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THERMAL UTILIZATION AND RESONANCE INTEGRALS IN HETEROGENEOUS REACTORS

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TABLE OF CONTENTS

TABLE OF CONTENTS (CONT'D)

LIST OF TABLES

LIST OF FIGURES

CHAPTER I

INTRODUCTION

A. Purpose and Statement of the Problem

In this thesis, a mathematical technique has been developed for solving the neutron (age) diffusion equation in "mixed" geometries. Thus, for example, we are able to obtain rigorous solutions to the diffusion equation for rectangular cells containing cylindrical fuel rods, cubical cells containing spherical fuel elements, etc.

The reason for developing this technique is that in this way a rigorous* expression for thermal utilization and resonance escape in heterogeneous reactors be obtained. Most calculations[1-3] of these quantities are based on approximations which attempt to avoid the complexities introduced by mixed geometry. Thus, for example, equivalent cylindrical cells replace the actual cells, and we attempt to evaluate the error introduced in this way. More seriously, in resonance escape calculations, it is assumed that the flux recovers completely to a constant in space and lethargy between resonances. Our method permits an accurate calculation of resonance escape without this approximation, and permits us to study the interference between closely spaced resonances.

The various approximations which we are attempting to evaluate are described in more detail below.

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* Rigorous in the context of the diffusion approximation.

1. It is generally assumed, in thermal utilization calculations, that rectangular cells with fuel rods at the center can be replaced by cylindrical cells for the purpose of flux calculations. The assumption that the reactor can be divided into cells, with a zero-current boundary condition at the surface, is clearly correct only for a very large system. However, we accept this assumption, but test the additional approximation that the actual cell shape may be replaced by a cylinder. We have done this by calculating the thermal utilization exactly for a system composed of rectangular cells, and comparing it with the same calculation carried out in the "equivalent cylindrical cell" approximation. The methods of our calculations are described in Chapters II and III. The numerical results, obtained via a code described in Appendix F, are presented in Chapter III. Our rigorous calculations have also been carried out for one- and three-dimensional systems, although no comparison with approximate methods are made for those cases. To anticipate the numerical results, we find that the "equivalent cell" approximation is rather good.

2. In resonance escape calculations, it is usually assumed that the flux feeding neutrons into the resonance is constant in space and lethargy. In case of closely spaced resonances the flux may not recover completely before reaching the next lower energy resonance and thus the flux may depend on both space and lethargy. When a resonance is in the transient region of the flux due to the presence of a higher energy resonance, the resonance integral of the lower energy resonance may differ from that calculated with the flat flux assumption. We investigate this effect,

the resonance interference, by quasi-rigorous calculations described in Chapters II, IV and V. Although our numerical results are restricted to the case of one-dimensional (i.e., slab) lattice, the two- or three-dimensional cases can be carried out using the expressions derived in Chapter II with proper first-collision probabilities. We find that resonance integral changes due to the presence of a nearby resonance at higher energy. Therefore, the flat flux assumption may not be valid for close resonances.

In order to understand better the purpose and methods of calculations described here, it is important to review the previous work which has been carried out.

B. Review of Previous Works

The thermal utilization, defined as the fractional number of thermal neutrons absorbed in the fuel, for a homogeneous system depends upon the composition of material, but in a heterogeneous system it depends also upon the size and shape of the unit cell. The effect of cell shape on thermal utilization has been considered by several authors.[4-11] The equivalent unit cell approximation of Wigner and Seitz,[12] which was originally conceived for the calculation of wave functions of crystal lattices, are usually used in calculating the flux in the unit cell. In this approximation it is assumed that the unit cell may be replaced by a cylinder (or a sphere, in case of a three-dimensional cell), although it is clear that a collection of cells cannot occupy the exact volume of any system unless the cells have straight sides. Nevertheless this assumption has become popular for the simplicity of the calculations. The heterogeneous method of Feinberg[13] also gained general acceptance.

Weinberg[4] investigated the thermal utilization of a square lattice and compared with that of a hexagonal lattice. He observed that the difference is negligibly small when the radius of the equivalent cell is small compared with the moderator diffusion length.

Clark and Newmarch[5] considered a finite-sized fuel rod in a square cell with a capturing moderator. They concluded that the thermal utilization calculated with the cylindrical cell approximation differs from their calculations only by a few tenths of a percent even for water lattices. Cohen[6] considered a square cell with a non-absorbing moderator and replaced the fuel rod by a line sink. He observed that the flux pattern is cylindrically symmetric to a radius of about one-fourth the lattice spacing and showed that a sufficiently accurate value for the thermal utilization is given by the equivalent cylindrical cell approximation.

Galanin[7] assumed a line fuel rod and small moderator absorption to find thermal flux patterns and thermal utilization of several lattices. He summed over lattice arrays of modified zero order Bessel functions of the second kind (i.e., the K_0, according to Watson's[14] notation). Galanin's approach is conceptually the same as the small source theory of Horning,[15] who showed that the small source theory holds good if the ratio of the fuel rod radius to the diffusion length is small.

Neumann[8] combined the lattice sum technique of Galanin[7] with the harmonic development of Clark and Newmarch[5] to obtain expressions for the thermal flux and the thermal utilization for several cells. He observed that the propagated effect of the fuel rod in a lattice will

be cylindrically symmetric when the lattice is highly symmetric, when the ratio of lattice spacing to moderator diffusion length is small, or when the ratio of fuel diameter to the lattice spacing is small.

Bailly du Bois[9] replaced the fuel rod by a cylindrically symmetric line sink, and solved the diffusion equation for a nonabsorbing moderator. He expressed the thermal utilization in terms of Jacobian theta functions. Pazy and Goshen[10] extended his method to the absorbing moderator case.

All the works mentioned above used diffusion theory and assumed a flat source distribution in the moderator. Horning and Galanin showed that these approximations should be satisfactory for all shapes.

Amouyal et al.[11] have developed an improved method for computing the thermal flux disadvantage factor and the thermal utilization in lattice cells in which the diffusion theory assumption was relaxed. They assumed that diffusion theory is applicable a few mean free paths away from the rod, and integral transport is used in the fuel. The method may break down if the moderator region is only one or two mean paths thick. Amouyal et al. concluded that the method gives results as good as those obtained from the P_3 or P_5 approximations.

Our approach has been somewhat different from all of those described above. We assumed diffusion theory to be valid in both fuel and moderator, and expand the flux in the cell in a Fourier series. In this sense, our work is similar to that of Clark and Newmarch,[5] although it is basically simpler. For example, we are able to satisfy the zero-current boundary condition exactly because of the form of our expansion. Thus, we do not need to consider, as Neumann[8] does, the contribution

of other cells. In this way, we are led to an infinite determinantal equation to solve for the expansion coefficients of the flux, although we find that a reasonably small number of terms represent a good approximation. The same technique is used both in the thermal utilization and resonance escape problems. The basic procedure is as follows:

In Chapter II we solve the age-diffusion equation for rectangular parallelepiped, rectangular, and slab cells. Fluxes have been expanded in terms of the solutions of the Helmholtz equation with lethargy dependent coefficients. In Chapter III expressions have been derived for the disadvantage factors and the thermal utilization for a rectangular parallelepiped cell with a spherical fuel element at the center, and a rectangular cell with a cylindrical fuel rod at the center. The disadvantage factors and the thermal utilizations have been computed for several rectangular cells and compared with those computed for the equivalent cylindrical cells. A flat source distribution of thermal neutrons in the moderator and no source of thermal neutrons in the fuel have been assumed.

Turning to the question of resonance escape, previous works have been along the following lines. Basically, we represent p, the resonance escape probability, by

$$p = e^{-\sum_i \frac{I_i}{\overline{\xi\sigma_s}}}$$

where I_i is the resonance integral of the i-th resonance.

$$\overline{\xi\sigma_s} = \xi_1\sigma_{p1} + \xi_2\sigma_{p2}$$

where σ_{p1} is the potential microscopic scattering cross section of the absorber, and σ_{s2} is the microscopic scattering cross section of the moderator per absorber nucleus. Therefore, we are interested in evaluating resonance integrals.

In order to simplify the resonance integral calculations Wigner[16] suggested two approximations: (i) the narrow resonance approximation (NR), in which it is assumed that the resonance line is extremely sharp and that a single collision with the absorber is enough for the neutron to pass the resonance region; (ii) the infinite mass (IM) approximation in which it is assumed that neutrons are not slowed down by the absorber. The distinction between these two extreme approximations is made with the help of the concept of practical width, defined as the interval over which the resonance cross section is larger than the potential cross section. When the practical width is small compared to the maximum energy change per collision, the NR approximation is used. The IM approximation is used in those cases where the practical width is larger than the maximum energy loss per collision. But there are cases where none of these approximations gives a satisfactory result.

First order corrections to the NR and IM approximations have been made by Spinney,[17] Chernick, and Vernon,[18] and Rothenstein.[19,20] Nordheim[21-23] has developed a numerical method of evaluating resonance integrals without the NR or IM approximation. Recently Goldstein and Cohen[24] developed an analytical method (λ-method) of evaluating resonance integrals for homogeneous media. The constant λ, which depends upon the resonance parameters only, has a value between 0 and 1 and the extreme values correspond to the usual NR and IM approximations. Hill and Schaefer[25] have extended the λ-method for the heterogeneous case. Sumner[26] has written a code based on the λ-method. In this work we have considered the NR and IM approximations only. However, the λ-method may

be used in our calculations for the resonances in which the maximum energy loss per collision of the absorber and the practical width have nearly the same value.

The usual procedure for calculating resonance integrals in heterogeneous systems has been to assume a flat neutron source (both spatially and lethargy wise) feeding into resonance. Then the resonance integral can be expressed simply in terms of constant source first collision probabilities such as those calculated by Case, et al.[29] While the flat source assumption is valid for resonances spaced sufficiently widely in lethargy, we are interested in finding corrections for closely spaced resonances. We are also interested in finding out whether the spacing in lethargy which may be considered wide depend upon the size of the cell.

Schermer and Corngold[27] investigated the interference between resonances in an infinite homogeneous media using a variational technique and concluded that the interference is negligibly small. Brown, et al.[48] performed an experiment to measure the interference between the resonances of gold, indium and rhenium. They placed a spherical lump of indium mixed with gold or rhenium in water. They observed a decrease in resonance integral of indium resonance due to the presence of interfering resonances. Foell, et al.[49] calculated the resonance absorption in gold and indium lumps using a Monte Carlo code[50] and compared with those calculated using the ZUT-MOD-3 code,[51] which considers two overlapping resonances. In our calculations we consider two non-overlapping resonances only.

In a recent paper Iijima investigated the resonance disadvantage factor due to (i) the incomplete recovery of flux at off resonance

energies and also due to (ii) failure of narrow resonance approximation in the moderator. The error, due to the first cause, of about two percent or less for rectangular uranium-graphite cell was observed. The error in the surface absorption due to the second cause may be sizeable. In this calculation Iijima used the Wigners rational approximation for the escape probability. No correction was made for non-flat source distribution.

In our investigation we study the interference between close resonances in a heterogeneous lattice. Our procedure has been to first find the neutron flux, as a function of energy and position, in the region of a given resonance due to the "negative source" contribution of a higher resonance. This is done by utilizing the solutions of the age-diffusion equation described in Chapter II. The results are given in Chapter V. In order to evaluate the resonance integral it is necessary to find the first-collision probabilities for spatial distributions of the neutrons above the resonance.

Case, et al.[29] have calculated the flat source collision probabilities for various geometries. Wigner's[16] rational approximation gives consistently higher value for the collision probability. Sauer[30] has developed an expression for collision probability based on Wigner's rational approximation and has shown that it gives a better result. Di Pasquantonio[31] used a different approach in developing analytic expressions for the collision probabilities for lattice cells composed of different media. But in all these calculations the flat source approximation has been used. Corrections for dense lattices, which were first investigated by Dancoff and Ginsburg,[32] and were later developed by

several other investigators,[33,34] are usually made in the calculation of resonance integrals. A simple equivalent expression for escape probabilities in dense lattices has been given by Bell.[35] Levine[36] computed resonance integrals by the Monte Carlo method and compared with those calculated from analytical expressions and concluded that the flat source collision probabilities are inapplicable in thick lumps.

In Chapter IV we derived expressions for first collision probabilities in a slab lattice due to flux distribution which is a sum of a spatially constant term and a cosine term. The cosine term appears due to the presence of first (higher energy) resonance absorption which has been replaced by a negative source. (We have shown that the next higher terms in the expansion of the flux may be neglected.) If the separation between resonances is large, the coefficient of cosine term vanishes. Since we have to compute the collision probabilities numerically, we cannot prove the reciprocity theorem.[29]

In Chapter V we have calculated the resonance integrals of two resonances separately, and also in the presence of one another. All resonances are assumed to be resolved. The unresolved resonances may be treated by the methods developed by Nordheim,[21] and Adler, et al.[37] The interference between resonances has been computed for several selected pair of resonances of U-238, Th-232, their oxides and their mixtures. We find that the resonance interference increases with the increase of thickness of fuel lump. We also observe that the resonance integral of the lower energy resonance increases for the graphite moderated lattices but it decreases for the heavy water moderated lattices,

although the total absorption by the lower energy resonance decreases due to the presence of higher energy resonance in both the cases. Hayes, Luming, and Zweifel[38] investigated the cases where the flux should take its asymptotic value below a resonance. This work is a further extension of their method. A part of our results was reported earlier.[39]

CHAPTER II

SOLUTION OF AGE-DIFFUSION EQUATION

In this chapter we solve the age-diffusion equation in (A) a rectangular parallelepiped cell with a spherical fuel element at the center of the cell, (B) a rectangular cell with a cylindrical fuel rod at the center of the cell and (C) a one dimensional slab cell. These cells are assumed to be one of many similar cells of infinite extent so that zero current boundary condition can be assumed at the surface of the cell. (In a reactor this assumption is valid for the cells not close to the boundary of the reactor.) It is also assumed that:

i. Age-diffusion theory holds both in the fuel and the moderator region.

ii. The cross sections are independent of energy and are region-wise constant.

iii. A uniform monoenergetic neutron source S_1 is present in region 1 (fuel) and S_2 in region 2 (moderator).

A. Rectangular Parallelepiped Cell with a Spherical Fuel Element at the Center of the Cell

Let us consider a rectangular parallelepiped cell of length 2a along the x-axis, 2b along the y-axis, and 2c along the z-axis with a spherical fuel element of radius ρ at the center of the cell (Figure 1). Let us choose the center of the cell as the origin of the coordinate system. For the steady state condition the age-diffusion equation can be written as

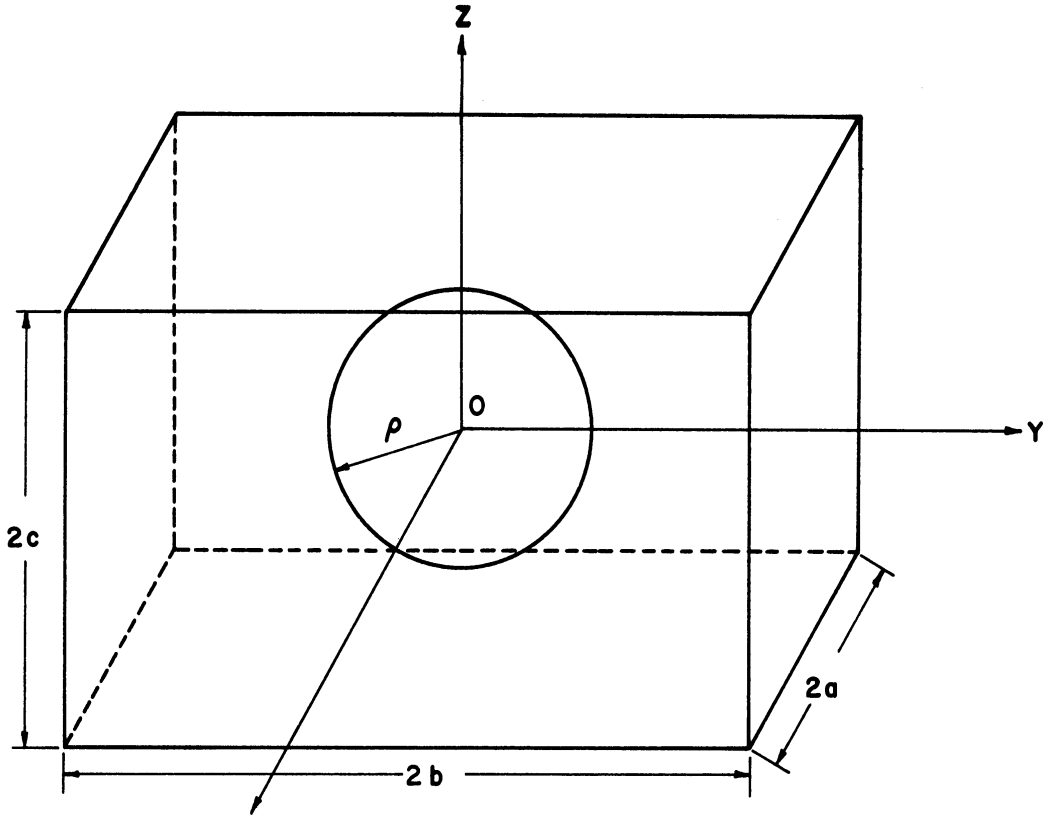

Figure 1. Sketch of a Rectangular Parallelepiped Cell.

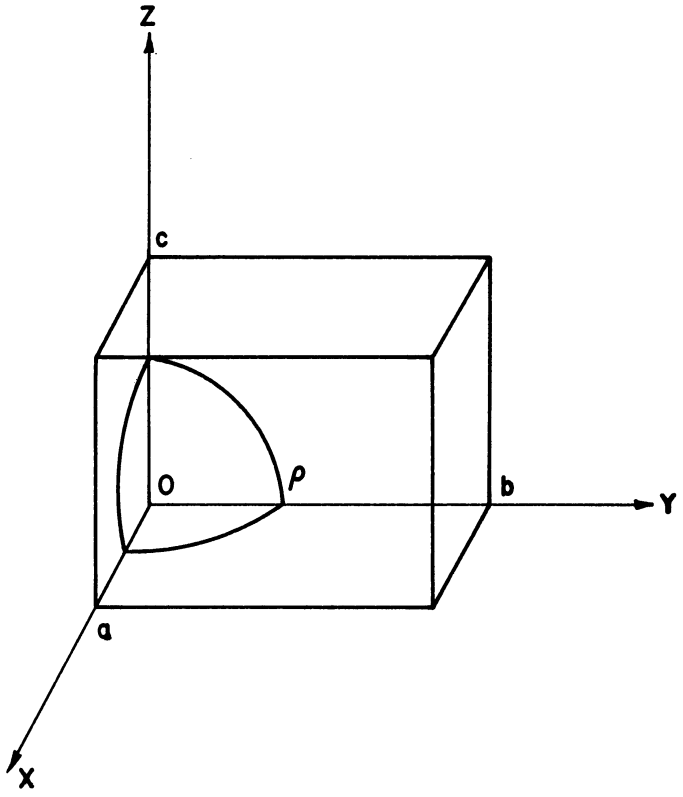

Figure 1a. Semiquadrant of a Rectangular Parallelepiped Cell.

$$-D\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}\right)-\frac{\partial D}{\partial x}\frac{\partial\phi}{\partial x}-\frac{\partial D}{\partial y}\frac{\partial\phi}{\partial y}$$

$$-\frac{\partial D}{\partial z}\frac{\partial\phi}{\partial z}+\Sigma_a\phi=-\xi\Sigma_s\frac{\partial\phi}{\partial u}+S(x,y,z)\,\delta(u) \qquad (2\text{-}1)$$

$\phi(x,y,z,u)$ is the usual neutron flux, $\Sigma_a(x,y,z)$ and $\Sigma_s(x,y,z)$ are macroscopic absorption and scattering cross sections respectively, $D(x,y,z)$ is the diffusion coefficient, $\xi(x,y,z)$ is the average logarithmic energy decrement per collision and $\delta(u)$ is Dirac's delta function. Thus $S(x,y,z)\,\delta(u)$ is a source of monoenergetic neutrons of lethargy $u = 0$, and is assumed constant in each region (as are the various material parameters).

Since we propose to enforce a zero current boundary condition at the surface of the cell, we expand the flux in the following way:

$$\phi(x,y,z,u)=\frac{a_{000}(u)}{8}+\sum_{i=0}^{\infty}{}'\sum_{j=0}^{\infty}{}'\sum_{k=0}^{\infty}{}' a_{ijk}(u)\cdot\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b}\cos\frac{k\pi z}{c} \qquad (2\text{-}2)$$

The summation $\sum'\sum'\sum'$ is over all values of i,j and k except the case $i = j = k = 0$. We insert Equation (2-2) in Equation (2-1), multiply by $\cos\frac{l\pi x}{a}\cos\frac{m\pi y}{b}\cos\frac{n\pi z}{c}$, divide by $\xi\Sigma_s$ and integrate over the cell. After taking a Laplace transform with respect to u, we obtain (λ = transform variable)

$$\bar{a}_{000}(\lambda)\Big[abc\left(\lambda+\frac{\Sigma a2}{\xi_2\Sigma_{s2}}\right)\delta_{ol}\,\delta_{om}\,\delta_{on}+\frac{\pi}{6}\rho^3\left(\frac{\Sigma a1}{\xi_1\Sigma_{s1}}-\frac{\Sigma a2}{\xi_2\Sigma_{s2}}\right)$$

$$\delta_{ol}\,\delta_{om}\,\delta_{on}+\frac{\bar{I}'(0,0,0,l,m,n)}{8}\Big]+\sum_{i=0}^{\infty}{}'\sum_{j=0}^{\infty}{}'\sum_{k=0}^{\infty}{}'\bar{a}_{ijk}(\lambda)$$

$$\Big[\left\{\left(\frac{i\pi}{a}\right)^2+\left(\frac{j\pi}{b}\right)^2+\left(\frac{k\pi}{c}\right)^2\right\}I(i,j,k,l,m,n)+$$

$$\left(\frac{i\pi}{a}\right)H_x(i,j,k,l,m,n)+\left(\frac{j\pi}{b}\right)H_y(i,j,k,l,m,n)$$

$$+\left(\frac{k\pi}{c}\right)H_z(i,j,k,l,m,n)+I'(i,j,k,l,m,n)$$

$$+\lambda abc\,\Lambda_{ijk}\,\delta_{il}\,\delta_{jm}\,\delta_{kn}\Big]=S(l,m,n) \tag{2-3}$$

where

$$\bar{I}(0,0,0,l,m,n)=\begin{cases}0, \text{ when } l=m=n=0\\ I'(0,0,0,l,m,n), \text{ otherwise}\end{cases} \tag{2-4}$$

δ_{ij} etc. are Kronecker deltas,

$l,m,n = 0,1,2,3,\ldots$ etc.

