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A STUDY OF FREE-SURFACE UNSTEADY GRAVITY FLOW THROUGH POROUS MEDIA

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PREFACE

The solution to the problem of unsteady free-surface gravity flow of liquids through a porous dam with vertical faces is the goal of this study. Since the results of the steady state case of this problem are to be used in the solution of the unsteady state case, Part I of this thesis is devoted to an extension of the existing results of the steady state case and their presentation in a more applicable form.

In Part II of this thesis, the differential equations governing the unsteady gravity flow of liquids through porous media are first derived. Next, the numerical solution of these equations is presented.

In Part III, the applicability of a viscous flow analogy to this problem is demonstrated. The experimental setup and the test data are discussed next. Finally, by comparing the experimental data with the results obtained through the numerical solution of the differential equations, the conclusions with regard to the limitations on the assumptions used in the theoretical part of this study are drawn.

Because of the large amount of numerical work, digital computers were used throughout this study. The computer programs written for the numerical solution of the differential equations of the unsteady state case are discussed and included in this thesis.

The author wishes to express his gratitude to Professor Victor L. Streeter for his suggestion of the topic and his valued advice throughout the course of this study. The author is also indebted to the members of his doctoral committee and would like to express his appreciation for their helpful counsel.

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NOMENCLATURE

Symbol	Unit	Description
b	ft	Width of the interspace in a Hélé-Shaw Model
f	--	Porosity, defined as volume of the voids/Total volume
g	ft/sec^2	Acceleration of gravity
$-h(X)$	ft	Depth of the liquid when motionless
H	ft	Depth of the liquid when in motion
H_e	ft	Depth of the liquid at the inflow surface
H_w	ft	Depth of the liquid at the outflow surface
H_s	ft	Height of the surface of seepage
H_o	ft	Depth of the liquid at the left boundary
H_L	ft	Depth of the liquid at the right boundary
$\bar{k}$	ft/sec	Coefficient of permeability
$\bar{k}_h$	ft/sec	Hypothetical coefficient of permeability
L	ft	Length of the porous medium
m	--	Slope of the impervious bottom
P	lb/ft	Pressure at any point
ΔR	lb	Resisting force on a liquid element
t	sec	Time
u	ft/sec	Velocity (horizontal component) of liquid through porous media
$\bar{u}$	ft/sec	Seepage velocity (horizontal component)
v	ft/sec	Velocity (vertical component) of liquid through porous media
V	ft/sec	Velocity of liquid through porous media

Symbol	Unit	Description
V_n	ft/sec	Velocity component normal to the impervious bottom
u_x, v_x	sec^{-1}	$\frac{\partial u}{\partial x}$, $\frac{\partial v}{\partial x}$
H_x	--	$\frac{\partial H}{\partial x}$
u_t, v_t	ft/sec^2	$\frac{\partial u}{\partial t}$, $\frac{\partial v}{\partial t}$
H_t	ft/sec	$\frac{\partial H}{\partial t}$
x	ft	abscissa
y	ft	ordinate
μ	slug/ft-sec	Viscosity of the liquid
ν	ft^2/sec	Kinematic viscosity of the liquid
ρ	$slug/ft^3$	Density of the liquid
η	ft	Height of undulations
σ	lb/ft	Surface tension of the liquid

INTRODUCTION

As mentioned in the preface, this study will deal only with a two-dimensional problem. It will also be assumed that compressibility effects are negligible. The effect of surface tension is neglected in the development of the fundamental equations. The capillary rise due to surface tension can, however, be estimated[10] and considered in the evaluation of the experimental results.

Other basic assumptions are:

1. Density of the liquid is constant
2. The porous medium has constant permeability and is isotropic
3. The region of flow is fully saturated
4. The temperature of the liquid is constant.

From Darcy's law and the equation of continuity, it is known that there must exist a velocity potential function that satisfies the Laplace equation. Although the existence of a velocity potential implies irrotational flow,[10] one must remember that since Darcy's law deals only with velocities in a macroscopic and not microscopic way, the flow through a porous medium is not irrotational.

Because of the existence of a velocity potential function, one may at first glance think that the problem reduces to one of finding a velocity potential function which satisfies both the Laplace equation and the boundary conditions. If, however, one recalls that one of these boundaries, namely the free surface, is the solution to the problem and hence is unknown a priori, the difficulty inherent in the nature of the problem becomes evident.

The solution to the problem of confined flow, whether steady or unsteady, of fluids through porous media is known.(3) It is a much simpler problem because the geometry of the region of interest is well-defined beforehand. In the problem of unconfined flow of liquids, however, the geometry of the free surface is not known. In these kinds of problems, the only condition assigned to the free surface is that the pressure at the free surface must equal atmospheric pressure. In the case of steady flow, the free surface is also a stream line.

Another difference between unconfined and confined flow is due to the effect of gravity. In the former case, gravity enters the equations explicitly, while it has only an implicit effect on the latter.

In the study of unconfined flow in porous media, it is difficult to take care of the capillary layer that necessarily overlies the main fluid body in the porous medium (e.g., sand). Although it might be thought at first that the capillary layer does not affect the flow, closer consideration shows that, on the contrary, it acts much like a siphon,(3) acting in the direction of the main flow and thus increasing it. The thickness of the capillary zone depends on the size of the particles of the porous medium. When the main fluid height is close to the top of the sand, however, so that the capillary layer is not completely developed, the flow in that layer may be neglected. When the thickness of the capillary zone is large compared to that of the main fluid body, the amount of the flow in the capillary zone can by no means be neglected.

Both the top and bottom surfaces of the capillary zone are at atmospheric pressure, and a well dug in the sand identifies the free surface of the main fluid body.

A Hélé-Shaw Model was used in this study in order to obtain experimental results since it is a good representation of a porous medium which meets most of the assumptions stated in the beginning of this part. When one chooses a sufficiently large value for the width of the channel in the model, the amount of capillary rise can be reduced to a negligible value.

In general, the steady unconfined flow problems can be easily solved by Dupuit's assumptions[3] (presented in 1863) which, essentially are based upon the hypotheses: (1) the liquid in a gravity flow system moves in shells (i.e., the stream lines are horizontal), the horizontal velocity being independent of the depth, and (2) the value of the horizontal velocity is proportional to the slope of the free surface. The above assumptions were questioned in 1927.[3] From that time on, attempts have been made to attack unconfined flow problems by direct potential theory methods. Special cases of such problems have been solved by the methods of conformal mapping and conjugate functions.[3] Relaxation methods can also be of use in solving unconfined flow problems.[12]

The problem of unsteady gravity flow of liquids through porous media has not yet been solved by exact potential theory, even for special cases of simple boundary geometry. Some examples of this problem have been solved only under certain assumptions.[4,16]

In this study, an attempt has been made to apply the equations for shallow water waves[7] to the problem of unsteady gravity flow of liquids through porous media. The main supposition underlying those equations is that the pressure is hydrostatic in any vertical section throughout the region of flow. It follows from the above hypothesis that the horizontal component of the velocity is independent of the depth. The resultant equations can then be solved by the method of characteristics.(6)

Because of the existence of a seepage surface(3) (see Figure 1) at the outflow boundaries, the above assumptions become fallacious near these boundaries. The following two assumptions are therefore made to compensate for the failure of previous assumptions near the outflow surfaces:

1. The elevation of the liquid in the reservoir adjacent to the outflow surface is assumed to be at the same elevation as the top of the surface of seepage.
2. The coefficient of permeability of the porous medium is increased in such a way as to keep the amount of discharge for the theoretical steady state case (see Section C, Part I) constant in spite of supposition 1. This is done because the fundamental equations of unsteady flow, derived in Part II, give the correct fluxes (see Section F, Part II).

Supposition 1 can be easily applied in the solution of unsteady gravity flow for the particular geometry concerning the problem in this study. The only assumption which has to be made is that the height of the surface of seepage in the unsteady case is the same as in the case of the steady state when the following two conditions are present:

1. The outflow height H_w (see Figure 1) in the unsteady state case is the same as the one in the steady state problem.
2. The discharge through the outflow surface of the unsteady state problem is the same as the discharge through the dam in a steady state.

Based on the above conditions, one can, as mentioned later in the text, solve the fundamental equations of unsteady gravity flow, derived in Part II of this study, by the method of characteristics as follows:

a. Disregarding the existence of a seepage surface at the outflow boundary, solve the fundamental equations of the unsteady gravity flow of liquids through porous media for the variables H (height of the free surface) and u (the horizontal component of the velocity). One can therefore compute the value of discharge through the outflow surface as a function of time t. One can also calculate the height of the surface of seepage as a function of time (see Section C, Part I).
b. Apply assumptions 1 and 2 mentioned above, and solve the fundamental equations of flow for the variables H and u.

As one can see from the results obtained, this theory provides a larger range of validity than the approximate theories suggested by other authors.[4,16,17]

I

THE STEADY STATE SOLUTION TO THE PROBLEM OF SEEPAGE THROUGH A DAM WITH VERTICAL FACES

A. The Present, Potential Theory

The analytical theory relating to this problem is presented in Reference 3. A diagrammatic representation of the problem is shown in Figure 1. The theory leads to a set of very complicated integral formulas for determining $H_w/\bar{k}$, $H_s/\bar{k}$, $Q/\bar{k}^2$, $L/\bar{k}$, $H_e/\bar{k}$, and other characteristics of the flow such as the velocity distributions along the outflow and inflow faces of the dam and the shape of the free surface according to the potential theory (see Appendix IV).

All of these formulas contain three arbitrary parameters a, b, and c. The third parameter c enters the formulas as a multiplying coefficient and therefore can be eliminated from the computations by calculating the quantities H_e/L, H_w/L, H_s/L and $Q/\bar{k}L$.

B. Extension of the Results for Practical Application

As mentioned in Section A, one must start with two arbitrary values for a and b $(a > b > 1)$ and then compute the values of H_w/L, H_s/L, H_e/L, $Q/\bar{k}L$ and the shape of the free surface. In practice, however, the value of H_s and the velocity distribution along the outlet and inlet faces are the ones which are desired. In this part of the study, an attempt has been made to provide a set of graphs by means of which one can solve the problem from a practical standpoint, i.e., having been given the values of L, H_e, and H_w, one can find the value of H_s and the shape of the free surface (Plates 13 - 23).

In order to accomplish the above task, one starts with a set of values for a and b and then computes the values of H_s/L, H_w/L, H_e/L and the shape of the free surface. The procedure used in these computations is found in Reference 2. By repeating this same procedure for other sets of values for a and b, one can plot the curves given in Plates 13 - 15. Then, if one starts with the values of H_e, H_w, and L as in a practical problem, one can find the values of a and b using the curves in Plates 13 and 14. Having discovered the values of a and b, the value of H_s can be computed by making use of the curves in Plate 15. The shape of the free surface can be plotted from the curves in Plates 16 - 23.

In Reference 13, one can find curves which give the values of H_s/H_e vs. L/H_e for different values of H_w/H_e. It was discovered that the results given in that reference are neither accurate nor complete. Since, as mentioned in the introduction, the results of the steady state solution are of use in the solution of the unsteady state problem of seepage through a dam with the same geometry as the one in the steady state problem, they are computed with greater precision as one part of this study (see Appendix IV).

As for the shape of the free surface, this study presents original results which are given in Plates 16 - 23.

C. Comparison of the Potential Theory with other Approximate Theories

An approximate solution to the problem of seepage through porous media can be obtained by the Dupuit-Forchheimer theory. In this case, the curve $\bar{h}$ (Figure 1) represents the free surface. The equation of this

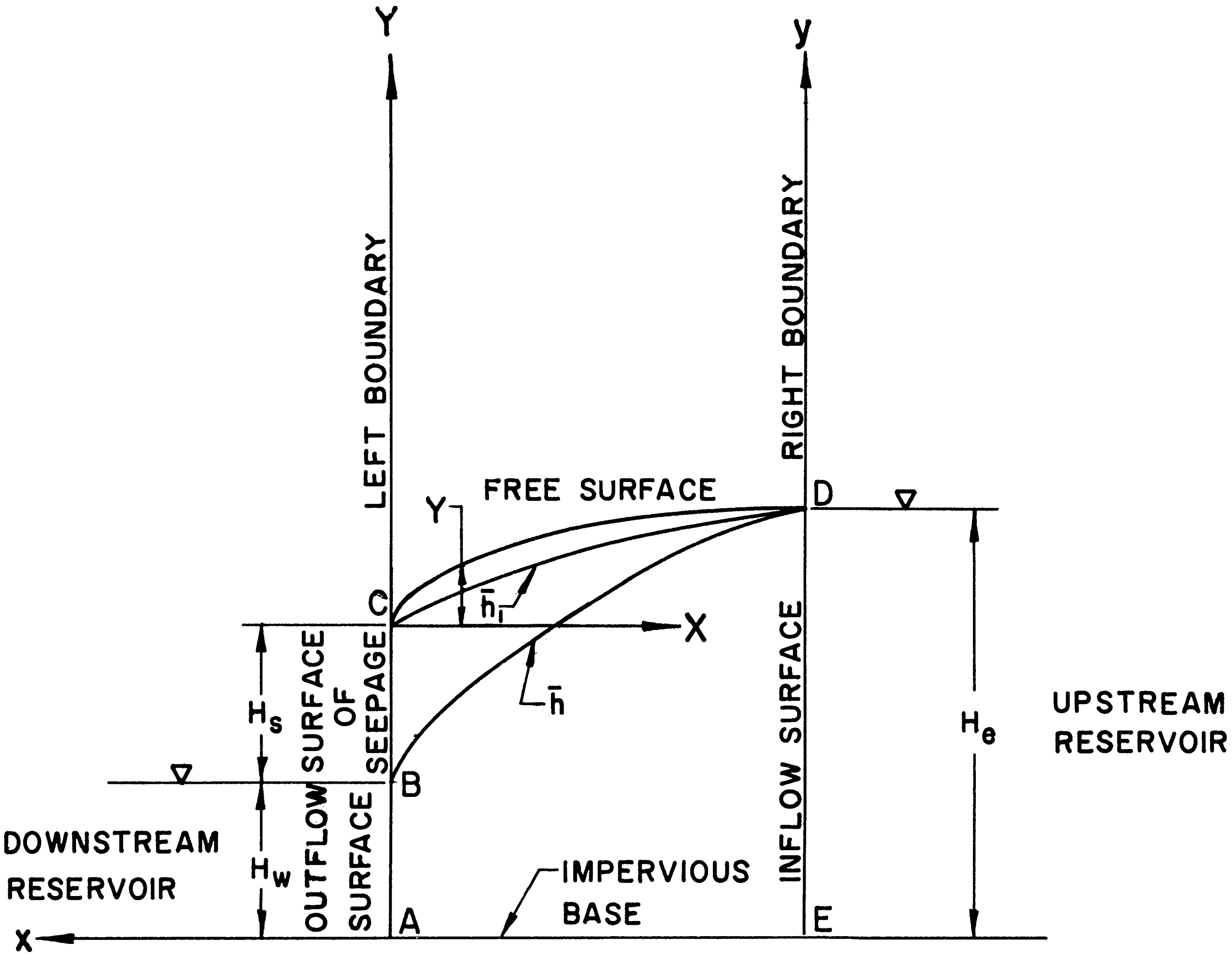

Figure 1. Diagram Showing the Region of Flow in the Steady State Case.

free surface is:

$$\bar{h}^2 = -(H_e^2 - H_w^2)\ x/L + H_e^2 \tag{1}$$

The main assumptions in this theory are as follows:

1. The streamlines can be taken as horizontal.
2. The velocities are independent of the depth.

These assumptions are basically the same as the assumptions of the existence of hydrostatic pressure on any vertical section throughout the region of flow. This assumption fails as one approaches the surface of seepage, BC in Figure 1. The only significant result of the calculations under these assumptions is that the correct flows (computed by the potential theory) are very closely reproduced by this theory. The equation for the flow Q, based on this approximate theory, is as follows:

$$Q = \bar{k}(H_e^2 - H_w^2)/2L \tag{2}$$

For practical purposes, therefore, it will suffice to compute the flux by the simple formula of Equation (2) and thus avoid the tedious calculations of the potential theory.* This fact is the basis for assuming that the fluxes predicted by the fundamental equations (derived in Part II) of the unsteady state problem are close to the actual unsteady fluxes. This fact is also used in determining the value of the height of

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* The study of erosion of the outflow face and its effect upon the stability of the dam, however, require the knowledge of the point of emergence of the free surface and the velocity distribution along the outflow face.

the surface of seepage in the unsteady state case as follows:

1. By making use of Equation (2), compute a theoretical steady state inflow height corresponding to the value of discharge through the outflow face. A theoretical steady state case is one in which the inflow height is equal to this theoretical inflow height.
2. By making use of the curves in Plates 13 - 15, find the height of the surface of seepage.

The curve $\bar{h}_1$ in Figure 1 represents a profile for the free surface which is much closer to the one obtained by the potential theory. It is based on Dupuit's assumption and a theoretical outflow height equal to the sum of the real outflow height and the height of the surface of seepage. It is worth mentioning that curve $\bar{h}_1$ is practically the same curve as the one which is obtained for the steady state solution of the problem of this study through the assumptions mentioned in the Introduction.

II

DERIVATION OF FUNDAMENTAL EQUATIONS OF UNSTEADY GRAVITY FLOW IN POROUS MEDIA

In the following paragraphs, the two equations of motion and continuity are applied to a fluid element inside the region of flow. Only the boundary conditions at the free surface and at the impervious base (see Figure 2) are used in deriving the fundamental equations of flow through a porous medium.

Let ox (see Figure 2) be the undisturbed surface of liquid in the porous medium. Let also

$$P = g\rho(\eta - y) \tag{3}$$

in which P is the pressure at any point (x, y) and ρ is the density of the liquid flowing in the medium. This follows from the assumption that pressure is hydrostatic on any vertical section throughout the region of flows. Then

$$P_x = g\rho\eta_x \tag{4}$$

It follows from (4) that the x-component of the acceleration of the liquid particles is independent of the depth y; hence, u, the x-component of the velocity, is also independent of y for all time t if it is assumed to be independent of y at some time, e.g., at $t = 0$. The assumption of hydrostatic pressure in a vertical line results from the hypothesis that the y-component of the acceleration of the liquid particles has only a minor effect on the pressure P.

A. Equation of Motion*

Let $\bar{u}$ be the seepage velocity, and let $\bar{k}$ be the coefficient of permeability of the porous medium. From Darcy's law,

$$\bar{u} = -\bar{k}\,\frac{P_x}{g\rho} \tag{5}$$

one can conclude that the resisting force ΔR, which is due to internal friction, is equal to

$$\Delta R = \rho g \bar{u} dA dx / \bar{k}\ . \tag{6}$$

One can then write the equation of motion in the x-direction for the fluid element shown in Figure 2:

$$\rho dAdx(u_t + uu_x) = (-(P + P_x dx) + P)\, dA - \rho g \bar{u} dA dx / \bar{k}$$

where u is the horizontal component of the velocity of the fluid element. Or, after simplification:

$$u_t + uu_x = -g\eta_x - g\,\frac{\bar{u}}{\bar{k}}$$

Or, since $\bar{u} = fu$, where f is the porosity of the porous material,

$$u_t + uu_x = -g\eta_x - gf\,\frac{u}{\bar{k}} \tag{7}$$

In deriving Equation (7), use has been made of $u_y = 0$, which is a direct consequence of the assumption of the existence of hydrostatic pressure on any vertical section throughout the region of flow.

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* The method employed in deriving the fundamental equations is in essence the same as the one used in deriving the equations of shallow water waves as described in Reference 7.

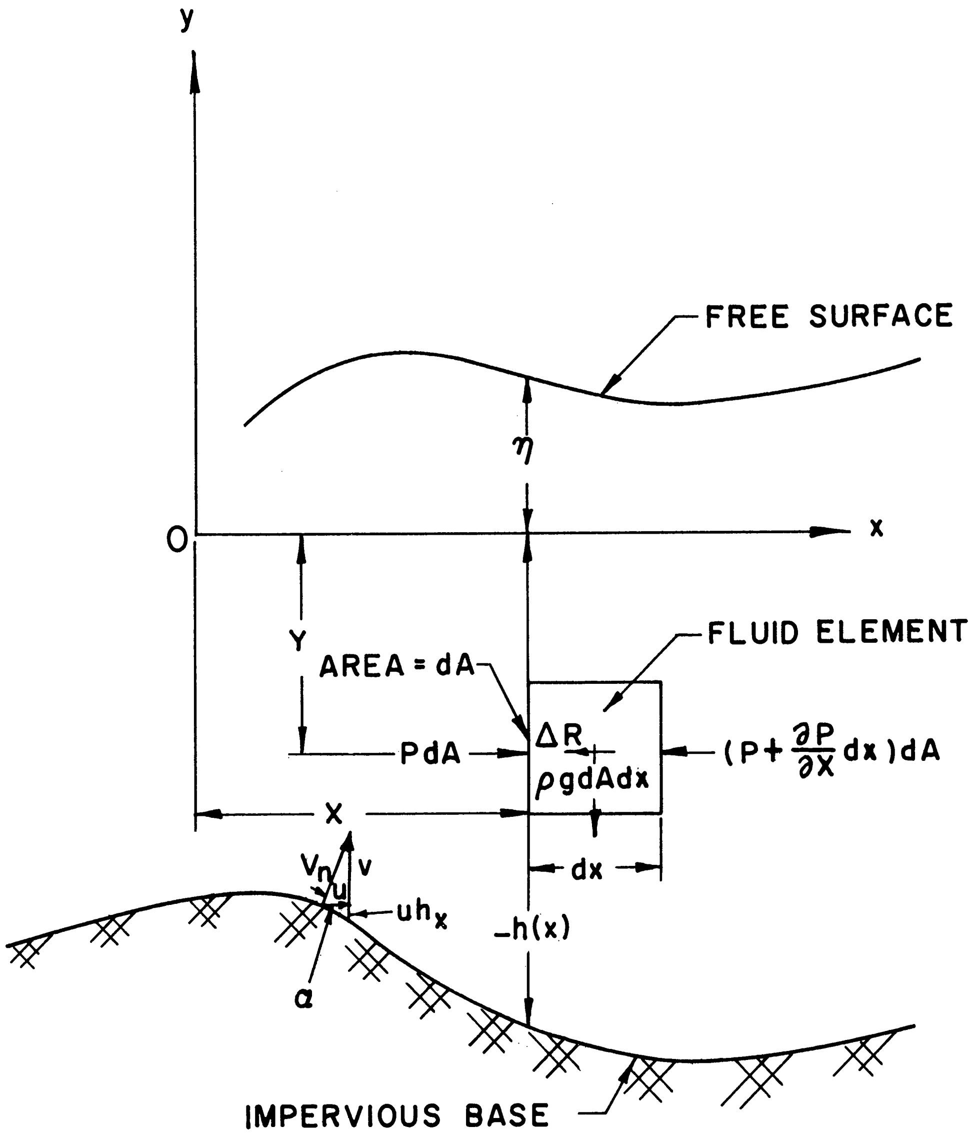

Figure 2. Diagram Showing the Region of Flow in the Unsteady State Case.

In deriving the equation of motion, use has been made of Darcy's law. One must therefore recall the limitations of that law, in particular the assumption of the existence of laminar flow.[3] This is assured if the Reynold's number is less than unity, i.e.,

$$\frac{VD}{\nu} \leq 1, \quad \text{or} \quad \frac{\overline{V}D}{f\nu} \leq 1.$$

In the above expressions, $\overline{V}$ is the seepage velocity, V is the velocity of a fluid particle, D is the average diameter of a sand grain, ν is the kinematic viscosity of the fluid, and f is the porosity of the porous material. In the case of a Hélé-Shaw model, D is the width of the interspace between the two walls of the channel in the model.

B. Equation Derived from Continuity and Boundary Conditions

The equation of continuity is

$$u_x + v_y = 0 \tag{8}$$

The kinematical condition to be satisfied at the free surface is

$$\frac{D\eta}{Dt} = \eta_t + u\eta_x = v \ , \quad \text{or}$$

$$(\eta_t + u\eta_x - v)_{y=\eta} = 0 \tag{9}$$

The dynamical condition at the free surface is

$$P\big|_{y=\eta} = 0 \tag{10}$$

At the impervious base, the condition on the component of velocity normal to the boundary is

$$V_n = 0, \quad \text{or}$$

$$V_n/\cos\alpha = (uh_x + v)_{y=-h} = 0 \tag{11}$$

in which α is the angle between the impervious bottom and a horizontal line, and

$$h_x = \frac{dh}{dx} = \tan\alpha \ .$$

From Equation (8) one can write

$$\int_{-h}^{\eta} u_x dy + v\Big|_{-h}^{\eta} = 0 \tag{12}$$

From Equations (9), (11), and (12), one can conclude

$$\int_{-h}^{\eta} u_x dy + \eta_t + u|_\eta \cdot \eta_x + u|_{-h} \cdot h_x = 0 \tag{13}$$

But in view of Leibnitz's formula

$$\frac{\partial}{\partial x} \int_{-h(x)}^{\eta(x)} u dy = u|_\eta \cdot \eta_x + u|_{-h} \cdot h_x + \int_{-h}^{\eta} u_x dy \tag{14}$$

The combination of Equations (14) and (13) leads to

$$\frac{\partial}{\partial x} \int_{-h}^{\eta} u dy = -\eta_t \tag{15}$$

Or, since u is independent of y,

$$[u(\eta + h)]_x = -\eta_t \tag{16}$$

Equations (7) and (16) are a system of two first-order differential equations for the functions $u(x,t)$ and $\eta(x,t)$. If in addition to sufficient side boundary conditions the values of u and η at the time $t = 0$ are given, the subsequent motion can be solved.

In deriving the above two fundamental equations, no restrictions were made on the side boundaries. One can, therefore, apply those two equations to regions having any shape or condition on the side boundaries.

C. The Numerical Solution of Equations (7) and (16) by the Method of Characteristic

Since the partial differential Equations (7) and (16) governing the flow are of the hyperbolic type, they can be solved numerically by the method of characteristics.*

1. Characteristic Equations

Since this study deals only with a region having a horizontal base, let

$$h = \text{const.}, \quad \text{or} \quad \frac{dh}{dx} = 0 \ .$$

Let also

$$\eta + h = H$$

Equations (7) and (16) can be written as follows:

$$L_1 = uu_x + u_t + gH_x + gf\,\frac{u}{k} = 0 \tag{17}$$

$$L_2 = Hu_x + uH_x + H_t = 0 \ , \tag{18}$$

where

$$u_x = \frac{\partial u}{\partial x} \ , \quad H_t = \frac{\partial H}{\partial t} \ , \ \ldots$$

* For a more thorough mathematical treatment of hyperbolic partial differential equations by the method of characteristics, see Reference 6.

Equations (17) and (18) are two simultaneous quasi-linear partial differential equations of the first order containing two independent and two dependent variables. Consider a linear combination of L_1 and L_2:

$$L = L_1 + \lambda L_2 = u_x(u + \lambda H) + u_t + (g + \lambda u)H_x + \lambda H_t$$
$$+ gf \frac{u}{k} = 0 \qquad (19)$$

One also knows that

$$du = u_x dx + u_t dt \ , \quad \text{and}$$
$$dH = H_x dx + H_t dt \ , \qquad (20)$$

or

$$\frac{du}{dt} = u_x \frac{dx}{dt} + u_t \ , \quad \text{and}$$

$$\frac{dH}{dt} = H_x \frac{dx}{dt} + H_t \qquad (21)$$

Now, by examination of Equation (19), with Equations (21) in mind, let

$$(u + \lambda H)u_x + u_t = u_x \frac{dx}{dt} + u_t = \frac{du}{dt} \ , \quad \text{and} \qquad (22)$$

$$(\frac{g}{\lambda} + u)H_x + H_t = H_x \frac{dx}{dt} + H_t = \frac{dH}{dt} \ . \qquad (23)$$

Then Equation (19) may be written as

$$dtL = du + \lambda dH + gf \frac{u}{k} dt \ . \qquad (24)$$

In order for Equation (19) to be in this form, the following must be true:

$$\frac{dx}{dt} = u + \lambda H \ , \quad \text{and} \quad \frac{dx}{dt} = \frac{g}{\lambda} + u \qquad (25)$$

from which one obtains

$$u + \lambda H = \frac{g}{\lambda} + u$$

or

$$\lambda = \pm\sqrt{\frac{g}{H}} \tag{26}$$

and the characteristic equations for u and H become

$$dtL = du + \sqrt{\frac{g}{H}}\, dH + gf\,\frac{u}{k}\, dt = 0 \ , \tag{27}$$

$$dtL = du - \sqrt{\frac{g}{H}}\, dH + gf\,\frac{u}{k}\, dt = 0 \tag{28}$$

By substituting λ from Equation (26) into Equations (25), the two different characteristic directions at the point (x,t) are given by

$$\xi_+ = \frac{dt}{dx} = \frac{1}{u + \sqrt{gH}} \ ; \ \xi_- = \frac{dt}{dx} = \frac{1}{u - \sqrt{gH}} \tag{29}$$

Equations (27) and (28) are two separate total differential equations with t as an independent variable and u and H as dependent variables. If $u = u(x,t)$ and $H = H(x,t)$ satisfy Equations (17) and (18), then Equations (29) become two separate ordinary differential equations of the first order. These determine two families of characteristic curves, or in short, "characteristics," C_+ and C_- in the (x,t) plane belonging to this solution $u(x,t)$, $H(x,t)$.

Equations (27) - (29) can be rewritten as follows:

along C_+

$$dt - \frac{dx}{u + \sqrt{gH}} = 0 \tag{30}$$

$$du + \sqrt{\frac{g}{H}}\, dH + gf\,\frac{u}{k}\, dt = 0 \tag{31}$$

$$\text{along } C_- \begin{cases} dt - \dfrac{dx}{u - \sqrt{gH}} = 0 & (32) \\ du - \sqrt{\dfrac{g}{H}}\, dH + gf\, \dfrac{u}{k}\, dt = 0 & (33) \end{cases}$$

Equations (30) - (33) are of a particularly simple form and are satisfied, according to the derivation, by every solution of the original system (17) and (18).

2. Finite Difference Approximation

Let first-order linear finite difference approximation be used for solving Equations (30) - (33); this finite difference approximation is expressed by

$$\int_{x_o}^{x_1} f(x)dx \approx f(x_o)(x_1 - x_o) \tag{34}$$

Referring to Figure 3, let it be assumed that x_A, t_A, u_A, H_A, x_B, t_B, u_B, and H_B are known, and that x_P, u_P, t_P and H_P are to be found. By applying Equation (34) to Equations (30) - (33), the following equations are obtained:

$$t_P - t_A - \left(\frac{1}{u + \sqrt{gH}}\right)_A (x_P - x_A) = 0 \tag{35}$$

$$u_P - u_A + \sqrt{\frac{g}{H_A}}\, (H_P - H_A) + gf\, \frac{u_A}{k}\, (t_P - t_A) = 0 \tag{36}$$

$$t_P - t_B - \left(\frac{1}{u - \sqrt{gH}}\right)_B (x_P - x_B) = 0 \tag{37}$$

$$u_P - u_B - \sqrt{\frac{g}{H_B}}\, (H_P - H_B) + gf\, \frac{u_B}{k}\, (t_P - t_B) = 0 \tag{38}$$

Equations (35) - (38) form a set of four linear equations with these unknowns:

x_P, t_P, H_P and u_P

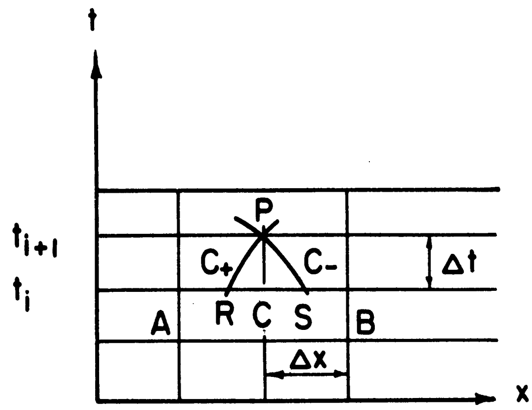

Figure 3 - a. Diagram Showing the Method of Specified Time Intervals.

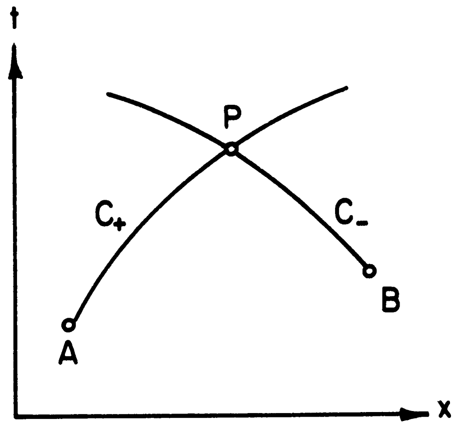

Figure 3. Intersection of Two Characteristics.

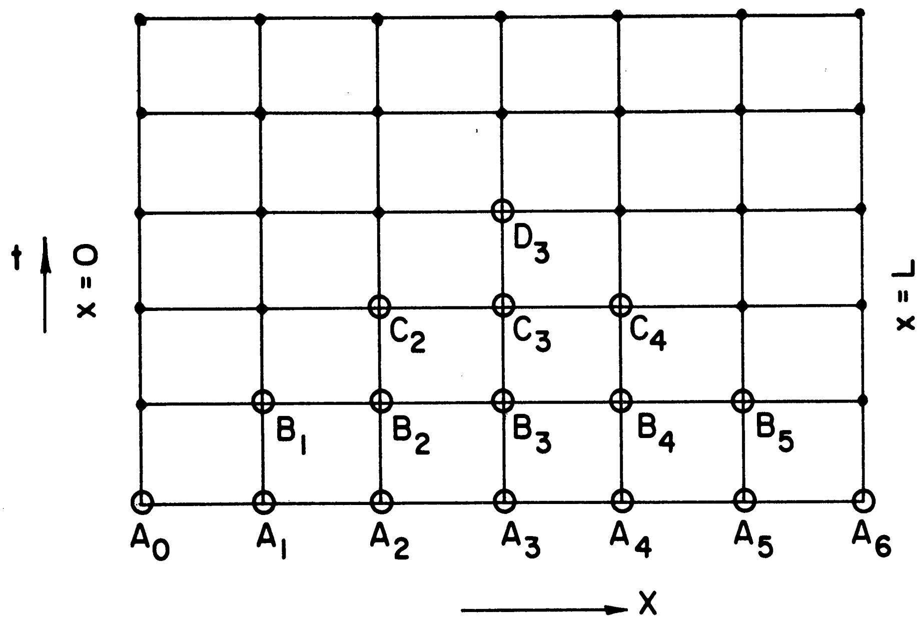

Figure 4. Characteristic Grid.

D. Specified Time Intervals

There are several ways by which a numerical solution of Equations (35) - (38) can be obtained. The one used in this study is called "Specified Time Intervals." This method is preferred because x_P and t_P can be pre-assigned so that only two values, H_P and u_P, are to be determined. The procedure can be described as follows: Let u and H be given at A_o, A_1 ... A_6 as shown in Figure 4. Then u and H can be computed at B_1, B_2 ... B_5; C_2, C_3, C_4, and D_3. If further information is given along $x = 0$ and $x = L$, then the values of u and H can also be computed for the points lying outside the triangle $A_oD_3A_6$ (Figure 4).

