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UNIVERSITY OF MICHIGAN
ENGINEERING RESEARCH INSTITUTE

ELECTRONIC DEFENSE GROUP TECHNICAL MEMORANDUM NO. 22

SUBJECT: Klinkhamer's Method of Determining Filter or Amplifier Transfer Functions

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I. INTRODUCTION

The process of network design breaks down logically into three parts: (1) specification of the network characteristics which would be ideal for the job, (2) determination of a transfer function approximating these ideal characteristics and (3) synthesizing one or more filters having the transfer function determined by step (2).

In filtering applications the engineering specifications can frequently be stated in terms of the minimum permissible attenuation in one or more stopbands and the maximum permissible attenuation in one or more passbands. Such a specification is given graphically in Figure 1 where it is indicated that the attenuation α must satisfy the

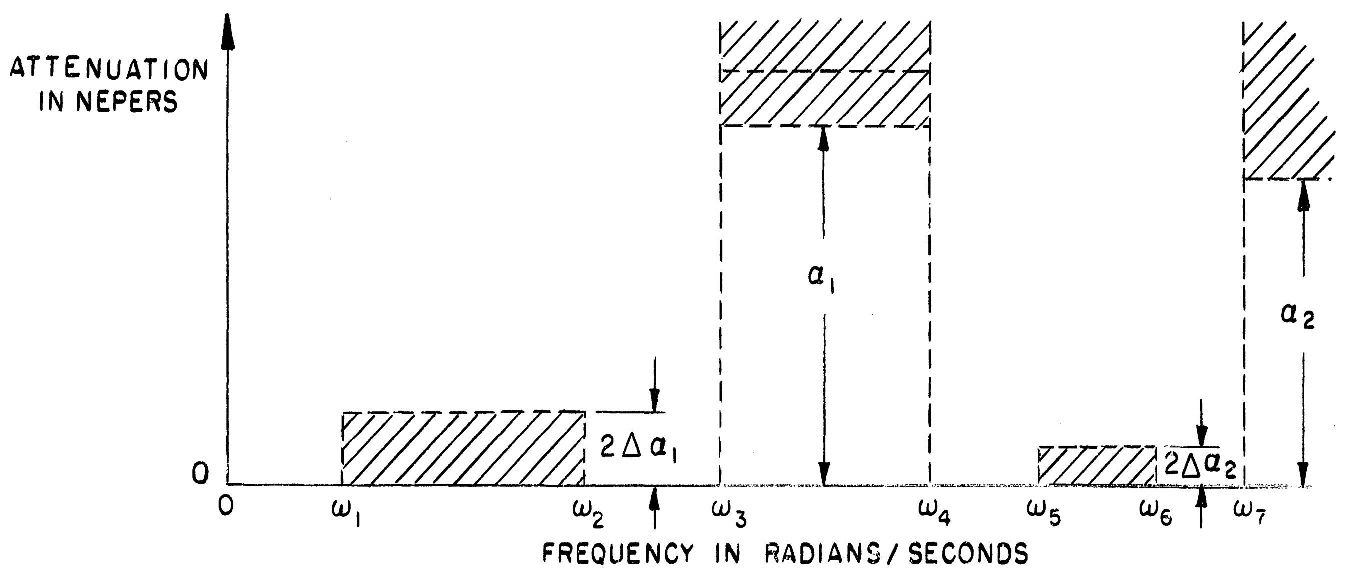

FIG. 1 A TYPICAL FILTER SPECIFICATION

following requirements.

$$\begin{aligned}
&\alpha < 2\Delta\alpha_1 && \text{for} \quad \omega_1 \lessgtr \omega \leq \omega_2 \\
&\alpha > \alpha_1 && \text{for} \quad \omega_3 \lessgtr \omega \leq \omega_4 \\
&\alpha < 2\Delta\alpha_2 && \text{for} \quad \omega_5 \lessgtr \omega \leq \omega_6 \\
&\alpha > \alpha_2 && \text{for} \quad \omega_7 \leq \omega.
\end{aligned} \tag{1}$$

The problem of finding a rational function

$$\frac{f(p)}{g(p)} = t(p),\ Z_{12}(p),\ \text{etc.} \tag{2}$$

of the complex frequency variable $p = \sigma + j\omega$ which meets the requirements of Eq. 1 in an optimum manner experimentally, has been very neatly solved by Klinkhamer.[1] Because this solution does not appear to have been used by many engineers in this country, who might profitably do so, this memorandum summarizes the approximation method and gives an example of its application to a particular problem. Klinkhamer's method of approximation enables one to determine experimentally, with the aid of an electrolytic tank or sheet of conducting paper, the poles and zeroes of a rational filter transfer function which approximates the given attenuation specification in a Tschebycheff manner in each pass- and stop-band. The pole-zero locations are determined uniquely, without any trial and error process (to the precision of the experimental set-up employed), and the particular rational function is optimum in the sense that it employs the minimum number of poles necessary to achieve the given specification. This, in general, means that the network synthesized to have this transfer function will require the minimum possible number of elements.

II. THE CHARACTERISTIC FUNCTION

The problem is to find a rational function $f(p)/g(p)$ having poles in the left half of the complex frequency p-plane and zeroes in this plane or on the axis of imaginaries ($j\omega$-axis), such that the attenuation

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[1] J. F. Klinkhamer, "Emperical Determination of Wave-Filter Transfer Functions with Specified Properties," Philips Research Reports, 3, pp. 60-80 and pp. 378-400.

$$\alpha = \ln \left|\frac{g(p)}{f(p)}\right| \text{ nepers} \tag{3}$$

satisfies the following requirements:

(a) in a finite number of pass-bands P_1, P_2, ---, P_n the attenuation lies between α_o and $\alpha_o + 2\Delta\alpha_1$, $\alpha_o + 2\Delta\alpha_2$, ..., $\alpha_o + 2\Delta\alpha_n$ (where the $\Delta\alpha$'s are less than 0.115 neper = 1 db) and

(b) in a finite number of stop-bands S_1, S_2, ..., S_m the attenuation lies between $\alpha_o + \alpha_1$, $\alpha_o + \alpha_2$, ..., $\alpha_o + \alpha_m$ and $\alpha_o + \alpha_1 + 2\Delta\alpha'_1$, $\alpha_o + \alpha_2 + 2\Delta\alpha'_2$, ..., $\alpha_o + \alpha_m + 2\Delta\alpha'_m$ (where α_1, α_2, ..., α_m are greater than 1.73 neper = 15 db, and $\Delta\alpha'_1$, ..., $\Delta\alpha'_m$ may range from 0 to ∞)

It is expedient to define a new function

$$\left|\frac{h(p)}{f(p)}\right|^2 = \left|\frac{g(p)}{f(p)}\right|^2 - e^{2\alpha_o} \tag{4}$$

which will be called a characteristic function. It is clear that the poles of this characteristic function $h|f$ are the same as the poles of the function $g|f$ which one wishes to determine, but if the attenuation α oscillates in an equal ripple fashion between α_o and $\alpha_o + 2\Delta\alpha_n$ in the pass-bands, the zeroes of h/f^2 all lie on the $j\omega$-axis, and they are all double. If one considers initially that the $\Delta\alpha'_m$ are all infinite[1], then h/f is a function having all its zeroes and poles on the $j\omega$-axis which characterizes the proposed filter function just as completely as g/f does. To make use of this characteristic function one must translate the given specifications on g/f, (a) and (b), to appropriate specifications on h/f.

III. SPECIFICATIONS ON THE CHARACTERISTIC FUNCTION

As long as the difference between the stop- and pass-band attenuations α_1, α_2, ..., α_m exceeds 15 db one may write in the stop-bands

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[1] This restriction of all the points of infinite loss to the $j\omega$-axis can be removed later.

$$\alpha - \alpha_o = \ln \left|\frac{g}{f}\right| - \alpha_o \geqq 1.73 \text{ nepers,} \tag{5}$$

or

$$\left|\frac{g}{f}\right| \; e^{-\alpha_o} \geqq 5.64 \tag{6}$$

Squaring both sides gives

$$\left|\frac{g}{f}\right|^2 \; e^{-2\alpha_o} \geqq 31.8 \; . \tag{7}$$

Therefore according to Eq. 4

$$\frac{\left|\frac{g}{f}\right|^2 - \left|\frac{h}{f}\right|^2}{\left|\frac{g}{f}\right|^2} = \frac{e^{+2\alpha_o}}{\left|\frac{g}{f}\right|^2} \leqq \frac{1}{31.8} \; . \tag{8}$$

From Eq. 8 one sees that in the stop-bands the attenuation obtained with $\frac{h}{f}$ differs from that with $\frac{g}{f}$ by at the most

$$\frac{1}{2} \ell n \left(1 - \frac{1}{31.8}\right) = .0155 \text{ nepers} = .135 \text{ db.}$$

Therefore as long as the stop-band attenuation is at least 15 db greater than the pass-band attenuation one can assume that in the stopband

$$\ell n \left|\frac{h(p)}{f(p)}\right| \approx \ell n \left|\frac{g(p)}{f(p)}\right| \tag{9}$$

In the passbands

$$\ell n \left|\frac{g}{f}\right| - \alpha_o = \delta \text{ nepers} \tag{10}$$

which may vary between 0 and $2\alpha\Delta_\nu$ nepers. Taking the exponential of both sides of this equation gives

$$\left|\frac{g}{f}\right| \; e^{-\alpha_o} = e^{\delta} = 1 + \delta + \frac{\delta^2}{2!} + \ldots \tag{11}$$

As long as δ is small compared to unity, only the first two terms in the series of Eq. 11 need be considered. Thus

$$\left|\frac{g}{f}\right|^2 \; e^{-2\alpha_o} \approx 1 + 2\delta \tag{12}$$

and substituting from Eq. 4 gives

$$\left|\frac{h}{f}\right|^2 e^{-2\alpha_o} \approx 2\delta \tag{13}$$

which varies between zero and $4\Delta\alpha_\nu$ over the passband. Therefore in the passband the attenuation of the characteristic function

$$\ell n \left|\frac{h}{f}\right| - \alpha_o \approx \frac{1}{2}\ \ell n(2\delta) \tag{14}$$

which varies between $-\infty$ and $\ell n2 + 1/2\ \ell n\Delta\alpha_\nu$. If the maximum passband attenuation $2\Delta\alpha_\nu$ does not exceed 0.1 nepers (.87 decibels) the error introduced by the approximation of Eq. 12 on the specification of $\left|\frac{h}{g}\right|$ will be at the most .0478 nepers (.41 db).