$\Lambda_{000} = 8$

$\Lambda_{i00}=\Lambda_{0j0}=\Lambda_{00k}=4$, for $i,j,k\neq 0$

$\Lambda_{ij0}=\Lambda_{i0k}=\Lambda_{0jk}=2$, for $i,j,k\neq 0$

$\Lambda_{ijk}=1$, for $i,j,k\neq 0$ (2-5)

and

$$I(i,j,k,l,m,n)=\iiint_{\text{cell}}\frac{D}{\xi\Sigma_s}\Big\{\cos\frac{i\pi x}{a}\cos\frac{j\pi y}{b}\cos\frac{k\pi z}{c}$$

$$\cos\frac{l\pi x}{a}\cos\frac{m\pi y}{b}\cos\frac{n\pi z}{c}\Big\}dx\,dy\,dz \tag{2-6}$$

$$H_x(i,j,k,l,m,n) = \iiint_{cell} \frac{1}{\xi \Sigma_s} \frac{\partial D}{\partial x} \sin\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \cos\frac{k\pi z}{c} \cos\frac{l\pi x}{a} \cos\frac{m\pi y}{b} \cos\frac{n\pi z}{c}\, dx\,dy\,dz \tag{2-7}$$

$$H_y(i,j,k,l,m,n) = \iiint_{cell} \frac{1}{\xi \Sigma_s} \frac{\partial D}{\partial y} \cos\frac{i\pi x}{a} \sin\frac{j\pi y}{b} \cos\frac{k\pi z}{c} \cos\frac{l\pi x}{a} \cos\frac{m\pi y}{b} \cos\frac{n\pi z}{c}\, dx\,dy\,dz \tag{2-8}$$

$$H_z(i,j,k,l,m,n) = \iiint_{cell} \frac{1}{\xi \Sigma_s} \frac{\partial D}{\partial z} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \sin\frac{k\pi z}{c} \cos\frac{l\pi x}{a} \cos\frac{m\pi y}{b} \cos\frac{n\pi z}{c}\, dx\,dy\,dz \tag{2-9}$$

$$S(l,m,n) = \iiint_{cell} \frac{S(x,y,z)}{\xi \Sigma_s} \cos\frac{l\pi x}{a} \cos\frac{m\pi y}{b} \cos\frac{n\pi z}{c}\, dx\,dy\,dz \tag{2-10}$$

The integrations defining I(i,j,k,l,m,n), Equation (2-6), have been carried out by dividing the cell into several parts such that in each

part $\frac{D}{\xi \Sigma_s}$ is constant and can be taken out of the integral.

$$I(i,j,k,l,m,n) = \frac{8 D_1}{\xi_1 \Sigma_{s1}} \int_0^{\rho} dx \int_0^{\sqrt{\rho^2 - x^2}} dy \int_0^{\sqrt{\rho^2 - x^2 - y^2}} dz \left\{ \quad \right\}$$

$$+ \frac{8 D_2}{\xi_2 \Sigma_{s2}} \left[\int_0^{a} dx \int_0^{b} dy \int_0^{c} dz - \int_0^{\rho} dx \int_0^{\sqrt{\rho^2 - x^2}} dy \int_0^{\sqrt{\rho^2 - x^2 - y^2}} \right]$$

$$\times \left\{ \cdots \cdots \right\}$$

These integrals are evaluated in Appendix A. We find

$$I(i,j,k,l,m,n) = \left(\frac{D_1}{\xi_1 \Sigma_{s1}} - \frac{D_2}{\xi_2 \Sigma_{s2}} \right) \left[F(i,j,k,l,m,n) + F(i,j,k,l,m,-n) \right.$$
$$+ F(i,j,k,l,-m,n) + F(i,j,k,-l,m,n) + F(i,j,k,-l,m,-n)$$
$$\left. + F(i,j,k,-l,-m,n) + F(i,j,k,l,-m,-n) + F(i,j,k,-l,-m,-n) \right]$$
$$+ \frac{D_2}{\xi_2 \Sigma_{s2}} abc \Lambda_{ijk} \delta_{il} \delta_{jm} \delta_{kn} \qquad (2\text{-}11)$$

Here $F(i,j,k,l,m,n)$ is given by

$$F(i,j,k,l,m,n) = J_{3/2} \left(\pi \rho \sqrt{\left(\frac{i+l}{a}\right)^2 + \left(\frac{j+m}{b}\right)^2 + \left(\frac{k+n}{c}\right)^2} \right)$$

$$\times \left(\frac{\rho}{2}\right)^{3/2} \left[\left(\frac{i+l}{a}\right)^2 + \left(\frac{j+m}{b}\right)^2 + \left(\frac{k+n}{c}\right)^2 \right]^{-3/2} \qquad (2\text{-}12)$$

$J_{3/2}$ is a Bessel function. To carry out the integrations defining H_x, H_y and H_z let us note that $\frac{1}{\xi \Sigma_s} \frac{\partial D}{\partial x}$ is a product of a step function

and a delta function

$$\frac{\partial D}{\partial x} = (D_2 - D_2)\,\delta(x - \sqrt{\rho^2 - y^2 - z^2}) \tag{2-13}$$

Thus, these integrations which contain $\frac{1}{\xi \Sigma_s}\frac{\partial D}{\partial x}$, $\frac{1}{\xi \Sigma_s}\frac{\partial D}{\partial y}$, and $\frac{1}{\xi \Sigma_s}\frac{\partial D}{\partial z}$ can be carried out easily. The integrations are carried out in Appendix B. We find

$$\begin{aligned} H_x(i,j,k,l,m,n) = \frac{1}{2}\left(\frac{1}{\xi_1\Sigma_{s1}} + \frac{1}{\xi_2\Sigma_{s2}}\right)(D_2 - D_1)\Bigg[& \frac{(i+l)\pi}{a}\Big\{F(i,j,k,l,m,n) \\ & + F(i,j,k,l,m,-n) + F(i,j,k,l,-m,n) \\ & + F(i,j,k,l,-m,-n)\Big\} + \frac{(i-l)\pi}{a}\Big\{F(i,j,k,-l,m,n) \\ & + F(i,j,k,-l,m,-n) + F(i,j,k,-l,-m,n) \\ & + F(i,j,k,-l,-m,-n)\Big\}\Bigg] \end{aligned} \tag{2-14}$$

Similarly we obtain

$$\begin{aligned} H_y(i,j,k,l,m,n) = \frac{1}{2}\left(\frac{1}{\xi_1\Sigma_{s1}} + \frac{1}{\xi_2\Sigma_{s2}}\right)(D_2 - D_1)\Bigg[& \frac{(k+m)\pi}{b}\Big\{F(i,j,k,l,m,n) \\ & + F(i,j,k,l,m,-n) + F(i,j,k,-l,m,n) \\ & + F(i,j,k,-l,m,-n)\Big\} + \frac{(j-m)\pi}{b}\Big\{F(i,j,k,l,-m,n) \\ & + F(i,j,k,l,-m,-n) + F(i,j,k,-l,-m,n) \\ & + F(i,j,k,-l,-m,-n)\Big\}\Bigg] \end{aligned} \tag{2-15}$$

and

$$\begin{aligned} H_z(i,j,k,l,m,n) = \frac{1}{2}\left(\frac{1}{\xi_1\Sigma_{s1}} + \frac{1}{\xi_2\Sigma_{s2}}\right)(D_2 - D_1)\Bigg[& \frac{(k+n)\pi}{c}\Big\{F(i,j,k,l,m,n) \\ & + F(i,j,k,l,-m,n) + F(i,j,k,-l,m,n) \\ & + F(i,j,k,-l,-m,n)\Big\} + \Big\{F(i,j,k,l,m,-n) \\ & + F(i,j,k,-l,m,-n) + F(i,j,k,l,-m,-n) \\ & + F(i,j,k,-l,-m,-n)\Big\}\frac{(k-n)\pi}{c}\Bigg] \end{aligned} \tag{2-16}$$

The integrations containing the source term can be performed in the same way as the integrations containing $\frac{D}{\xi \Sigma_s}$. Thus we get

$$S(l,m,n) = 8\left(\frac{S_1}{\xi_1 \Sigma_{s1}} - \frac{S_2}{\xi_2 \Sigma_{s2}}\right) \bar{F}(0,0,0,l,m,n) + \frac{S_2}{\xi_2 \Sigma_{s2}} 8abc\, \delta_{0l}\delta_{om}\delta_{on}$$

$$- \left(\frac{S_1}{\xi_1 \Sigma_{s1}} - \frac{S_2}{\xi_2 \Sigma_{s2}}\right) \frac{4\pi}{3}\rho^3 \delta_{0l}\delta_{om}\delta_{on} \qquad (2\text{-}17)$$

where

$$\bar{F}(0,0,0,l,m,n) = \begin{cases} 0, \text{ when } l = m = n = 0 \\ F(0,0,0,l,m,n), \text{ otherwise} \end{cases} \qquad (2\text{-}18)$$

$I'(i,j,k,l,m,n)$ is exactly same as $I(i,j,k,l,m,n)$, (Equation (2-6)), except that D_1 and D_2 should be replaced by Σ_{a1} and Σ_{a2} respectively.

We can solve the matrix Equation (2-3) and obtain the expansion coefficients $a_{ijk}(u)$. In order to calculate the thermal utilization and disadvantage factor, we may assume that $S_1 = 0$, $\xi\Sigma_s = 1$ and $\lambda = 0$. In other words it is assumed that monoenergetic neutrons are produced uniformly in the moderators only and there is no slowing down. Absorption and diffusion are the processes by which neutrons are removed from the system. The energy of the neutrons is the average thermal energy and the measured valued of material parameters can be used in the calculations of disadvantage factor and thermal utilization.

B. Rectangular Cell with a Fuel Rod at the Center of the Cell

Let us consider a rectangular cell of length 2a along the x-axis and width 2b along the y-axis with a fuel rod of radius ρ at the center

of the cell (Figure 2). The cell is assumed to be infinite along the z-axis. For the steady state condition the age-diffusion equation is written as

$$-D(x,y)\left(\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right)-\frac{\partial D}{\partial x}\frac{\partial\phi}{\partial x}-\frac{\partial D}{\partial y}\frac{\partial\phi}{\partial y}+\Sigma_a(x,y)\phi(x,y,u)$$

$$=-\xi(x,y)\Sigma_s(x,y)\frac{\partial\phi}{\partial u}+S(x,y)\delta(u) \qquad (2\text{-}19)$$

All the quantities are defined as in the previous section.

In order to satisfy the boundary conditions, we expand the flux $\phi(x,y,u)$ in the following way:

$$\phi(x,y,u)=\frac{a_{00}(u)}{4}+\sum_{k=0}^{\infty}{}'\sum_{l=0}^{\infty}{}' a_{kl}(u)\cos\frac{k\pi x}{a}\cos\frac{l\pi y}{b} \qquad (2\text{-}20)$$

The summation $\sum'\sum'$ is over all values of k and l except the pair $l = k = 0$. We insert Equation (2-20) in Equation (2-19), multiply by $\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}$ and divide by $\xi\Sigma_s$. Then upon integrating over the cell, we find after taking Laplace transform (λ = transform variable) the following equations for the expansion coefficients $\bar{a}_{kl}$.

$$\bar{a}_{00}(\lambda)\left[ab\left(\lambda+\frac{\Sigma_{a2}}{\xi_2\Sigma_{s2}}\right)\delta_{om}\delta_{on}+\frac{\pi\rho^2}{4}\left(\frac{\Sigma_{a1}}{\xi_1\Sigma_{s1}}-\frac{\Sigma_{a2}}{\xi_2\Sigma_{s2}}\right)\delta_{om}\delta_{on}\right.$$

$$\left.-\frac{\bar{I}'(0,0,m,n)}{4}\right]+\sum_{k=0}^{\infty}{}'\sum_{l=0}^{\infty}{}'\bar{a}_{kl}(\lambda)\left[\left\{\left(\frac{k\pi}{a}\right)^2+\left(\frac{l\pi}{b}\right)^2\right\}I(k,l,m,n)\right.$$

$$+\left(\frac{k\pi}{a}\right)H_x(k,l,m,n)+\left(\frac{l\pi}{b}\right)H_y(k,l,m,n)+I'(k,l,m,n)$$

$$\left.+ab\lambda\Lambda_{kl}\delta_{mk}\delta_{nl}\right]=S(m,n) \qquad (2\text{-}21)$$

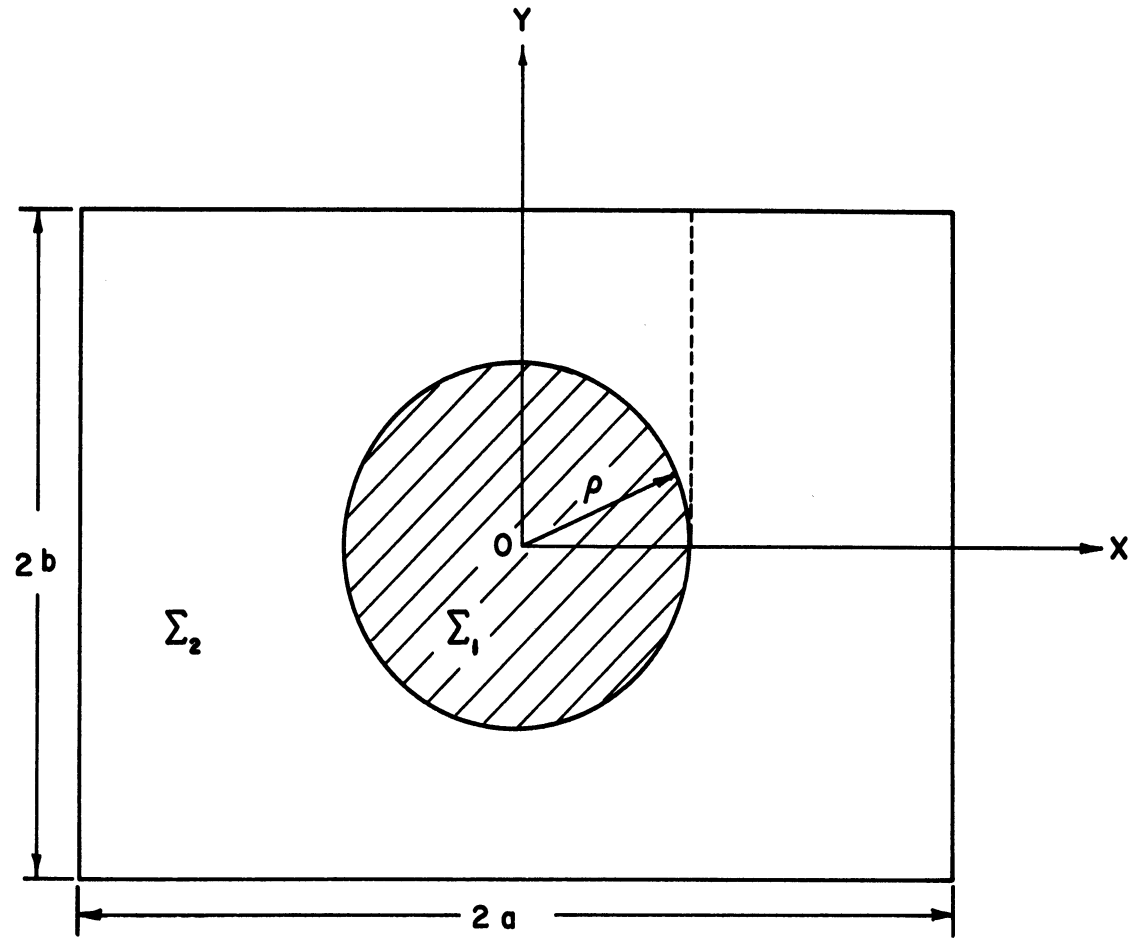

Figure 2. Cross Section of a Rectangular Cell.

where

$$\bar{I}'(0,0,m,n) = \begin{cases} 0, \text{ when } m = n = 0 \\ I'(0,0,m,n), \text{ otherwise} \end{cases} \tag{2-22}$$

$m,n = 0,1,2,3,\ldots$ etc.

$$\Lambda_{oo} = 4$$

$$\Lambda_{ko} = \Lambda_{ol} = 2, \quad \text{for } k,l \neq 0$$

$$\Lambda_{kl} = 1, \quad \text{for } k,l \neq 0 \tag{2-23}$$

and

$$I(k,l,m,n) = \iint_{cell} \frac{D(x,y)}{\xi \Sigma_s} \left\{ \cos \frac{k\pi x}{a} \cos \frac{l\pi y}{b} \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \right\} dx dy$$

$$= 4 \left[\frac{D_1}{\xi_1 \Sigma_{s1}} \int_0^{\rho} dx \int_0^{\sqrt{\rho^2 - x^2}} dy \{ \cdots \} + \frac{D_2}{\xi_2 \Sigma_{s2}} \int_0^{\rho} dx \int_{\sqrt{\rho^2 - x^2}}^{b} dy \{ \cdots \} \right.$$

$$\left. + \frac{D_2}{\xi_2 \Sigma_{s2}} \int_{\rho}^{a} dx \int_0^{b} dy \{ \cdots \} \right] \tag{2-24}$$

$$H_x(k,l,m,n) = \iint_{cell} \frac{1}{\xi \Sigma_s} \frac{\partial D}{\partial x} \sin \frac{k\pi x}{a} \cos \frac{l\pi y}{b}$$

$$\cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} dx dy \tag{2-25}$$

$$H_y(k,l,m,n) = \iint_{cell} \frac{1}{\xi \Sigma_s} \frac{\partial D}{\partial y} \cos \frac{k\pi x}{a} \sin \frac{l\pi y}{b}$$

$$\cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} dx dy \tag{2-26}$$

$$S(m,n) = \iint \frac{S(x,y)}{\xi \Sigma_s} \cos\frac{m\pi x}{a} \cos\frac{n\pi y}{b}\, dx\, dy \qquad (2\text{-}27)$$

After performing the integrals in Equation (2-24) (see Appendix C), we find

$$I(k,l,m,n) = \left(\frac{D_1}{\xi_1 \Sigma_{s1}} - \frac{D_2}{\xi_2 \Sigma_{s2}}\right)\Big[F(k,l,m,n) + F(k,l,-m,n) + F(k,l,m,-n) + F(k,l,-m,-n)\Big] + ab\Lambda_{kl}\delta_{mk}\delta_{nl} \qquad (2\text{-}28)$$

where

$$F(k,l,m,n) = J_1\left(\pi\rho\sqrt{\left(\frac{k+m}{a}\right)^2 + \left(\frac{l+n}{b}\right)^2}\right)\frac{\rho}{2}\left[\left(\frac{k+m}{a}\right)^2 + \left(\frac{l+n}{b}\right)^2\right]^{-1/2} \qquad (2\text{-}29)$$

$J_1(..)$ is a first order Bessel function of the first kind. As before let us note that $\frac{1}{\xi\Sigma_s}\frac{\partial D}{\partial x}$ is a product of a step function and a delta function.

$$\frac{1}{\xi\Sigma_s} = \frac{1}{\xi_1\Sigma_{s1}} + \left(\frac{1}{\xi_2\Sigma_{s2}} - \frac{1}{\xi_1\Sigma_{s1}}\right)\mathcal{S}\left(x - \sqrt{\rho^2 - y^2}\right) \qquad (2\text{-}30)$$

Here $\mathcal{S}$ is the unit step function and

$$\frac{\partial D}{\partial x} = (D_1 - D_2)\,\delta\left(x - \sqrt{\rho^2 - y^2}\right) \qquad (2\text{-}31)$$

Evaluating the integrals in Equation (2-25) we find

$$H_x(k,l,m,n) = \left(\frac{1}{\xi_1\Sigma_{s1}} + \frac{1}{\xi_2\Sigma_{s2}}\right)(D_2 - D_1)\left[\frac{(k+m)\pi}{a}\Big\{F(k,l,m,n) + F(k,l,m,-n)\Big\} + \frac{(k-m)\pi}{a}\Big\{F(k,l,-m,n) + F(k,l,-m,-n)\Big\}\right] \qquad (2\text{-}32)$$

Similarly, we get

$$H_y(k,l,m,n) = \frac{1}{2}\left(\frac{1}{\xi_1\Sigma_{s1}} + \frac{1}{\xi_2\Sigma_{s2}}\right)(D_2 - D_1)\left[\frac{(l+n)\pi}{b}\left\{F(k,l,m,n) + F(k,l,-m,n)\right\} + \frac{(l-n)\pi}{b}\left\{F(k,l,m,-n) + F(k,l,-m,-n)\right\}\right] \tag{2-33}$$

$I'(k,l,m,n)$ has the same expression as $I(k,l,m,n)$, Equation (2-28), except that D_1 and D_2 should be replaced by Σ_{a1} and Σ_{a2} respectively.

$$S(m,n) = 4\left(\frac{S_1}{\xi_1\Sigma_{s1}} - \frac{S_2}{\xi_2\Sigma_{s2}}\right)\bar{F}(0,0,m,n) + \left\{\left(\frac{S_1}{\xi_1\Sigma_{s1}} - \frac{S_2}{\xi_2\Sigma_{s2}}\right)\pi\rho^2 + 4ab\frac{S_2}{\xi_2\Sigma_{s2}}\right\}\delta_{om}\,\delta_{on} \tag{2-34}$$

where

$$\bar{F}(0,0,m,n) = \begin{cases} 0, & \text{when } m = n = 0 \\ F(0,0,m,n), & \text{otherwise} \end{cases} \tag{2-35}$$

Solving the matrix Equation (2-21) and taking inverse Laplace transform, we get the expansion coefficients $a_{kl}(u)$.

C. <u>One Dimensional Slab Cell</u>

Let us consider a slab cell of width 2b with a central fuel lump of width 2a (Figure 3). For the steady state condition we write the age-diffusion equation as

$$-\frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}\phi(x,u)\right) + \Sigma_a(x)\phi(x,u) = -\xi(x)\Sigma_s(x)\frac{\partial\phi}{\partial u} + S(x)\,\delta(u) \tag{2-36}$$

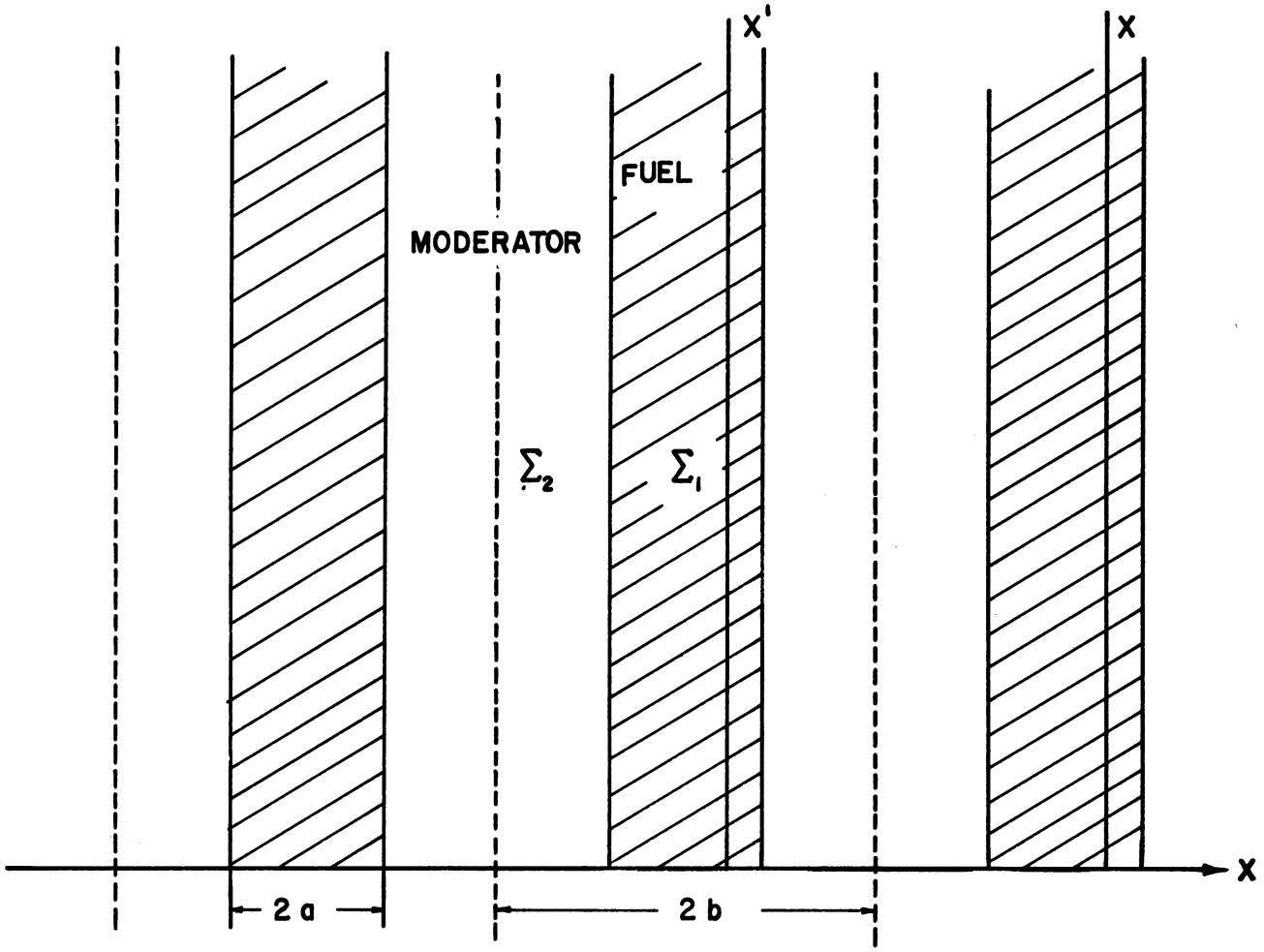

Figure 3. Cross Section of a Slab Lattice.

The one-dimensional case can be treated either by the method used in the two and three dimensional cases, i.e., by integrating over the singularity in $\frac{\partial D}{\partial x}$ or alternately, by measuring the distance in units of diffusion coefficient, D(x). In the present section we adopt the latter approach. Introducing a new variable

$$y = \int_0^x \frac{dx'}{D(x')} \tag{2-37}$$

we write Equation (2-36) in the following form

$$-\frac{3(1-\bar{\mu}_i)}{\xi(y)}\frac{\partial^2}{\partial y^2}\phi(y,u) + \frac{\Sigma_a}{\xi\Sigma_s}\phi(y,u)$$

$$= -\frac{\partial\phi}{\partial u} + \frac{S(y)}{\xi(y)\Sigma_s(y)}\delta(u) \tag{2-38}$$

Here we use the relation

$$D_i = \frac{1}{3\Sigma_{si}(1-\bar{\mu}_i)} \tag{2-39}$$

where $\bar{\mu}$ is the average cosine of the scattering angle per collision in the laboratory coordinate system. The boundaries x = a and x = b correspond to $y = \alpha$ and $y = \beta$. They are given by

$$\alpha = \frac{a}{D_1}$$

$$\beta = \frac{a}{D_1} - \frac{b-a}{D_2} \tag{2-40}$$

As before we expand the flux

$$\phi(y,u) = \frac{a_0(u)}{2} + \sum_{n=1}^{\infty} a_n(u) \cos \frac{n\pi y}{\beta} \qquad (2\text{-}41)$$

We insert Equation (2-41) in Equation (2-38), multiply by $\cos \frac{m\pi y}{\beta}$ and integrate over y from $-\beta$ to $+\beta$. Taking the Laplace transform, we get

$$\bar{a}_0(\lambda)\left[(\xi_1 - \xi_2)\frac{\sin(m\pi\alpha/\beta)}{m\pi} + (\xi_2 + \lambda)\,\delta_{om}\right]$$

$$+\sum_{n=1}^{\infty} \bar{a}_n(\lambda)\left[\left\{\xi_1 - \xi_2 + (w_1 - w_2)\left(\frac{n\pi}{\beta}\right)^2\right\}\frac{\sin(m+n)\pi\alpha/\beta}{(m+n)}\right.$$

$$+\left\{\xi_1 - \xi_2 + (w_1 - w_2)\left(\frac{n\pi}{\beta}\right)^2\right\}\frac{\alpha}{\beta}\,\delta_{mn}$$

$$\left.+\left\{\xi_2 + w_2\left(\frac{n\pi}{\beta}\right)^2\right\}\delta_{mn} + \lambda\,\delta_{mn}\right] = \bar{s}(m) \qquad (2\text{-}42)$$

where

$$m = 0,1,2,3, \text{ etc.}$$

$$g_i = \frac{\Sigma a_i}{\xi_i \Sigma s_i}$$

$$w_i = \frac{3(1-\bar{\mu}_i)}{\xi_i} \qquad (2\text{-}43)$$

and

$$\bar{S}(m) = \frac{2}{\pi}\left(\frac{S_1}{\xi_1 \Sigma_{s1}} - \frac{S_2}{\xi_2 \Sigma_{s2}}\right)\frac{\sin(m\pi\alpha/\beta)}{m} + \frac{2S_2}{\xi_2 \Sigma_{s2}}\delta_{om} \quad (2\text{-}44)$$

For the particular case when there is no absorption during slowing down, $\Sigma_a = 0$ and Equation (2-42) becomes

$$\bar{a}_0(\lambda)\lambda\,\delta_{0m} + \sum \bar{a}_n(\lambda)\left[A_{mn} + \lambda\,\delta_{mn}\right] = \bar{S}(m) \quad (2\text{-}45)$$

where

$$A_{mn} = \pi\left(\frac{\eta}{\beta}\right)^2\left[(w_1 - w_2)\frac{\sin(m+n)\pi\alpha/\beta}{(m+n)} + \frac{\pi\alpha}{\beta}\delta_{mn} + w_2\pi\,\delta_{mn}\right] \quad (2\text{-}46)$$

We can take account of resonance absorption during slowing down by replacing the resonances by delta function sinks. Solving Equation (2-45) for $\bar{a}_n(\lambda)$ and taking the inverse Laplace transform we obtain the expansion coefficients for the case in which there is no non-resonance absorption during slowing down. Since at high energy resonance absorption is the most dominant absorption term during slowing down, the neglect of non-resonance absorption is a good approximation.

CHAPTER III

THERMAL UTILIZATION

In this chapter we obtain expressions for the thermal utilization and disadvantage factors for (A) a rectangular parallelepiped cell, (A') an equivalent spherical cell having the same volume as the rectangular parallelepiped cell, (B) a rectangular cell, and (B') an equivalent cylindrical cell having same cross sectional area as the rectangular cell. For the cases (A) and (B) the fluxes obtained in Chapter II are used in the calculations of the average fluxes. But for the cases (A') and (B') the diffusion equation is solved separately in the fuel and the moderator regions with the boundary conditions that the current and the flux of thermal neutrons are continuous at the fuel-moderator interface. In all cases a flat source of thermal neutrons in the moderator is assumed.

