuity when $H = H_c$. This alternative does not seem physically reasonable.

following the assumptions (45) when the depth equals or very nearly equals either of the two second-order critical depths. The order-of-magnitude assumptions (45) then become

$$
\begin{aligned}
A_{11} &= \mathcal{O}(A_{11}) \\
B_{11} &\equiv 0 \\
A_{2\ell} \text{ and/or } B_{2\ell} &= \mathcal{O}(A_{11})
\end{aligned}
\tag{46}
$$

All other A_{mn} and B_{mn} = $\mathcal{O}(A_{11}^2)$ or higher. It is understood that $\ell = 3$ when H $\approx$ 0.19811 and $\ell = 4$ when H $\approx$ 0.34698.

The substitution of ϕ (22) and η (23) into the free-surface boundary conditions (17) and (18) and into (19), and the evaluation of the individual A_{mn}, B_{mn}, and $\mathcal{J}_{\pm m}$ have been carried out in the manner described in the previous section through the second approximation.[6]

The results of the first approximation are

$$A_{11} = A_{11} \quad , \quad B_{11} \equiv 0 \tag{47}$$

$$A_{2\ell}\left[1 - \frac{K_\ell\, th_\ell}{4K_1\, th_1}\right] = 0 \tag{48}$$

$$B_{2\ell}\left[1 - \frac{K_\ell\, th_\ell}{4K_1\, th_1}\right] = 0 \tag{49}$$

$$\mathcal{J}_1(\nu) = A_{11}\, K_1\, th_1\, J_{01} \tag{50}$$

[6] When the depth is approximately equal to one of the j th order critical depths ($j \geq 3$), the method of solution is conceptually identical to that outlined here; however, the labor involved is considerably greater because the solution must be carried to the j th approximation in order to obtain the relationship between $A_{8\ell}$ and/or $B_{8\ell}$ and A_{11}.

$$\mathcal{J}_2(\Omega) = \frac{1}{2} A_{2\ell} K_\ell \, \mathrm{th}_\ell \, J_{0\ell} \tag{51}$$

$$\mathcal{J}_{-2}(\Omega) = -\frac{1}{2} B_{2\ell} K_\ell \, \mathrm{th}_\ell \, J_{0\ell} \tag{52}$$

$$\frac{1}{G} = K_1 \, \mathrm{th}_1 \quad . \tag{53}$$

C_1 and all other A_{mn}, B_{mn}, and $\mathcal{J}_{\pm m}$ are either zero or of order higher than $\mathcal{O}(A_{11})$. From (48) and (49) it is seen that if the depth is exactly critical, $A_{2\ell}$ and $B_{2\ell}$ are arbitrary to the first or linear approximation. That is, when $H = H_c$, $A_{2\ell}$ and $B_{2\ell}$ are linearly independent of A_{11}. However, if the depth is not exactly critical $A_{2\ell}$ and $B_{2\ell}$ will be zero.

The results of the second approximation are given in full in Appendix IV. As in the general-depth solution it is found that $B_{2\ell}$ (and indeed all B_{mn}, $m \geq 1$) is proportional to B_{11} and is hence identically zero. The most interesting result is the relation between $A_{2\ell}$ and A_{11}. This relation is of the form

$$A_{2\ell} = \frac{C_2 \pm \sqrt{C_2^2 + C_4}}{C_3} \tag{54}$$

in which C_2, C_3, and C_4 are functions of H and A_{11} given by

$$C_2 = 1 - \frac{K_\ell \, \mathrm{th}_\ell}{4 K_1 \, \mathrm{th}_1} \tag{55}$$

$$\begin{aligned} C_3 = \frac{K_\ell^2 \, \mathrm{th}_\ell}{8 \, \mathrm{th}_1} \Bigg[& \left(3 \, \mathrm{th}_1 \mathrm{th}_\ell + \frac{\mathrm{th}_\ell}{\mathrm{th}_1} - 2 \frac{K_\ell}{K_1} \right) \alpha_1 (J_{01}, J_{0\ell}) \; - \\ & - \frac{K_\ell \, \mathrm{th}_\ell}{K_1 \, \mathrm{th}_1} \alpha_1 (J_{11}, J_{1\ell}) \Bigg] \end{aligned} \tag{56}$$

$$C_4 = \frac{A_{11}^2 K_1^2}{4}\left[(3\varpi_1^2 - 1)\alpha_\ell(J_{01}^2) + 2\alpha_\ell(J_{11}^2)\right]C_3 \quad . \tag{57}$$

The quantity $C_2^2 + C_4$ under the radical in equation (54) has been shown (analytically for $\ell = 4$ and numerically for $\ell = 3$) to be positive for both second-order critical depths.

At first glance $A_{2\ell}$ appears to be double-valued. However the possiblility of double-valuedness is quickly dispelled. For any depth nearly but not quite critical (that is, $|C_2|$ very small but non-zero) let A_{11} become infinitesimal. By the results of the first approximation $A_{2\ell}$ must approach zero as A_{11} becomes infinitesimal. This requires that the negative sign in equation (54) be taken when $C_2 > 0$, and the positive sign taken when $C_2 < 0$. Only when C_2 is exactly zero is the sign of $A_{2\ell}$ not uniquely determined. The absolute magnitude of $A_{2\ell}$ is determined then; however, the motion due to $A_{2\ell}$ is either in phase with the primary A_{11} motion or is 180° out of phase. For a depth differing in the slightest amount from the critical depth, the phase between $A_{2\ell}$ and A_{11} is also uniquely determined. Thus equation (54) may be written as

$$\left.\begin{aligned} A_{2\ell} &= \frac{C_2 - \sqrt{C_2^2 + C_4}}{C_3} \qquad (C_2 \geq 0) \\[2ex] A_{2\ell} &= \frac{C_2 + \sqrt{C_2^2 + C_4}}{C_3} \qquad (C_2 \leq 0) \quad . \end{aligned}\right\} \tag{58}$$

The magnitude $\left|\frac{A_{2\ell}}{A_{11}}\right|$ is plotted against H for $A_{11} = .01$ in Figure 2. This figure illustrates the transition of $A_{2\ell}$ from order $\mathcal{O}(A_{11}^2)$ to $\mathcal{O}(A_{11})$ as H approaches the critical depth.

The purpose of this section has been to point out the existence of the critical depths and to show that the wave motion when the depth is critical or nearly critical can be analyzed by the procedure of the previous section if the order-of-magnitude assumptions (25) for non-critical depths are appropriately modified. This dissertation is primarily concerned with the motion when the depth is non-critical; nevertheless, certain aspects of the critical-depth solution will be discussed in sections 2 and 4 of the next chapter.

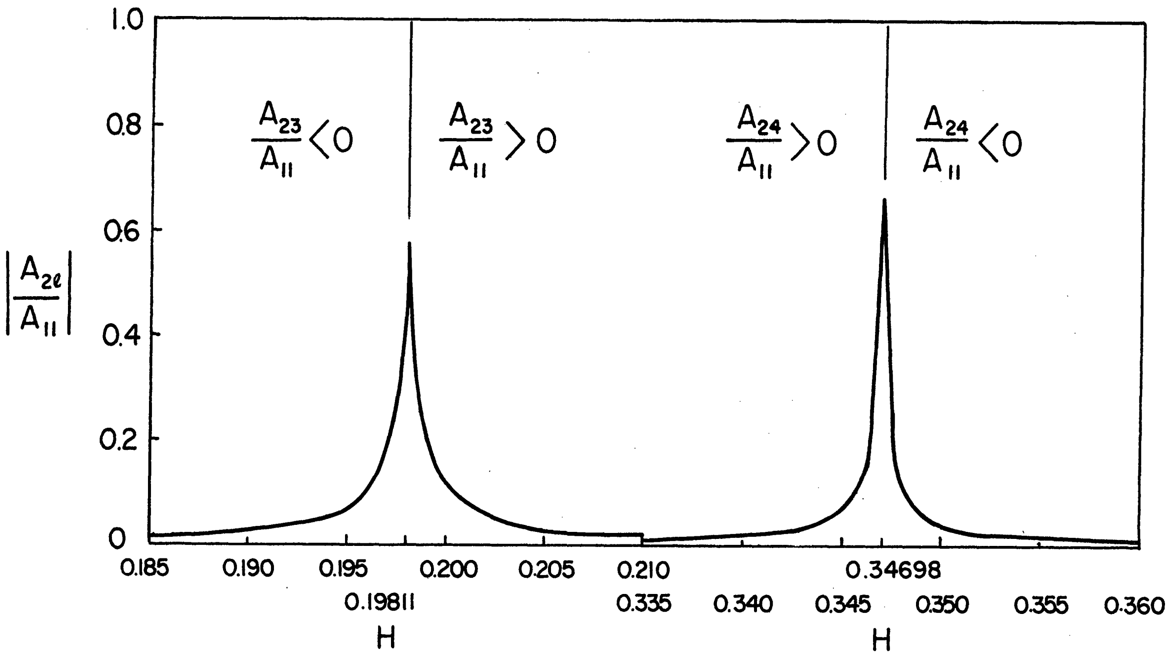

Figure 2. The Magnitude $\left|\frac{A_{2\ell}}{A_{11}}\right|$ versus H for $A_{11} = .01$

CHAPTER III

ANALYSIS OF THE SOLUTION

1. The Surface Profile

The surface elevation $\eta(r,t)$ of a periodic, axisymmetric gravity wave in fluid of general depth, as given to the third approximation in Appendix III, equations (III-6) through (III-9), may be examined more conveniently in the form,

$$\eta^*(r,t) = A_{11}K_1^2\,J_0^*(r) + J_{01}\cos t + (A_{11}K_1^2)^2\,J_1^*(r)\cos t + \tag{59}$$
$$+ A_{11}K_1^2\,J_2^*(r)\cos 2t + (A_{11}K_1^2)^2\,J_3^*(r)\cos 3t,$$

where the starred functions are defined by

$$A_{11}K_1 th_1\,\eta^*(r,t) = \eta(r,t)$$
$$A_{11}K_1 th_1\,(A_{11}K_1^2)\,J_0^*(r) = J_0(r)$$
$$A_{11}K_1 th_1\,(A_{11}K_1^2)^2\,J_1^*(r) = J_1(r) - A_{11}K_1 th_1\,J_{01} \tag{60}$$
$$A_{11}K_1 th_1\,(A_{11}K_1^2)\,J_2^*(r) = J_2(r)$$
$$A_{11}K_1 th_1\,(A_{11}K_1^2)^2\,J_3^*(r) = J_3(r) \quad .$$

In the form (59) the effects on η^* of the parameters A_{11} and H have been separated since the $J_m^*(r)$, $(m = 0, 1,2,3)$, depend only on H.

A number of observations may be made illustrating how the characteristics of a finite-amplitude wave differ from those of a wave of infinitesimal amplitude. For waves of infinitesimal amplitude, the temporal mean elevation of the surface is identically zero at all radii. There is a nodal circle at r = .628. The surface is horizontal twice during each period, at $t = \frac{\pi}{2}$ and $t = \frac{3\pi}{2}$. The crests and troughs are identical in shape; that is $\eta(r,0) = -\eta(r,\pi)$.

In contrast to the infinitesimal wave, the finite wave does not have these properties. Because of the J_0^* term which is independent of time, the temporal mean elevation of the surface at any given radius is in general not zero.[7] There is no nodal circle. The surface is never horizontal. Probably the most striking difference between the infinitesimal and the finite wave is the marked alteration in the latter of the shape of the crests and troughs. The crests become higher and narrower and the troughs become broader and shallower as the amplitude is increased.

Penney and Price[(49)], in their discussion of periodic, finite-amplitude standing waves in rectangular co-ordinates for water of infinite depth, noticed the effects of finite amplitude consistent with their geometrical configuration, equivalent to the effects mentioned above. The narrowing and heightening of the crests and the broadening and flattening of the troughs have been observed both experimentally and analytically over a wide range of relative depths for both standing and progressive waves of finite amplitude.[(11)(24)(41)(46)(49)(58)]

If we define the amplitude of the surface displacement $\eta(r,t)$ to be $\frac{1}{2}[\eta(0,0) - \eta(0,\pi)]$, and define the quantity N, dependent solely on the parameters A_{11} and H, as

$$N = \frac{\eta^*(0,0) - \eta^*(0,\pi)}{2}, \qquad (61)$$

7 J_0^*, and hence the temporal mean surface elevation, is zero at two particular values of r, which values are functions of H.

then the function $\frac{\eta^*(r,t)}{N}$ will have an "amplitude" of unity regardless of what values are assigned to A_{11} and H. $\frac{\eta^*(r,t)}{N}$ thus indicates the shape of the free surface. This function, evaluated at t = 0 and t = π, is plotted in Figures 3 through 6 for both finite and infinitesimal waves for four different depths, including the two extremes H = 0 and H = ∞. For each depth the amplitude parameter A_{11} of the finite wave has the value $\frac{.2}{K_1^2}$. It is understood that when we speak of a function evaluated when A_{11} is zero and/or H is zero or infinity we really mean the limit of that function as A_{11} approaches zero and/or H approaches zero or infinity.

Figures 3 through 6 clearly show the difference in profile shape between a finite-amplitude wave and an infinitesimal wave. The heightening of the crests and the broadening of the troughs of the finite wave is most pronounced for the small depths; in the limiting case H = 0, the tip of the crest in the center of the basin as given by the non-linear theory is 29.7% higher than that predicted by the linear solution.

In calculating the shape of the finite wave for H = ∞ and H = .3 the third-order terms made negligible contributions. For the shallower depths, however, the third-order terms were significant, their contributions for H = 0 being almost half as large as those of the second-order terms. This observation is in keeping with the predition that, if our method is convergent, the convergence will be least rapid for very shallow depths.