To compute u and H at P from u_B, H_B, u_A, H_A, u_C, and H_C first obtain the values of u_R, H_R, u_S, and H_S (see Figure 3-a) by making use of linear interpolation:

$$(\xi_+)_R = \frac{\Delta t}{\Delta x \dfrac{u_C - u_R}{u_C - u_A}} = \frac{\Delta t}{\Delta x \dfrac{H_C - H_R}{H_C - H_A}}$$

$$(\xi_-)_S = \frac{\Delta t}{\Delta x \dfrac{u_C - u_S}{u_C - u_B}} = \frac{\Delta t}{\Delta x \dfrac{H_C - H_S}{H_C - H_B}} \quad , \text{ or if } \Theta = \frac{\Delta t}{\Delta x}$$

then

$$u_R = u_C \, [1 - \Theta(\xi_+)_R^{-1}] + u_A \Theta(\xi_+)_R^{-1} \tag{39}$$

$$H_R = H_C \, [1 - \Theta(\xi_+)_R^{-1}] + H_A \Theta(\xi_+)_R^{-1} \tag{40}$$

$$u_S = u_C \, [1 + \Theta(\xi_-)_S^{-1}] - u_B \Theta(\xi_-)_S^{-1} \tag{41}$$

$$H_S = H_C \, [1 + \Theta(\xi_-)_S^{-1}] - H_B \Theta(\xi_-)_S^{-1} \tag{42}$$

It has been assumed that Δt is sufficiently small so that PR and PS are straight lines with slopes of $(\xi_+)_R$ and $(\xi_-)_S$ respectively. u_P and H_P can now be calculated from

$$u_P - u_R + \sqrt{\frac{g}{H_R}} (H_P - H_R) + gf \frac{u_R}{\bar{k}} (t_P - t_R) = 0 \qquad (43)$$

and

$$u_P - u_S - \sqrt{\frac{g}{H_S}} (H_P - H_S) + gf \frac{u_S}{\bar{k}} (t_P - t_S) = 0 \qquad (44)$$

or

$$u_P = u_R - \sqrt{\frac{g}{H_R}} (H_P - H_R) - gf \frac{u_R}{\bar{k}} dt \qquad (45)$$

$$u_P = u_S + \sqrt{\frac{g}{H_S}} (H_P - H_S) - gf \frac{u_S}{\bar{k}} dt \qquad (46)$$

A simultaneous solution of Equations (45) and (46) will result in the values of H_P and u_P.

E. Boundary Conditions

At the left boundary (see Figure 5), $x = 0$ and $H = H_o(t)$. One can therefore find u_S, H_S, and u_P from Equations (41), (42), and (46) respectively. Having found u_P, one can then proceed to obtain the values of u and H at the points lying outside the triangle $A_oD_3A_6$ (Figure 4).

At the right boundary (see Figure 6), $x = L$ (width of the medium) and $H = H_L(t)$; one can therefore find u_R, H_R and u_P from Equations (39), (40) and (45) respectively. Having found u_P, one can then proceed to obtain the values of u and H at the points lying outside the triangle $A_oD_3A_6$ (Figure 4).

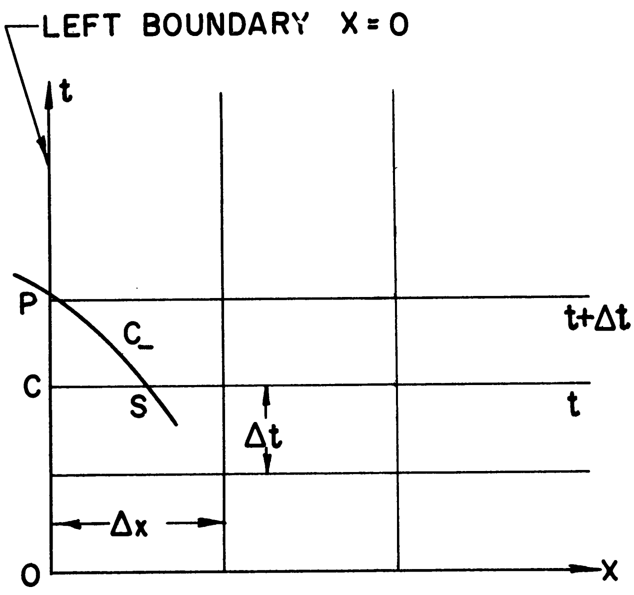

Figure 5. Left Boundary
Conditions.

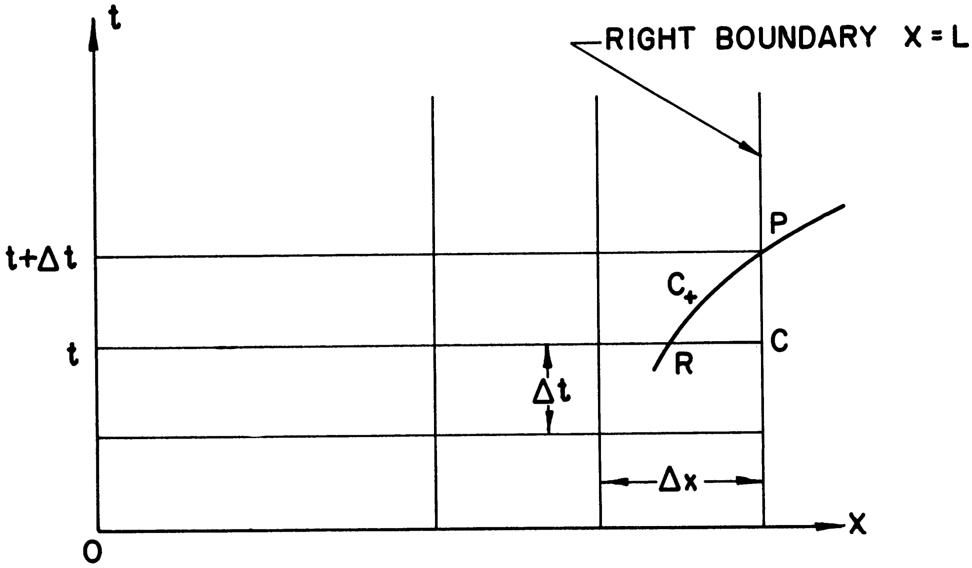

Figure 6. Right Boundary
Conditions .

F. Application of the Method of Characteristic to the Solution of Unsteady Gravity Flow of Liquids Through a Porous Dam with Vertical Sides

By making use of the results in Part I of this study, and by following the steps outlined below, one can obtain a solution to this problem:

1. By assuming that hydrostatic pressure prevails on any vertical section throughout the region of flow, make use of the fundamental Equations (7) and (16) and the characteristic Equations (45) and (46).
2. Vary the given quantities of H_e and H_w (see Figure 1) with time as required by the problem; these values are used as the boundary conditions in applying the method of characteristics.
3. Compute by the method of characteristics the values of H and u at any section and for any time.
4. Compute the quantity of discharge Q at any time at the boundary or boundaries with a surface of seepage, i.e., a boundary from which fluid flows to an adjacent reservoir.
5. Using Plates 13 - 15, obtain the values of H_s for any time (see Section C, Part I).
6. Replace the value of H_w at the boundary having a surface of seepage with $H_w + H_s$ for any time.
7. Recompute, by using the method of characteristics and the new boundary conditions stated in Step 6, the values of H and u at any section and for any time.

Since, as one can tell by looking at the curves in Plates 13 - 15, in a general case it is difficult to find any empirical formula which relates H_s to Q and to carry out Step 5 along with the rest of the computations, this step will be handled as follows:

A. Follow Steps 1 - 4 for the entire time before the steady state condition is established.

B. Use the values of Q, computed in Step 4 of the first computer program, to find H_s vs. time by making use of Plates 13 - 15 (see Section C, Part I).

Only when the height of the outflow surface remains constant and the height of the inflow surface is changed (Experiment 1), can one carry out Step 5 along with the rest of the computations. This may be done as follows: Calculate the height of the surface of seepage for some arbitrary values of discharge which fall within the range of the values of discharge encountered in the computations. Then, interpolate for the values of the height of the surface of seepage from the above information which is available as part of the read-in data of the program.

As is discussed later, however, in order to avoid the restrictions on the solution by the method of characteristics, one must carry out Step 5 along with the rest of the computations when $H_w = 0$ or when $H_e \gg H_w$.

Even when H_w changes with time, one can try to carry out Step 5 simultaneously with the rest of the computations. In this case, however, one must make use of double-interpolation on the read-in data for different sets of values of Q, H_e and H_w.

Note:

In Step 7, use a hypothetical coefficient of permeability computed by the following equation:

$$\bar{k}_h = \bar{k} \frac{H_{e_{th}}^2 - H_w^2}{H_{e_{th}}^2 - (H_w + H_s)^2} \tag{47}$$

in which $H_{e_{th}}$ is the theoretical steady state inflow height (see Section C, Part I).

This value of the coefficient of permeability allows one to obtain as a steady solution a parabolic free surface passing through points C and D (curve $\bar{h}_1$ in Figure 1) and yet permits the same flux as the one obtained by curve $\bar{h}$. Thus, this change in permeability is a way of compensating for the incorrect assumption of the existence of hydrostatic pressure on any vertical section. One should therefore remember that since $H_{e_{th}}$, H_w, and H_s are only functions of time, $\bar{k}_h$ is also dependent on it. If the free surface profiles encountered in the unsteady state flow are of the type shown in Figure 7-a, $\bar{k}_h$ does not depend on x. In this case, the value of $\bar{k}_h$ computed by Equation (47) is the same for the entire region of Figure 7-a. The only time $\bar{k}_h$ could depend on x is when the unsteady free surfaces have shapes such as the ones shown in Figures 7-b and 7-c. In these cases, the hypothetical coefficient of permeability in different regions of flow is taken to be as follows:

for region $A_1C_1DD_1$ of Figure 7-b,

$$\bar{k}_h = \bar{k} \frac{H_{e_{th}}^2 - H_{w_1}^2}{H_{e_{th}}^2 - (H_{w_1} + H_{s_1})^2} \tag{48a}$$

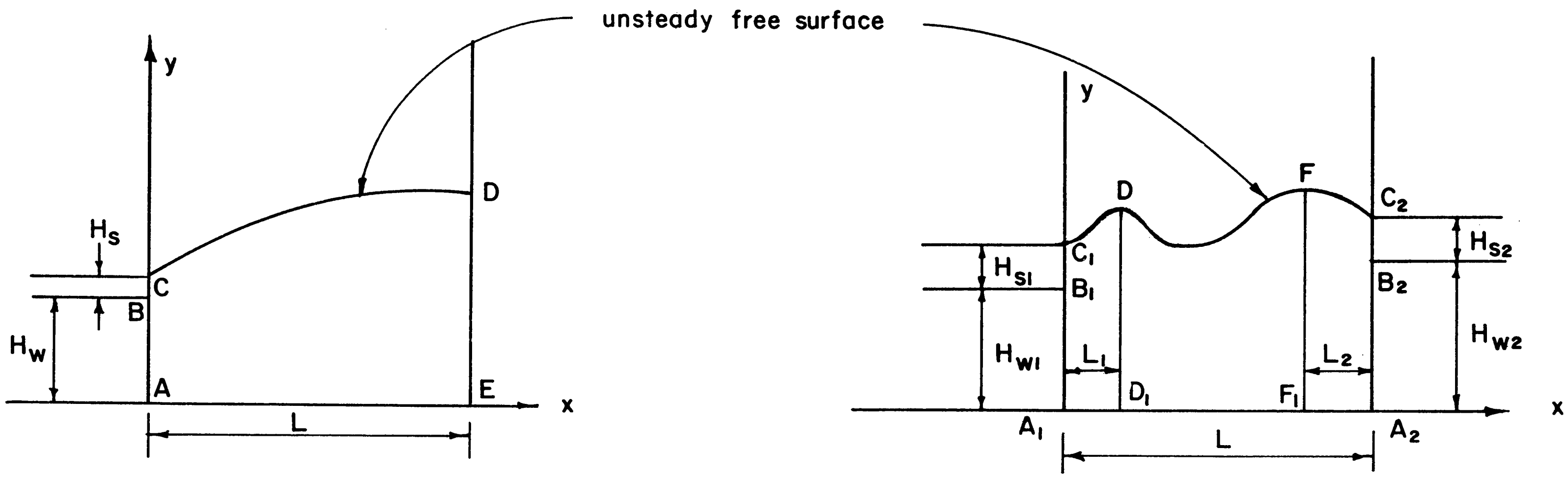

Figure 7 - a. Diagram Showing the Type of an Unsteady Free Surface Referred to in Equation 47.

Figure 7 - b. Diagram Showing the Different Regions Referred to in Equations 48 - a and 48 - b.

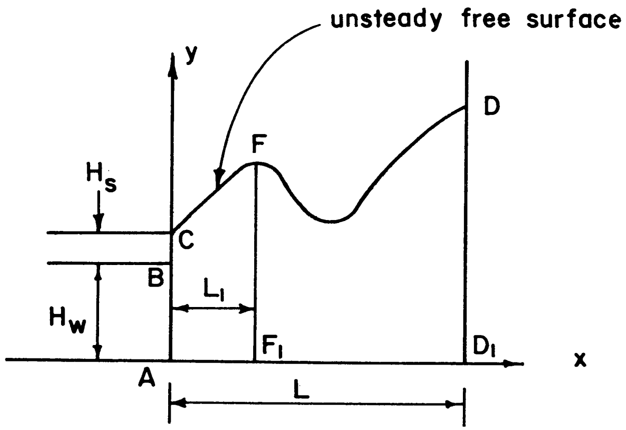

Figure 7 - c. Diagram Showing the Flow Regions Referred to in Equation 48 - c.

for region $A_2C_2FF_1$ of Figure 7-b

$$\bar{k}_h = \bar{k}\,\frac{H_{e_{th}}^2 - H_{w_2}^2}{H_{e_{th}}^2 - (H_{w_2} + H_{s_2})^2} \tag{48b}$$

for region $ACFF_1$ of Figure 7-c,

$$\bar{k}_h = \bar{k}\,\frac{H_{e_{th}}^2 - H_w^2}{H_{e_{th}}^2 - (H_w + H_s)^2}$$

and in all the other regions of Figures 7-b and 7-c,

$$\bar{k}_h = \bar{k}$$

In Equations (48a) - (48c), $H_{e_{th}}$ is computed by means of Equation (2) on the basis of discharge through the outflow boundary, the width of the region (e.g., L_1 in region $A_1C_1DD_1$), and the outflow height.

In the computer programs written for the two experiments of this study, the method of specified time intervals was used (see Section D of Part II). The slope of the characteristic lines at any point of the characteristic grid depend on the value of the variables u and H at that point. One can, therefore, in order to save machine time, make use of the following:

Since the required number of points (see Section G of Part III) along the x-axis of the characteristic grid depends on the curvature of the free surface, one can change this number during the execution of the program to match the expected curvature of the free surface. The value of the time interval Δt will, therefore, also be changed accordingly.

One can even choose points unevenly spaced along the entire length between the two boundaries of the characteristic grid. The distribution of the points along the x-axis will depend on the distribution of the expected values of the curvature of the free surface along that axis.

In other words, since the slopes of the characteristic lines depend on H and u, and hence on x and t, one can let Δt and Δx depend on x and t as well.

III

VISCOUS FLOW ANALOGY (HÉLÉ-SHAW MODEL)

In order to show that the laminar flow between two closely-spaced vertical plates is analogous to the two-dimensional flow of liquids through porous media, it is necessary to prove that the former is a potential flow.[8,9] It is shown in the following paragraphs that only the average velocity of the fluid particles (averaged in the direction perpendicular to the plane of the Hélé-Shaw model) can be derived from a potential function. One can therefore conclude that the flow of liquids in a Hele-Shaw model is irrotational only in a macroscopic sense and is not irrotational microscopically. The flow of a viscous fluid in a narrow slit belongs to the group of "creeping" flows.[5] One can then neglect the inertia terms in the Navier-Stokes equations. One can also neglect

$$\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 v}{\partial x^2} \text{ and } \frac{\partial^2 v}{\partial y^2}$$

The Navier-Stokes equations may then be written in the following form:

$$0 = -g \frac{\partial \varphi}{\partial x} + \nu \frac{\partial^2 u}{\partial z^2} \tag{48}$$

$$0 = -g \frac{\partial \varphi}{\partial y} + \nu \frac{\partial^2 v}{\partial z^2} \tag{49}$$

$$0 = \frac{\partial \varphi}{\partial z}, \quad (\varphi = y + \frac{P}{\rho g}) \tag{50}$$

where u, v are the components of velocity at a point (x,y,z), ν is the kinematic viscosity of the liquid, and b is the width of the interspace (Figure 8).

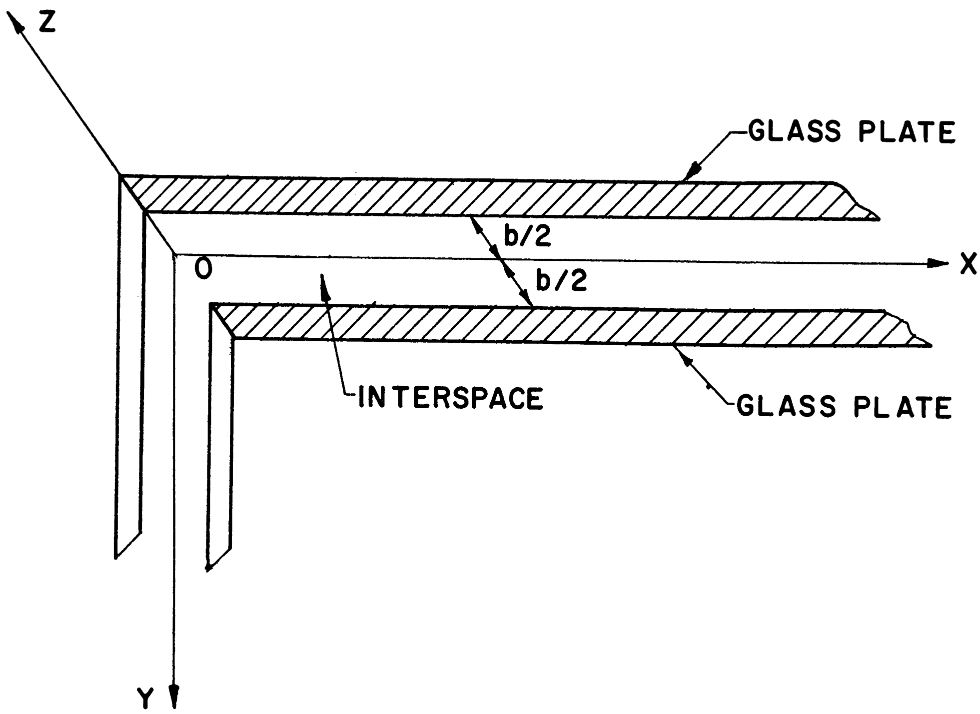

Figure 8. Schematic Diagram of
a Hélé-Shaw Model.

From Equation (50), one concludes that φ is not a function of z. Hence, from Equation (48) one obtains:

$$-gz \frac{\partial \varphi}{\partial x} + \nu \frac{\partial u}{\partial z} = C_1 \text{ , and}$$

$$-g \frac{z^2}{2} \frac{\partial \varphi}{\partial x} + \nu u = C_1 z + D_1 \text{ ;}$$

but $\frac{\partial u}{\partial z} = 0$, for $z = 0$; and $u = 0$, for $Z = \pm \frac{b}{2}$,

therefore

$$u = \frac{g}{2\nu} (z^2 - \frac{b^2}{4}) \frac{\partial \varphi}{\partial x} \text{ , and}$$

$$u_m = \frac{\frac{g}{2\nu} \frac{\partial \varphi}{\partial x} \int_{-b/2}^{+b/2} (z^2 - \frac{b^2}{4}) \, dz}{b}$$

$$= \frac{g \frac{\partial \varphi}{\partial x}}{2\nu b} \left| \frac{z^3}{3} - \frac{b^2}{4} z \right|_{-b/2}^{+b/2} \text{ , or}$$

$$u_m = -g \frac{b^2}{12\nu} \frac{\partial \varphi}{\partial x} \tag{51}$$

By the same procedure it can be shown that

$$v_m = -g \frac{b^2}{12\nu} \frac{\partial \varphi}{\partial y} \tag{52}$$

From this, it is clear that the average component of velocity of the flow of a fluid in a narrow slit does possess a potential.

A. Derivation of Model Scales

Let $F(x,y,t) = 0$ describe the phreatic surface. For the phreatic surface $(P = 0)$, the potential $\varphi = y + \frac{P}{\rho g} = y$, therefore

$F = \varphi - y$. Hence

$$\frac{DF}{Dt} = \frac{\partial\varphi}{\partial t} + \frac{\partial\varphi}{\partial x}\frac{dx}{dt} + \frac{\partial\varphi}{\partial y}\frac{dy}{dt} - \frac{dy}{dt} = 0 \ ; \tag{53}$$

$\frac{dx}{dt}$ and $\frac{dy}{dt}$ describe u and v respectively. By Darcy's law,

$$\begin{aligned} u &= -\frac{\bar{k}}{f}\frac{\partial\varphi}{\partial x} \\ v &= -\frac{\bar{k}}{f}\frac{\partial\varphi}{\partial y} \end{aligned} \tag{54}$$

The substitution of (54) into (53) yields

$$\bar{k}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \left(\frac{\partial\varphi}{\partial y}\right)^2 - \frac{\partial\varphi}{\partial y}\right] = f\,\frac{\partial\varphi}{\partial t} \tag{55}$$

In the case of laminar flow between two parallel plates, the equation corresponding to Equation (55) is

$$\bar{k}_M\left[\left(\frac{\partial\varphi_M}{\partial x_M}\right)^2 + \left(\frac{\partial\varphi_M}{\partial y_M}\right)^2 - \frac{\partial\varphi_M}{\partial y_M}\right] = f_M\,\frac{\partial\varphi_M}{\partial t_M} \tag{56}$$

where

$$\bar{k}_M = \frac{1}{12}\,g\,\frac{b^2}{\nu}\ , \quad \text{and} \quad f_M = 1 \ ;$$

M is a subscript describing model quantities. By substituting

$$\left\{\begin{aligned} k &= \frac{\bar{k}_M}{\bar{k}_r}\ ; \quad x = \frac{x_M}{x_r}\ ; \quad y = \frac{y_M}{y_r}\ ; \\ \varphi &= \frac{\varphi_M}{\varphi_r}\ ; \quad f = \frac{f_M}{f_r}\ ; \quad t = \frac{t_M}{t_r}\ ; \end{aligned}\right. \tag{57}$$

into (54), one gets

$$\frac{\bar{k}_M}{\bar{k}_r}\left[\frac{x_r^2}{\varphi_r^2}\left(\frac{\partial\varphi_M}{\partial x_M}\right)^2 + \frac{y_r^2}{\varphi_r^2}\left(\frac{\partial\varphi_M}{\partial y_M}\right)^2 - \frac{y_r}{\varphi_r}\frac{\partial\varphi_M}{\partial y_M}\right] = \frac{1}{f_r} f_M \frac{t_r}{\varphi_r}\frac{\partial\varphi_M}{\partial t_M} \tag{58}$$

By comparing Equations (43) and (58), the following relationships are obtained:

$$\frac{x_r^2}{\bar{k}_r\varphi_r^2} = \frac{y_r^2}{\bar{k}_r\varphi_r^2} = \frac{y_r}{\bar{k}_r\varphi_r} = \frac{t_r}{f_r\varphi_r} \tag{59}$$

or

$$x_r = y_r = \varphi_r = L_r$$

$$t_r = f_r x_r/\bar{k}_r \tag{60}$$

where

$$f_r = \frac{f_M}{f} = \frac{1}{f}$$

Time Scale

From Equation (60), one can obtain

$$t = \frac{t_M g f b_M^2}{12\bar{k}x_r\nu_M} \tag{61}$$

Equation (61) indicates that an appropriate time scale can be selected for a given length scale by manipulating the width of the interspace b_M and the kinematic viscosity of the liquid

Discharge Scale

The discharge scale is obtained from Darcy's law,

$$Q_{xr} = \frac{Q_x}{Q_{xM}} = \frac{-\bar{k}b\int\frac{\partial\varphi}{\partial x}\,dy}{-\bar{k}_M b_M\int\frac{\partial\varphi_M}{\partial x_M}\,dy_M} = \bar{k}_r b_r x_r \tag{62}$$

similarly

$$Q_{yr} = Q_{xr} = \bar{k}_r b_r x_r \tag{63}$$

Velocity Scale

From Equation (60), and (54)

$$\bar{u} = -\bar{k}\frac{\partial\varphi}{\partial x} = -\frac{\bar{k}_M}{\bar{k}_r}\frac{x_r}{\varphi_r}\frac{\partial\varphi_M}{\partial x_M} = \frac{u_M}{\bar{k}_r}, \tag{64}$$

and

$$\bar{v} = -\bar{k}\frac{\partial\varphi}{\partial y} = -\frac{\bar{k}_M}{\bar{k}_r}\frac{y_r}{\varphi_r}\frac{\partial\varphi_M}{\partial y_M} = \frac{v_M}{\bar{k}_r} \tag{65}$$

B. The Experimental Setup

Plate 1 shows the experimental setup. A schematic representation of this apparatus is shown in Figure 9.

The equipment is made up of the following main elements:

1. Two pieces of 1/4" plate glass, 24" x 20", spaced .112" apart. The two pieces of plate glass are fastened with C-clamps along the top of the channel and with thumb screws along the bottom. The spacing is maintained by strips of opaque plexiglass.
2. Two regulating reservoirs B1 and B2.
3. Two regulating overflows C1 and C2 which maintain the desired heights of fluid along the upstream and downstream sides of the channel.
4. Two head reservoirs D1 and D2, connected to each other by a gate valve G1. Gate valve G2 connects D2 to B2, and

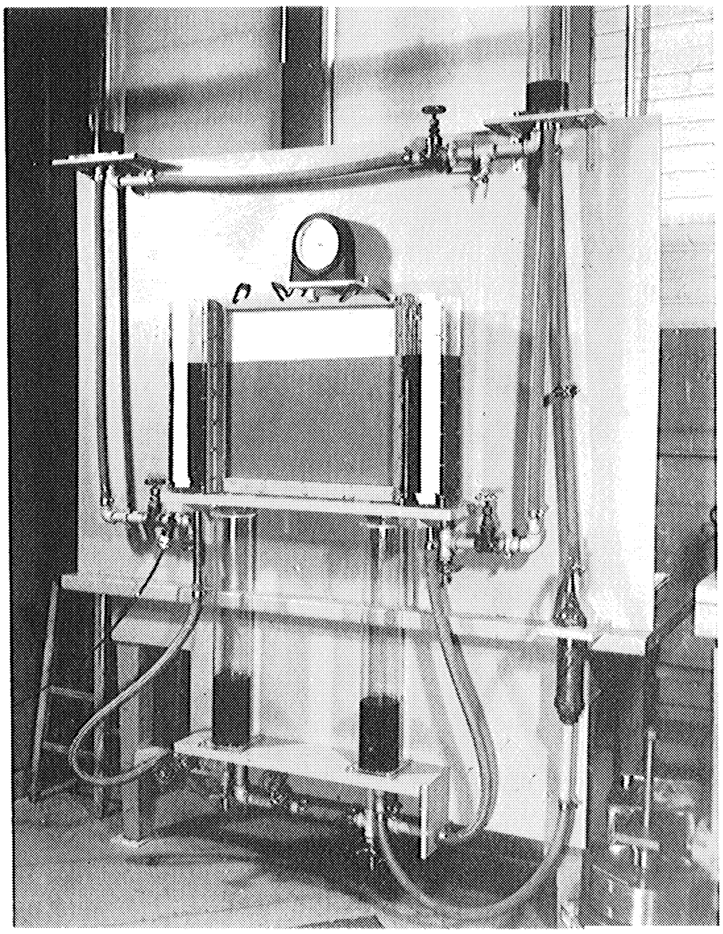

Plate 1
Experimental Setup

gate valve G3 connects D1 to B1. When gate valve G1 is open, the two head reservoirs act as one.

5. Two sump reservoirs E1 and E2 with connections to each other and to the two regulating reservoirs. When gate valve G4 is open, the two sump reservoirs act as one. By supplying two sump reservoirs, the discharge through the apparatus can be measured by closing the valves G1 and G4 and by using the storage equation. The two head reservoirs make possible the elimination of long connections which are undesirable because they create large head losses which in turn reduce the rate at which the height of the fluid in the regulating reservoirs is raised.

6. A hand pump which raises the fluid from the sump to the head reservoir.

Castor oil is the fluid used in this setup for the following reasons:

I. Castor oil is highly viscous. High viscosity is necessary in order to obtain an appropriate time scale, i.e., enough time to observe and take photographs of the unsteady state, after a disturbance has been introduced and before the re-establishment of the steady state.

II. Glycerin, though highly viscous, absorbs moisture, and thus its viscosity is lowered. It also requires measurement of its viscosity at the time of each experiment.

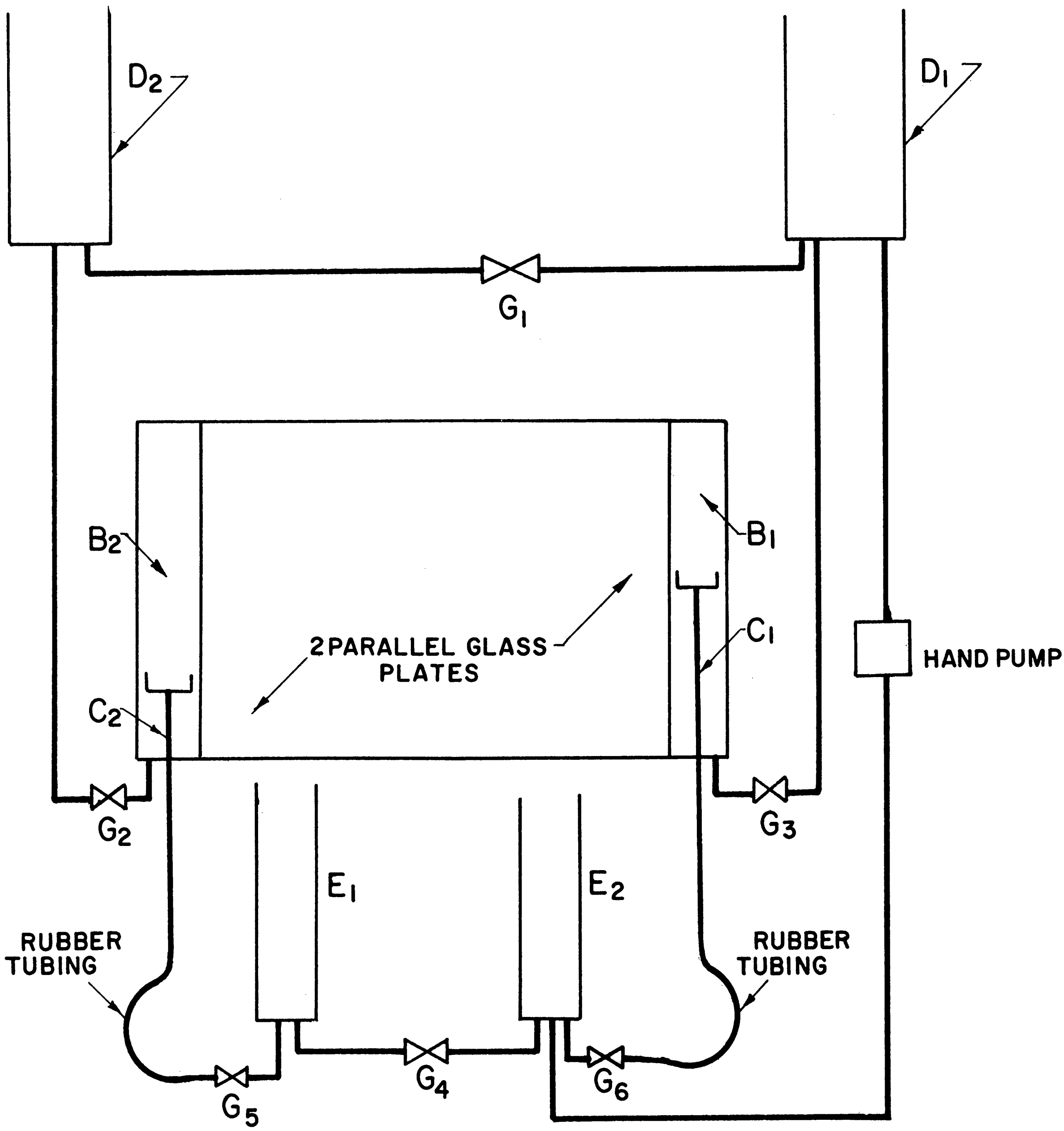

Figure 9. A Schematic Representation of
The Setup (Hélé-Shaw Model).

III. Castor oil has a smaller value of surface tension than glycerin; thus by using castor oil, the effects of capillary rise are lessened.

The larger the width of the channel, the smaller the value of the capillary rise but the larger the coefficient of permeability of the model. As mentioned in I, however, smaller values of the coefficient of permeability are desired. Thus, the width of the channel is chosen to be equal to 0.112". This value of the width creates a capillary rise equal to 0.10".

In order to be able to photograph the surfaces of the liquid, the castor oil is dyed with Sudan III. A sheet of graph paper is attached to the back of the channel so that one can read the coordinates of the free surfaces from the photographs.

C. Experiment No. 1

In this experiment, the starting steady state free surface is a horizontal line with a height of two inches above the bottom of the channel. The unsteady state is introduced by raising the elevation of the liquid in the reservoir B1 by opening valve G3. The top of the regulating overflow in the regulating reservoir B1 is kept at an elevation of 12" above the bottom of the channel. The top of the regulating spillway in the regulating reservoir B2 is maintained at an elevation of 2" above the bottom of the channel. In other words, the elevation of the liquid in the regulating reservoir B2 is kept constant while the elevation of the liquid in the reservoir B1 is raised to 12" above the bottom of the channel with a net rise of 12" - 2" = 10" above the starting horizontal free surface.

TABLE I

EXPERIMENTAL DATA

Experiment No. 1					Experiment No. 2				
Model Properties	Boundary Conditions				Model Properties	Boundary Conditions			
	Right		Left			Right		Left	
	Time Sec	El. in.	Time Sec	El. in.		Time Sec	El. in.	Time Sec	El. in.
width of the interspace					The same as in				
= b = 0.112";					Experiment No. 1				
temperature of the	0	2		2	except for:		18	0	18
liquid = 78.8°F;	11.95	4.5		2	Temperature of		18	10.3	16.5
ν = kinematic viscosity	24.1	7	$t \geq o$	2	the liquid = 84.5° /F;	$t \geq o$	18	23.2	15
of the liquid = 0.00555	38.9	9.5		2	ν = kinematic viscosity		18	37.6	13.5
ft^2/sec ; $\bar{k}_M = \frac{gb^2}{12\nu}$ =	53.9	12		2	of the liquid = 0.00425		18	50.1	12.15
0.04215 '/sec.;					ft^2/sec; $\bar{k}_M$ = .04845 '/sec				
σ = surface tension									
of the liquid = .0024									
lb/ft; capillary rise									
= h_c = 0.1"									

NOTE: In both experiments, times at which the photographs (see plates 2 - 12) are taken are as follows: 20, 40, 60, 80, 100 and 160 secs. after the unsteady state begins.

The elevations of the liquid in the regulating reservoir B1 and the corresponding times during the experiment at which the liquid reached those levels are shown in Table I. Photographs (see Plates 2 - 7) have been taken at the times shown in Table I.

D. Experiment No. 2

In this experiment the starting steady state free surface is a horizontal line with a height of 18 inches above the bottom of the channel. The unsteady state is introduced by lowering the elevation of the liquid in reservoir B2 by opening valve G5. The top of the regulating spillway C2 is kept at an elevation of 12.15" above the bottom of the channel. Having opened the valve G5, the elevation of the top of the liquid in the regulating reservoir B2 is lowered to 12.15". The elevation of the liquid in the regulating reservoir B1 is, however, kept at the initial elevation of 18" -- this can be accomplished by adjusting valve G3. The elevations of the liquid in the regulating reservoir B2 and the corresponding times at which the liquid has reached the levels are shown in Table I.