In Section C numerical values of the thermal utilization and disadvantage factors are computed for a rectangular cell and compared with those computed for the equivalent cylindrical cell. The disadvantage factor is defined as the ratio of the average flux in the moderator region to the average flux of thermal neutrons in the fuel region of the cell.

A. Rectangular Parallelepiped Cell

The flux for this cell is given by Equation (2-2)

$$\phi(x,y,z,u) = \frac{a_{000}(u)}{8} + \sum_{i=0}^{\infty,}\sum_{j=0}^{\infty,}\sum_{k=0}^{\infty,} a_{ijk}(u)\cos\frac{i\pi x}{a} \cdot\cos\frac{j\pi y}{b}\cos\frac{k\pi z}{c} \qquad (3\text{-}1)$$

In order to calculate the thermal utilization a_{ijk} are obtained by solving the matrix Equation (2-3) with $\lambda = 0$, $\xi \Sigma_s = 1$ and $S_1 = 0$. When the expansion coefficients, a_{ijk}, are known, the average fluxes in the fuel and the moderator region can be found by integrating over and dividing by the volume of fuel and the moderator respectively. Thus, the average flux, $\bar{\Phi}_1$, in the spherical fuel element of radius ρ is

$$\bar{\Phi}_1 = \frac{a_{000}}{8} + \frac{3}{4\pi\rho^3} \sum_{i=0}^{\infty\,\prime} \sum_{j=0}^{\infty\,\prime} \sum_{k=0}^{\infty\,\prime} a_{ijk} \iiint_{\text{fuel}} \cos\frac{i\pi x}{a} \times \cos\frac{j\pi y}{b} \cos\frac{k\pi z}{c}\, dx\, dy\, dz$$

$$= \frac{a_{000}}{8} + \frac{6}{\pi\rho^3} \sum_{i=0}^{\infty\,\prime} \sum_{j=0}^{\infty\,\prime} \sum_{k=0}^{\infty\,\prime} a_{ijk} F(i,j,k,0,0,0) \qquad (3\text{-}2)$$

Here $F(i,j,k,0,0,0)$ is defined as in Equation (2-8).

The average flux, $\bar{\Phi}_2$, in the moderator is

$$\bar{\Phi}_2 = \frac{a_{000}}{8} + \frac{1}{\left(8abc - \frac{4\pi}{3}\rho^3\right)} \sum_{i=0}^{\infty\,\prime} \sum_{j=0}^{\infty\,\prime} \sum_{k=0}^{\infty\,\prime} a_{ijk} \iiint_{\text{moderator}} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \cos\frac{k\pi z}{c}\, dx\, dy\, dz$$

$$= \frac{a_{000}}{8} - \frac{1}{\left(abc - \frac{\pi}{6}\rho^3\right)} \sum_{i=0}^{\infty\,\prime} \sum_{j=0}^{\infty\,\prime} \sum_{k=0}^{\infty\,\prime} a_{ijk} F(i,j,k,0,0,0) \qquad (3\text{-}3)$$

The average flux on the surface of the fuel element is

$$\Phi_s = \frac{a_{000}}{8} + \frac{1}{4\pi\rho^2}\sum_{i=0}^{\infty}{}'\sum_{j=0}^{\infty}{}'\sum_{k=0}^{\infty}{}' a_{ijk} \iint_{\text{surface}} \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \cos\frac{k\pi z}{c}\, ds$$

$$= \frac{a_{000}}{8} + \sum_{i=0}^{\infty}{}'\sum_{j=0}^{\infty}{}'\sum_{k=0}^{\infty}{}' a_{ijk} J_{1/2}\left(\pi\rho\sqrt{\left(\frac{i}{a}\right)^2+\left(\frac{j}{b}\right)^2+\left(\frac{k}{c}\right)^2}\right) \times\left(\frac{1}{2\rho}\right)^{1/2}\left[\left(\frac{i}{a}\right)^2+\left(\frac{j}{b}\right)^2+\left(\frac{k}{c}\right)^2\right]^{-1/4} \tag{3-4}$$

The above integral can be evaluated easily.

The thermal utilization, f, is the ratio of thermal neutrons absorbed in the uranium to the total number absorbed in the cell. Let us consider that the fuel element contains natural uranium only. The thermal utilization is usually expressed by the following relation

$$\frac{1}{f} - 1 = \frac{V_2 \Sigma_{a2}}{V_1 \Sigma_{a1}} F + E - 1 \tag{3-5}$$

V_1 and V_2 are the volumes occupied by the fuel and the moderator in the cell. F is the ratio of the average flux on the surface of the fuel sphere to the average flux in the fuel element, i.e., $F = \bar{\Phi}_s / \bar{\Phi}_1$. E is characteristic of flux distribution in the moderator, and is given by

$$E - 1 = \frac{\Sigma_{a2} V_2 \bar{\Phi}_2 - \Sigma_{a2} V_2 \Phi_s}{V_1 \bar{\Phi}_1 \Sigma_{a1}} \tag{3-6}$$

The absorption in the fuel, $V_1 \Sigma_{a1} \bar{\Phi}_1$, is equal to the net current into the spherical fuel element. (E-1) is indeed that part of the absorption in the moderator which is due to the excess of the neutron flux in the moderator over the value of the flux at the fuel-moderator interface, divided by the absorption in the fuel. This is sometimes called excess absorption.

The disadvantage factor, d, is defined as

$$d = \bar{\Phi}_2 / \bar{\Phi} \tag{3-7}$$

The thermal utilization, f, can be expressed in terms of the disadvantage factor, d.

$$\frac{1}{f} = 1 + \frac{\Sigma_{a2} V_2}{\Sigma_{a1} V_1} d \tag{3-8}$$

The thermal utilization depends upon the atomic ratio and the volume ratio of the fuel and moderator. It also depends upon the size and shape of the fuel and moderator separately and is introduced by the presence of disadvantage factor, d, in Equation (3-8)

A'. Equivalent Spherical Cell

The rectangular parallelepiped cell is approximated by an equivalent spherical cell having equal volume. The normal derivative of flux is assumed to vanish on the surface of the cell. This assumption, which underlies the method for computing the cohesive energy in a crystal lattice,[12] is introduced to simplify the mathematics. The radius, ρ_1 , of the equivalent cell is

$$\rho_1 = \left(\frac{6abc}{\pi}\right)^{1/3} \tag{3-9}$$

In order to find the flux in the cell, we must solve the following diffusion equations:

$$D_1 \nabla^2 \phi_1(r) - \Sigma_{a1} \phi_1(r) = 0 \qquad \text{(in fuel)} \qquad (3\text{-}10a)$$

$$D_2 \nabla^2 \phi_2(r) - \Sigma_{a2} \phi_2(r) + S_2 = 0 \qquad \text{(in moderator)} \qquad (3\text{-}10b)$$

The boundary conditions are

$$\phi_1(\rho) = \phi_2(\rho) \qquad (3\text{-}11)$$

$$D_1 \left.\frac{d\phi_1}{dr}\right|_{r=\rho} = D_2 \left.\frac{d\phi_2}{dr}\right|_{r=\rho} \qquad (3\text{-}12)$$

Solving Equations (3-10a) and (3-10b) with above boundary conditions, the average fluxes in the fuel and the moderator can be found easily. Thus, the expressions for E and F can be obtained[2] in the following form:

$$F = \frac{\rho^2 \chi_1^2 \tanh \chi_1 \rho}{3(\chi_1 \rho - \tanh \chi_1 \rho)} \qquad (3\text{-}12)$$

$$E = \frac{\chi_2^2(\rho_1^2 - \rho^2)}{3\rho}\left[\frac{1 - \chi_2 \rho_1 \coth \chi_2(\rho_1 - \rho)}{1 - \chi_2^2 \rho_1 \rho - \chi_2(\rho_1 - \rho)\coth \chi_2(\rho_1 - \rho)}\right] \qquad (3\text{-}13)$$

where

$$\chi_1^2 = \frac{\Sigma_{a1}}{D_1} \quad \text{and} \quad \chi_2^2 = \frac{\Sigma_{a2}}{D_2} \qquad (3\text{-}14)$$

B. Rectangular Cell

For the rectangular cell the flux has been shown in Chapter II to be

$$\phi(x,y) = \frac{a_{00}}{4} + \sum{}' \sum{}' a_{kl} \cos\frac{k\pi x}{a} \cos\frac{l\pi y}{b} \qquad (3\text{-}15)$$

a_{kl} are obtained by solving the matrix Equation (2-17) with $\lambda = 0$, $\xi\Sigma_s = 1$ and $S_1 = 0$. Thus, the average flux in the fuel rod is given by

$$\bar{\phi}_1 = \frac{a_{00}}{4} + \frac{1}{\pi\rho^2} \sum_{k=0}^{\infty}{}' \sum_{l=0}^{\infty}{}' a_{kl} \iint_{fuel} \cos\frac{k\pi x}{a} \cdot \cos\frac{l\pi y}{b}\, dx\, dy$$

$$= \frac{a_{00}}{4} + \frac{4}{\pi\rho^2} \sum_{k=0}^{\infty}{}' \sum_{l=0}^{\infty}{}' a_{kl} F(k,l,0,0) \qquad (3\text{-}16)$$

F(k,l,0,0) is defined as in Equation (2-22). The average flux in the moderator is

$$\bar{\phi}_2 = \frac{a_{00}}{4} + \frac{1}{(4ab - \pi\rho^2)} \sum_{k=0}^{\infty}{}' \sum_{l=0}^{\infty}{}' a_{kl} \iint_{mod} \cos\frac{k\pi x}{a} \cos\frac{l\pi y}{b}\, dx\, dy$$

$$= \frac{a_{00}}{4} - \frac{4}{(4ab - \pi\rho^2)} \sum_{k=0}^{\infty}{}' \sum_{l=0}^{\infty}{}' a_{kl} F(k,l,0,0) \qquad (3\text{-}17)$$

The average flux, $\bar{\phi}_s$, on the surface of the fuel rod is

$$\bar{\phi}_s = \frac{a_{00}}{4} + \frac{1}{2\pi\rho}\sum_{k=0}^{\infty}{}'\sum_{l=0}^{\infty}{}' a_{kl} \int_{\text{circle}} \cos\frac{k\pi x}{a} \cos\frac{l\pi y}{b}\, ds$$

$$= \frac{a_{00}}{4} + \sum_{k=0}^{\infty}{}'\sum_{l=0}^{\infty}{}' a_{kl} J_0\left(\pi\rho\sqrt{\left(\frac{k}{a}\right)^2 + \left(\frac{l}{b}\right)^2}\right) \qquad (3\text{-}18)$$

The above integral can be evaluated easily following the method as shown in Appendix C.

B'. Equivalent Cylindrical Cell

In this case the rectangular cell is approximated by an equivalent cylindrical cell having the same cross-sectional area. The diffusion Equations (3-10a) and (3-10b) are solved with flux and current continuity condition at the fuel-moderator interface. It is also assumed that the normal derivative of flux vanishes at the surface of the cell. The radius, ρ_1, of the equivalent cylindrical cell is

$$\rho_1 = \left(\frac{4ab}{\pi}\right)^{1/2} \qquad (3\text{-}19)$$

Solving Equations (3-10a) and (3-10b) in cylindrical coordinate system it is found[37] that the flux, $\phi_1(r)$, in the fuel rod is

$$\phi_1(r) = A\, I_0(\chi r) \qquad (3\text{-}20)$$

and the flux, $\phi_2(r)$, in the moderator is

$$\Phi_2(r) = C\left[I_0(\chi_2 r) K_1(\chi_2 \rho_1) + K_0(\chi_2 r) I_1(\chi_2 \rho_1) \right] + \frac{S_2}{\Sigma_{a2}} \quad (3\text{-}21)$$

I_0, I_1, K_0 and K_1 are zero and first order, modified Bessel functions of the first and second kind respectively.[14] The constants A and C may be evaluated by means of the boundary conditions.

$$A I_0(\chi_1 \rho) = C\left[I_0(\chi_1 \rho) K_1(\chi_2 \rho_1) + K_0(\chi_2 \rho) I_1(\chi_2 \rho_1) \right] + \frac{S_2}{\Sigma_{a2}} \quad (3\text{-}22)$$

and

$$A D_1 \chi_1 I_1(\chi_1 \rho) = C D_2 \chi_2 \left[I_1(\chi_2 \rho) K_1(\chi_2 \rho_1) - K_1(\chi_2 \rho) I_1(\chi_2 \rho_1) \right] \quad (3\text{-}23)$$

The thermal utilization, f, in the equivalent cylindrical cell is given by

$$\frac{1}{f} - 1 = \frac{V_2 \Sigma_{a2}}{V_1 \Sigma_{a1}} F + E - 1 \quad (3\text{-}5)$$

where

$$F = \frac{\chi_1 \rho}{2} \frac{I_0(\chi_1 \rho)}{I_1(\chi_1 \rho)} \quad (3\text{-}24)$$

and

$$E = \frac{\chi_2(\rho_1^2 - \rho^2)}{2\rho} \left[\frac{I_0(\chi_2 \rho) K_1(\chi_2 \rho_1) + K_0(\chi_2 \rho) I_1(\chi_2 \rho_1)}{I_1(\chi_2 \rho_1) K_1(\chi_2 \rho) - K_1(\chi_2 \rho_1) I_1(\chi_2 \rho)} \right] \quad (3\text{-}25)$$

χ_1 and χ_2 are defined as in Equation (3-14).

C. Comparison between a Rectangular and an Equivalent Cylindrical Cell

A computer program (Appendix F) has been written for IBM-7090 using MAD[42] language to solve Equation (2-21) for a_{kl} and to compute the average fluxes in the fuel and moderator, the disadvantage factor and thermal utilization. Let us note that Equation (2-21) is an infinity by infinity matrix. Tables III-1 and III-2 show the average fluxes, disadvantage factors, and thermal utilization of two different cells calculated keeping different terms in the expansion of the flux. From the tables we find that the expansion coefficients converge rapidly. We get an excellent approximation for disadvantage factor and thermal utilization if we keep the expansion coefficients for $k, l = 0,1,2,3,4$.

In Tables III-3, 4 and 5 we have calculated the disadvantage factors and the thermal utilizations for three different cells varying the radius of the fuel rod and compared with those calculated with the equivalent cylindrical cell approximation. The computer program written for the equivalent cylindrical cell is given in Appendix F. From these tables we observe that the equivalent cylindrical cell approximation is accurate except for very tight lattices. The thermal cross sections and the diffusion coefficients used in these calculations has been taken from ANL-5800.

TABLE III-1

THERMAL UTILIZATION OF A NATURAL URANIUM-GRAPHITE CELL FOR VARIOUS FLUX APPROXIMATIONS

a = 2 cm, b = 2.5 cm, ρ = 1 cm

k	ℓ	Average flux* in the fuel $\bar{\phi}_1$	Average flux* in the moderator $\bar{\phi}_2$	$\bar{\phi}_2/\bar{\phi}_1$	Thermal Utilization f
1	1	14.039043	16.524909	1.177068	0.995510
2	2	14.060333	16.703461	1.187985	0.995469
3	3	14.075512	16.736275	1.189035	0.995465
4	4	14.090440	16.762192	1.189615	0.995463
5	5	14.099829	16.774762	1.189714	0.995462
6	6	14.105524	16.782078	1.189752	0.995462
7	7	14.111894	16.789801	1.189762	0.995462

* Arbitrary unit

TABLE III-2

THERMAL UTILIZATION OF A NATURAL URANIUM-HEAVY WATER CELL FOR VARIOUS FLUX APPROXIMATIONS

a = 2 cm, b = 2 cm, ρ = 1 cm

k	ℓ	Average flux* in the fuel $\bar{\phi}_1$	Average flux* in the moderator $\bar{\phi}_2$	$\bar{\phi}_2/\bar{\phi}_1$	Thermal Utilization f
1	1	9.936120	12.057761	1.213528	0.999550
2	2	9.893984	12.055554	1.218473	0.999548
3	3	9.856714	12.039424	1.221444	0.999547
4	4	9.849240	12.034834	1.221905	0.999547
5	5	9.840412	12.028025	1.222309	0.999547
6	6	9.837569	12.025747	1.222431	0.999547
7	7	9.833622	12.022149	1.222556	0.999547

* Arbitrary unit

TABLE III-3

COMPARISON OF A RECTANGULAR NATURAL URANIUM-GRAPHITE CELL WITH AN EQUIVALENT CYLINDRICAL CELL

$a = 2$ cm, $b = 2.5$ cm, $k = \ell = 5$

ρ (cm)	$(\bar{\Phi}_2/\bar{\Phi}_1)_{RC}$	$(\bar{\Phi}_2/\bar{\Phi}_1)_{ECC}$	f_{RC}	f_{ECC}
1.8	1.435236	1.300985	0.999012	0.998699
1.6	1.353071	1.273241	0.998566	0.998568
1.4	1.292296	1.242156	0.997930	0.998404
1.2	1.239316	1.208218	0.996982	0.998189
1.0	1.189714	1.171716	0.995462	0.997893
0.8	1.142149	1.133084	0.992756	0.997454
0.6	1.096617	1.093305	0.987104	0.996727
0.4	1.054568	1.054131	0.971620	0.995273

RC = Rectangular cell

ECC = Equivalent cylindrical cell

TABLE III-4

COMPARISON OF A RECTANGULAR NATURAL URANIUM-HEAVY WATER CELL WITH AN EQUIVALENT CYLINDRICAL CELL

$a = 2$ cm, $b = 2.5$ cm, $k = \ell = 5$

ρ (cm)	$(\bar{\Phi}_2/\bar{\Phi}_1)_{RC}$	$(\bar{\Phi}_2/\bar{\Phi}_1)_{ECC}$	f_{RC}	f_{ECC}
1.8	1.595902	1.325893	0.999861	0.999832
1.6	1.484790	1.303616	0.999800	0.999814
1.4	1.401965	1.272824	0.999714	0.999792
1.2	1.328663	1.238422	0.999588	0.999764
1.0	1.259559	1.199773	0.999388	0.999726
0.8	1.193202	1.157289	0.999033	0.999669
0.6	1.130091	1.111750	0.998294	0.999576
0.4	1.072665	1.065726	0.996243	0.999391

TABLE III-5

COMPARISON OF A SQUARE NATURAL URANIUM-HEAVY WATER CELL WITH AN EQUIVALENT CYLINDRICAL CELL

ρ (cm)	$(\bar{\Phi}_2/\bar{\Phi}_1)_{RC}$	$(\bar{\Phi}_2/\bar{\Phi}_1)_{ECC}$	f_{RC}	f_{ECC}
0.8	1.085149	1.070877	0.999903	0.999863
0.6	1.061513	1.053869	0.999756	0.999820
0.4	1.039283	1.035871	0.999345	0.999735
0.2	1.016171	1.015347	0.997169	0.999481

CHAPTER IV

COLLISION PROBABILITIES IN A SLAB LATTICE

The resonance integral, I, is defined as

$$I = \frac{1}{\phi_{10}} \int_{Res} \phi_1(u)\sigma_a(u)du \tag{4-1}$$

where ϕ_{10} is the flux in the fuel element which would exist at the resonance energy in the absence of the resonance. $\phi_1(u)$ is the average flux in the fuel element in the resonance region. The lethargy integral in Equation (4-1) is over the resonance interval. In order to get an expression for $\phi_1(u)$ in the resonance we write the integral equations for $\phi(u)$ in two media, following Chernick,[43] in the form given below:

$$\sum_i V_1\phi_1(u)\Sigma_i(u) = \sum_i P_1(u) \int_{u-\Delta_i}^{u} \frac{V_1\Sigma_{si}\phi_1(u')e^{u'-u}}{1-\alpha_i}\, du' + \sum_j (1-P_2(u)) \int_{u-\Delta_j}^{u} \frac{V_2\Sigma_{sj}\phi_2(u')e^{u'-u}}{1-\alpha_j}\, du' \tag{4-2a}$$

$$\sum_j V_2E_j(u)\phi_2(u) = \sum_j P_2(u) \int_{u-\Delta_j}^{u} \frac{V_2\Sigma_{sj}\phi_2(u')e^{u'-u}}{1-\alpha_j}\, du' + \sum_i (1-P_1(u)) \int_{u-\Delta_i}^{u} \frac{V_1\Sigma_{si}\phi_1(u')e^{u'-u}}{1-\alpha_i}\, du' \tag{4-2b}$$

The summations $\sum_i$ and $\sum_j$ are over the atoms in the fuel and moderator respectively. Σ_i and Σ_{si} are the total and scattering macroscopic cross sections respectively of the i-th atom. $\phi_1(u)$ and $\phi_2(u)$ are the fluxes in the fuel and moderator respectively. The maximum fractional energy loss of a neutron in an elastic collision with an atom of mass, A_i, is $(1 - \alpha_i)$, where

$$\alpha_i = \left(\frac{A_i - 1}{A_i + 1}\right)^2 \tag{4-3}$$

$$\Delta_i = \ell n(1/\alpha_i) \tag{4-4}$$

The collision probabilities $P_1(u)$ and $P_2(u)$, which enable us to relate the collision densities $\Sigma_1(u)\phi_1(u)$ and $\Sigma_2(u)\phi_2(u)$ to the previous collisions at higher energy, i.e., at lower lethargy, are defined for the lattice in the following way:

$P_k(u)$ is the probability that the neutrons of lethargy u originating in region k (k = 1 for fuel, k = 2 for moderator) will make their first-collision in the same region k (probably after traversing other regions). The integrals on the right hand sides of Equations (4-1a) and (4-2b) can be evaluated easily in the NR and IM approximations. The details of the calculations of the resonance integrals will be shown in Chapter V. In this chapter we derive expressions for the first-collision probabilities.

Where two resonances are close together the flat-source collision probability is inapplicable in the calculation of second (lower energy) resonance. In our calculations we replace the first (higher energy) resonance by a delta function sink and solve the age-diffusion equation

to obtain the energy and spatial dependent flux above the second resonance. In Section A we obtain the spatial and energy dependent flux. In Section B we derive expressions for the first-collision probabilities using the flux obtained in Section A as the source distribution of neutrons.

A. Solution of Age-Diffusion Equation with Delta Function Source and Sink of Neutrons

The age-diffusion equation is exactly the same as Equation (2-29) with $\Sigma_a = 0$ and a modified source term. A delta function negative source, S_o, at lethargy $u = u_o$ is added to the source term.

$$-\frac{\partial}{\partial x}\left(D(x)\frac{\partial}{\partial x}\phi(x,u)\right) = -\xi(x)\Sigma_s(x)\frac{\partial\phi}{\partial u} + S(x)\delta(u) - S_o(x)\delta(u-u_o) \qquad (4\text{-}5)$$

As in Section C, Chapter II, the flux is expanded in the following way

$$\phi(y,u) = \frac{a_o(u)}{2} + \sum_{n=1}^{\infty} a_n(u)\cos\frac{n\pi y}{\beta} \qquad (4\text{-}6)$$

All the terms have been defined in Chapter II. The matrix equation for the expansion coefficients becomes

$$\bar{a}_o(\lambda)\lambda\,\delta_{om} + \sum_{n=1}^{\infty}\bar{a}_n(\lambda)\left[A_{mn} + \lambda\delta_{mn}\right] = \bar{S}(m) - \bar{S}_o(m)e^{-\lambda u_o} \qquad (4\text{-}7)$$

Equation (4-7) is exactly the same as Equation (2-38) except for the modified source term.

In order to find the expansion coefficients, $a_n(u)$, we have to solve Equation (4-7) and take the inverse Laplace transform. Let us

note that the roots of the following determinant, Equation (4-8), multiplied by u or $(u-u_o)$ will appear in the expansion coefficients as negative exponent.

$$\begin{vmatrix} \lambda & A_{01} & A_{02} & \cdots & A_{0n} & \cdots \\ 0 & A_{11}+\lambda & A_{12} & \cdots & A_{1n} & \cdots \\ 0 & A_{21} & A_{22}+\lambda & \cdots & A_{2n} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & A_{m1} & A_{m2} & A_{mn}+\lambda \cdots & A_{mn} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & A_{n1} & A_{n2} & \cdots & A_{mn}+\lambda & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0 \qquad (4\text{-}8)$$

Equation (4-8) is an infinity by infinity determinant. Taking different values of $m=n$ we have solved Equation (4-8) for λ. Table IV-1 shows that the roots converge rapidly and the smallest root may be approximated by A_{11}, i.e., by the solution of a 2 x 2 determinant. The actual value of the smallest root, λ_1, is about three percent less than A_{11}. Let us also note that the next higher root, λ_2, is about five times as large as λ_1. Since λ_i appear in the expansion coefficients as negative exponent, we can neglect all the terms involving λ_i except for $i = 1$.

Solving Equation (4-7) for $\bar{a}_n(\lambda)$ and then taking the inverse Laplace transform we obtain the expansion coefficients $a_n(u)$ for different approximation. Solving a $(n+1)$ x $(n+1)$ matrix equation we find the expansion coefficients $a_i(u)$, where $i \leq n$, have the following form

TABLE IV-1

ROOTS OF DETERMINANTS OF SEVERAL NATURAL URANIUM-GRAPHITE SLAB LATTICES

Half-width of fuel = a cm
Half-width of Moderator = (b-a) cm

σ_p^U = 10.75 barns σ_p^C = 4.5 barns

a	b	λ_1	λ_2	λ_3	λ_4
.5	1.0	1126.798			
		1114.54	5905.7		
		1093.19	5905.1	10999.9	
		1092.75	5865.9	10954.9	21876.0
1.0	2	281.7			
		278.6	1476.4		
		273.3	1476.2	2749.9	
		273.2	1466.4	2738.7	5469.0
1.5	3	125.2			
		123.8	656.2		
		121.5	656.1	1222.2	
		121.4	651.7	1217.2	2430.7
2.0	4	70.4			
		69.6	369.1		
		68.3	369.0	687.5	
		68.29	366.6	684.7	1367.2
2.5	6	33.1			
		31.7	143.5		
		31.6	140.7	301.5	
		31.2	140.6	298.1	571.8

$$a_i(u) = C_{n1}\, e^{-\lambda_1(u-u_o)} + C_{n2}\, e^{-\lambda_2(u-u_o)} + \ldots + C_{nn}\, e^{-\lambda_n(u-u_o)}$$

$$\simeq C_{n1}\, e^{-\lambda_1(u-u_o)}$$

where $C_{n1},\ldots,C_{nn}$ are constants. We found in the one-speed case, i.e., the thermal energy group, that the first two terms in the expansion of the flux are correct to about two percent. The expansion coefficients for the higher terms are very small. We, therefore, keep only the first two terms in the expansion of the flux $\phi(y,u)$, Equation (4-6), and write

$$\phi(y,u) = \frac{a_o(u)}{2} + a_1(u)\, \cos \frac{\pi y}{\beta} \tag{4-9}$$

where

$$y = \begin{cases} x/D_1 & , \quad \text{for} \quad -a < x < a \\ a/D_1 + x-a/D_2, & \quad \text{for} \quad a < x < b \end{cases} \tag{4-10}$$

The expansion coefficients are

$$a_o(u) = S(0) - \frac{A_{01}S(1)}{A_{11}}\, (1-e^{-\lambda_1 u}) - \mathcal{S}(u-u_o) \left\{ S_o(0) - \frac{A_{01}S_o(1)}{A_{11}} \left(1-e^{-\lambda_1(u-u_o)}\right) \right\} \tag{4-11}$$

and

$$a_1(u) = S(1)\, e^{-\lambda_1 u} - \mathcal{S}(u-u_o)\, S_o(1)\, e^{-\lambda_1(u-u_o)} \tag{4-12}$$

where

$$\mathcal{J}(u-u_o) = \begin{cases} 1 & \text{for} \quad u \geq u_o \\ 0 & \text{for} \quad u \leq u_o \end{cases} \tag{4-13}$$

is a unit step function.