In summary, if one wishes a transfer function g/f such that the attenuation given by Eq. 3 will satisfy conditions (a) and (b), it is necessary to choose the characteristic function h f such that:

$$\boxed{\begin{array}{ll} \text{in the pass-bands } P_\nu,\ \ell n\,\frac{h}{f}\ -\alpha_o \text{ must be } \leqq\ \ell n\,(2) + 1/2\ \ell n\Delta\alpha_\nu & (15) \\ \text{in the stop-bands } S_\nu,\ \ell n\,\frac{h}{f}\ -\alpha_o \text{ must be } \geqq\ \alpha_\nu & (16) \end{array}}$$

The approximation problem is thus reduced to finding a rational function h/f having all its zeroes and poles on the $j\omega$-axis and satisfying the Eqs. 15 and 16.

IV. THE ELECTROLYTIC TANK ANALOG FOR DETERMINING h/f

The analog between the potentials and currents in a electrolytic tank (sheet of uniform conducting paper) and the attenuation and phase of rational function is well known.[1,2,3,4] This is illustrated in Figure 2. A sheet of conducting paper is used as

[1] A. R. Boothroyd, E. C. Cherry, R. Maker, Journ. I.E.E., Part I, Vol. 96, pp. 163-177.
[2] R. E. Scott, Proc. I.R.E., Vol. 40, pp. 970-973.
[3] J. F. Klinkhammer, Philips Res. Repts., Vol. 3, pp. 60-80, pp. 378-400.
[4] W. H. Huggins, Proc. I.R.E., Vol. 36, pp. 421-424.

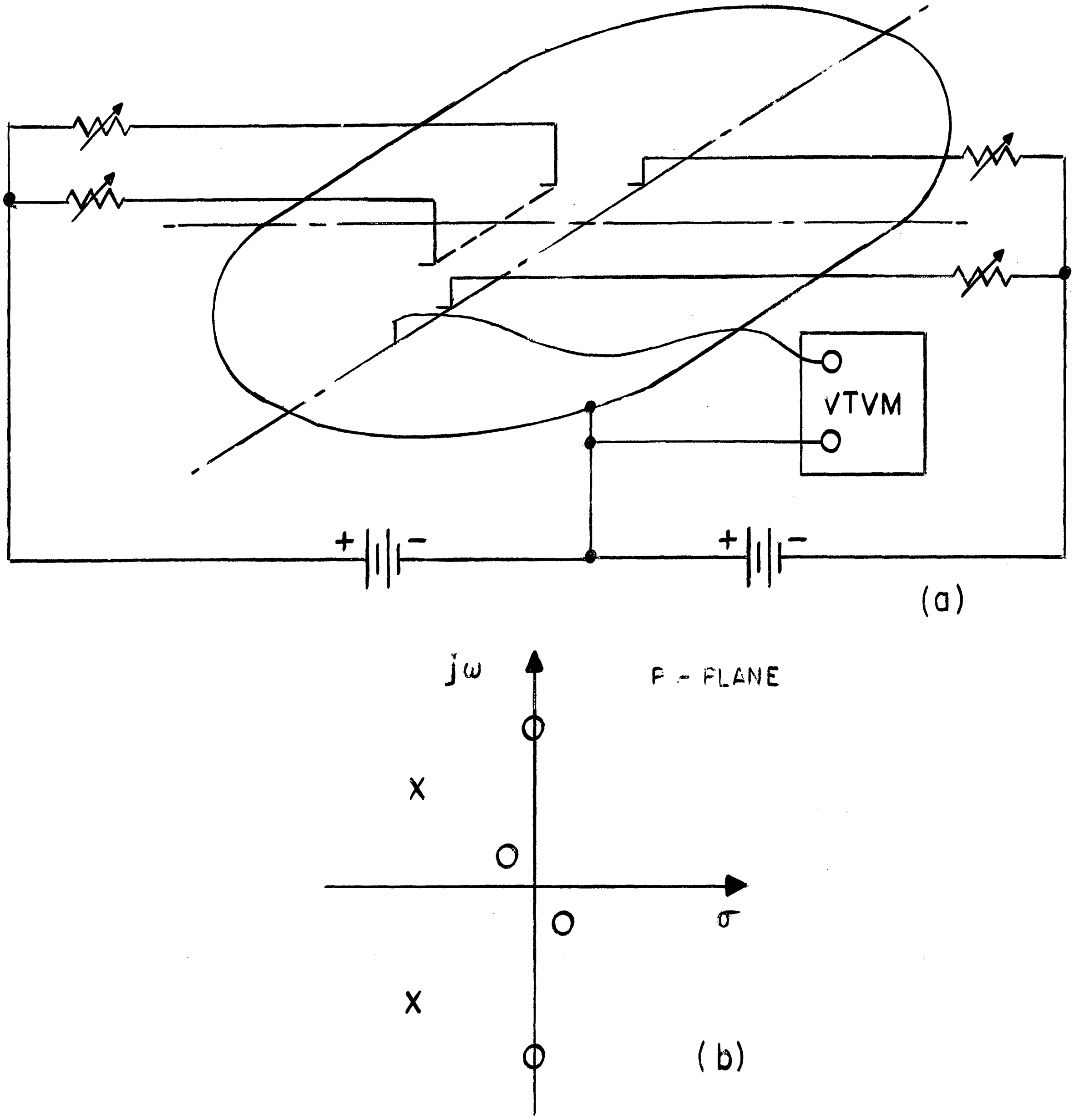

FIG. 2 THE POTENTIAL ANALOG OF A RATIONAL FUNCTION IN TERMS OF ITS POLES AND ZEROS IN THE COMPLEX FREQUENCY PLANE.

an analog of the complex frequency p-plane. This sheet should be infinite in extent, but in practice a circular electrode at same distance from, and concentic with, the origin can be used to represent the point at infinity. It is also possible to use a conformal transformation to map the infinite p-plane into a finite area. The variable resistors in Figure 2(a) are adjusted so that the current flow in each lead is the same. Positive unit currents are connected to flow into the paper at the points analogous to the pole locations of the function being similated, and negative unit currents are withdrawn at the zero locations. It can then be shown that the potential along the line on the conducting paper corresponding to the $j\omega$-axis is proportional to the logarithm of the magnitude of the rational function whose poles and zeroes are located as indicated in Figure 2(b). This potential can be measured experimentally with a moving probe and a vacuum-tube voltmeter as indicated. It is also possible to show that the current flow at right angles to the $j\omega$-axis in Figure 2(a) is proportional to the rate of change of the angle of the rational function having the poles and zeroes indicated in Figure 2(b).

The general approach to determining h/f with the potential analog is:

(1) Set-up a non-analytic function meeting the specifications (15) and (16) by introducing band electrodes into the electrolytic tank at the locations of the pass- and stop-bands and adjusting the potentials of these electrodes to meet the specifications; and

(2) replace these band electrodes by a row of needle electrodes carrying the same currents as the bands.

