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The theory of interaction between wave and basic flow*

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This paper investigates the interaction between transient wave and non-stationary and non-conservative basic flow. An interaction equation is derived from the zonally symmetric and non-hydrostatic primitive equations in Cartesian coordinates by using the Momentum–Casimir method. In the derivation, it is assumed that the transient disturbances satisfy the linear perturbation equations and the basic states are non-conservative and slowly vary in time and space. The diabatic heating composed of basic-state heating and perturbation heating is also introduced. Since the theory of wave–flow interaction is constructed in non-hydrostatic and ageostrophic dynamical framework, it is applicable to diagnosing the interaction between the meso-scale convective system in front and the background flow.

It follows from the local interaction equation that the local tendency of pseudomomentum wave-activity density depends on the combination of the perturbation flux divergence second-order in disturbance amplitude, the local change of basic-state pseudomomentum density, the basic-state flux divergence and the forcing effect of diabatic heating. Furthermore, the tendency of pseudomomentum wave-activity density is opposite to that of basic-state pseudomomentum density. The globally integrated basic-state pseudomomentum equation and wave-activity equation reveal that the global development of basic-state pseudomomentum is only dominated by the basic-state diabatic heating while it is the forcing effect of total diabatic heating from which the global evolution of pseudomomentum wave activity results. Therefore, the interaction between the transient wave and the non-stationary and non-conservative basic flow is realized in virtue of the basic-state diabatic heating.

Keywords: wave–flow interaction, pseudomomentum wave activity, diabatic heating, Momentum–Casimir method
PACC: 9260, 9260D, 9260V

1. Introduction

The interaction between the transient wave (or 'eddy') and the basic flow is an important content of atmospheric dynamics. As a powerful tool to investigate this interaction, the E–P flux firstly proposed by Eliassen and Palm[1] has been widely used in the studies of wave propagation,[2−4] wave–mean flow interaction[5−8] and stratosphere warming.[9,10] Andrews and McIntyre (1976, 1978) developed and generalized the E–P flux theory by defining residual circulation components in the transformed Eulerian mean equations.[11,12] Edmon *et al* (1980) established 'Eliassen–Palm Cross Section' method to measure the total forcing of zonal-mean state by eddies and a net wave propagation from one height and latitude to another.[13] Huang and Gambo (1983) studied the E–P theorem in a three-dimensional spherical atmosphere and brought forward the dynamics theory of the quasi-stationary planetary wave that propagates alongside the two waveguides.[14] Gao *et al* (1990) employed the E–P flux to diagnose the acceleration and deceleration of zonal mean flow,[15] and they discussed the interaction between transient wave and westerly jet stream at upper level and gave a very comprehensive explanation to the mechanism of upper-level jet stream acceleration.[16] It is for the first time that the E–P flux is creatively extended into upper-level jet stream. This is an important progress in the application of E–P flux. Gao *et al* (2004) formularized a new expression of E–P flux in baroclinic atmosphere and constructed an ageostrophic generalized E–P flux theory that is a complement to their previous pertinent investigations.[17] Pfeffer (1992) examined the dynamics of eddy-induced acceleration of the zonal-mean flow in the troposphere and lower stratosphere

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from the perspective of conventional and transformed Eulerian diagnostics.[18]

In the most of previous E–P flux theories, the hydrostatic balance or quasi-geostrophic approximation is often used. Due to this, the E–P flux theories are suitable for large-scale weather systems and cannot exactly present the interaction between non-hydrostatic, ageostrophic meso-scale waves and basic flow. In fact, the interaction between wave and basic flow includes two aspects, namely, the feedback of basic flow to wave and the forcing effect of wave on basic flow. In this sense, E–P flux represents the effect of wave on basic flow, but cannot reveal the feedback of basic flow to wave.

Many studies have shown that the concept of wave activity can theoretically underlie the wave–flow interaction.[19−21] Haynes (1988) took into account the effects of forcing and dissipation to investigate the finite-amplitude, local wave–activity relations for disturbances to zonal and nonzonal basic-state flows in isentropic coordinates by using the Momentum–Casimir and Energy–Casimir methods.[22] Scinocca and Shepherd (1992) constructed the non-hydrostatic finite-amplitude wave–activity conservation laws for pseudomomentum and pseudoenergy from the two-dimensional anelastic and Boussinesq equations.[23] Ran and Gao (2007) derived a three-dimensional, non-hydrostatic and ageostrophic local wave–activity relation for pseudomomentum from the non-hydrostatic primitive equations in Cartesian coordinates by using the Momentum–Casimir method.[24] This form of wave–activity relation is constructed for the first time and may be used as a diagnostic for mesoscale flows poorly described by the quasi-geostrophic or hydrostatic approximation. In the previous studies of wave–activity relation, it is generally assumed that the basic states are stationary, which means that there is not the effect of wave on basic flow. This is a shortcoming in the investigation of wave–flow interaction.

It is pertinent to ask whether we are able to construct an equation which may present the interaction between non-hydrostatic, ageostrophic and meso-scale transient wave and non-stationary basic flow and synchronously link the two aspects of wave–flow interaction. The studies of Haynes (1988)[22] and Scinocca and Shepherd (1992)[23] enlighten us on this question. They showed that for the pseudomomentum wave–activity law derived in Hamiltonian system with the Momentum–Casimir method, the basic states may not be stationary, but must be symmetric with regards to some coordinate axis.