The terms containing $e^{-\lambda_1 u}$ can be neglected. The terms containing $e^{-\lambda_1(u-u_o)}$ will be important only when $\lambda_1(u-u_o) \lesssim 1$. For widely separated resonances $\lambda_1(u-u_o)$ at the second resonance would be large and consequently $e^{-\lambda_1(u-u_o)} \to 0$, i.e., $a_1 \to 0$ and as expected flux would be constant in lethargy and in space. Therefore, we find that the flux above the second resonance becomes spatially and lethargy dependent due to the presence of a close first (higher energy) resonance only.

B. Calculation of First Collision Probabilities

The first collision probabilities are usually calculated assuming a flat source distribution as given by Case et al.[29] We derive expressions for the first collision probabilities for source distributions represented by the spatially dependent flux, $\phi(x,u)$, shown in Equation (4-5). The single collision kernel at x' due to a plane source of one neutron per cm^2 per sec located at x' is given by[1]

$$\frac{1}{2} E_1(\Sigma|x-x'|) \tag{4-14}$$

where $E_n(x)$ is defined as[44]

$$E_n(x) = \int_1^\infty e^{-xt} t^n dt$$

$$= \int_0^1 e^{-x/\mu} \mu^{n-2} d\mu \tag{4-15}$$

for positive values of $x \cdot \Sigma$ is the total macroscopic cross section of the medium. For different media we have to take the optical distance between the plane source and the point under consideration. Therefore, the probability, $P_1(u)$, that the neutrons of lethargy u originating in the lump $(-a < x' < a)$ of region 1 will make their first collision anywhere in the same region 1 of the lattice is given by (see Figure 3)

$$
\begin{aligned}
P_1(u) = {} & \frac{\Sigma_1}{2} \int_{-a}^{a} dx' \phi(x') \int_{-a}^{a} E_1(\Sigma_1 |x-x'|) dx \Big/ \int_{-a}^{a} \phi(x') dx' \\
& + \sum_{n=1}^{\infty} \frac{\Sigma_1}{2} \int_{-a}^{a} dx' \phi(x') \int_{2nb-a}^{2nb+a} E_1 \Big[n2(b-a)(\Sigma_2-\Sigma_1) \\
& + \Sigma_1(x-x') \Big] dx \Big/ \int_{-a}^{a} \phi(x') dx' \\
& + \sum_{n=1}^{\infty} \frac{\Sigma_1}{2} \int_{-a}^{a} dx' \phi(x') \int_{-2nb-a}^{-2nb+a} dx \; E_1 \Big[n2(b-a)(\Sigma_2-\Sigma_1) \\
& + \Sigma_1(x'-x) \Big] \Big/ \int_{-a}^{a} \phi(x') dx
\end{aligned}
\tag{4-16}
$$

The above integrations have been evaluated in Appendix D. Finally for $P_1(u)$ we get

$$
P_1(u) = 1 - \frac{a_o(u)}{2d_1 \Sigma_1 \bar{\phi}_1} \left[0.5 - \sum_{n=0}^{\infty} \{ E_3(\ell_1(n)) - 2E_3(\ell_2(n)) + E_3(\ell_3(n)) \} \right]
$$

$$- \frac{a_1(u)}{d_1\bar{\phi}_1}\left[\frac{\sin \alpha' a}{\alpha'}\left\{1 - \frac{\Sigma_1}{2}\tan^{-1}\left(\frac{\alpha'}{\Sigma_1}\right)\right\} + \frac{\Sigma_1 \cos \alpha' a}{2\alpha'^2}\log_e\left(1 + \frac{\alpha'^2}{\Sigma_1^2}\right)\right.$$

$$+ \sum_{n=0}^{\infty} \frac{a_1(u)}{d_1\bar{\phi}_1}\left[\Sigma_1 \cos \alpha' a\{\Psi_1(n) - 2\Psi_2(n) + \Psi_3(n)\}\right.$$

$$\left.\left. + \alpha' \sin \alpha' a\{\Psi_1'(n) - \Psi_3'(n)\}\right]\right. \tag{4-17}$$

where

$$d_1 = 2a$$

$$d_2 = 2(b-a) \tag{4-18}$$

$$\Psi_i(n) = \int_0^1 \frac{e^{-\ell_i(n)/t} t\,dt}{\Sigma_2^2 + \alpha'^2 t^2}, \qquad i = 1,2,3 \tag{4-19}$$

$$\Psi_i'(n) = \int_0^1 \frac{e^{-\ell_i(n)/t} t^2\,dt}{\Sigma_2^2 + \alpha'^2 t^2} \tag{4-20}$$

$$\alpha' = \frac{\pi D_2}{D_2 a + D_1(b-a)} \tag{4-21}$$

$$\bar{\phi}_1 = \frac{1}{2a}\int_{-a}^{a} \phi(x')\,dx'$$

and

$$\begin{aligned} \ell_1(n) &= (n+1)d_2\Sigma_2 + nd_1\Sigma_1 \\ \ell_2(n) &= (n+1)(d_2\Sigma_2 + d_1\Sigma_1) \\ \ell_3(n) &= nd_2\Sigma_2 + (n+1)d_1\Sigma_1 \end{aligned} \tag{4-22}$$

Similarly, we find the probability, $P_2(u)$, that the neutrons of lethargy u originating in the moderator lump $(a < x' < 2b-a)$ of region 2 will make their first collision in the same region 2 anywhere in the lattice.

$$\begin{aligned} P_2(u) = {} & \frac{\Sigma_2}{2} \int_a^{2b-a} dx'\phi(x') \int_a^{2b-a} E_1\left[\Sigma_2|x-x'|\right]dx \Big/ \int_a^{2b-a} \phi(x')dx' \\ & + \sum_{n=1}^{\infty} \frac{\Sigma_2}{2} \int_a^{2b-a} dx'\phi(x') \int_{2nb+a}^{2(n+1)b+a} E_1\left[2na(\Sigma_1-\Sigma_2)+(x-x')\Sigma_2\right]dx \Big/ \int_a^{2b-a} \phi(x')dx' \\ & + \sum_{n=1}^{\infty} \frac{\Sigma_2}{2} \int_a^{2b-a} dx'\phi(x') \int_{-2nb+a}^{-2(n-1)b-a} E_1\left[2na(\Sigma_1-\Sigma_2)+(x'-x)\Sigma_2\right]dx \Big/ \int_a^{-2b-a} \phi(x')dx' \end{aligned} \tag{4-23}$$

The above integrals have been evaluated in Appendix E. We find

$$P_2(u) = 1 - \frac{a_0(u)}{2d_2\bar{\phi}_2\Sigma_2}\left[0.5-\sum_{n=0}^{\infty} E_3(\ell_1'(n))-2E_3(\ell_2'(n))+E_3(\ell_3'(n))\right]$$

$$- \frac{a_1(u)\cos\Theta}{2d_2\bar{\phi}_2}\left[\frac{\Sigma_2}{2\beta'^2}\log_e\left(1+\frac{\beta'^2}{\Sigma_2^2}\right)(\cos\beta'(2b-a)+\cos\beta'a)\right.$$

$$\left.+\frac{1}{\beta'}\left(1-\frac{\Sigma_2}{\beta'}\tan^{-1}\frac{\beta'}{\Sigma_2}\right)(\sin\beta'(2b-a)-\sin\beta'a)\right]$$

$$+ \frac{a_1(u)\sin\Theta}{2d_2\bar{\phi}_2}\left[\frac{\Sigma_2}{2\beta'^2}\log_e\left(1+\frac{\beta'^2}{\Sigma_2^2}\right)(\sin\beta'(2b-a)+\sin\beta'a)\right.$$

$$\left.+\frac{1}{\beta'}\left(1-\frac{\Sigma_2}{\beta'}\tan^{-1}\frac{\beta'}{\Sigma_2}\right)(\cos\beta'(2b-a)+\cos\beta'a)\right]$$

$$+\sum_{n=0}^{\infty}\frac{a_1(u)\cos\Theta}{2d_2\bar{\phi}_2}\left[\Sigma_2\{\cos\beta'(2b-a)+\cos\beta'a\}\{\Pi_1(n)-2\Pi_2(n)\right.$$

$$\left.+\Pi_3(n)\}+\beta'\{\sin\beta'(2b-a)-\sin\beta'a\}\{\Pi_1'(n)-\Pi_3'(n)\}\right]$$

$$-\sum_{n=0}^{\infty}\frac{a_1(u)\sin\Theta}{2d_2\bar{\phi}_2}\left[\Sigma_2\{\sin\beta'(2b-a)+\sin\beta'a\}\{\Pi_1(n)-2\Pi_2(n)\Pi_3(n)\}\right.$$

$$\left.-\beta'\{\cos\beta'(2b-a)-\cos\beta'a\}\{\Pi_1'(n)-\Pi_3'(n)\}\right] \qquad (4\text{-}24)$$

where

$$\Theta = \frac{\pi a}{\beta}\left(\frac{1}{D_1}-\frac{1}{D_2}\right) \qquad (4\text{-}25)$$

$$\beta' = \frac{\pi}{\beta D_2} \qquad (4\text{-}26)$$

$$\bar{\phi}_2 = \frac{1}{2(b-a)} \int_a^{2b-a} \phi(x')dx' \tag{4-27}$$

$$II_i(n) = \int_0^1 \frac{e^{-\ell_i'(n)/t} t\,dt}{\Sigma_2^2+\beta'^2 t^2}, \qquad i = 1,2,3 \tag{4-28}$$

$$II_i'(n) = \int_0^1 \frac{e^{-\ell_i(n)/t} t^2 dt}{\Sigma_2^2+\beta'^2 t^2} \tag{4-29}$$

$$\ell_1'(n) = (n+1)d_1\Sigma_1 + nd_2\Sigma_2$$

$$\ell_2'(n) = (n+1)(d_1\Sigma_1 + d_2\Sigma_2)$$

$$\ell_3'(n) = nd_1\Sigma_1 + (n+1)d_2\Sigma_2$$

Let us note that the first two terms of $P_1(u)$ (also $P_2(u)$) are exactly the same as derived by Rotherstein,[19],[20] with a flat source assumption. The remaining terms are due to the presence of the cosine term in the flux, $\phi(x,u)$.

Hence, for the flat source assumption we get

$$P_1(u) \to P_1^o(u) = 1 - \frac{1}{d_2\Sigma_2}\left\{0.5 - \sum_{n=0}^{\infty}\left[E_3(\ell_1(n)-2E_3(\ell_2(n))+E_3(\ell_3(n))\right]\right\} \tag{4-31}$$

and

$$P_2(u) \to P_2^o(u) = 1 - \frac{1}{d_2\Sigma_2}\left\{0.5 - \sum_{n=0}^{\infty}\left[E_3(\ell_1'(n))-2E_3(\ell_2'(n))+E_3(\ell_3'(n))\right]\right\} \tag{4-32}$$

From the reciprocity theorem it can be proved[29] that

$$\Sigma_1 V_1 (1 - P_1^{\circ}(u)) = \Sigma_2 V_2 (1 - P_2^{\circ}(u)) \qquad (4\text{-}33)$$

CHAPTER V

INTERFERENCE BETWEEN RESONANCES IN A SLAB LATTICE

In this chapter the interference between two close resonances is investigated. It is assumed that the flux takes its asymptotic value before reaching the first (higher energy) resonance. The flat source collision probability is used in the calculation of the first resonance integral. The second resonance (lower energy) integral is evaluated in the absence and also in the presence of first resonance. The lethargy and spatially dependent flux, due to the presence of the first resonance, and the corresponding first-collision probabilities are used in the calculations of the second resonance integral.

In Section A, the first resonance integral and the strength of the sink due to resonance absorption are obtained. In Section B, the second resonance integral is calculated in the absence of first resonance and in the presence of a delta function sink due to the higher energy resonance. The NR and IM approximations are made in these calculations. It has been assumed that the Breit-Wigner one-level resonance cross-section formula can be used for all resonances. In Section C, the computational details are discussed and the numerical values of several pair of resonance integrals are shown, Throughout these calculations it has been assumed that the change of temperature is inconsequential in the evaluation of resonance integrals.

A. Number of Neutrons Absorbed in the Fuel Element Per Unit Volume Per Second by the First Resonance

In this case we assume that the flux takes its asymptotic value before reaching the resonance, i.e., it is constant in space and in lethargy within at least one collision interval above the resonance. It is also assumed that there is no absorption during slowing down. The asymptotic flux in a reactor lattice is given[45] by

$$\phi_{asy} = \frac{q(V_1+V_2)}{V_1\xi_1\Sigma_{p1}+V_2\xi_2\Sigma_{p2}} \qquad (5\text{-}1)$$

where q is the slowing down density, ξ_1 and ξ_2 are the average logarithmic energy decrement per collision in the fuel and the moderator region respectively. Σ_{p1} and Σ_{p2} are the macroscopic potential scattering cross sections in the fuel and the moderator respectively.

The average number of neutrons absorbed in the fuel lump per unit volume per second, S_o, is[46]

$$S_o = N_o I \phi_{asy} \qquad (5\text{-}2)$$

Where N_o is the number of absorber nuclei per unit volume of the fuel (region 1), I is the resonance integral defined in Equation (4-2). In this case $\phi_{10} = \phi_{asy}$. In order to evaluate the resonance integral, I, we make the NR or IM approximation and solve the integral Equation (4-2a) to obtain $\phi_1(u)$ at the resonance lethargy. For the higher resonance the escape probabilities given by Equations (4-31), (4-32) and (4-33) can be used.

1. Narrow Resonance Approximation

When the maximum energy loss in a collision with an absorber atom is greater than the practical width of the resonance, the narrow resonance (NR) approximation is applied. Usually the absorber is the heaviest atom in a reactor system. If the NR approximation can be used for the absorber atom, it is expected that the NR approximation is also applicable for the atoms mixed with the fuel element and for the moderator atoms. In other words we can use the asymptotic values of $\phi_1(u')$ and $\phi_2(u')$ inside the integrals in Equation (4-2a) and evaluate the flux, $\phi_1(u)$, at the resonance. The asymptotic value of the flux is given by Equation (5-1), i.e.,

$$\phi_1(u') = \phi_2(u') = \phi_{asy} \tag{5-3}$$

From Equations (4-2a) and (5-3), we find

$$\sum_i V_1\phi_1(u)\Sigma_T^i(u) = \phi_{asy}V_1P_1(u)\sum_i \Sigma_s^i + \phi_{asy}V_2(1-P_2(u))\sum_j \Sigma_s^j \tag{5-4}$$

Let us note that $\sum_i \Sigma_T^i = \Sigma_1(u)$ and $\sum_i \Sigma_s^i = \Sigma_{s1}$ are the macroscopic total and scattering cross sections respectively in the fuel element; and $\sum_j \Sigma_s^j = \Sigma_{s2}$ is the macroscopic scattering cross section in the moderator lump. Let us assume that the fuel element may contain U^{238}, Th^{232} and O^{16}-atoms. From Equations (4-1) and (5-4) the resonance integral for the i-th absorber, I_{NR}^i, is obtained

$$I^i_{NR} = \frac{1}{\phi_{asy}} \int_{Res} \phi_1(u)\,\sigma^i_a(u)\,du$$

$$= (\Sigma^U_s + \Sigma^{Th}_s + \Sigma^O_s) \int_{Res} \frac{\sigma^i_a(u)\,P^O_1(u)\,du}{(\Sigma^U_T+\Sigma^{Th}_T+\Sigma^O_T)} - \int_{Res} \sigma^i_a(u)\,P^O_1(u)\,du$$

$$+ \int_{Res} \sigma^i_a(u)\,du \qquad (5\text{-}5)$$

(Here i = Th or U), the relation, Equation (4-33) between $P^O_1(u)$ and $P^O_2(u)$ has been used to obtain Equation (5-5).

2. Infinite Mass (IM) Approximation

When the practical width is larger than the maximum energy loss in a collision with the absorber atom, the IM approximation is employed. It is assumed that the scattering by the absorber atoms changes only the direction of motion of the neutron but not its energy. Therefore, for this particular case, we assume $\alpha_U, \alpha_{Th} \to 1$, which make the integrals due to the absorbers delta functions. But the NR approximation is used for the moderator and the moderating material present in the fuel element, for example, the oxygen in the uranium oxide. As before, we assume that the fuel element contains U,Th and O . Thus, in the infinite mass approximation Equation (4-2a) becomes

$$V_1\phi_1(u)(\Sigma_T^U + \Sigma_T^{Th} + \Sigma_T^O) = V_1\Sigma_s^U P_1^O(u)\phi_1(u) + \Sigma_s^{Th} V_1 P_1^O(u)\phi_1(u)$$

$$+ V_1\phi_{asy}\Sigma_s^O P_1^O(u) + \Sigma_{s2} V_2 \phi_{asy}(1-P_2^O(u))$$

(5-6)

Assuming that there is no absorption in the moderator and using the reciprocity relation, Equation (4-33), we obtain the resonance integral for the i-th absorber in the following form:

$$I_{IM}^i = \int_{Res} \frac{(\Sigma_T^U + \Sigma_T^{Th} + \Sigma_T^O)\,\sigma_a^i(u)\,du}{(\Sigma_T^U + \Sigma_T^{Th} + \Sigma_T^O) - P_1^O(u)(\Sigma_s^{Th} + \Sigma_s^U)}$$

$$- \int \frac{(\Sigma_T^U + \Sigma_T^{Th})\,\sigma_a^i(u)\,P_1^O(u)\,du}{(\Sigma_T^U + \Sigma_T^{Th} + \Sigma_T^O) - P_1^O(u)(\Sigma_s^U + \Sigma_s^{Th})}$$

(5-7)

here i = U or Th. Let us note that $\Sigma_T^O \approx \Sigma_s^O$, and when i = Th, $\Sigma_T^U = \Sigma_p^U$, the potential scattering cross section of uranium, because we assumed that there is no non-resonance absorption during slowing down. Similary, when i = uranium, $\Sigma_T^{Th} = \Sigma_p^{Th}$. Now, from Equations

(5-1), (5-2), (5-5) and (5-7) we can compute the strength of the sink S_o , due to the higher energy resonance at lethargy u_o .

B. Interference Between Two Closely Spaced Resonances

In this section, we calculate the interference between two close resonances. The absorption rate by the first (higher energy) resonance has been calculated in Section A. In order to calculate the second (lower energy) resonance integral in the presence of the first resonance, we consider the first resonance as a delta function sink. The solution of the age-diffusion equation with this approximation has been obtained in Chapter IV. The average fluxes $\bar{\phi}_1(u)$ and $\bar{\phi}_2(u)$ in the fuel and moderator respectively are used in the calculations of the second resonance integral. The average first-collision probabilities $P_1(u)$ and $P_2(u)$, derived in Chapter IV, due to spatially dependent source distribution are used in these calculations. In the absence of the first resonance integral, the uninterfered second resonance integral will have the same form as shown in Section A. The NR and IM approximations have been used in these calculations.

We write the neutron balance Equation (4-2a) assuming that the fuel element is a homogeneous mixture of U, Th and O atoms and that the moderator may contain one or more light atoms in the following way:

$$V_1\bar{\phi}_1(u)(\Sigma_T^U + \Sigma_T^{Th} + \Sigma_T^O) = V_1\Sigma_s^O P_1(u) \int_{u-\Delta_O}^{u} \bar{\phi}_1(u') \frac{e^{u'-u}}{1-\alpha_O} du'$$

$$+ V_1\Sigma_s^U P_1(u) \int_{u-\Delta_U}^{u} \bar{\phi}_1(u') \frac{e^{u'-u}}{1-\alpha_U} du'$$

$$+ V_1\Sigma_s^{Th} P_1(u) \int_{u-\Delta_{Th}}^{u} \bar{\phi}_1(u') \frac{e^{u'-u}}{1-\alpha_{Th}}$$

$$+ V_2(1 - P_2(u)) \sum_j \Sigma_s^j \int_{u-\Delta_j}^{u} \bar{\phi}_2(u') \frac{e^{u'-u}}{1-\alpha_j} du \qquad (5\text{-}8)$$

The average fluxes $\bar{\phi}_1(u')$ and $\bar{\phi}_2(u')$ above the second (lower energy) resonance is

$$\bar{\phi}_1(u') = \phi_0 + \phi_1' e^{-A_{11}(u'-u_o)}, \quad \text{for} \quad u' \leq u_o \qquad (5\text{-}9a)$$

$$\bar{\phi}_2(u') = \phi_0 + \phi_2' e^{-A_{11}(u'-u_o)}, \quad u' < u_o \qquad (5\text{-}9b)$$

ϕ_o , ϕ_1' , ϕ_2' are defined in Equations (D-2), (D-3), and (E-7) respectively. In order to solve Equation (5-8) we make the usual NR and IM approximations.

1. Narrow Resonance Approximation

In the NR approximations we replace $\tilde{\phi}_1(u')$ and $\tilde{\phi}_2(u')$ in Equation (5-8) by their values in Equations (5-9) and (5-10) when $\Delta_i = \ell n\,(1/\alpha_i)$ is less than or equal to $\Delta u = U_r - U_o$. Here U_r is the lethargy at the second resonance. And when $\ell n(1/\alpha_i) > (U_r - U_o)$, $\bar{\phi}_1 = \bar{\phi}_2 = \phi_{asy}$ in the interval $\ell n(1/\alpha_i) - (U_r - U_o)$ and $\bar{\phi}_1$ and $\bar{\phi}_2$ are given by Equations (5-9) and (5-10) in the interval $(U_r - U_o)$. Let us evaluate a typical integral.

$$R_i(u) = \int_{u-\ell n(1/\alpha_i)}^{u} \bar{\phi}_1(u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du' \qquad (5\text{-}10)$$

For the case $\ell n(1/\alpha_i) < (U_r - U_o)$

$$R_i(u) = \phi_o + \phi'\left[e^{(A_{11}-1)\ell n(1/\alpha_i)} - 1\right]\frac{e^{A_{11}(u-u_o)}}{(1-\alpha_i)(A_{11}-1)} \qquad (5\text{-}11)$$

For the case $\ell n(1/\alpha_i) > (U_r - U_o)$

$$R_i(u) = \int_{u-\ell n(1/\alpha_i)}^{u} \bar{\phi}_1(u')\,\frac{e^{u'-u}}{1-\alpha_i}\,du'$$

$$= \int_{u-\ell n(1/\alpha_i)}^{u_o} \phi_{asy}\,\frac{e^{u'-u}}{1-\alpha_i} + \int_{u_o}^{u}\left[\phi_o + \phi_1' e^{-A_{11}(u'-u_o)}\right]\frac{e^{u'-u}}{1-\alpha_i}\,du'$$

$$= \frac{\phi_{asy}}{1-\alpha_i}\left[e^{-(u-u_o)} - e^{-\ell n(1/\alpha_i)}\right] + \frac{\phi_o}{1-\alpha_i}\left[1 - e^{-(u-u_o)}\right]$$

$$+ \phi_1' \frac{\left[e^{-(u-u_o)} - e^{-A_{11}(u-u_o)}\right]}{(1-\alpha_i)(A_{11}-1)} \qquad (5\text{-}12)$$

Similarly

$$R_j(u) = \phi_o + \phi_2' \left[e^{(A_{11}-1)\ell n(1/\alpha_j)} - 1\right] \frac{e^{-A_{11}(u-u_o)}}{(1-\alpha_j)(A_{11}-1)} \qquad (5\text{-}13)$$

for the case $\ell n(1/\alpha_j) < (U_r - U_o)$

and

$$R_j(u) = \frac{\phi_{asy}}{1-\alpha_j}\left[e^{-(u-u_o)} - e^{-\ell n(1/\alpha_j)}\right] + \frac{\phi_o}{1-\alpha_j}\left[1-e^{-(u-u_o)}\right]$$

$$+ \phi_2' \frac{e^{-(u-u_o)} - e^{-A_{11}(u-u_o)}}{(1-\alpha_j)(A_{11}-1)} \qquad (5\text{-}14)$$

for the case $\ell n(1/\alpha_j) > (U_r - U_o)$.