2. The Frequency of Oscillation

The frequency of oscillation ω of an axisymmetric infinitesimal-amplitude first-mode gravity wave in a circular tank of constant depth has been given by previous investigators(19)(11)(46) in the dimensional form

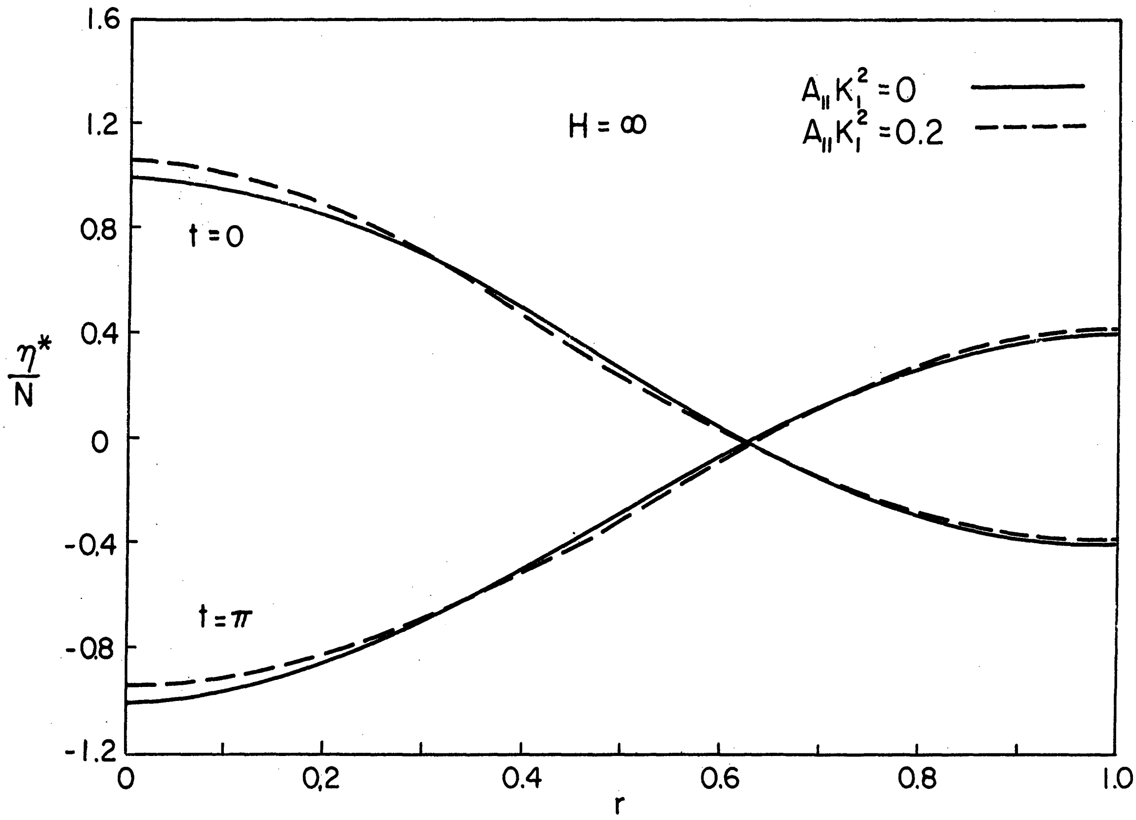

Figure 3. Configuration of the Free Surface for H = ∞

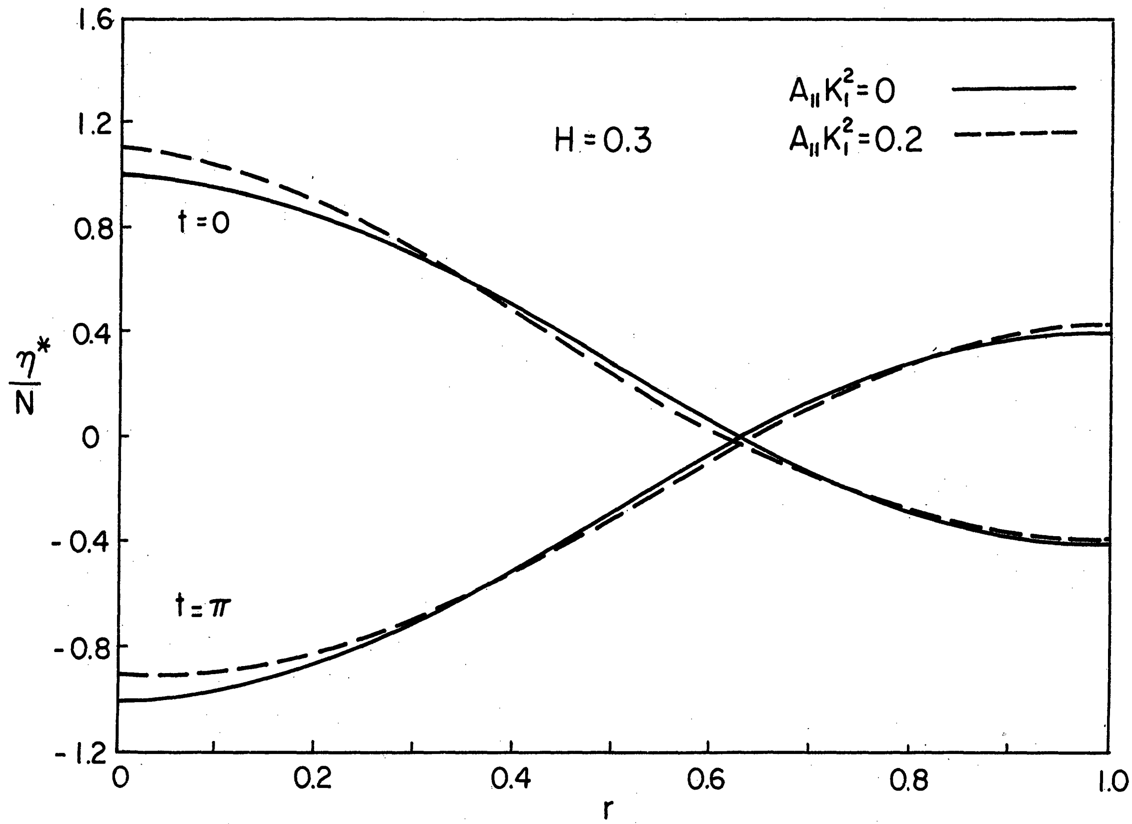

Figure 4. Configuration of the Free Surface for H = .3

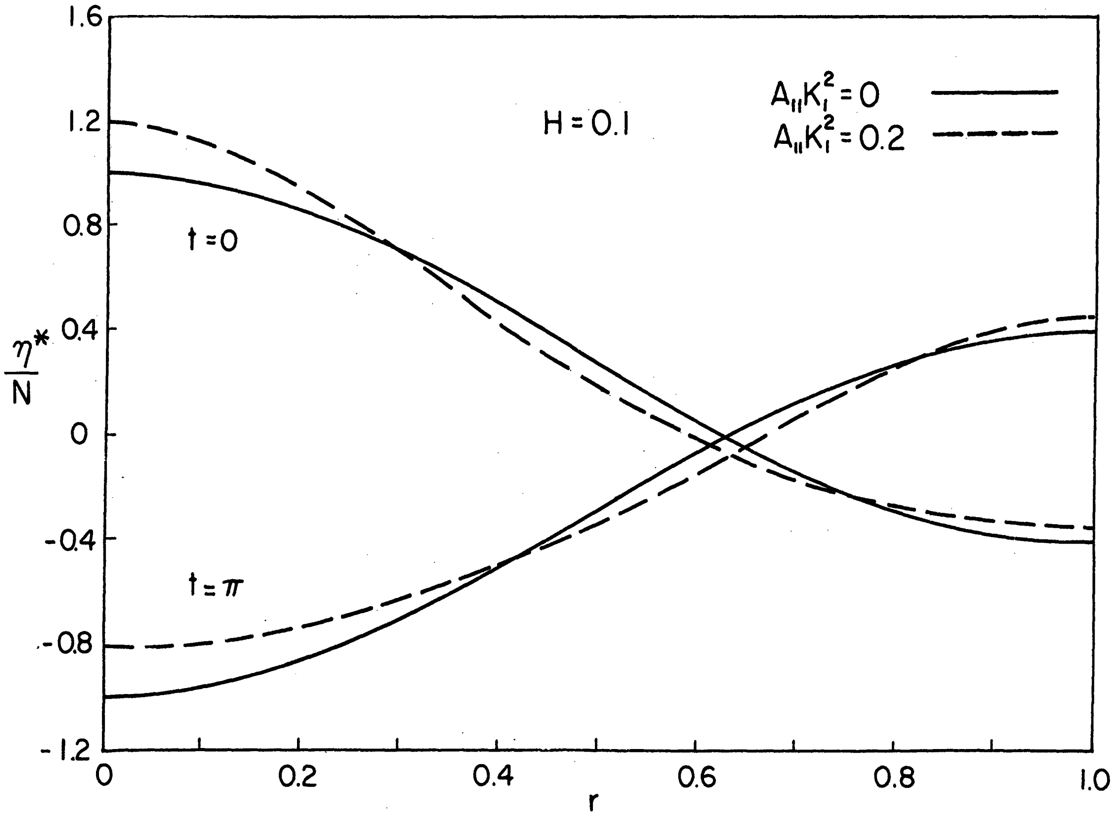

Figure 5. Configuration of the Free Surface for H = .1

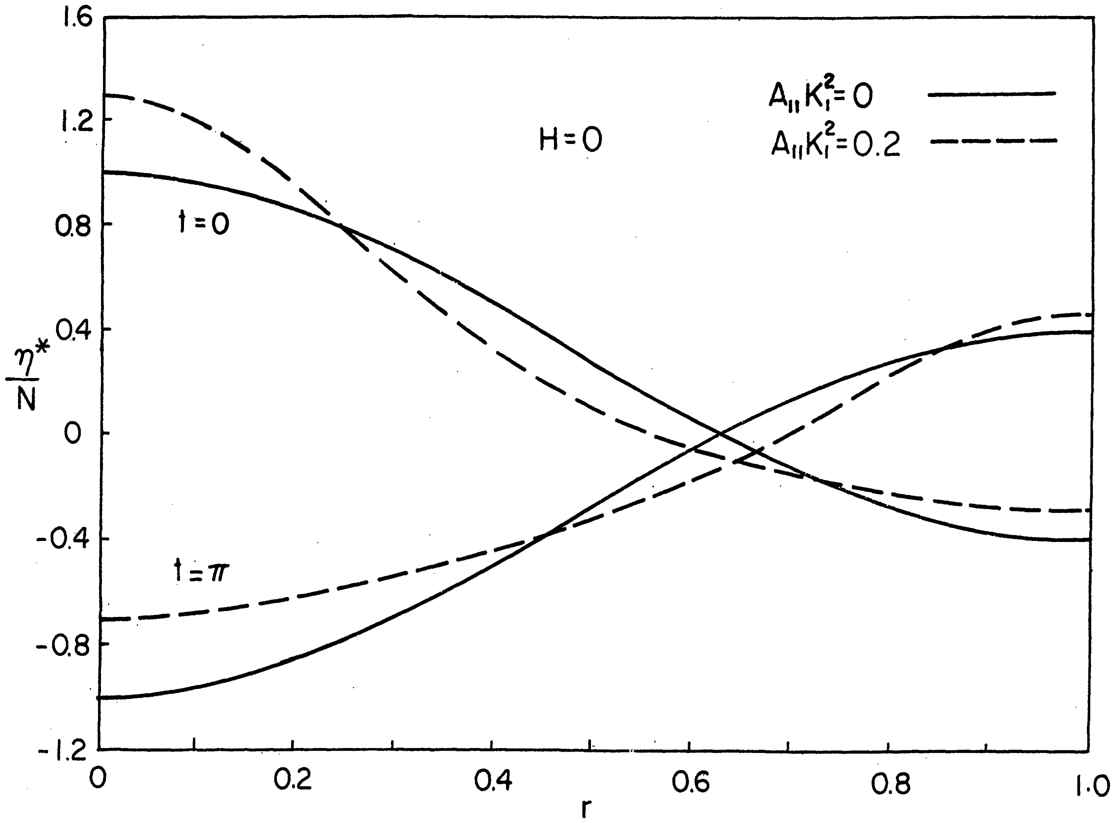

Figure 6. Configuration of the Free Surface for H = 0

$$\omega^2 = \bar{g}\,\frac{K_1}{R}\tanh\frac{K_1}{R}\bar{H} \quad . \tag{62}$$

In the non-dimensional notation of this study (62) becomes equation (41)

$$\frac{1}{G} = K_1\,th_1$$

which is the frequency equation to both the first and second approximations when the depth is non-critical. In the third approximation the frequency equation is

$$\frac{1}{G} = K_1\,th_1\left\{1 + \frac{(A_{11}K_1^2)^2}{4}\,G_c\right\} \tag{63}$$

where the correction factor G_c is a function of H only and is given by

$$\begin{aligned} G_c = \Big[& -\tfrac{1}{2}\left(2\,th_1^4 - th_1^2 + 1\right)\alpha_1(J_{01}^3) - \\ & -\tfrac{1}{2}\left(1 + th_1^2\right)\alpha_1\left(J_{01}\,J_{11}^2\right) + \alpha_1\left(\frac{J_{11}^3}{K_1 r}\right) + \\ & + \left(1 - th_1^2\right)^2 J_0^2(K_1) - \frac{1}{4}\sum_{p=1}^{\infty}\frac{K_p^2}{K_1^2}\,\Gamma_p\,\alpha_1\left(J_{01}\,J_{0p}\right) + \\ & + \frac{1}{8}\left(1 + 3\,th_1^2\right)\sum_{p=1}^{\infty}\frac{K_p\,th_p}{K_1\,th_1}\,\Gamma_p\,\alpha_1\left(J_{01}\,J_{0p}\right) - \\ & - \frac{1}{8}\sum_{p=1}^{\infty}\frac{K_p^2\,th_p}{K_1^2\,th_1}\,\Gamma_p\,\alpha_1\left(J_{11}\,J_{1p}\right)\Big] . \end{aligned} \tag{64}$$

Since the derivatives with respect to H of th_1^4, th_1^2, and $\frac{th_p}{th_1}$, (p = 1,2...), and consequently also those of Γ_p vanish at H = 0 and H = ∞, it follows from (64) that $\frac{dG_c}{dH} = 0$ when H = 0 and H = ∞. G_c has been plotted against

H in Figure 7. Since G_c in the range $1 \leq H \leq \infty$ differs from $G_c\big|_{H=1}$ by less than one-fourth of one percent, only the range $0 \leq H \leq 1$ is shown in Figure 7.[8]

Equation (64) contains the second-order critical-depth factors of the type (43) which cause Γ_3 and Γ_4 to become infinite when H = .19811 and H = .34698, respectively. Thus for depths nearly equal to (say within $\pm$ .002 of) either of these depths, the critical-depth solution must be used. Since (64) is not applicable in the immediate vicinity of the critical depths, the curve in Figure 7 has been drawn smoothly through these depths. The location of the second-order critical depths is indicated in Figure 7 by the vertical dashed lines.

A most interesting feature of Figure 7 is that G_c changes sign; it is positive for small depths and negative for large depths. The value of H at which G_c passes through zero might even be selected as the division between moderately shallow and moderately deep water. Thus for large depths the frequency of oscillation is decreased and the period increased in comparison with those of an infinitesimal-amplitude wave. Conversely, for small depths the frequency is increased and the period decreased when compared with the values predicted by the linear theory. This result for very large depths is qualitatively the same as that obtained by Penney and Price (49) for standing waves in a rectangular co-ordinate system.

If the period of oscillation is represented by $\mathcal{T}$ (dimensional),[9] and the subscript o is used to denote the values of ω and $\mathcal{T}$ given by the

[8] For similar reason only the range $0 \leq H \leq 1$ is shown in Figure 8 also.
[9] The non-dimensional period is 2π.