Photographs of the varying shapes of the free surface, taken at different intervals (see Table I) between the introduction of the disturbance and the establishment of a steady state, are shown in Plates 8 - 12.

E. Comments on the Computer Programs of the Unsteady State Solution

In order to reduce machine time, one can use a large value for the length scale and thus carry out the computations for a computational model much smaller than the Hele-Shaw model used in the experiments.

This has the advantage of increasing the slope of the characteristic lines and hence one can use a much larger value for Δt in the characteristic grid.

The resulting computations obtained for the computational model correspond, however, to those obtainable by using the full-scale model in the equations derived for the model scales at the beginning of this chapter.

Because of the problem of instability, Δt has to be smaller than $\bar{k}_M/g$. Therefore, if the length scale were equal to unity, the number of iterations along the t-axis of the characteristic grid would be as large as:

total time taken in the experiment before the steady state is establish$/(\bar{k}_M/g)$.

The use of a large length scale, however, reduces the number of iterations along the t-axis considerably. The maximum value of Δx in the characteristic grid is limited by the minimum number of points required along the x-axis. One can therefore conclude that the use of a large length scale minimizes the number of iterations along the t-axis in the following ways:

1. A large length scale reduces the value of Δx, and thus the value of Δt (which must be smaller than or equal to the minimum slope of the characteristic lines times Δx) can be assumed less than $\bar{k}_M/g$ without increasing the number of iterations along the t-axis.
2. A large length scale increases the value of the minimum slope of the characteristic lines and thus increases the value of Δt.

The maximum value of the length scale is, however, limited by the maximum of the velocities of the fluid particles. This is so because of the restriction imposed on the slope of the characteristic lines, i.e., $\xi_+ > 0$ and $\xi_- < 0$. The above restriction leads to a maximum value for the scale length x_r as follows:

$$x_{r_{max}} = \frac{gH_{min}}{u^2_{max}} \qquad (66)$$

Another way of reducing machine time is to let the coefficient of permeability of the computational model be much larger than that of the experimental model. By doing so, the maximum value of the time interval Δt -- as far as the stability is concerned -- is increased and thus machine time is reduced. Also, as it follows from the equations derived for model scales, by choosing a large value for the coefficient of permeability for the computational model, the amount of time before which the steady state is reached is reduced and hence machine time will decrease. By this method, however, since the value of the velocities increase, the nature of flow in the computational model may become turbulent. This is contrary to the assumptions under which the equations for model scales were derived, and hence they cannot be used to correlate the results of the computational model to those of the experimental model.

F. Discussion of the Theoretical and Experimental Results

The results of the two experiments stated previously are shown both on the photographs of Plates 2 - 12 and by the solid curves of Figures 10 - 20.

In order to obtain the theoretical results, the same data used in the experiments are fed into the digital computer. These data are shown in Table I. The following sources of error is present in the results:

1. Because of the heat produced by the camera lights the temperature of the viscous liquid in the Hélé-Shaw model rises during the course of the experiment and an average temperature, as considered in the computation, produces a source of error in the theoretical results.
2. Because of mechanical difficulties, the width of the channel changes by five per cent and an average value of this width, as considered in the computations, does create a source of error in the theoretical results.
3. Errors involved in measuring the rate of raising or lowering the elevations of the liquid in the reservoirs effects the theoretical results.
4. The use of incorrect assumptions such as the presence of hydrostatic pressure on any vertical section and hypothetical coefficient of permeability is another cause of the small differences between the theoretical and experimental results.
5. Although in plotting the curves of Figures 10 - 20, the amount of capillary rise which equals .1 inch has been deducted from the elevation of the corresponding curves in Plates 2 - 12, due to the unsteady nature of the flow and also due to the curvature of the varying free surface,

.1 inch is only an average value for the amount of the capillary rise and depends both on the location of the point on the free surface and time.

G. Limitations to the Solution by the Method of Characteristics

When applying the method of characteristics to the solution of unsteady gravity flow through a dam with vertical faces, these factors must be considered:

1. If a) H_w (height of the outflow surface) equals zero, and b) the computations are carried out in two computer programs so that the existence of seepage surfaces is disregarded in the first program, then the horizontal component of the velocity near the outflow surface approaches infinity. This makes the slope of the characteristic lines equal to zero and hence the solution by the method of characteristics impossible.

   Furthermore, when $H_w = 0$, the nature of the flow may become turbulent since the velocities near the outflow surface increase very rapidly. This is again contrary to the assumption under which the fundamental equations were derived.

2. If a) H_e (inflow height) is much greater than H_w (outflow height), and b) all the calculations are not done in a single computer program so that the existence of the seepage surfaces is ignored in the first program, then the curvature of the free surfaces near the outflow

faces become very large at or about the time of steady condition. Hence the required number of points along the x-axis of the characteristic grid increase rapidly and thus as mentioned before, machine time will increase considerably. This was noticed in the program written for the first experiment where $H_e = 6H_w$.

The required number of points along the x-axis of the characteristic grid can be estimated from the following equation:

$$\left|\frac{\frac{d\bar{h}}{dx} - \frac{\Delta\bar{h}}{\Delta x}}{\frac{d\bar{h}}{dx}}\right|_{x=0} \leq \epsilon \; ,$$

or

$$\frac{\left.\frac{d\bar{h}}{dx}\right|_{x=0} - \frac{\bar{h}|_{x=L/N} - \bar{h}|_{x=0}}{L/N}}{\left.\frac{d\bar{h}}{dx}\right|_{x=0}} \leq \epsilon \; . \qquad (67)$$

Where $\bar{h}$ is the steady state solution of the free surface based on Dupuit's assumptions and computed by means of Equation (1), N is the required number of points along the x-axis, and ϵ is the required accuracy for the value of the discharge computed by the method of characteristics. If the computations are carried out in a single computer program, $\bar{h}_1$ should be substituted for $\bar{h}$ (see Figure 1).

In the above equation, it is assumed that the maximum curvature of the free surfaces occurs at the time of the steady state and at the outflow surface ($x = 0$). The above assumption is well justified in the case of the two experiments of this study.

If, however, one deals with a problem involving boundary conditions which do not justify the assumption underlying Equation (67), one must estimate the required number of points along the x-axis on the basis of the maximum curvature of the free surfaces encountered in the course of the unsteady state.

The number of points along the x-axis of the characteristic grid in Experiment No. 1 were taken to be equal to 200. This introduces a maximum error of 3.4% in the value of discharges computed in that program. In Experiment No. 2, however, 40 points along the x-axis of the characteristic grid are sufficient and the maximum error introduced in the value of the discharges does not exceed 1%.

The seriousness of the above limitations is, however, reduced when one considers the use of the supposition of a theoretical outflow height (equal to the sume of the real outflow height plus the height of the surface of seepage) along with the rest of the computations in a single computer program. By doing so, the danger of failure of the method of characteristics when the above-mentioned conditions are present will disappear or decrease to a great extent.

H. Conclusions

In the first part of this study, after a short presentation of the existing theory of the solution of steady free surface seepage through a porous bank with vertical faces, numerical results concerning the height of the seepage surface as a function of inflow height, outflow height, and the thickness of the porous bank are given. Although similar numerical results, presented in Reference 13, are already available, the results given in this study were obtained in much more detail and show some discrepancies with the results in Reference 13. Furthermore, results concerning the shape of the free surface are presented in this study (see Plates 16 - 23).

The results of the second part of this study (see Figures 10 - 20) are highly satisfactory and demonstrate the workability of the various assumptions introduced in the course of this study. One must, however, in solving problems of this kind, be aware of the limitations mentioned in Section G of this part.

In spite of the incorrect assumptions made in deriving the fundamental equations of flow in porous media, the experimental free surfaces check very closely with the theoretical ones. Determination of the other features of flow, such as the pressure or the velocity of a fluid particle at any point, has not been the goal of this study. Obviously, the value of pressure at any point, based on the previous assumptions, is different from the true pressure, especially at the points near the outflow surfaces. The horizontal component of the velocity at any vertical section, as computed in this study, is only an average value for the horizontal components of the velocities of all the points over that section. The vertical component of the velocity of the fluid particles was eliminated from the fundamental equations and was not studied. As for the shape of the free surface and the amount of discharge, one can claim that the fundamental equations derived in this study are valid for any shape of the region of flow and the method of characteristics can be applied for a numerical solution of these equations. Even when the surfaces of seepage exist, one can solve any two-dimensional unsteady state problem of seepage through porous media if the solution of the steady state case of the problem is known.

The existence of a capillary zone above the water table was neglected in this study. The theoretical results are compared with the experimental results obtained in a Hele-Shaw model. As mentioned in the introduction, comparison of the theoretical results with the actual field measurements is meaningless because of the existence of a capillary layer. Not only does the capillary layer increase the amount of discharge, but because of the unsteady nature of the flow, it is not easy to estimate the amount of discharge in the capillary layer and thus the correction of the actual discharges measured in field.

A more realistic distribution of the pressure and the velocity at any time may be determined by drawing a flow net in the region of flow for that time.(18) This can be accomplished as follows:

1. From the successive profiles of the free surface, the direction of the stream lines at the point of their intersection with the free surface will be found.
2. By the relaxation method, the flow net for the entire region of flow can be drawn.

It is believed that under the same assumptions as were used in deriving the two-dimensional fundamental equations, the differential equations of flow through axi-symmetric porous regions are also of the hyperbolic type. They can therefore be solved numerically by the method of characteristics. Unlike the case of this study, however, the lack of a steady state solution makes unsteady state problems of flow through axi-symmetrical regions difficult. This difficulty can be overcome if one finds either an empirical or an analytical formula which relates the height of a seepage surface to other properties of the flow such as the outflow height, amount of discharge, etc. Even in the absence of such formulas, experimental data gathered by means of models [e.g., electrical conduction models[19]] can serve as the solution of the steady state case.

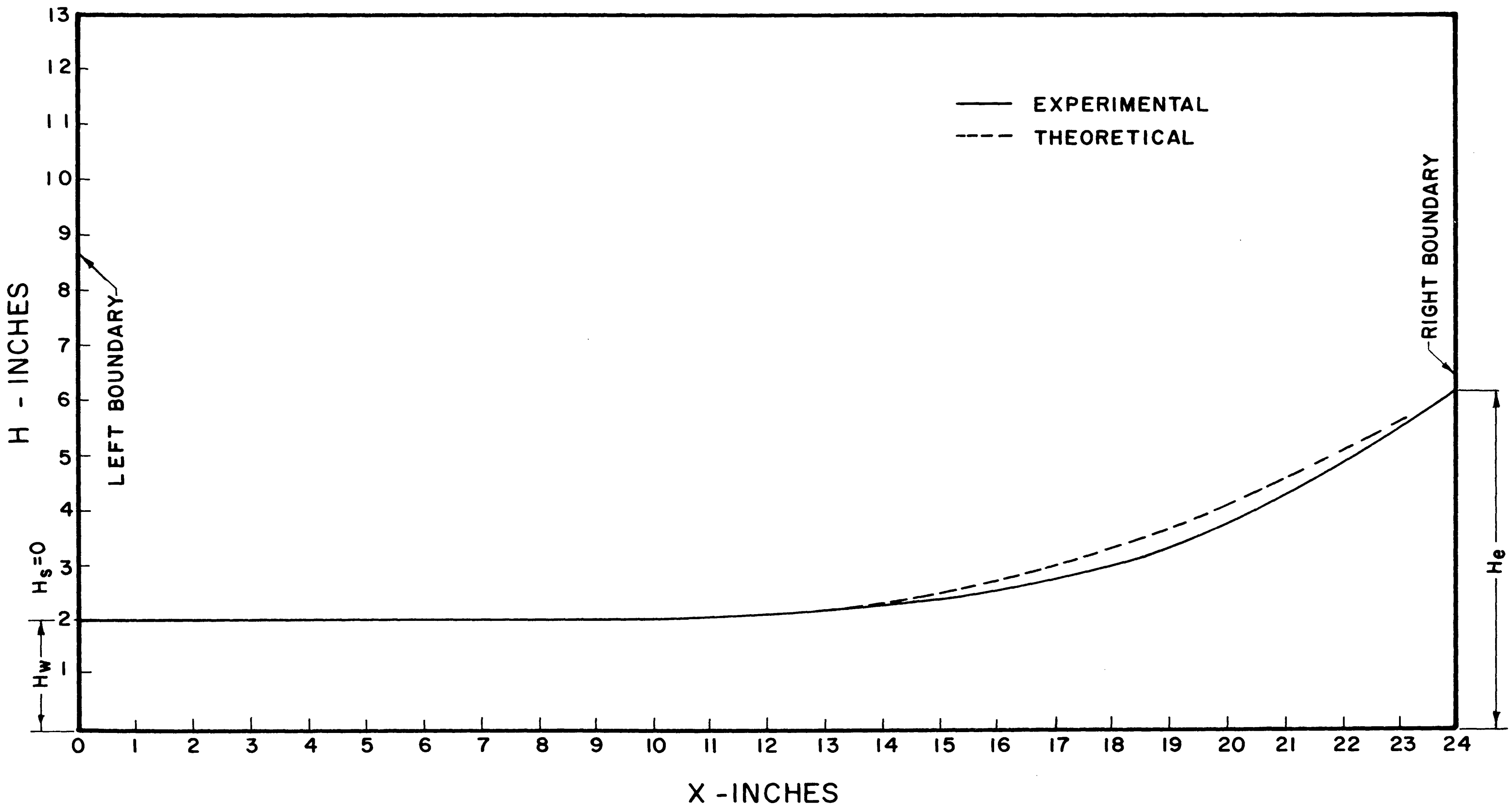

Figure 10. Free Surface at t = 20 Sec. in Experiment 1.

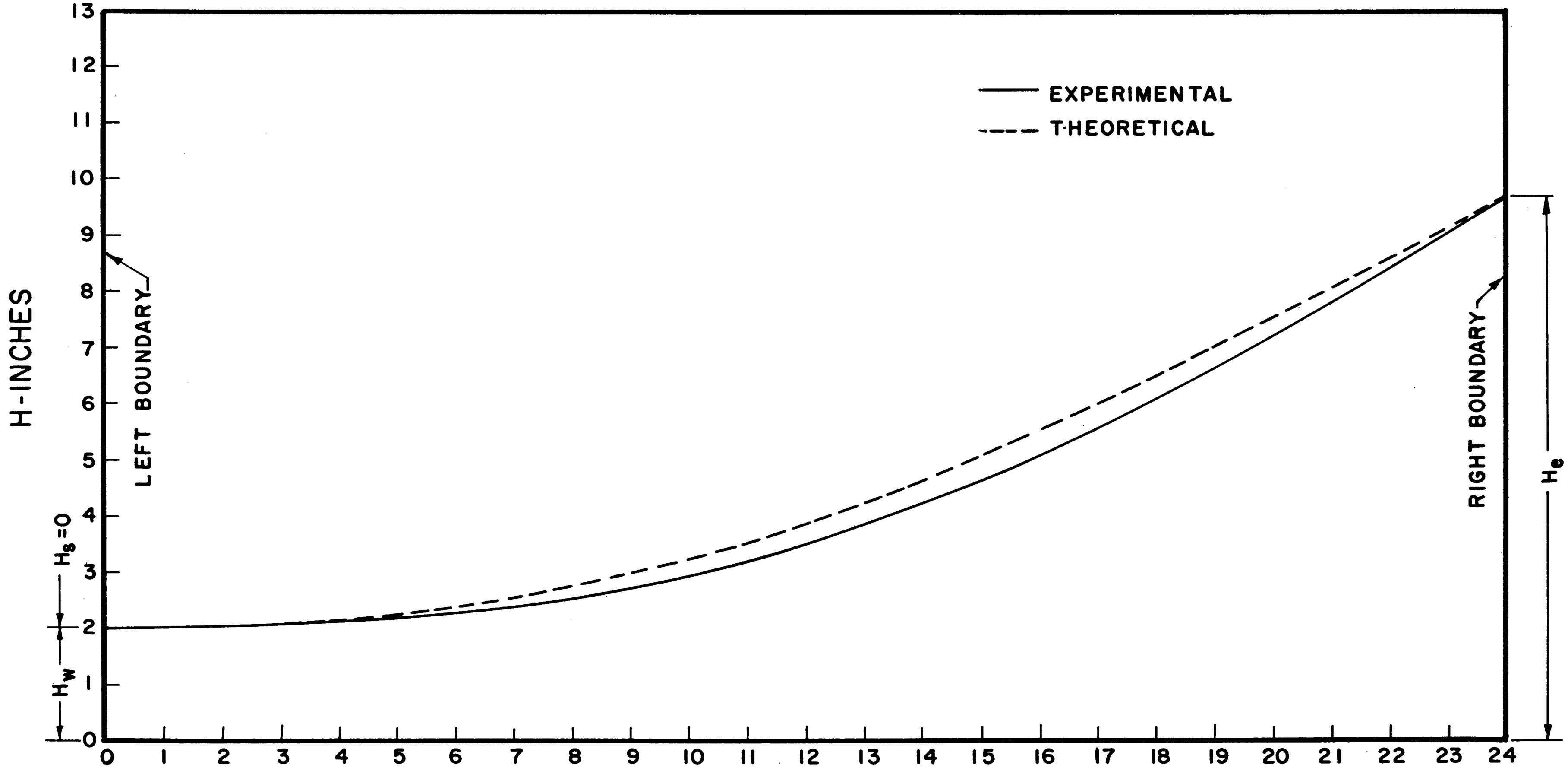

Figure 11. Free Surface at t = 40 Sec. in Experiment 1.

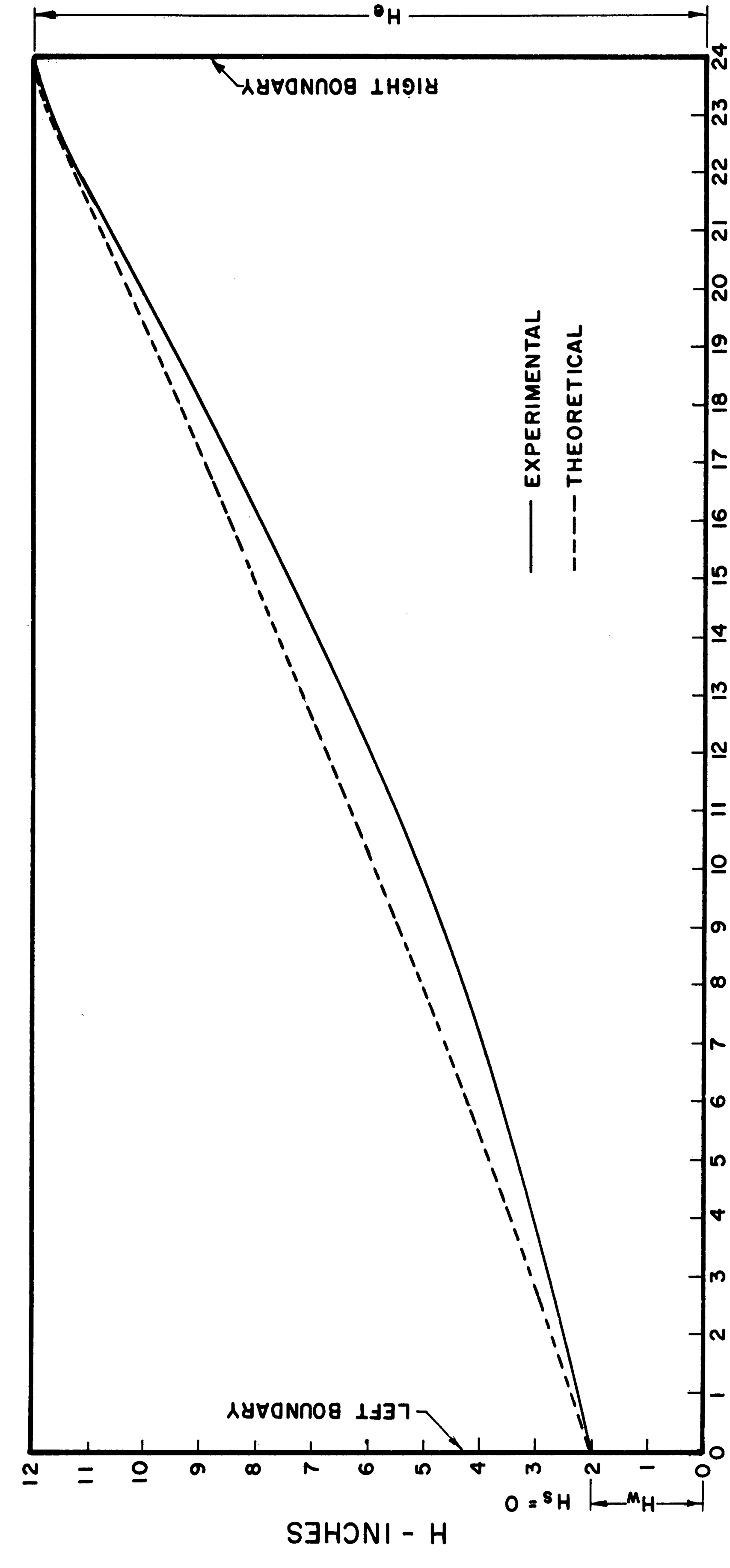

Figure 12. Free Surface at t = 60 Sec. in Experiment 1.

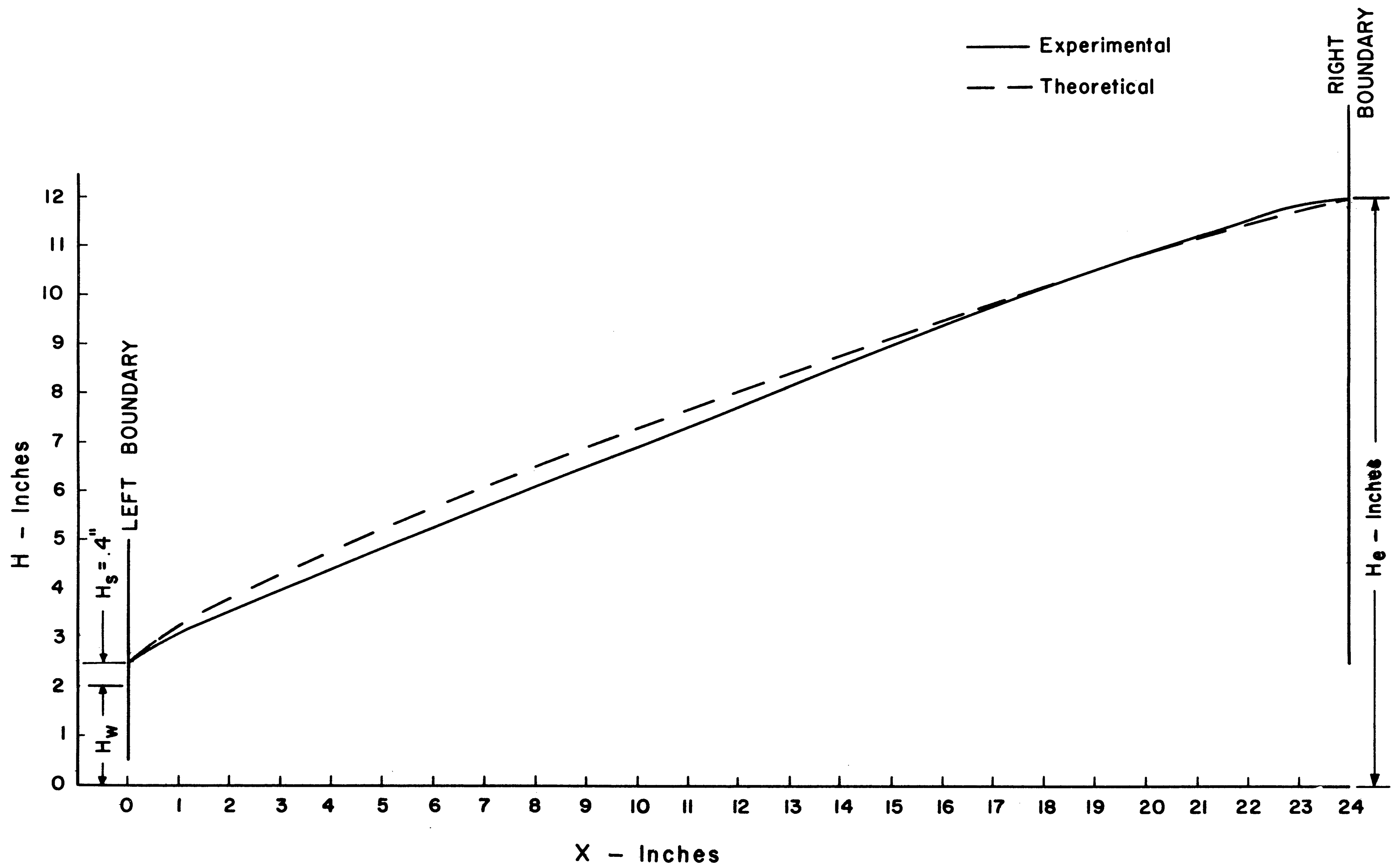

Figure 13. Free Surface at t = 80 Sec. in Experiment 1.

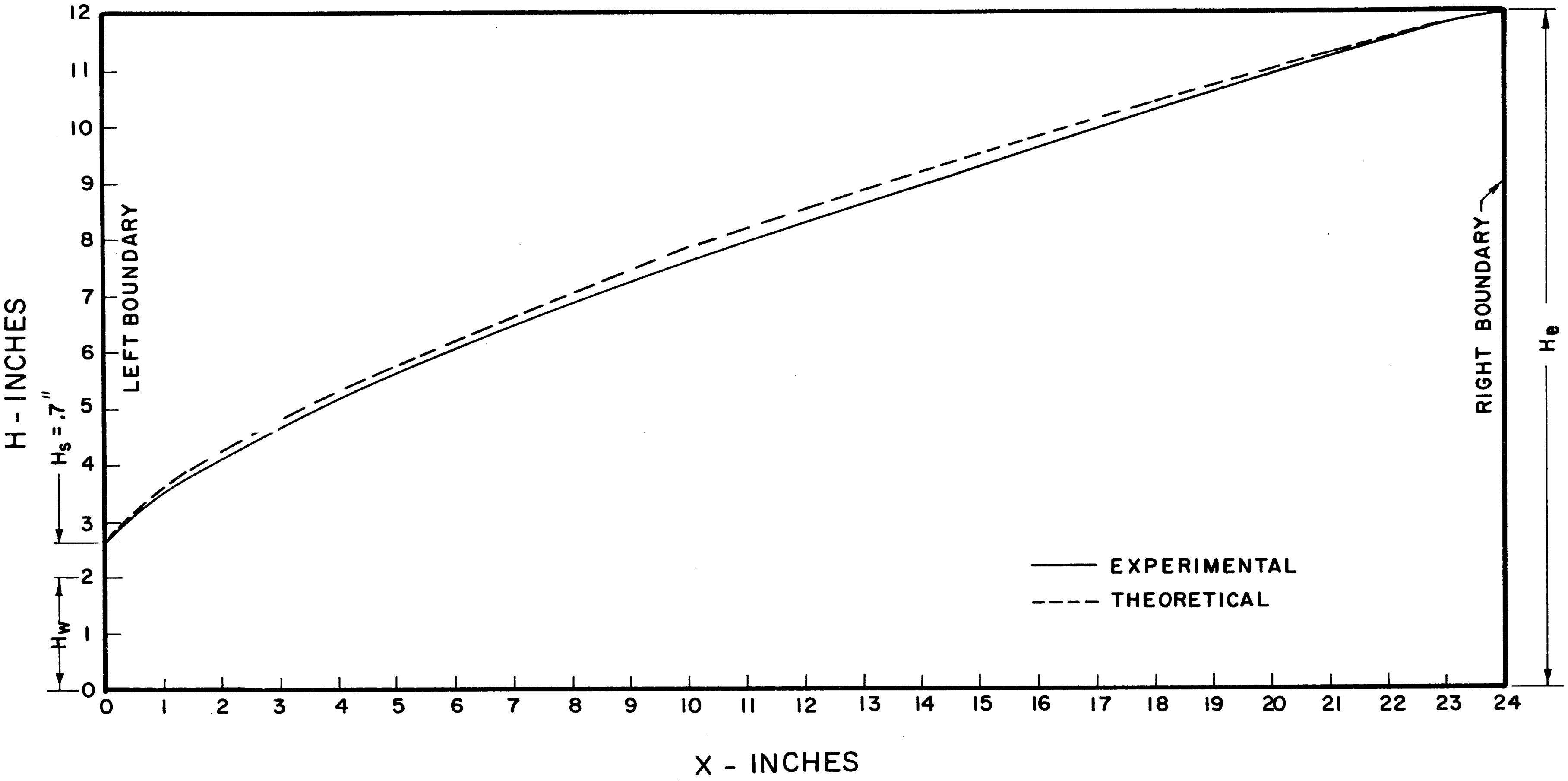

Figure 14. Free Surface at t = 100 Sec. in Experiment

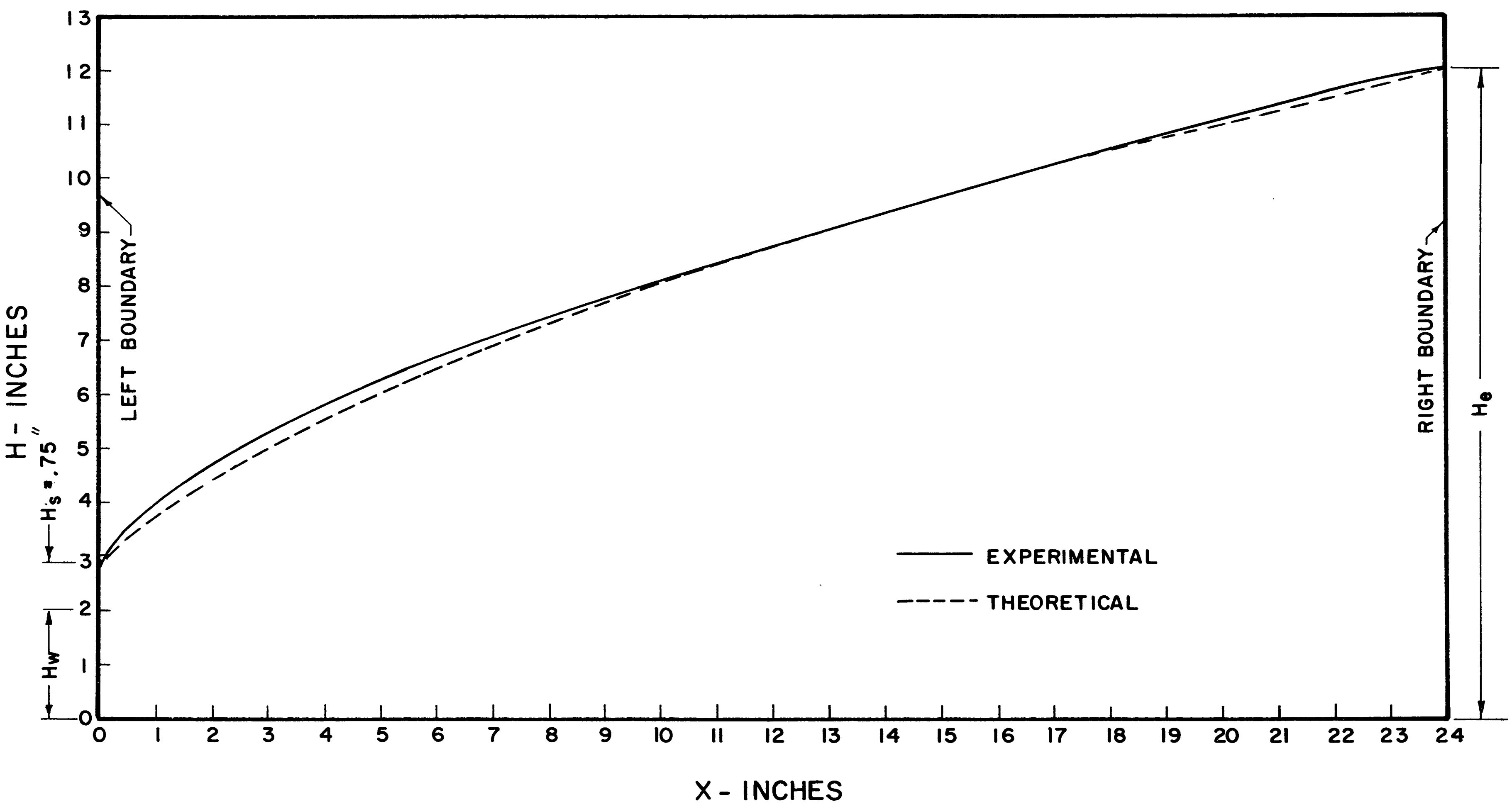

Figure 15. Steady State Free Surface in Experiment 1.

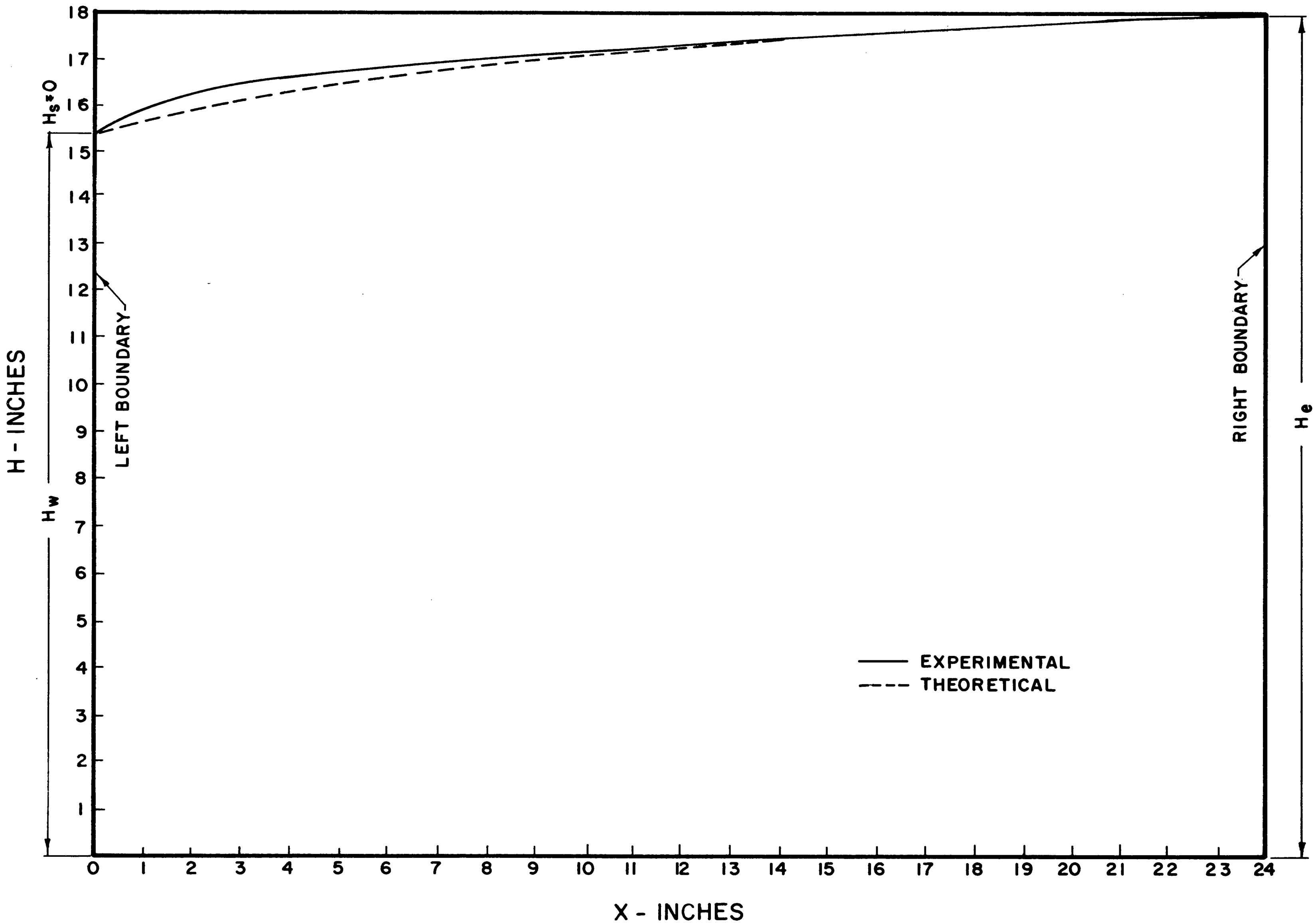

Figure 16. Free Surface at t = 20 Sec. in Experiment 2.