The changes of $R_i(u)$ and $R_j(u)$ are very small in the resonance region compared to the change of cross-sections. So, we take $R_i(u)$ and $R_j(u)$ out of the integral and put $R_i(u) = R_i(U_r)$ and $R_j(u) = R_j(U_r)$ when the integration is over the resonance. Finally, we obtain the interfered resonance integral in the NR approximation for the i-th absorber as

$$(I^i_{NR})_{int} = \frac{1}{\phi_1(U_r)} \int_{Res} \bar{\phi}_1(u)\sigma^i_a(u)du$$

$$= \frac{1}{\bar{\phi}_2(U_r)} \left[\Sigma^U_s R_U(U_r) + \Sigma^{Th}_s R_{Th}(U_r) + \Sigma^O_s R_O(U_r)\right]$$

$$x \int_{Res} \frac{P_1(u)\sigma^i_a(u)du}{\Sigma^U_T+\Sigma^{Th}_T+\Sigma^O_T} + \frac{1}{\bar{\phi}_1(U_r)} \sum_j \left[\Sigma^j_s R_j(U_r)\right] \frac{V_2}{V_1}$$

$$x \int_{Res} \frac{(1-P_2(u))\sigma^i_a(u)du}{(\Sigma^U_T + \Sigma^{Th}_T + \Sigma^O_T)} \qquad (5\text{-}15)$$

2. Infinite Mass Approximation

Now we set $\alpha_U, \alpha_{Th} \rightarrow 1$ in Equation (5-8) and use the narrow resonance approximation for the other integrals, i.e., we replace $\bar{\phi}_1(u')$ and $\bar{\phi}_2(u')$ by their values in Equations (5-9a) and (5-9b). Finally we get

$$(I^i_{IM})_{int} = \int_{Res} \bar{\phi}_1(u)\sigma^i_a(u)du \Big/ \phi_1(U_r)$$

$$= \frac{\Sigma^O_s}{\bar{\phi}_1(U_r)} R_O(U_r) \int_{Res} \frac{P_1(u)\sigma^i_a(u)du}{(\Sigma^U_T+\Sigma^{Th}_T+\Sigma^O_T) - P_1(u)(\Sigma^{Th}_s+\Sigma^U_s)}$$

$$+ \frac{V_2}{V_1} \sum_j \frac{\Sigma^j_s R_j(U_r)}{\bar{\phi}_1(U_r)} \int \frac{(1-P_2(u))\sigma^i_a(u)du}{(\Sigma^U_T+\Sigma^{Th}_T+\Sigma^O_T) - P_1(u)(\Sigma^{Th}_s+\Sigma^U_s)} \qquad (5\text{-}16)$$

C. Computational Details and Results

The cross sections at the resonance have been calculated using the Breit-Wigner one-level formula. The absorption and the scattering cross sections are

$$\sigma_a(E) = \sigma_o \frac{\Gamma_r}{\Gamma}\sqrt{E_o/E} \; \frac{1}{4(E-E_o)^2/\Gamma^2+1} \tag{5-17}$$

$$\sigma_s(E) = \sigma_o \frac{\Gamma_n}{\Gamma} \frac{1}{4(E-E_o)^2/\Gamma^2+1} + (\sigma_o\sigma_p g \frac{\Gamma_n}{\Gamma})^{1/2} \frac{2(E-E_o)/\Gamma}{4(E-E_o)^2/\Gamma^2+1} + \sigma_p \tag{5-18}$$

The first term in Equation (5-18) represents the resonance scattering, the second term is due to interference between the resonance and potential scattering, and the third term is the potential scattering cross section of the absorber. Γ_r , Γ_n and Γ are the radiation, neutron and the total width at the resonance respectively.

$$\sigma_o = g 4\pi \lambda^2 \frac{\Gamma_n}{\Gamma} \tag{5-19}$$

where λ_o is the wavelength associated with a neutron of energy E_o . Here E_o is the energy at the peak of the resonance. A simple relation between λ_o and E_o is

$$\lambda_o \simeq 4.55 \times 10^{-10} \; E_o^{-1/2} \tag{5-20}$$

where E_o is in electron volts and λ_o is in cm. The statistical weight factor, g, is given by

$$g = \frac{2J+1}{2(2I+1)} \qquad (5\text{-}21)$$

where I and J are the nuclear and channel spins respectively. For resolved resonances in U^{238} and Th^{232} we may take $I = 0$ and $J = \frac{1}{2}$. But in the computer program (Appendix F) provisions have been made for selecting any value of I and J .

The integrations over the resonances have been performed numerically using Simpson's rule.(47) The range of integration has been taken six times the practical width of the resonance. No correction has been made for the absorption in the wings of the resonance. The error due to the finite interval of integration is negligible. It has been shown(21) that even when the range of integration is five-times the practical width, the wing correction is less than a few percent of the main part of the absorption.

In the calculations of collision probabilities the values of the function $E_3(x)$ have been interpolated from the tables given in Reference 44. The integrations in Equations (4-19), (4-20), (4-28) and (4-29) have been evaluated numerically using the Simpson's rule.

The potential scattering cross sections and the resonance parameters used in the calculations are shown in Tables V-1, V-2, and V-3. The practical width, Γ_{prac} , has been calculated using the relation

$$\Gamma_{prac} = \Gamma\sqrt{\sigma_o/\sigma_p} \tag{5-22}$$

where all the terms in the above equation have already been defined.

In the Tables V-4 through V-10 we have shown the resonance integrals of several pairs of resonances for different lattices. The second (lower energy) resonances have two values for the resonance integral, one assuming no absorption of neutrons during slowing down before reaching the resonance and the other assuming a delta function sink above it. In order to find the rate of absorption of neutrons by the lower energy resonance in the absence of the first (higher energy) resonance, we have to multiply the resonance integral by the density of the absorber atoms and also by the lethargy dependent flux, which is a constant and has the same value as the flux above the first (higher energy) resonance when it is present. Therefore, the absorption rate by the second resonance in the absence of the first resonance is

$$\rho_1' = \phi_{10}\, Na(I)_{unint} \tag{5-23}$$

where ϕ_{10} is the asymptotic flux above the first resonance. The absorption by the second resonance in the presence of the first resonance is

$$\rho_2' = \bar{\phi}_2\, Na(I)_{int} \tag{5-24}$$

where $\bar{\phi}_2$ is the average flux which would be present at the second resonance if the second resonance were not there; $(I)_{int}$ is the interfered resonance integral given by Equation (5-15) or Equation (5-16). Let us define

$$\rho' = \frac{\rho_2' - \rho_1'}{\rho_1'} \qquad (5\text{-}25)$$

ρ' is the ratio of capture due to interference to non-interference capture. ρ' has been computed for all pairs of resonances under consideration.

In Table V-4 we have shown the interference between the 116.5 ev and 102.5 ev resonances in several U^{238} - graphite lattices. It is found that the interference is larger in thicker lumps. In Table V-5 we have shown the interference between several selected pairs of resonances of a U^{238} - graphite lattice having 2 cm thick fuel lump and 18 cm thick moderator lump. It is found that the interference depends upon the resonance integral of the higher resonance, which is proportional to the strength of the sink, and also upon the lethargy separation, Δu, between the first and second resonances. In Table V-6 we have shown the interference between several pairs of resonances in a UO_2 - heavy water lattice; the thickness of the fuel being 1 cm and that of the moderator 3 cm. In Tables V-7 and V-8 we have shown the interference between pairs of resonances of a Th^{232} - graphite and a Th^{232} - heavy water lattice respectively. In Tables V-9 and V-10 we have shown the interference between resonances when the absorber lumps contain a homogeneous

mixture of U^{238} and Th^{232} having graphite and heavy water as moderator respecticely. From all the tables mentioned above we notice that the interfered resonance integral is larger than the resonance integral with a flat flux approximation in the graphite moderated cells, i.e., defining

$$\Delta I = (I)_{int} - (I)_{unint} \qquad (5\text{-}26)$$

ΔI is positive for the graphite moderated lattices. But for the heavy water moderated lattices ΔI is a negative quantity. But this should not mislead us. The presence of a high energy resonance, of course, decreases the absorption by the second (lower energy) resonance in both the graphite moderated and heavy water moderated lattices. The negative value of ρ' shows this effect clearly.

TABLE V-1

POTENTIAL SCATTERING CROSS SECTIONS

Elements	σ^P (barns)
D	3.4
C	4.5
O	3.8
Th^{232}	12.0
U^{238}	10.75

TABLE V-2

RESONANCE PARAMETERS FOR U^{238}*

(In Electron Volts)

E_o	Γ_n	Γ_r	Γ_p (Calculated)	Approximation
6.7	1.52×10^{-3}	2.46×10^{-2}	1.080339	IM
21.0	8.9×10^{-3}	2.46×10^{-2}	1.853616	IM
36.9	3.25×10^{-2}	2.46×10^{-2}	3.488660	IM
81.3	2.1×10^{-3}	2.46×10^{-2}	0.408538	NR
90.0	9.0×10^{-5}	2.46×10^{-2}	0.077299	NR
102.5	6.5×10^{-2}	2.46×10^{-2}	3.708180	IM
116.5	1.5×10^{-2}	2.46×10^{-2}	1.110817	NR
189.6	1.35×10^{-1}	2.46×10^{-2}	5.244164	IM
208.5	5.50×10^{-2}	2.46×10^{-2}	2.254223	NR
264.5	2.3×10^{-4}	2.46×10^{-2}	0.072286	NR
274.0	2.7×10^{-2}	2.46×10^{-2}	1.109286	NR
398.5	1.00×10^{-2}	2.46×10^{-2}	0.458391	NR
411.0	1.7×10^{-2}	2.46×10^{-2}	0.645301	NR

* Taken from Reference 21.

TABLE V-3

RESONANCE PARAMETERS FOR Th^{232}*

(In Electron Volts)

E_o	Γ_n	Γ_r	Γ_{prac} (Calculated)	Approximation
21.84	2.40×10^{-3}	3.0×10^{-2}	0.878572	IM
23.48	4.0×10^{-3}	3.0×10^{-2}	1.120588	IM
113.15	1.10×10^{-2}	4.2×10^{-2}	1.056990	NR
121.00	1.85×10^{-2}	4.1×10^{-2}	1.404356	NR
128.50	1.00×10^{-4}	4.1×10^{-2}	0.083271	NR
129.40	3.10×10^{-3}	4.1×10^{-2}	0.478585	NR
196.8	1.3×10^{-4}	4.10×10^{-2}	0.076747	NR
199.8	9.0×10^{-3}	4.10×10^{-2}	0.698770	NR
203.00	1.88×10^{-2}	4.10×10^{-2}	1.095739	NR
366.70	3.5×10^{-2}	4.10×10^{-2}	1.255238	NR
371.00	2.4×10^{-2}	4.10×10^{-2}	0.954775	NR
402.8	9.0×10^{-3}	4.10×10^{-2}	0.492138	NR
413.4	2.68×10^{-2}	4.10×10^{-2}	0.976164	NR
456.4	2.82×10^{-2}	4.1×10^{-2}	0.962788	NR
465.0	4.5×10^{-2}	4.1×10^{-2}	1.343243	NR

* Taken from Reference 21.

TABLE V-4

RESONANCE INTERFERENCE BETWEEN THE 116.5ev AND 102.5ev RESONANCES FOR U^{238} - GRAPHITE CELLS

Thickness of the Absorber Lump = 2a
Thickness of the Moderator Lump = 2(b-a)

2a (cm)	2(b-a) (cm)	E_o (ev)	$(I)_{unint}$ (barns)	$(I)_{int}$ (barns)	$\frac{(I)_{int}-(I)_{unint}}{(I)_{int}} \times 100$	ρ'
1.0	9.0	116.5	0.136624		2.900018	-0.043485
		102.5	0.175556	0.180799		
2.0	18.0	116.5	0.117069		6.197581	-0.072165
		102.5	0.134082	0.142941		
2.0	28.0	116.5	0.117076		6.592241	-0.108977
		102.5	0.134096	0.143560		
4.0	26	116.5	0.106046		9.220431	-0.109864
		102.5	0.126165	0.138980		
4.0	36.0	116.5	0.106046		11.637581	-0.105127
		102.5	0.126164	0.142780		
5.0	45.0	116.5	0.103780		18.543231	-0.044865
		102.5	0.150339	0.184563		

TABLE V-5

RESONANCE INTERFERENCE BETWEEN SEVERAL
PAIRS OF RESONANCES IN A U^{238} - GRAPHITE CELL

Thickness of the Absorber = 2cm
Thickness of the Moderator = 18cm

E_o (ev)	Δu	$(I)_{unint}$ (barns)	$(I)_{int}$ (barns)	$\frac{(I)_{int}-(I)_{unint}}{(I)_{int}} \times 100$	ρ'
90.0		0.015188			
	0.101664			0.493352	-0.014496
81.3		0.088780	0.089220		
208.5		0.068054			
	0.095022			4.862141	-0.043569
189.6		0.059805	0.062861		
274.0		0.038607			
	0.035287			1.393043	-0.059245
264.5		4.757314×10^{-3}	4.835669×10^{-3}		
411.0		0.01833			
	0.030886			0.801507	-0.028683
398.5		0.015957	0.016086		

TABLE V-6

RESONANCE INTERFERENCE BETWEEN SEVERAL PAIRS OF
RESONANCES IN A UO_2 - HEAVY WATER CELL

Thickness of the Absorber = 1.0cm
Thickness of the Moderator = 3.0cm

E_o (ev)	Δu	$(I)_{unint}$ (barns)	$(I)_{int}$ (barns)	$\frac{(I)_{int}-(I)_{unint}}{(I)_{int}} \times 100$	ρ'
90.0		0.022161			
	0.101664			-3.796641	-0.037345
81.3		0.138441	0.133377		
208.5		0.102702			
	0.095022			-4.759987	-0.048970
189.6		0.112919	0.107788		
291.6		0.048700			
	0.062255			-2.251708	-0.023812
274.0		0.058837	0.057541		
411.0		0.028091			
	0.030886			-1.085112	-0.012130
398.5		0.024579	0.024315		

TABLE V-7

RESONANCE INTERFERENCE BETWEEN SEVERAL PAIRS OF RESONANCES IN A Th^{232} - GRAPHITE CELL

Thickness of the Absorber = 20cm
Thickness of the Moderator = 18.0cm

E_o (ev)	Δu	$(I)_{unint}$ (barns)	$(I)_{int}$ (barns)	$\frac{(I)_{int}-(I)_{unint}}{(I)_{int}} \times 100$	ρ'
23.48		0.650028			
	0.072406			43.312819	-0.383695
21.84		0.568972	1.003704		
121.0		0.189870			
	0.067253			4.763815	-0.155140
113.15		0.174218	0.182933		
129.4		0.079632			
	6.979465×10^{-3}			7.213855	-0.047351
128.5		0.012501	0.013473		
203.0		0.087732			
	0.015889			3.601960	-0.088312
199.8		0.067155	0.069664		
413.4		0.033996			
	0.025975			1.23086	-0.032289
402.8		0.023351	0.023642		
465.0		0.033140			
	0.018668			1.277404	-0.032649
456.4		0.029775	0.030160		

TABLE V-8

INTERFERENCE BETWEEN RESONANCES IN A Th^{232} - HEAVY WATER CELL

Thickness of the Absorber Lump = 2.0cm
Thickness of the Moderator Lump = 6.0cm

E_o (ev)	Δu	$(I)_{unint}$ (barns)	$(I)_{int}$ (barns)	$\frac{(I)_{int}-(I)_{unint}}{(I)_{int}} \times 100$	ρ'
23.48		0.630786			
	0.072406			-2.837988	-0.057473
21.84		0.551959	0.536727		
121.0		0.187481			
	0.067253			-1.723887	-0.025939
113.15		0.171990	0.169076		
129.4		0.078597			
	6.979465×10^{-3}			0.483585	-6.309174×10^{-3}
128.5		0.012371	0.012432		
203.0		0.086629			
	0.015889			-0.410150	-0.011506
119.8		0.066294	0.066023		
371.0		0.038097			
	0.013569			-0.270201	-6.34224×10^{-3}
366.0		0.043781	0.043663		
413.4		0.033575			
	0.025975			-0.728074	-9.293795×10^{-3}
402.8		0.023053	0.022887		
465.0		0.032739			
	0.018668			-0.522162	-7.695124×10^{-3}
456.4		0.029407	0.029254		

TABLE V-9

RESONANCE INTERFERENCE IN A ($U^{238}Th^{232}$) - GRAPHITE CELL

Thickness of the Absorber = 2cm
Thickness of the Moderator = 18cm

Number of U^{238} atoms per unit vol: 0.023915×10^{24}
Number of Th^{232} atoms per unit vol: 0.014650×10^{24}

Absorber	E_o (ev)	Δu	$(I)_{unint}$ (barns)	$(I)_{int}$ (barns)	$\frac{(I)_{int}-(I)_{unint}}{(I)_{int}} \times 100$	ρ'
U^{238}	116.5		0.157977			3.233714
		0.029177				
Th^{232}	113.15		0.265604	0.274480	-0.133932	
U^{238}	208.5		0.090544			
		0.026733				2.059386
Th^{232}	203.0		0.133127	0.135926	-0.075256	

TABLE V-10

RESONANCE INTERFERENCE IN A ($U^{238}Th^{232}$) - HEAVY WATER CELL

Thickness of the Absorber = 2cm
Thickness of the Moderator = 18cm

Number of U^{238} atoms per unit vol: 0.023915×10^{24}
Number of Th^{232} atoms per unti vol: 0.014650×10^{24}

Absorber	E_o (ev)	Δu	$(I)_{unint}$ (barns)	$(I)_{int}$ (barns)	$\frac{(I)_{int}-(I)_{unint}}{(I)_{int}} \times 100$	ρ'
U^{238}	116.5		0.157975			
		9.929177				-0.444601
Th^{232}	113.15		0.265600	0.264425	-0.013552	
U^{238}	208.5		0.090542			
		0.026733				
Th^{232}	203.0		0.133125	0.132400	-0.010764	-0.547137

CHAPTER VI

CONCLUDING REMARKS

Solving the age-diffusion equation in reactor lattices by a Fourier series expansion of the flux, and observing that the series converges rapidly, we have been able to treat several problems of reactor physics satisfactorily. From the investigations made in this thesis we can make the following observations.

1. The equivalent cell approximation used in the calculation of the disadvantage factor and the thermal utilization in diffusion theory is good for most practical lattice cells. Similar observations have been made earlier by several investigators. They used different methods of calculation. This confirms the usefulness of the method of calculation employed here.

2. In the case of a slab lattice, it has been observed that a resonance is influenced by the presence of another resonance above it, i.e., at higher energy. Since there is depletion of flux due to absorption by the higher energy resonance, the absorption of neutrons by the lower energy resonance decreases. The resonance integral does not depend upon the magnitude of the flux if the flux is constant spatially and lethargywise. But in the case of closely spaced resonances, the resonance integral is different from that calculated by the flat flux approximation. The difference between the resonance integrals is a measure of the resonance interference which arises due to the incomplete flux recovery after the absorption by the higher energy resonance. In the transient region the flux is spatially and lethargy dependent and

the flux in the moderator is higher than the flux in the fuel lump. Depending upon the excess flux in the moderator above the lower energy resonance the resonance integral may be greater or smaller than that calculated by the flat flux approximation. The lethargy interval of the transient region depends upon the strength of the higher energy resonance and also upon the size of the fuel lump. We have observed that the ratio of the interfered resonance integral to the flat flux resonance integral is greater than unity in the case of graphite moderated cells but it is less than unity in the case of heavy water moderated cells. The resonance interference in the two- and three-dimensional lattices may be treated in the same way.

In the calculation of resonance interference, the higher energy resonance has been replaced by a delta function sink. This approximation is expected to be quite good for narrow resonances. The delta function sink approximation may not be satisfactory for a wide resonance. In these calculations, we have assumed two non-overlapping resonances. But there may be cases where two resonances may overlap each other, for example, in the case of a mixture of two absorbers. The 21.84 ev and 23.48 ev resonances of Th^{232} are wide resonances and also they overlap each other. The results obtained in Table V-7 and V-8 for those cases are not reliable. A Monte Carlo calculation is desirable in order to investigate those overlapping cases. Also, ZUT-MOD-3 code[51] can be used to treat the overlapping resonances.

The technique of solving the age diffusion equation may also be used to treat lattices with hydrogeneous moderator. In that case we can solve the Selengut-Goertzel equations.

APPENDIX A

THE INTEGRAL I(i,j,k,l,m,n)

Here we evaluate the integration shown in Equation (2-6)

$$I(i,j,k,l,m,n) = \iiint_{cell} \frac{D}{\xi \Sigma_s} \left\{ \cos \frac{i\pi x}{a} \cos \frac{j\pi y}{b} \cos \frac{k\pi z}{c} \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{b} \cos \frac{n\pi z}{c} \right\} dxdydz \qquad (2\text{-}6)$$

Since it is assumed that $\frac{D}{\xi \Sigma_s}$ is constant in each region of the cell, namely the spherical fuel element (region 1) and the outer moderator lump (region 2), we divide the entire volume of the cell into several parts such that in each part $\frac{D}{\xi \Sigma_s}$ is constant and integrate over the parts separately. Let us note that the integrand has eight-fold symmetry so that we can integrate over one-eighth of the cell only, (see Figures 1 and 1a). The integration over the cell can be carried out in the following way:

$$\iiint_{Cell} \frac{D}{\xi \Sigma_s} \left\{ \cos \frac{i\pi x}{a} \cos \frac{j\pi y}{b} \cos \frac{k\pi z}{c} \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{b} \cos \frac{n\pi z}{c} \right\} dxdydz = 8 \left[\frac{D_1}{\xi_1 \Sigma_{s1}} \int_0^{\rho} dx \int_0^{\sqrt{\rho^2-x^2}} dy \int_0^{\sqrt{\rho^2-x^2-y^2}} dz \left\{ \text{----} \right\} \right.$$

$$+ \frac{D_2}{\xi_2 \Sigma_{s2}} \int_0^{a} dx \int_0^{b} dy \int_0^{c} dz \left\{ \text{-----} \right\}$$

$$\left. - \frac{D_2}{\xi_2 \Sigma_{s2}} \int_0^{\rho} dx \int_0^{\sqrt{\rho^2-x^2}} dy \int_0^{\sqrt{\rho^2-x^2-y^2}} dz \left\{ \text{-----} \right\} \right] \qquad (A\text{-}1)$$

Note that

$$\left\{------\right\} = \frac{1}{8}\left(\cos\frac{(i+l)\pi x}{a} + \cos\frac{(i-l)\pi x}{a}\right)\left(\cos\frac{(j+m)\pi y}{b} + \cos\frac{(j-m)\pi y}{b}\right)\left(\cos\frac{(k+n)\pi z}{c} + \cos\frac{(k-n)\pi z}{c}\right) \qquad \text{(A-2)}$$

Let us consider the first term of the above expression after performing the multiplications. The remaining seven terms will give similar results except that the signs of l,m,n will be different in each case. Let us define $F(i,j,k,l,m,n)$ as

$$F(i,j,k,l,m,n) = \int_0^{\rho} dx \int_0^{\sqrt{\rho^2-x^2}} dy \int_0^{\sqrt{\rho^2-x^2-y^2}} dz \cos\frac{(i+l)\pi x}{a} \cos\frac{(j+m)\pi y}{b} \cos\frac{(k+n)\pi z}{c} = \frac{1}{\gamma}\int_0^{\rho} dx \int_0^{\sqrt{\rho^2-x^2}} dy \cos(\alpha x)\cos(\beta y)\sin\left(\gamma\sqrt{\rho^2-x^2-y^2}\right) \qquad \text{(A-3)}$$

where

$$\alpha = \frac{(i+l)\pi}{a}\ ;\quad \beta = \frac{(j+m)\pi}{b} \quad \text{and} \quad \gamma = \frac{(k+n)\pi}{c} \qquad \text{(A-4)}$$

Let

$$\begin{aligned} y &= \sqrt{\rho^2-x^2}\ \sin\theta \\ dy &= \sqrt{\rho^2-x^2}\ \cos\theta\, d\theta \\ x &= \rho\ \sin\varphi \\ dx &= \rho\ \cos\varphi\, d\varphi \end{aligned} \qquad \text{(A-5)}$$

We now expand the sines and cosines of Equation (A-3) in terms of its arguments and substitute the values of x and y from Equation (A-5) and get

$$F(i,j,k,l,m,n) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{p+q+r}\frac{\alpha^{2p}}{(2p)!}\frac{\beta^{2q}}{(2q)!}\frac{\gamma^{2r}}{(2r+1)!}\rho^{2(p+q+r)+3}$$

$$\int_0^{\frac{\pi}{2}}\sin^{2p}\varphi\cos^{2(q+r)+3}\varphi\,d\varphi\int_0^{\frac{\pi}{2}}\sin^{2q}\theta\cos^{2(r+1)}\theta\,d\theta \qquad \text{(A-6)}$$

The definite integrals in Equation (A-6) are known. Thus, we get

$$F(i,j,k,l,m,n) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{p+q+r}\frac{\alpha^{2p}}{(2p)!}\frac{\beta^{2q}}{(2q)!}\frac{\gamma^{2r}}{(2r+1)!}\rho^{2(p+q+r)+3}$$

$$\frac{\Gamma\left(\frac{2p+1}{2}\right)\Gamma(q+r+2)}{2\Gamma\left(p+q+r+\frac{5}{2}\right)}\;\frac{\Gamma\left(\frac{2q+1}{2}\right)\Gamma\left(\frac{2r+3}{2}\right)}{2\Gamma(q+r+2)}$$

$$= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}(-1)^{p+q+r}\frac{\rho^{2(p+q+r)+3}}{\Gamma\left(p+q+r+\frac{5}{2}\right)}\;\frac{\pi^{\frac{3}{2}}\alpha^{2p}\beta^{2q}\gamma^{2r}}{2^{2(p+q+r)+3}\,p!\,q!\,r!} \qquad \text{(A-7)}$$

we know

$$\frac{\Gamma\left(\frac{2p+1}{2}\right)\Gamma\left(\frac{2q+1}{2}\right)\Gamma\left(\frac{2r+3}{2}\right)}{(2p)!\,(2q)!\,(2r+1)!} = \frac{\pi^{\frac{3}{2}}}{2^{2(p+q+r)+1}\,p!\,q!\,r!} \qquad \text{(A-8)}$$

Let us make the following substitutions in Equation (A-7)

$$s = p + q + r$$

$$t = q + r$$

and obtain

$$F(i,j,k,l,m,n) = \sum_{S=0}^{\infty} (-1)^S \frac{\rho^{2S+3}}{\Gamma(s+\frac{5}{2})} \frac{\pi^{\frac{3}{2}}}{2^{2S+3}} \left[\sum_{t=0}^{S} \sum_{r=0}^{t} \frac{(\alpha^2)^{s-t} (\beta^2)^{t-r} (\gamma^2)^r \, S!}{(s-t)!\,(t-r)!\,r!} \right] \frac{1}{S!}$$

$$= \sum_{S=0}^{\infty} (-1)^S \frac{\rho^{2S+3}}{\Gamma(s+\frac{5}{2})} \frac{\pi^{\frac{3}{2}}}{2^{2S+3}} \frac{(\alpha^2+\beta^2+\gamma^2)^S}{S!} \qquad \text{(A-9)}$$

We know[40] that

$$J_n(z) = \sum_{S=0}^{\infty} \frac{(-1)^S \, z^{n+2S}}{2^{n+2S} \, S! \, \Gamma(n+S+1)} \quad , \text{ where } n \text{ is any number.} \qquad \text{(A-10)}$$

Using above relation we can express Equation (A-9) as

$$F(i,j,k,l,m,n) = J_{3/2}\left(\rho\sqrt{\alpha^2+\beta^2+\gamma^2}\right) \left(\frac{\pi\rho}{2}\right)^{3/2} \left(\alpha^2+\beta^2+\gamma^2\right)^{-3/2}$$

$$= J_{3/2}\left(\pi\rho\sqrt{\left(\frac{i+l}{a}\right)^2+\left(\frac{j+m}{b}\right)^2+\left(\frac{k+n}{c}\right)^2}\right)$$

$$\left(\frac{\rho}{2}\right)^{3/2} \left[\left(\frac{i+l}{a}\right)^2+\left(\frac{j+m}{b}\right)^2+\left(\frac{k+n}{c}\right)^2\right]^{-\frac{3}{2}} \qquad \text{(A-11)}$$

$J_{3/2}$ is a Bessel function.