A band electrode carrying n unit currents into an electrolytic tank can be divided into n sections each carrying unit current as indicated in Figure 3(a). The potential along

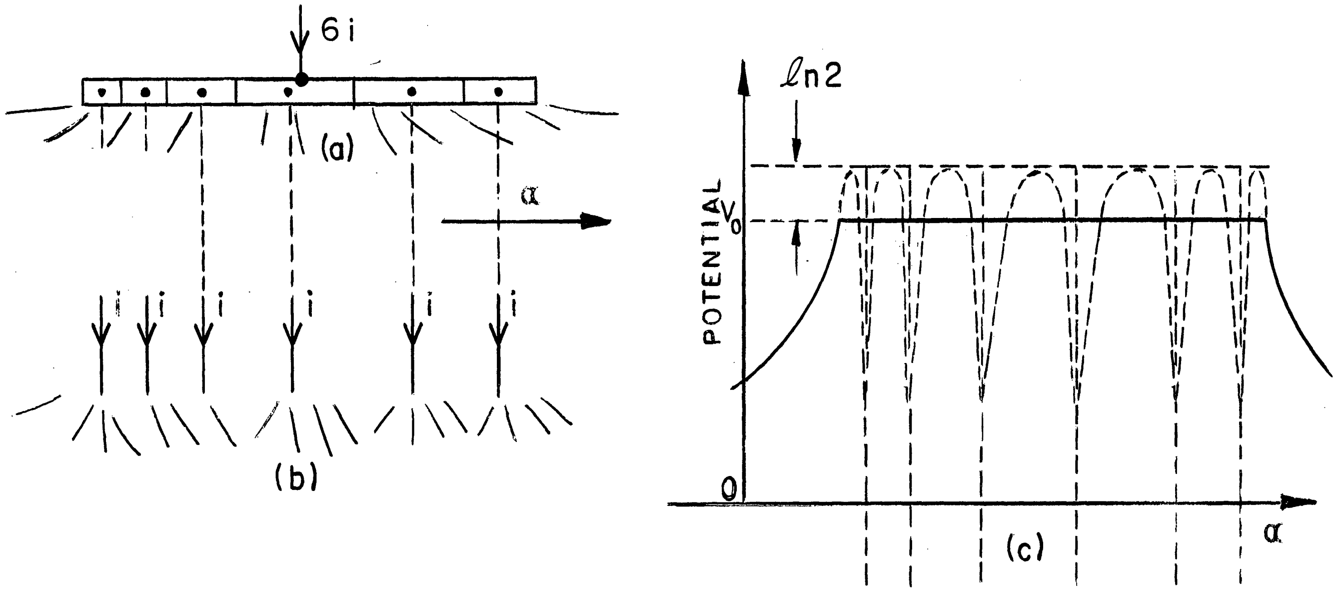

FIG. 3 (a) A BAND ELECTRODE, (b) AN EQUIVALENT ARRAY OF NEEDLE SOURCES, AND (c) POTENTIAL ALONG BAND AND ALONG THE NEEDLE ARRAY.

the line of such an electrode might be as indicated by the solid curve of Figure 3(c). If now this band electrode is replaced by a row of needle electrodes each carrying unit current and located at the midpoints of the equal current divisions of the band electrode as shown in Figure 3(b), it can be shown that the potential due to these needles will approximate that due to the band as indicated by the dashed curve in Figure 3(c).[1] Along the line where the electrode was located at a potential V_o the potential will now vary between $-\infty$ and $V_o + \ell n 2$ as indicated. At other points in the electrolytic tank the approximation will, of course, be much better.

Designating the non-analytic function set-up by the band electrodes as $\ell n|F(p)|$, the conditions that $F(p)$ must satisfy in order that when the band electrodes are replaced by rows of needle electrodes the conditions of Eqs. 15 and 16 will be met are:

$$\text{in the passbands, } P_v,\ \ell n|F|-\alpha \text{ must be } \leqq \frac{1}{2}\ell n\,(\Delta\alpha_v) \tag{17}$$

$$\text{in the stopbands, } S_v,\ \ell n|F|-\alpha_o \text{ must be } \geqq \alpha_v + \ell n\,(2) \tag{18}$$

The steps of a feasible, although not the most direct, experimental procedure are indicated in Figures 4 and 5.

[1]Klinkhammer, loc. cit., Section 3.4.

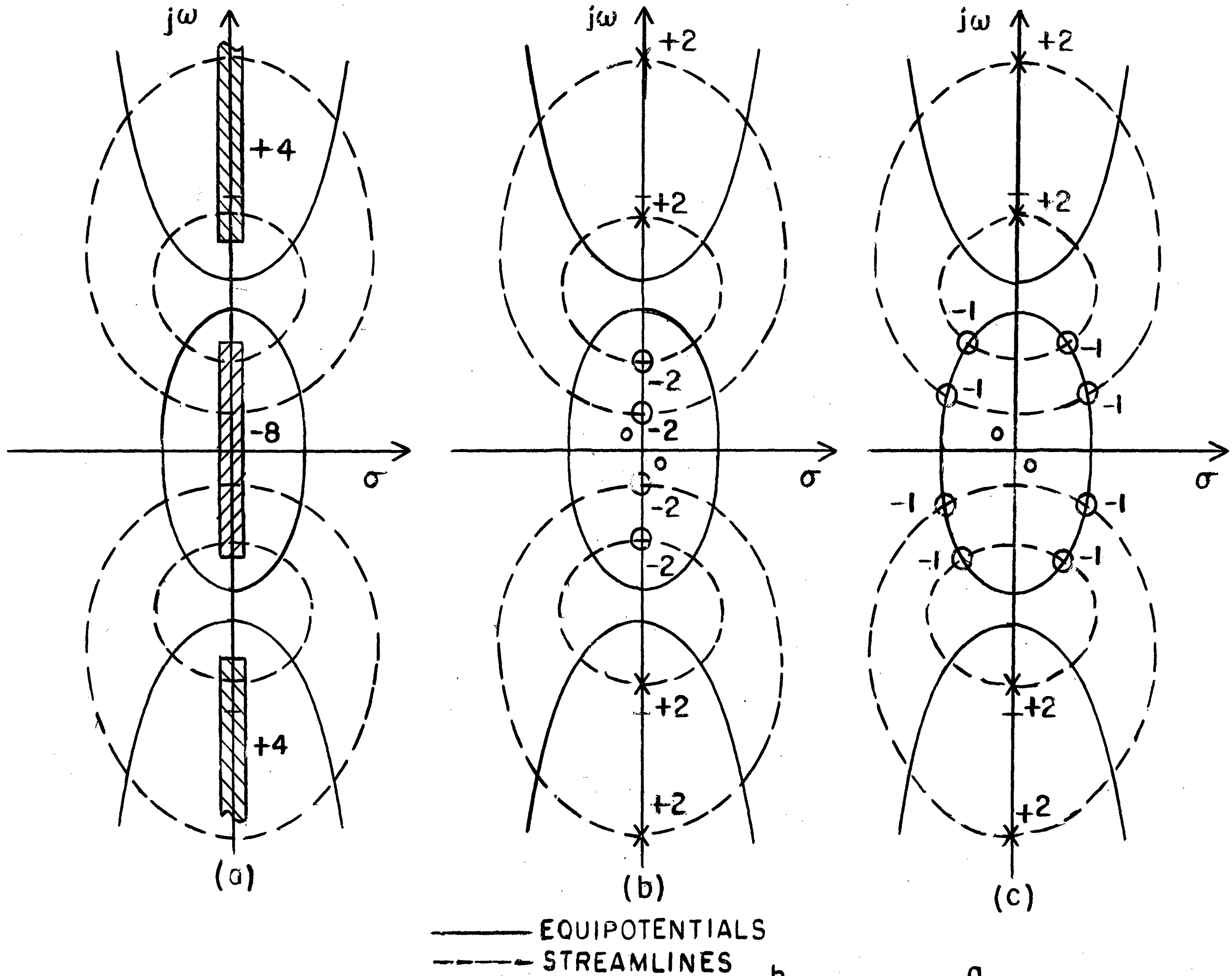

FIG. 4 POTENTIAL ANALOGS OF (a) $F(p)$, (b) $\frac{h}{f}(p)$ AND (c) $\frac{g}{f}(p)$ IN THE COMPLEX FREQUENCY PLANE

(1) Electrodes are introduced into the potential analog (electrolytic tank or conducting paper) at the locations where pass- and stop-bands are desired, which might be as indicated in Figure 3(a). A sufficient number of double unit currents are then permitted to flow to and from these electrodes to establish their potentials as required by Eqs. 17 and 18. The potential as measured along the jω-axis might then appear as shown in Figure 5(a); it is hypothesized, for the purposes of this example, that in order to establish these potentials it is necessary to force four unit currents into each of the stop-band electrodes and to withdraw eight unit currents from the pass-band electrode.

(2) The band electrodes are divided into elements each of which delivers or withdraws two unit currents from the potential tank or paper. In principle the distribution of current along the electrodes can be formed by tracing an equipotential curve running very close to the electrode. The distance between any point on this curve and the electrode is then proportional to the current per length flowing at that point. An alternate procedure will be developed shortly.