The purpose of this paper is to construct an equation representing the interaction between non-hydrostatic, ageostrophic and meso-scale wave and basic flow with the Momentum–Casimir method on the bases of Haynes (1988),[22] Scinocca and Shepherd (1992)[23] and Ran *et al* (2007).[24] The zonally symmetric governing equations on f-plane in Cartesian coordinates are presented in the next section. The derivation of wave–flow interaction equation for disturbances to non-stationary and non-conservative basic states is addressed in Section 3. In Section 4, the discussion is given.

2. Governing equations

We start with the momentum, continuity, thermodynamic and state equations in Cartesian coordinates on an f-plane, under zonally symmetric, diabatic and frictionless conditions

$$\frac{\partial u}{\partial t} + \boldsymbol{v}\cdot\nabla u - f_0 v = 0, \tag{1}$$

$$\frac{\partial v}{\partial t} + \boldsymbol{v}\cdot\nabla v + f_0 u = -\frac{1}{\rho}\frac{\partial p}{\partial y}, \tag{2}$$

$$\frac{\partial w}{\partial t} + \boldsymbol{v}\cdot\nabla w = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g, \tag{3}$$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\boldsymbol{v}) = 0, \tag{4}$$

$$\frac{\partial \theta}{\partial t} + \boldsymbol{v}\cdot\nabla\theta = Q, \tag{5}$$

$$p = \rho R T. \tag{6}$$

In this system, $\boldsymbol{v} = (u, v, w)$, is the velocity vector in which u, v, w are the eastward, northward and upward components of wind; f_0 is the Coriolis parameter (assumed constant), p is the pressure, ρ is the density, R is the air constant, g is the gravitational acceleration, Q is the diabatic heating, $\theta = T\left(\frac{p_{\mathrm{s}}}{p}\right)^{\frac{R}{c_{\mathrm{p}}}}$ in which T is the temperature, p_{s} is the reference surface pressure, c_{p} is the specific heat at constant pressure, is the potential temperature, and $\nabla = \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}$. Equations (1) and (4) may be combined to derive an equation which expresses conservation of zonal angular momentum about rotation axis, being

$$\frac{\partial}{\partial t}(\rho U) + \nabla\cdot(\rho\boldsymbol{v}U) = 0, \tag{7}$$

where $U = u - f_0 y$.

An equation for Ertel potential vorticity may be formed from equations (1)–(5), being

$$\frac{\partial q}{\partial t}+\boldsymbol{v}\cdot\nabla q=\frac{1}{\rho}\omega\cdot\nabla\boldsymbol{Q}, \quad (8)$$

where $q=\omega\cdot\nabla\theta/\rho$ is the Ertel potential vorticity with $\omega=(\partial w/\partial y-\partial v/\partial z,\partial u/\partial z,-\partial u/\partial y+f_0)$. In the two-dimensional flow, we exploited the fact that in the conservative case any function, $C(q,\theta)$ say, of Ertel potential vorticity and potential temperature is constant following the fluid motion. If we multiply Eq.(5) by $\dfrac{\partial C}{\partial\theta}$ and Eq.(8) by $\dfrac{\partial C}{\partial q}$ and then add the results, we obtain

$$\frac{\partial}{\partial t}(\rho C)+\nabla\cdot(\rho\boldsymbol{v}C)=\frac{\partial C}{\partial q}\omega\cdot\nabla Q+\rho\frac{\partial C}{\partial\theta}Q, \quad (9)$$

in which the continuity equation (4) has been used. The sum of Eqs.(7) and (9) may give

$$\frac{\partial}{\partial t}[\rho(U+C)]+\nabla\cdot[\rho\boldsymbol{v}(U+C)] =\frac{\partial C}{\partial q}\omega\cdot\nabla Q+\rho\frac{\partial C}{\partial\theta}Q. \quad (10)$$

We now proceed to calculate wave–flow interaction relations for flows described by these equations.

3. Equation for interaction between wave and basic flow

In order to determine the interaction between wave and basic flow, we consider a basic state in which the various flow quantities (denoted by the subscript '0') are functions of y, z and t, but slowly vary in space and time compared with the disturbance quantities (denoted by the subscript 'e'). Thus, all quantities are composed of two parts: the basic-state part and the disturbance part, being

$$\begin{aligned}
u&=u_0(y,z,t)+u_{\rm e}(y,z,t),\\
v&=v_0(y,z,t)+v_{\rm e}(y,z,t),\\
w&=w_0(y,z,t)+w_{\rm e}(y,z,t),\\
p&=p_0(y,z,t)+p_{\rm e}(y,z,t),\\
\rho&=\rho_0(y,z,t)+\rho_{\rm e}(y,z,t),\\
T&=T_0(y,z,t)+T_{\rm e}(y,z,t),\\
\theta&=\theta_0(y,z,t)+\theta_{\rm e}(y,z,t),\\
q&=q_0(y,z,t)+q_{\rm e}(y,z,t),\\
Q&=Q_0(y,z,t)+Q_{\rm e}(y,z,t). \quad (11)
\end{aligned}$$

Note that the diabatic heating in basic state Q_0 does not vanish.