The second integral of Equation (A-1) is relatively simple.

$$\int_0^a dx \int_0^b dy \int_0^c dz \cos\frac{i\pi x}{a} \cos\frac{j\pi y}{b} \cos\frac{k\pi z}{c} \cos\frac{l\pi x}{a} \cos\frac{m\pi y}{b} \cos\frac{n\pi z}{c}$$

$$= \Lambda_{ijk}\delta_{il}\delta_{jm}\delta_{kn} \qquad \text{(A-12)}$$

Λ_{ijk} has been defined in Equation (2-5). The third integral of Equation (A-1) is exactly the same as the first one. Taking all the terms of Equation (A-2), we finally get

$$\begin{aligned} I(i,j,k,l,m,n) = &\left(\frac{D_1}{\xi_1 \Sigma_{s1}} - \frac{D_2}{\xi_2 \Sigma_{s2}}\right) F(i,j,k,l,m,n) + F(i,j,k,l,m,-n) \\ &+ F(i,j,k,l,-m,n) + F(i,j,k,-l,m,n) \\ &+ F(i,j,k,-l,-m,n) + F(i,j,k,-l,m,-n) \\ &+ F(i,j,k,l,-m,-n) + F(i,j,k,-l,-m,-n) \\ &+ \frac{D_2}{\xi_2 \Sigma_{s2}} abc \Lambda_{ijk}\delta_{il}\delta_{jm}\delta_{kn} \end{aligned} \qquad \text{(A-13)}$$

APPENDIX B

THE INTEGRAL $H_Z(i,j,k,l,m,n)$

Let us perform an integration containing terms like $\frac{1}{\xi \Sigma_s} \frac{\partial D}{\partial x}$

We know that $\frac{1}{\xi \Sigma_s}$ is constant in each region of the cell except at the fuel-moderator interface where it is a step function. Whereas $\frac{\partial D}{\partial x}$ is a delta function at the fuel-moderator interface but zero everywhere else. Therefore, we can take the average value of $\frac{1}{\xi \Sigma_s}$ at the interface.

$$\frac{\partial D}{\partial x} = (D_2 - D_1)\,\delta\left(x - \sqrt{\rho^2 - y^2 - z^2}\right) \qquad \text{(B-1)}$$

$$\frac{\partial D}{\partial y} = (D_2 - D_1)\,\delta\left(y - \sqrt{\rho^2 - x^2 - z^2}\right) \qquad \text{(B-2)}$$

and

$$\frac{\partial D}{\partial z} = (D_2 - D_1)\,\delta\left(z - \sqrt{\rho^2 - x^2 - y^2}\right) \qquad \text{(B-3)}$$

Let us consider the following integral

$$H_Z(i,j,k,l,m,n) = \iiint_{cell} \frac{1}{\xi \Sigma_s} \frac{\partial D}{\partial z} \left\{ \cos \frac{i\pi x}{a} \cos \frac{j\pi y}{b} \sin \frac{k\pi z}{c} \cos \frac{l\pi x}{a} \cos \frac{m\pi y}{b} \cos \frac{n\pi z}{c} \right\} dx\,dy\,dz \qquad \text{(B-4)}$$

The term inside the bracket can be written as

$$\left\{----\right\} = \frac{1}{8} \left(\cos \frac{(i+l)\pi x}{a} + \cos \frac{(i-l)\pi x}{a} \right) \left(\cos \frac{(j+m)\pi y}{b} + \cos \frac{(j-m)\pi y}{b} \right) \left(\sin \frac{(k+n)\pi z}{c} + \sin \frac{(k-n)\pi z}{c} \right) \qquad \text{(B-5)}$$

Let us take the first term of Equation (B-5), i.e.,

$$\cos \frac{(i+l)\pi x}{a} \quad \cos \frac{(j+m)\pi y}{b} \quad \sin \frac{(k+n)\pi z}{c}$$

and carry out the integration shown in Equation (B-4). The remaining

seven terms of Equation (B-5) will give similar results except for the change of sign of l, m, and n.

$$\iiint_{cell} \frac{1}{\xi \Sigma_s} \frac{\partial D}{\partial z} \cos\alpha x \cos\beta y \sin\gamma z \, dxdydz$$

$$= 8 \cdot \frac{1}{2} \cdot \left(\frac{1}{\xi_1 \Sigma_{s1}} + \frac{1}{\xi_2 \Sigma_{s2}} \right) (D_2 - D_1) \int_0^{\rho} dx \int_0^{\sqrt{\rho^2-x^2}} dy \int_{\sqrt{\rho^2-x^2-y^2}-\epsilon}^{\sqrt{\rho^2-x^2-y^2}+\epsilon} dz \cos\alpha x \cos\beta y$$

$$\sin\gamma z \; \delta(z - \sqrt{\rho^2 - x^2 - y^2})$$

$$= 8 \cdot \frac{1}{2} \cdot \left(\frac{1}{\xi_1 \Sigma_{s1}} + \frac{1}{\xi \Sigma_{s2}} \right) (D_2 - D_1) \int_0^{\rho} dx \int_0^{\sqrt{\rho^2-x^2}} dy \cos\alpha x \cos\beta y \sin\gamma\sqrt{\rho^2-x^2-y^2} \tag{B-6}$$

α, β, γ are defined as in Equation (A-4). Observing that the above integral has exactly the same form as in Equation (A-3), we can easily write

$$\iiint_{cell} \frac{1}{\xi \Sigma_s} \frac{\partial D}{\partial z} \cos\alpha x \cos\beta y \sin\gamma z \, dxdydz$$

$$= 8 \cdot \frac{1}{2} \cdot \left(\frac{1}{\xi_1 \Sigma_{s1}} + \frac{1}{\xi \Sigma_{s2}} \right) (D_2 - D_1) \frac{(k+n)\pi}{c} \; F(i,j,k,l,m,n) \tag{B-7}$$

Taking all the terms of Equation (B-5), we finally get

$$\begin{aligned} H_z(i,j,k,l,m,n) = \frac{1}{2} \left(\frac{1}{\xi_1 \Sigma_{s1}} + \frac{1}{\xi_2 \Sigma_{s2}} \right) (D_2 - D_1) \Bigg[& \frac{(k+n)\pi}{c} \\ & \Big\{ F(i,j,k,l,m,n) + F(i,j,k,-l,m,n) \\ & + F(i,j,k,l,-m,n) + F(i,j,k,-l,-m,n) \Big\} \\ & + \frac{(k-n)\pi}{c} \Big\{ F(i,j,k,l,m,-n) + F(i,j,k,l,-m,-n) \\ & + F(i,j,k,-l,m,-n) + F(i,j,k,-l,-m,-n) \Big\} \Bigg] \end{aligned} \tag{B-8}$$

Similarly we can derive expressions for $H_x(i,j,k,l,m,n)$ and $H_y(i,j,k,l,m,n)$.

APPENDIX C

THE INTEGRAL I(k,l,m,n)

The integration in Equation (2-24) is very similar to the integration carried out in Appendix A except that here it is a surface integral. In this case also the entire area of the cell has been divided into several parts in each of which $\frac{D}{\xi \Sigma_s}$ is constant (see Figure 2) and integrations have been carried out separately.

$$I(k,l,m,n) = \int_{Cell}\int \frac{D}{\xi\Sigma_s}\left\{\cos\frac{k\pi x}{a}\cos\frac{l\pi y}{b}\cos\frac{m\pi x}{a}\cos\frac{n\pi z}{c}\right\}dxdy$$

$$= 4\left[\frac{D_1}{\xi_1\Sigma_{s1}}\int_0^{\rho}dx\int_0^{\sqrt{\rho^2-x^2}}dy\left\{----\right\} + \frac{D_2}{\xi_2\Sigma_{s2}}\int_0^{\rho}dx\int_0^{\sqrt{\rho^2-x^2}}dy\left\{---\right\}\right.$$

$$\left. + \frac{D_2}{\xi_2\Sigma_{s2}}\int_{\rho}^{a}dx\int_0^{b}dy\left\{----\right\}\right] \qquad (2\text{-}20)$$

We know

$$\cos\frac{k\pi x}{a}\cos\frac{l\pi y}{b}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} = \left(\cos\frac{(m+k)\pi x}{a}\right.$$

$$\left. + \cos\frac{(k-m)\pi x}{a}\right)\left(\cos\frac{(l+n)\pi y}{b} + \cos\frac{(l-n)\pi y}{b}\right) \qquad (C\text{-}1)$$

Let us consider the first term on the right hand side of Equation (C-1) and perform the integrations shown in Equation (2-24).

$$\int_0^{\rho}dx\int_0^{\sqrt{\rho^2-x^2}}dy\cos\alpha x\cos\beta y = F(k,l,m,n)$$

where

$$\alpha = \frac{(k+m)\pi}{a} \quad ; \text{ and } \beta = \frac{(l+n)\pi}{b} \qquad (C\text{-}2)$$

$$F(k,l,m,n) = \int_0^{\rho}dx\left(\cos\alpha x\right)\frac{1}{\beta}\sin\beta\sqrt{\rho^2-x^2} \qquad (C\text{-}3)$$

As in Appendix A let us expand the sine and cosine in Equation (C-3) and then make the following substitutions:

$$\begin{aligned} x &= \rho \sin\theta \\ dx &= \rho \cos\theta d\theta \end{aligned} \tag{C-4}$$

$$\cos \alpha x = \sum_{p=0}^{\infty} (-1)^p \frac{\alpha^{2p} x^{2p}}{(2p)!} \tag{C-5}$$

$$\sin \beta\sqrt{\rho^2 - x^2} = \sum_{q=0}^{\infty} (-1)^q \frac{\beta^{2q+1} (\rho^2 - x^2)^{\frac{2q+1}{2}}}{(2q+1)!} \tag{C-6}$$

Inserting Equation (C-4), (C-5) and (C-6) in Equation (C-3) we get

$$\begin{aligned} F(k,l,m,n) &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (-1)^{p+q} \frac{\alpha^{2p}\beta^{2q}\rho^{2(p+q+1)}}{(2p)!\,(2q+1)!} \int_0^{\frac{\pi}{2}} \sin^{2p}\theta \cos^{2(q+1)}\theta \, d\theta \\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (-1)^{p+q} \frac{\alpha^{2p}\beta^{2q}\rho^{2(p+q+1)}}{(2p)!\,(2q)!} \frac{\Gamma\left(\frac{2p+1}{2}\right)\Gamma\left(\frac{2q+3}{2}\right)}{2\Gamma(p+q+2)} \\ &= \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} (-1)^{p+q} \frac{\alpha^{2p}\beta^{2q}\rho^{2(p+q+1)}}{2\Gamma(p+q+2)} \frac{\pi}{2^{2(p+q)+1}\, p!\, q!} \end{aligned} \tag{C-7}$$

let $s = p + q$, and get

$$\begin{aligned} F(k,l,m,n) &= \sum_{s=0}^{\infty} (-1)^s \frac{\rho^{2s+2}\,\pi}{2^{2s+2}\,\Gamma(s+2)} \left[\sum_{q=0}^{\infty} \frac{(\alpha^2)^{s-q}(\beta)^q\, s!}{(s-q)!\, q!} \right] \frac{1}{s!} \\ &= \sum_{s=0}^{\infty} (-1)^s \frac{\left[\rho\sqrt{\alpha^2+\beta^2}\right]^{2s+1}}{2^{2s+1}\, s!\,\Gamma(s+2)} \frac{\rho\pi}{2(\alpha^2+\beta^2)^{1/2}} \end{aligned}$$

$$= J_1\left(\rho\sqrt{\alpha^2+\beta^2}\right)\frac{\pi\rho}{2}\left[\alpha^2+\beta^2\right]^{-\frac{1}{2}} \qquad \text{(C-8)}$$

We used the relation in Equation (A-10) in the above equation. Substituting the values of α and β, we get

$$F(k,l,m,n) = J_1\left(\rho\pi\sqrt{\left(\frac{k+m}{a}\right)^2+\left(\frac{l+n}{b}\right)^2}\right)\frac{\rho}{2}\left[\left(\frac{k+m}{a}\right)^2+\left(\frac{l+n}{b}\right)^2\right]^{-\frac{1}{2}} \qquad \text{(C-9)}$$

In the same way we integrate the second and third term of the right hand side of Equation (2-24) and finally get

$$I(k,l,m,n) = \left(\frac{D_1}{\xi_1\sum_{s1}} - \frac{D_2}{\xi_2\sum_{s2}}\right)\Big[F(k,l,m,n) + F(k,l,m,-n) + F(k,l,-m,n) + F(k,l,-m,-n)\Big] + ab\Lambda_{kl}\delta_{mk}\delta_{nl} \qquad \text{(C-10)}$$

Λ_{kl} has been defined in Equation (2-23).

Following the same method as has been shown in Appendix B we can obtain the expressions for $H_x(k,l,m,n)$ and $H_y(k,l,m,n)$ easily.

APPENDIX D

DERIVATION OF THE FIRST-COLLISION PROBABILITY IN A FUEL LUMP

$P_1(u)$ is the probability that the neutrons of lethargy u originating in the fuel lump ($-a < x' < a$) of region 1 will make their first collision in the same region 1 (probably after crossing other regions such as the moderator in this case) anywhere in the lattice. From Equation (4-16) we write

$$
\begin{aligned}
P_1(u) = {} & \frac{\Sigma_1}{2}\int_{-a}^{a} dx'\phi(x')\int_{-a}^{x'} E_1(\Sigma_1\{x'-x\})\,dx \Big/ \int_{-a}^{a}\phi(x,u)\,dx' \\
& + \frac{\Sigma_1}{2}\int_{-a}^{a} dx'\phi(x',u)\int_{x'}^{a} E_1(\Sigma_1\{x-x'\})\,dx \Big/ \int_{-a}^{a}\phi(x',u)\,x' \\
& + \sum_{n=1}^{\infty}\frac{\Sigma_1}{2}\int_{-a}^{a} dx'\phi(x',u)\int_{2nb-a}^{2nb+a} E_1(2n\{b-a\}\{\Sigma_2-\Sigma_1\}+\Sigma_1\{x-x'\})\,dx \Big/ \int_{-a}^{a}\phi(x',u)\,dx' \\
& + \sum_{n=1}^{\infty}\frac{\Sigma_1}{2}\int_{-a}^{a} dx'\phi(x',u)\int_{-2nb-a}^{-2nb+a} E_1[2n(b-a)(\Sigma_2-\Sigma_1) \\
& + \Sigma_1(x'-x)]\,dx \Big/ \int_{-a}^{a}\phi(x',u)\,dx' \qquad \text{(D-1)}
\end{aligned}
$$

The flux, $\phi(x',u)$, in the fuel lump is given by Equations (4-9) and (4-10)

$$\phi(x',u) = \frac{a_o(u)}{2} + a_1(u)\cos \alpha' x', \qquad \text{for } -a < x' < a$$

$$\int_{-a}^{a} \phi(x',u)dx' = 2a\bar{\phi}_1(u)$$

where

$$\bar{\phi}_1(u) = \frac{1}{2a}\int_{-a}^{a} \phi(x',u)dx'$$

$$= \frac{a_o(u)}{2} + a_1(u)\,\frac{1}{\alpha' a}\sin(\alpha' a)$$

$$= \frac{1}{2}\left[S(0) - \frac{A_{o1}S(1)}{A_{11}} - \mathcal{S}(u-u_o)\left\{S_o(0) - \frac{A_{o1}S_o(1)}{A_{11}}\right\}\right]$$

$$+ \frac{1}{2}\,\mathcal{S}(u-u_o)\left\{-\frac{A_{o1}S_o(1)}{A_{11}}\right\}e^{-A_{11}(u-u_o)}$$

$$- \mathcal{S}(u-u_o)S_o(1)\,\frac{1}{\alpha' a}\sin(\alpha' a)e^{-A_{11}(u-u_o)}$$

$$= \phi_o + \phi_1' \, e^{-A_{11}(u-u_o)} \qquad \text{(D-2)}$$

Here

$$\phi_o = \frac{1}{2}\left[S(0) - \frac{A_{o1}S(1)}{A_{11}} - \mathcal{S}(u-u_o)\left\{S_o(0) - \frac{A_{o1}S_o(1)}{A_{11}}\right\}\right] \qquad \text{(D-3)}$$

$$\phi_1' = -\mathcal{S}(u-u_o)\left[\frac{A_{o1}S_o(1)}{2A_{11}} + \frac{S_o(1)}{\alpha' a}\sin(\alpha' a)\right] \qquad \text{(D-4)}$$

By definition

$$E_n(x) = \int_1^\infty e^{-x\mu}\,\mu^n\,d\mu \qquad \text{for } x \geq 0 \text{ and}$$

$$= \int_0^1 e^{-x/t}\,t^{n-2}\,dt \qquad n, \text{ a non-negative integer} \tag{D-5}$$

We also know[44] that

$$\int E_n(Ax+B)dx = -\frac{1}{A}\,E_{n+1}(Ax+b) \tag{D-6}$$

where A and B are constants.

Integrating Equation (D-1) with respect to x, we obtain

$$P_1(u) = \frac{1}{4a\bar{\phi}_1}\int_{-a}^{a} dx' \left[\frac{a_0(u)}{2} + a_1(u)\cos\alpha'x'\right]\left[E_2(0)-E_2(\Sigma_1x'+\Sigma_1a)\right]$$

$$+ \frac{1}{4a\bar{\phi}_1}\int_{-a}^{a} dx' \left[\frac{a_0(u)}{2} + a_1(u)\cos\alpha'x'\right]\left[E_2(0)-E_2(\Sigma_1a-\Sigma_1x')\right]$$

$$+ \sum_{n=1}^{\infty}\frac{1}{4a\bar{\phi}_1}\int_{-a}^{a} dx' \left[\frac{a_0(u)}{2} + a_1(u)\cos\alpha'x'\right]\left[E_2\{nd_2\Sigma_2\right.$$

$$\left. + (2n-1)a\Sigma_1-\Sigma_1x'\} - E_2\{nd_2\Sigma_2 + (2n+1)a\Sigma_1-\Sigma_1x'\}\right]$$

$$+ \sum_{n=1}^{\infty}\frac{1}{4a\bar{\phi}_1}\int_{-a}^{a} dx' \left[\frac{a_0(u)}{2} + a_1(u)\cos\alpha'x'\right]\left[E_2\{nd_2\Sigma_2\right.$$

$$\left. + (2n-1)a\Sigma_1+\Sigma_1x'\} - E_2\{nd_2\Sigma_2 + (2n+1)a\Sigma_1+\Sigma_1x'\}\right] \tag{D-7}$$

where $d_2 = 2(b-a)$, the thickness of the moderator.

We use the integral form, Equation (D-5), for $E_2\{\cdots\}$ when it is multiplied by $\cos \alpha' x'$ and perform the integrations in Equation (D-7) and find

$$
\begin{aligned}
P_1(u) &= 1 + \frac{1}{4a\bar{\phi}_1}\{E_3(2a\Sigma_1) - E_3(0)\}\frac{a_0(u)}{\Sigma_1} \\
&+ \sum_{n=1}^{\infty} \frac{1}{4a\bar{\phi}_1}\frac{a_0(u)}{2\Sigma_1}\{E_3(nd_2\Sigma_2+(n-1)d_1\Sigma_1) - E_3(nd_2\Sigma_2+nd_1\Sigma_1) \\
&- E_3(nd_2\Sigma_2+nd_1\Sigma_1) + E_3(nd_2\Sigma_2+(n+1)d_1\Sigma_1)\} \\
&+ \sum_{n=1}^{\infty} \frac{1}{4a\bar{\phi}_1}\frac{a_0(u)}{2\Sigma_1}\{E_3(nd_2\Sigma_2+(n-1)d_1\Sigma_1 - 2E_3(nd_2\Sigma_2+nd_1\Sigma_1) \\
&+ E_3(nd_2\Sigma_2+(n+1)d_1\Sigma_1)\} \\
&- \frac{1}{4a\bar{\phi}_1} a_1(u)\left[\int_{-a}^{a} dx' \cos \alpha' x' \int_0^1 e^{-\Sigma_1(x'+a)/t} dt\right. \\
&\left. + \int_{-a}^{a} dx' \cos \alpha' x' \int_0^1 e^{-\Sigma_1(a-x')/t}\, dt\right] \\
&+ \sum_{n=1}^{\infty} \frac{a_1(u)}{4a\bar{\phi}_1}\left[\int_{-a}^{a} dx' \cos \alpha' x' \int_0^1 e^{-\left[nd_2\Sigma_2+(2n-1)a\Sigma_1-\Sigma_1 x'\right]/t} dt\right. \\
&\left. - \int_{-a}^{a} dx' \cos \alpha' x' \int_0^1 e^{-\left[nd_2\Sigma_2+(2n+1)a\Sigma_1-\Sigma_1 x'\right]/t} dt\right. \qquad \text{(D-8)}
\end{aligned}
$$

$$+\int_{-a}^{a} dx' \cos \alpha' x' \int_{0}^{1} e^{-\left[nd_2\Sigma_2+(2n-1)a\Sigma_1+\Sigma_1 x'\right]/t} dt$$

$$-\int_{-a}^{a} dx' \cos \alpha' x' \int_{0}^{1} e^{-\left[nd_2\Sigma_2+(2n+1)a\Sigma_1+\Sigma_1 x'\right]/t} dt \Bigg] \qquad \text{(D-8) (Cont'd)}$$

here $d_1 = 2a$, the thickness of the fuel lump.

In Equation (D-8) we have integrations of the following types

$$\int_{-a}^{a} \cos \alpha' x' dx' \int_{0}^{1} e^{-\left[K+\Sigma_1 x'\right]/t} dt$$

$$= \int_{0}^{1} e^{-K/t} dt \int_{-a}^{a} \cos \alpha' x' e^{-\Sigma_1 x'/t} dx'$$

$$= \int_{0}^{1} dt e^{-K/t} \left[\frac{e^{-\Sigma_1 a/t}}{\left(\frac{\Sigma_1}{t}\right)^2 + (\alpha')^2} \left\{ \left(-\frac{\Sigma_1}{t}\right) \cos \alpha' a + \alpha' \sin \alpha' a \right\} \right.$$

$$\left. - \frac{e^{\Sigma_1 a/t}}{\left(\frac{\Sigma_1}{t}\right)^2 + \alpha'^2} \left\{ \left(-\frac{\Sigma_1}{t}\right) \cos \alpha' a - \alpha' \sin \alpha' a \right\} \right]$$

$$= \alpha' \sin \alpha' a \left[\int_{0}^{1} \frac{e^{-\left[K+\Sigma_1 a\right]/t} t^2 dt}{\Sigma_1^2 + \alpha'^2 t^2} + \int_{0}^{1} \frac{e^{-\left[K-\Sigma_1 a\right]/t} t^2 dt}{\Sigma_1^2 + \alpha'^2 t^2} \right]$$

$$+ \Sigma_1 \cos \alpha' a \left[\int_{0}^{1} \frac{e^{-\left[K-\Sigma_1 a\right]/t} t dt}{\Sigma_1^2 + \alpha'^2 t^2} - \int_{0}^{1} \frac{e^{-\left[K+\Sigma_1 a\right]/t} t dt}{\Sigma_1^2 + \alpha'^2 t^2} \right] \qquad \text{(D-9)}$$

and

$$\int_{-a}^{a} \cos \alpha' x' \int_{0}^{1} e^{-\left[K-\Sigma_1 x'\right]/t} \, dt = \alpha' \sin \alpha' a \left[\int_{0}^{1} \frac{e^{-\left[K+\Sigma_1 a\right]/t} t^2 dt}{\Sigma_1^2+\alpha'^2 t^2} + \int_{0}^{1} \frac{e^{-\left[K-\Sigma_1 a\right]/t} t^2 dt}{\Sigma_1^2+\alpha'^2 t^2} \right] + \Sigma_1 \cos \alpha' a \left[\int_{0}^{1} \frac{e^{-\left[K-\Sigma_1 a\right]/t} t dt}{\Sigma_1^2+\alpha'^2 t^2} - \int_{0}^{1} \frac{e^{-\left[K+\Sigma_1 a\right]/t} t dt}{\Sigma_1^2+\alpha'^2 t^2} \right] \tag{D-10}$$

Let us also note that

$$\int_{-a}^{a} dx' \cos \alpha' x' \int_{0}^{1} e^{-\Sigma_1 (a+x')/t} dt = \int_{-a}^{a} dx' \cos \alpha' x' \int_{0}^{1} e^{-\Sigma_1 (a-x')/t} dt$$

$$\mp \alpha' \sin \alpha' a \left[\int_{0}^{1} \frac{e^{-2a\Sigma_1/t} t^2 dt}{\Sigma_1^2+\alpha'^2 t^2} + \int_{0}^{1} \frac{t^2 dt}{\Sigma_1^2+\alpha'^2 t^2} \right]$$

$$+ \Sigma_1 \cos \alpha' a \left[\int_{0}^{1} \frac{t dt}{\Sigma_1^2+\alpha'^2 t^2} - \int_{0}^{1} \frac{e^{-2a\Sigma_1/t} t dt}{\Sigma_1^2+\alpha'^2 t^2} \right] \tag{D-11}$$

We know that

$$\int_0^1 \frac{t^2 dt}{\Sigma_1^2+\alpha'^2 t^2} = \frac{1}{\alpha'^2}\left[1 - \frac{\Sigma_1}{\alpha'}\tan^{-1}\left(\frac{\alpha'}{\Sigma_1}\right)\right] \tag{D-12}$$

and

$$\int_0^1 \frac{t\,dt}{\Sigma_1^2+\alpha'^2 t^2} = \frac{1}{2\alpha'^2}\log_e\left(1 + \frac{\alpha'^2}{\Sigma_1^2}\right) \tag{D-13}$$