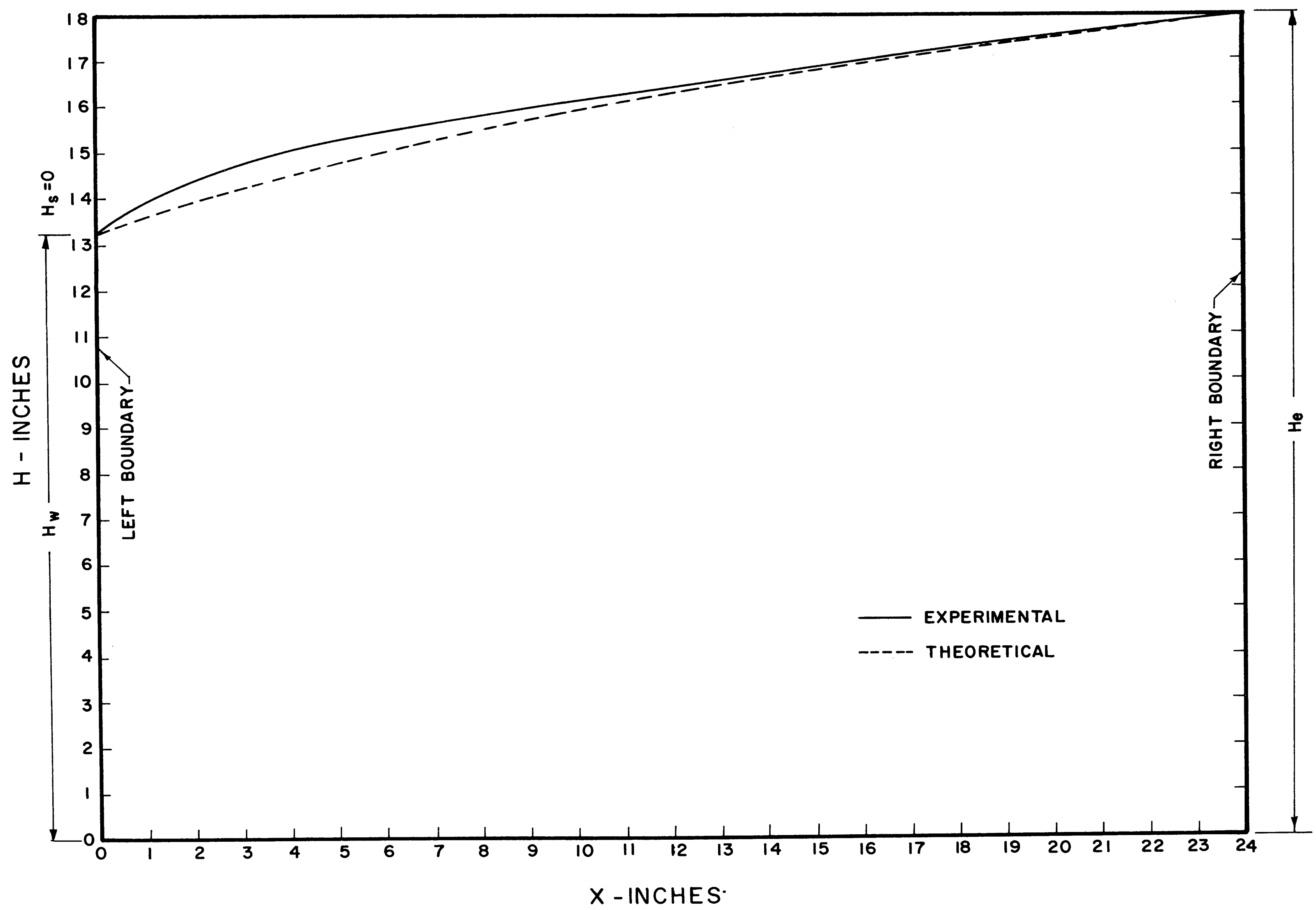

Figure 17. Free Surface at t = 40 Sec. in Experiment 2.

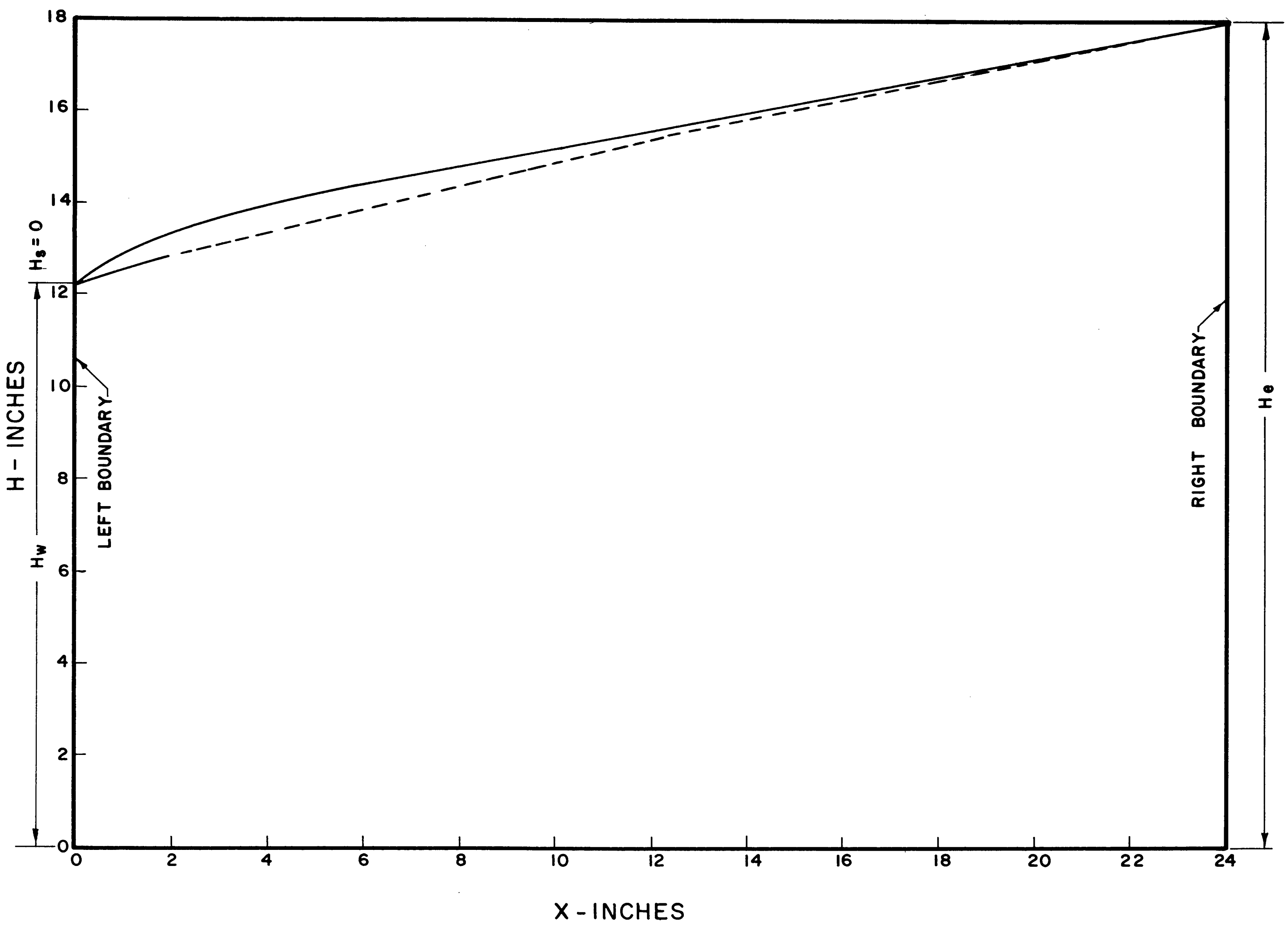

Figure 18. Free Surface at t = 60 Sec. in Experiment 2.

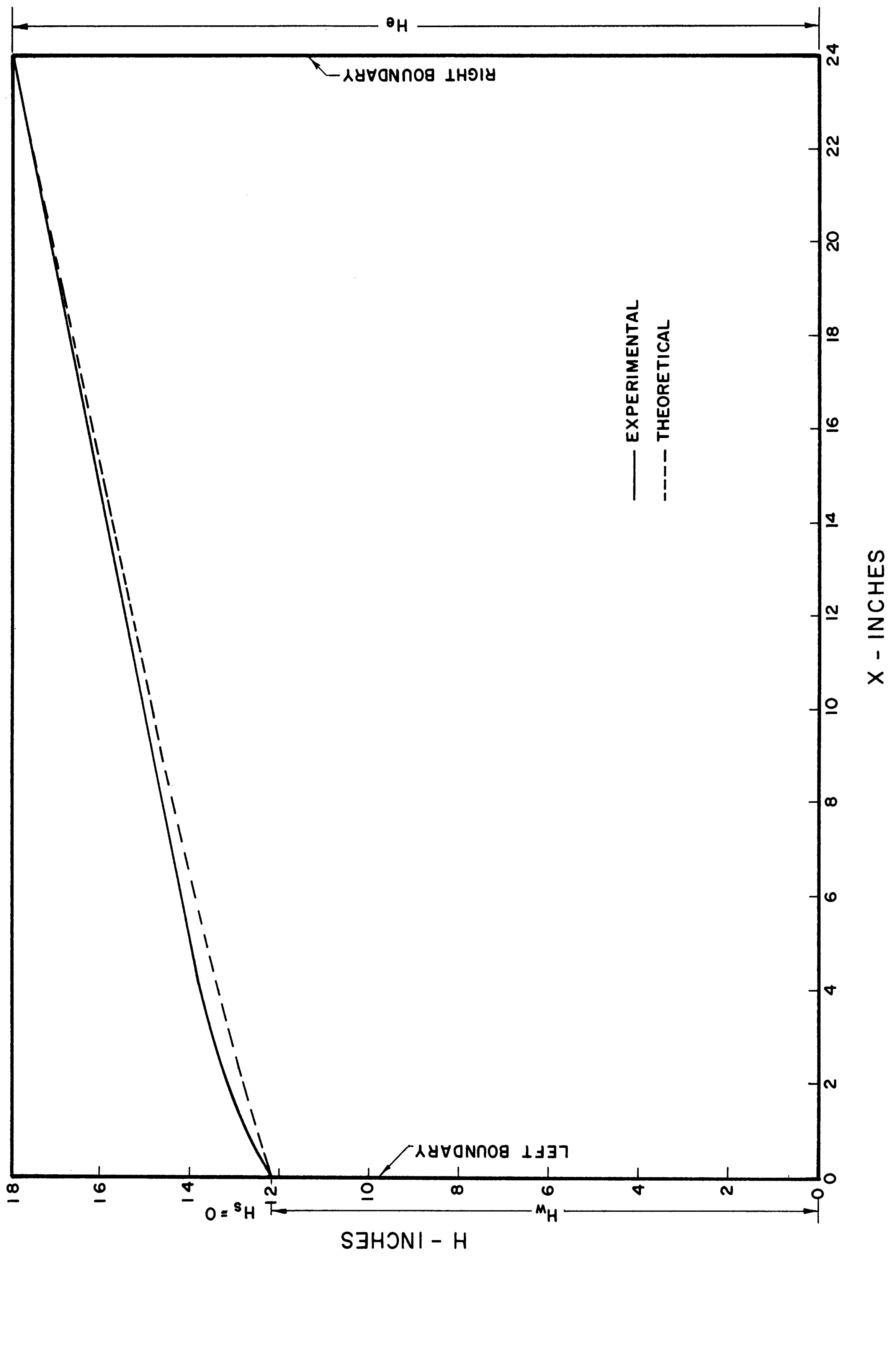

Figure 19. Free Surface at t = 80 Sec. in Experiment 2.

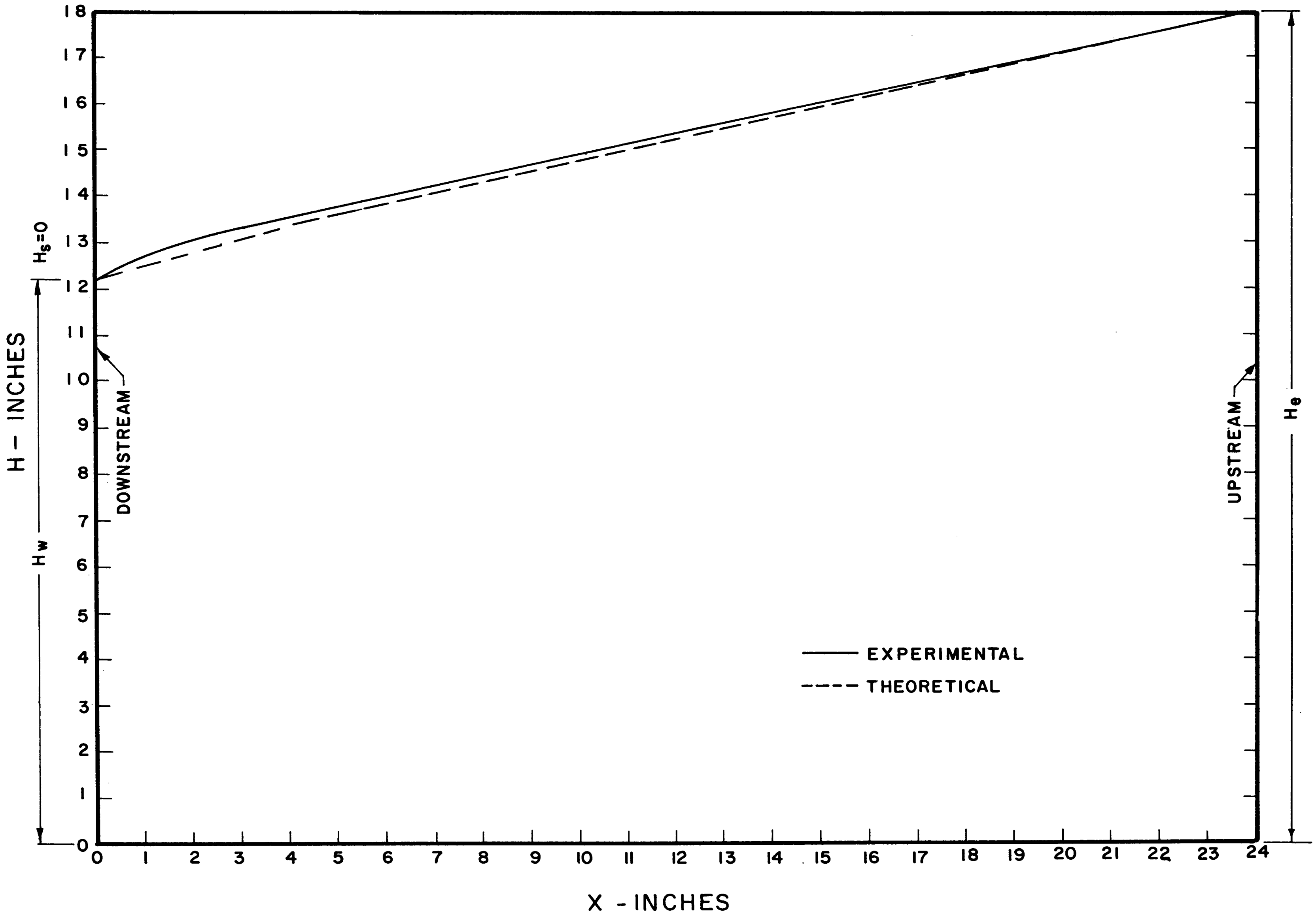

Figure 20. Steady State Free Surface in Experiment 2.

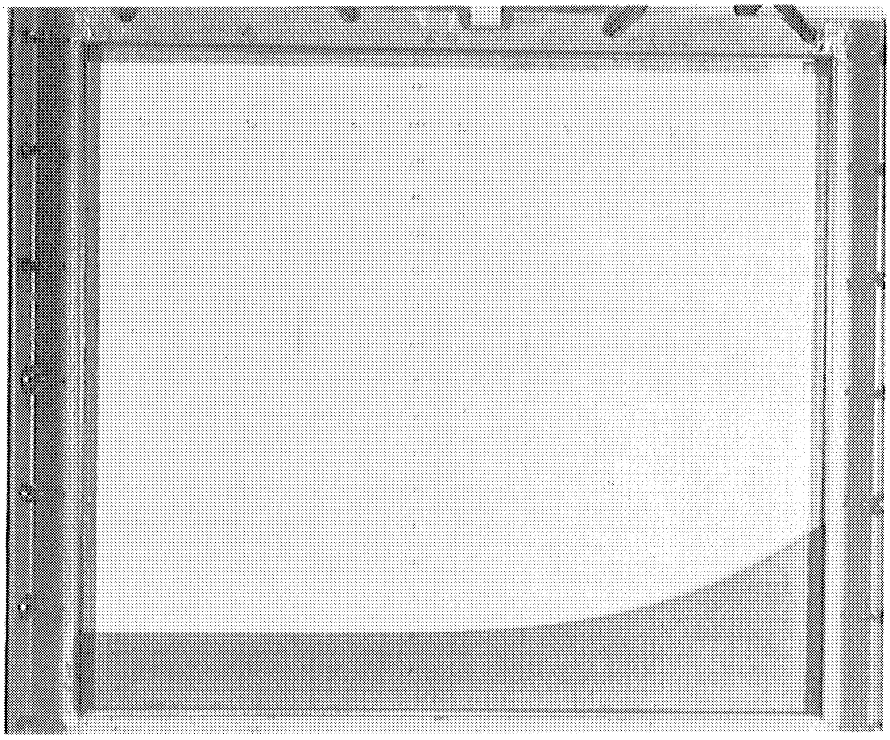

Plate 2
Experimental Free Surface (Experiment No. 1, T = 20 Sec.)

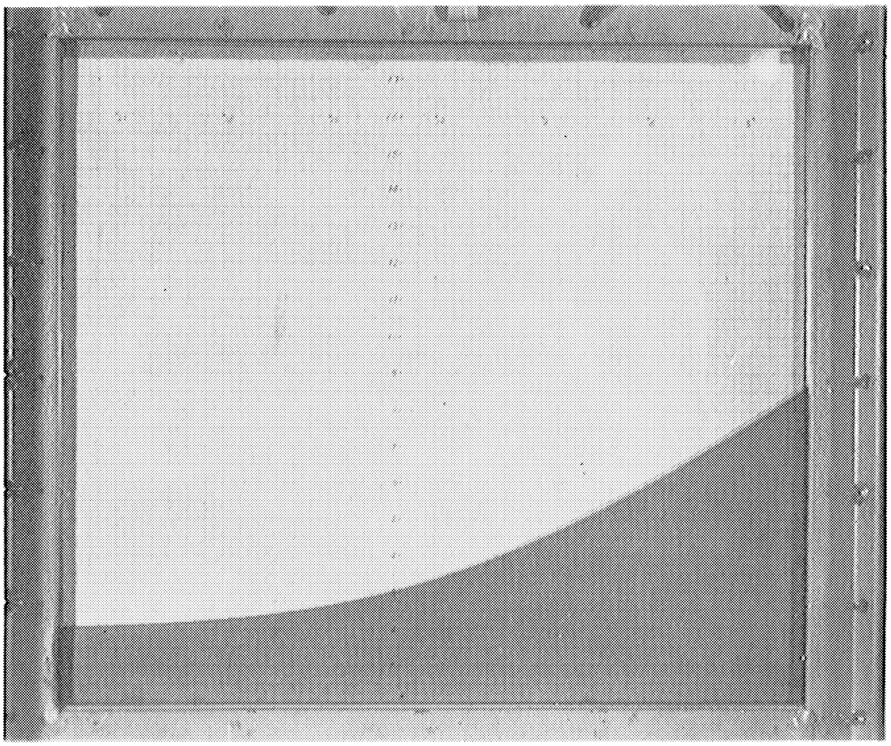

Plate 3
Experimental Free Surface (Experiment No. 1, T = 40 Sec.)

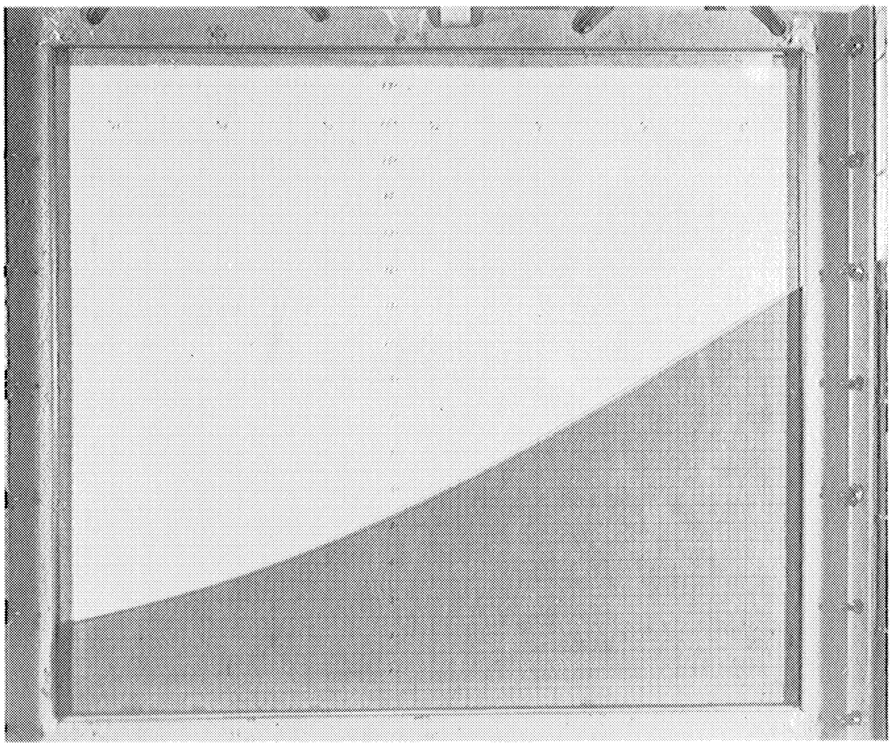

Plate 4
Experimental Free Surface (Experiment No. 1, T = 60 Sec.)

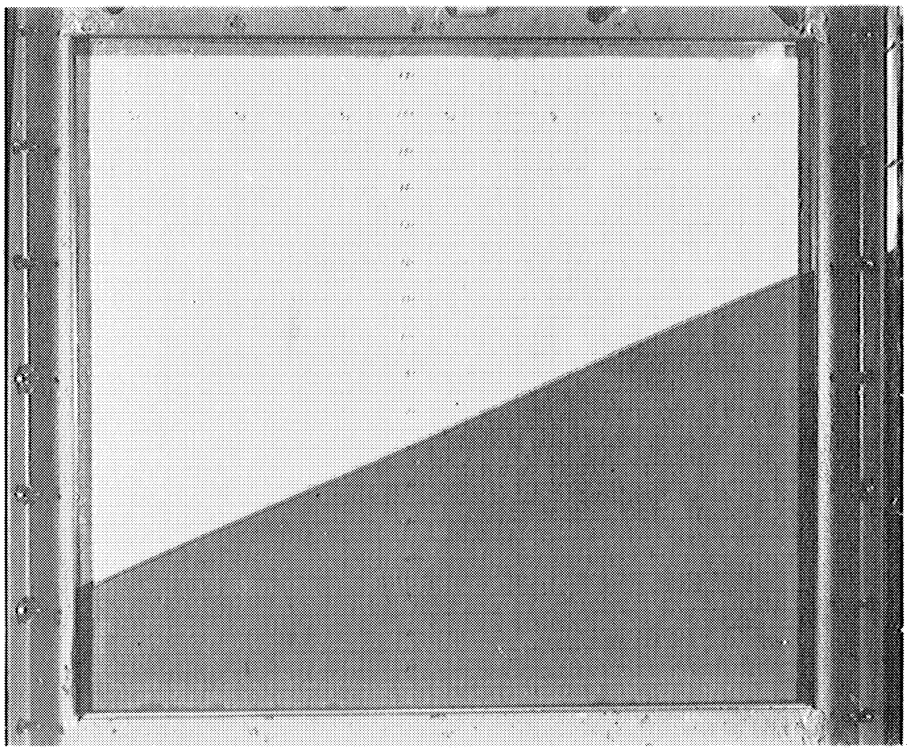

Plate 5
Experimental Free Surface (Experiment No. 1, T = 80 Sec.)

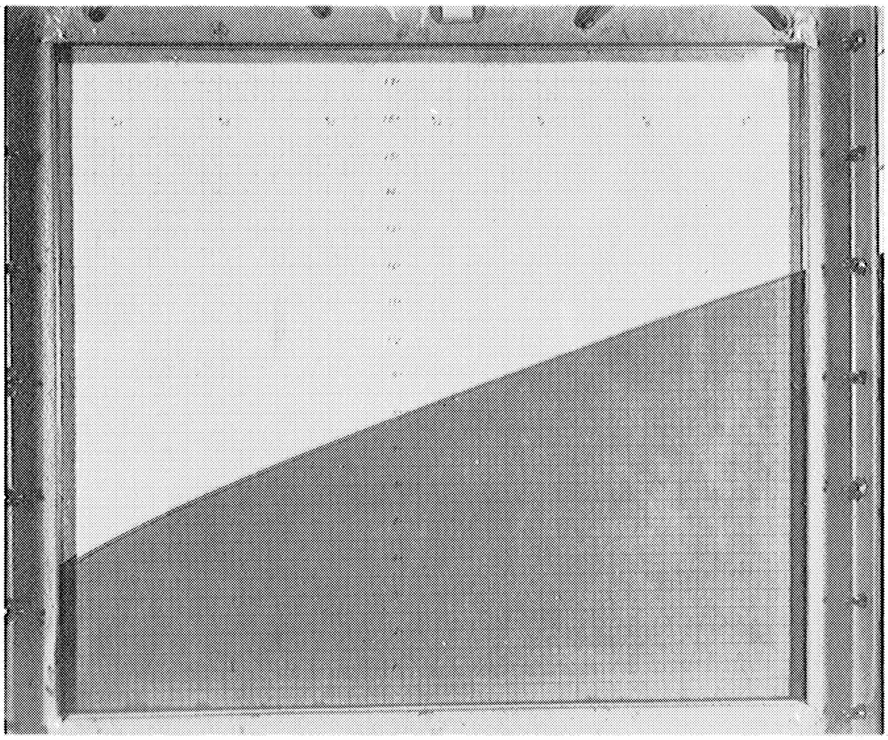

Plate 6
Experimental Free Surface (Experiment No. 1, T = 100 Sec.)

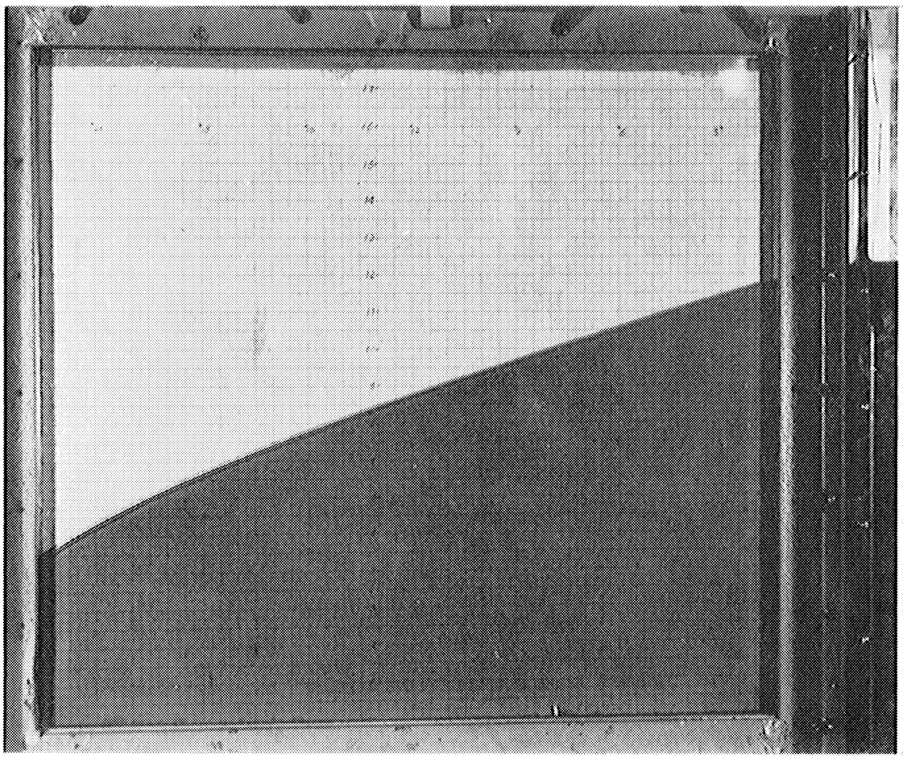

Plate 7
Experimental Free Surface (Experiment No. 1, Steady State)

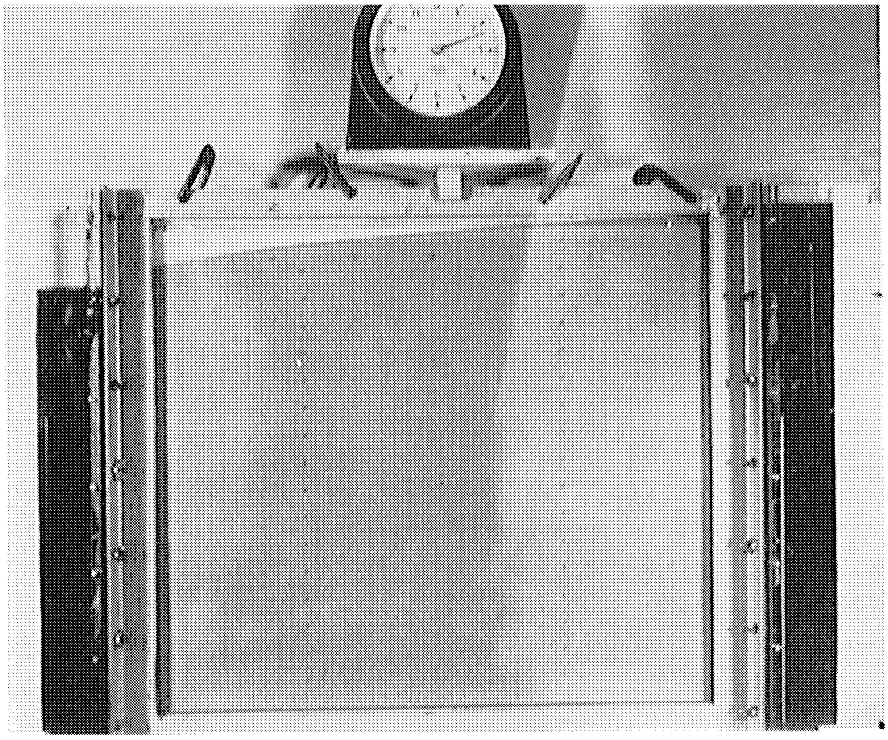

Plate 8
Experimental Free Surface (Experiment No. 2, T = 20 Sec.)

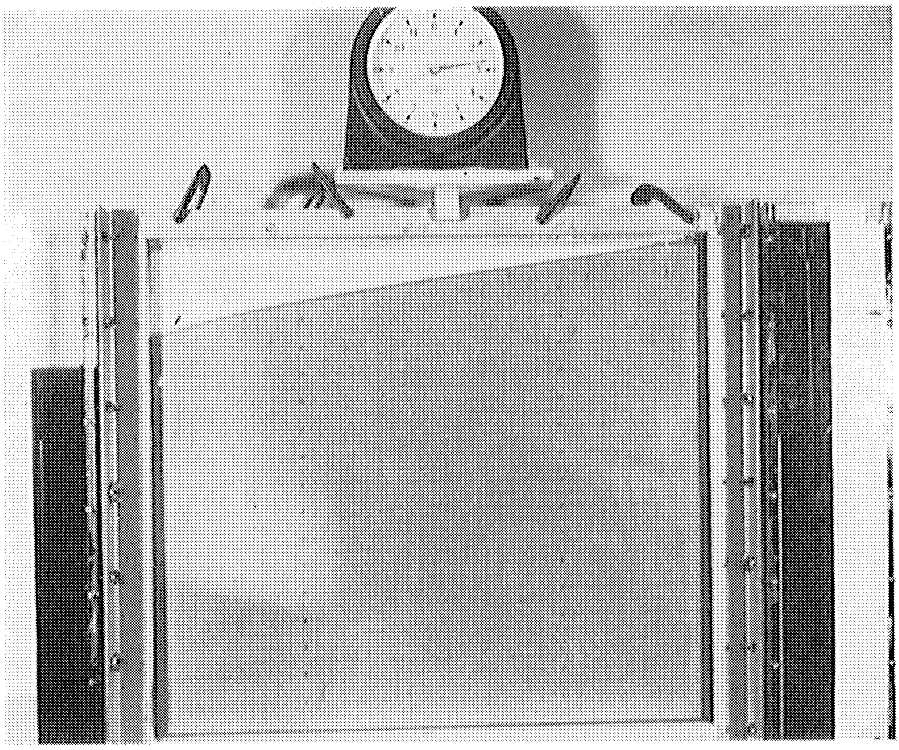

Plate 9
Experimental Free Surface (Experiment No. 2, T = 40 Sec.)

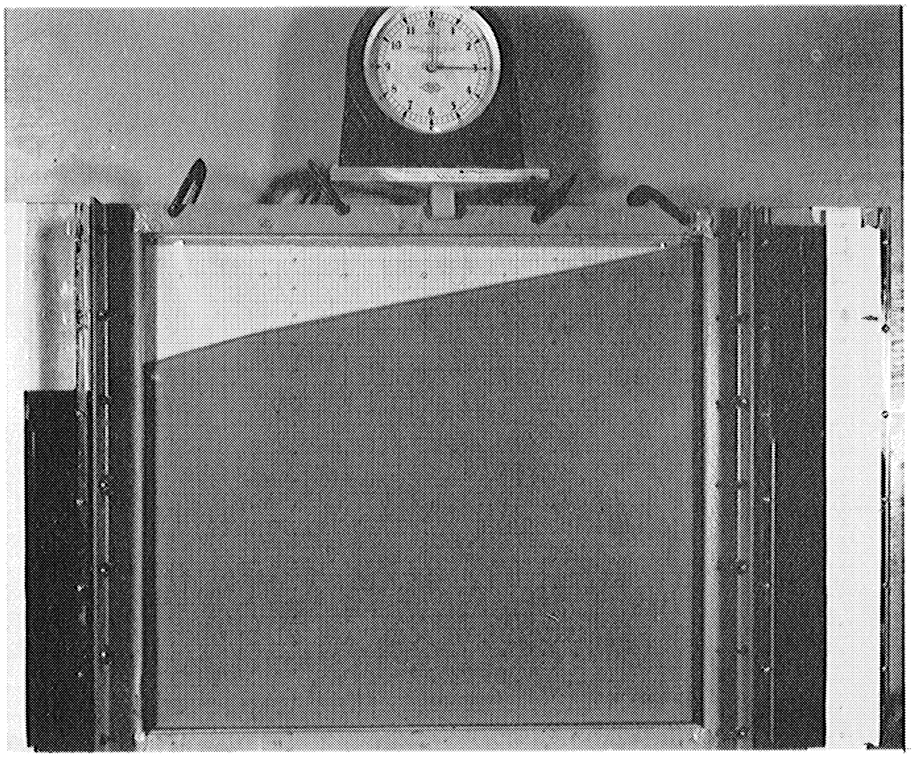

Plate 10
Experimental Free Surface (Experiment No. 2, T = 60 Sec.)