Under the assumption of small-amplitude disturbance, we may obtain the linearized form of disturbance equations by substituting Eq.(11) into Eqs.(1)-(5)

$$\frac{\partial u_{\rm e}}{\partial t}=-\boldsymbol{v}_0\cdot\nabla u_{\rm e}-v_{\rm e}\frac{\partial u_0}{\partial y}-w_{\rm e}\frac{\partial u_0}{\partial z}+f_0v_{\rm e}, \quad (12)$$

$$\frac{\partial v_{\rm e}}{\partial t}=-\boldsymbol{v}_0\cdot\nabla v_{\rm e}-v_{\rm e}\frac{\partial v_0}{\partial y}-w_{\rm e}\frac{\partial v_0}{\partial z}-f_0u_{\rm e} -\frac{1}{\rho_0}\frac{\partial p_{\rm e}}{\partial y}+\frac{\rho_{\rm e}}{\rho_0^2}\frac{\partial p_0}{\partial y}, \quad (13)$$

$$\frac{\partial w_{\rm e}}{\partial t}=-\boldsymbol{v}_0\cdot\nabla w_{\rm e}-v_{\rm e}\frac{\partial w_0}{\partial y}-w_{\rm e}\frac{\partial w_0}{\partial z} -\frac{1}{\rho_0}\frac{\partial p_{\rm e}}{\partial z}-g\frac{\rho_{\rm e}}{\rho_0}, \quad (14)$$

$$\frac{\partial\rho_{\rm e}}{\partial t}=-\nabla\cdot(\boldsymbol{v}_0\rho_{\rm e})-\nabla\cdot(\boldsymbol{v}_{\rm e}\rho_0), \quad (15)$$

$$\frac{\partial\theta_{\rm e}}{\partial t}=-\boldsymbol{v}_0\cdot\nabla\theta_{\rm e}-v_{\rm e}\frac{\partial\theta_0}{\partial y}-w_{\rm e}\frac{\partial\theta_0}{\partial z}+Q_{\rm e}, \quad (16)$$

where $\boldsymbol{v}_0=(u_0,v_0,w_0)$ is the basic-state velocity vector and $\boldsymbol{v}_{\rm e}=(u_{\rm e},v_{\rm e},w_{\rm e})$ is the perturbation velocity vector. In deriving these disturbance equations, we have used the series expansion that

$$1/\rho=1/\rho_0-\rho_{\rm e}/\rho_0^2+\rho_{\rm e}^2/\rho_0^3+\cdots. \quad (17)$$

Subtraction of the disturbance equations from the primitive equations results in the basic-state equations

$$\frac{\partial u_0}{\partial t}=-v_0\frac{\partial u_0}{\partial y}-w_0\frac{\partial u_0}{\partial z}+fv_0-\boldsymbol{v}_{\rm e}\cdot\nabla u_{\rm e}, \quad (18)$$

$$\frac{\partial v_0}{\partial t}=-v_0\frac{\partial v_0}{\partial y}-w_0\frac{\partial v_0}{\partial z}-f_0u_0-\frac{1}{\rho_0}\frac{\partial p_0}{\partial y}-\boldsymbol{v}_{\rm e}\cdot\nabla v_{\rm e} -\frac{\rho_{\rm e}^2}{\rho_0^3}\frac{\partial p_0}{\partial y}+\frac{\rho_{\rm e}}{\rho_0^2}\frac{\partial p_{\rm e}}{\partial y}, \quad (19)$$

$$\frac{\partial w_0}{\partial t}=-v_0\frac{\partial w_0}{\partial y}-w_0\frac{\partial w_0}{\partial z}-\frac{1}{\rho_0}\frac{\partial p_0}{\partial z}-g -\boldsymbol{v}_{\rm e}\cdot\nabla w_{\rm e}-\frac{\rho_{\rm e}^2}{\rho_0^3}\frac{\partial p_0}{\partial z}+\frac{\rho_{\rm e}}{\rho_0^2}\frac{\partial p_{\rm e}}{\partial z}, \quad (20)$$

$$\frac{\partial\rho_0}{\partial t}=-\frac{\partial}{\partial y}(\rho_0v_0)-\frac{\partial}{\partial z}(\rho_0w_0)-\nabla\cdot(\rho_{\rm e}\boldsymbol{v}_{\rm e}), \quad (21)$$

$$\frac{\partial\theta_0}{\partial t}=-v_0\frac{\partial\theta_0}{\partial y}-w_0\frac{\partial\theta_0}{\partial z}+Q_0-\boldsymbol{v}_{\rm e}\cdot\nabla\theta_{\rm e}. \quad (22)$$

The above basic-state equations are distinct in that they contain the disturbance quantities quadric in small disturbance amplitude. This implies two things: one thing is that the basic-state quantities are not a steady solution to the primitive equations, and another thing is that the spacial and temporal variations of basic-state quantities are smaller than those of disturbance quantities. For a given data, the common way to obtain this kind of basic states is band-pass filter of time and space.

If the quadric disturbance quantities are neglected in Eqs.(18)–(22), the basic states become a steady solution, which indicates that the transient wave does not impose on the basic flow. Therefore, the inclusion of the quadric disturbance quantities in Eqs.(18)–(22) is quite important in the investigation of wave–basic flow interaction.