Evaluating all the integrals of Equation (D-8) and rearranging the terms we finally get

$$P_1(u) = 1 - \frac{a_0(u)}{2d_1\Sigma_1\bar{\phi}_1}\left[0.5 - \sum_{n=0}^{\infty}\{E_3(\ell_1(n)) - 2E_3(\ell_2(n)) + E_3(\ell_3(n))\}\right]$$

$$- \frac{a_1(u)}{d_1\bar{\phi}_1}\left[\frac{\sin\alpha' a}{\alpha'}\left\{1 - \frac{\Sigma_1}{\alpha'}\tan^{-1}\left(\frac{\alpha'}{\Sigma_1}\right)\right\} + \frac{\Sigma_1\cos\alpha' a}{2\alpha'^2}\log_e\left(1 + \frac{\alpha'^2}{\Sigma_1^2}\right)\right]$$

$$+ \sum_{n=0}^{\infty}\frac{a_1(u)}{d_1\bar{\phi}_1}\left[\Sigma_1\cos\alpha' a\left\{\Psi_1(n) - 2\Psi_2(n) + \Psi_3(n)\right\}\right.$$

$$\left. + \alpha'\sin\alpha' a\left\{\Psi'_1(n) - \Psi'_3(n)\right\}\right] \tag{D-14}$$

where

$$\Psi_i(n) = \int_0^1 \frac{e^{-\ell_i(n)/t}\,t\,dt}{\Sigma_1^2+\alpha'^2 t^2}, \qquad i = 1,2,3 \tag{D-15}$$

$$\Psi'_j(n) = \int_0^1 \frac{e^{-\ell_j(n)/t} t^2 dt}{\Sigma_1^2 + \alpha'^2 t^2}, \qquad j = 1,3 \tag{D-16}$$

and

$$\begin{aligned} \ell_1(n) &= n d_1 \Sigma_1 + (n+1) d_2 \Sigma_2 \\ \ell_2(n) &= (n+1)(d_1 \Sigma_1 + d_2 \Sigma_2) \\ \ell_3(n) &= (n+1) d_1 \Sigma_1 + n d_2 \Sigma_2 \end{aligned} \tag{D-17}$$

APPENDIX E

DERIVATION OF THE FIRST-COLLISION PROBABILITY IN THE MODERATOR LUMP

$P_2(u)$ is the probability that the neutrons of lethargy u originating in the moderator lump ($a < x' < 2b-a$) of region 2 will make their first collision in the same region 2 (probably after traversing other regions) anywhere in the lattice. From Equation (4-28) we know

$$P_2(u) = \frac{\Sigma_{e2}}{2} \int_a^{2b-a} dx'\phi(x') \int_a^{x'} E_1\left[\Sigma_{e2}(x'-x)\right]dx \Big/ \int_a^{2b-a} \phi(x')dx'$$

$$+ \frac{\Sigma_{e2}}{2} \int_a^{2b-a} dx'\phi(x') \int_{x'}^{2b-a} E_1\left[\Sigma_{e2}(x-x')\right]dx \Big/ 2(b-a)\bar{\phi}_2$$

$$+ \sum_{n=1}^{\infty} \frac{\Sigma_{e2}}{2} \int_a^{2b-a} dx'\phi(x') \int_{2nb+a}^{2(n+1)b-a} E_1\left[2na(\Sigma_1-\Sigma_{e2})+(x-x')\Sigma_{e2}\right]dx \Big/ 2(b-a)\bar{\phi}_2$$

$$+ \sum_{n=1}^{\infty} \frac{\Sigma_{e2}}{2} \int_a^{2b-a} dx'\phi(x') \int_{-2nb+a}^{-2(n-1)b-a} E_1\left[2na(\Sigma_1-\Sigma_{e2})+(x'-x)\Sigma_{e2}\right]dx \Big/ 2(b-a)\bar{\phi}_2 \qquad \text{(E-1)}$$

where

$$\bar{\phi}_2 = \frac{1}{2(b-a)} \int_a^{2b-a} \phi(x')dx' \qquad \text{(E-2)}$$

From Equations (4-9) and (4-10) we know that

$$\phi(x') = \frac{a_0(u)}{2} + a_1(u)\cos\left[\left(\frac{a}{D_1} + \frac{x'-a}{D_2}\right)\frac{\pi}{\beta}\right] \quad \text{for } \alpha < x' < 2b-a$$

Therefore, we find

$$\bar{\phi}_2(u) = \frac{1}{2(b-a)}\int_a^{2b-a}\left[\frac{a_0(u)}{2} + a_1(u)\{\cos\theta\cos\beta'x' - \sin\theta\sin\beta'x'\}\right]dx' \tag{E-3}$$

where

$$\theta = \frac{\pi a}{\beta}\left(\frac{1}{D_1} - \frac{1}{D_2}\right)$$

$$\beta' = \frac{\pi}{\beta D_2} \tag{E-4}$$

Performing the integrations in Equation (E-3) we find

$$\bar{\phi}_2(u) = \frac{a_0(u)}{2} + \frac{a_1(u)\beta}{2\pi(\beta-\alpha)}\left[\sin\frac{\pi}{\beta}(2\beta-\alpha) - \sin\frac{\pi\alpha}{\beta}\right] \tag{E-5}$$

α and β have been defined in Equation (2-40). $\bar{\phi}_2(u)$ can also be written as

$$\bar{\phi}_2(u) = \phi_0 + \phi_2' e^{-A_{11}(u-u_0)} \tag{E-6}$$

where ϕ_0 has same value in Equation (D-3) and

$$\phi_2' = -\mathcal{A}(u-u_0)\left[\frac{A_{01}S_0(1)}{2A_{11}} - \frac{S_0(1)\beta}{\pi(\beta-\alpha)}\sin\left(\frac{\pi\alpha}{\beta}\right)\right] \tag{E-7}$$

Integrating Equation (E-1) with respect to x we find

$$P_2(u) = \frac{1}{4(b-a)\bar{\phi}_2}\left[2E_2(0)\int_a^{2b}\phi(x')dx' - \int_a^{2b-a}dx'\left\{\frac{a_0(u)}{2}\right.\right.$$

$$\left.\left. + a_1(u)(\cos\Theta\cos\beta'x' - \sin\Theta\sin\beta'x')\right\}E_2\left\{\Sigma_e(2b-a) - \Sigma_e x'\right\}\right.$$

$$- \int_a^{2b-a}dx'\left\{\frac{a_0(u)}{2} + a_1(u)(\cos\Theta\cos\beta'x' - \sin\Theta\sin\beta'x')\right\}E_2(\Sigma_e x' - \Sigma_e a)$$

$$+ \sum_{n=1}^{\infty}\int_a^{2b-a}dx'\left\{\frac{a_0(u)}{2} + a_1(u)(\cos\Theta\cos\beta'x' - \sin\Theta\sin\beta'x')\right\}$$

$$\left\{E_2(nd_1\Sigma_1 + nd_2\Sigma_e + a\Sigma_e - \Sigma_e x') - E_2(nd_1\Sigma_1 + (n+1)d_2\Sigma_e + a\Sigma_e - \Sigma_e x')\right\}$$

$$+ \sum_{n=1}^{\infty}\int_a^{2b-a}dx'\left\{\frac{a_0(u)}{2} + a_1(u)(\cos\Theta\cos\beta'x' - \sin\Theta\sin\beta'x')\right\}$$

$$\left\{E_2(nd_1\Sigma_1 + (n-1)d_2\Sigma_e - a\Sigma_e + \Sigma_e x') - E_2(nd_1\Sigma_1 + nd_2\Sigma_e - a\Sigma_e + \Sigma_e x')\right\} \quad \text{(E-8)}$$

As in Appendix D we express $E_2(\ldots)$ in integral form, Equation (D-5), when it is multiplied by $\cos \beta' x'$ or $\sin \beta' x'$ and get

$$P_2(u) = 1 - \frac{a_0(u)}{2d_2\bar{\phi}_2\Sigma_e}\left[0.5-E_3(d_2\Sigma_e) - \sum_{n=1}^{\infty}\left\{E_3(nd_1\Sigma_1+(n-1)d_2\Sigma_e) - 2E_3(nd_1\Sigma_1+nd_2\Sigma_e) + E_3(nd_1\Sigma_1+(n+1)d_2\Sigma_e)\right\}\right]$$

$$- \frac{a_1(u)\cos\theta}{2d_2\bar{\phi}_2}\int_a^{2b-a}\cos\beta' x'\int_0^1 e^{-\left[\Sigma_e(2b-a)-\Sigma_e x'\right]/t}\,dt$$

$$+ \frac{a_1(u)\sin\theta}{2d_2\bar{\phi}_2}\int_a^{2b-a}\sin\beta' x'\int_0^1 e^{-\left[\Sigma_e(2b-a)-\Sigma_e x'\right]/t}\,dt$$

$$- \frac{a_1(u)\cos\theta}{2d_2\bar{\phi}_2}\int_a^{2b-a}\cos\beta' x'\int_0^1 e^{-\left[\Sigma_e x'-\Sigma_e a\right]/t}\,dt$$

$$+ \frac{a_1(u)\sin\theta}{2d_2\bar{\phi}_2}\int_0^{2b-a}\sin\beta' x'\int_0^1 e^{-\left[\Sigma_e x'-\Sigma_e a\right]/t}\,dt$$

$$+ \sum_{n=1}^{\infty}\frac{a_1(u)}{2d_2\bar{\phi}_2}\left\{\cos\theta\int_a^{2b-a}\cos\beta' x'\int_0^1 e^{-\left[nd_1\Sigma_1+nd_2\Sigma_e+a\Sigma_e-\Sigma_e x'\right]/t}\,dt\right.$$

$$- \sin\theta\int_a^{2b-a}\sin\beta' x'\int_0^1 e^{-\left[nd_1\Sigma_1+nd_2\Sigma_e+a\Sigma_e-\Sigma_e x'\right]/t}\,dt$$

$$- \cos\theta\int_a^{2b-a}\cos\beta' x'\int_0^1 e^{-\left[nd_1\Sigma_1+(n+1)d_2\Sigma_e+a\Sigma_e-\Sigma_e x'\right]/t}\,dt$$

$$\left.+ \sin\theta\int_a^{2b-a}\sin\beta' x'\int_0^1 e^{-\left[nd_1\Sigma_1+(n+1)d_2\Sigma_e+a\Sigma_e-\Sigma_e x'\right]/t}\,dt\right\} \qquad \text{(E-9)}$$

$$+ \cos\Theta \int_a^{2b-a} \cos\beta' x' \int_0^1 e^{-\left[nd_1\Sigma_1+(n-1)d_2\Sigma_e-a\Sigma_e+\Sigma_e x'\right]/t} dt$$

$$- \sin\Theta \int_a^{2b-a} \sin\beta' x' \int_0^1 e^{-\left[nd_1\Sigma_1+(n-1)d_2\Sigma_e-a\Sigma_e+\Sigma_e x'\right]/t} dt$$

$$- \cos\Theta \int_a^{2b-a} \cos\beta' x' \int_0^1 e^{-\left[nd_1\Sigma_1+nd_2\Sigma_e-a\Sigma_e+\Sigma_e x'\right]/t} dt$$

$$+ \sin\Theta \int_a^{2b-a} \sin\beta' x' \int_0^1 e^{-\left[nd_1\Sigma_1+nd_2\Sigma_e-a\Sigma_e+\Sigma_e x'\right]/t} dt \Bigg\} \qquad \text{(E-9) (Cont'd)}$$

In Equation (E-9) we have integrations of the following types:

$$\int_a^{2b-a} \cos\beta' x' \int_0^1 e^{-\left[K+\Sigma_e x'\right]/t} dt$$

$$= \beta' \sin\beta'(2b-a) \int_0^1 \frac{e^{-\left[K+\Sigma_e(2b-a)\right]/t} t^2 dt}{\Sigma_e^2+\beta'^2 t^2} - \beta' \sin\beta' a \int_0^1 \frac{e^{-\left[K+\Sigma_e a\right]/t} t^2 dt}{\Sigma_e^2+\beta'^2 t^2}$$

$$+ \Sigma_e \cos\beta' a \int_0^1 \frac{e^{-\left[K+\Sigma_e a\right]/t} t\, dt}{\Sigma_e^2+\beta'^2 t^2} - \Sigma_e \cos\beta'(2b-a) \int_0^1 \frac{e^{-\left[K+(2b-a)\Sigma_e\right]/t} t\, dt}{\Sigma_e^2+\beta'^2 t^2} \qquad \text{(E-10)}$$

and

$$\int_a^{2b-a} \sin\beta' x' \int_0^1 e^{-\left[K+\Sigma_e x'\right]/t} dt = \int_0^1 dt\, e^{-K/t} \int_a^{2b-a} \sin\beta' x'\, e^{-\Sigma_e x'/t} dx'$$

$$= \int_0^1 e^{-K/t} \left[\frac{e^{-\Sigma_e(2b-a)/t} t^2}{\Sigma_e^2+\beta'^2 t^2} \left\{ \left(-\frac{\Sigma_e}{t}\right) \sin\beta'(2b-a) - \beta' \cos\beta'(2b-a) \right\} \right.$$

$$\left. - \frac{e^{-\Sigma_e a/t} t^2}{\Sigma_e^2+\beta'^2 t^2} \left\{ \left(-\frac{\Sigma_e}{t}\right) \sin\beta' a - \beta' \cos\beta' a \right\} \right] dt \qquad \text{(E-11)}$$

$$= \Sigma_e \sin \beta' a \int_0^1 \frac{e^{-\left[K+\Sigma_e a\right]/t} t\,dt}{\Sigma_e^2+\beta'^2 t^2} - \Sigma_e \sin \beta'(2b-a) \int_0^1 \frac{e^{-\left[K+(2b-a)\Sigma_e\right]/t} t\,dt}{\Sigma_e^2+\beta'^2 t^2}$$

$$+ \beta' \cos \beta' a \int_0^1 \frac{e^{-\left[K+\Sigma_e a\right]/t} t^2 dt}{\Sigma_e^2+\beta'^2 t^2}$$

$$- \beta' \cos \beta'(2b-a) \int_0^1 \frac{e^{-\left[K+(2b-a)\Sigma_1\right]/t} t^2 dt}{\Sigma_e^2+\beta'^2 t^2} \qquad \text{(E-11)}$$

(Cont'd)

Evaluating all the integrals of Equation (E-9) we finally get

$$P_2(u) = 1 - \frac{a_0(u)}{2d_2\Sigma_e\bar{\phi}_2}\left[0.5 - \sum_{n=0}^{\infty}\left\{E_3(\ell_1'(n)) - 2E_3(\ell'_2(n)) + E_3(\ell_3'(n))\right\}\right]$$

$$- \frac{a_1(u)\cos\Theta}{2d_2\bar{\phi}_2}\left[\frac{\Sigma_e}{2\beta'^2}\log_e\left(1+\frac{\beta'^2}{\Sigma_e^2}\right)(\cos\beta'(2b-a)+\cos\beta' a)\right.$$

$$\left. + \frac{1}{\beta'}\left(1 - \frac{\Sigma_e}{\beta'}\tan^{-1}\frac{\beta'}{\Sigma_e}\right)(\sin\beta'(2b-a)-\sin\beta' a)\right]$$

$$+ \frac{a_1(u)\sin\Theta}{2d_2\bar{\phi}_2}\left[\frac{\Sigma_e}{2\beta'^2}\log_e\left(1+\frac{\beta'^2}{\Sigma_e^2}\right)(\sin\beta'(2b-a)+\sin\beta' a)\right. \qquad \text{(E-12)}$$

$$+\frac{1}{\beta'}\left(1-\frac{\Sigma_2}{\beta'}\tan^{-1}\frac{\beta'}{\Sigma_2}\right)(\cos\beta'(2b-a)+\cos\beta'a)\Bigg]$$

$$+\sum_{n=0}^{\infty}\frac{a_1(u)\cos\theta}{2d_2\phi_2}\Bigg[\Sigma_2\{\cos\beta'(2b-a)+\cos\beta'a\}\{\Pi_1(n)-2\Pi_2(n)+\Pi_3(n)\}$$

$$+\beta'\{\sin\beta'(2b-a)-\sin\beta'a\}\{\Pi_1'(n)-\Pi_3'(n)\}\Bigg]$$

$$-\sum_{n=0}^{\infty}\frac{a_1(u)\sin\theta}{2d_2\phi_2}\Bigg[\Sigma_2\{\sin\beta'(2b-a)+\sin\beta'a\}\{\Pi_1(n)-2\Pi_2(n)+\Pi_3(n)\}$$

$$-\beta'\{\cos\beta'(2b-a)-\cos\beta'a\}\{\Pi_1'(n)-\Pi_3'(n)\}\Bigg] \qquad \text{(E-12) (Cont'd)}$$

where

$$\Pi_i(n) = \int_0^1 \frac{e^{-\ell_i'(n)/t}t\,dt}{\Sigma_2^2+\beta'^2t^2}, \qquad i = 1,2,3 \qquad \text{(E-13)}$$

$$\Pi_i' = \int_0^1 \frac{e^{-\ell_i'(n)/t}t^2\,dt}{\Sigma_2^2+\beta'^2t^2} \qquad \text{(E-14)}$$

$$\begin{aligned}\ell_1'(n) &= (n+1)d_1\Sigma_1+nd_2\Sigma_2\\ \ell_2'(n) &= (n+1)(d_1\Sigma_1+\Sigma_2d_2)\\ \ell_3'(n) &= nd_1\Sigma_1+(n+1)d_2\Sigma_2\end{aligned} \qquad \text{(E-15)}$$

APPENDIX F

COMPUTER PROGRAMS

The computer programs used in obtaining the tables of results given in the text are listed in the following pages. The programs, written in MAD,[42] are for the IBM-7090 computer. They are written mainly for checking the calculations used in this thesis. A brief description of the input and output variables are given below:

Program No. 1

This program calculates the average fluxes in the fuel and the moderator lumps, the disadvantage factor and the thermal utilization of a rectangular cell with a cylindrical fuel rod at the center of the cell.

Input

HK,HL,HM,HN = the highest value of k,l,m,n respectively

SA = half-width of the cell (in cm) along the x-axis (a)

SB = half-width of the cell (in cm) along the y-axis (b)

SOR2 = source density in the moderator (S_2)

SA1 = macroscopic absorption cross section of the fuel element (Σ_{a1}) for thermal neutrons

SA2 = macroscopic absorption cross section of the moderator material (Σ_{a2}) for thermal neutrons

D1 = diffusion coefficient of thermal neutrons in the fuel element (D_1)

D2 = diffusion coefficient of thermal neutrons in the moderator (D_2)

DF = an arbitrary constant by which each term of the matrix are divided so that the determinant does not become too large for the machine

Output

AFLUX1 = average flux in the fuel rod ($\overline{\phi}_1$)

AFLUX2 = average flux in the moderator ($\overline{\phi}_2$)

RATIO = the disadvantage factor (d)

THERU = the thermal utilization (f)

PROGRAM NO. 1

```
$ COMPILE MAD,EXECUTE,PRINT OBJECT,PUNCH OBJECT,DUMP
          DIMENSION A(6650,ADIM),DEL(6650,DIM),PHI(100,PDIM),
         1 PHIF(100,PDIM),PHIM(100,PDIM)
          VECTOR VALUES PDIM = 1,1
          VECTOR VALUES ADIM = 2,1,0
          VECTOR VALUES DIM =4,0,0,0,0
START     READ DATA
          PRINT RESULTS HM,HN,HK,HL,SA,SB,SA1,SA2,D1,D2,R,DF,SOR2
          LC = HM+HN+HM*HN+2
          LR = (HK+1)*(HL+1)
          ADIM(2) = LC
          DIM(1) = (HK+1)*(HN+1)*(HL+1)+(HN+1)*(HL+1)+HL+2
          DIM(2) = HK+1
          DIM(3) =HN+1
          DIM(4)= HL+1
          PI = 3.142592
          RP = PI*R
          D= D1-D2
          S= SA1-SA2
         R
         R CALCULATION OF FIRST AND LAST COLUMN OF THE MATRIX
         R
          PRINT COMMENT $0 FIRST AND LAST COLUMN OF THE MATRIX$
          A(1,1) = (S*RP*R/4.0 + SA2*SA*SB)/DF
          A(1,LC) = (4.0*SA*SB-RP*R)*SOR2/DF
          PRINT RESULTS A(1,1),A(1,LC)
          I = 1
          THROUGH LOOP9,FOR L=1,1,L.G.HL
          I = I+1
          Z9 = L/SB
          X9 = RP*Z9
          Y = BSL1.(X9,1,1,B9,0)
          WHENEVER Y.E.2.0,TRANSFER TO COMM
          A(I,1) = (S*R*B9/(Z9*2.0))/DF
          A(I,LC) = (-2.0*R*B9/Z9)*SOR2/DF
LOOP9     PRINT RESULTS A(I,1),A(I,LC)
          I = HL+1
          THROUGH LOOP1,FOR K=1,1, K.G.HK
          THROUGH LOOP1,FOR L=0,1, L.G.HL
          I = I+1
          Z = SQRT.((K/SA).P.2+ (L/SB).P.2)
          X = RP*Z
          Y = BSL1.(X,1,1,B,0)
          WHENEVER Y.E.2.0,TRANSFER TO COMM
          A(I,1) = (S*R*B/(Z*2))/DF
          A(I,LC) = (-2*R*B/Z)*SOR2/DF
LOOP1     PRINT RESULTS A(I,1),A(I,LC)
          I = 0
          THROUGH LOOP2, FOR K=0,1, K.G.HK
          THROUGH LOOP2, FOR L=0,1, L.G.HL
          I = I+1
          J = 1
          THROUGH LOOP3, FOR N=1,1, N.G.HN
```

```
            J = J+1
            WHENEVER N.E.L .AND. K.E.0
            DEL(0,K,N,L) = SA*SB*2
            OTHERWISE
            DEL(0,K,N,L) =0
            END OF CONDITIONAL
            Z1 = SQRT.((K/SA).P.2 + ((N+L)/SB).P.2)
            X1 =RP*Z1
            Y = BSL1.(X1,1,1,B1,0)
            WHENEVER Y.E.2.0,TRANSFER TO COMM
            Z2 = SQRT.((K/SA).P.2 +((N-L)/SB).P.2)
            X2 = RP*Z2
            Y = BSL1.(X2,1,1,B2,0)
            WHENEVER Y.E.2.0,TRANSFER TO COMM
            A(I,J) = (((N*PI/SB).P.2*(D*R*(B1/Z1+B2/Z2)+D2*DEL(0,K,N,L))
           1 +N*PI/SB*(-D)*R*PI/SB*((N+L)*B1/Z1+(N-L)*B2/Z2)+S*R*(B1/Z1+
           2 B2/Z2)+SA2*DEL(0,K,N,L))/2)/DF
LOOP3       CONTINUE
LOOP2       PRINT RESULTS A(I,2)...A(I,HN+1)
            I = 0
            THROUGH LOOP4, FOR K=0,1, K.G.HK
            THROUGH LOOP4, FOR L=0,1, L.G.HL
            I = I+1
            J = HN+1
            THROUGH LOOP5,FOR M=1,1, M.G.HM
            J = J+1
            WHENEVER L.E.0 .AND. M.E.K
            DEL(M,K,0,L) =SA*SB*2
            OTHERWISE
            DEL(M,K,0,L) =0
            END OF CONDITIONAL
            Z3 = SQRT.(((M+K)/SA).P.2 + (L/SB).P.2)
            X3 = RP*Z3
            Y = BSL1.(X3,1,1,B3,0)
            WHENEVER Y.E.2.0,TRANSFER TO COMM
            Z4 = SQRT.(((M-K)/SA).P.2 + (L/SB).P.2)
            X4 = RP*Z4
            Y = BSL1.(X4,1,1,B4,0)
            WHENEVER Y.E.2.0,TRANSFER TO COMM
            A(I,J) = (((M*PI/SA).P.2.0*(D*R*(B3/Z3+B4/Z4)+D2*DEL(M,K,0,L)
           1 )+ (M*PI/SA)*(-D)*R*PI/SA*((M+K)*B3/Z3+(M-K)*B4/Z4)+S*R*(B3/
           2 Z3+B4/Z4)+ SA2*DEL(M,K,0,L))/2)/DF
LOOP5       CONTINUE
LOOP4       PRINT RESULTS A(I,HN+2)...A(I,HN+HM+1)
            I = 0
            THROUGH LOOP6,FOR K=0,1, K.G.HK
            THROUGH LOOP6, FOR L=0,1, L.G.HL
            I = I+1
            J = HM+HN+1
            THROUGH LOOP7,FOR M=1,1, M.G.HM
            THROUGH LOOP7,FOR N=1,1, N.G.HN
            J = J+1
            WHENEVER L.E.N .AND. K.E.M
            DEL(M,K,N,L) = SA*SB
            OTHERWISE
            DEL(M,K,N,L) = 0
            END OF CONDITIONAL
            Z5 = SQRT.(((M+K)/SA).P.2 +((N+L)/SB).P.2)
            X5 = RP*Z5
```

```
          Y = BSL1.(X5,1,1,B5,0)
          WHENEVER Y.E.2.0,TRANSFER TO COMM
          Z6 = SQRT.(((M+K)/SA).P.2 +((N-L)/SB).P.2)
          X6 = RP*Z6
          Y = BSL1.(X6,1,1,B6,0)
          WHENEVER Y.E.2.0,TRANSFER TO COMM
          Z7 = SQRT.(((M-K)/SA).P.2 +((N+L)/SB).P.2)
          X7 = RP*Z7
          Y = BSL1.(X7,1,1,B7,0)
          WHENEVER Y.E.2.0,TRANSFER TO COMM
          Z8 = SQRT.(((M-K)/SA).P.2 +((N-L)/SB).P.2)
          X8 = RP*Z8
          Y = BSL1.(X8,1,1,B8,0)
          WHENEVER Y.E.2.0,TRANSFER TO COMM
          A(I,J) = (((M*PI/SA).P.2.0 +(N*PI/SB).P.2.0)*(D*R/2.0*(B5/Z5
         1 +B6/Z6+B7/Z7+B8/Z8)+D2*DEL(M,K,N,L))+ M*PI/SA*(-D)*PI/SA*R/
         2 2.0*((M+K)*(B5/Z5+B6/Z6)+(M-K)*(B7/Z7+B8/Z8)) + N*PI/SB*
         3 (-D)*PI/SB*R/2.0*((N+L)*(B5/Z5+B7/Z7)+(N-L)*(B6/Z6+B8/Z8))+
         4 S*R/2.0*(B5/Z5+B6/Z6+B7/Z7+B8/Z8) + SA2*DEL(M,K,N,L))/DF
LOOP7     CONTINUE
LOOP6     PRINT RESULTS A(I,HM+HN+2)...A(I,LC-1)
          PRINT COMMENT $0 THE MATRIX$
          PRINT RESULTS A(1,1)...A(LR,LC)
         R
         R CALL FOR GJRDT  SUBROUTINE
         R
          PRINT COMMENT $0 CALCULATION OF COEFFIENTS$
          G = GJRDT.(LR,LC,A(1,1),DE)
          WHENEVER G .E. 0.0,TRANSFER TO OUT
          PRINT RESULTS DE
          THROUGH LOOP8, FOR I=1,1,I .G. LR
LOOP8     PRINT RESULTS A(I,LC)
          PRINT COMMENT $0 AVERAGE FLUX IN FUEL$
          I = 1
          PHIF(I) = A(I,LC)/4
          THROUGH LP1, FOR N=1,1, N.G.HN
          I = I+1
          S1 = N/SB
          T1 = RP*S1
          Y = BSL1.(T1,1,1,W1,0)
          WHENEVER Y.E.2.0,TRANSFER TO COMM
LP1       PHIF(I) = PHIF(I-1)+ A(I,LC)/RP*W1/S1
          I = HN+1
          THROUGH LP2, FOR M=1,1, M.G.HM
          I = I+1
          S2 = M/SA
          T2 = RP*S2
          Y = BSL1.(T2,1,1,W2,0)
          WHENEVER Y.E.2.0,TRANSFER TO COMM
LP2       PHIF(I) = PHIF(I-1) + A(I,LC)/RP*W2/S2
          I = HM+HN+1
          THROUGH LP3, FOR M=1,1, M.G.HM
          THROUGH LP3, FOR N=1,1, N.G.HN
          I = I+1
          S3 = SQRT.((M/SA).P.2 + (N/SB).P.2)
          T3 = RP*S3
          Y = BSL1.(T3,1,1,W3,0)
          WHENEVER Y.E.2.0,TRANSFER TO COMM
LP3       PHIF(I) = PHIF(I-1) + 2*A(I,LC)/RP*W3/S3
```

```
            AFLUX1= PHIF(I)
            PRINT RESULTS AFLUX1
            PRINT COMMENT $0 AVERAGE FLUX IN MODERATORS$
            F = 4.0*SA*SB - PI*R*R
            I = 1
            PHIM(I) = A(I,LC)/4
            THROUGH LP4, FOR N=1,1, N.G.HN
            I = I+1
            S4 = N/SB
            T4 = RP*S4
            Y = BSL1.(T4,1,1,W4,0)
            WHENEVER Y.E.2.0,TRANSFER TO COMM
LP4         PHIM(I) = PHIM(I-1) -R*A(I,LC)/F*W4/S4
            I = HN+1
            THROUGH LP5, FOR M=1,1, M.G.HM
            I = I+1
            S5 = M/SA
            T5 = RP*S5
            Y = BSL1.(T5,1,1,W5,0)
            WHENEVER Y.E.2.0,TRANSFER TO COMM
LP5         PHIM(I) = PHIM(I-1) -R*A(I,LC)/F*W5/S5
            I = HM+HN+1
            THROUGH LP6, FOR M=1,1, M.G.HM
            THROUGH LP6, FOR N=1,1, N.G.HN
            I = I+1
            S6 = SQRT.((M/SA).P.2 + (N/SB).P.2)
            T6 = RP*S6
            Y = BSL1.(T6,1,1,W6,0)
            WHENEVER Y.E.2.0,TRANSFER TO COMM
LP6         PHIM(I) = PHIM(I-1) -2.0*R*A(I,LC)/F*W6/S6
            AFLUX2= PHIM(I)
            PRINT RESULTS AFLUX2
            RATIO = AFLUX2/AFLUX1
            INVDIS = 1+(4*SA*SB-PI*R*R)*SA2/(PI*R*R*SA1)*RATIO
            THERU = 1/INVDIS
            PRINT RESULTS RATIO,THERU
            INTEGER HM,HN,HK,HL,LC,LR,I,K,L,J,N,M
            TRANSFER TO START
COMM        PRINT COMMENT $0 ARGUMENT OF BSL1 TOO LARGE$
            TRANSFER TO START
OUT         PRINT COMMENT $0 ERROR IN GJRDT  OPERATIONS$
            TRANSFER TO START
            END OF PROGRAM
$DATA
```

Program No. 2

This program calculates the average fluxes in the fuel and the moderator lump, the disadvantage factor, the thermal utilization and flux at different points in the equivalent cylindrical cell.