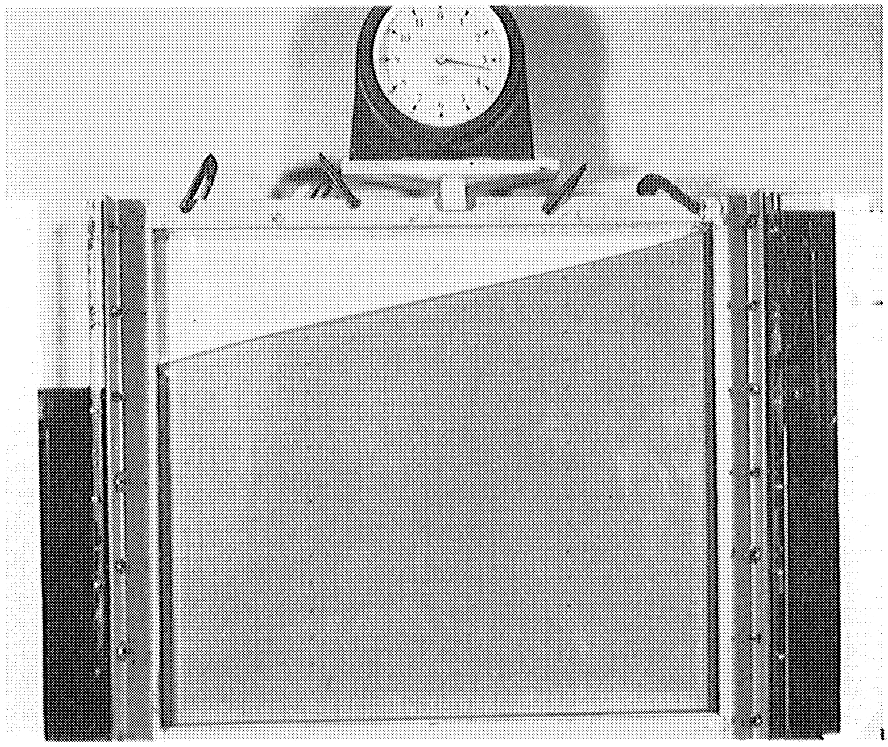

Plate 11
Experimental Free Surface (Experiment No. 2, T = 80 Sec.)

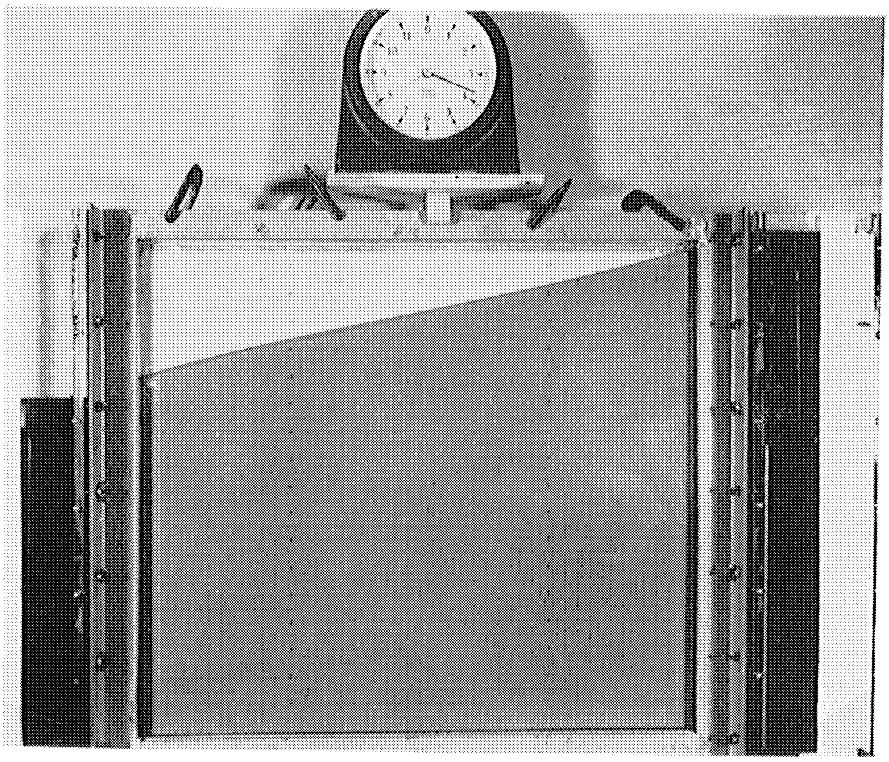

Plate 12
Experimental Free Surface (Experiment No. 2, Steady State)

APPENDIX I

MAD LANGUAGE PROGRAMS
FOR EXPERIMENT NO. 1

The MAD language programs written for Experiment No. 1 are presented on pages 70 and 75. In the first program on pages 70 - 71, the existence of the seepage surface was disregarded while the computations were carried out. In the second program (see pages 75 - 76), however, the height of the surface of seepage was taken into consideration (see Section F of Part II). Use has been made of the results obtained in the first program (see Figures 21 and 22) in order to compute the height of the seepage surface versus time (see Section C, Part I). Then, in the second program, the sum of this height and the actual outflow height is substituted for the value of the variable H at the boundary with the surface of seepage.

A COMPUTER PROGRAM WRITTEN FOR EXPERIMENT NO. 1
(The Existence of the Seepage Surface is not Taken Into Consideration)

```
        EXECUTE  FTRAP.(0)
        DIMENSION  U(300),U1(300),H(300),H1(300),ELR(10),TMR(10),
       1TCR(10),TSC(10),TSM(10),KP(10)
        PRINT  FORMAT  TITLE
        VECTOR  VALUES  TITLE=$1H ,S30,54HUNSTEADY  GRAVITY  FLOW OF
       1 LIQUIDS  IN  POROUS  MEDIA//*$
        INTEGER  I,JJ,LL,FFI,K,M,N,II,K0,N0,LLL
        II=4
        JJ=6
START   READ  FORMAT  DATA,KM,TTM,XR,ELR(0)...ELR(II),TMR(0)...TMR(II
       1),TSM(1)...TSM(JJ),L, DT, DX, KP(1)...KP(JJ),MM,NN
        KC=KM
        DD=DT/KC
        TTC=TTM/XR
        VECTOR  VALUES  DATA=$(6F12.5)*$
        THROUGH  A1,FOR  M=0,1,M.G.II
        ELR(M)=ELR(M)/XR
A1      TCR(M)=TMR(M)/XR
        LL=L/DX
        LLL=(LL/10)+.01
        FF=TTC/DT
        FFI=FF+.5
        TH=DT/(DX/XR)
        READ  FORMAT  DATA1,H(1)...H(LL),  U(1)...U(LL), H(0),U(0)
        VECTOR  VALUES  DATA1=$(10F7.5)*$
        THROUGH  A3,FOR  I=0,1,I.G.LL
A3      H(I)=H(I)/XR
        READ  FORMAT  DATA2,K0,N0
        VECTOR  VALUES  DATA2=$2I10*$
        K=K0
        N=N0
BEGIN   T=K*DT
        THROUGH  ANN,  FOR I=LL,-1,I.L.0
        KESPC=1./(U(I)+SQRT.(32.2*H(I)))
        BB=TH/KESPC
        WHENEVER  I.E.0,TRANSFER  TO  A10
        UR=U(I)*(1.-BB)+U(I-1)*BB
        HR=H(I)*(1.-BB)+H(I-1)*BB
        EEE=SQRT.(HR)
        WHENEVER  I.E.LL,TRANSFER  TO  RIGHT
A10     KESMC=1./(U(I)-SQRT.(32.2*H(I)))
        CC=TH/KESMC
        US=U(I)*(1.+CC)-U(I+1)*CC
        HS=H(I)*(1.+CC)-H(I+1)*CC
        EE=SQRT.(HS)
        WHENEVER  I.E.0, TRANSFER  TO  LEFT
        H1(I)=EE*EEE*(1.+(((US-UR)*(5.67*DD-.1765))/(EE+EEE)))
        U1(I)=UR-32.2*UR*DD-(H1(I)-HR)*(5.67/EEE)
ANN     CONTINUE
        THROUGH  A9,FOR  I=0,LLL,I.G.LL
A9      WHENEVER  .ABS.((H1(I)-H(I))/H(I)).G.MM,TRANSFER  TO  ON
        TRANSFER  TO  PRINT1
ON      WHENEVER K.E.KP (N),TRANSFER TO RESUL1
        WHENEVER  K.E.FFI
        T=T*XR
        THROUGH  A4, FOR  I=0,1,I.G.LL
        X=I*DX
        H1(I)=H1(I)*XR
A4      PRINT  RESULTS  U1(I),H1(I),X,T
```

```
               TRANSFER  TO  START
               OTHERWISE
               TRANSFER  TO COUNT
               END  OF  CONDITIONAL
  COUNT        K=K+1
               THROUGH  LOOP,  FOR  I=0,1,I.G.LL
               H(I)=H1(I)
               WHENEVER  U1(I).G.0.
               U(I)=0.
               OTHERWISE
               U(I)=U1(I)
  LOOP         END  OF  CONDITIONAL
               TRANSFER  TO  BEGIN
  RESUL1       T=T*XR
               N=N+1
               THROUGH  PRINT, FOR  I=0,LLL,I.G.LL
               X=I*DX
               H1(I)=H1(I)*XR
  PRINT        PRINT  RESULTS  T,X,U1(I),H1(I)
               QOUTP=-H1(0)*U1(0)/(L*KC)
               HECS=L*SQRT.(2.*QOUTP+(H1(0)/L).P.2)
               LPCHE=L/HECS
               HWPCHE=H1(0)/HECS
               HEPCL=1./LPCHE
               HWPCL=H1(0)/L
               HW=H1(0)
               PRINT  RESULTS  T,QOUTP,HECS,L,HW,LPCHE,HWPCHE,HEPCL,HWPCL
               THROUGH  A2,FOR  I=0,LLL,I.G.LL
  A2           H1(I)=H1(I)/XR
               TRANSFER  TO  COUNT
  RIGHT        WHENEVER  T.GE.TCR(II)
               H1(I)=ELR(II)
               TRANSFER  TO  A8
               OTHERWISE
               M=0
               TRANSFER  TO  A7
               END  OF  CONDITIONAL
  A7           WHENEVER  T.LE.TCR(M+1)
               H1(I)=ELR(M)+(T-TCR(M))*(ELR(M+1)-ELR(M))/(TCR(M+1)-TCR(M))
               TRANSFER  TO  A8
               OTHERWISE
               M=M+1
               END  OF  CONDITIONAL
               TRANSFER  TO  A7
  A8           U1(I)=UR-32.2*UR*(  DD  )-(H1(I)-HR)*(5.67/EEE)
               TRANSFER  TO  ANN
  LEFT         H1(I)=H(0)
               U1(I)=US+(H1(I)-HS)*(5.67/EE)-(32.2*US*DD)
               TRANSFER  TO  ANN
  PRINT1       THROUGH  A0,FOR  I=0,LLL,I.G.(LL-LLL)
  A0           WHENEVER  .ABS.(((U1(I+LLL)*H1(I+LLL)-U1(I)*H1(I))/(U1(I)*
              1H1(I)))).G.NN,TRANSFER  TO  ON
               TRANSFER  TO  PRINT4
  PRINT4       T=T*XR
               THROUGH  PRINT2,FOR  I=0,LLL,I.G.LL
               H1(I)=H1(I)*XR
               X=DX*I
  PRINT2       PRINT  RESULTS  X,T,H1(I),U1(I)
               QOUTP=-H1(0)*U1(0)/(L*KC)
               HECS=L*SQRT.(2.*QOUTP+(H1(0)/L).P.2)
               LPCHE=L/HECS
               HWPCHE=H1(0)/HECS
               HEPCL=1./LPCHE
               HWPCL=H1(0)/L
               HW=H1(0)
               PRINT  RESULTS  T,QOUTP,HECS,L,HW,LPCHE,HWPCHE,HEPCL,HWPCL
               TRANSFER  TO  START
               END  OF  PROGRAM
```

THEORETICAL RESULTS OF EXPERIMENT NO. 1
(The Existence of the Seepage Surface is Not Taken Into Consideration)

T = 20.00

X =	.000000,	U1(0) =	3.502364E-08,	H1(0) =	.166670
X =	.200000,	U1(20) =	-2.749190E-06,	H1(20) =	.166675
X =	.400000,	U1(40) =	-2.617029E-05,	H1(40) =	.166746
X =	.600000,	U1(60) =	-1.586491E-04,	H1(60) =	.167197
X =	.800000,	U1(80) =	-7.232803E-04,	H1(80) =	.169355
X =	1.000000.	U1(100) =	-2.538783E-03,	H1(100) =	.177383
X =	1.200000,	U1(120) =	-6.553812E-03,	H1(120) =	.199930
X =	1.400000,	U1(140) =	-.012107,	H1(140) =	.246031
X =	1.600000,	U1(160) =	-.017170,	H1(160) =	.317253
X =	1.800000,	U1(180) =	-.020778,	H1(180) =	.408369
X =	2.000000,	U1(200) =	-.023107,	H1(200) =	.513009
QOUTP =	-6.924543E-08,	HECS =	.166668,	L =	2.000000
HW =	.166670,	HWPCL =	.083335		
LPCHE =	11.999880,	HWPCHE =	1.000010,	HEPCL =	.083334

T = 40.00

X =	.000000,	U1(0) =	-1.195539E-03,	H1(0) =	.166670
X =	.200000,	U1(20) =	-1.840494E-03,	H1(20) =	.174052
X =	.400000,	U1(40) =	-3.983651E-03,	H1(40) =	.188301
X =	.600000,	U1(60) =	-7.582569E-03,	H1(60) =	.216662
X =	.800000,	U1(80) =	-.011734,	H1(80) =	.263749
X =	1.000000,	U1(100) =	-.015363,	H1(100) =	.329151
X =	1.200000,	U1(120) =	-.018066,	H1(120) =	.409202
X =	1.400000,	U1(140) =	-.019946,	H1(140) =	.499812
X =	1.600000,	U1(160) =	-.021229,	H1(160) =	.597735
X =	1.800000,	U1(180) =	-.022102,	H1(180) =	.700662
X =	2.000000,	U1(200) =	-.022684,	H1(200) =	.806938
QOUTP =	2.363707E-03,	HECS =	.216075,	L =	2.000000
HW =	.166670,	HWPCL =	.083335	HEPCL =	.108038
HWPCHE =	.771351,	LPCHE =	9.256031,		

T = 60.00

X =	.000000,	U1(0) =	-.016686,	H1(0) =	.166670
X =	.200000,	U1(20) =	-.014113,	H1(20) =	.240308
X =	.400000,	U1(40) =	-.015305,	H1(40) =	.311290
X =	.600000,	U1(60) =	-.017075,	H1(60) =	.389043
X =	.800000,	U1(80) =	-.018521,	H1(80) =	.473934
X =	1.000000,	U1(100) =	-.019355,	H1(100) =	.563610
X =	1.200000,	U1(120) =	-.019582,	H1(120) =	.655323
X =	1.400000,	U1(140) =	-.019294,	H1(140) =	.746579
X =	1.600000,	U1(160) =	-.018605,	H1(160) =	.835374
X =	1.800000,	U1(180) =	-.017623,	H1(180) =	.920201
X =	2.000000,	U1(200) =	-.016443,	H1(200) =	1.000000
QOUTP =	.032990,	HECS =	.540088,	L =	2.000000
HW =	.166670,	HWPCL =	.083335	HEPCL =	.270044
HWPCHE =	.308598,	LPCHE =	3.703097,		

T = 80.00

X =	.000000,	U1(0) =	-.043962,	H1(0) =	.166670
X =	.200000,	U1(20) =	-.024358,	H1(20) =	.323482
X =	.400000,	U1(40) =	-.019848,	H1(40) =	.428677
X =	.600000,	U1(60) =	-.017754,	H1(60) =	.518143
X =	.800000,	U1(80) =	-.016484,	H1(80) =	.599452
X =	1.000000,	U1(100) =	-.015552,	H1(100) =	.675403
X =	1.200000,	U1(120) =	-.014767,	H1(120) =	.747197
X =	1.400000,	U1(140) =	-.014045,	H1(140) =	.815370
X =	1.600000,	U1(160) =	-.013350,	H1(160) =	.880162
X =	1.800000,	U1(180) =	-.012667,	H1(180) =	.941686
X =	2.000000,	U1(200) =	-.011991,	H1(200) =	1.000000
QOUTP =	.086917,	HECS =	.850360,	L =	2.000000
HW =	.166670,	HWPCL =	.083335	HEPCL =	.425180
HWPCHE =	.195999,	LPCHE =	2.351944,		

T = 100.00

X =	.000000,	U1(0) =	-.053725,	H1(0) =	.166670
X =	.200000,	U1(20) =	-.026879,	H1(20) =	.349786
X =	.400000,	U1(40) =	-.020798,	H1(40) =	.463133
X =	.600000,	U1(60) =	-.017748,	H1(60) =	.554882
X =	.800000,	U1(80) =	-.015834,	H1(80) =	.634767
X =	1.000000,	U1(100) =	-.014479,	H1(100) =	.706824
X =	1.200000,	U1(120) =	-.013440,	H1(120) =	.773148
X =	1.400000,	U1(140) =	-.012595,	H1(140) =	.834966
X =	1.600000,	U1(160) =	-.011878,	H1(160) =	.893057
X =	1.800000,	U1(180) =	-.011250,	H1(180) =	.947944
X =	2.000000,	U1(200) =	-.010688,	H1(200) =	1.000000
QOUTP =	.106220,	HECS =	.936770,	L =	2.000000
HW =	.166670,	HWPCL =	.083335	HEPCL =	.468385
HWPCHE =	.177920,	LPCHE =	2.134995,		

T = 160.00

X =	.000000,	U1(0) =	-.057221,	H1(0) =	.166670
X =	.200000,	U1(20) =	-.027711,	H1(20) =	.359029
X =	.400000,	U1(40) =	-.021128,	H1(40) =	.475123
X =	.600000,	U1(60) =	-.017766,	H1(60) =	.567716
X =	.800000,	U1(80) =	-.015634,	H1(80) =	.647211
X =	1.000000,	U1(100) =	-.014125,	H1(100) =	.718013
X =	1.200000,	U1(120) =	-.012984,	H1(120) =	.782490
X =	1.400000,	U1(140) =	-.012082,	H1(140) =	.842092
X =	1.600000,	U1(160) =	-.011345,	H1(160) =	.897787
X =	1.800000,	U1(180) =	-.010728,	H1(180) =	.950256
X =	2.000000,	U1(200) =	-.010202,	H1(200) =	1.000000
QOUTP =	.113132,	HECS =	.965835,	L =	2.000000
HW =	.166670,	HWPCL =	.083335	HEPCL =	.482917
HWPCHE =	.172566,	LPCHE =	2.070747,		

NOTE: The above results are used (see Figures 21 and 22) to determine the height of the seepage surface H_s as a function of time in Experiment No. 1.

A COMPUTER PROGRAM WRITTEN FOR EXPERIMENT NO. 1
(The Existence of the Seepage Surface is Taken Into Consideration)

```
          EXECUTE  FTRAP.(0)
          DIMENSION  U(300),U1(300),H(300),H1(300),ELR(10),TMR(10),
         1TCR(10),TSC(10),TSM(10),KP(10),TML(10),TCL(10),ELL(10)
          PRINT  FORMAT  TITLE
          VECTOR  VALUES  TITLE=$1H ,S30,54HUNSTEADY  GRAVITY  FLOW OF
         1 LIQUIDS  IN  POROUS  MEDIA//*$
          INTEGER  I,JJ,LL,FFI,K,M,N,II,KO,NO,LLL
          II=4
          JJ=6
START     READ  FORMAT  DATA,KM,TTM,XR,ELR(0)...ELR(II),TMR(0)...TMR(II
         1),TSM(1)...TSM(JJ),L,  DT, DX,KP(1)...KP(JJ),MM,NN,HO,
         1TML(0)...TML(JJ),ELL(0)...ELL(JJ)
          KC=KM
          DD=DT/KC
          TTC=TTM/XR
          HO=HO/XR
          VECTOR  VALUES  DATA=$(6F12.5)*$
          THROUGH  A1,FOR  M=0,1,M.G.II
          ELR(M)=ELR(M)/XR
A1        TCR(M)=TMR(M)/XR
          THROUGH   A2,FOR  M=0,1,M.G.JJ
          ELL(M)=ELR(L)/XR
A2        TCL(M)=TML(M)/XR
          LL=L/DX
          LLL=(L/10)+.01
          FF=TTC/DT
          FFI=FF+.5
          TH=DT/(DX/XR)
          READ  FORMAT  DATA1,H(1)...H(LL),U(1)...U(LL),H(0),U(0)
          VECTOR  VALUES  DATA1=$(10F7.5)*$
          THROUGH  A3,FOR  I=0,1,I.G.LL
A3        H(I)=H(I)/XR
          READ  FORMAT  DATA2,KO,NO
          VECTOR  VALUES  DATA2=$2I10*$
          K=KO
          N=NO
BEGIN     T=K*DT
          THROUGH  ANN,  FOR I=0,1,I.G.LL
          KESPC=1./(U(I)+SQRT.(32.2*H(I)))
          BB=TH/KESPC
          WHENEVER  I.E.0,TRANSFER  TO   A10
          UR=U(I)*(1.-BB)+U(I-1)*BB
          HR=H(I)*(1.-BB)+H(I-1)*BB
          EEE=SQRT.(HR)
          WHENEVER  I.E.LL,TRANSFER  TO  RIGHT
A10       KESMC=1./(U(I)-SQRT.(32.2*H(I)))
          CC=TH/KESMC
          US=U(I)*(1.+CC)-U(I+1)*CC
          HS=H(I)*(1.+CC)-H(I+1)*CC
          EE=SQRT.(HS)
          WHENEVER  I.E.0, TRANSFER  TO  LEFT
          H1(I)=EE*EEE*(1.+(((US-UR)*(5.67*DD-.1765))/(EE+EEE)))
          U1(I)=UR-32.2*UR*DD-(H1(I)-HR)*(5.67/EEE)
ANN       CONTINUE
          THROUGH  A9,FOR  I=0,LLL,I.G.LL
A9        WHENEVER  .ABS.((H1(I)-H(I))/H(I)).G.MM,TRANSFER  TO  ON
          TRANSFER  TO  PRINT1
ON        WHENEVER K.E.KP (N),TRANSFER TO RESUL1
          WHENEVER  K.E.FFI
          T=T*XR
          THROUGH  A4,FOR  I=0,1,I.G.LL
          X=I*DX
          H1(I)=H1(I)*XR
A4        PRINT  RESULTS  U1(I),H1(I),X,T
          TRANSFER  TO  START
          END  OF  CONDITIONAL
```

```
COUNT    K=K+1
         DD=DD*((H1(LL).P.2)-(H1(0).P.2))/((H1(LL).P.2)-(H0.P.2))
         THROUGH  LOOP,  FOR  I=0,1,I.G.LL
         H(I)=H1(I)
         WHENEVER  U1(I).G.0.
         U(I)=0.
         OTHERWISE
         U(I)=U1(I)
LOOP     END  OF  CONDITIONAL
         TRANSFER  TO  BEGIN
RESUL1   T=T*XR
         N=N+1
         THROUGH  PRINT, FOR  I=0,LLL,I.G.LL
         H1(I)=H1(I)*XR
         X=I*DX
PRINT    PRINT  RESULTS  T,X,U1(I),H1(I)
         QOUTP=-H1(0)*U1(0)/(L*KC)
         PRINT  RESULTS   QOUTP,T
         THROUGH  A5,  FOR  I=0,LLL,I.G.LL
A5       H1(I)=H1(I)/XR
         TRANSFER  TO  COUNT
RIGHT    WHENEVER  T.GE.TCR(II)
         H1(I)=ELR(II)
         TRANSFER  TO  A8
         OTHERWISE
         M=0
         TRANSFER  TO  A7
         END  OF  CONDITIONAL
A7       WHENEVER  T.LE.TCR(M+1)
         H1(I)=ELR(M)+(T-TCR(M))*(ELR(M+1)-ELR(M))/(TCR(M+1)-TCR(M))
         TRANSFER  TO  A8
         OTHERWISE
         M=M+1
         END  OF  CONDITIONAL
         TRANSFER  TO  A7
A8       U1(I)=UR-32.2*UR*(  DD )-(H1(I)-HR)*(5.67/EEE)
         TRANSFER  TO  ANN
LEFT     WHENEVER   T.GE.TCL(JJ)
         H1(I)=ELL(JJ)
         TRANSFER  TO  A80
         OTHERWISE
         M=0
         TRANSFER  TO  A70
         END  OF  CONDITIONAL
A70      WHENEVER  T.LE.TCL(M+1)
         H1(I)=ELL(M)+(T-TCL(M))*(ELL(M+1)-ELL(M))/(TCL(M+1)-TCL(M))
         TRANSFER  TO  A80
         OTHERWISE
         M=M+1
         TRANSFER  TO  A70
         END  OF  CONDITIONAL
A80      U1(I)=US+(H1(I)-HS)*(5.67/EE)-(32.2*US*DD)
         TRANSFER  TO  ANN
PRINT1   THROUGH  A0,FOR  I=0,LLL,I.G.(LL-LLL)
     A0  WHENEVER .ABS.(((U1(I+LLL)*H1(I+LLL)-U1(I)*H1(I))/(U1(I)*
        1H1(I)))).G.NN,TRANSFER  TO  ON
         TRANSFER  TO  PRINT4
PRINT4   T=T*XR
         THROUGH  PRINT2,FOR  I=0,LLL,I.G.LL
         H1(I)=H1(I)*XR
         X=DX*I
PRINT2   PRINT  RESULTS  X,T,H1(I),U1(I)
         QOUTP=-H1(0)*U1(0)/(L*KC)
         PRINT  RESULTS   QOUTP,T
         TRANSFER  TO  START
         END  OF  PROGRAM
```

THEORETICAL RESULTS OF EXPERIMENT NO. 1
(The Existence of the Seepage Surface is Taken Into Consideration)

T = 20.00

X =	.000000,	U1(0) =	3.615343E-08,	H1(0) =	.166670
X =	.200000,	U1(20) =	-2.758953E-06,	H1(20) =	.166675
X =	.400000,	U1(40) =	-2.616679E-05,	H1(40) =	.166746
X =	.600000,	U1(60) =	-1.586478E-04,	H1(60) =	.167197
X =	.800000,	U1(80) =	-7.232857E-04,	H1(80) =	.169355
X =	1.000000,	U1(100) =	-2.538786E-03,	H1(100) =	.177383
X =	1.200000,	U1(120) =	-6.553812E-03,	H1(120) =	.199930
X =	1.400000,	U1(140) =	-.012107,	H1(140) =	.246031
X =	1.600000,	U1(160) =	-.017170,	H1(160) =	.317253
X =	1.800000,	U1(180) =	-.020778,	H1(180) =	.408369
X =	2.000000,	U1(200) =	-.023107,	H1(200) =	.513009

T = 40.00

X =	.000000,	U1(0) =	-1.195536E-03,	H1(0) =	.166670
X =	.200000,	U1(20) =	-1.840495E-03,	H1(20) =	.174052
X =	.400000,	U1(40) =	-3.983645E-03,	H1(40) =	.188301
X =	.600000,	U1(60) =	-7.582561E-03,	H1(60) =	.216662
X =	.800000,	U1(80) =	-.011734,	H1(80) =	.263749
X =	1.000000,	U1(100) =	-.015363,	H1(100) =	.329151
X =	1.200000,	U1(120) =	-.018066,	H1(120) =	.409202
X =	1.400000,	U1(140) =	-.019946,	H1(140) =	.499812
X =	1.600000,	U1(160) =	-.021229,	H1(160) =	.597735
X =	1.800000,	U1(180) =	-.022102,	H1(180) =	.700662
X =	2.000000,	U1(200) =	-.022684,	H1(200) =	.806938

T = 60.00

X =	.000000,	U1(0) =	-.016054,	H1(0) =	.170270
X =	.200000,	U1(20) =	-.013902,	H1(20) =	.241951
X =	.400000,	U1(40) =	-.015218,	H1(40) =	.312148
X =	.600000,	U1(60) =	-.017045,	H1(60) =	.389528
X =	.800000,	U1(80) =	-.018518,	H1(80) =	.474233
X =	1.000000,	U1(100) =	-.019364,	H1(100) =	.563809
X =	1.200000,	U1(120) =	-.019596,	H1(120) =	.655460
X =	1.400000,	U1(140) =	-.019310,	H1(140) =	.746673
X =	1.600000,	U1(160) =	-.018621,	H1(160) =	.835432
X =	1.800000,	U1(180) =	-.017639,	H1(180) =	.920228
X =	2.000000,	U1(200) =	-.016458,	H1(200) =	1.000000

NOTE: The above results were used in plotting the dotted curves in Figures 10 - 15.

T = 80.00

X = .000000,	U1(0) = -.036268,	H1(0) = .196670
X = .200000,	U1(20) = -.023185,	H1(20) = .334149
X = .400000,	U1(40) = -.019442,	H1(40) = .434504
X = .600000,	U1(60) = -.017616,	H1(60) = .521679
X = .800000,	U1(80) = -.016469,	H1(80) = .601707
X = 1.000000,	U1(100) = -.015600,	H1(100) = .676879
X = 1.200000,	U1(120) = -.014847,	H1(120) = .748167
X = 1.400000,	U1(140) = -.014141,	H1(140) = .815991
X = 1.600000,	U1(160) = -.013453,	H1(160) = .880528
X = 1.800000,	U1(180) = -.012770,	H1(180) = .941852
X = 2.000000,	U1(200) = -.012092,	H1(200) = 1.000000

T = 100.00

X = .000000,	U1(0) = -.039016,	H1(0) = .225670
X = .200000,	U1(20) = -.024864,	H1(20) = .371995
X = .400000,	U1(40) = -.020080,	H1(40) = .476393
X = .600000,	U1(60) = -.017469,	H1(60) = .563642
X = .800000,	U1(80) = -.015758,	H1(80) = .640798
X = 1.000000,	U1(100) = -.014509,	H1(100) = .711034
X = 1.200000,	U1(120) = -.013529,	H1(120) = .776061
X = 1.400000,	U1(140) = -.012718,	H1(140) = .836907
X = 1.600000,	U1(160) = -.012018,	H1(160) = .894232
X = 1.800000,	U1(180) = -.011398,	H1(180) = .948488
X = 2.000000,	U1(200) = -.010836,	H1(200) = 1.000000

T = 160.00

X = .000000,	U1(0) = -.042920,	H1(0) = .227670
X = .200000,	U1(20) = -.025955,	H1(20) = .385102
X = .400000,	U1(40) = -.020437,	H1(40) = .492808
X = .600000,	U1(60) = -.017420,	H1(60) = .580734
X = .800000,	U1(80) = -.015444,	H1(80) = .657029
X = 1.000000,	U1(100) = -.014020,	H1(100) = .725403
X = 1.200000,	U1(120) = -.012931,	H1(120) = .787924
X = 1.400000,	U1(140) = -.012061,	H1(140) = .845885
X = 1.600000,	U1(160) = -.011347,	H1(160) = .900161
X = 1.800000,	U1(180) = -.010746,	H1(180) = .951378
X = 2.000000,	U1(200) = -.010230,	H1(200) = 1.000000

APPENDIX II

MAD LANGUAGE PROGRAMS
FOR EXPERIMENT NO. 2

The MAD Language programs written for Experiment No. 2 are presented on Pages 81 and 86. The computations, as in Experiment No. 1, were planned to be carried out in two programs (on pages 81 - 82 and 86 - 87, respectively).

The results of the computations of the first program, however, suggest that their refinement by the second program is not necessary, (this can be noted in Figures 23 and 24) the reason being that the height of the surface of seepage is very small (of the order of .03").