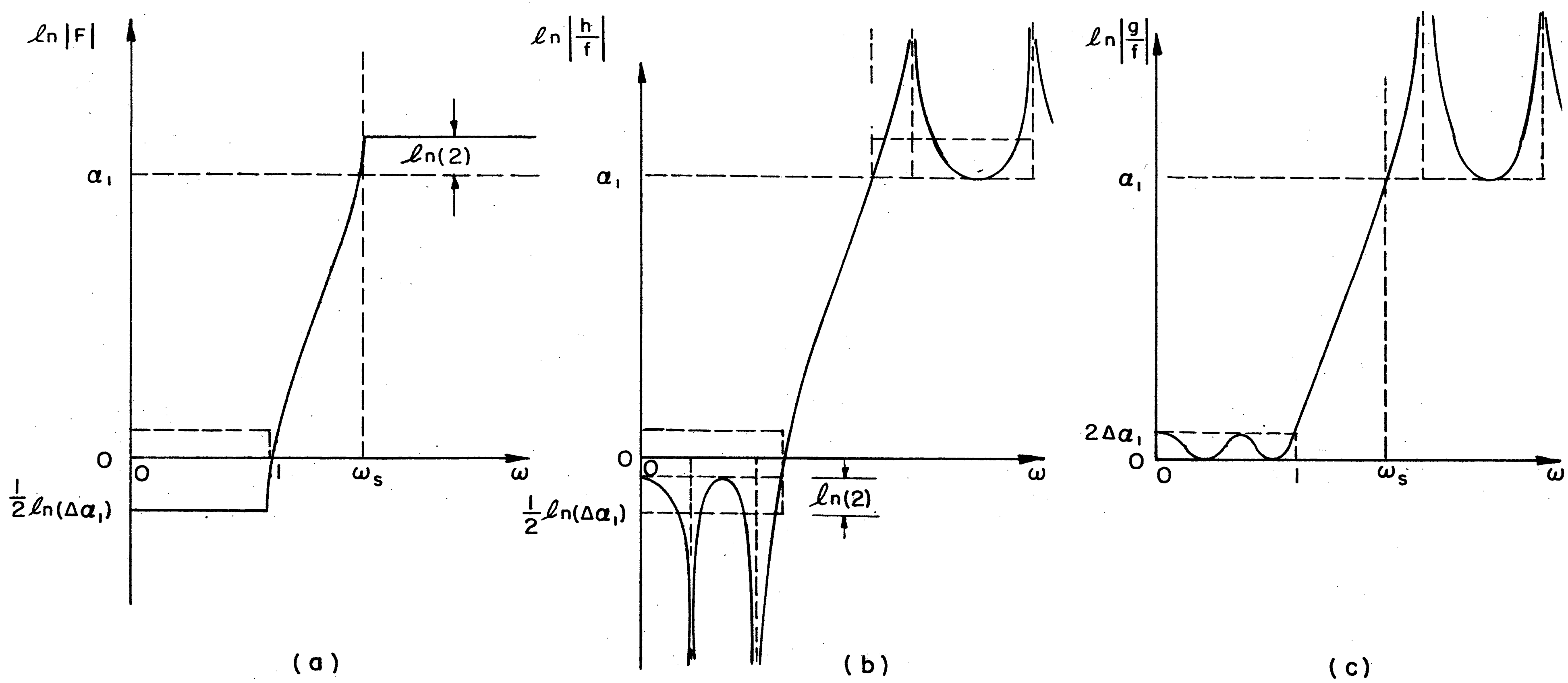

FIG. 5. POTENTIAL OR ATTENUATION FUNCTIONS ALONG THE $j\omega$-AXIS FOR A TYPICAL EXAMPLE OF: (a) THE NON-ANALYTIC FUNCTION F(p), (b) THE CHARACTERISTIC FUNCTION $\frac{h}{f}$(p), AND (c) THE TRANSFER FUNCTION $\frac{g}{f}$(p).

2262 A-SI-105 BM 11-15-55

(3) Needle electrodes are then located at the centers of the elements determined in (2), and double unit currents are forced to flow into or out of these needles. This is illustrated in Figure 4(b) where four needle electrodes are indicated in the pass-band and two in each stop-band. The potential distribution along the $j\omega$-axis corresponding to this new electrode configuration is shown in Figure 5(b). This function is the potential analog of the squared magnitude of the characteristic function $\frac{h}{f}$; the zeroes of this function are double and are located wherever a needle is withdrawing a double current from the analog while the poles are located where currents flow into the analog.

(4) The squared magnitude of transfer functions $\frac{g}{f}$ is related to the characteristic function $\frac{h}{f}$ by Eq. 4. Therefore adding the constant $e^{2\alpha_o}$ to the function in Figure 4(b) gives the potential analog of $|\frac{g}{f}|^2$ as shown in Figure 4(c). One can show that the only effect of adding a constant to $|\frac{h}{f}|^2$ is to cause the zeroes to move along the streamlines from their original positions toward the poles. The zeroes of $|\frac{g}{f}|^2$ will lie where

$$\frac{h(p)}{f(p)} \cdot \frac{h(-p)}{f(-p)} = e^{-2\alpha_o} \tag{19}$$

Taking the logarithm of both rides gives

$$\ln \left|\frac{h}{f}\right|^2 = -2\alpha_o \tag{20}$$

which means that the zeroes of $\frac{g}{f}$ are located on the $-2\alpha_o$ equipotential in the $\frac{h}{f}(p)$ or the F(p) potential plane. In particular, if the minimum passband attenuation is zero nepers this is the zero equipotential. The potential along the $j\omega$-axis for the pole-zero configuration of Figure 4(c) is shown in Figure 5(c).

Comparing Figures 4(a) and (c) one sees that the poles and zeroes of the transfer function $|\frac{g}{f}(p)|^2$ are located at the intersections of certain equipotentials and streamlines of the potential analog of the function F(p). If the minimum pass-band attenuation is zero, the zeroes (points of infinite gain) are located on the zero equipotential; and the poles (points of infinite loss) are located on the $\alpha_1 + \ln(2)$ equipotential - on the $j\omega$-axis. If one sets up the potential analog of F(p), it is a straight forward process to experimentall determine the location of these equipotentials. To complete the locations of the poles and zeroes, however, the stream lines in this potential analog must also be located.

For the function

$$\ell n\ F(p) = \ell n\ |F(p)| + j \arg F(p) \qquad (21)$$

illustrated in Figure 4(a) the real and imaginary part would vary along the jω-axis as indicated in Figure 6. In our potential analog the potential along the jω-axis is proportional to $\ell n\ |F|$, and the current flow into the tank is proportional to arg F. Thus

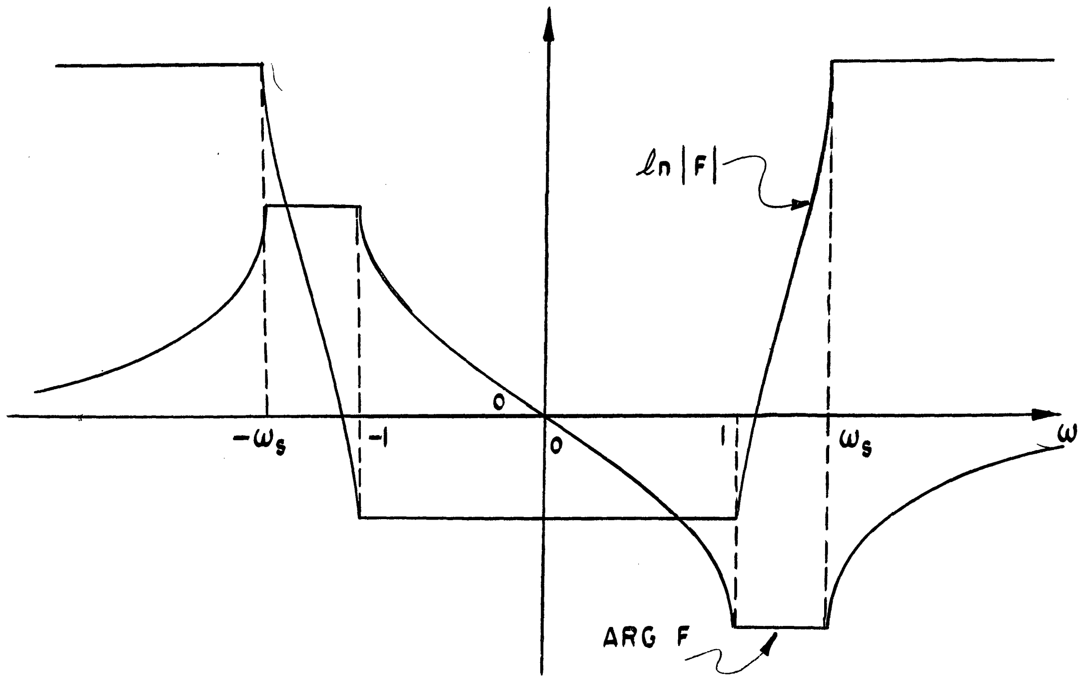

FIG. 6 REAL AND IMAGINARY PARTS OF THE LOGARITHM OF THE NON-ANALYTIC FUNCTIONS F (jω).

between the equipotential bands, there is no current flow into the tank, and the arg F is necessarily a constant. A method of measuring arg F(p) directly is to set up the conjugate potential analog of F(p). In this conjugate configuration the equipotentials are the stream lines of the original configuration. We know that the stream function is a constant along the jω-axis in every transition region between a pass-band and a stop-band. The stream function will change by an amount proportional to the current flow from the band electrodes as one traverses a band between two transition regions. Experimentally then, one can set up the conjugate potential map by withdrawing the original set of band electrodes, corresponding to the pass- and stop-bands of the proposed filter design, and inserting a new set of band electrodes located in all the transition regions of the filter specifications. The potentials between these new electrodes should be made proportional to the number of double unit currents flowing to the band electrode which was originally located between them.