Furthermore, the dynamic and thermodynamic fields in basic state are assumed to also satisfy the ideal gas law and definitions of potential temperature and Ertel potential vorticity

$$p_0 = \rho_0 R T_0, \tag{23}$$

$$\theta_0 = T_0 \left(\frac{p_\mathrm{s}}{p_0}\right)^{R/c_\mathrm{p}}, \tag{24}$$

$$q_0 = \frac{\omega_0 \cdot \nabla\theta_0}{\rho_0}, \tag{25}$$

where $\omega_0 = (\partial w_0/\partial y - \partial v_0/\partial z,\ \partial u_0/\partial z,\ f_0 - \partial u_0/\partial y)$ is the basic-state absolute vorticity. In a similar manner, if one takes the individual derivative to Eq.(25) and employs the basic-state equations to eliminate the local time changes of u_0, v_0, w_0 and θ_0, then an equation in a flux form for q_0 may be given by

$$\begin{aligned}&\frac{\partial}{\partial t}(\rho_0 q_0) + \frac{\partial}{\partial y}(\rho_0 v_0 q_0) + \frac{\partial}{\partial z}(\rho_0 w_0 q_0)\\ =&\frac{\partial}{\partial y}\left(\frac{\partial\theta_0}{\partial z}\boldsymbol{v}_\mathrm{e}\cdot\nabla u_\mathrm{e} - \frac{\partial u_0}{\partial z}\boldsymbol{v}_\mathrm{e}\cdot\nabla\theta_\mathrm{e}\right)\\ &-\frac{\partial}{\partial z}\left[\frac{\partial\theta_0}{\partial y}\boldsymbol{v}_\mathrm{e}\cdot\nabla u_\mathrm{e} + \left(f_0 - \frac{\partial u_0}{\partial y}\right)\boldsymbol{v}_\mathrm{e}\cdot\nabla\theta_\mathrm{e}\right]\\ &+\nabla\cdot(\omega_0 Q_0),\end{aligned} \tag{26}$$

which suggests that the global integration of $\rho_0 q_0$ is conservative for periodic boundary conditions.

At small disturbance amplitude, we perform a Taylor series expansion of $C(q,\theta)$ about (q_0,θ_0). Considering the low-order contribution, we determine the expression for $C(q,\theta)$ to be

$$\begin{aligned}C(q,\theta) =& C_0 + \frac{\partial C_0}{\partial q_0}q_\mathrm{e} + \frac{\partial C_0}{\partial\theta_0}\theta_\mathrm{e}\\ &+\frac{1}{2}\left(\frac{\partial^2 C_0}{\partial q_0^2}q_\mathrm{e}^2 + \frac{\partial^2 C_0}{\partial\theta_0^2}\theta_\mathrm{e}^2\right)\\ &+\frac{\partial^2 C_0}{\partial q_0\partial\theta_0}q_\mathrm{e}\theta_\mathrm{e},\end{aligned} \tag{27}$$

where $C_0 = C(q_0,\theta_0)$ is the evaluation of C at (q_0,θ_0) and

$$q_\mathrm{e} = \frac{1}{\rho}(\omega_\mathrm{e}\cdot\nabla\theta_0 + \omega_\mathrm{e}\cdot\nabla\theta_\mathrm{e} + \omega_0\cdot\nabla\theta_\mathrm{e} - \rho_\mathrm{e} q_0) \tag{28}$$

is the perturbation potential vorticity in which $\omega_\mathrm{e} = (\partial w_\mathrm{e}/\partial y - \partial v_\mathrm{e}/\partial z,\ \partial u_\mathrm{e}/\partial z,\ -\partial u_\mathrm{e}/\partial y)$ is the relative perturbation vorticity.

Employing Eqs.(11), (27) and (28), the quantity $\rho(U+C)$ may be expanded to be

$$\begin{aligned}\rho(U+C) =& \frac{\partial}{\partial y}\left[\frac{\partial C_0}{\partial q_0}\left(\frac{\partial u_0}{\partial z}\theta_\mathrm{e} - \frac{\partial\theta_0}{\partial z}u_\mathrm{e}\right)\right] + \frac{\partial}{\partial z}\left\{\frac{\partial C_0}{\partial q_0}\left[\frac{\partial\theta_0}{\partial y}u_\mathrm{e} + \left(f_0 - \frac{\partial u_0}{\partial y}\right)\theta_\mathrm{e}\right]\right\}\\ &+u_\mathrm{e}\left[\rho_0 + \frac{\partial\theta_0}{\partial z}\frac{\partial}{\partial y}\left(\frac{\partial C_0}{\partial q_0}\right) - \frac{\partial\theta_0}{\partial y}\frac{\partial}{\partial z}\left(\frac{\partial C_0}{\partial q_0}\right)\right]\\ &-\theta_\mathrm{e}\left[\frac{\partial u_0}{\partial z}\frac{\partial}{\partial y}\left(\frac{\partial C_0}{\partial q_0}\right) + \left(f_0 - \frac{\partial u_0}{\partial y}\right)\frac{\partial}{\partial z}\left(\frac{\partial C_0}{\partial q_0}\right) - \rho_0\frac{\partial C_0}{\partial\theta_0}\right]\\ &+\rho_\mathrm{e}\left(U_0 + C_0 - q_0\frac{\partial C_0}{\partial q_0}\right) + \rho_0(U_0 + C_0) + J,\end{aligned} \tag{29}$$