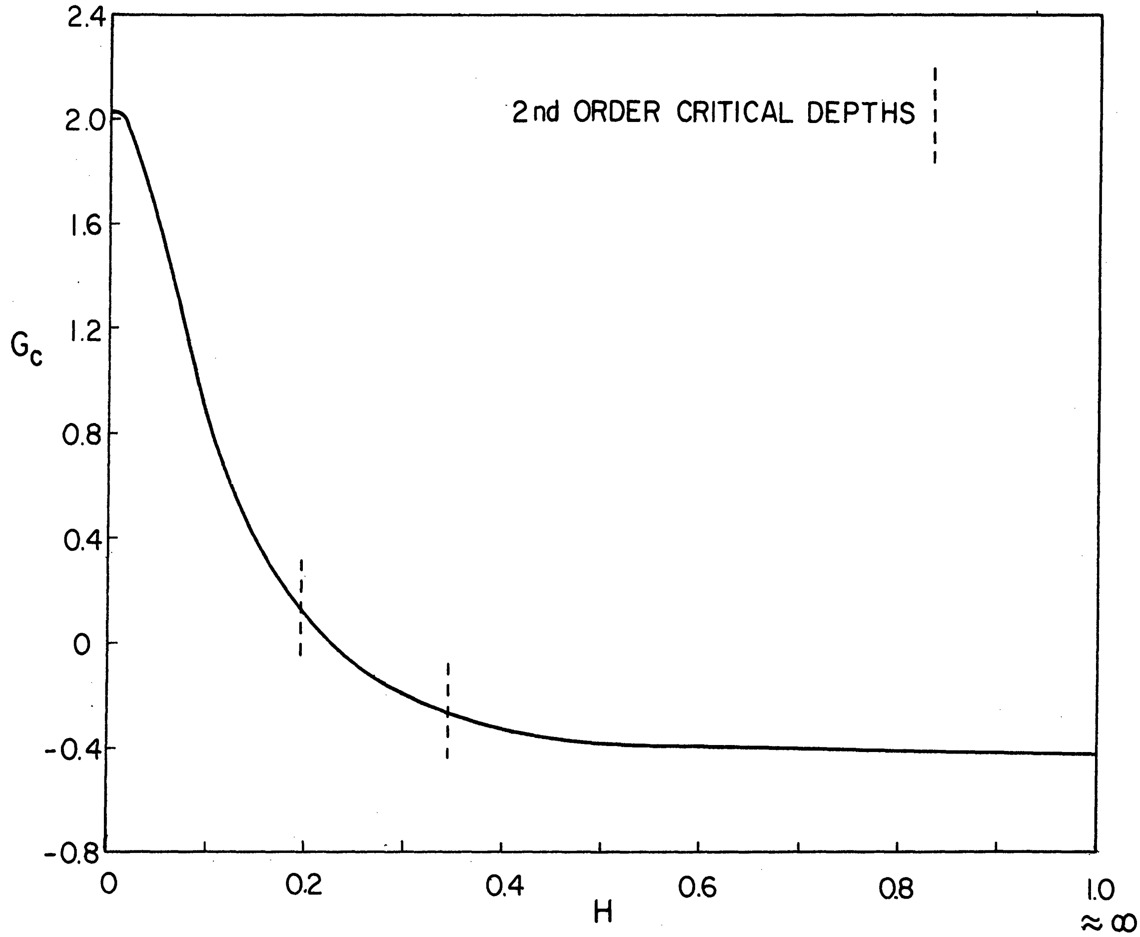

Figure 7. The Frequency Correction Factor G_c versus H

linear theory, then $\frac{\omega-\omega_0}{\omega_0}$ and $\frac{\tau-\tau_0}{\tau_0}$ will be the relative corrections to the linear theory due to finite amplitude. $\frac{\omega-\omega_0}{\omega_0}$ and $\frac{\tau-\tau_0}{\tau_0}$ have been calculated for the extreme cases H = 0 and H = ∞ for waves of approximately the maximum amplutude.[10] These values are shown in Table 3.

TABLE 3

$\frac{\omega-\omega_0}{\omega_0}$ AND $\frac{\tau-\tau_0}{\tau_0}$ FOR WAVES OF APPROXIMATELY

THE MAXIMUM AMPLUTUDE

H	$A_{11}K_1^2$	$\frac{\omega-\omega_0}{\omega_0}$	$\frac{\tau-\tau_0}{\tau_0}$
0	.6	+ 8.8%	-8.1%
∞	.8	- 3.4%	+3.3%

When the depth is very nearly equal to either of the second-order critical depths, the frequency is given to the second approximation by

$$\frac{1}{G} = K_1 \mathrm{th}_1 \left\{ 1 + \frac{A_{2\ell} K_1 K_\ell}{4} \left[\left(2\frac{K_\ell}{K_1} - \frac{\mathrm{th}_\ell}{\mathrm{th}_1} - 3\,\mathrm{th}_1 \mathrm{th}_\ell \right) \alpha_1 (J_{01}\, J_{0\ell}) + \frac{K_\ell \mathrm{th}_\ell}{K_1 \mathrm{th}_1} \alpha_1 (J_{11}\, J_{1\ell}) \right] \right\} \tag{65}$$

in which it is understood that $\ell = 3$ when H $\approx$ .19811 and $\ell = 4$ when H $\approx$ .34698. Since the sign of $A_{2\ell}$ when $H = H_c^-$ is opposite to that when $H = H_c^+$, it is noted from (65) that the sign of the frequency correction when $H = H_c^-$ is opposite to that when $H = H_c^+$. The predicted corrections to the frequency at the second-order critical depths have been computed for an amplitude of A_{11} = .01 and are shown in Table 4.

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10 The wave of maximum amplitude is discussed in section 4 of this chapter.

TABLE 4

$\frac{\omega-\omega_0}{\omega_0}$ AT THE SECOND-ORDER CRITICAL DEPTHS FOR $A_{11} = .01$

H	$\frac{\omega-\omega_0}{\omega_0}$
$.19811^-$	-.05%
$.19811^+$	+.05%
$.34698^-$	-.01%
$.34698^+$	+.01%

Although these corrections are very small in magnitude, it should be remembered that the general-depth solution does not predict any correction until the third approximation. Since $\frac{1}{G}$ is linear in $A_{2\ell}$ and since the correction term is very small, $\frac{\omega-\omega_0}{\omega_0}$ will also be linear in $A_{2\ell}$; hence to a suitable scale a plot of $\left|\frac{\omega-\omega_0}{\omega_0}\right|$ versus H for H nearly critical will have the same shape as the curves in Figure 2, if it is assumed that the function of H in square brackets in (65) is constant for values of H sufficiently near H_c.[11]

3. The Pressure and Velocity Distributions

The pressure at any point in the fluid may be obtained through the use of the Bernoulli equation. Merging the F(t) in (7) with ϕ as was done before writing the dynamic surface boundary condition (8), we obtain the dimensionless pressure equation

$$p - p_0 = \frac{\partial \phi}{\partial t} + C_1 - Gz - \frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial z}\right)^2 . \qquad (66)$$

11 The variation of this function with H is much much less than the variation of $\left|\frac{A_{2\ell}}{A_{11}}\right|$ when H is nearly critical.

The radial and vertical components of the velocity, v_r and v_z, respectively, are obtained by differentiating ϕ in the usual manner

$$v_r = -\frac{\partial\phi}{\partial r} \,, \qquad v_z = -\frac{\partial\phi}{\partial z} \,. \tag{67}$$

Because other aspects of the solution are of greater interest, no attempt has been made toward detailed analysis of the pressure or velocity distributions. One observation in connection with the velocity, however, should be made. Twice during each period ($t = 0$ and $t = \pi$) the fluid is everywhere momentarily at rest. This implies that the motion may be generated by giving the free surface the configuration $\eta(r, 0)$ or $\eta(r, \pi)$ and releasing the fluid from rest.

4. The Wave of Maximum Amplitude

Michell[15] and Havelock[9] have discussed the maximum amplitude of progressive waves in deep water. If the amplitude exceeds this limiting value, breaking will occur at the crests and the waves cannot be propagated with constant form. Penney and Price[49], following a different approach have obtained the maximum amplitude of stationary waves in a rectangular co-ordinate system when the fluid depth is infinite. With their criterion it can be shown that for axisymmetric standing waves there is a maximum value of A_{11}, this maximum being a function of H.

Since the motion is most extreme at the center of the tank, it is assumed that the condition of impending breaking will occur at the tip of the crest at $r = 0$ at the instant when $\eta(0,t)$ reaches its greatest positive elevation.[12]

---

[12] $\eta(0,t)$ will in general be maximum when $t = 0$. However, when the depth is nearly critical <u>and</u> $A_{q\ell}/A_{11} < 0$, there is the possibility that $\eta(0,t)$ may reach its greatest value when t is approximately $\frac{\pi}{q}$; that is, close to the time when that component of the surface elevation due to $A_{q\ell}$ makes its greatest contribution.

The criterion limiting the amplitude of the waves is based on the postulate that the fluid cannot withstand tension.[13] If the atmospheric pressure p_o is zero, the pressure just inside the liquid must be positive or zero and consequently at the crest[14]

$$\frac{\partial p}{\partial z} \leq 0 \quad . \tag{68}$$

The dimensionless Euler equation of motion in the z-direction is, in mixed notation,

$$-\frac{\partial^2 \phi}{\partial z \, \partial t} + v_r \frac{\partial v_z}{\partial r} + v_z \frac{\partial v_z}{\partial z} = -G - \frac{\partial p}{\partial z} \quad . \tag{69}$$

At the crest $v_r = v_z = 0$, and hence from (68)

$$\frac{\partial^2 \phi}{\partial z \, \partial t} \leq G \quad . \tag{70}$$

Consequently, the criterion limiting the amplitude of the waves is that the downward acceleration at the crest must not exceed the gravitational acceleration (G in non-dimensional units).

Now suppose that p_o is not zero, but is positive. If $\frac{\partial p}{\partial z}$ is positive at the crest, the liquid is not in tension, but from the equation of motion (69) the downward acceleration in the liquid just below the top of the crest is greater than it is at the crest. This is physically unacceptable. Thus (68) is true at the crest and the criterion (70) applies regardless of weather or not p_o is zero. At the tip of the crest of a maximum wave the equality sign in (68), and hence in (70), will hold.

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13 This criterion may also be obtained(49) through stability considerations.
14 In the remainder of this section and in the next section the term "crest" refers only to a crest at r = 0 at the instant of its greatest elevation.

To the first approximation (70) yields

$$(A_{11} K_1^2)_{max} = \frac{1}{th_1^2} \quad . \tag{71}$$

This permits A_{11} to become very large when the depth is very small. However, the first approximation to $(A_{11} K_1^2)_{max}$ cannot be expected to be at all reasonable since it is an attempt to use the infinitesimal-amplitude wave solution for predicting the breaking wave. The second and third approximations have also been obtained and all three are plotted in Figure 8.[15] No attempt has been made to show the effects of the critical depths in this plot. The maximum amplitude when the depth is critical or nearly critical will be discussed later in this section.

The value of $(A_{11} K_1^2)_{max}$, as predicted by the third approximation is about .8 for large depths and about .75 for small depths. It may readily be seen from Figure 8 that the convergence of the successive approximations (if, indeed, they do converge) is much slower for shallow depths. This is in agreement with what was anticipated at the end of section 2, Chapter II.

In the general-depth solution were assumed valid when the depth is critical, application of (70) would lead to the conclusion that $(A_{11} K_1^2)_{max} = 0$ when the depth is critical. Use of the critical-depth solution in (70) for depths critical or nearly critical results in a less stringent value of $(A_{11} K_1^2)_{max}$; however, $(A_{11} K_1^2)_{max}$ when the depth is critical is considerably smaller than at non-critical depths. By use of

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15 The third approximation to $(A_{11}K_1^2)_{max}$ was evaluated for only four values of H, namely ∞, .50, .25, and .10.

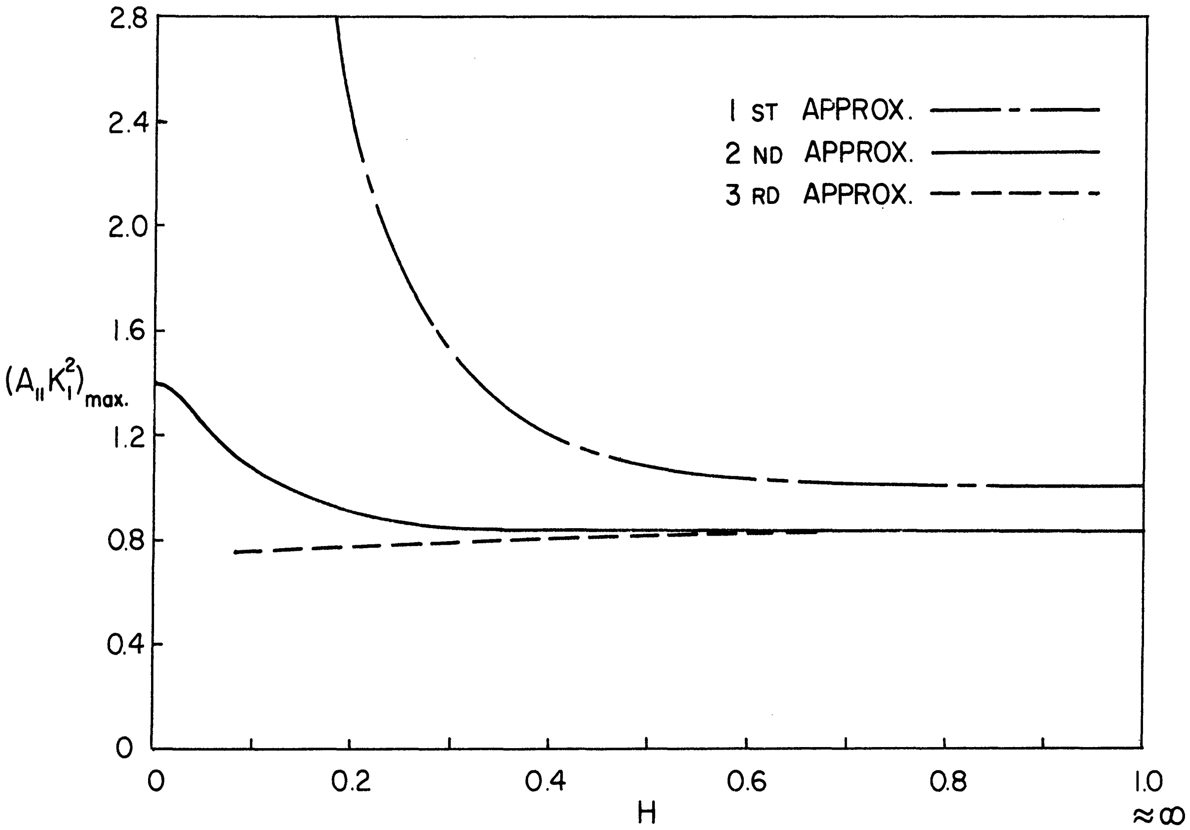

Figure 8. Maximum Amplitude versus H for H not Critical

the relation (58) between $A_{2\ell}$ and A_{11}, the maximum amplitude when H equals either of the second-order critical depths has been computed to the first approximation and is shown in Table 5.