This is illustrated in Figure 7. With the electrode configuration shown in Figure 7(a) it is established that four unit currents must flow to each stop-band electrode and eight unit currents be withdrawn from the passband electrode to satisfy the specifications of Eqs. 17 and 18. The equipotential V=0 can then be traced as indicated. Next the conjugate electrode configuration is set up as indicated in Figure 7(b). There is only a single pair of transition bands in the example illustrated. If the potentials of these electrodes are made $+\frac{V_o}{2}$ and $-\frac{V_o}{2}$ respectively, then V_o is proportional to the four double unit currents flowing from the pass-band electrode in Figure 7(a). The equipotentials in 7(b) corresponding to the streamlines which pass through the centers of the section of the pass band electrode in 7(a) carrying double unit currents are at

$$V = -\frac{V_o}{2} + \frac{(2k-1)}{2n} V_o \qquad k = 1, 2, \ldots, n \tag{22}$$

where n is the number of double unit currents. In the example illustrated n = 4.

Now the poles of g/f lie at the intersections of the "streamline" equipotentials in Figure 7(b) with the $\alpha_1 + \ln(2)$ equipotentials in 7(a) and the zeroes lie at the intersection of these same streamlines with the V = 0 equipotential in 7(a). For the example in Figure 7 the pole locations are indicated by crosses and the zero locations by circles.

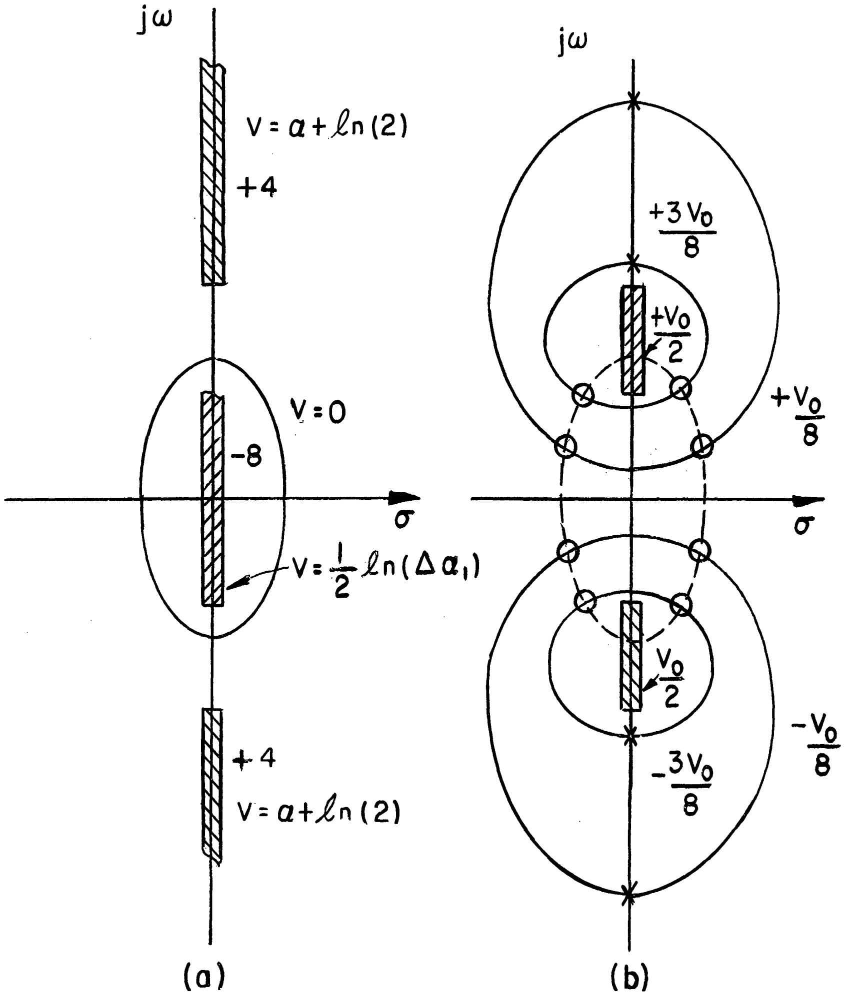

FIG. 7 (a) AN ELECTRODE CONFIGURATION ESTABLISHING THE FUNCTION $\ln F(p)$, AND (b) THE CONJUGATE ELECTRODE CONFIGURATION

V. SUMMARY OF THE EXPERIMENTAL PROCEDURE

The experimental procedure necessary to determine the poles and zeroes of the transfer function $\frac{g(p)}{f(p)}$ such that the attenuation, Eq. 3, satisfies the specifications that, in the passbands, P_v

$$0 \leq \ln \left|\frac{g}{f}\right| - \alpha_o \leq \Delta\alpha_v \; , \tag{23}$$

and in the stopband, S_v

$$\alpha_v = \ln \left|\frac{g}{f}\right| - \alpha_o \leq \infty \; , \tag{24}$$

can be summarized as follows:

(1) Find the proportionality constant C of the electrolytic tank or conducting paper. To do this one introduces two needle electrodes into the electrolytic tank, one carrying a positive unit current and the other a negative unit current, located at let us say, $p = \pm a$ as indicated in Figure 8. Then the potential difference V_o between the

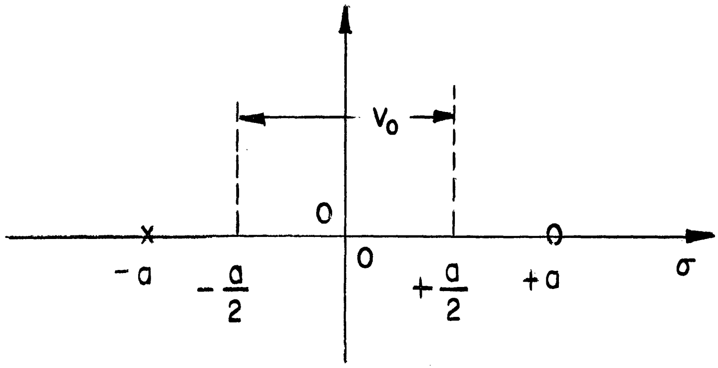

FIG. 8 SET-UP FOR CALIBRATION OF THE POTENTIAL ANALOG OF THE COMPLEX FREQUENCY PLANE.

points $\pm \frac{a}{2}$ is given by

$$V_o = C \ln \left(- \frac{a}{2} + a\right) - C \ln \left(- \frac{a}{2} - a\right) - C \ln \left(\frac{a}{2} + a\right) + C \ln \left(\frac{a}{2} - a\right) = -C \ln (9) \tag{24}$$

and the proportionality constant of the tank is

$$C = - \frac{V_o}{\ln(9)} \tag{25}$$

(2) Put band electrodes into the tank (or paint band electrodes on the conducting paper) along those parts of the $j\omega$-axis where pass- and stop- bands are desired. Then feed an integral number of unit currents to each band until:

(a) the potential of each passband electrode where an attenuation ripple of $\pm \Delta\alpha_v$ is desired becomes

$$V_{p_v} = \frac{C}{2} \ln \Delta\alpha_v ; \tag{26}$$

and (b) the potential of each stop band electrode where a minimum attenuation of α_v is required satisfies the relation

$$V_{s_v} = C\left[\alpha_v + \ln (2)\right]. \tag{27}$$

If the attenuation in the stopbands is further restricted to vary between $\alpha_v - \Delta\alpha_v$ and $\alpha_v + \Delta\alpha_v$, the stopband electrode potentials should be

$$V_{s_v} = C\left[\alpha_v - \frac{1}{2}\ \ln\ (e^{4\Delta\alpha_v'} - 1) + \ln\ (2) + 2\Delta\alpha_v'\right] \qquad (28)$$

which simplifies to

$$V_{s_v} = C\left[\alpha_v - \frac{1}{2}\ \ln\ \Delta\alpha_v'\right] \qquad (29)$$

if $\Delta\alpha_v \leq 0.1$.

(3) Check whether there are portions of any of the stopband electrodes at a potential below the electrolyte potential in the immediate vicinity of the electrodes. If this occurs, shorten these electrodes until this is no longer true. Make the same check for the passband electrodes.

(4) Check the band potentials with these shortened electrodes and determine whether the potentials required in (2) can be achieved with fewer unit currents.

(5) Plot the equipotential

$$V = 0\ ; \qquad (30)$$

and if $\Delta\alpha_v'$ is not infinite, the equipotential

$$V_v = \alpha_v \qquad (31)$$

(6) Remove the band electrodes inserted in (2) and replace them by the conjugate set. There will now be band electrodes in the tank wherever transition regions in the specified attenuation characteristic occur. Adjust the currents flowing to these new electrodes so as to make the potential difference between any adjacent electrodes equal to N times the current taken by the intervening electrode in the original electrode configuration (N is an arbitrary constant).

(7) Designating the lowest potential in this conjugate configuration as the zero reference potential, plot out the equipotentials

$$V_m = N(\frac{1}{2} + m) \qquad (32)$$

where m = 0, 1, ... until the highest potential in the tank is exceeded.

(8) The intersections of the equipotentials of Eq. 32 with that of Eq. 30 locate the zeroes of

$$\frac{g(p)}{f(p)} = \frac{1}{t(p)} \; ; \; \frac{1}{Z_{12}(p)} \; , \text{ etc.}$$

while the intersections with the equipotential of Eq. 31 locate the poles.