where $U_0 = u_0 - f_0 y$ is the zonal basic-state absolute momentum density, and

$$J = \rho_\mathrm{e}\left(u_\mathrm{e} + \frac{\partial C_0}{\partial\theta_0}\theta_\mathrm{e}\right) + \frac{\partial C_0}{\partial q_0}(\omega_\mathrm{e}\cdot\nabla)\theta_\mathrm{e} + \rho_0\left[\frac{1}{2}\left(\frac{\partial^2 C_0}{\partial q_0^2}q_\mathrm{e}^2 + \frac{\partial^2 C_0}{\partial\theta_0^2}\theta_\mathrm{e}^2\right) + \frac{\partial^2 C_0}{\partial q_0\partial\theta_0}q_\mathrm{e}\theta_\mathrm{e}\right]$$

is interpreted as the pseudomomentum wave–activity density which is quadratic in small disturbance amplitude.

Following the Momentum–Casimir method used by Haynes (1988),[22] we should choose that function C which makes the third, fourth and fifth terms on the right-hand side of Eq.(29) associated with the leading-order wave vanish. It is therefore required that

$$\rho_0+\frac{\partial\theta_0}{\partial z}\frac{\partial}{\partial y}\left(\frac{\partial C_0}{\partial q_0}\right)-\frac{\partial\theta_0}{\partial y}\frac{\partial}{\partial z}\left(\frac{\partial C_0}{\partial q_0}\right)=0, \quad (30)$$

$$\frac{\partial u_0}{\partial z}\frac{\partial}{\partial y}\left(\frac{\partial C_0}{\partial q_0}\right)+\left(f_0-\frac{\partial u_0}{\partial y}\right)\frac{\partial}{\partial z}\left(\frac{\partial C_0}{\partial q_0}\right) -\rho_0\frac{\partial C_0}{\partial\theta_0}=0, \quad (31)$$

$$U_0+C_0-q_0\frac{\partial C_0}{\partial q_0}=0. \quad (32)$$

The three conditions (30), (31) and (32) may be shown to be equivalent by firstly taking partial derivatives of Eq.(32) with respect to y and z, respectively, and then substituting the results into the left-hand sides of Eqs.(30) and (31).

Under the conditions (30)–(32), Eq.(29) becomes

$$\begin{aligned}&\rho\left(U+C\right)\\ =&\frac{\partial}{\partial y}\left[\frac{\partial C_0}{\partial q_0}\left(\frac{\partial u_0}{\partial z}\theta_\mathrm{e}-\frac{\partial\theta_0}{\partial z}u_\mathrm{e}\right)\right]\\ &+\frac{\partial}{\partial z}\left\{\frac{\partial C_0}{\partial q_0}\left[\frac{\partial\theta_0}{\partial y}u_\mathrm{e}+\left(f_0-\frac{\partial u_0}{\partial y}\right)\theta_\mathrm{e}\right]\right\}\\ &+\rho_0\left(U_0+C_0\right)+J. \end{aligned} \quad (33)$$

Employing Eqs.(30)–(32) again, we may further simplify Eq.(33), being

$$\begin{aligned}&\rho\left(U+C\right)\\ =&\rho_0\left(u_\mathrm{e}+\frac{\partial C_0}{\partial\theta_0}\theta_\mathrm{e}\right)\\ &+\frac{\partial C_0}{\partial q_0}\left(\omega_\mathrm{e}\cdot\nabla\theta_0+\omega_0\cdot\nabla\theta_\mathrm{e}+\omega_0\cdot\nabla\theta_0\right)+J. \end{aligned} \quad (34)$$

We may express the time derivative of the local wave–activity density J in flux form by substituting Eq.(33) into the first term and Eq.(34) into the second term on the left-hand side of Eq.(10), and then eliminate the local time changes of disturbance quantities with Eqs.(12)–(16)

$$\begin{aligned}&\frac{\partial J}{\partial t}+\frac{\partial}{\partial y}\left\{v_0J+v_\mathrm{e}\left[\rho_0\left(u_\mathrm{e}+\frac{\partial C_0}{\partial\theta_0}\theta_\mathrm{e}\right)+\frac{\partial C_0}{\partial q_0}(\omega_\mathrm{e}\cdot\nabla\theta_0+\omega_0\cdot\nabla\theta_\mathrm{e})\right]\right\}\\ &+\frac{\partial}{\partial z}\left\{w_0J+w_\mathrm{e}\left[\rho_0\left(u_\mathrm{e}+\frac{\partial C_0}{\partial\theta_0}\theta_\mathrm{e}\right)+\frac{\partial C_0}{\partial q_0}(\omega_\mathrm{e}\cdot\nabla\theta_0+\omega_0\cdot\nabla\theta_\mathrm{e})\right]\right\}\\ =&-\frac{\partial}{\partial t}[\rho_0(U_0+C_0)]-\frac{\partial}{\partial y}[\rho_0v_0(U_0+C_0)]-\frac{\partial}{\partial z}[\rho_0w_0(U_0+C_0)]+S, \end{aligned} \quad (35)$$