A COMPUTER PROGRAM WRITTEN FOR EXPERIMENT NO. 2
(The Existence of the Seepage Surface is Not Taken Into Consideration)

```
          EXECUTE  FTRAP.(0)
          DIMENSION  U(300),U1(300),H(300),H1(300),ELL(10),TML(10),
          1TCL(10),TSC(10),TSM(10),KP(10)
          PRINT  FORMAT  TITLE
          VECTOR  VALUES  TITLE=$1H ,S30,54HUNSTEADY  GRAVITY  FLOW OF
          1 LIQUIDS  IN  POROUS  MEDIA//*$
          INTEGER  I,JJ,LL,FFI,K,M,N,II,K0,N0,LLL
          II=4
          JJ=6
START     READ  FORMAT  DATA,KM,TTM,XR,ELL(0)...ELL(II),TML(0)...TML(II
          1),TSM(1)...TSM(JJ),L,  DT,DX,KP(1)...KP(JJ),MM,NN
          KC=KM
          DD=DT/KC
          TTC=TTM/XR
          VECTOR  VALUES  DATA=$(6F12.5)*$
          THROUGH  A1,FOR  M=0,1,M.G.II
          ELL(M)=ELL(M)/XR
A1        TCL(M)=TML(M)/XR
          LL=L/DX
          LLL=(LL/10)+.01
          FF=TTC/DT
          FFI=FF+.5
          TH=DT/(DX/XR)
          READ  FORMAT  DATA1,H(1)...H(LL),U(1)...U(LL),H(0),U(0)
          VECTOR  VALUES  DATA1=$(10F7.5)*$
          THROUGH  A3, FOR  I=0,1,I.G.LL
A3        H(I)=H(I)/XR
          READ  FORMAT  DATA2,K0,N0
          VECTOR  VALUES  DATA2=$2I10*$
          K=K0
          N=N0
BEGIN     T=K*DT
          THROUGH  ANN,  FOR I=0,1,I.G.LL
          KESPC=1./(U(I)+SQRT.(32.2*H(I)))
          BB=TH/KESPC
          WHENEVER  I.E.0,TRANSFER  TO  A10
          UR=U(I)*(1.-BB)+U(I-1)*BB
          HR=H(I)*(1.-BB)+H(I-1)*BB
          EEE=SQRT.(HR)
          WHENEVER  I.E.LL,TRANSFER  TO  RIGHT
A10       KESMC=1./(U(I)-SQRT.(32.2*H(I)))
          CC=TH/KESMC
          US=U(I)*(1.+CC)-U(I+1)*CC
          HS=H(I)*(1.+CC)-H(I+1)*CC
          EE=SQRT.(HS)
          WHENEVER  I.E.0, TRANSFER  TO  LEFT
          H1(I)=EE*EEE*(1.+(((US-UR)*(5.67*DD-.1765))/(EE+EEE)))
          U1(I)=UR-32.2*UR*DD-(H1(I)-HR)*(5.67/EEE)
ANN       CONTINUE
          THROUGH  A9,FOR  I=0,LLL,I.G.LL
A9        WHENEVER  .ABS.((H1(I)-H(I))/H(I)).G.MM,TRANSFER  TO  ON
          TRANSFER  TO  PRINT1
ON        WHENEVER K.E.KP (N),TRANSFER TO RESUL1
          WHENEVER  K.E.FFI
          T=T*XR
          THROUGH  A4, FOR  I=0,1,I.G.LL
          X=I*DX
          H1(I)=H1(I)*XR
A4        PRINT  RESULTS  U1(I),H1(I),X,T
```

```
                TRANSFER TO START
                OTHERWISE
                TRANSFER TO COUNT
                END OF CONDITIONAL
    COUNT       K=K+1
                THROUGH LOOP, FOR I=0,1,I.G.LL
                H(I)=H1(I)
                WHENEVER U1(I).G.0.
                U(I)=0.
                OTHERWISE
                U(I)=U1(I)
    LOOP        END OF CONDITIONAL
                TRANSFER TO BEGIN
    RESUL1      T=T*XR
                N=N+1
                THROUGH PRINT, FOR I=0,LLL,I.G.LL
                X=I*DX
                H1(I)=H1(I)*XR
    PRINT       PRINT RESULTS T,X,U1(I),H1(I)
                QOUTP=-H1(0)*U1(0)/(L*KC)
                HECS=L*SQRT.(2.*QOUTP+(H1(0)/L).P.2)
                LPCHE=L/HECS
                HWPCHE=H1(0)/HECS
                HEPCL=1./LPCHE
                HWPCL=H1(0)/L
                HW=H1(0)
                PRINT RESULTS T,QOUTP,HECS,L,HW,LPCHE,HWPCHE,HEPCL,HWPCL
                THROUGH A2,FOR I=0,LLL,I.G.LL
    A2          H1(I)=H1(I)/XR
                TRANSFER TO COUNT
    LEFT        WHENEVER T.GE.TCL(II)
                H1(I)=ELL(II)
                TRANSFER TO A8
                OTHERWISE
                M=0
                TRANSFER TO A7
                END OF CONDITIONAL
    A7          WHENEVER T.LE.TCL(M+1)
                H1(I)=ELL(M)+(T-TCL(M))*(ELL(M+1)-ELL(M))/(TCL(M+1)-TCL(M))
                TRANSFER TO A8
                OTHERWISE
                M=M+1
                END OF CONDITIONAL
                TRANSFER TO A7
    A8          U1(I)=US+(H1(I)-HS)*(5.67/EE)-(32.2*US*DD)
                TRANSFER TO ANN
    RIGHT       H1(I)=H(I)
                U1(I)=UR-32.2*UR*( DD )-(H1(I)-HR)*(5.67/EEE)
                TRANSFER TO ANN
    PRINT1      THROUGH A0,FOR I=0,LLL,I.G.(LL-LLL)
    A0          WHENEVER .ABS.(((U1(I+LLL)*H1(I+LLL)-U1(I)*H1(I))/(U1(I)*
              1H1(I)))).G.NN,TRANSFER TO ON
                TRANSFER TO PRINT4
    PRINT4      T=T*XR
                THROUGH PRINT2,FOR I=0,LLL,I.G.LL
                X=DX*I
    PRINT2      PRINT RESULTS X,T,H1(I),U1(I)
                QOUTP=-H1(0)*U1(0)/(L*KC)
                HECS=L*SQRT.(2.*QOUTP+(H1(0)/L).P.2)
                LPCHE=L/HECS
                HWPCHE=H1(0)/HECS
                HEPCL=1./LPCHE
                HWPCL=H1(0)/L
                HW=H1(0)
                PRINT RESULTS T,QOUTP,HECS,L,HW,LPCHE,HWPCHE,HEPCL,HWPCL
                TRANSFER TO START
                END OF PROGRAM
```

THEORETICAL RESULTS OF EXPERIMENT NO. 2
(The Existence of the Seepage Surface is Not Taken Into Consideration)

T = 20.00

X =	.000000,	U1(0) =	-9.905467E-03,	H1(0) =	1.281008
X =	.200000,	U1(10) =	-8.196860E-03,	H1(10) =	1.323226
X =	.400000,	U1(20) =	-6.775485E-03,	H1(20) =	1.358292
X =	.600000,	U1(30) =	-5.577906E-03,	H1(30) =	1.387444
X =	.800000,	U1(40) =	-4.561073E-03,	H1(40) =	1.411636
X =	1.000000,	U1(50) =	-3.694367E-03,	H1(50) =	1.431625
X =	1.200000,	U1(60) =	-2.967504E-03,	H1(60) =	1.448222
X =	1.400000,	U1(70) =	-2.399962E-03,	H1(70) =	1.463211
X =	1.600000,	U1(80) =	-1.989871E-03,	H1(80) =	1.476649
X =	1.800000,	U1(90) =	-1.744364E-03,	H1(90) =	1.488694
X =	2.000000,	U1(100) =	-1.656149E-03,	H1(100) =	1.500000
QOUTP =	.130949,	HECS =	1.639687,	L =	2.000000
HW =	1.281008,	HWPCL =	.640504	HEPCL =	.819844
HWPCHE =	.781251,	LPCHE =	1.219745,		

T = 40.00

X =	.000000,	U1(0) =	-.014995,	H1(0) =	1.103400
X =	.200000,	U1(10) =	-.012750,	H1(10) =	1.167520
X =	.400000,	U1(20) =	-.011052,	H1(20) =	1.221920
X =	.600000,	U1(30) =	-9.727736E-03,	H1(30) =	1.269176
X =	.800000,	U1(40) =	-8.677352E-03,	H1(40) =	1.311004
X =	1.000000,	U1(50) =	-7.845240E-03,	H1(50) =	1.348582
X =	1.200000,	U1(60) =	-7.193926E-03,	H1(60) =	1.382837
X =	1.400000,	U1(70) =	-6.697672E-03,	H1(70) =	1.414515
X =	1.600000,	U1(80) =	-6.336263E-03,	H1(80) =	1.444257
X =	1.800000,	U1(90) =	-6.090462E-03,	H1(90) =	1.472620
X =	2.000000,	U1(100) =	-5.940947E-03,	H1(100) =	1.500000
QOUTP =	.170747,	HECS =	1.607316,	L =	2.000000
HW =	1.103400,	HWPCL =	.551700	HEPCL =	.803658
HWPCHE =	.686486,	LPCHE =	1.244310,		

T = 60.00

X =	.000000,	U1(0) =	-.015159,	H1(0) =	1.012500
X =	.200000,	U1(10) =	-.014293,	H1(10) =	1.072080
X =	.400000,	U1(20) =	-.013504,	H1(20) =	1.128414
X =	.600000,	U1(30) =	-.012779,	H1(30) =	1.181881
X =	.800000,	U1(40) =	-.012112,	H1(40) =	1.232788
X =	1.000000,	U1(50) =	-.011484,	H1(50) =	1.281552
X =	1.200000,	U1(60) =	-.010775,	H1(60) =	1.329496
X =	1.400000,	U1(70) =	-.010099,	H1(70) =	1.375592
X =	1.600000,	U1(80) =	-9.575599E-03,	H1(80) =	1.418994
X =	1.800000,	U1(90) =	-9.183768E-03,	H1(90) =	1.460286
X =	2.000000,	U1(100) =	-8.903473E-03,	H1(100) =	1.500000
QOUTP =	.158395,	HECS =	1.514039,	L =	2.000000
HW =	1.012500,	HWPCL =	.506250	HEPCL =	.757019
HWPCHE =	.668741,	LPCHE =	1.320970,		

T = 80.00

X =	.000000,	U1(0) =	-.014423,	H1(0) =	1.012500
X =	.200000,	U1(10) =	-.013644,	H1(10) =	1.071540
X =	.400000,	U1(20) =	-.012989,	H1(20) =	1.127318
X =	.600000,	U1(30) =	-.012425,	H1(30) =	1.180356
X =	.800000,	U1(40) =	-.011933,	H1(40) =	1.231078
X =	1.000000,	U1(50) =	-.011497,	H1(50) =	1.279792
X =	1.200000,	U1(60) =	-.011107,	H1(60) =	1.326732
X =	1.400000,	U1(70) =	-.010753,	H1(70) =	1.372084
X =	1.600000,	U1(80) =	-.010430,	H1(80) =	1.416000
X =	1.800000,	U1(90) =	-.010131,	H1(90) =	1.458603
X =	2.000000,	U1(100) =	-9.854734E-03,	H1(100) =	1.500000
QOUTP =	.150706,	HECS =	1.493588,	L =	2.000000
HW =	1.012500,	HWPCL =	.506250	HEPCL =	.746794
HWPCHE =	.677898,	LPCHE =	1.339058,		

T = 100.00

X =	.000000,	U1(0) =	-.014498,	H1(0) =	1.012500
X =	.200000,	U1(10) =	-.013706,	H1(10) =	1.071539
X =	.400000,	U1(20) =	-.013031,	H1(20) =	1.127401
X =	.600000,	U1(30) =	-.012448,	H1(30) =	1.180556
X =	.800000,	U1(40) =	-.011936,	H1(40) =	1.231366
X =	1.000000,	U1(50) =	-.011483,	H1(50) =	1.280121
X =	1.200000,	U1(60) =	-.011079,	H1(60) =	1.327055
X =	1.400000,	U1(70) =	-.010714,	H1(70) =	1.372361
X =	1.600000,	U1(80) =	-.010384,	H1(80) =	1.416200
X =	1.800000,	U1(90) =	-.010083,	H1(90) =	1.458708
X =	2.000000,	U1(100) =	-9.806606E-03,	H1(100) =	1.500000
QOUTP =	.151488,	HECS =	1.495680,	L =	2.000000
HW =	1.012500,	HWPCL =	.506250	HEPCL =	.747840
HWPCHE =	.676950,	LPCHE =	1.337184,		

T = 120.00

X =	.000000,	U1(0) =	-.014493,	H1(0) =	1.012500
X =	.200000,	U1(10) =	-.013702,	H1(10) =	1.071527
X =	.400000,	U1(20) =	-.013028,	H1(20) =	1.127379
X =	.600000,	U1(30) =	-.012446,	H1(30) =	1.180528
X =	.800000,	U1(40) =	-.011936,	H1(40) =	1.231337
X =	1.000000,	U1(50) =	-.011484,	H1(50) =	1.280095
X =	1.200000,	U1(60) =	-.011081,	H1(60) =	1.327034
X =	1.400000,	U1(70) =	-.010717,	H1(70) =	1.372346
X =	1.600000,	U1(80) =	-.010387,	H1(80) =	1.416191
X =	1.800000,	U1(90) =	-.010086,	H1(90) =	1.458704
X =	2.000000,	U1(100) =	-9.809406E-03,	H1(100) =	1.500000
QOUTP =	.151438,	HECS =	1.495547,	L =	2.000000
HW =	1.012500,	HWPCL =	.506250	HEPCL =	.747774
HWPCHE =	.677010,	LPCHE =	1.337303,		

~~.999998~~

NOTE: Since, as it is noticed in Figures 23 and 24, H_S is negligible, the above results are used in plotting the dotted curves of Figures 16 - 20.

A COMPUTER PROGRAM WRITTEN FOR EXPERIMENT NO. 2
(The Existence of the Seepage Surface is Not Taken Into Consideration)

```
        EXECUTE  FTRAP.(0)
        DIMENSION  U(300),U1(300),H(300),H1(300),ELL(10),TML(10),
       1TCL(10),TSC(10),TSM(10),KP(10),ELLL(9),TCLL(9),TMLL(9),H11(1)
        PRINT  FORMAT  TITLE
        VECTOR  VALUES  TITLE=$1H ,S30,54HUNSTEADY  GRAVITY  FLOW OF
       1 LIQUIDS  IN  POROUS  MEDIA//*$
        INTEGER  I,JJ,LL,FFI,K,M,N,II,KO,NO,LLL
        II=4
        JJ=6
START   READ  FORMAT  DATA,KM,TTM,XR,ELL(0)...ELL(II),TML(0)...TML(II
       1),TSM(1)...TSM(JJ),L,   DT,DX,KP(1)...KP(JJ),MM,NN,TMLL(0)...
       1TMLL(JJ),ELLL(0)...ELLL(JJ)
        KC=KM
        DDO=DT/KC
        DD=DDO
        TTC=TTM/XR
        VECTOR  VALUES  DATA=$(6F12.5)*$
        THROUGH  A1,FOR  M=0,1,M.G.II
        ELL(M)=ELL(M)/XR
 A1     TCL(M)=TML(M)/XR
        THROUGH A20,FOR  M=0,1,M.G.JJ
        ELLL(M)=ELLL(M)/XR
A20     TCLL(M)=TMLL(M)/XR
        LL=L/DX
        LLL=(LL/10)+.01
        FF=TTC/DT
        FFI=FF+.5
        TH=DT/(DX/XR)
        READ  FORMAT  DATA1,H(1)...H(LL),U(1)...U(LL),H(0),U(0)
        VECTOR  VALUES  DATA1=$(10F7.5)*$
        THROUGH  A3, FOR  I=0,1,I.G.LL
 A3     H(I)=H(I)/XR
        READ  FORMAT  DATA2,KO,NO
        VECTOR  VALUES  DATA2=$2I10*$
        K=KO
        N=NO
BEGIN   T=K*DT
        THROUGH  ANN,  FOR I=0,1,I.G.LL
        KESPC=1./(U(I)+SQRT.(32.2*H(I)))
        BB=TH/KESPC
        WHENEVER  I.E.0,TRANSFER  TO   A10
        UR=U(I)*(1.-BB)+U(I-1)*BB
        HR=H(I)*(1.-BB)+H(I-1)*BB
        EEE=SQRT.(HR)
        WHENEVER  I.E.LL,TRANSFER  TO  RIGHT
 A10    KESMC=1./(U(I)-SQRT.(32.2*H(I)))
        CC=TH/KESMC
        US=U(I)*(1.+CC)-U(I+1)*CC
        HS=H(I)*(1.+CC)-H(I+1)*CC
        EE=SQRT.(HS)
        WHENEVER  I.E.0, TRANSFER  TO  LEFT
        H1(I)=EE*EEE*(1.+(((US-UR)*(5.67*DD-.1765))/(EE+EEE)))
        U1(I)=UR-32.2*UR*DD-(H1(I)-HR)*(5.67/EEE)
 ANN    CONTINUE
        THROUGH  A9,FOR  I=0,LLL,I.G.LL
 A9     WHENEVER  .ABS.((H1(I)-H(I))/H(I)).G.MM,TRANSFER  TO  ON
        TRANSFER  TO  PRINT1
 ON     WHENEVER K.E.KP (N),TRANSFER TO RESUL1
        WHENEVER  K.E.FFI
```

```
          T=T*XR
          THROUGH  A4, FOR  I=0,1,I.G.LL
          X=I*DX
          H1(I)=H1(I)*XR
 A4       PRINT  RESULTS  U1(I),H1(I),X,T
          TRANSFER  TO  START
          OTHERWISE
          TRANSFER  TO  COUNT
          END  OF  CONDITIONAL
 COUNT    K=K+1
          DD=DD0*((H1(LL).P.2)-(H1(0).P.2))/((H1(LL).P.2)-(H11(0).P.2))
          THROUGH  LOOP,  FOR  I=0,1,I.G.LL
          H(I)=H1(I)
          WHENEVER  U1(I).G.0.
          U(I)=0.
          OTHERWISE
          U(I)=U1(I)
 LOOP     END  OF  CONDITIONAL
          TRANSFER  TO  BEGIN
RESUL1    T=T*XR
          N=N+1
          THROUGH  PRINT, FOR  I=0,LLL,I.G.LL
          X=I*DX
          H1(I)=H1(I)*XR
PRINT     PRINT  RESULTS  T,X,U1(I),H1(I)
          THROUGH  A2,FOR  I=0,LLL,I.G.LL
 A2       H1(I)=H1(I)/XR
          TRANSFER  TO  COUNT
 LEFT     WHENEVER  T.GE.TCL(II)
          H11(I) =ELL(II)
          TRANSFER  TO  A8
          OTHERWISE
          M=0
          TRANSFER  TO  A7
          END  OF  CONDITIONAL
 A7       WHENEVER  T.LE.TCL(M+1)
          H11(I)=ELL(M)+(T-TCL(M))*(ELL(M+1)-ELL(M))/(TCL(M+1)-TCL(M))
          TRANSFER  TO  A8
          OTHERWISE
          M=M+1
          END  OF  CONDITIONAL
          TRANSFER  TO  A7
 A8       WHENEVER  T.GE.TCLL(JJ)
          H1(I)=H11(I)+ELLL(JJ)
          TRANSFER  TO  A11
          OTHERWISE
          M=0
          TRANSFER  TO  A12
          END  OF  CONDITIONAL
  A12     WHENEVER  T.LE.TCLL(M+1)
          H1(I)=H11(I)+(T-TCLL(M))*(ELLL(M+1)-ELLL(M))/(TCLL(M+1)-TCLL
         1(M))+ELLL(M)
          TRANSFER  TO  A11
          OTHERWISE
          M=M+1
          END  OF  CONDITIONAL
          TRANSFER  TO  A12
 A11      U1(I)=US+(H1(I)-HS)*(5.67/EE)-(32.2*US*DD)
          TRANSFER TO  ANN
RIGHT     H1(I)=H(I)
          U1(I)=UR-32.2*UR*(  DD )-(H1(I)-HR)*(5.67/EEE)
          TRANSFER  TO  ANN
 PRINT1   THROUGH  A0,FOR  I=0,LLL,I.G.(LL-LLL)
 A0       WHENEVER  .ABS.(((U1(I+LLL)*H1(I+LLL)-U1(I)*H1(I))/(U1(I)*
         1H1(I)))).G.NN,TRANSFER  TO  ON
          TRANSFER  TO  PRINT4
 PRINT4   T=T*XR
          THROUGH  PRINT2,FOR  I=0,LLL,I.G.LL
          X=DX*I
 PRINT2   PRINT  RESULTS  X,T,H1(I),U1(I)
          TRANSFER TO  START
          END  OF  PROGRAM
```

APPENDIX III

MAD LANGUAGE PROGRAMS FOR THE STEADY STATE SOLUTION OF THE PROBLEM OF SEEPAGE THROUGH A DAM WITH VERTICAL FACES

Steady State Solution of Seepage Through a Porous Bank with Vertical Faces and a Horizontal Impervious Base

The main assumptions in the analytical theory of this problem are:

1. Two-dimensional flow
2. Isotropic medium
3. Existence of full saturation in the flow region
4. Negligible capillary forces
5. Validity of Darcy's law.

This problem has been analysed by Muskat(3,14) by means of a conformal mapping of the hodograph representation (Figure 1-a) onto an infinite half-plane (Figure 1-c). The horizontal and vertical components of the seepage velocity $\bar{u}, \bar{v}$ at a point (x, y) referred to axes through E (see Figure 1) are given by the following equation:

$$z = x + iy = c \int e^{\tau + i\Theta} \, d(\bar{u} - i\bar{v})$$

$$= c \int \sqrt{\frac{\lambda - 1}{\lambda(a-\lambda)(\lambda-b)}} \; e^{\alpha + i\Theta} \, d(\bar{u} - i\bar{v}) \tag{68}$$

where Θ is the sum of the inclinations of the velocity and the acceleration with the x-axis, λ is the modular elliptic function,(3,14,15) α is equal to

$$\frac{-3}{2\pi} \int_0^1 \frac{\beta(t)}{t-\lambda} \, dt \, , \quad \text{and} \quad \tau = \log_e \frac{|(\text{velocity})|}{|(\text{acceleration})|} \, .$$

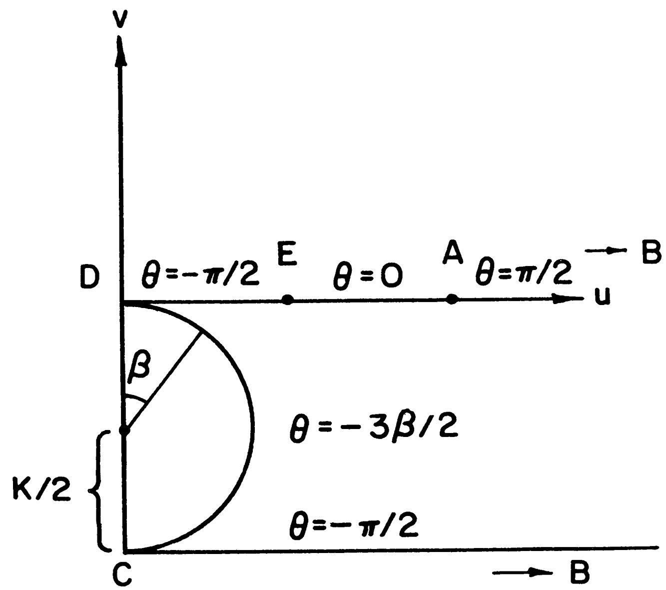

Figure 1 - a. The Hodograph Plane Representation of Figure 1.

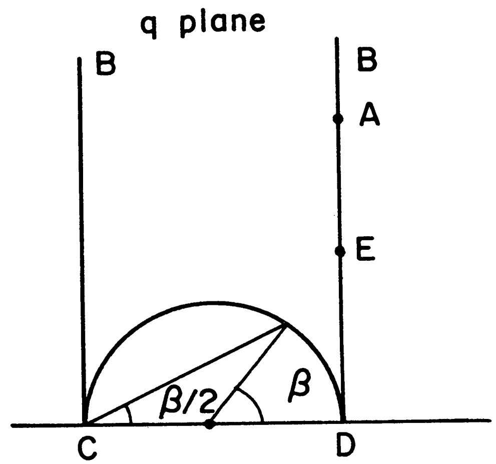

Figure 1 - b. The q Plane Diagram Corresponding to Figure 1 - a.

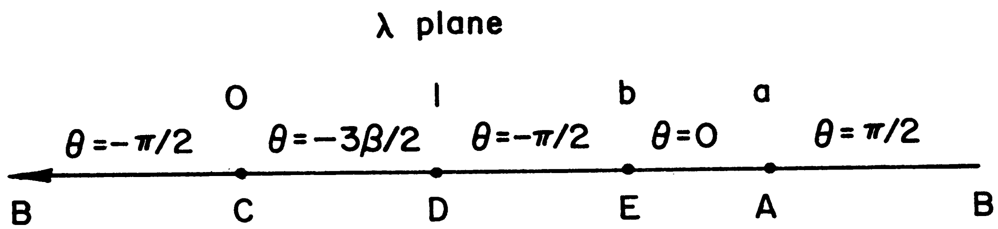

Figure 1 - c. The λ - Plane Map of Figure 1 - b.

The following is a summary of the relationships between the different functions encountered in the theoretical analysis of the problem:

$$q = 1 + \frac{i}{\bar{k}}(\bar{u} - i\bar{v}) \;;$$

$$\bar{u} = \bar{k} \sin \beta/2 \cos \beta/2 \text{ , for any point on the free surface;}$$

$$\bar{v} = -\bar{k} \sin^2 \beta/2 \text{ , for any point on the free surface;}$$

$$-\frac{\pi K'}{K} = G(\bar{\bar{k}}^2) = 2 \log_e \frac{\bar{\bar{k}}}{4} + \frac{\bar{\bar{k}}^2}{2} + \frac{13}{64}\bar{\bar{k}}^4 + \frac{23}{192}\bar{\bar{k}}^6 + \ldots$$

$$\frac{\bar{u}}{k} = \frac{K'}{K} = -\frac{1}{\pi} G(\frac{1}{1-\lambda}) \quad \text{for} \quad -\infty \leq \lambda \leq 0 \;;$$

$$\pi i(\frac{q-1}{q}) = -\pi \frac{K'}{K} = G(\lambda) \quad \text{for} \quad 0 \leq \lambda \leq 1 \;;$$

$$\frac{\bar{u}}{\bar{k}} = \frac{K'}{K} = -\frac{1}{\pi} G(\frac{1}{\lambda}) \quad \text{for} \quad 1 \leq \lambda \leq \infty \;.$$

In the above relations, K and K' are complete elliptic integrals of the first kind with modulus $\bar{\bar{k}}$ and $\sqrt{1 - \bar{\bar{k}}^2}$ respectively.*

From Equation (68) and with the aid of the first three of the above relations and Figures 1-a through 1-c, one can obtain the formulas for the determination of the dimensions of the flow system of Figure 1:

$$\frac{L}{c} = -\frac{\bar{k}}{2} \int_{\beta=\pi}^{\beta=0} \sqrt{\frac{\lambda - 1}{\lambda(a-\lambda)(\lambda-b)}}\, e^{\alpha} \cos \frac{\beta}{2}\, d\beta \qquad (69)$$

---

* The modulus of the elliptic integral K is here taken as $\bar{\bar{k}}$ in order to avoid confusion with the symbols k and $\bar{k}$ already used to denote the hydraulic conductivity and the coefficient of permeability of the porous medium.

$$\frac{H_W}{c} = \int_{u(A)}^{u(B)} \sqrt{\frac{\lambda - 1}{\lambda(a-\lambda)(\lambda-b)}}\; e^{\alpha}\; d\bar{u} \tag{70}$$

$$\frac{H_S}{c} = -\int_{o}^{u(B)} \sqrt{\frac{\lambda - 1}{\lambda(a-\lambda)(\lambda-b)}}\; e^{\alpha}\, d\bar{u} \tag{71}$$

$$\frac{H_e}{c} = \frac{H_W + H_S}{c} - \frac{\bar{k}}{2} \int_{\beta=\pi}^{\beta=0} \sqrt{\frac{\lambda - 1}{\lambda(a-\lambda)(\lambda-b)}}\; e^{\alpha} \sin\frac{\beta}{2}\, d\beta \tag{72}$$

The determination of the free surface is carried out by the following two integrals:

$$X = -\frac{\bar{k}}{2} \int_{\beta=\pi}^{\beta} \sqrt{\frac{\lambda - 1}{\lambda(a-\lambda)(\lambda-b)}}\; e^{\alpha} \cos\frac{\beta}{2}\, d\beta \tag{73}$$

$$Y = -\frac{\bar{k}}{2} \int_{\beta=\pi}^{\beta} \sqrt{\frac{\lambda - 1}{\lambda(a-\lambda)(\lambda-b)}}\; e^{\alpha} \sin\frac{\beta}{2}\, d\beta \tag{74}$$

where X and Y are the coordinates of a point on the free surface (see Figure 1)

Note:

For $a = \infty$, the term $a - \lambda$ is left out of the integrands of the above equations.[14]

A numerical integration of the integrals of Equations (69) - (74) can be accomplished by making use of the link between u, β and λ in a single variable λ. In evaluating the integrals for infinite values of λ, as in Equations (70) and (71), the integration in λ beyond

+20 and -20 is carried out analytically (see Reference 2). Also, in the integration of Equation (70) at the singular point ($\lambda = a$), the integration is carried out analytically from $\lambda = a$ to $\lambda = a$ plus a small value. Since the limits of the integrals of Equations (69) and (72) - (74) are singularities of the integral for α, those limits are taken to be not equal to the singular values ($\lambda = 0$ and $\lambda = 1$) but equal to ϵ and $1 - \epsilon$, where ϵ is a very small positive value. This can be justified if one proves that the value of the integrands of the integrals of Equations (69) and (72) - (74) approaches zero for $\lambda = 0$ and $\lambda = 1$, which is done as follows:

$$\left| \sqrt{\frac{\lambda - 1}{\lambda(a-\lambda)(\lambda-b)}} \; e^{\alpha} \right|_{\lambda=0} = e^{\tau_C}$$

but

$$\tau_C = \log_e \frac{|\text{velocity}|_C}{|\text{acceleration}|_C} = \log_e \frac{\bar{k}}{\infty} = -\infty ,$$

and

$$e^{\tau_C} = 0 .$$

Similarly, at $\lambda = 1$, since

$$\tau_D = \log_e \frac{0}{|\text{Accel.}|_D} = -\infty ,$$

since $|\text{Accel.}|_D$ is non-zero; then $e^{\tau_D} = 0$.

As $t = \lambda$ is the point of singularity of the integrand in the integral for α, the numerical integration of that integral, when $0 < \lambda < 1$, is done in the following manner:

$$\int_0^1 \frac{\beta(t)}{t-\lambda}\, dt = \int_0^1 \frac{\beta(t) - \beta(\lambda)}{t - \lambda}\, dt + \int_0^1 \frac{\beta(\lambda)}{t-\lambda}\, dt$$

$$= \int_0^1 \frac{\beta(t) - \beta(\lambda)}{t - \lambda}\, dt + \beta(\lambda) \log_e \left|\frac{1 - \lambda}{\lambda}\right| .$$

Since one can prove that the integrand of this integral is always bounded, its numerical integration is possible. The proof can be demonstrated as follows:

$$\lim_{\substack{t=\lambda \\ 0 < \lambda < 1}} \frac{\beta(t) - \beta(\lambda)}{t - \lambda} = \frac{d\beta}{dt}\Big|_{\substack{t=\lambda \\ 0 < \lambda < 1}} ;$$

but $\frac{d\beta}{dt}$ is bounded for $t = \lambda$ and $\epsilon \leq \lambda \leq 1 - \epsilon$, where ϵ is a small but finite positive number.

The MAD language program written for the computations of the dimensions of the flow system (namely H_e, H_W, L, and H_S) is presented on pages 95 - 97. The computations of the shape of the free surface are carried out in a separate program immediately following the first program in this appendix (see page 98).

For a more complete explanation of the procedures used in these computer programs, one can refer to Reference 2. In this reference are (for $a = 10$ and $b = 5$) all the necessary numerical and analytical integrations involved in the computations of H_e, H_w, H_s, L and the shape of the free surface.

A COMPUTER PROGRAM WRITTEN FOR THE COMPUTATIONS OF THE GEOMETRIC DIMENSIONS OF THE SYSTEM OF FIGURE 1

```
          EXECUTE  FTRAP.
          DIMENSION AB(200),BC(200),X(300,DIM),Y(300,DIM),BETA(200),
         1LAM(200),EPS(200),GAMT(200),GAML(200),T(200),B(20)
          INTEGER I,J,K,M
          READ FORMAT DATA1, T(0)...T(50),LAM(1)...LAM(15),LAM(30)...
         1LAM(102),BETA(30)...BETA(40),BETA(86)...BETA(96),EPS(30)...
         1EPS(33),GAMT(46)...GAMT(50),GAML(13)...GAML(15)
          THROUGH TS,FOR I=0,1,I.G.50
 TS       T(100-I)=1.-T(I)
          THROUGH LAMS1,FOR I=1,1,I.G.14
 LAMS1    LAM(30-I)=1.-LAM(I)
          GAMT(0)=3.1416
          THROUGH GAMTS1,FOR I=1,1,I.G.45
 GAMTS1   GAMT(I)=2.*ATAN.((-1./3.1416)*(ELOG.(T(I)/16.)+.5*T(I)+(13./
         164.)*( T(I).P.2)+(23./192.)*( T(I).P.3)))
          THROUGH GAMTS2,FOR I=0,1,I.G.49
 GAMTS2   GAMT(100-I)=3.1416-GAMT(I)
          GAML(1)=3.1416
          THROUGH GAMLS1,FOR I=2,1,I.G.12
 GAMLS1   GAML(I)=2.*ATAN.((-1./3.1416)*(ELOG.(LAM(I)/16.)+.5*LAM(I)+(1
         13./64.)*(LAM(I).P.2)+(23./192.)*(LAM(I).P.3)))
          THROUGH GAMLS2,FOR I=1,1,I.G.14
 GAMLS2   GAML(30-I)=3.1416-GAML(I)
          THROUGH EP, FOR J=1,1,J.G.102
          WHENEVER J.E.30,J=34
          WHENEVER J.E.86
          THROUGH EP1,FOR I=86,1,I.G.89
 EP1      EPS(I)=EPS(I-56)*((LAM(I)/(LAM(I)-1.)).P.1.5)
          J=I
          END OF CONDITIONAL
          WHENEVER J.L.29.AND.J.G.1,TRANSFER TO BACK
          S=0.
          THROUGH SUM,FOR I=0,1,I.G.99
 BACK2    GAMTA=(GAMT(I)+GAMT(I+1))/2.
          TA=(T(I)+T(I+1))/2.
          DT=T(I+1)-T(I)
          WHENEVER J.L.29.AND.J.G.1,TRANSFER TO BACK1
 SUM      S=S+(GAMTA/(TA-LAM(J)))*DT
 EP       EPS(J)=EXP.((-3./2./3.1416)*S)
          THROUGH BET1,FOR I=41,1,I.G.62
 BET1     BETA(I)=(1./3.1416)*(ELOG.(16.*LAM(I))-(.5/LAM(I))-(13./64./
         1((LAM(I)).P.2))-(23./192./((LAM(I)).P.3)))
          THROUGH BET2,FOR I=97,1,I.G.102
 BET2     BETA(I)=(1./3.1416)*(ELOG.(16.*(1.-LAM(I)))-(.5/(1.-LAM(I)))
         1-(13./64./((1.-LAM(I)).P.2))-(23./192./((1.-LAM(I)).P.3)))
          TRANSFER TO START
 BACK     S=GAML(J)*ELOG.((1.-LAM(J))/LAM(J))
          THROUGH BACK1,FOR I=0,1,I.G.99
           TRANSFER TO BACK2
 BACK1    S=S+((GAMTA-GAML(J))/(TA-LAM(J)))*DT
          TRANSFER TO EP
 START    READ FORMAT DATA2,A,B(1)...B(7)
          THROUGH FS,FOR K=1,1,K.G.7
          THROUGH FS,FOR M=2,1,M.G.29
          X(K,M)=0.
          Y(K,M)=0.
          THROUGH FS,FOR I=1,1,I.G.(M-1)
          LAMA=(LAM(I)+LAM(I+1))/2.
          EPSA=(EPS(I)+EPS(I+1))/2.
```

```
                WHENEVER A.E.999.,TRANSFER TO A2
                RAD=SQRT.((LAMA-1.)/((LAMA-A)*(B(K)-LAMA)*LAMA))
       A1       CGAMLA=(COS.(GAML(I)/2.)+COS.(GAML(I+1)/2.))/2.
                SGAMLA=(SIN.(GAML(I)/2.)+SIN.(GAML(I+1)/2.))/2.
                DGAML=GAML(I+1)-GAML(I)
                Y(K,M)=Y(K,M)-EPSA*RAD*SGAMLA*DGAML*.5
       FS       X(K,M)=X(K,M)-EPSA*RAD*CGAMLA*DGAML*.5
                TRANSFER TO ABS
       A2       RAD=SQRT.((LAMA-1.)/(LAMA*(LAMA-B(K))))
                TRANSFER TO A1
       ABS      WHENEVER A.E.999.
                THROUGH ABS1,FOR K=1,1,K.G.7
      ABS1      AB(K)=0.
                TRANSFER  TO  BCS
                END OF CONDITIONAL
                I=30
       A3       WHENEVER A.E.LAM(I),TRANSFER TO A5
                WHENEVER I.G.85,TRANSFER TO BCS
                I=I+1
                TRANSFER TO A3
       A5       THROUGH ABS2,FOR K=1,1,K.G.7
                EPSA=(EPS(I)+EPS(I+1))/2.
                LAMA=(LAM(I)+LAM(I+1))/2.
                AA1=SQRT.(LAM(I+1)-B(K))
                AA2=SQRT.(LAM(I+1)-A)
                AA3=SQRT.(A-B(K))
                COEF=2.*ELOG.((AA1+AA2)/AA3)
                WHENEVER  I.L.39
                BETAA=(BETA(I)+BETA(I+1))/2.
                EL=(1./3.1416)*SQRT.((LAMA-1.)/LAMA)*EPSA*((1./(LAMA-1.)-.5
               1-(13./32.)*(1.-LAMA)-(23./64.)*(1.-LAMA).P.2)/(BETAA.P.-2))
               1AB(I+1)=EL*COEF
                OTHERWISE
                EL1=(1./3.1416)*(1./LAMA)*SQRT.((LAMA-1.)/LAMA)*EPSA
                EL2=(.5/LAMA)*EL1
                EL3=(13./16./LAMA)*EL2
                EL4=(23./32./(LAMA.P.2))*EL2
                AB(I+1)=COEF*(EL1+EL2+EL3+EL4)
                END  OF  CONDITIONAL
                WHENEVER I.G.60,TRANSFER TO A8
                THROUGH A6,FOR J=(I+1),1,J.G.61
                LAMA=(LAM(J)+LAM(J+1))/2.
                EPSA=(EPS(J)+EPS(J+1))/2.
                DBETA=BETA(J+1)-BETA(J)
       A6       AB(J+1)=AB(J)+EPSA*SQRT.((LAMA-1.)/((LAMA-B(K))*(LAMA-A)*LAMA
               1))*DBETA
                AA1=1./SQRT.(A*B(K))
                AA2=(SQRT.(A)+SQRT.(B(K)))/(SQRT.(B(K))-SQRT.(A))
                 AA3=SQRT.(B(K))-SQRT.(A*(20.-B(K))/(20.-A))
                AA4=SQRT.(B(K))+SQRT.(A*(20.-B(K))/(20.-A))
                AB(K)=(1.04*AA1/3.1416)*.ABS.ELOG.(AA2*AA3/AA4)+AB(62)
     ABS2       CONTINUE
                TRANSFER TO BCS
       A8       AA1=1./SQRT.(A*B(K))
                AA2=(SQRT.(A)+SQRT.(B(K)))/(SQRT.(B(K))-SQRT.(A))
                AA3=SQRT.(B(K))-SQRT.(A*(LAM(I+1)-B(K))/(LAM(I+1)-A))
                AA4=SQRT.(B(K))+SQRT.(A*(LAM(I+1)-B(K))/(LAM(I+1)-A))
                AB(K)=AB(I+1)+(1.04*AA1/3.1416)*.ABS.ELOG.(AA2*AA3/AA4)
```

```
                TRANSFER  TO  ABS2
BCS             THROUGH  BCS1,FOR J=1,1,J.G.7
                WHENEVER  A.E. 999.
                AA1=SQRT.(B(J)+20.)-SQRT.(B(J))
                AA2=SQRT.(B(J)+20.)+SQRT.(B(J))
                BC(86)=(1./SQRT.(B(J)))*(.96/3.1416)*ELOG.(AA2/AA1)
                OTHERWISE
                AA1=1./SQRT.(B(J)*A)
                AA2=(SQRT.(B(J))+SQRT.(A))/(SQRT.(B(J))-SQRT.(A))
                AA3=SQRT.(B(J))-SQRT.(A*(20.+B(J))/(20.+A))
                AA4=SQRT.(B(J))+SQRT.(A*(20.+B(J))/(20.+A))
                BC(86)=(.96*AA1/3.1416)*.ABS.ELOG.(AA2*AA3/AA4)
                END  OF  CONDITIONAL
                THROUGH BCS2,FOR I=86,1,I.G.101
                LAMA=(LAM(I)+LAM(I+1))/2.
                WHENEVER A.E.999.
                RAD=SQRT.((LAMA-1.)/((LAMA)*(B(J)-LAMA)))
                OTHERWISE
                RAD=SQRT.((LAMA-1.)/((LAMA-A)*(LAMA-B(J))*LAMA))
                END OF CONDTIONAL
                EPSA=(EPS(I)+EPS(I+1))/2.
                DBETA=BETA(I+1)-BETA(I)
BCS2            BC(I+1)=BC(I)+RAD*EPSA*DBETA
BCS1            BC(J)=BC(102)
                THROUGH RESUL1,FOR J=1,1,J.G.7
                B(J)=(-1./2.3)*ELOG.(B(J)-1.)
                L=X(J,29)
                HW=AB(J)
                HS=BC(J)
                HE=HW+HS+Y(J,29)
                HWPCHE=HW/HE
                HWPCL=HW/L
                HSPCL=HS/L
                HSPCHE=HS/HE
                HEPCL=HE/L
                LPCHE=L/HE
RESUL1          PRINT  RESULTS  A,B(J),HWPCHE,HWPCL,HSPCL,HSPCHE,HEPCL,LPCHE
                TRANSFER TO START
                VECTOR VALUES DIM=2,1,30
                VECTOR VALUES DATA1=$8F9.6*$
                VECTOR VALUES DATA2=$8F9.6*$
                END OF PROGRAM
```

Note: The results obtained by the above computer program are used in plotting the curves in Plates 13 - 15.