VI. AN EXAMPLE OF THE EXPERIMENTAL PROCEDURE

In applying Klinkhammer's method it is convenient to use Teledeltos conducting paper for the potential analog.[1] Electrodes are painted on this paper with an air-drying conducting silver paint.[2] A typical pole-zero configuration for a network function is shown in Figure 9(a). By symmetry one sees that the current flow across the line corresponding to the σ-axis will always be zero. It is therefore always possible for experimental purposes to replace the entire p-plane analog by a semi-circular analog corresponding to the upper or lower half of the p-plane only. If in this process any current sources on the σ-axis are halved, no change in the potential or current flow across the $j\omega$-axis is introduced. This process can be carried one step further when one recalls that adding poles and zeroes in the right half plane at the mirror images of those in the left half plane has the effect of doubling the attenuation (analog: potential) along the $j\omega$ axis and reducing the phase shift to zero. This same effect can be obtained by cutting our half-plane potential analog in half again leaving the quarter plane analog as illustrated in Figure 9(c). If the current sources corresponding to poles and zeroes within the plane are halved and those on the axes are reduced to one quarter of their values in Figure 9(a), the potentials measured in the Figure 9(c) analog will be exactly the same as those in 9(a).

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[1] Teledeltos paper is manufactured by the Western Union Telegraph Co., 60 Hudson Street, New York 13, N. Y. The paper used, type L-39, here has a resistance of 1500 ohms/square.

[2] A suitable paint is General Cement Mfg. Co's. "Silver Print", No. 21-2.

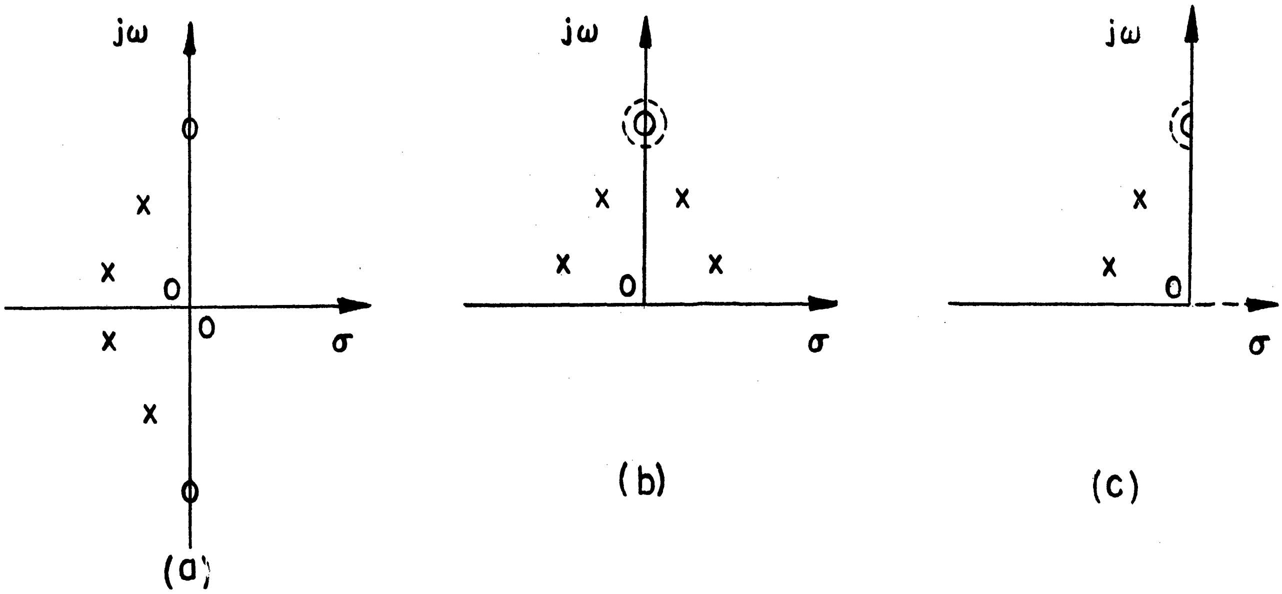

FIG 9. REDUCTION OF THE POTENTIAL ANALOG TO AN EQUIVALENT QUARTER PLANE ANALOG FOR THE MAGNITUDE OF A NETWORK FUNCTION.

The reason for wishing to reduce the potential analog to a quarter plane lies in the desirability of reducing errors due to the finite size of the analog plane. Normally in the potential analog the point at infinity is replaced by a circular electrode at a radius large compared to the distance from the origin to the region where measurements are to be made. Experimentally it has been determined that if the radius of the tank is ten times the radius of that portion used for measurements the attenuation errors introduced do not exceed 0.4 db. For a given maximum dimension of the electrolytic tank or conducting paper use of the quarter plane allows one to double the frequency scale (inches/rad/sec.) employed. For example, teledeltes paper is available in rolls 31 inches wide. With a radius of 31 inches for the quarter plane analog measurements can be made from the origin cut to a circle 3.1 inches from the origin.

If a larger useful region is desired for the analog, some scheme for overcoming the error due to finite tank size must be employed. One approach is to employ a conformal transformation which maps the entire p-plane into a rectangle.[1,2,3] In this case infinity maps into a single point at a finite distance from the origin; the drawback of this approach is the necessity of transforming the locations of the poles and zeroes in the rectangular ω-plane back into the p-plane through the transformation

$$p = j\ \mathrm{sn}(w,k) \tag{33}$$

When w is complex, evaluation of p from Eq. 33 requires making six entries into elliptic function tables in which interpolation in two directions is required.

Another approach, is to employ a double-layer construction of the potential tank.[3] The lower layer is connected to the upper along the circular boundary and represents an inversion of that part of the p-plane lying outside the boundary. The points corresponding to p=0 and $p=\infty$ are located at the centers of the upper and lower layers of the tank as indicated in Figure 10(a). This same idea might be employed with two sheets of conducting paper insulated everywhere except at the boundary as indicated in Figure 10(b).

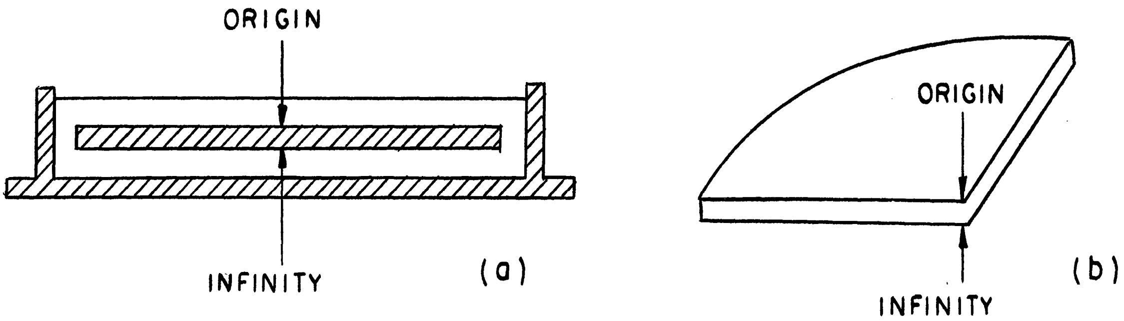

FIG. 10 TECHNIQUE FOR SIMULATING THE INFINITE P-PLANE WITH A FINITE (a) ELECTROLYTIC TANK OR (b) CONDUCTING SHEET.

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1 Fano, R. M. "A Note on the Solution of Certain Approximation Problems in Network Synthesis", Jan. Frankl. Inst. , Vol. 249, No. 3, pp. 189-205, March, 1950.

2 G. L. Matthaei, "Conformal Mappings for Filter Transfer Function Synthesis", Proc. IRE, Vol. 41, pp. 1658-1664.

3 A. R. Boothroyd, E. C. Cherry, and R. Makar, "An Electrolytic for the Measurement of Steady-state Response, Transient Response, and Allied Properties of Network," Proc. IEE, Part 1, Vol. 196, p. 163, May 1949.

As an illustration of this approximation method, consider a filter specified as follows:

$0 \leq \omega < 1.0$	no specification
$1.0 \leq \omega \leq 1.546$	$0 < \alpha < 0.5$ db
$1.546 < \omega < 1.728$	no specification
$1.728 \leq \omega \leq 1.91$	$20 \text{ db} < \alpha$
$1.91 < \omega \leq \infty$	no specification

Converting the specification to nepers gives

$$\Delta\alpha_1 = .0289 \text{ nepers}$$

for $1 \leq \omega \leq 1.546$ radians sec., and

$$\alpha_1 > 2.31 \text{ nepers}$$

for $1.728 \leq \omega \leq 1.91$ rad./sec.