TABLE 5

THE MAXIMUM AMPLITUDE AT THE SECOND-ORDER CRITICAL DEPTHS

H	sign of $\frac{A_{2\ell}}{A_{11}}$	$(A_{11}K_1^2)_{max}$
$.19811^-$	-	.523
$.19811^+$	+	.428
$.34698^-$	+	.210
$.34698^+$	-	.249

It is noticed that at either H_c $(A_{11}K_1^2)_{max}$ is smaller when $\frac{A_{2\ell}}{A_{11}}$ is positive than when this ratio is negative. This is to be expected since when $\frac{A_{2\ell}}{A_{11}}$ is positive the A_{11} and $A_{2\ell}$ components of the motion are in phase and both make their greatest contribution to the downward acceleration simultaneously at t = 0. On the other hand when $\frac{A_{2\ell}}{A_{11}}$ is negative the two components are out of phase and the greatest downward acceleration during a period will be less than that for the same A_{11} when $\frac{A_{2\ell}}{A_{11}}$ is positive. It follows that $(A_{11}K_1^2)_{max}$ is larger when $\frac{A_{2\ell}}{A_{11}}$ is negative than when $\frac{A_{2\ell}}{A_{11}}$ is positive, both at the same H_c.

The component of downward acceleration due to $A_{2\ell}$ is of order $\mathcal{O}(A_{11})$ when $H = H_c$ but becomes of order $\mathcal{O}(A_{11}^2)$ as $|H-H_c|$ increases. If the critical-depth solution is substituted into (70) to the first approximation and the resulting equation applied as $|H-H_c|$ increases, then the $A_{2\ell}$ term of that equation becomes of order $\mathcal{O}(A_{11}^2)$ but the other terms of order $\mathcal{O}(A_{11}^2)$ have not been taken into account. Since terms of order $\mathcal{O}(A_{11}^2)$ are of comparable magnitude with terms of order $\mathcal{O}(A_{11})$ in determining the maximum amplitude, $(A_{11}K_1^2)_{max}$ from the first approximation when $H \approx H_c$ will not approach, as $|H-H_c|$ increases, $(A_{11}K_1^2)_{max}$ from either the first or second approximations as plotted in Figure 8 for H not critical. In order to investigate the manner in which, as $|H-H_c|$ increases, $(A_{11}K_1^2)_{max}$ for depths nearly critical approaches the value predicted from the general-depth theory requires that both the general-depth solution and the critical-depth solution be carried to a sufficiently high order of approximation that the apparent convergence of each has been secured.

5. The Angle at the Crest of a Maximum Wave

Stokes(24) showed that, if progressive waves exist having a discontinuity of slope at the crest, the angle enclosed there is 120°. Experimental work(58) has indeed shown that such a wave apparently can exist. Penney and Price(49) recently concluded that the crest of the maximum stable standing wave in rectangular co-ordinates is also pointed, enclosing an angle of 90°. Following the general procedure used by Penney and Price leads to the conclusion that the maximum axisymmetric standing wave has at r = 0 a crest enclosing an angle of approximately 109.5°.

Because the eigenfunctions J_{0n} all have first derivatives with respect to r which are zero at r = 0, the solution for $\eta(r,t)$, when taken to any finite order, must necessarily give a wave profile having a horizontal tangent at the crest. However, consideration of the free surface as the isobar, $p = p_0$, does yield a non-zero slope at the crest of a maximum wave at the instant of its greatest elevation.[16]

The equation of the free surface of the maximum wave at the instant $t = 0$ may be regarded as given by the implicit relation

$$p(r, z) = p_0 \quad , \tag{72}$$

where the curves $p(r,z) = C$, $(C \geq p_0)$, are the isobars in the $r - z$ plane. Assuming $p(r,z)$ to be continuous throughout the fluid, then for an infinitesimal displacement (dr,dz) from the point (r,z) we have

$$dp = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz \quad .$$

Taking the point (r,z) at the tip of the crest, and choosing a displacement (dr,dz) such that the new point is also in the free surface,

$$0 = \frac{\partial p}{\partial r} dr + \frac{\partial p}{\partial z} dz \quad . \tag{73}$$

From the previous section $\frac{\partial p}{\partial z}$ is zero at the crest of a maximum wave. From (73) we perceive that $\frac{\partial p}{\partial r}$ is also zero. Hence the tip of the crest is a singular point.

Proceeding to the second order in dr and dz for an infinitesimal displacement from the crest along the free surface,

$$0 = \frac{\partial^2 p}{\partial r^2}(dr)^2 + 2\frac{\partial^2 p}{\partial r \partial z} dr\, dz + \frac{\partial^2 p}{\partial z^2}(dz)^2 \quad . \tag{74}$$

Since p is obtained from the velocity potential by the Bernoulli equation (66)

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16 For the rest of this section, this instant of greatest elevation is taken as $t = 0$. Because of the reservation of footnote 12, page 32, it is understood that "$t = 0$" in this section is not necessarily the same time origin used elsewhere in this paper.

$$p - p_0 = \frac{\partial \phi}{\partial t} + C_1 - Gz - \frac{1}{2}\left(\frac{\partial \phi}{\partial r}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi}{\partial z}\right)^2 ;$$

and $\frac{\partial \phi}{\partial r}$ and $\frac{\partial \phi}{\partial z}$ are identically zero at the crest when t = 0, p must satisfy $\nabla_1^2 p = 0$ at the crest when t = 0:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{\partial^2 p}{\partial z^2} = 0 \quad . \tag{75}$$

At r = 0 the middle term of (75) is indeterminate. However one application of l' Hospital's rule yields,

$$\lim_{r \to 0} \frac{\frac{\partial p}{\partial r}}{r} = \lim_{r \to 0} \frac{\frac{\partial^2 p}{\partial r^2}}{1} = \frac{\partial^2 p}{\partial r^2} \quad . \tag{76}$$

Thus at the tip of the crest

$$\frac{\partial^2 p}{\partial r^2} = -\frac{1}{2}\frac{\partial^2 p}{\partial z^2} \quad . \tag{77}$$

Since ϕ must have the form (22) at least as far as its dependence on r and z is concerned, and since $\frac{d J_{0m}}{dr}$ is zero at r = 0, $\frac{\partial \phi}{\partial r}$ = 0 at r = 0 for all z, A_{11}, and t. It follows that not only $\frac{\partial}{\partial r}$ but also $\frac{\partial^{n+1}}{\partial r \, \partial z^n}$ of $\frac{\partial \phi}{\partial t}$, $\left(\frac{\partial \phi}{\partial r}\right)^2$, and $\left(\frac{\partial \phi}{\partial z}\right)^2$ are zero at r = 0, assuming unlimited differentiability of ϕ. Since p_0, C_1, and Gz are independent of r, it follows from (66) that

$$\frac{\partial p}{\partial r} = 0 \qquad \text{at} \quad r = 0 \tag{78}$$

and

$$\frac{\partial^{n+1} p}{\partial r \, \partial z^n} = 0 \qquad \text{at} \quad r = 0 \; . \tag{79}$$

From (79) it is seen that the middle term of (74) is zero and hence that

$$\frac{\partial^2 p}{\partial r^2}(dr)^2 + \frac{\partial^2 p}{\partial z^2}(dz)^2 = 0 \quad . \tag{80}$$

From (77) and (80)

$$\frac{\partial^2 p}{\partial z^2}\left[\frac{1}{2}(dr)^2 - (dz)^2\right] = 0 \tag{81}$$

at the tip of the crest of the maximum wave.

Therefore

$$dz = \pm \frac{1}{\sqrt{2}}\, dr \, . \tag{82}$$

The negative root is the one associated with a crest. If the angle between the negative z-axis and the tangent to the surface is denoted by γ then the total enclosed angle 2γ at the crest of a maximum wave is

$$2\gamma = 2 \arctan \sqrt{2} \approx 109^\circ 28' \, . \tag{83}$$

This conclusion is valid unless $\frac{\partial^2 p}{\partial z^2}$ and consequently $\frac{\partial^2 p}{\partial r^2}$ are zero.

That the angle at the crest of the maximum axisymmetric wave should be greater than the 90° of the maximum two-dimensional standing wave of Penney and Price might be anticipated from the difference in geometry. Let us view both wave types from above. In the latter case there is a "line crest" parallel to the y-axis; as the crest rises toward its greatest elevation, the fluid particles approach it only from the positive and negative x-directions. In the axisymmetric case there is a "point crest" at $r = 0$; as the crest rises toward its greatest elevation, the fluid particles approach radially from all angles θ . If each system possesses a given amplitude less than the maximum amplitude of either, and if the amplitude of each is gradually increased, it seems logical, because of the geometric difference noted above, for the axisymmetric wave to reach an unstable condition earlier, that is, at a less sharply pointed crest. This is indeed the result just found.

At first glance it appears that (82) is valid for a depression as well as for an elevation at r = 0. However at the bottom of a trough at the instant of its greatest depression the acceleration is upward and hence from (69) $\frac{\partial p}{\partial z}$ is not zero. From (78) $\frac{\partial p}{\partial \eta}$ is zero there at all times. Thus we conclude from (73) that dz = 0 for a depression and the tangent to the surface at the bottom of the trough is horizontal.

6. The Energy of the Wave Motion

In a progressive wave of infinitesimal amplitude the total energy of the motion is one-half kinetic and one-half potential. Similarly in an axisymmetric standing wave, the mean kinetic energy equals the mean potential energy to the first or linear approximation. What is true for an axisymmetric standing wave of finite amplitude?

The potential energy V of the wave motion is

$$V = \frac{1}{2} G \iint_A \eta^2 \, dA \tag{84}$$

in which dA is an element of area in a plane normal to the z-axis. Setting dA = rdr dθ and performing the integration with respect to θ, we obtain

$$V = \pi G \int_0^1 r \eta^2 \, dr . \tag{85}$$

If a bar denotes the dimensional energies, then V and T have been made non-dimensional by

$$V = \frac{\bar{V}}{\rho\omega^2 R^5} \quad , \quad T = \frac{\bar{T}}{\rho\omega^2 R^5} \quad . \tag{86}$$

The kinetic energy T of the wave motion is

$$T = \frac{1}{2} \iiint\limits_{\mathcal{V}} \left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 \right] d\mathcal{V} \tag{87}$$

in which $d\mathcal{V}$ is an element of volume of the fluid. Upon integration with respect to θ, (87) may be written as

$$T = \pi \int\limits_{r=0}^{1} r \left\{ \int\limits_{z=-H}^{\eta} \left[\left(\frac{\partial\phi}{\partial r}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2 \right] dz \right\} dr \quad . \tag{88}$$

However, a more convenient form for the evaluation of T may be obtained by using Green's theorem to transform the triple integration (87) into a double integration

$$T = -\frac{1}{2} \iint\limits_{\mathcal{S}} \phi \frac{\partial\phi}{\partial n} \, d\mathcal{S} \tag{89}$$

in which $d\mathcal{S}$ is an element of area in the boundaries of the fluid region, and dn is an increment of length along the normal to $d\mathcal{S}$, positive when directed into the fluid region. Since $\frac{\partial\phi}{\partial n}$ is zero on the solid boundaries, only the integral over the free surface will make a contribution to (89). Setting $d\mathcal{S} = r d\theta \, ds$ and performing the θ integration,

$$T = -\pi \int\limits_{C} r \phi \frac{\partial\phi}{\partial n} \, ds \tag{90}$$

where C is the curve formed by the intersection of the free surface $z = \eta$ with a plane containing the z-axis, and ds is an increment of length along C.

On C, $dz = \frac{\partial \eta}{\partial r} dr$, and

$$\frac{\partial \phi}{\partial n} ds = \left[\frac{\partial \phi}{\partial r} \frac{\partial r}{\partial n} + \frac{\partial \phi}{\partial z} \frac{\partial z}{\partial n} \right] ds =$$

$$= \frac{\partial \phi}{\partial r} dz - \frac{\partial \phi}{\partial z} dr = \left[\frac{\partial \phi}{\partial r} \frac{\partial \eta}{\partial r} - \frac{\partial \phi}{\partial z} \right] dr .$$

By the kinematic free-surface boundary condition (17)

$$\frac{\partial \phi}{\partial r} \frac{\partial \eta}{\partial r} - \frac{\partial \phi}{\partial z} = \frac{\partial \eta}{\partial t}$$

at $z = \eta$. Thus

$$\frac{\partial \phi}{\partial n} ds = \frac{\partial \eta}{\partial t} dr$$

on C and we finally arrive at the expression

$$T = -\pi \int_0^1 r \left[\phi \frac{\partial \eta}{\partial t} \right]_{z=\eta} dr \quad . \qquad (91)$$

Performing the integrations (85) and (91) on the solution to the third approximation at general depth, we obtain

$$\begin{aligned} \frac{1}{\pi} V = {} & \frac{A_{11}^2 K_1 th_1}{4} J_0^2(K_1) \left[1 + \cos 2t \right] + \\ & + \frac{A_{11}^3 K_1^3 th_1}{4} \left[C_5 \cos t + C_6 \cos 3t \right] + \\ & + \frac{A_{11}^4 K_1^5 th_1}{96} \left[C_7 + C_8 \cos 2t + C_9 \cos 4t \right] , \end{aligned} \qquad (92)$$

and

$$\frac{1}{\pi} T = \frac{A_{11}^2 K_1 \mathrm{th}_1}{4} J_0^2(K_1)\left[1 - \cos 2t\right] + $$
$$+ \frac{A_{11}^3 K_1^3 \mathrm{th}_1}{4}\left[C_{10} \cos t - C_{10} \cos 3t\right] + \tag{93}$$
$$+ \frac{A_{11}^4 K_1^5 \mathrm{th}_1}{96}\left[C_{11} + C_{12} \cos 2t + C_{13} \cos 4t\right] .$$

The C_n, $(5 \leq n \leq 13)$, are rather long functions of H and are presented in Appendix V.

Because the system is conservative and no work is done at the boundaries, the total energy T + V must be constant. This requires that the sums $(C_5 + C_{10})$, $(C_6 - C_{10})$, $(C_8 + C_{12})$, and $(C_9 + C_{13})$ each be zero. Certain identities, involving integrals of products of Bessel functions, are sufficient to prove that the first three of these sums are each identically zero. These identities have been proved analytically in Appendix II as equations (II-15), (II-16), (II-17), (II-18), and (II-21). $C_9 + C_{13}$ will also be identically zero if two expressions (II-22), (II-23) are identities. The identity of these expressions has not been proved, but strong evidence for it has been presented in Appendix II. Thus the total energy of the wave is given by

$$\frac{1}{\pi}(T+V) = \frac{A_{11}^2 K_1 \mathrm{th}_1}{2} J_0^2(K_1) + \frac{A_{11}^4 K_1^5 \mathrm{th}_1}{96}\left(C_7 + C_{11}\right) . \tag{94}$$

Platzman[(50)] in his investigation of the partition of energy in periodic progressive waves of finite amplitude and permanent form in water of infinite depth found that the kinetic energy exceeded the potential energy. For the wave of maximum amplitude he found $\frac{T-V}{V} \approx \frac{1}{8}$, the first term in his expression yielding a value for $\frac{T-V}{V}$ of 9.93%.