A COMPUTER PROGRAM WRITTEN FOR THE COMPUTATIONS OF THE FREE SURFACE IN FIGURE 1

```
          EXECUTE  FTRAP.
          DIMENSION  X(30),Y(30),XP(30),YP(30),YPP(20),
         1LAM(200),EPS(200),GAMT(200),GAML(200),T(200)
          INTEGER I,J,K,M
          READ FORMAT DATA1, T(0)...T(50),LAM(1)...LAM(15),GAMT(46)...G
         1AMT(50),GAML(13)...GAML(15)
          THROUGH TS,FOR I=0,1,I.G.50
TS        T(100-I)=1.-T(I)
          THROUGH LAMS1,FOR I=1,1,I.G.14
LAMS1     LAM(30-I)=1.-LAM(I)
          GAMT(0)=3.1416
          THROUGH GAMTS1,FOR I=1,1,I.G.45
GAMTS1    GAMT(I)=2.*ATAN.((-1./3.1416)*(ELOG.(T(I)/16.)+.5*T(I)+(13./
         164.)*( T(I).P.2)+(23./192.)*( T(I).P.3)))
          THROUGH GAMTS2,FOR I=0,1,I.G.49
GAMTS2    GAMT(100-I)=3.1416-GAMT(I)
          GAML(1)=3.1416
          THROUGH GAMLS1,FOR I=2,1,I.G.12
GAMLS1    GAML(I)=2.*ATAN.((-1./3.1416)*(ELOG.(LAM(I)/16.)+.5*LAM(I)+(1
         13./64.)*(LAM(I).P.2)+(23./192.)*(LAM(I).P.3)))
          THROUGH GAMLS2,FOR I=1,1,I.G.14
GAMLS2    GAML(30-I)=3.1416-GAML(I)
          THROUGH EP, FOR J=1,1,J.G.29
          WHENEVER J.L.29.AND.J.G.1,TRANSFER TO BACK
           S=0.
          THROUGH SUM,FOR I=0,1,I.G.99
BACK2     GAMTA=(GAMT(I)+GAMT(I+1))/2.
          TA=(T(I)+T(I+1))/2.
          DT=T(I+1)-T(I)
          WHENEVER J.L.29.AND.J.G.1,TRANSFER TO BACK1
SUM       S=S+(GAMTA/(TA-LAM(J)))*DT
EP        EPS(J)=EXP.((-3./2./3.1416)*S)
          TRANSFER TO START
BACK      S=GAML(J)*ELOG.((1.-LAM(J))/LAM(J))
          THROUGH BACK1,FOR I=0,1,I.G.99
           TRANSFER TO BACK2
BACK1     S=S+((GAMTA-GAML(J))/(TA-LAM(J)))*DT
          TRANSFER TO EP
START     READ  FORMAT  DATA2,A,B,HEPCL,HWPCL
          PRINT  RESULTS  HWPCL,HEPCL
          THROUGH FS,FOR M=2,1,M.G.29
          X(M)=0.
          Y(M)=0.
          THROUGH FS,FOR I=1,1,I.G.(M-1)
          LAMA=(LAM(I)+LAM(I+1))/2.
          EPSA=(EPS(I)+EPS(I+1))/2.
          WHENEVER A.E.999.,TRANSFER TO A2
          RAD=SQRT.((LAMA-1.)/((LAMA-A)*(B-LAMA)*LAMA))
A1        CGAMLA=(COS.(GAML(I)/2.)+COS.(GAML(I+1)/2.))/2.
          SGAMLA=(SIN.(GAML(I)/2.)+SIN.(GAML(I+1)/2.))/2.
          DGAML=GAML(I+1)-GAML(I)
          Y(M) = Y(M)  -EPSA*RAD*SGAMLA*DGAML*.5
FS        X(M) = X(M)  -EPSA*RAD*CGAMLA*DGAML*.5
          TRANSFER  TO  OUT
A2        RAD=SQRT.((LAMA-1.)/(LAMA*(LAMA-B)))
          TRANSFER  TO  A1
OUT       XP(1)=0.
          YP(1)=0.
          THROUGH  A4,FOR  M=2,1,M.G.29
          XP(M)=X(M)/X(29)
A4        YP(M)=Y(M)/X(29)
          M=2
          THROUGH  A11,FOR  K=1,1,K.G.10
A12       WHENEVER  XP(M).L.(K/10.)
          M=M+1
          TRANSFER  TO  A12
          OTHERWISE
          YPP(K)=YP(M-1)+(YP(M)-YP(M-1))*((K/10.-XP(M-1))/(XP(M)-XP(
         1M-1)))
          PRINT  RESULTS      K,YPP(K)
A11       END  OF  CONDITIONAL
          TRANSFER  TO  START
          VECTOR  VALUES  DATA1=$8F9.6*$
          VECTOR  VALUES  DATA2=$4F18.8*$
          END  OF  PROGRAM
```

Note: The results obtained by the above computer program are used in plotting the curves in Plates 16 - 23.

APPENDIX IV

STEADY STATE SOLUTION TO THE PROBLEMS OF SEEPAGE THROUGH A DAM WITH VERTICAL FACES

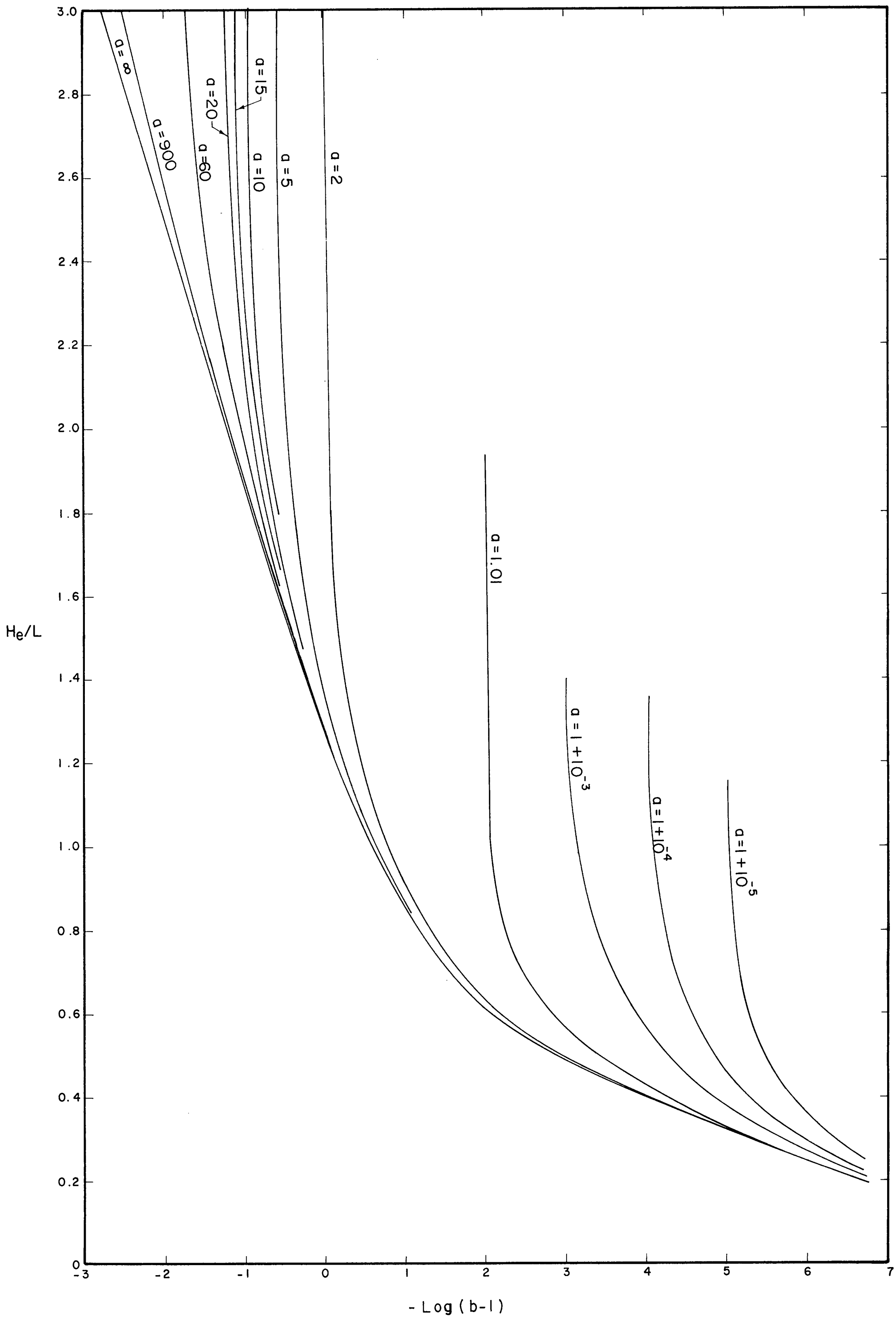

Plate 13

$\frac{H_e}{L} = \frac{\text{Outflow Height}}{\text{Width}}$ vs Parameter b for Different Values of Parameter a.

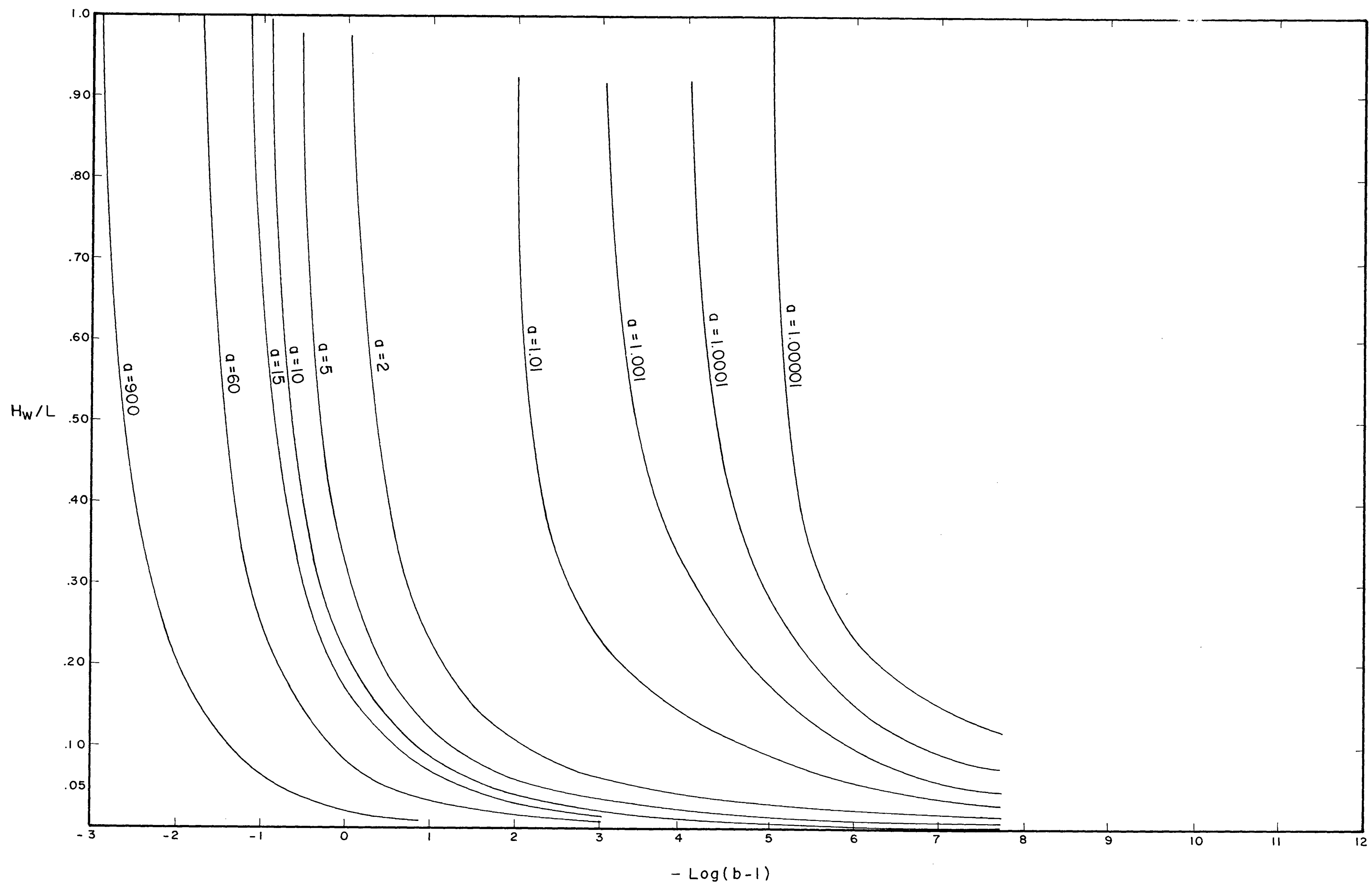

Plate 14

$\frac{H_e}{L} = \frac{\text{Outflow Height}}{\text{Width}}$ vs Parameter b for Different Values of Parameter a.

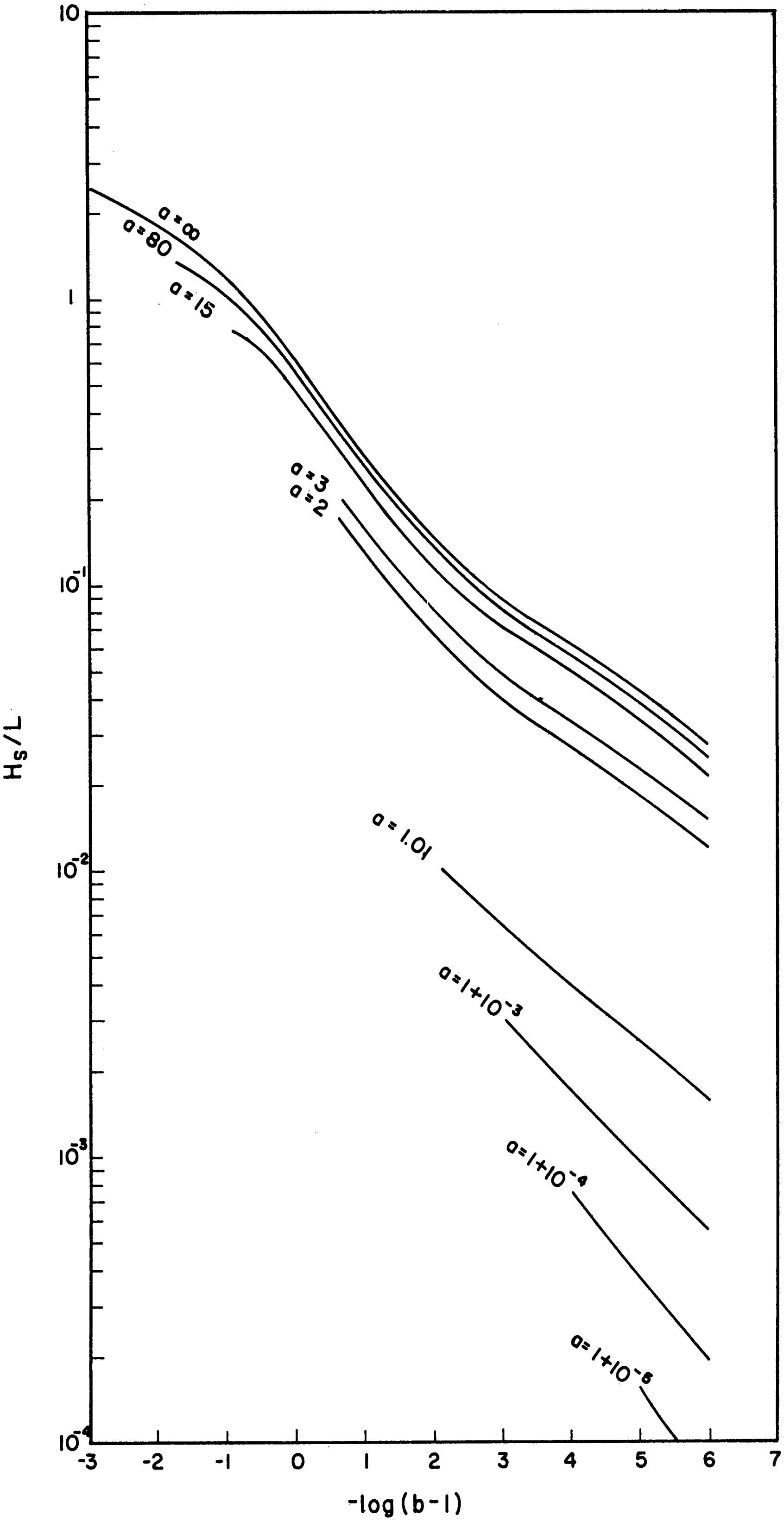

Plate 15

$\frac{H_s}{L} = \frac{\text{Height of Seepage Surface}}{\text{Width of the Dam}}$ vs Parameter b for Different Values of Parameter a

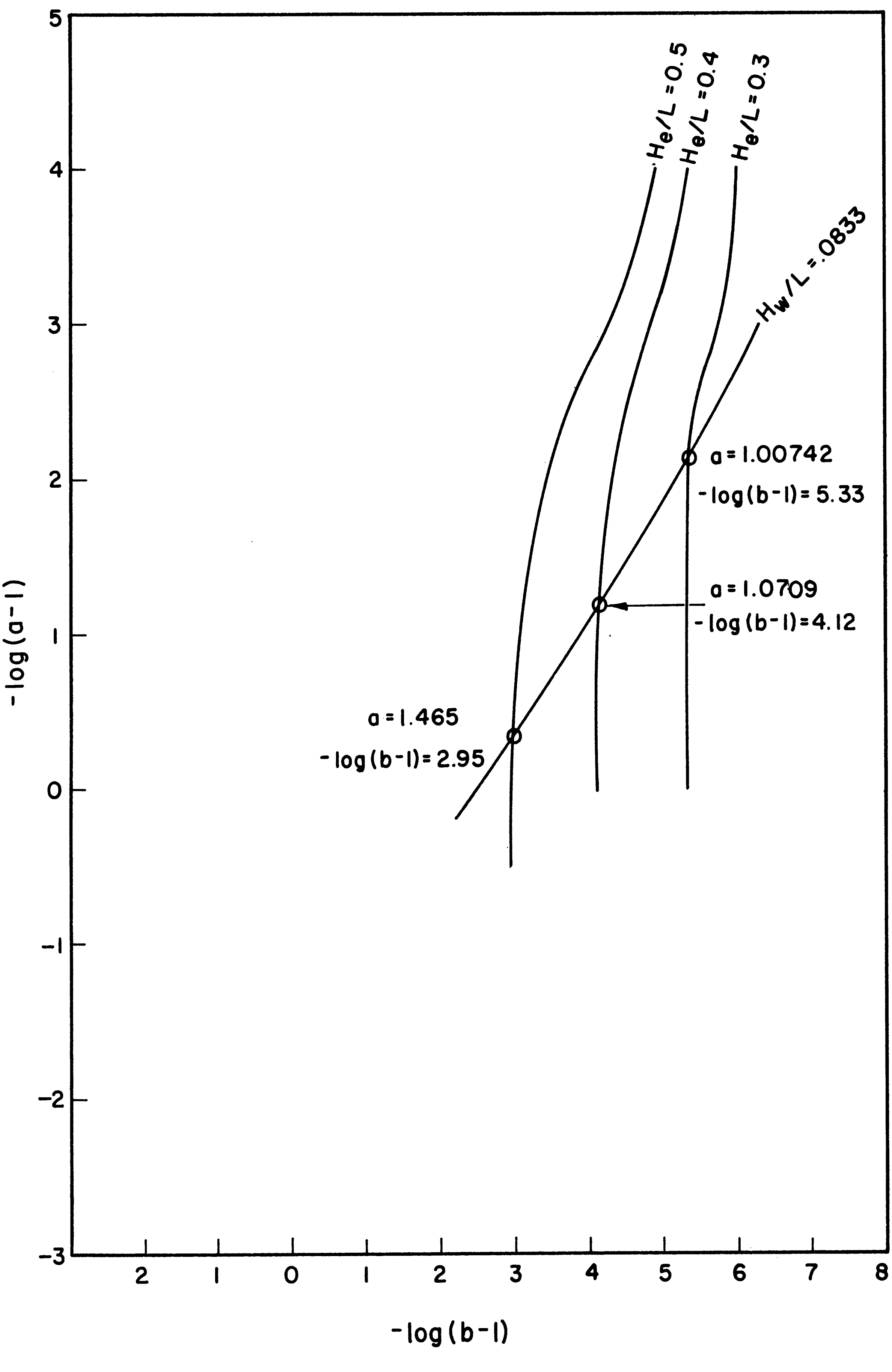

Figure 21. Curves Showing the Values of a and b for Specific Values of H_e/L and H_W/L Encountered in Experiment 1.

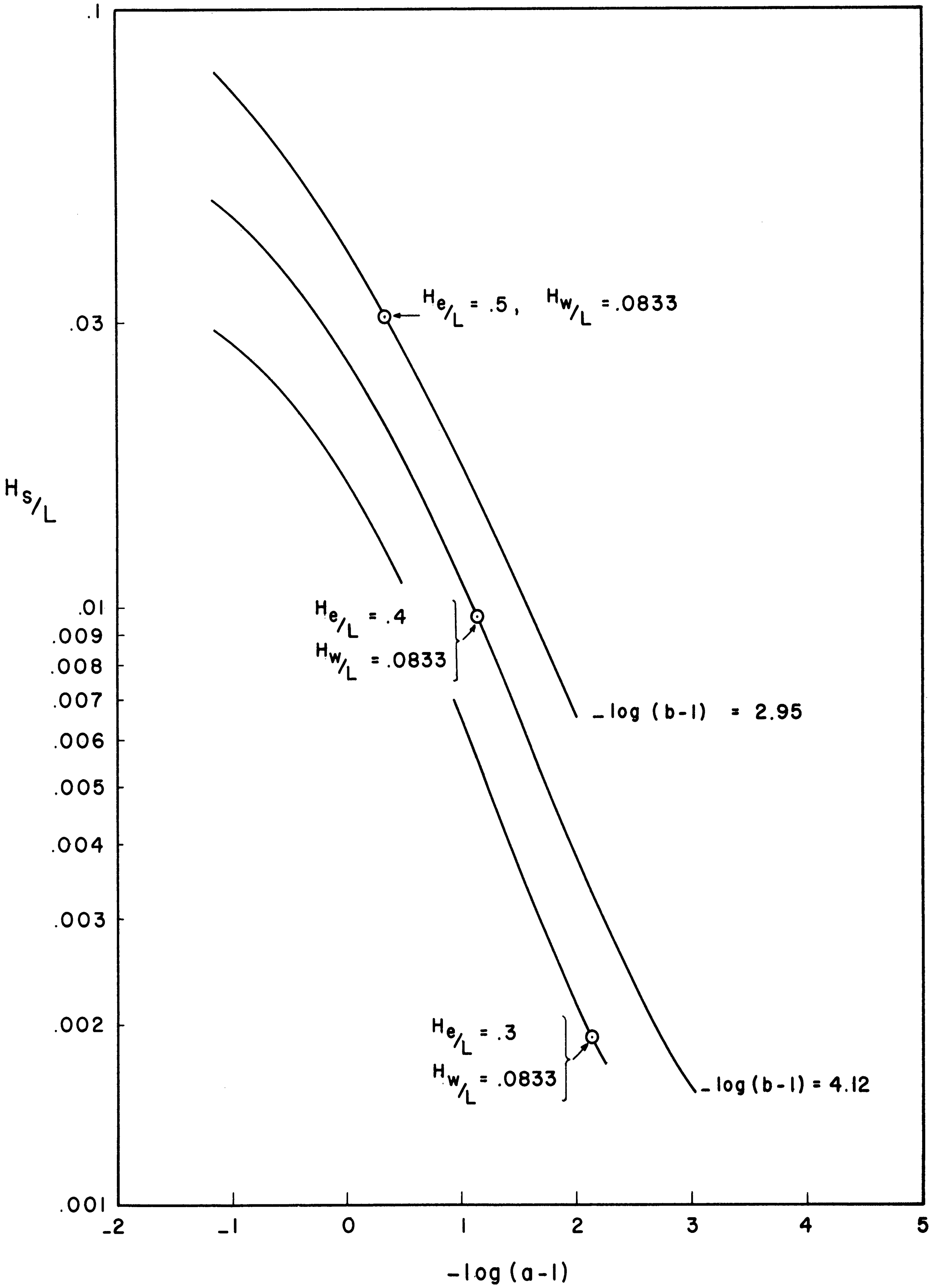

Figure 22. Curves Showing H_s/L vs a for Specific Values of b Encountered in Figure 21.

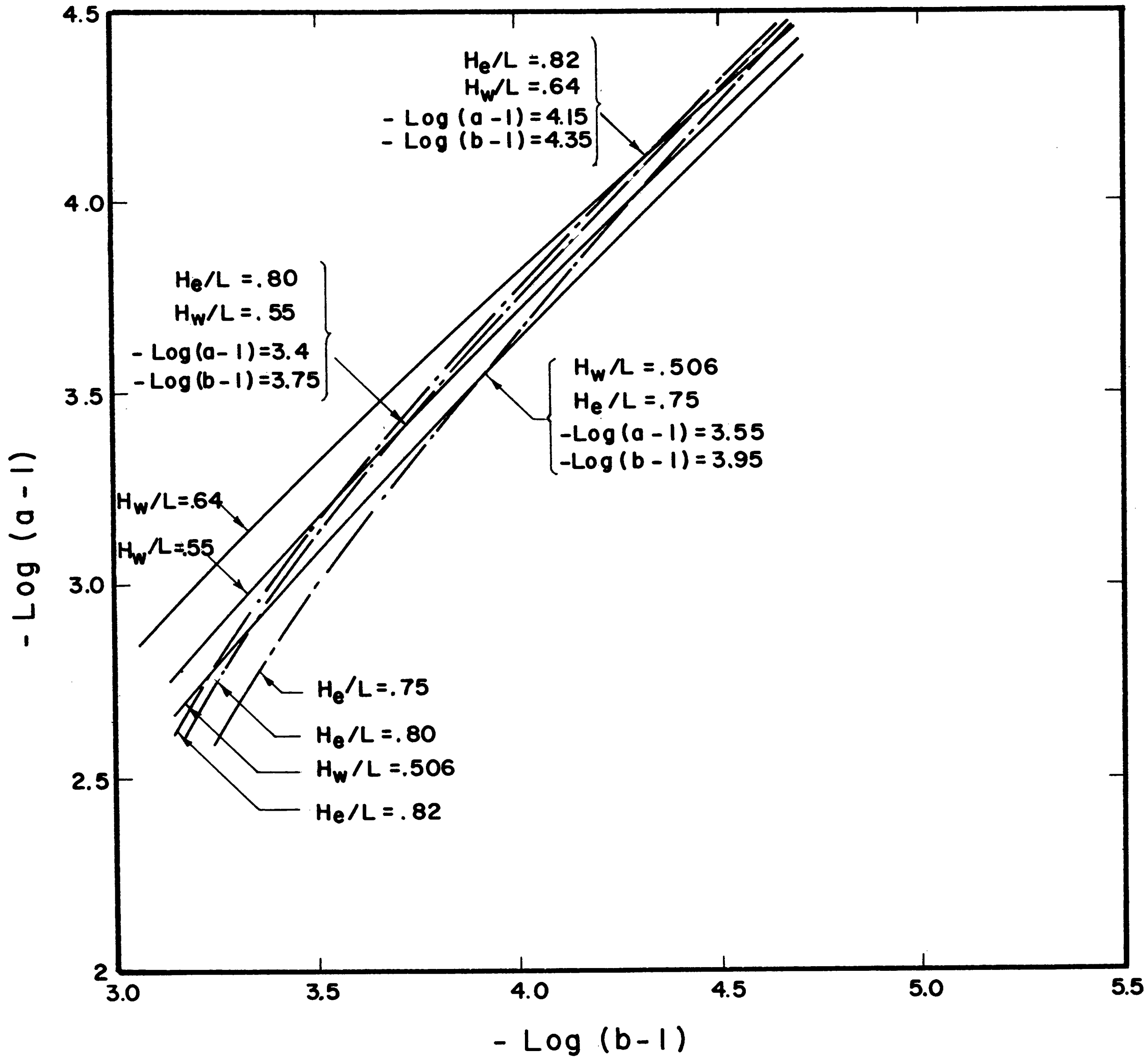

Figure 23. Curves Showing the Values of a and b for Specific Values of H_e/L and H_w/L Encountered in Experiment 2.

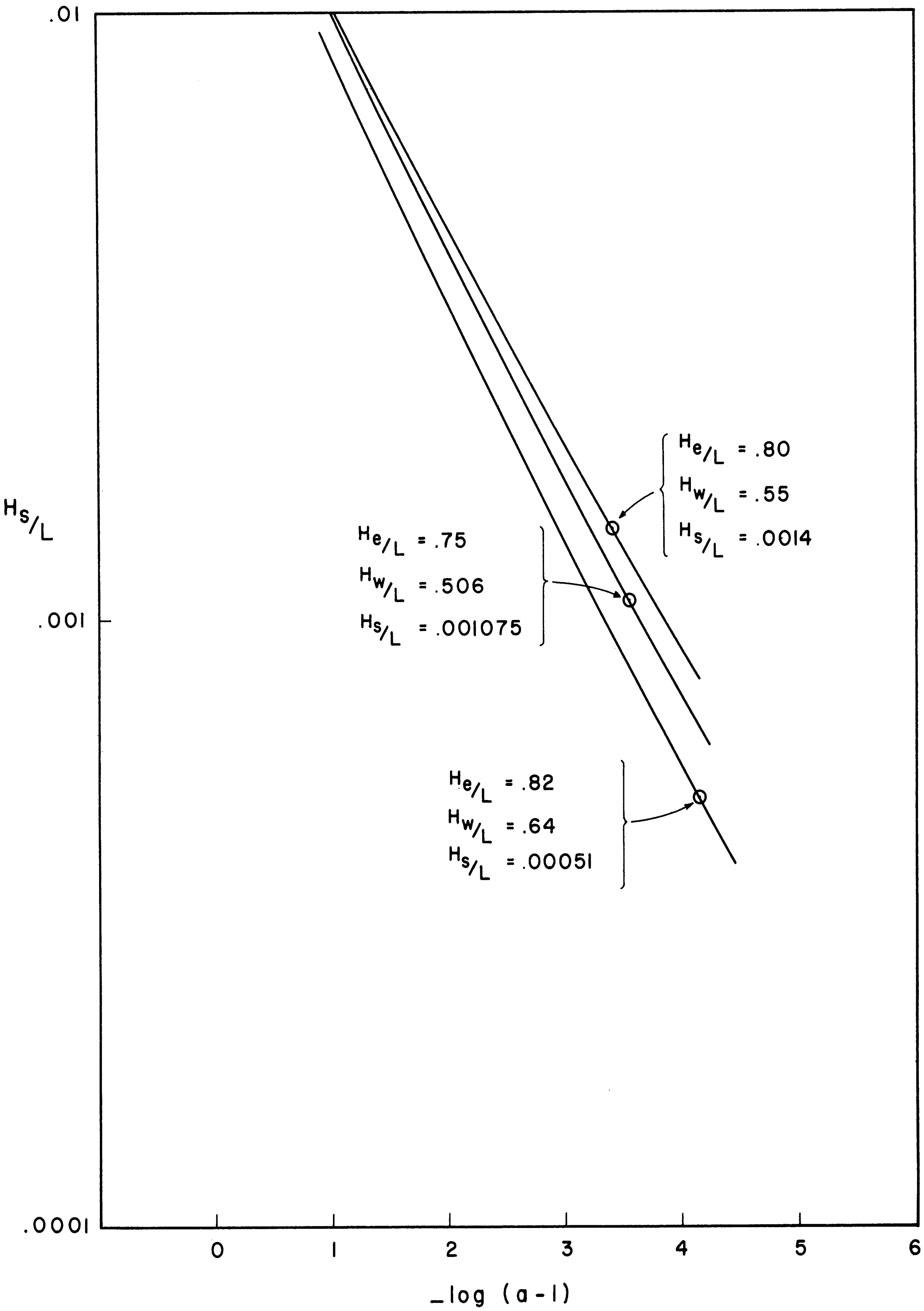

Figure 24. Curves Showing H_s/L vs a for Specific Values of b Encountered in Figure 23.