Using the teledeltors paper type L-39 and a 22 1/2 volt battery in series with a 34,000 ohm resistor as a unit current source, the constant of the tank was found to be -0.91 units on the particular vacuum tube voltmeter employed. Then according to Eq. 26 the potential of the passband electrode should be

$$V_p \geq \frac{-0.91}{2} \ln(.0289) = 1.61,$$

and from Eq. 27 the potential of the stopband electrode should be

$$V_s \leq -0.91(2.31 + .693) = -2.76$$

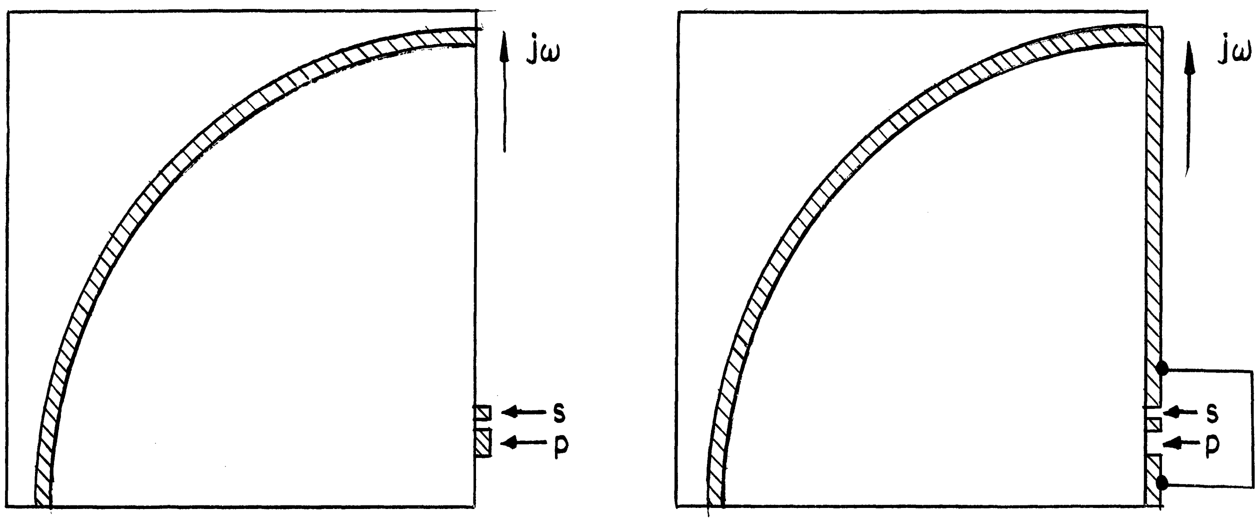
Since, for fixed currents, using the quarter tank analog doubles the potentials everywhere, these requirements become

$$V_p \geq 3.22$$

$$V_s \leq -5.52$$

or that the difference in potential between the pass- and stop-band electrodes must exceed 8.74.

Electrodes were painted on a piece of teledeltos paper as indicated in Figure 11(a); the radius of the quarter sheet employed was 18 inches and the frequency scale was 2 inches per radian per second



(a) (b)

FIG. 11 (a) QUARTER PLANE ANALOG OF EXAMPLE WITH PASS- AND STOP-BAND ELECTRODES AND (b) THE CONJUGATE ELECTRODE CONFIGURATION.

The painted areas are crosshatched. Connecting a single positive unit current lead to the passband electrode (p) and a single negative unit current to the stop electrode (s), the measured potential was $V_p - V_s = 7.0$. This is less than the required value of 8.74, but when two unit currents are used to each electrode $V_p - V_s = 14$.

At this stage one would normally trace out the equipotential $V = V_p - 3.22$ on which the poles should be located in accordance with Eq. 30. Since, however, the observed potential difference far exceeds the minimum required by the specification, some modifications of the design is possible. Since there is no pass- or stop-band electrode at the origin, odd numbers of poles were not considered worthwhile. One could, arbitrarily locate a pole on the negative real axis and a zero at infinity, but for the configuration being considered one would not expect such a pole-zero pair to effect the potential between the pass- and stop-band electrodes appreciably. It was therefore decided to take advantage of the observed excess potential to exceed the given specifications. The equipotential

$$V = V_p - 5$$

was traced out as indicated by the solid line in Figure 12. This means the ripple in the pass-band should be reduced to

$$2\Delta\alpha_1 = 2\,\mathrm{Exp}\left[\frac{2V_p}{C}\right] = .0081 = .07 \text{ db},$$

and the stop-band attenuation should be increased to

$$\alpha_1 = \frac{V_s}{C} - \ln(2) = 4.26 \text{ nepers} = 37 \text{ db}$$

The conjugate electrode configuration was then painted on a second sheet of teledeltos paper as indicated in Figure 10(b). Current was injected into the transition band electrode (between s and p in Figure 10(b)) and withdrawn from the electrode along the rest of the $j\omega$-axis. The magnitude of this current was adjusted until the transition band electrode was at a potential of 4.0 relative to the electrode along the balance of the axis. The equipotentials $V = V_t - 1$ and $V_t - 3$ were then traced. These are the dashed lines in Figure 12.

The intersections of the dashed and solid lines in Figure 12 locate the poles of the transfer function $f(p)/g(p)$ at

$$p = -.55 \pm j\,1.57\ ,$$

$$p = -.085 \pm j\,1.583\ ;$$

and the zeroes of the function should lie at the intersection of the dashed lines with the $j\omega$-axis:

$$p = \pm j\,1.745$$

$$p = \pm j\,1.88\ .$$

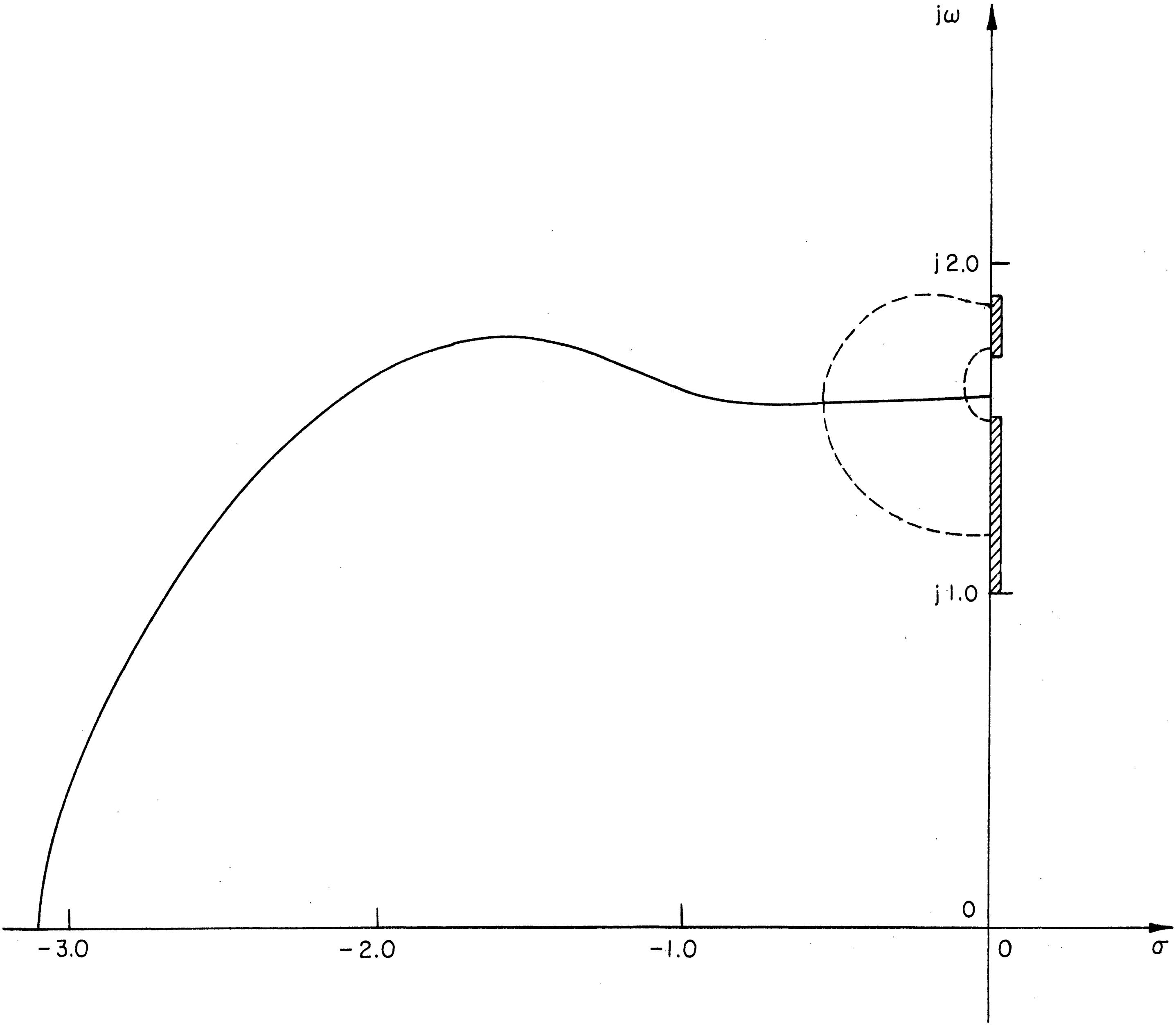


FIG. 12. POTENTIAL PLOTS FOR POLE-ZERO DETERMINATIONS.

The complete transfer function can be written as

$$\frac{f(p)}{g(p)} = K \quad \frac{(p^2 + 3.045)(p^2 + 3.534)}{(p^2 + 1.11p + 2.773)(p^2 + .17p + 2.533)}$$

where the constant K determines the reference gain. Choosing K = 0.621 gives a reference gain of zero decibels; for this case the calculated attenuation is plotted in Figure 13. One finds, from this plot, that the attenuation does not meet the specification (dotted lines) at the high frequency end of the pass-band.

From the position of the poles it is apparent that the poles at $p = -.085 \pm j1.583$ have the greatest influence on the attenuation at the edge of the passband. Moving this pair of poles to

$$p = -.077 \pm j\,1.583$$

and choosing a new multiplying constant K = 0.595 yields the attenuation characteristic plotted in Figure 14.