Representing the temporal mean kinetic and potential energy by T_M and V_M, respectively, and making use of several of the identities of Appendix II we obtain[17]

$$\frac{T_M - V_M}{V_M} = \frac{(A_{11}K_1^2)^2}{16}\Bigg\{-2\left(3\mathit{th}_1^4 - 4\mathit{th}_1^2 + 5\right)\alpha_1\left(J_{01}\,J_{11}^2\right) + 6\alpha_1\left(\frac{J_{11}^3}{K_1\pi}\right) + \frac{(3\mathit{th}_1^2+1)}{4}\sum_{n=1}^{\infty}\frac{K_n\,\mathit{th}_n}{K_1\,\mathit{th}_1}\Gamma_n\,\alpha_1\left(J_{01}\,J_{0n}\right) + \left(1-\mathit{th}_1^2\right)^2 J_0^4(K_1) - 2\sum_{n=1}^{\infty}\frac{K_n}{K_1}\Gamma_n\,\alpha_1\left(J_{11}\,J_{1n}\right)\Bigg\}. \qquad (95)$$

For a wave of approximately maximum amplitude ($A_{11}K_1^2 = .8$) when $H = \infty$, (95) yields $\frac{T_M - V_M}{V_M} = -3.4\%$. Thus for an axisymmetric standing wave in fluid of infinite depth the mean potential energy exceeds the mean kinetic energy. Although one would hesitate to predict the sign of $T_M - V_M$ in advance, this result does seem reasonable since V is always positive while T equals zero twice during each period; consequently both $V_{min} > T_{min} = 0$ and $V_{max} > T_{max}$. The differences between the present problem (standing wave in cylindrical co-ordinates) and Platzman's problem (progressive wave in rectangular co-ordinates) cause the algebraic sign of our result for

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[17] Note that from (92) and (93) $T_M - V_M$ is of order $\mathcal{O}(A_{11}^4)$. This is in qualitative agreement with the result of Rayleigh(20) who first pointed out that the difference between the kinetic and potential energy of an oscillatory progressive wave of finite amplitude and permanent form in water of infinite depth is of fourth order.

$T_M - V_M$ to be opposite to his, both for the case $H = \infty$. However, as Starr (53, p. 185) in his article on energy integrals for gravity waves observed,

> So far as the writer has been able to find there appears to be no simple means for obtaining the magnitude or algebraic sign of ....ϵ[18]..... from general considerations without making use of the detailed solution to the wave problem.

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18 Starr's ϵ is defined as the difference between the kinetic and potential energy of the waves.

CHAPTER IV

CONCLUSIONS AND RECOMMENDATIONS

1. Concluding Remarks

The exact equations governing the free oscillations of finite-amplitude axisymmetric gravity waves are presented. These equations include two types: a linear group, and two non-linear free-surface boundary conditions. The eigenfunctions are determined from the linear equations. To represent a periodic first-mode motion a linear combination of these eigenfunctions is taken. An iteration procedure is followed to find the coefficients in this combination in terms of an amplitude parameter, A_{11}, such that the two non-linear boundary conditions are satisfied. Because of the complicated nature of the problem neither the existence of this motion nor the convergence of the solution procedure has been proved.

No a priori limitations have been made on the depth of the liquid. It is found, however, that there are certain discrete depths at which a higher mode at a frequency equal to an integral multiple of the basic frequency is of the same order of magnitude as the first mode. The motion when the depth is approximately equal to one of these critical depths is treated by an appropriate modification of the procedure used in obtaining the general solution.

Heightening of the crests and broadening of the troughs, when compared to the linear solution for the surface configuration, result from the analysis and are typical of all finite-amplitude wave solutions available so far. The period of oscillation, compared to that of infinitesimally small oscillations, is increased for large depths but decreased for small depths. The maximum amplitude for which an axisymmetric wave will remain stable has been investigated; it is found that the maximum wave has a pointed crest at $r = 0$ enclosing an angle of approximately 109.5°. The potential energy of the motion is greater than the kinetic energy (at least when the depth is infinite), and the difference between the potential and kinetic energy is proportional to the fourth power of the amplitude.

Detailed results pertaining to the surface configuration, the frequency of oscillation, the wave of maximum amplitude, and the energy of the wave motion are presented in Chapter III.

2. Suggestions for Further Study

Further study in several areas both directly and indirectly related to the present work seems desirable. These areas fall into two classes: those suggested by the results of this work, and more difficult problems toward whose solutions this work is but a first step.

In this dissertation the main interest has been in a first-mode wave when the depth is non-critical. Mention of the critical depths has been for the purpose of noting their existence and of showing that the motion when $H \approx H_c$ may be treated by the general procedure outlined in Chapter II. A detailed study of the coupled motion carrying the solution to higher approximations when $H \approx H_c$ would undoubtedly reveal many interesting features not noted here.

By a judicious permutation of subscripts the motion of axisymmetric waves of modes higher than the first may be obtained from the general-depth solution of this paper. However, there will be a different set of critical depths pertaining to the higher mode. For modes higher than the second there will also be the possibility of exciting lower-mode oscillations at frequencies equal to the basic frequency divided by an integer. For example, a third-mode wave at frequency ω will excite a first mode at frequency $\frac{1}{2}\omega$ when $H = .19811$. Caution should thus be exercised in applying the results of this study to higher modes.

An experimental study of the maximum-amplitude axisymmetric wave would be useful in verifying the prediction that such a wave has a crest angle of 109.5° as well as in gaining a greater understanding of the mechanism of breaking.

Of considerable practical interest are those modes of oscillation which are not axisymmetric. If the motion varies with θ as well as with r, z, and t, there are nodal diameters as well as nodal circles in the linear solution, which is discussed in some detail by Lamb[11]. Conceptually, a non-linear solution for unsymmetric gravity waves in a circular basin is only slightly more difficult than the work of this paper. However, a second infinite set of eigenvalues is introduced by the angular variation and the additional summation over this set makes the problem extremely involved from a computational standpoint.

Only free oscillations have been considered in this dissertation. The results obtained may be of use in attempting to solve the more difficult, but very important, problem of forced vibrations. Both resonant and non-resonant cases should be studied.

Very closely related to the forced motion is the problem of determining the motion following release of the fluid from rest with an initial arbitrary axisymmetric configuration of the free surface. This motion, which is easily analyzed if the problem is linearized, will in general be non-periodic.

APPENDIX I

THE EXPANSION OF FUNCTIONS IN DINI SERIES

The expansion of an arbitrary function F(r) of the real variable r in the form

$$F(r) = \sum_{n=1}^{\infty} b_n \, J_\nu(\lambda_n r) \quad , \tag{I-1}$$

where $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots\ldots\ldots$ denote the positive zeros in ascending order of magnitude of the function

$$x^{-\nu}\left\{x\, J_\nu'(x) + M\, J_\nu(x)\right\}$$

when $\nu \geq -1/2$ and M is any given constant, was first investigated by Dini(61). The coefficients in the expansion are given by the formula

$$\left\{(\lambda_n^2 - \nu^2)\, J_\nu^2(\lambda_n) + \lambda_n^2\, J_\nu'^2(\lambda_n)\right\} b_n = \tag{I-2}$$

$$= 2\lambda_n^2 \int_0^1 r\, F(r)\, J_\nu(\lambda_n r)\, dr \quad .$$

Dini noted that the expansion (I-1) must be modified by the insertion of an initial term, $\mathcal{B}_0(r)$, when $M + \nu = 0$ [19], although he gave its value incorrectly(66, p. 597). This initial term is given by

$$\mathcal{B}_0(r) = 2(\nu+1)\, r^{\nu} \int_0^1 r^{\nu+1}\, F(r)\, dr \quad . \tag{I-3}$$

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[19] A different initial term must also be inserted when $M + \nu < 0$, (66, p.597).

Of particular interest in this paper is the special case $M = \nu = 0$. Then

$$F(r) = B_0(r) + \sum_{n=1}^{\infty} b_n J_0(\lambda_n r) = \tag{I-4}$$

$$= 2\int_0^1 r F(r)\, dr + \sum_{n=1}^{\infty} b_n J_0(\lambda_n r)$$

where λ_n are the positive zeros of

$$x J_0'(x)$$

and the coefficients, b_n, are given by

$$b_n = \frac{\int_0^1 r F(r) J_0(\lambda_n r)\, dr}{\frac{1}{2} J_0^2(\lambda_n)} . \tag{I-5}$$

It is seen that the λ_n are precisely K_n. It is also seen that, if K_0 be defined as zero, the initial term $B_0(r)$ in (I-4) is the zeroth term of the summation with its coefficient b_0 given by (I-5) for $n = 0$. Because there are many different functions $F(r)$ which we wish to expand in Dini series, let us adopt the nomenclature $\alpha_n(F)$ for the b_n associated with a particular $F(r)$. Then

$$F(r) = \sum_{n=0}^{\infty} \alpha_n(F) J_{0n} , \tag{I-6}$$

where $K_0 \equiv 0$ and $K_1, K_2, K_3, \ldots\ldots\ldots$ are the positive zeros of $J_1(K)$ arranged in ascending order of magnitude and the coefficients $\alpha_n(F)$ are given by

$$\alpha_n(F) = \frac{\int_0^1 r F(r) J_{0n}\, dr}{\frac{1}{2} J_0^2(K_n)} . \tag{I-7}$$

If F(r) is continuous and has limited total fluctuation in the interval $0 \leq r \leq 1$ and if the integral $\int_0^1 r^{1/2} F(r)\, dr$ exists and converges absolutely, then the Dini series (I-6), is uniformly convergent and uniformly summable in the interval $0 \leq r \leq 1$ (66, pp. 598-616). All functions F(r) which we encounter, at least through the third approximation, meet these conditions.

Except in the very simplest instances, the integral in (I-7) cannot be evaluated analytically but must be done by numerical means. Those α's which have been computed through numerical integration are presented in Table 6. It is believed that the first three decimal places are accurate, while the fourth decimal place, particularly for the higher values of n, is known to be unreliable. The $\alpha_n(J_{01})$, whose values may be obtained analytically, were also computed numerically for comparison with the exact values.

For the discussion in Chapter III, section 6, of the energy of the wave motion, a number of integrals $S(F)$ are needed, where the operator $S(F)$ is defined as

$$S(F) \equiv \int_0^1 r\, F(r)\, dr \quad . \tag{I-8}$$

It is seen that the α's and S's are not independent but are related by

$$\alpha_n(F) = \frac{S(J_{0n} F)}{\frac{1}{2} J_0^2(K_n)} = \frac{S(J_{0n} F)}{S(J_{0n}^2)} \quad . \tag{I-9}$$

Table 7 gives the integrals which have been computed. The first three decimal places are believed accurate, while the fourth decimal place is unreliable.

TABLE 6

$\alpha_n(F)$ FOR VARIOUS FUNCTIONS $F(r)$

$F(r)$	J_{01} exact α's	J_{01} computed α's	J_{01}^2	J_{11}^2	J_{01}^3
$\alpha_0(F)$	zero	0.0000	0.1622	0.1622	0.0572
$\alpha_1(F)$	1.0000	1.0000	0.3523	0.1761	0.4139
$\alpha_2(F)$	zero	0.0001	0.4794	-0.3241	0.3157
$\alpha_3(F)$	zero	0.0001	0.0070	-0.0174	0.2082
$\alpha_4(F)$	zero	0.0002	-0.0007	0.0043	0.0060
$\alpha_5(F)$	zero	0.0002	0.0004	-0.0018	-0.0004
$\alpha_6(F)$	zero	0.0003	0.0002	0.0009	0.0004
$\alpha_7(F)$	zero	0.0003	0.0004	-0.0005	0.0002
$\alpha_8(F)$	zero	0.0007	0.0003	0.0003	0.0004
$\alpha_9(F)$	zero	0.0003	0.0003	-0.0002	0.0003
$\alpha_{10}(F)$	zero	0.0004	0.0004	0.0001	0.0004

$F(r)$	$J_{01}J_{11}^2$	$J_{11}^3/K_1 r$	$J_{01}J_{02}$	$J_{01}J_{03}$	$J_{01}J_{04}$
$\alpha_0(F)$	0.0286	0.0429	zero	zero	zero
$\alpha_1(F)$	0.1380	0.0811	0.2662	0.0027	-0.0002
$\alpha_2(F)$	-0.0185	-0.0775	0.3020	0.2945	0.0040
$\alpha_3(F)$	-0.1404	-0.0483	0.4254	0.2929	0.3087
$\alpha_4(F)$	-0.0089	0.0026	0.0076	0.4037	0.2897
$\alpha_5(F)$	0.0017	-0.0011	-0.0008	0.0076	0.3919
$\alpha_6(F)$	-0.0006	0.0005	0.0005	-0.0008	0.0076
$\alpha_7(F)$	0.0003	-0.0003	0.0002	0.0006	-0.0008
$\alpha_8(F)$	-0.0002	0.0002	0.0004	0.0002	0.0006
$\alpha_9(F)$	0.0001	0.0001	0.0003	0.0004	0.0002
$\alpha_{10}(F)$	-0.0001	0.0001	0.0004	0.0004	0.0005

TABLE 6, CON'T.

$F(r)$	$J_{01}J_{05}$	$J_{01}J_{06}$	$J_{01}J_{07}$	$J_{01}J_{08}$	$J_{01}J_{09}$
$\alpha_0(F)$	zero	zero	zero	zero	zero
$\alpha_1(F)$	0.0001	0.0000	0.0001	0.0000	0.0000
$\alpha_2(F)$	-0.0003	0.0002	0.0001	0.0001	0.0001
$\alpha_3(F)$	0.0047	-0.0004	0.0003	0.0001	0.0001
$\alpha_4(F)$	0.3173	0.0052	-0.0004	0.0003	0.0001
$\alpha_5(F)$	0.2882	0.3230	0.0056	-0.0005	0.0004
$\alpha_6(F)$	0.3846	0.2873	0.3272	0.0057	-0.0005
$\alpha_7(F)$	0.0078	0.3795	0.2868	0.3303	0.0059
$\alpha_8(F)$	-0.0007	0.0075	0.3758		
$\alpha_9(F)$	0.0007	-0.0007	0.0075		
$\alpha_{10}(F)$	0.0003	0.0007	-0.0006		

$F(r)$	$J_{01}J_{010}$	$J_{11}J_{12}$	$J_{11}J_{13}$	$J_{11}J_{14}$	$J_{11}J_{15}$
$\alpha_0(F)$	zero	zero	zero	zero	zero
$\alpha_1(F)$	0.0001	0.2436	0.0032	-0.0005	0.0001
$\alpha_2(F)$	0.0001	0.0824	0.2604	0.0054	-0.0008
$\alpha_3(F)$	0.0001	-0.3132	0.0551	0.2681	0.0066
$\alpha_4(F)$	0.0002	-0.0157	-0.3071	0.0416	0.2716
$\alpha_5(F)$	0.0002				
$\alpha_6(F)$	0.0004				
$\alpha_7(F)$	-0.0004				

TABLE 6, CON'T.