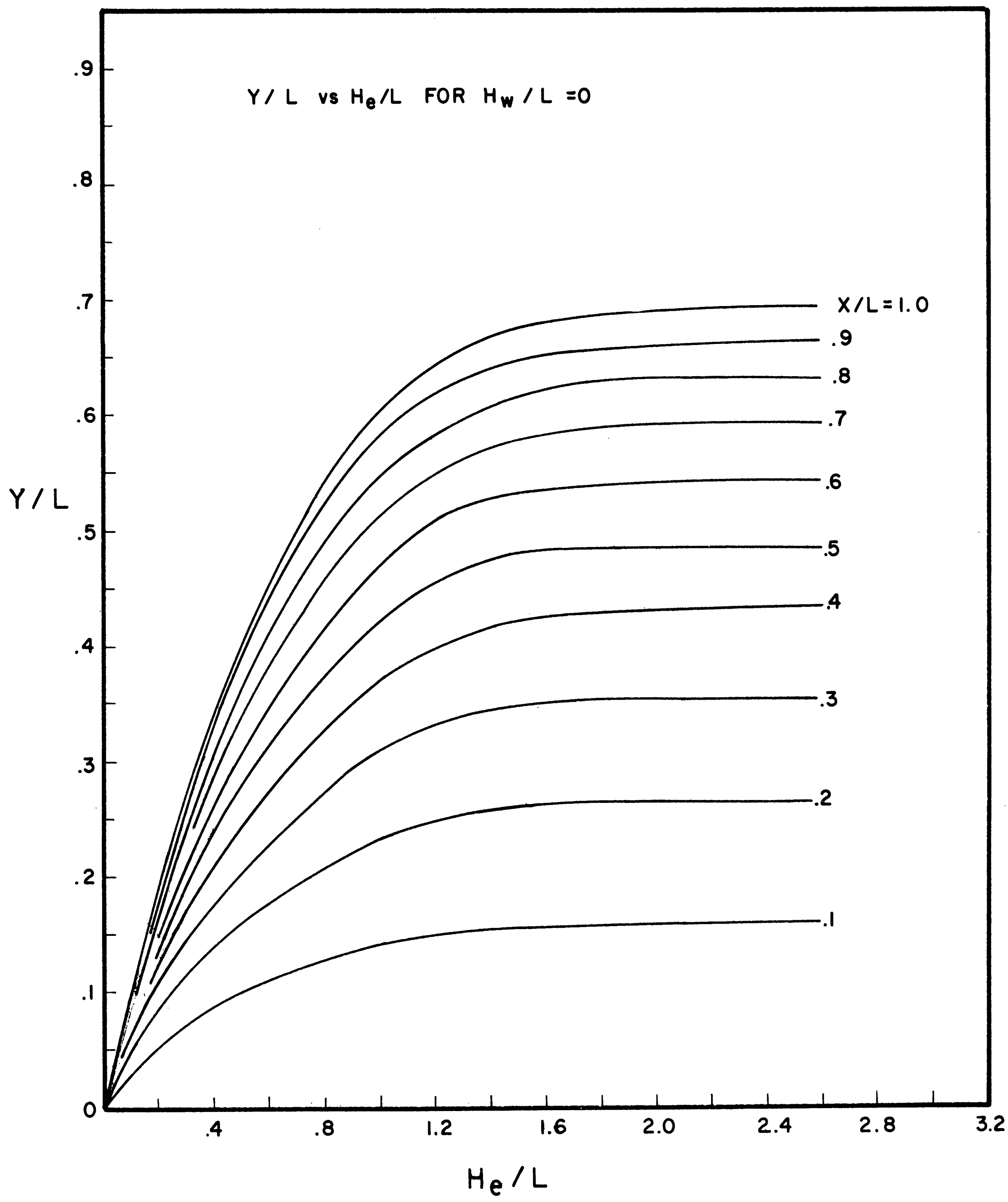

Plate 16
Curves Which Determine the Shape of the Free Surface
($H_W/L = 0$)

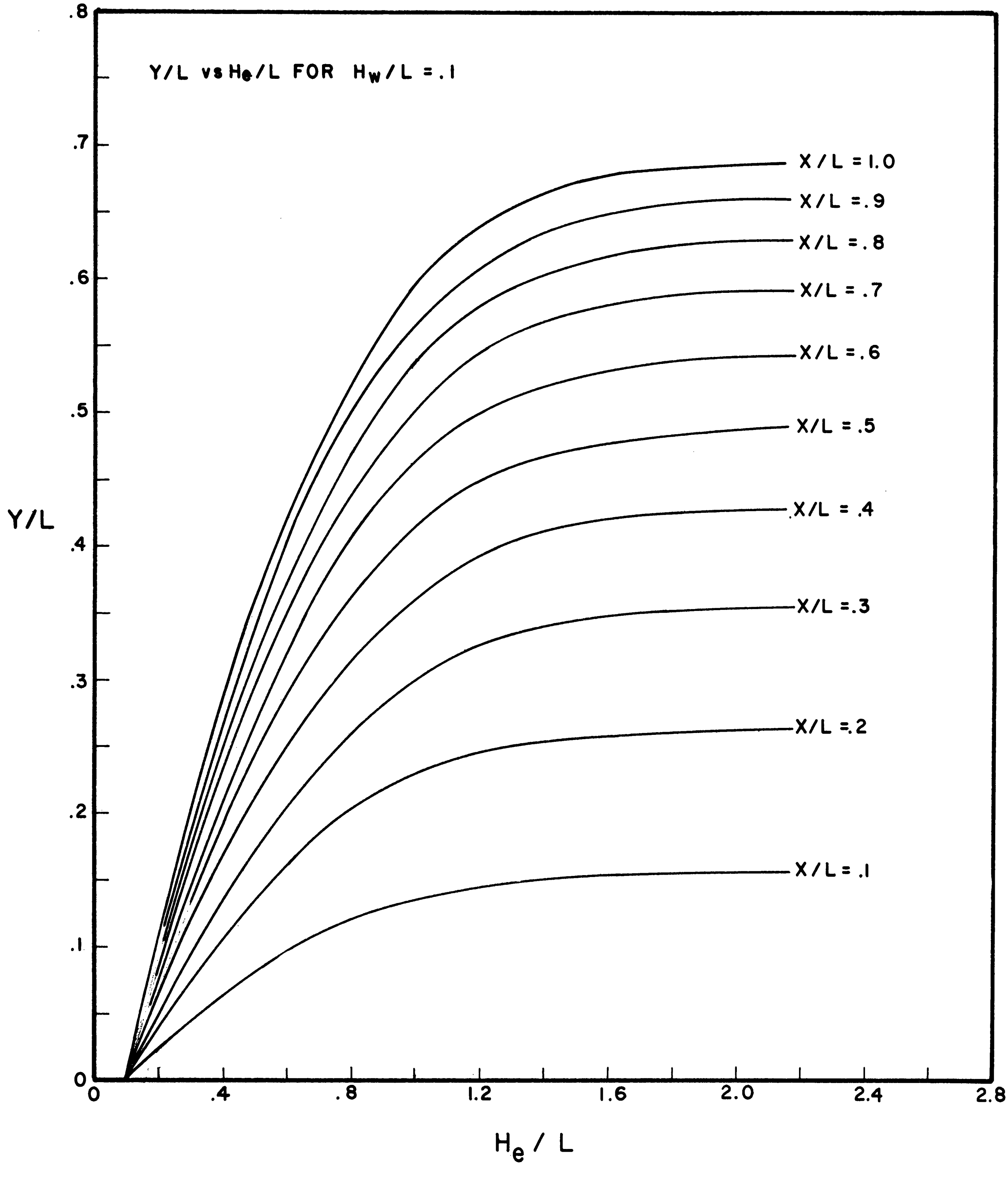

Plate 17
Curves Which Determine the Shape of the Free Surface
(H_w/L = .1)

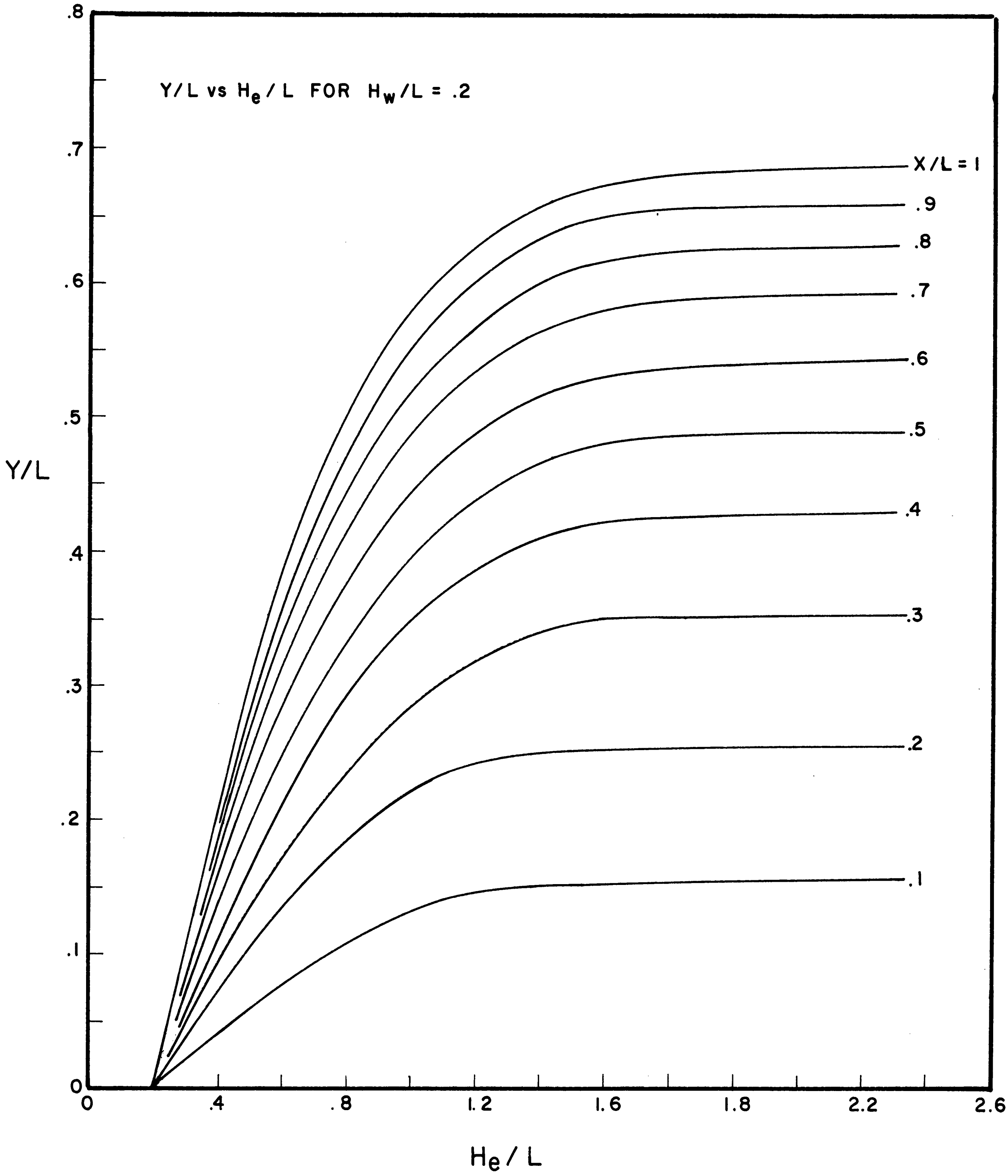

Plate 18
Curves Which Determine the Shape of the Free Surface
(H_w/L = .2)

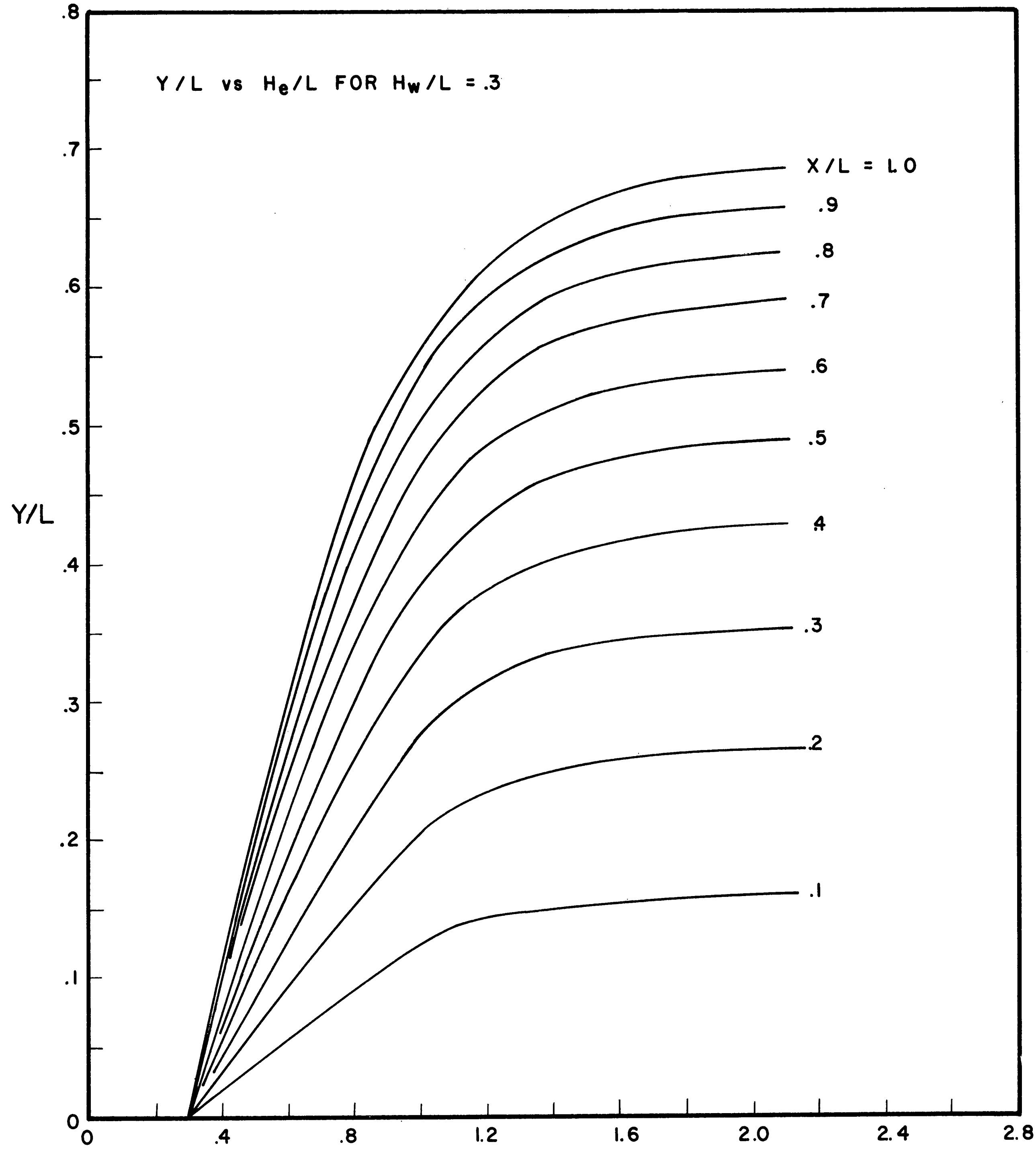

Plate 19
Curves Which Determine the Shape of the Free Surface
(H_w/L = .3)

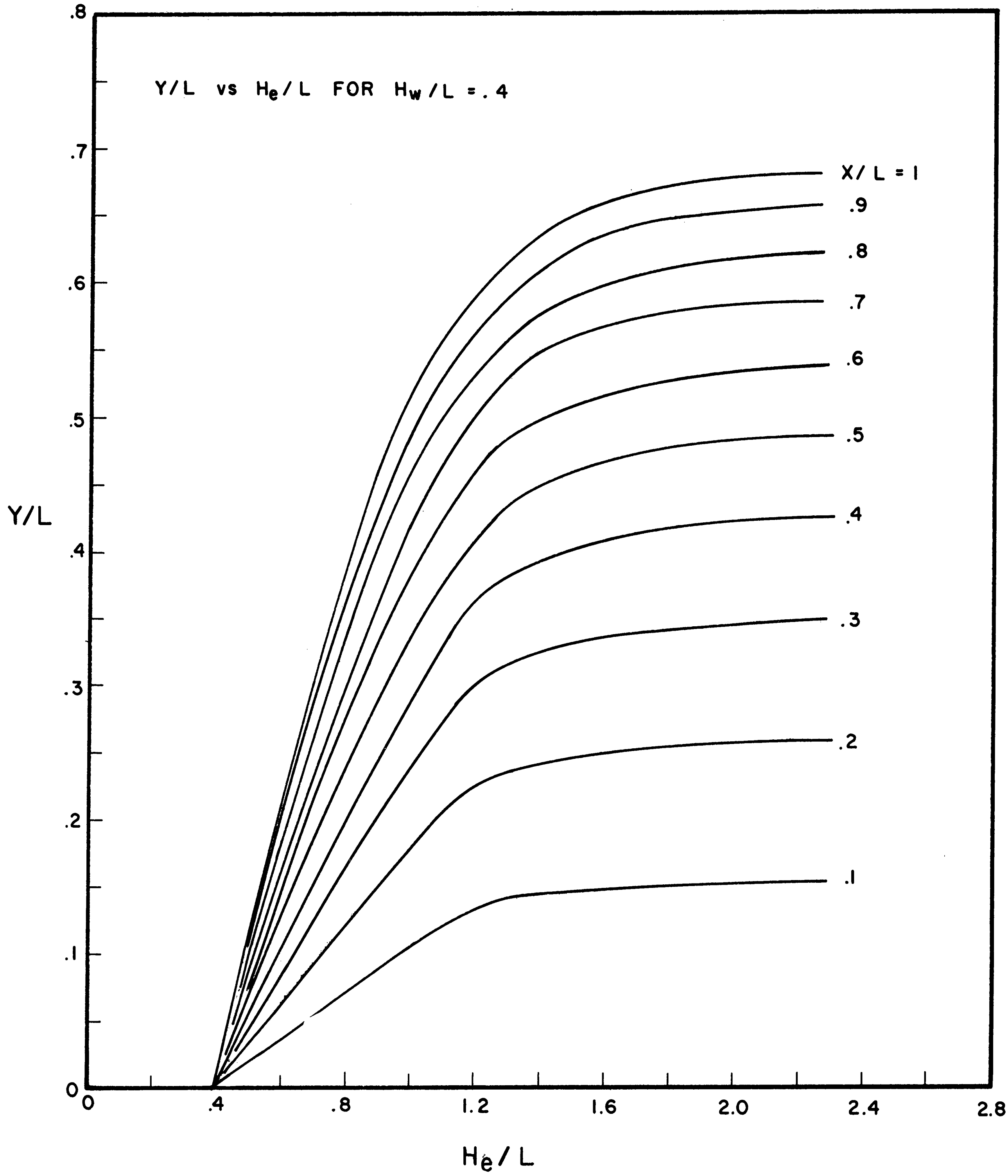

Plate 20
Curves Which Determine the Shape of the Free Surface
($H_w/L = .4$)

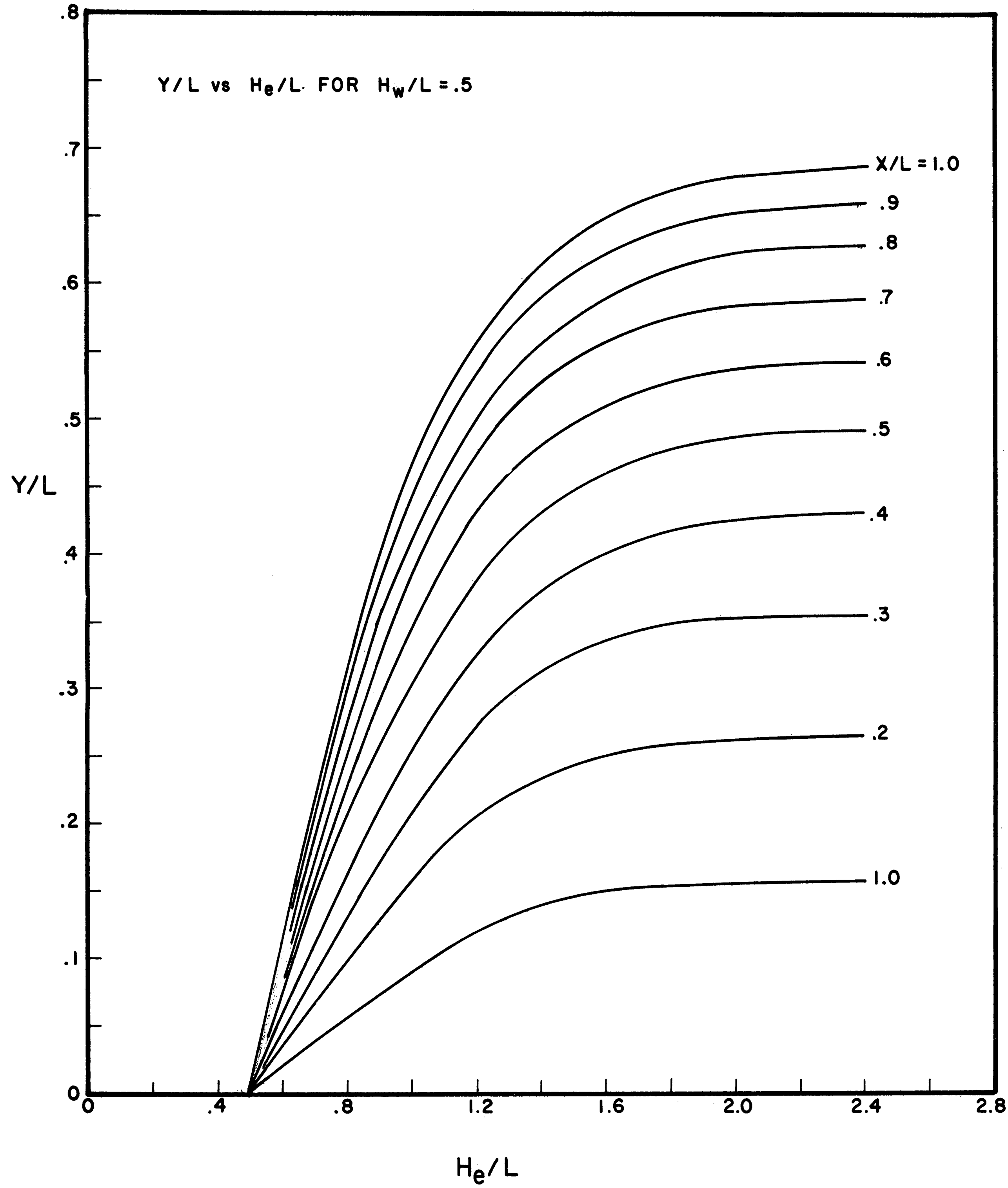

Plate 21
Curves Which Determine the Shape of the Free Surface
$H_w/L = .5$)

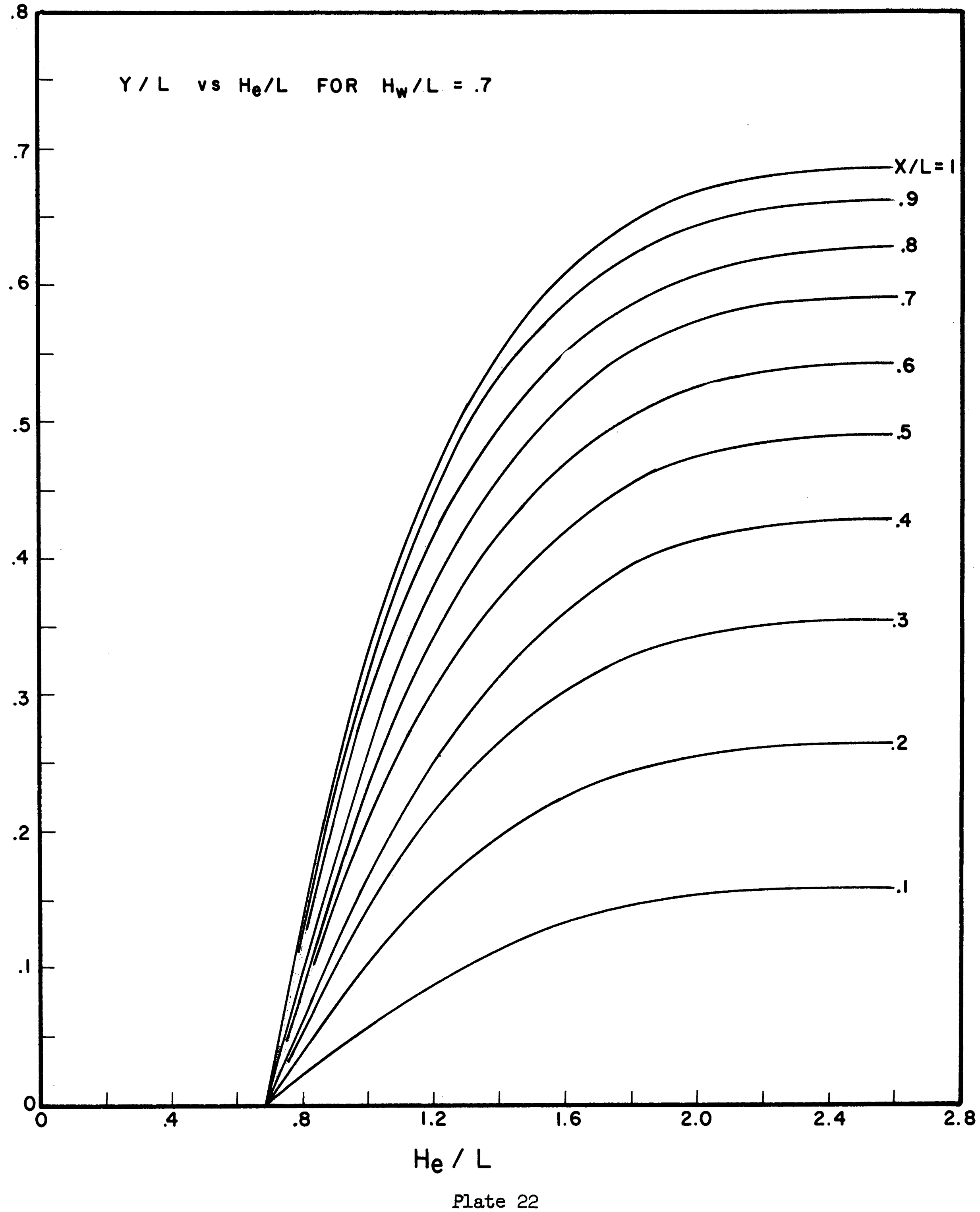

Plate 22
Curves Which Determine the Shape of the Free Surface
($H_w/L = .7$)

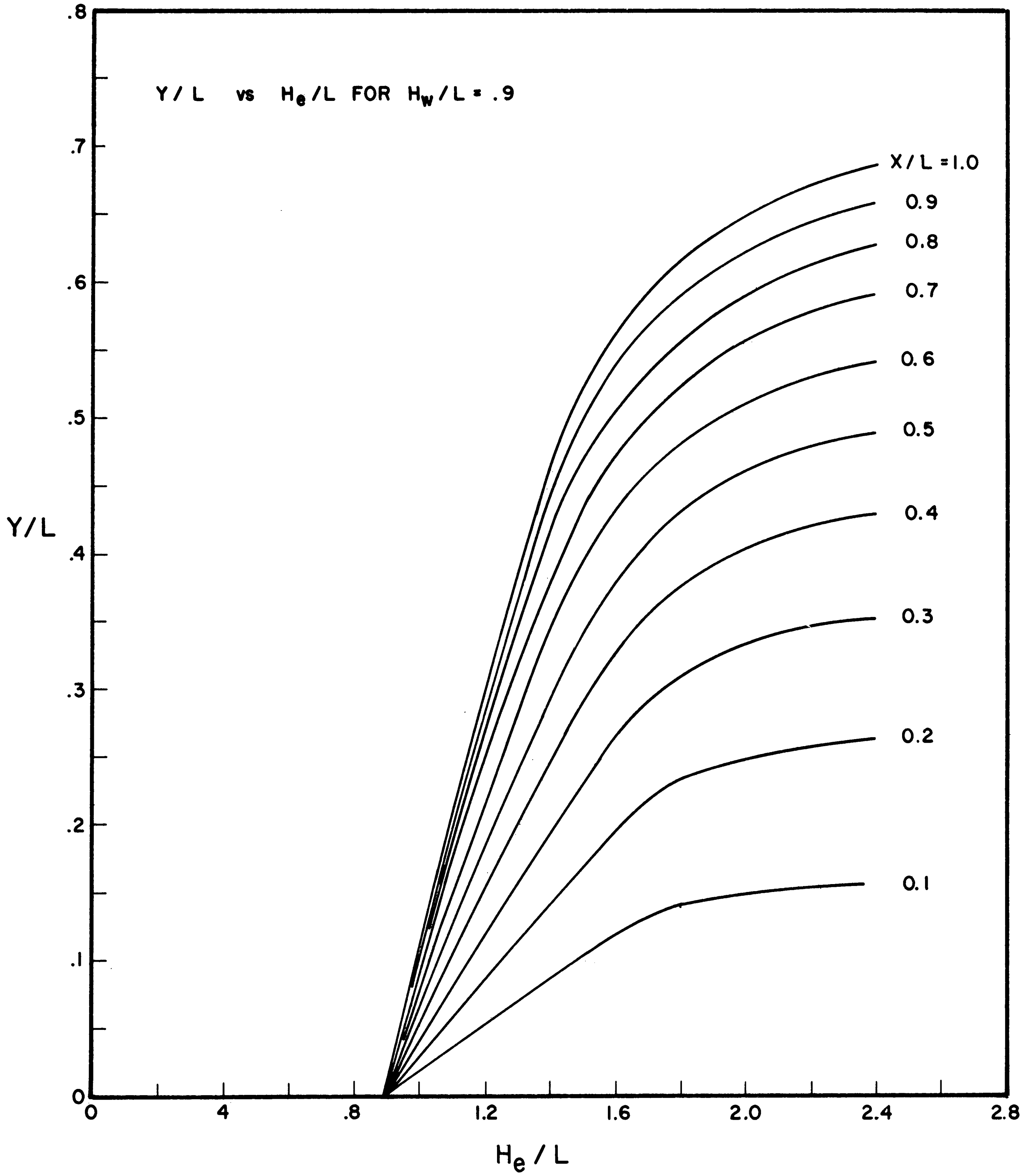

Plate 23
Curves Which Determine the Shape of the Free Surface
($H_w/L = .9$)

APPENDIX V

TABLES II AND III
LIST OF SYMBOLS USED IN THE MAD STATEMENTS

TABLE II

LIST OF SYMBOLS USED IN THE MAD STATEMENTS
OF THE UNSTEADY STATE
PROGRAMS

Symbol	Equivalent Symbol in the MAD Statements	Description
u	U	Horizontal comp. of velocity at the beginning of an interval (Δ t)
u	U1	Horizontal comp. of velocity at the end of an interval (Δ t)
H	H	Elevation of the free surface at the beginning of an interval (Δ t)
H	H1	Elevation of the free surface at the end of an interval (Δ t)
H_L	ELR	Elevation of the top of the liquid at the right boundary
$t_{L,M}$	TMR	Time at which the fluid height at the right boundary of the actual model reaches ELR
$t_{c,L}$	TCR	Time at which the fluid height at the right boundary of the computational model reaches ELR/XR
$\bar{k}$	KM	Coefficient of permeability of the model
x_r	XR	Length scale = $\frac{\text{Length of experimental model}}{\text{Length of computational model}}$
$t_{M,o}$	TML	Time at which the fluid height at the left boundary of the actual model reaches ELL
$t_{c,o}$	TCL	Time at which the fluid height at the left boundary of the computational model reaches ELL/XR
$(\xi_+)_C$	KESPC	Slope of C_+ curve at C

TABLE II
(CONT'D)

Symbol	Equivalent Symbol in the MAD Statements	Description
$(\xi_-)_C$	KESMC	Slope of C_- curve at C
u_S, H_S, . . .	US, HS, . . .	u, H, etc., at S
u_R, H_R, . . .	UR, HR, . . .	u, H, etc., at R
$t_{s,M}$	TSM	Time at which the photographs are taken
$t_{s,c}$	TSC	Time (corresponding to the computational model) at which the photographs are taken
t	T	Time
Δx	DELX	Distance between two adjacent points on the x-axis of the characteristic grid
Δt	DELT	Time interval in the characteristic grid for the computational model
Θ	TH	
L	L	Length of the model
$H_{e_{th}}$	HECS	The value of the inflow height which corresponds to the same amount of discharge in a steady state case
	QOUTP	$Q/L\bar{k}_M$
$\bar{k}_M$	KM	The coefficient of permeability of the model

TABLE III

LIST OF SYMBOLS USED IN THE MAD STATEMENTS
OF THE STEADY STATE PROGRAMS

Conventional Symbol	Equivalent Symbol in the MAD Statements	Description
t	T	A variable taking values between 0 and 1
λ	LAM	Modular elliptic function
$\beta(\lambda)$	GAML	Angle B in radians (see Figure 1a and 1b
	BETA	$\frac{q-1}{i}$ for $\infty \geq \lambda \geq 1$ or $-\infty \leq \lambda \leq 0$
a	A	A parameter involved in the integral formulas of the steady state solution
b	B	A parameter involved in the integral formulas of the steady state solution
H_w/L	HWFCL	Height of outflow surface/width of the dam
H_w/H_e	HWPCHE	Height of outflow surface/height of inflow surface
H_s/L	HSPCL	Height of seepage surface/width of the dam
H_s/H_e	HSPCHE	Height of seepage surface/height of inflow surface
H_e/L	HEPCL	Height of inflow surface/width of the dam
L/H_e	LPCHE	Width of the dam/Height of inflow surface
$H_s/c\bar{k}$	BC	Height of seepage surface
$H_w/c\bar{k}$	AB	Height of outflow surface

TABLE III
(CONT'D)

Conventional Symbol	Equivalent Symbol in the MAD Statements	Description
X	X	x-coordinate of a point on the free surface
Y	Y	y-coordinate of a point on the free surface
X/L	XP	Dimensionless x-coordinate of a point on the free surface
Y/L	YP	Dimensionless y-coordinate of a point on the free surface
$\beta(t)$	GAMT	Angle β in radians (see Figures 1-a and 1-b
$\frac{-2\pi\alpha}{3}$	S	$\int_0^1 \frac{\beta(t)}{t - \lambda}\, dt$
	EPS	e^{α}
	YPP	YP when XP= .1, .2, .3, . . and 1

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