VII. CONCLUSIONS

Klinkhamer has developed a practical experimental procedure for determining the poles and zeroes of a rational network function satisfying a specification of constant attenuation levels in a number of pass- and stop-bands. This procedure, although experimental in nature, is not a cut and try process; the poles and zeroes are located at the intersection of certain well defined equipotential lines. These equipotentials are conveniently located with the aid of an electrolytic tank or a sheet of conducting paper.

An important feature of the network functions obtained by this method is that they are optimum in that they meet the specifications with the minimum number of poles and zeroes, which in turn means that the networks synthesized to have these characteristics will have a minimum number of elements.[1] If the specifications of the example of Section 6 were met with a bandpass filter derivable from a low-pass filter, for example,

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[1] Klinkhamer, loc. cit. p. 395.

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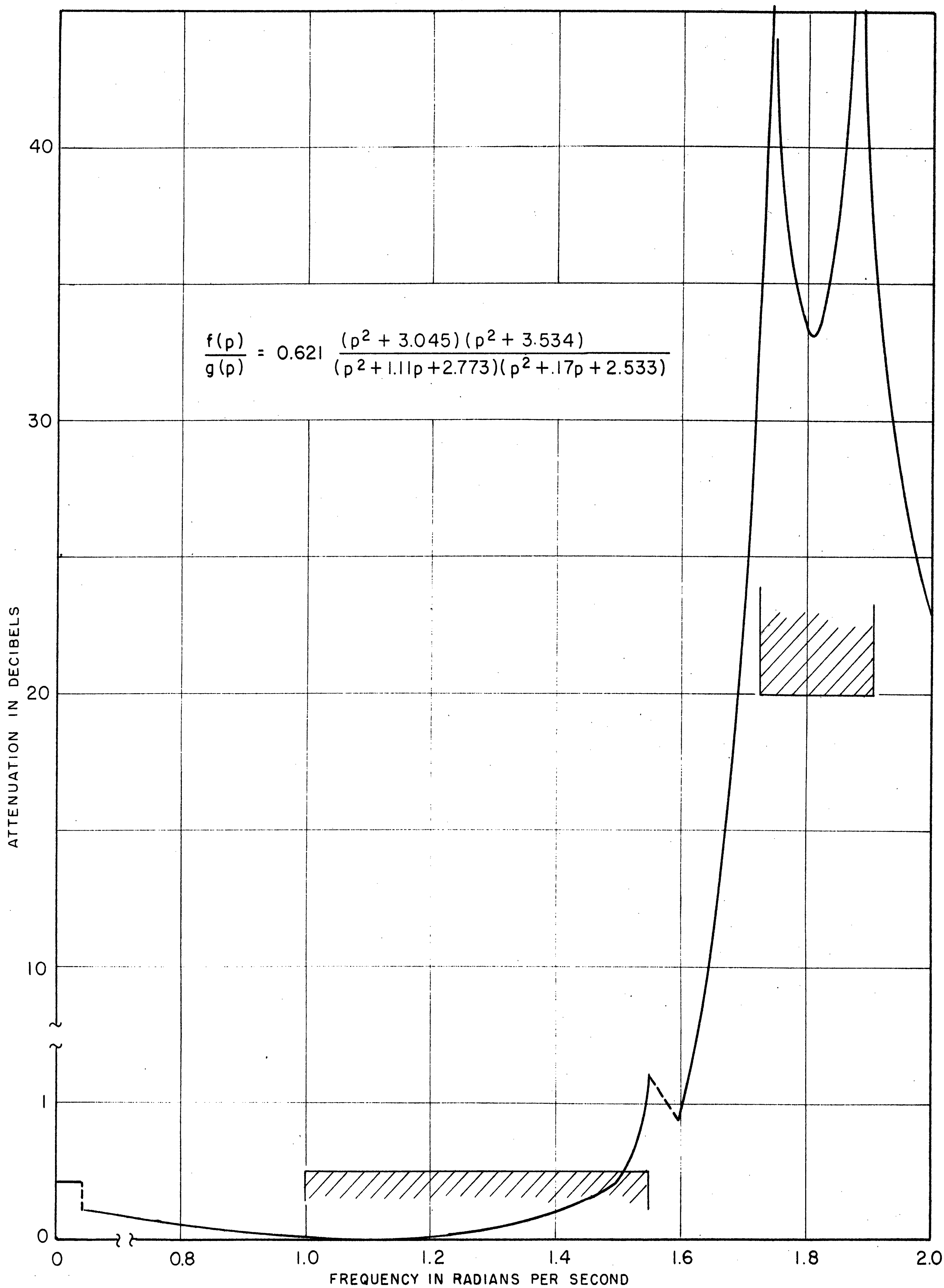

FIG. 13. RESULTS OF EXAMPLE

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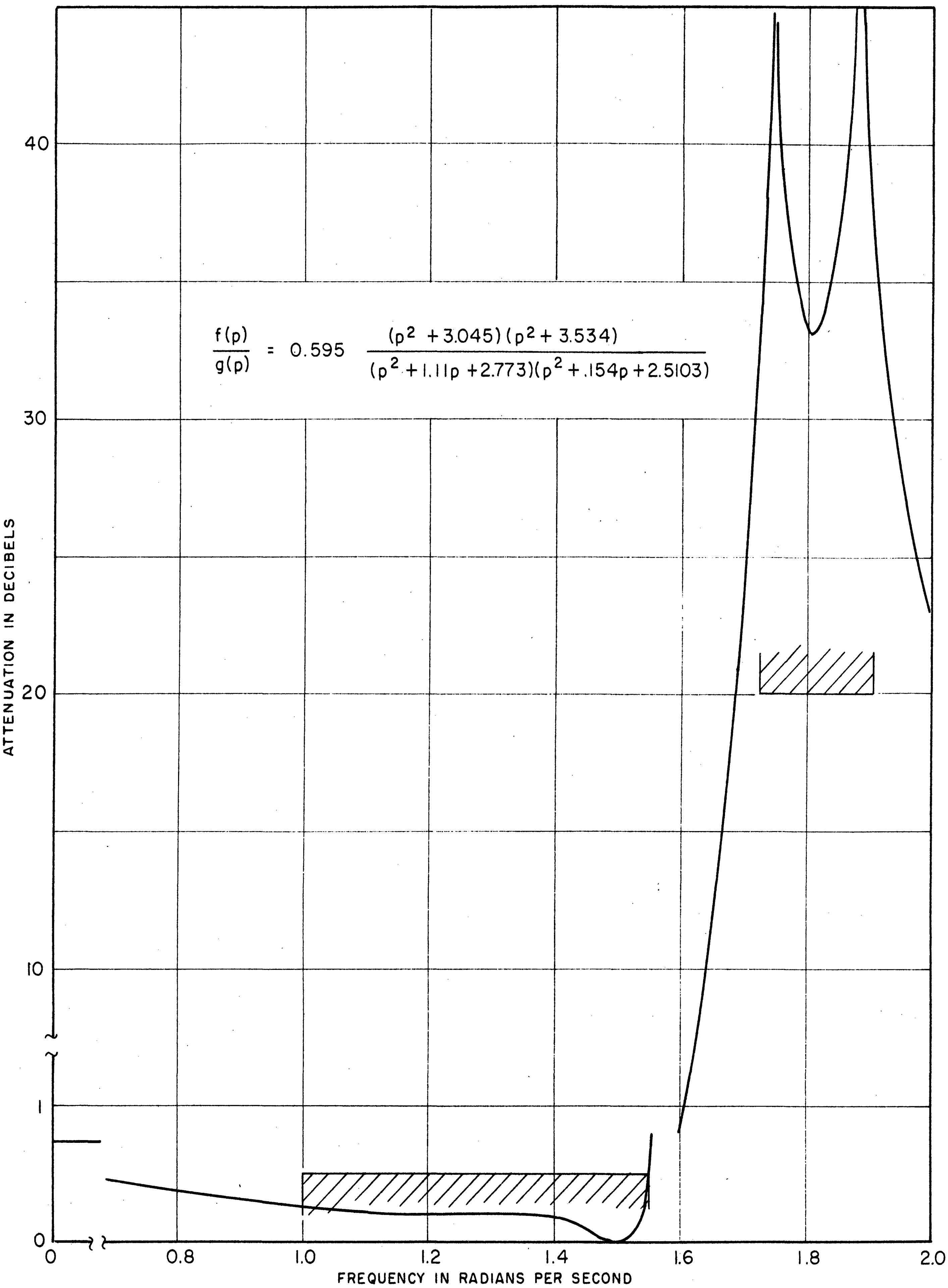

FIG. 14. EXAMPLE AFTER POLE READJUSTMENT

a 6 pole function would be required. Such a filter would have a stop-band s from $\omega = 1.73$ rad/sec to infinity and from zero to 0.895 rad/sec. If on the other hand the specification is met with a low pass design having a passband from zero to 1.55 rad/sec and a stop band from 1.73 rad/sec to infinity, four poles are sufficient. The function obtained here, however, provides much more attenuation in the specified band from 1.73 to 1.91 rad/sec with the same number of poles.

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