F(r)	$J_{11}J_{16}$	$J_{11}J_{17}$	$J_{11}J_{18}$	$J_{11}J_{19}$	$J_{11}J_{110}$
$\alpha_0(F)$	zero	zero	zero	zero	zero
$\alpha_1(F)$	-0.0000	0.0000	-0.0000	0.0000	-0.0000
$\alpha_2(F)$	0.0002	-0.0001	0.0000	-0.0000	0.0000
$\alpha_3(F)$	-0.0011	0.0003	-0.0001	0.0001	-0.0000
$\alpha_4(F)$	0.0073	-0.0013	0.0004	-0.0002	0.0001

TABLE 7

S(F) FOR VARIOUS FUNCTIONS F(r)

F(r)	100 S(F)	F(r)	100 S(F)	F(r)	100 S(F)
J_{01}^3	2.8577	$J_{01}J_{02}J_{06}$	0.0009	$J_{01}J_{04}J_{05}$	0.7564
$J_{01}^2J_{02}$	2.1589	$J_{01}J_{02}J_{07}$	0.0003	$J_{01}J_{04}J_{06}$	0.0123
$J_{01}^2J_{03}$	0.0219	$J_{01}J_{02}J_{08}$	0.0005	$J_{01}J_{04}J_{07}$	-0.0011
$J_{01}^2J_{04}$	-0.0016	$J_{01}J_{02}J_{09}$	0.0003	$J_{01}J_{04}J_{08}$	0.0008
$J_{01}^2J_{05}$	0.0008	$J_{01}J_{02}J_{010}$	0.0004	$J_{01}J_{04}J_{09}$	0.0003
$J_{01}^2J_{06}$	0.0003	$J_{01}J_{03}^2$	0.9130	$J_{01}J_{04}J_{010}$	0.0005
$J_{01}^2J_{07}$	0.0005	$J_{01}J_{03}J_{04}$	0.9624	$J_{01}J_{05}^2$	0.5561
$J_{01}^2J_{08}$	0.0004	$J_{01}J_{03}J_{05}$	0.0147	$J_{01}J_{05}J_{06}$	0.6234
$J_{01}^2J_{09}$	0.0004	$J_{01}J_{03}J_{06}$	-0.0013	$J_{01}J_{05}J_{07}$	0.0108
$J_{01}^2J_{010}$	0.0004	$J_{01}J_{03}J_{07}$	0.0008	$J_{01}J_{05}J_{08}$	-0.0009
$J_{01}J_{02}^2$	1.3602	$J_{01}J_{03}J_{08}$	0.0003	$J_{01}J_{05}J_{09}$	0.0007
$J_{01}J_{02}J_{03}$	1.3263	$J_{01}J_{03}J_{09}$	0.0004	$J_{01}J_{05}J_{010}$	0.0003
$J_{01}J_{02}J_{04}$	0.0181	$J_{01}J_{03}J_{010}$	0.0004	$J_{01}J_{06}^2$	0.4658
$J_{01}J_{02}J_{05}$	-0.0015	$J_{01}J_{04}^2$	0.6906	$J_{01}J_{06}J_{07}$	0.5304

TABLE 7 CON'T.

F(r)	100 S(F)	F(r)	100 S(F)	F(r)	100 S(F)
$J_{01}J_{06}J_{08}$	0.0092	$J_{01}J_{11}J_{18}$	-0.0000	$J_{03}J_{11}J_{12}$	-0.9765
$J_{01}J_{06}J_{09}$	-0.0007	$J_{01}J_{11}J_{19}$	0.0000	$J_{03}J_{11}J_{13}$	0.1719
$J_{01}J_{06}J_{010}$	0.0007	$J_{01}J_{11}J_{110}$	-0.0000	$J_{03}J_{11}J_{14}$	0.8357
$J_{01}J_{07}^2$	0.4009	$J_{02}J_{11}^2$	-1.4594	$J_{03}J_{11}J_{15}$	0.0206
$J_{01}J_{07}J_{08}$	0.4616	$J_{02}J_{11}J_{12}$	0.3709	$J_{03}J_{11}J_{16}$	-0.0034
$J_{01}J_{07}J_{09}$	0.0082	$J_{02}J_{11}J_{13}$	1.1728	$J_{03}J_{11}J_{17}$	0.0010
$J_{01}J_{07}J_{010}$	-0.0006	$J_{02}J_{11}J_{14}$	0.0243	$J_{03}J_{11}J_{18}$	-0.0004
$J_{01}J_{11}^2$	1.4286	$J_{02}J_{11}J_{15}$	-0.0036	$J_{03}J_{11}J_{19}$	0.0002
$J_{01}J_{11}J_{12}$	1.9761	$J_{02}J_{11}J_{16}$	0.0010	$J_{03}J_{11}J_{110}$	-0.0001
$J_{01}J_{11}J_{13}$	0.0263	$J_{02}J_{11}J_{17}$	-0.0004	$J_{04}J_{11}^2$	0.0102
$J_{01}J_{11}J_{14}$	-0.0037	$J_{02}J_{11}J_{18}$	0.0002	$J_{04}J_{11}J_{12}$	-0.0373
$J_{01}J_{11}J_{15}$	0.0009	$J_{02}J_{11}J_{19}$	-0.0001	$J_{04}J_{11}J_{13}$	-0.7321
$J_{01}J_{11}J_{16}$	-0.0003	$J_{02}J_{11}J_{110}$	0.0000	$J_{04}J_{11}J_{14}$	0.0992
$J_{01}J_{11}J_{17}$	0.0001	$J_{03}J_{11}^2$	-0.0542	$J_{04}J_{11}J_{15}$	0.6474

TABLE 7 CON'T.

F(r)	100 S(F)	F(r)	100 S(F)	F(r)	100 S(F)
$J_{04}J_{11}J_{16}$	0.0175	$J_{01}^3J_{04}$	0.0144	$J_{01}J_{08}J_{11}^2$	-0.0002
$J_{04}J_{11}J_{17}$	-0.0030	$J_{01}^3J_{05}$	-0.0008	$J_{01}J_{09}J_{11}^2$	0.0001
$J_{04}J_{11}J_{18}$	0.0009	$J_{01}^3J_{06}$	0.0007	$J_{01}J_{010}J_{11}^2$	-0.0001
$J_{04}J_{11}J_{19}$	-0.0004	$J_{01}^3J_{07}$	0.0003	$\frac{J_{11}^3}{K_1r}$	2.1428
$J_{04}J_{11}J_{110}$	0.0002	$J_{01}^3J_{08}$	0.0005	$J_{01}\frac{J_{11}^3}{K_1r}$	0.6578
$J_{05}J_{11}^2$	-0.0034	$J_{01}^3J_{09}$	0.0003	$J_{02}\frac{J_{11}^3}{K_1r}$	-0.3490
$J_{06}J_{11}^2$	0.0014	$J_{01}^3J_{010}$	0.0004	$J_{03}\frac{J_{11}^3}{K_1r}$	-0.1506
$J_{07}J_{11}^2$	-0.0006	$J_{01}^2J_{11}^2$	1.1189	$J_{04}\frac{J_{11}^3}{K_1r}$	0.0061
$J_{08}J_{11}^2$	0.0004	$J_{01}J_{02}J_{11}^2$	-0.0834	$J_{05}\frac{J_{11}^3}{K_1r}$	-0.0021
$J_{09}J_{11}^2$	-0.0002	$J_{01}J_{03}J_{11}^2$	-0.4377	$J_{06}\frac{J_{11}^3}{K_1r}$	0.0009
$J_{010}J_{11}^2$	0.0001	$J_{01}J_{04}J_{11}^2$	-0.0211	$J_{07}\frac{J_{11}^3}{K_1r}$	-0.0004
J_{01}^4	3.3572	$J_{01}J_{05}J_{11}^2$	0.0033	$J_{08}\frac{J_{11}^3}{K_1r}$	0.0002
$J_{01}^3J_{02}$	1.4217	$J_{01}J_{06}J_{11}^2$	-0.0010	$J_{09}\frac{J_{11}^3}{K_1r}$	0.0002
$J_{01}^3J_{03}$	0.6490	$J_{01}J_{07}J_{11}^2$	0.0004	$J_{010}\frac{J_{11}^3}{K_1r}$	0.0001
				J_{11}^4	2.0407

APPENDIX II

CERTAIN INTEGRALS OF BESSEL FUNCTIONS

1. Orthogonality Relations

Provided that κ_n, $(n \geq 0)$ are the eigenvalues for which $J_1(\kappa_n) = 0$, it follows from McLachlan(63, pp. 102-104) that

$$S\left(J_{0m} J_{0n}\right) = 0 \qquad m \neq n \qquad \text{(II-1)}$$

$$S\left(J_{1m} J_{1n}\right) = 0 \qquad m \neq n \qquad \text{(II-2)}$$

$$S\left(J_{0n}^2\right) = \frac{1}{2} J_0^2(\kappa_n) \qquad n \geq 0 \qquad \text{(II-3)}$$

$$S\left(J_{1n}^2\right) = \frac{1}{2} J_0^2(\kappa_n) \qquad n \geq 1 \qquad \text{(II-4)}$$

where the operator $S(F)$ has been defined as

$$S(F) \equiv \int_0^1 r\, F(r)\, dr \; . \qquad \text{(I-8)}$$

2. Some Identities

The differential equation

$$y'' + \frac{1}{r} y' + \kappa^2 y = 0 \qquad \text{(II-5)}$$

where prime denotes differentiation with respect to r is satisfied by $y = J_0(\kappa r)$ for which $y' \equiv \frac{dy}{dr} = -\kappa J_1(\kappa r)$. Multiply (II-5) by $r v w \psi$, where v, w, and ψ are any single-valued and differentiable functions of

r in the range $0 \leq r \leq 1$.

$$r u w t y'' + u w t y' + r K^2 u w t y = 0 \tag{II-6}$$

Noting that

$$[r u w t y']' = r u w t y'' + r u w t' y' + r u w' t y' + r u' w t y' + u w t y' \;, \tag{II-7}$$

we may rewrite (II-6) as

$$r u w t' y' + r u w' t y' + r u' w t y' - r K^2 u w t y = [r u w t y']' \;. \tag{II-8}$$

Multiply by dr and integrate from 0 to r; then

$$\int_0^r r\left[u w t' y' + u w' t y' + u' w t y' - K^2 u w t y\right] dr = r u w t y' \;. \tag{II-9}$$

This result is very general. $u, w,$ and t have not been restricted to any particular functions of r. y has been restricted to being $J_0(Kr)$, but K is not necessarily an eigenvalue. If we now require that K be one of the positive eigenvalues K_n, $(n \geq 1)$, and perform the integration from 0 to 1, then, upon substituting for y and y' and dividing by K,

$$\int_0^1 r\left[(u w t' + u w' t + u' w t) J_{1n} + K_n u w t J_{0n}\right] dr = 0 \;. \tag{II-10}$$

The special cases of (II-10) of interest in this paper are obtained by choosing u, w, t from the functions unity, J_{0n}, and J_{1n}, and by combining certain results obtained directly from (II-10). These identities are, $(n \geq 1)$,

$$S(J_{0n}^2) - S(J_{1n}^2) = 0 \tag{II-11}$$

$$S(J_{01}^3) - 2\,S(J_{01}\,J_{11}^2) = 0 \tag{II-12}$$

$$3\,S(J_{01}\,J_{11}^2) - 2\,S\left(\frac{J_{11}^3}{K_1 r}\right) = 0 \tag{II-13}$$

$$3\,S(J_{01}^3) - 4\,S\left(\frac{J_{11}^3}{K_1 r}\right) = 0 \tag{II-14}$$

$$S(J_{01}^4) - 3\,S(J_{01}^2\,J_{11}^2) = 0 \tag{II-15}$$

$$3\,S(J_{01}^2 J_{11}^2) - 2\,S\left(J_{01}\,\frac{J_{11}^3}{K_1 r}\right) - S(J_{11}^4) = 0 \tag{II-16}$$

$$S(J_{01}^2 J_{0n}) - S(J_{11}^2 J_{0n}) - \frac{K_n}{K_1}\,S(J_{01}\,J_{11}\,J_{1n}) = 0 \tag{II-17}$$

$$\frac{K_n}{K_1}\,S(J_{01}^2\,J_{0n}) - 2\,S(J_{01}\,J_{11}\,J_{1n}) = 0 \tag{II-18}$$

$$\left(\frac{K_n^2}{K_1^2} - 2\right) S(J_{01}^2\,J_{0n}) + 2\,S(J_{11}^2\,J_{0n}) = 0 \tag{II-19}$$

$$\frac{K_n}{K_1}\,S(J_{11}^2\,J_{0n}) + \left(\frac{K_n^2}{K_1^2} - 2\right) S(J_{01}\,J_{11}\,J_{1n}) = 0 \;. \tag{II-20}$$

From the definitions of Γ_n (42), α_n (I-7), and $S(F)$ (I-8) and the identity (II-12),

$$8\,th_1^2\, S(J_{01}^3) = \Gamma_1\, J_0^2(K_1) \quad . \tag{II-21}$$

Two expressions of a somewhat different form from the above identities have been verified numerically:

$$2\,S(J_{01}^2 J_{11}^2) = \sum_{n=1}^{\infty}\left\{\left[\alpha_n(J_{01}^2) - \alpha_n(J_{11}^2)\right] S(J_{01}^2 J_{0n})\right\} \tag{II-22}$$

and

$$\begin{aligned} &3\,S(J_{01}^2 J_{11}^2) - 2\,S\left(J_{01}\,\frac{J_{11}^3}{K_1 r}\right) = \\ &= \frac{1}{2}\sum_{n=1}^{\infty}\left\{\left[\alpha_n(J_{01}^2) - \alpha_n(J_{11}^2)\right]\left[S(J_{01}^2 J_{0n}) - 2\,S(J_{11}^2 J_{0n})\right]\right\} . \end{aligned} \tag{II-23}$$

Use of the numerical integrations tabulated in Tables 6 and 7 yields for (II-22) a left side of .02238 and a right side of .02239; for (II-23), a left side of .02041 and a right side of .02041. In each case the first two terms of the infinite sum provide over 99.9% of the total right side. These numerical results strongly indicate the identity of (II-22) and (II-23). Furthermore, the physical consideration that the total energy of the wave motion must be independent of time requires it.