where

$$\begin{aligned}S=&-\frac{\partial}{\partial y}\left\{\left[\frac{\partial C_0}{\partial q_0}\frac{\partial u_\mathrm{e}}{\partial z}+\frac{1}{q_0}\frac{\partial u_0}{\partial z}\left(\frac{\partial C_0}{\partial\theta_0}\theta_\mathrm{e}+u_\mathrm{e}\right)\right]Q_0\right\}\\ &+\frac{\partial}{\partial z}\left\{\left[\frac{\partial C_0}{\partial q_0}\frac{\partial u_\mathrm{e}}{\partial y}-\frac{1}{q_0}\left(f_0-\frac{\partial u_0}{\partial y}\right)\left(\frac{\partial C_0}{\partial\theta_0}\theta_\mathrm{e}+u_\mathrm{e}\right)\right]Q_0\right\}\\ &-\frac{\partial C_0}{\partial q_0}\omega_0\cdot\nabla Q_\mathrm{e}-\rho_0\frac{\partial C_0}{\partial\theta_0}Q_\mathrm{e}+\frac{\partial C}{\partial q}\omega\cdot\nabla Q+\rho\frac{\partial C}{\partial\theta}Q \end{aligned} \quad (36)$$

is the source or sink which involves the forcing of diabatic heating. The equation is quite important because it associates the transient wave (the terms on the left-hand side) with the basic flow (the first three terms on the right-hand side) and concisely describes the two-way local interaction between wave and basic flow.

If Eq.(35) is globally integrated over a two-dimensional space σ on whose boundaries the normal complement of flux vanishes, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\sigma} J\mathrm{d}y\mathrm{d}z = -\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\sigma} \rho_0(U_0+C_0)\mathrm{d}y\mathrm{d}z + \iint_{\sigma} S\mathrm{d}y\mathrm{d}z. \tag{37}$$

The equation represents the global interaction between wave and basic flow with non-conservative effect. In the equation the global tendencies of pseudomomentum wave activity and basic-state pseudomomentum are opposite, which means that the decrease of basic-state pseudomomentum may result in the increase of pseudomomentum wave activity and vice versa.

We now proceed to analyse how the interaction takes place. Employing the basic-state equations (18)–(25), we may rewrite the first three terms on the right-hand side of Eq.(35) as

$$\begin{aligned}&\frac{\partial}{\partial t}\left[\rho_0\left(U_0+C_0\right)\right]+\frac{\partial}{\partial y}\left[\rho_0 v_0\left(U_0+C_0\right)\right]+\frac{\partial}{\partial z}\left[\rho_0 w_0\left(U_0+C_0\right)\right]\\ =&\frac{\partial}{\partial y}\left[\frac{\partial C_0}{\partial q_0}\left(\frac{\partial \theta_0}{\partial z}\boldsymbol{v}_{\mathrm{e}}\cdot\nabla u_{\mathrm{e}}-\frac{\partial u_0}{\partial z}\boldsymbol{v}_{\mathrm{e}}\cdot\nabla\theta_{\mathrm{e}}\right)\right]\\ &-\frac{\partial}{\partial z}\left\{\frac{\partial C_0}{\partial q_0}\left[\frac{\partial \theta_0}{\partial y}\boldsymbol{v}_{\mathrm{e}}\cdot\nabla u_{\mathrm{e}}+\left(f_0-\frac{\partial u_0}{\partial y}\right)\boldsymbol{v}_{\mathrm{e}}\cdot\nabla\theta_{\mathrm{e}}\right]\right\}+\frac{\partial C_0}{\partial q_0}\omega_0\cdot\nabla Q_0+\rho_0\frac{\partial C_0}{\partial \theta_0}Q_0.\end{aligned} \tag{38}$$

Besides the basic-state flux divergence (the second and third terms on the left-hand side of Eq.(38)) and the basic-state diabatic heating (the last two terms on the right-hand side of Eq.(38)), the transports of perturbation momentum and potential temperature (the first two terms on the right-hand side of Eq.(38)) contribute to the local evolution of basic-state pseudomomentum, representing the influence of wave on basic flow. Globally integrating the equation with simple boundary conditions, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint \rho_0\left(U_0+C_0\right)\mathrm{d}y\mathrm{d}z = \iint_{\sigma}\left[\frac{\partial C_0}{\partial q_0}\omega_0\cdot\nabla Q_0+\rho_0\frac{\partial C_0}{\partial \theta_0}Q_0\right]\mathrm{d}y\mathrm{d}z. \tag{39}$$

It is evident from Eq.(39) that in the global sense, the basic-state pseudomomentum is not conservative, uniquely being determined by the basic-state diabatic heating.