APPENDIX III

THE SOLUTION AT GENERAL DEPTH TO THE THIRD APPROXIMATION

The results of the third approximation to the solution at general depth are:

$$A_{11} = A_{11} \quad , \qquad B_{11} \equiv 0 \tag{III-1}$$

$$A_{1n} = \frac{A_{11}^{3}\, K_1^{4}}{4\left[1 - \frac{K_n \mathrm{th}_n}{K_1 \mathrm{th}_1}\right]} \Bigg\{ -\frac{1}{2}\left(2\,\mathrm{th}_1^4 - \mathrm{th}_1^2 + 1\right) \alpha_n\left(J_{01}^3\right) -$$

$$-\frac{1}{2}\left(1+\mathrm{th}_1^2\right) \alpha_n\left(J_{01} J_{11}^2\right) + \alpha_n\left(\frac{J_{11}^3}{K_1 r}\right) -$$

$$-\frac{1}{4} \sum_{p=1}^{\infty} \frac{K_p^2}{K_1^2} \Gamma_p\, \alpha_n\left(J_{01} J_{0p}\right) + \frac{1}{8}\left(1+3\,\mathrm{th}_1^2\right) \sum_{p=1}^{\infty} \frac{K_p \mathrm{th}_p}{K_1 \mathrm{th}_1} \Gamma_p\, \alpha_n\left(J_{01} J_{0p}\right) - \tag{III-2}$$

$$-\frac{1}{8} \sum_{p=1}^{\infty} \frac{K_p^2 \mathrm{th}_p}{K_1^2 \mathrm{th}_1} \Gamma_p\, \alpha_n\left(J_{11} J_{1p}\right) \Bigg\} \qquad (n = 0), \; (n \geq 2)$$

$$A_{2n} = -\frac{A_{11}^2 K_1^2}{8} \Gamma_n \qquad (n \geq 0) \tag{III-3}$$

$$A_{3n} = -\frac{A_{11}^{3}\, K_1^{4}}{36\left[1 - \frac{K_n \mathrm{th}_n}{9 K_1 \mathrm{th}_1}\right]} \Bigg\{ \frac{1}{2}\left(11\,\mathrm{th}_1^2 - 1\right) \alpha_n\left(J_{01}^3\right) +$$

$$+\frac{5}{2}\left(1+\mathrm{th}_1^2\right) \alpha_n\left(J_{01} J_{11}^2\right) - \alpha_n\left(\frac{J_{11}^3}{K_1 r}\right) -$$

$$-\frac{1}{8}\left(21\,\mathrm{th}_1^2 - 1\right) \sum_{p=1}^{\infty} \frac{K_p \mathrm{th}_p}{K_1 \mathrm{th}_1} \Gamma_p\, \alpha_n\left(J_{01} J_{0p}\right) + \tag{III-4}$$

$$+\frac{1}{4}\sum_{p=1}^{\infty}\frac{K_p^2}{K_1^2}\Gamma_p\,\alpha_n(J_{01}J_{0p})-\frac{1}{8}\sum_{p=1}^{\infty}\frac{K_p^2\,th_p}{K_1^2\,th_1}\Gamma_p\,\alpha_n(J_{11}J_{1p})-$$

$$-\sum_{p=1}^{\infty}\frac{K_p}{K_1}\Gamma_p\,\alpha_n(J_{11}J_{1p})\Bigg\}\qquad (n\geq 0)$$

$$C_1=\frac{A_{11}^2K_1^2}{4}(1-th_1^2)\,J_0^2(K_1)\qquad\text{(III-5)}$$

$$J_0(r)=\frac{A_{11}^2K_1^3th_1}{4}\left\{th_1^2J_{01}^2-J_{11}^2+(1-th_1^2)\,J_0^2(K_1)\right\}\qquad\text{(III-6)}$$

$$J_1(r)=A_{11}K_1th_1J_{01}+\frac{A_{11}^3K_1^5th_1}{4}\Bigg\{\frac{1}{2}(3th_1^2-1)\,J_{01}^3-$$

$$-\frac{1}{2}(6th_1^2+1)\,J_{01}J_{11}^2+\frac{J_{11}^3}{K_1r}+$$

$$+\frac{1}{8}\sum_{n=1}^{\infty}\frac{K_n\,th_n}{K_1\,th_1}\Gamma_n\,J_{01}J_{0n}-\frac{1}{4}\sum_{n=1}^{\infty}\frac{K_n^2}{K_1^2}\Gamma_n\,J_{01}J_{0n}-$$

$$-\frac{1}{8}\sum_{n=1}^{\infty}\frac{K_n^2\,th_n}{K_1^2\,th_1}\Gamma_n\,J_{11}J_{1n}+\frac{1}{4}\sum_{n=1}^{\infty}\frac{K_n}{K_1}\Gamma_n\,J_{11}J_{1n}+$$

$$+(1-th_1^2)\,J_0^2(K_1)\,J_{01}+$$

$$+\sum_{n=2}^{\infty}\frac{\frac{K_n\,th_n}{K_1\,th_1}J_{0n}}{\left[1-\frac{K_n\,th_n}{K_1\,th_1}\right]}\Bigg[-\frac{1}{2}(2th_1^4-th_1^2+1)\,\alpha_n(J_{01}^3)-\qquad\text{(III-7)}$$

$$-\frac{1}{2}(th_1^2+1)\,\alpha_n(J_{01}J_{11}^2)+\alpha_n\left(\frac{J_{11}^3}{K_1r}\right)-$$

$$-\frac{1}{4}\sum_{p=1}^{\infty}\frac{K_p^2}{K_1^2}\Gamma_p\,\alpha_n(J_{01}J_{0p})+\frac{1}{8}(1+3th_1^2)\sum_{p=1}^{\infty}\frac{K_p\,th_p}{K_1\,th_1}\Gamma_p\,\alpha_n(J_{01}J_{0p})-$$

$$-\frac{1}{8}\sum_{p=1}^{\infty}\frac{K_p^2\,th_p}{K_1^2\,th_1}\Gamma_p\,\alpha_n(J_{11}J_{1p})\Bigg]\Bigg\}$$

$$J_2(r) = \frac{A_{11}^2 K_1^3 th_1}{4}\left\{ J_{01}^2 - J_{11}^2 - \frac{1}{4}\sum_{n=1}^{\infty} \frac{K_n th_n}{K_1 th_1} \Gamma_n J_{on} \right\} \tag{III-8}$$

$$\begin{aligned} J_3(r) = \frac{A_{11}^3 K_1^5 th_1}{12}\Bigg\{ & \frac{1}{2}(1+th_1^2) J_{01}^3 - \left(\frac{5}{2}+th_1^2\right) J_{01} J_{11}^2 + \\ & + \frac{J_{11}^3}{K_1 r} - \frac{1}{8}\sum_{n=1}^{\infty} \frac{K_n th_n}{K_1 th_1} \Gamma_n J_{01} J_{on} - \frac{1}{4}\sum_{n=1}^{\infty} \frac{K_n^2}{K_1^2} \Gamma_n J_{01} J_{on} + \\ & + \frac{1}{8}\sum_{n=1}^{\infty} \frac{K_n^2 th_n}{K_1^2 th_1} \Gamma_n J_{11} J_{1n} + \frac{1}{4}\sum_{n=1}^{\infty} \frac{K_n}{K_1} \Gamma_n J_{11} J_{1n} - \\ & - \frac{1}{9}\sum_{n=1}^{\infty} \frac{\frac{K_n th_n}{K_1 th_1} J_{on}}{\left[1 - \frac{K_n th_n}{9 K_1 th_1}\right]} \Bigg[\frac{1}{2}(11 th_1^2 - 1)\, \alpha_n (J_{01}^3) + \\ & + \frac{5}{2}(1+th_1^2)\alpha_n (J_{01} J_{11}^2) - \alpha_n \left(\frac{J_{11}^3}{K_1 r}\right) - \\ & - \frac{1}{8}(21 th_1^2 - 1) \sum_{p=1}^{\infty} \frac{K_p th_p}{K_1 th_1} \Gamma_p \alpha_n (J_{01} J_{op}) + \\ & + \frac{1}{4}\sum_{p=1}^{\infty} \frac{K_p^2}{K_1^2} \Gamma_p \alpha_n (J_{01} J_{op}) - \frac{1}{8}\sum_{p=1}^{\infty} \frac{K_p^2 th_p}{K_1^2 th_1} \Gamma_p \alpha_n (J_{11} J_{1p}) - \\ & - \sum_{p=1}^{\infty} \frac{K_p}{K_1} \Gamma_p \alpha_n (J_{11} J_{1p}) \Bigg] \Bigg\} \end{aligned} \tag{III-9}$$

$$
\begin{aligned}
\frac{1}{G} = K_1 \,\mathrm{th}_1 \Bigg\{ 1 + \frac{A_{11}^2 K_1^4}{4} \Bigg[& - \frac{1}{2}\left(2\,\mathrm{th}_1^4 - \mathrm{th}_1^2 + 1\right) \alpha_1 \left(J_{01}^3\right) - \\
& - \frac{1}{2}\left(1 + \mathrm{th}_1^2\right) \alpha_1 \left(J_{01} J_{11}^2\right) + \alpha_1 \left(\frac{J_{11}^3}{K_1 \Omega}\right) + \\
& + \left(1 - \mathrm{th}_1^2\right)^2 J_0^2(K_1) - \frac{1}{4} \sum_{p=1}^{\infty} \frac{K_p^2}{K_1^2} \Gamma_p \, \alpha_1 \left(J_{01} J_{0p}\right) + \\
& + \frac{1}{8}\left(1 + 3\,\mathrm{th}_1^2\right) \sum_{p=1}^{\infty} \frac{K_p \,\mathrm{th}_p}{K_1 \,\mathrm{th}_1} \Gamma_p \, \alpha_1 \left(J_{01} J_{0p}\right) - \qquad \text{(III-10)} \\
& - \frac{1}{8} \sum_{p=1}^{\infty} \frac{K_p^2 \,\mathrm{th}_p}{K_1^2 \,\mathrm{th}_1} \Gamma_p \, \alpha_1 \left(J_{11} J_{1p}\right) \Bigg] \Bigg\} \quad .
\end{aligned}
$$

All other A_{mn}, B_{mn}, and $\mathcal{J}_{\pm m}$ are either zero or of order higher than $\mathcal{O}(A_{11}^3)$. The quantity Γ_n is a function of H only and has been defined previously by equation (42) on page 14.

APPENDIX IV

THE SOLUTION AT THE SECOND-ORDER CRITICAL DEPTHS TO THE SECOND APPROXIMATION

The results of the second approximation to the solution when the depth is equal to or very nearly equal to either of the second-order critical depths are given below. When $H \approx .19811$, $\ell = 3$ and when $H \approx .34698$, $\ell = 4$ in these expressions.

$$A_{11} = A_{11} \quad , \quad B_{11} \equiv 0 \tag{IV-1}$$

$$A_{1n} = \frac{A_{11} A_{2\ell} K_1 K_\ell}{4\left[1 - \frac{K_n \mathrm{th}_n}{K_1 \mathrm{th}_1}\right]} \left\{ \left(2\frac{K_\ell}{K_1} - \frac{\mathrm{th}_\ell}{\mathrm{th}_1} - 3\,\mathrm{th}_1 \mathrm{th}_\ell \right) \alpha_n (J_{01} J_{0\ell}) + \right. \tag{IV-2}$$

$$\left. + \frac{K_\ell \mathrm{th}_\ell}{K_1 \mathrm{th}_1} \alpha_n (J_{11} J_{1\ell}) \right\} \qquad \begin{matrix} (n = 0) \\ (n \geq 2) \end{matrix}$$

$$A_{2\ell} = \frac{C_2 - \sqrt{C_2^2 + C_4}}{C_3} \qquad (C_2 \geq 0) \tag{IV-3}$$

$$A_{2\ell} = \frac{C_2 + \sqrt{C_2^2 + C_4}}{C_3} \qquad (C_2 \leq 0) \tag{IV-4}$$

$$A_{2n} = -\frac{A_{11}^2 K_1^2}{8} \Gamma_n \qquad \begin{array}{l} (0 \le n \le \ell - 1) \\ (n \ge \ell + 1) \end{array} \tag{IV-5}$$

$$\begin{aligned} A_{3n} = \frac{A_{11} A_{2\ell} K_1 K_\ell}{36\left[1 - \frac{K_n \mathrm{th}_n}{9 K_1 \mathrm{th}_1}\right]} \Bigg\{ & \left(-21 \mathrm{th}_1 \mathrm{th}_\ell + \frac{\mathrm{th}_\ell}{\mathrm{th}_1} + 2\frac{K_\ell}{K_1}\right) \alpha_n (J_{01} J_{0\ell}) - \\ & - \left(8 + \frac{K_\ell \mathrm{th}_\ell}{K_1 \mathrm{th}_1}\right) \alpha_n (J_{11} J_{1\ell}) \Bigg\} \qquad (n \ge 0) \end{aligned} \tag{IV-6}$$

$$\begin{aligned} A_{4n} = -\frac{A_{2\ell}^2 K_\ell^2}{16\left[1 - \frac{K_n \mathrm{th}_n}{16 K_1 \mathrm{th}_1}\right]} \Bigg\{ & \left(3 \mathrm{th}_\ell^2 - \frac{K_\ell \mathrm{th}_\ell}{4 K_1 \mathrm{th}_1}\right) \alpha_n (J_{0\ell}^2) + \\ & + \left(1 + \frac{K_\ell \mathrm{th}_\ell}{4 K_1 \mathrm{th}_1}\right) \alpha_n (J_{1\ell}^2) \Bigg\} \qquad (n \ge 0) \end{aligned} \tag{IV-7}$$

$$\begin{aligned} C_1 = & \frac{A_{11}^2 K_1^2}{4} (1 - \mathrm{th}_1^2) J_0^2(K_1) + \\ & + \frac{A_{2\ell}^2 K_\ell^2}{4} (1 - \mathrm{th}_\ell^2) J_0^2(K_\ell) \end{aligned} \tag{IV-8}$$

$$\mathcal{J}_0(r) = \frac{A_{11}^2 K_1^3 th_1}{4}\left\{th_1^2 J_{01}^2 - J_{11}^2 + (1 - th_1^2) J_0^2(K_1)\right\} +$$

$$+ \frac{A_{2\ell}^2 K_1 K_\ell^2 th_1}{4}\left\{th_\ell^2 J_{0\ell}^2 - J_{1\ell}^2 + (1 - th_\ell^2) J_0^2(K_\ell)\right\} \qquad \text{(IV-9)}$$

$$\mathcal{J}_1(r) = A_{11} K_1 th_1 J_{01} +$$

$$+ \frac{A_{11} A_{2\ell} K_1^2 K_\ell th_1}{4}\left\{\left(2\frac{K_\ell}{K_1} - \frac{th_\ell}{th_1}\right) J_{01} J_{0\ell} + \left(\frac{K_\ell th_\ell}{K_1 th_1} - 2\right) J_{11} J_{1\ell} +\right.$$

$$+ \sum_{n=2}^{\infty} \frac{\frac{K_n th_n}{K_1 th_1} J_{0n}}{\left[1 - \frac{K_n th_n}{K_1 th_1}\right]}\left[\left(2\frac{K_\ell}{K_1} - \frac{th_\ell}{th_1} - 3 th_1 th_\ell\right) \alpha_n (J_{01} J_{0\ell}) + \right. \qquad \text{(IV-10)}$$

$$\left.\left. + \frac{K_\ell th_\ell}{K_1 th_1} \alpha_n (J_{11} J_{1\ell})\right]\right\}$$

$$\mathcal{J}_2(r) = \frac{1}{2} A_{2\ell} K_\ell th_\ell J_{0\ell} +$$

$$+ \frac{A_{11}^2 K_1^3 th_1}{4}\left\{J_{01}^2 - J_{11}^2 - \frac{1}{4} \sum_{\substack{n=1 \\ (n \neq \ell)}}^{\infty} \frac{K_n th_n}{K_1 th_1} \Gamma_n J_{0n}\right\} \qquad \text{(IV-11)}$$

$$\mathcal{J}_3(r) = \frac{A_{11} A_{2\ell} K_1^2 K_\ell\, th_1}{12} \Bigg\{ \left(2\frac{K_\ell}{K_1} + \frac{th_\ell}{th_1} \right) J_{01} J_{0\ell} -$$

$$- \left(\frac{K_\ell\, th_\ell}{K_1\, th_1} + 2 \right) J_{11} J_{1\ell} +$$

$$+ \frac{1}{9} \sum_{n=1}^{\infty} \frac{\frac{K_n\, th_n}{K_1\, th_1} J_{0n}}{\left[1 - \frac{K_n\, th_n}{9 K_1\, th_1} \right]} \Bigg[\left(2\frac{K_\ell}{K_1} + \frac{th_\ell}{th_1} - 21\, th_1\, th_\ell \right) \alpha_n (J_{01} J_{0\ell}) -$$

$$- \left(8 + \frac{K_\ell\, th_\ell}{K_1\, th_1} \right) \alpha_n (J_{11} J_{1\ell}) \Bigg] \Bigg\} \qquad \text{(IV-12)}$$

$$\mathcal{J}_4(r) = \frac{A_{2\ell}^2 K_\ell^3\, th_\ell}{16} \Bigg\{ J_{0\ell}^2 - J_{1\ell}^2 +$$

$$+ \frac{K_1\, th_1}{4 K_\ell\, th_\ell} \sum_{n=1}^{\infty} \frac{\frac{K_n\, th_n}{K_1\, th_1} J_{0n}}{\left[1 - \frac{K_n\, th_n}{16 K_1\, th_1} \right]} \Bigg[\left(3\, th_\ell^2 - \frac{K_\ell\, th_\ell}{4 K_1\, th_1} \right) \alpha_n (J_{0\ell}^2) +$$

$$+ \left(1 + \frac{K_\ell\, th_\ell}{4 K_1\, th_1} \right) \alpha_n (J_{1\ell}^2) \Bigg] \Bigg\} \qquad \text{(IV-13)}$$

$$\frac{1}{G} = K_1 \mathrm{th}_1 \Bigg\{ 1 +$$

$$+ \frac{A_{2l} K_1 K_l}{4} \Bigg[\left(2 \frac{K_l}{K_1} - \frac{\mathrm{th}_l}{\mathrm{th}_1} - 3\, \mathrm{th}_1 \mathrm{th}_l \right) \alpha_1 \left(J_{01}\; J_{0l} \right) + \tag{IV-14}$$

$$+ \frac{K_l \mathrm{th}_l}{K_1 \mathrm{th}_1} \alpha_1 \left(J_{11}\; J_{1l} \right) \Bigg] \Bigg\}$$

in which the functions C_2, C_3, and C_4 of H and A_{11} in equations (IV-3) and (IV-4) have been defined previously by equations (55), (56), and (57) on pp. 19-20, and the function Γ_n of H has been defined by equation (42) on page 14. All other A_{mn}, B_{mn}, and $J_{Im}(\eta)$ are either zero or of order higher than $\mathcal{O}(A_{11}^2)$.

APPENDIX V

FUNCTIONS OF H APPEARING IN THE ENERGY EXPRESSIONS

The functions C_n of H, $(5 \leq n \leq 13)$, in the expressions for the potential (92) and kinetic (93) energy are:

$$C_5 = 2\varkappa_1^2 \, S(J_{01}^3) - S(J_{01} J_{11}^2) - \frac{1}{8} \Gamma_1 \, J_0^2(K_1) \tag{V-1}$$

$$C_6 = S(J_{01} J_{11}^2) - \frac{1}{8} \Gamma_1 \, J_0^2(K_1) \tag{V-2}$$

$$\begin{aligned} C_7 = {} & 18\varkappa_1^4 \, S(J_{01}^4) + 12\varkappa_1^2 \, S(J_{01}^2 J_{11}^2) - \\ & - \frac{9}{2}\varkappa_1^2 \sum_{n=1}^{\infty} \frac{K_n \varkappa_n}{K_1 \varkappa_1} \Gamma_n \, S(J_{01}^2 J_{0n}) + 6\, S(J_{01}^2 J_{11}^2) - \\ & - 6\, S\left(J_{01} \frac{J_{11}^3}{K_1 \pi}\right) - 3 \sum_{n=1}^{\infty} \frac{K_n^2}{K_1^2} \Gamma_n \, S(J_{01}^2 J_{0n}) + \\ & + \frac{3}{2} \sum_{n=1}^{\infty} \frac{K_n \varkappa_n}{K_1 \varkappa_1} \Gamma_n \, S(J_{11}^2 J_{0n}) + \\ & + 6 \sum_{n=1}^{\infty} \frac{K_n}{K_1} \Gamma_n \, S(J_{01} J_{11} J_{1n}) - \frac{3}{2} \sum_{n=1}^{\infty} \frac{K_n^2 \varkappa_n}{K_1^2 \varkappa_1} \Gamma_n \, S(J_{01} J_{11} J_{1n}) + \\ & + \frac{3}{32} \sum_{n=1}^{\infty} \frac{K_n^2 \varkappa_n^2}{K_1^2 \varkappa_1^2} \Gamma_n^2 \, J_0^2(K_n) + 3(1 + 3\varkappa_1^2)(1 - \varkappa_1^2) \, J_0^4(K_1) \end{aligned} \tag{V-3}$$

$$
\begin{aligned}
C_8 = {} & 12\, \mathrm{th}_1^4\, S\left(J_{01}^4\right) + 33\, \mathrm{th}_1^2\, S\left(J_{01}^2 J_{11}^2\right) - \\
& - \frac{39}{8}\, \mathrm{th}_1^2 \sum_{n=1}^{\infty} \frac{K_n\, \mathrm{th}_n}{K_1\, \mathrm{th}_1}\, \Gamma_n\, S\left(J_{01}^2 J_{0n}\right) - \\
& - 9\, S\left(J_{01}^2 J_{11}^2\right) - 3\, S\left(J_{01} \frac{J_{11}^3}{K_1 \pi}\right) + \\
& + \frac{3}{8} \sum_{n=1}^{\infty} \frac{K_n\, \mathrm{th}_n}{K_1\, \mathrm{th}_1}\, \Gamma_n\, S\left(J_{01}^2 J_{0n}\right) - \frac{21}{4} \sum_{n=1}^{\infty} \frac{K_n^2}{K_1^2}\, \Gamma_n\, S\left(J_{01}^2 J_{0n}\right) + \\
& + 3 \sum_{n=1}^{\infty} \frac{K_n\, \mathrm{th}_n}{K_1\, \mathrm{th}_1}\, \Gamma_n\, S\left(J_{11}^2 J_{0n}\right) + 9 \sum_{n=1}^{\infty} \frac{K_n}{K_1}\, \Gamma_n\, S\left(J_{01} J_{11} J_{1n}\right) - \\
& - \frac{3}{8} \sum_{n=1}^{\infty} \frac{K_n^2\, \mathrm{th}_n}{K_1^2\, \mathrm{th}_1}\, \Gamma_n\, S\left(J_{01} J_{11} J_{1n}\right) + 6\left(1 - \mathrm{th}_1^4\right) J_0^4(K_1)
\end{aligned}
\qquad \text{(V-4)}
$$

$$
\begin{aligned}
C_9 = {} & -15\, \mathrm{th}_1^2\, S\left(J_{01}^2 J_{11}^2\right) + \frac{21}{8}\, \mathrm{th}_1^2 \sum_{n=1}^{\infty} \frac{K_n\, \mathrm{th}_n}{K_1\, \mathrm{th}_1}\, \Gamma_n\, S\left(J_{01}^2 J_{0n}\right) + \\
& + 3\, S\left(J_{01}^2 J_{11}^2\right) + 3\, S\left(J_{01} \frac{J_{11}^3}{K_1 \pi}\right) - \\
& - \frac{21}{8} \sum_{n=1}^{\infty} \frac{K_n\, \mathrm{th}_n}{K_1\, \mathrm{th}_1}\, \Gamma_n\, S\left(J_{01}^2 J_{0n}\right) - \frac{9}{4} \sum_{n=1}^{\infty} \frac{K_n^2}{K_1^2}\, \Gamma_n\, S\left(J_{01}^2 J_{0n}\right) + \\
& + \frac{3}{2} \sum_{n=1}^{\infty} \frac{K_n\, \mathrm{th}_n}{K_1\, \mathrm{th}_1}\, \Gamma_n\, S\left(J_{11}^2 J_{0n}\right) + \frac{9}{8} \sum_{n=1}^{\infty} \frac{K_n^2\, \mathrm{th}_n}{K_1^2\, \mathrm{th}_1}\, \Gamma_n\, S\left(J_{01} J_{11} J_{1n}\right) + \\
& + 3 \sum_{n=1}^{\infty} \frac{K_n}{K_1}\, \Gamma_n\, S\left(J_{01} J_{11} J_{1n}\right) + \frac{3}{32} \sum_{n=1}^{\infty} \frac{K_n^2\, \mathrm{th}_n^2}{K_1^2\, \mathrm{th}_1^2}\, \Gamma_n^2\, J_0^2(K_n)
\end{aligned}
\qquad \text{(V-5)}
$$

$$C_{10} = th_1^2 S(J_{01}^3) + S(J_{01} J_{11}^2) - \frac{1}{4} \Gamma_1 J_0^2(K_1) \tag{V-6}$$

$$\begin{aligned}
C_{11} = {} & 12\, th_1^4 S(J_{01}^4) + 36\, th_1^2 S(J_{01}^2 J_{11}^2) - \\
& - \frac{9}{2} th_1^2 \sum_{n=1}^{\infty} \frac{K_n th_n}{K_1 th_1} \Gamma_n S(J_{01}^2 J_{0n}) - 24 S(J_{01}^2 J_{11}^2) + \\
& + 12 S\left(J_{01} \frac{J_{11}^3}{K_1 \eta}\right) + \frac{3}{2} \sum_{n=1}^{\infty} \frac{K_n th_n}{K_1 th_1} \Gamma_n S(J_{01}^2 J_{0n}) - \\
& - 3 \sum_{n=1}^{\infty} \frac{K_n^2}{K_1^2} \Gamma_n S(J_{01}^2 J_{0n}) - \\
& - \frac{3}{2} \sum_{n=1}^{\infty} \frac{K_n^2 th_n}{K_1^2 th_1} \Gamma_n S(J_{01} J_{11} J_{1n}) + 3 \sum_{n=1}^{\infty} \frac{K_n}{K_1} \Gamma_n S(J_{01} J_{11} J_{1n}) - \\
& - 3 \sum_{n=1}^{\infty} \Gamma_n S(J_{01}^2 J_{0n}) + 3 \sum_{n=1}^{\infty} \Gamma_n S(J_{11}^2 J_{0n}) + \\
& + \frac{3}{8} \sum_{n=1}^{\infty} \frac{K_n th_n}{K_1 th_1} \Gamma_n^2 J_0^2(K_n) + 6 (1 - th_1^4) J_0^4(K_1)
\end{aligned} \tag{V-7}$$

$$
\begin{aligned}
C_{12} = & -12\, \mathrm{th}_1^4\, S(J_{01}^4) - 33\, \mathrm{th}_1^2\, S(J_{01}^2 J_{11}^2) + \\
& + \frac{39}{8} \mathrm{th}_1^2 \sum_{n=1}^{\infty} \frac{K_n \mathrm{th}_n}{K_1 \mathrm{th}_1} \Gamma_n\, S(J_{01}^2 J_{0n}) + 9 S(J_{01}^2 J_{11}^2) + \\
& + 3 S\left(J_{01} \frac{J_{11}^3}{K_1 \eta}\right) - \frac{27}{8} \sum_{n=1}^{\infty} \frac{K_n \mathrm{th}_n}{K_1 \mathrm{th}_1} \Gamma_n\, S(J_{01}^2 J_{0n}) - \\
& - \frac{3}{4} \sum_{n=1}^{\infty} \frac{K_n^2}{K_1^2} \Gamma_n\, S(J_{01}^2 J_{0n}) + \frac{27}{8} \sum_{n=1}^{\infty} \frac{K_n^2 \mathrm{th}_n}{K_1^2 \mathrm{th}_1} \Gamma_n\, S(J_{01} J_{11} J_{1n}) + \\
& + 3 \sum_{n=1}^{\infty} \frac{K_n}{K_1} \Gamma_n\, S(J_{01} J_{11} J_{1n}) - 6(1 - \mathrm{th}_1^4)\, J_0^4(K_1)
\end{aligned}
\qquad \text{(V-8)}
$$

$$
\begin{aligned}
C_{13} = & -3\, \mathrm{th}_1^2\, S(J_{01}^2 J_{11}^2) - \frac{3}{8} \mathrm{th}_1^2 \sum_{n=1}^{\infty} \frac{K_n \mathrm{th}_n}{K_1 \mathrm{th}_1} \Gamma_n\, S(J_{01}^2 J_{0n}) + \\
& + 15\, S(J_{01}^2 J_{11}^2) - 15\, S\left(J_{01} \frac{J_{11}^3}{K_1 \eta}\right) + \\
& + \frac{15}{8} \sum_{n=1}^{\infty} \frac{K_n \mathrm{th}_n}{K_1 \mathrm{th}_1} \Gamma_n\, S(J_{01}^2 J_{0n}) + \frac{15}{4} \sum_{n=1}^{\infty} \frac{K_n^2}{K_1^2} \Gamma_n\, S(J_{01}^2 J_{0n}) - \\
& - \frac{15}{8} \sum_{n=1}^{\infty} \frac{K_n^2 \mathrm{th}_n}{K_1^2 \mathrm{th}_1} \Gamma_n\, S(J_{01} J_{11} J_{1n}) - \\
& - 6 \sum_{n=1}^{\infty} \frac{K_n}{K_1} \Gamma_n\, S(J_{01} J_{11} J_{1n}) + 3 \sum_{n=1}^{\infty} \Gamma_n\, S(J_{01}^2 J_{0n}) - \\
& - 3 \sum_{n=1}^{\infty} \Gamma_n\, S(J_{11}^2 J_{0n}) - \frac{3}{8} \sum_{n=1}^{\infty} \frac{K_n \mathrm{th}_n}{K_1 \mathrm{th}_1} \Gamma_n^2\, J_0^2(K_n) \, .
\end{aligned}
\qquad \text{(V-9)}
$$

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