Replacement of the first three terms on the right-hand side of Eq.(35) with Eq.(38) may result in a two-dimensional ageostrophic non-hydrostatic pseudomomentum wave–activity relation

$$\frac{\partial J}{\partial t}+\nabla\cdot\boldsymbol{F}=S_{\mathrm{J}}, \tag{40}$$

where $\boldsymbol{F} = (F_y,\ F_z)$ is interpreted as the pseudomomentum wave–activity flux which is explicitly of quadratic order in small disturbance amplitude, and whose components are

$$\begin{aligned}F_y =&v_0 J+v_{\mathrm{e}}\left[\rho_0\left(u_{\mathrm{e}}+\frac{\partial C_0}{\partial \theta_0}\theta_{\mathrm{e}}\right)+\frac{\partial C_0}{\partial q_0}\left(\omega_{\mathrm{e}}\cdot\nabla\theta_0+\omega_0\cdot\nabla\theta_{\mathrm{e}}\right)\right]\\ &+\frac{\partial C_0}{\partial q_0}\left(\frac{\partial \theta_0}{\partial z}\boldsymbol{v}_{\mathrm{e}}\cdot\nabla u_{\mathrm{e}}-\frac{\partial u_0}{\partial z}\boldsymbol{v}_{\mathrm{e}}\cdot\nabla\theta_{\mathrm{e}}\right),\end{aligned} \tag{41}$$

$$\begin{aligned}F_z =&w_0 J+w_{\mathrm{e}}\left[\rho_0\left(u_{\mathrm{e}}+\frac{\partial C_0}{\partial \theta_0}\theta_{\mathrm{e}}\right)+\frac{\partial C_0}{\partial q_0}\left(\omega_{\mathrm{e}}\cdot\nabla\theta_0+\omega_0\cdot\nabla\theta_{\mathrm{e}}\right)\right]\\ &-\frac{\partial C_0}{\partial q_0}\left[\frac{\partial \theta_0}{\partial y}\boldsymbol{v}_{\mathrm{e}}\cdot\nabla u_{\mathrm{e}}+\left(f_0-\frac{\partial u_0}{\partial y}\right)\boldsymbol{v}_{\mathrm{e}}\cdot\nabla\theta_{\mathrm{e}}\right],\end{aligned} \tag{42}$$

and

$$S_J = S - \left(\frac{\partial C_0}{\partial q_0}\omega_0\cdot\nabla Q_0 + \rho_0\frac{\partial C_0}{\partial \theta_0}Q_0\right) \tag{43}$$

is the source or sink for wave activity. Globally integrating Eq.(40) with the same boundary conditions as Eq.(37) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\iint_\sigma J\mathrm{d}y\mathrm{d}z = \iint_\sigma S_J\mathrm{d}y\mathrm{d}z, \tag{44}$$

which reveals that the global non-conservation of pseudomomentum wave activity results from the total diabatic heating. Comparing Eq.(44) with Eq.(39), we may infer that in the global sense, the interaction between wave and basic flow is realized in virtue of the basic-state diabatic heating Q_0 which is a key linking tie between wave and basic flow.

It is clear that the diabatic heating in basic state Q_1 plays a crucial role in the wave-basic flow interaction. We now provide a proof that Q_0 cannot be chosen arbitrarily and should satisfy the certain physical restriction. Employing Eqs.(21), (22) and (26), we get an equation for $\rho_0 q_0\dfrac{\partial C_0}{\partial q_0}$:

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho_0 q_0\frac{\partial C_0}{\partial q_0}\right) + \frac{\partial}{\partial y}\left(v_0\rho_0 q_0\frac{\partial C_0}{\partial q_0}\right) + \frac{\partial}{\partial z}\left(w_0\rho_0 q_0\frac{\partial C_0}{\partial q_0}\right)\\
=&q_0^2\frac{\partial^2 C_0}{\partial q_0^2}\nabla\cdot(\rho_\mathrm{e}\boldsymbol{v}_\mathrm{e}) + \left(q_0\frac{\partial^2 C_0}{\partial q_0^2} + \frac{\partial C_0}{\partial q_0}\right)\left\{\frac{\partial}{\partial y}\left(\frac{\partial\theta_0}{\partial z}\boldsymbol{v}_\mathrm{e}\cdot\nabla u_\mathrm{e} - \frac{\partial u_0}{\partial z}\boldsymbol{v}_\mathrm{e}\cdot\nabla\theta_\mathrm{e}\right)\right.\\
&\left.-\frac{\partial}{\partial z}\left[\frac{\partial\theta_0}{\partial y}\boldsymbol{v}_\mathrm{e}\cdot\nabla u_\mathrm{e} + \left(f_0 - \frac{\partial u_0}{\partial y}\right)\boldsymbol{v}_\mathrm{e}\cdot\nabla\theta_\mathrm{e}\right]\right\} - \rho_0 q_0\frac{\partial^2 C_0}{\partial q_0\partial\theta_0}\boldsymbol{v}_\mathrm{e}\cdot\nabla\theta_\mathrm{e}\\
&+\rho_0 q_0\frac{\partial^2 C_0}{\partial q_0\partial\theta_0}Q_0 + \left(q_0\frac{\partial^2 C_0}{\partial q_0^2} + \frac{\partial C_0}{\partial q_0}\right)\omega_0\cdot\nabla Q_0.
\end{aligned} \tag{45}$$

Subtracting Eq.(45) from Eq.(38), and then using Eq.(32), we are left with an equation for Q_0:

$$\begin{aligned}
&\rho_0\left(\frac{\partial C_0}{\partial\theta_0} - q_0\frac{\partial^2 C_0}{\partial q_0\partial\theta_0}\right)Q_0 - q_0\frac{\partial^2 C_0}{\partial q_0^2}\omega_0\cdot\nabla Q_0\\
=&q_0\frac{\partial^2 C_0}{\partial q_0^2}\left[\frac{\partial\theta_0}{\partial z}\frac{\partial}{\partial y}(\boldsymbol{v}_\mathrm{e}\cdot\nabla u_\mathrm{e}) - \frac{\partial\theta_0}{\partial y}\frac{\partial}{\partial z}(\boldsymbol{v}_\mathrm{e}\cdot\nabla u_\mathrm{e}) - \omega_0\cdot\nabla(\boldsymbol{v}_\mathrm{e}\cdot\nabla\theta_\mathrm{e}) + q_0\nabla\cdot(\rho_\mathrm{e}\boldsymbol{v}_\mathrm{e})\right]\\
&+\rho_0\left[\left(\frac{\partial C_0}{\partial\theta_0} - q_0\frac{\partial^2 C_0}{\partial q_0\partial\theta_0}\right)\boldsymbol{v}_\mathrm{e}\cdot\nabla\theta_\mathrm{e} + \boldsymbol{v}_\mathrm{e}\cdot\nabla u_\mathrm{e}\right].
\end{aligned} \tag{46}$$

The equation, which in nature is an alternative representation of Eq.(32), shows that Q_0 is associated with the disturbance quantities. For a specific form of C, Q_0 may be evaluated from other given basic-state quantities and disturbance quantities through Eq.(46).

On the bases of equations (39), (44) and (46), we may give a simple physical procedure of wave–basic flow interaction. The transient wave indirectly force the basic flow to develop through Eq.(39) by directly influencing on Q_0 in Eq.(46). On the other hand, the developed basic flow and Q_0 carry out a feedback on the transient wave through Eq.(44).

4. Discussion

The zonally symmetric, non-hydrostatic system of equations in Cartesian coordinates that is widely applied to meso-scale phenomena, such as squall line and conditional symmetric instability is employed. With the assumption of small amplitude, the disturbance quantities are subjected to the linear perturbation equations and the perturbation terms quadric in disturbance amplitude are kept down in the basic-state equations, implying that the basic states vary more slowly than the transient disturbances in time and space. Then on the bases of these equations, we employ the Momentum–Casimir method used by Haynes (1988) to derive the local and global equations presenting the interaction between transient wave and non-stationary, non-conservative basic flow.

It follows from the local equation (35) that the local tendency of pseudomomentum wave–activity density is mainly dominated by the combination of perturbation flux divergence quadric in disturbance amplitude, local change of basic-state pseudomomentum density, basic-state flux divergence, and forcing effect of diabatic heating. The tendencies of pseudomomentum wave–activity density and basic-state pseudomomentum density are opposite, which indicates that the decaying basic-state pseudomomentum density is propitious to prompt the growth of pseudomomentum wave activity density and vice versa. For a closed system with vanishing flux components normal to boundaries, the globally integrated equation (37) shows that the global development of pseudomomentum wave activity depends on the global tendency of basic-state pseudomomentum and the diabatic heating in basic state.

Using the basic-state pseudomomentum equation (38), we derive the wave–activity equation (40) on the base of the wave–basic flow interaction equation (37). It is shown that in the global sense, the basic-state diabatic heating uniquely gives rise to the evolution of basic-state pseudomomentum and the pseudomomentum wave activity develops under the influence of the total diabatic heating. Compared the two equations, it may be inferred that the basic-state diabatic heating is a key tie linking wave and basic flow. On the one hand, the disturbances effect on the basic-state heating in Eq.(46) to indirectly force the development of basic-state pseudomomentum through Eq.(39). On the other hand, the developed basic flow and basic-state heating carry out a feedback on the pseudomomentum wave activity through Eq.(44).

In the derivation, a single-valued function $C(q,\theta)$ of Ertel potential vorticity and potential temperature is introduced and its expression is not specified. One may choose different forms of $C(q,\theta)$ with all kinds of intentions, which gives a great freedom to discuss the interaction between wave and various basic flows. It is worth emphasizing that the specific expression of $C(q,\theta)$ should guarantee the reasonable evaluation of Q_0 in Eq.(46).

The theory of wave–flow interaction obtained here may prove to be an useful diagnostic in many situations. Although the effects of dissipation are not included in this paper, if desired, one may generalize the present result to consider the effects by taking into account the terms of turbulent dissipation in the primitive governing equations, as Haynes (1988) did. As an alternative way, one may calculate the effects as a residual of the two sides of Eq.(35). When the lateral conditions are nonperiodic and the components of flux normal to the bottom and upper boundaries do not vanish, the globally integrated equation of wave–flow interaction can provide a way of presenting the role of boundary condition in the dynamics.

When the specific humidity ($q_{\rm v}$) is introduced, the wave–flow interaction theory may be extended to the moist atmosphere. In this situation, the state equation (6) should be replaced by $p = \rho RT\,(1 + 0.61q_{\rm v})$, and the equivalent potential temperature and moist potential vorticity should be used instead of potential temperature and potential vorticity.

Built up in non-hydrostatic and ageostrophic dynamic framework, the wave–flow interaction theory is applicable to diagnose the interaction between the meso-scale convective system in front, which often induces heavy rainfall, and the background flow. Although the zonal symmetry is employed in the derivation, the two-dimensional theory is still accurate with a small error when the variations in the along-front direction are small compared to the variations across the front in the middle latitudes and the meridional regional range is not too large. In addition, it seems that there is no obstruction to generalize the present result to β-plane approximation from f-plane approximation.

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