Input

A = half-width (in cm) of the rectangular cell whose equivalent cell calculation is need, along the x-axis (a)

B = half-width (in cm) of the rectangular cell along the y-axis (b)

R1 = radium (in cm) of the fuel rod (ρ)

SA1 = macroscopic absorption cross section (Σ_{a1}) in the fuel for thermal neutrons

SA2 = macroscopic absorption cross section (Σ_{a2}) in the moderator for thermal neutrons

D1 = thermal diffusion coefficient (D_1) of the fuel material

D2 = thermal diffusion coefficient (D_2) of the moderator material

SEP = intervals of the radius at which the fluxes are calculated

Output

APHI1 = average flux in the fuel rod ($\bar{\phi}_1$)

APHI2 = average flux in the moderator lump ($\bar{\phi}_2$)

RATIO = the disadvantage factor (d)

THERU = thermal utilization (f)

```
$COMPILE MAD,EXECUTE,DUMP,PRINT OBJECT,I/O DUMP,PUNCH OBJECT
 START     READ DATA
           PRINT RESULTS A,B,R1,SA1,SA2,D1,D2,SEP
           K1 = SQRT.(SA1/D1)
           K2 = SQRT.(SA2/D2)
           R2 = 2*SQRT.(A*B/3.141592)
           Z1 = K1*R1
           Z3 = K2*R1
           Z4 = K2*R2
           PRINT RESULTS Z1,Z3,Z4
          R
          R  ZERO ORDER MODIFIED BESSEL FUNCTION OF FIRST KIND
          R
           L = BSL1.(Z1,2,0,B1,0)
           WHENEVER L .E. 2.0,TRANSFER TO OUT
           L = BSL1.(Z3,2,0,B2,0)
           WHENEVER L .E. 2.0,TRANSFER TO OUT
          R
          R  FIRST ORDER MODIFIED BESSEL FUNCTION OF FIRST KIND
          R
           L = BSL1.(Z1,2,1,B3,0)
           WHENEVER L .E. 2.0,TRANSFER TO OUT
           L = BSL1.(Z3,2,1,B4,0)
           WHENEVER L .E. 2.0,TRANSFER TO OUT
           L = BSL1.(Z4,2,1,B5,0)
           WHENEVER L .E. 2.0,TRANSFER TO OUT
          R
          R  ZERO ORDER MODIFIED BESSEL FUNCTION OF SECOND KIND
          R
           L = BSL1.(Z3,3,0,B6,0)
           WHENEVER L .E. 2.0,TRANSFER TO OUT
          R
          R  FIRST ORDER MODIFIED BESSEL FUNCTION OF SECOND KIND
          R
           L = BSL1.(Z3,3,1,B7,0)
           WHENEVER L .E. 2.0,TRANSFER TO OUT
           L = BSL1.(Z4,3,1,B8,0)
           WHENEVER L .E. 2.0,TRANSFER TO OUT
          R
          R CALCULATION OF Q/(SA2*A)
          R
           CONS = B1-D1*K1*B3*(B2*B8-B6*B5)/(D2*K2*(B4*B8-B7*B5))
           PRINT RESULTS CONS
          R
          R  CALCULATION OF C/A
          R
           CBYA = (B1-CONS)/(B2*B8+B6*B5)
           PRINT RESULTS CBYA
          R
          R  CALCULATION OF FLUX IN FUEL
          R
           PRINT COMMENT $0 FLUX IN THE FUEL$
           THROUGH LOOP1,FOR R=0,SEP, R .G. R1
```

```
          F = R*K1
          L = BSL1.(F,2,0,F1,0)
          WHENEVER L .E. 2.0,TRANSFER TO OUT
          PH1BYA = F1
LOOP1     PRINT RESULTS R,PH1BYA
         R
         R CALCULATION OF FLUX IN THE MODERATOR
         R
          PRINT COMMENT $0 FLUX IN THE MODERATOR$
          THROUGH LOOP2, FOR R =R1,SEP, R .G. R2
          M = R*K2
          L = BSL1.(M,2,0,M1,0)
          WHENEVER L .E. 2.0,TRANSFER TO OUT
          L = BSL1.(M,3,0,M2,0)
          WHENEVER L .E. 2.0,TRANSFER TO OUT
          PH2BYA =CBYA*(M1*B8+M2*B5)+CONS
LOOP2     PRINT RESULTS R,PH2BYA
          PRINT COMMENT $0 AVERAGE FLUXES$
          APHI1 = 2*B3/(K1*R1)
          APHI2 = 2*CBYA/(K2*(R2*R2-R1*R1))*(B8*(R2*B5-R1*B4) +B5*(R1*
         1 B7-R2*B8))+CONS
          PRINT COMMENT $0 DISADVANTAGE FACTOR$
          RATIO = APHI2/APHI1
          INDIS = 1+ R2*SA2/(R1*SA1)*RATIO
          THERU = 1/INDIS
          PRINT RESULTS APHI1,APHI2,RATIO,THERU
          TRANSFER TO START
OUT       PRINT COMMENT $0 ERROR IN BSL1 SUBROUTINE$
          TRANSFER TO START
          END  OF PROGRAM
$DATA
```

Program No. 3

This program calculates the roots of the determinant, Equation (4-8).

Input

SA = half-width of the fuel lump (a)

SB = half-width of the one-dimensional cell (b)

SIGS1 = macroscopic scattering cross section of the fuel material (Σ_{S1})

SIGS2 = macroscopic scattering cross section of the moderator (Σ_{S2})

MEU1 = the average cosine of the scattering angle per collision in the fuel in the Laboratory Coordinate System ($\bar{\mu}_1$)

MEU2 = the average cosine of the scattering angle per collision with the moderator atom in the Laboratory Coordinate System ($\bar{\mu}_2$)

CI1 = the average logarithmic energy decrement of neutrons per collision in the fuel element (ξ_1)

CI2 = the average logarithmic energy decrement of neutrons per collision in the moderator lump (ξ_2)

S1 = source density in the fuel region (S_1)

HN = the maximum value of n in the summation of the flux. (HN = 4 in this program)

Output

R(1),R(2), etc. = are the roots of the determinantal equation. R(1) + iR(2) is the first root, but R(2) = R(4) = ... = 0, i.e., the imaginary part of the root becomes zero, the roots appear with a negative sign.

PROGRAM NO. 3

```
$COMPILE MAD,EXECUTE,PRINT OBJECT,DUMP,PUNCH OBJECT
          DIMENSION A(100,ADIM),S(10),B(20),R(20)
          VECTOR VALUES ADIM = 2,0,0
READ      READ DATA
          PRINT COMMENT $0 INPUT DATA FOR NO ABSORPTION CASES$
          PRINT RESULTS SA,SB,SIGS1,SIGS2,MEU1,MEU2,CI1,CI2,S1,HN
          ADIM(1) = HN+2
          ADIM(2) = HN+1
          D1 = 1.0/(3*SIGS1*(1-MEU1))
          D2 = 1.0/(3*SIGS2*(1-MEU2))
          W1 = 3.0*(1-MEU1)/CI1
          W2 = 3.0*(1-MEU2)/CI2
          W = W1-W2
          ALP = SA/D1
          BETA = ALP+(SB-SA)/D2
          PI = 3.1415926
          T = PI*ALP/BETA
          A(0,0) = 0
          THROUGH LOOP1,FOR M=0,1, M.G.HN
          A(M,0) = 0
          THROUGH LOOP1,FOR N=1,1, N.G.HN
          WHENEVER M.E.N
          A(M,N) =PI*(N/BETA).P.2*(W*(SIN.((N+M)*T)/(M+N)+T)+W2*PI)
          OTHERWISE
          A(M,N) = PI*(N/BETA).P.2*W*SIN.((M+N)*T)/(M+N)
          END OF CONDITIONAL
          WHENEVER M.E.0
          S(M) = 2*S1*T/(PI*CI1*SIGS1)
          OTHERWISE
          S(M) = 2*S1*SIN.(M*T)/(PI*CI1*SIGS1*M)
          END OF CONDITIONAL
LOOP1     CONTINUE
          PRINT RESULTS A(0,0)...A(HN,HN)
          PRINT RESULTS S(0)...S(HN)
          WHENEVER HN.G.2, TRANSFER TO HN3
          B(0) = 1
          B(1) = 0
          B(2) = A(1,1) + A(2,2)
          B(3) = 0
          B(4) = A(1,1)*A(2,2) - A(2,1)*A(1,2)
          B(5) = 0
          Q = ZER2.(2,B(0),R(1))
          WHENEVER Q.E.2.0, TRANSFER TO OUT2
          WHENEVER Q.E.3.0, TRANSFER TO OUT3
          PRINT RESULTS R(1)...R(4)
          TRANSFER TO READ
HN3       WHENEVER HN.G.3,TRANSFER TO HN4
          B(0)=1
          B(1) = 0
          B(2) = A(1,1)+A(2,2)+A(3,3)
          B(3) = 0
          B(4) = A(1,1)*A(2,2)+A(1,1)*A(3,3)+A(2,2)*A(3,3)-A(3,2)*
         1 A(2,3)-A(1,2)*A(2,1)-A(1,3)*A(3,1)
```

```
          B(5) = 0
          B(6) = A(1,1)*(A(2,2)*A(3,3)-A(3,2)*A(2,3))-A(1,2)*(A(2,1)*
         1 A(3,3)-A(3,1)*A(2,3))+A(1,3)*(A(2,1)*A(3,2)-A(2,2)*A(3,1))
          B(7) = 0
          Q = ZER2.(3,B(0),R(1))
          WHENEVER Q.E.2.0,TRANSFER TO OUT2
          WHENEVER Q.E.3.0,TRANSFER TO OUT3
          PRINT RESULTS R(1)...R(6)
          TRANSFER TO READ
HN4       WHENEVER HN.G.4, TRANSFER TO HN5
          B(0) = 1
          B(1) = 0
          B(2) = A(1,1) + A(2,2) + A(3,3) + A(4,4)
          B(3) = 0
          B(4) = A(1,1)*(A(2,2) +A(3,3) +A(4,4)) +A(2,2)*(A(3,3)+
         1 A(4,4)) +A(3,3)*A(4,4) -A(2,3)*A(3,2)- A(2,4)*A(4,2)-A(3,4)*
         2 A(4,3)-A(1,2)*A(2,1)-A(1,3)*A(3,1)-A(1,4)*A(4,1)
          B(5) = 0
          B(6) = A(1,1)*(A(2,2)*A(3,3) +A(2,2)*A(4,4)+A(3,3)*A(4,4)-
         1 A(2,3)*A(3,2) -A(2,4)*A(4,2)-A(3,4)*A(4,3)) +A(2,2)*(A(3,3)*
         2 A(4,4)-A(3,4)*A(4,3)) -A(2,3)*(A(3,2)*A(4,4)-A(3,4)*A(4,2))+
         3 A(2,4)*(A(3,2)*A(4,3) -A(4,2)*A(3,3)) -A(1,2)*(A(2,1)*A(3,3)
         4 +A(2,1)*A(4,4)-A(2,3)*A(3,1) -A(2,4)*A(4,1))+A(1,3)*(A(2,1)*
         5 A(3,2)+A(3,4)*A(4,1)-A(2,2)*A(3,1)-A(3,1)*A(4,4)) -A(1,4)*
         6 (A(2,2)*A(4,1)-A(3,1)*A(4,3)+A(4,1)*A(3,3)-A(2,1)*A(4,2))
          B(7) = 0
          B(8) =A(1,1)*(A(2,2)*(A(3,3)*A(4,4)-A(3,4)*A(4,3)) -A(2,3)*(
         1 A(3,2)*A(4,4)-A(3,4)*A(4,2)) +A(2,4)*(A(3,2)*A(4,3)-A(4,2)*
         2 A(3,3))) -A(1,2)*(A(2,1)*(A(3,3)*A(4,4)-A(3,4)*A(4,3)) -
         3 A(2,3)*(A(3,1)*A(4,4)-A(3,4)*A(4,1)) +A(2,4)*(A(3,1)*A(4,3)-
         4 A(4,1)*A(3,3))) +A(1,3)*(A(2,1)*(A(3,2)*A(4,4)-A(3,4)*A(4,2)
         5 ) -A(2,2)*(A(3,1)*A(4,4)-A(3,4)*A(4,1)) +A(2,4)*(A(3,1)*
         6 A(4,2)-A(3,2)*A(4,1))) -A(1,4)*(A(2,1)*(A(3,2)*A(4,3)-A(4,2)
         7 *A(3,3)) -A(2,2)*(A(3,1)*A(4,3)-A(4,1)*A(3,3)) +A(2,3)*(
         8 A(3,1)*A(4,1)-A(3,2)*A(4,1)))
          B(9) = 0
          Q = ZER2.(4,B(0),R(1))
          WHENEVER Q.E.2.0, TRANSFER TO OUT2
          WHENEVER Q.E.3.0, TRANSFER TO OUT3
          PRINT RESULTS R(1)...R(8)
HN5       TRANSFER TO READ
OUT2      PRINT COMMENT $0 ARGUMENTS OF ZER2 ARE OUT OF RANGE$
          TRANSFER TO READ
OUT3      PRINT COMMENT $0 IMPOSSIBLE TO LOCATE THE ROOTS$
          TRANSFER TO READ
          INTEGER M,N ,HN
          END OF PROGRAM
$DATA
```

Program No. 4

This program computes the resonance interference between a pair of resonances in a slab lattice. The fuel lump contains U^{238}, Th^{232} and O^{16} but the moderator may contain any monoatomic moderator.

Input

SA = half-width of the fuel slab (in cm) (a)

SB = half-width of the cell (in cm) (b)

NU = number of U^{238} atoms in fuel/barn-cm

NTH = number of Th^{232} atoms in fuel/barn-cm

NOX = number of oxygen atoms/barn-cm

SIG2 = total macroscopic cross section of the moderator (Σ_2)

MEU2 = the average cosine of the scattering angle per collision with the moderator atom in the Laboratory Coordinate System ($\bar{\mu}_M$)

CI2 = the average energy decreasement per collision (ξ_2)

SORS = source density at higher energy

SPINJ = channel spin (J)

SPINI = nuclear spin (I)

ALPM = $\alpha_M = \left(\frac{A_M-1}{A_M+1}\right)^2$

EFR = energy of the first (higher energy) resonance (E_o)

GNF = the neutron width (in ev) at the first resonance (Γ_n)

GGF = the radiation width (in ev) at the first resonance (Γ_r)

IMF = $\begin{cases} 0, & \text{NR approximation for the first resonance} \\ 1, & \text{IM approximation for the first resonance} \end{cases}$

SURAN = { 0, when the first resonance is due to Th^{232}
        { 1, when the first resonance is due to U^{238}

HM,HN = the highest value of the m,n in the expansion of the flux, here HM = HN = 2 will do

NFR = one half the number of divisions used in integrating over the first resonance (the range of integration is six times the practical width)

ESR = energy at the peak of the second (lower energy) resonance

SURAN = { 0, when the second resonance is due to Th^{232}
        { 1, when the second resonance is due to U^{238}

IMS = { 0, when NR approximation is used for the second resonance
      { 1, when IM approximation is used for the second resonance

GNS = the neutron width (in ev) at the second resonance (Γ_n)

GGS = the radiation width (in ev) at the second resonance (Γ_r)

NSR = one half the number of divisions used in integrating over the second resonance (range of integration is six times the practical width)

Output

GAMP = the practical width (in ev) of the first resonance (Γ_{prac})

FRI = resonance integral of the first resonance (I)

GAMPS = the practical width (in ev) of the second resonance (Γ_{prac})

UNIRI = the uninterfered second resonance integral, (I) unint

DELU = the lethargy difference between the first and second resonance, $\Delta u = u_r - u_o$

RI = the interfered second resonance integral $(I)_{int}$

$$PCERR = \frac{(I)_{int} - (I)_{unint}}{(I)_{int}} \times 100$$

$$PCINA = \frac{\rho_2' - \rho_1'}{\rho_1'}$$

PROGRAM NO. 4

```
$COMPILE MAD, EXECUTE, PRINT OBJECT, DUMP, PUNCH OBJECT
          DIMENSION A(81,ADIM),S(10),AS(10),XT(282),YT(282),F(250),
         1 FN(250),IN(100),JN(100)
          EXECUTE FTRAP.
          VECTOR VALUES ADIM = 2,0,0
          READ DATA
          SIGPU = 10.75
          SIGPTH = 12.0
          SIGPO = 3.8
          MEUU = 0.0028
          MEUTH = 0.0029
          MEUO=0.0317
          CIU = 0.0084
          CITH = 0.0086
          CIOX = 0.12
 START    READ DATA
          PRINT RESULTS SA, SB, NU, NTH, NOX, SIG2,MEU2,CI2,SORS,
         2 EFR, NFR, SPINJ, SPINI, GNF, GGF,    IMF, FURAN, HM,HN,
         3 ESR, NSR, GNS, GGS,ALPM, SURAN,IMS
          ADIM(1) =HM+2
          ADIM(2) = HM+1
          SIGSU = NU*SIGPU
          SIGSTH = NTH*SIGPTH
          SIGSO = NOX*SIGPO
          SIGS1 = SIGSU+SIGSTH+SIGSO
          MEU1 =(MEUU*SIGSU + MEUTH*SIGSTH + MEUO*SIGSO)/SIGS1
          D1 = 1.0/(3*SIGS1*(1-MEU1))
          D2 = 1.0/(3*SIG2*(1-MEU2))
          CI1 = (CIU*SIGSU+CITH*SIGSTH+CIOX*SIGSO)/SIGS1
          W1 = 3.0*(1-MEU1)/CI1
          W2 = 3.0*(1-MEU2)/CI2
          W = W1-W2
          ALPH = SA/D1
          BETA = ALPH + (SB-SA)/D2
          PI = 3.1415926
          PIT = PI*CI2*SIG2
          PIC = PI*CI1*SIGS1
          T = PI*ALPH/BETA
          TM = 2*(SB-SA)
          TF = 2*SA
          THROUGH LOOP, FOR M=0,1, M.G.HM
          A(M,0) =0
          WHENEVER M.E.0
          S(M) = 2*SORS*T*(1/PIC-1/PIT) + 2*SORS/(CI2*SIG2)
          OTHERWISE
          S(M) = 2*SORS*SIN.(M*T)/M*(1/PIC-1/PIT)
          END OF CONDITIONAL
          THROUGH LOOP, FOR N=1,1, N.G.HM
          WHENEVER M.E.N
          A(M,N) = PI*(N/BETA).P.2*(W*(SIN.((M+N)*T)/(M+N)+T)+W2*PI)
          OTHERWISE
          A(M,N) = PI*(N/BETA).P.2*W*SIN.((M+N)*T)/(M+N)
          END OF CONDITIONAL
```

```
LOOP     CONTINUE
         PRINT RESULTS A(0,0)...A(HM,HM)
         PRINT RESULTS S(0)...S(HM)
        R CALCULATION OF THE STRENGTH OF THE SINK
        R EFR IS THE ENERGY AT THE PEAK OF THE FIRST RESONANCE
         PHIASH = SORS*SB/(SA*CI1*SIGS1 + (SB-SA)*CI2*SIG2)
         PRINT RESULTS PHIASH
         G = (2*SPINJ+1)/(2*(2*SPINI+1))
         GTF = GNF+GGF
         LAMH = (4.55E-10)/SQRT.(EFR)*(1.0E12)
         C1 = 4*PI*LAMH*LAMH*GNF/GTF*G
         C2 = C1*GGF/GTF*SQRT.(EFR)
         C3 = C1*GNF/GTF
         PRINT RESULTS C1,C2,C3
         WHENEVER FURAN .E.1
         GAMP = GTF*SQRT.(C1/SIGPU)
         PRINT COMMENT $0FIRST RESONANCE ATEFR IS DUE TO URANIUM$
         OTHERWISE
         PRINT COMMENT $0FIRST RESONANCE AT EFR IS DUE TO THORIUM$
         GAMP = GTF*SQRT.(C1/SIGPTH)
         END OF CONDITIONAL
         FEL = 3.0*GAMP
         PRINT RESULTS GAMP ,FEL
         HE = EFR+FEL
         LE = EFR-FEL
         H =FEL/NFR
        R
        R2FEL IS ENERGY INTERVAL OF INTEGRATION OF FIRST RESONANCE
        R 2NFR IS NUMBER OF DIVISION USED IN INTEGRATING FIRST RESON
         WHENEVER IMF.E.1
         PRINT COMMENT $0 INFINITE MASS APPROX FOR FIRST RESONANCES$
         OTHERWISE
         PRINT COMMENT $0 NARROW RESONANCE APPROX FOR FIRST RESONANCES$
         END OF CONDITIONAL
         J = 0
         THROUGH LOOP1, FOR E=LE,H, E.G.HE
         X = 2*(E-EFR)/GTF
         RT = 1+X*X
         SIGA = C2/(SQRT.(E)*RT)
         SIGSR = C3/RT
         WHENEVER FURAN .E.1
         SIGSI = SQRT.(C3*G*SIGPU)*X/RT
         SIGT = NU*(SIGA+ SIGSR+ SIGSI+ SIGPU)
         SIGT1 = SIGT + SIGSTH + SIGSO
         TRANSFER TO KHAL
         OTHERWISE
         SIGSI = SQRT.(C3*G*SIGPTH)*X/RT
         SIGT = NTH*(SIGA+ SIGSR+ SIGSI+ SIGPTH)
         SIGT1 = SIGT+ SIGSU+ SIGSO
         END OF CONDITIONAL
        R CALCULATION OF FIRST COLLISION PROBABILITY IN FUEL P1
KHAL     Y =0
         THROUGH EDA ,    FOR N=0,1, N.G.HN
         L1 = (N+1)*TM*SIG2 + N*TF*SIGT1
         L2 = (N+1)*(TM*SIG2 + TF*SIGT1)
         L3 = N*TM*SIG2 + (N+1)*TF*SIGT1
         WHENEVER L1.G.10.0
         Y1 = 0
         OTHERWISE
```

```
          Y1 = TAB.(L1,XT,YT,1,1,3,281,SW)
          END OF CONDITIONAL
          WHENEVER SW.E.2.0,TRANSFER TO OUT1
          WHENEVER L2.G.10.0
          Y2 = 0
          OTHERWISE
          Y2 = TAB.(L2,XT,YT,1,1,3,281,SW)
          END OF CONDITIONAL
          WHENEVER SW.E.2.0,TRANSFER TO OUT1
          WHENEVER L3.G.10.0
          Y3 = 0
          OTHERWISE
          Y3 = TAB.(L3,XT,YT,1,1,3,281,SW)
          END OF CONDITIONAL
          WHENEVER SW.E.2.0,TRANSFER TO OUT1
EDA       Y = Y+Y1+Y3-2*Y2
          P1 = 1-(1-2*Y)/(2*SIGT1*TF)
          WHENEVER IMF.E.1
          F(J) = (SIGT1-(SIGT1-SIGSO)*P1)*SIGA/(E*(SIGT1-P1*(SIGSU+
         1 SIGSTH)))
          J = J+1
          OTHERWISE
          F(J) = SIGA/E - (SIGT1-SIGS1)*P1/(SIGT1*E)*SIGA
          J = J+1
          END OF CONDITIONAL
LOOP1     CONTINUE
          F1 = 0
          THROUGH JAHAN, FOR N=1,1, N.G.(J-1)/2
JAHAN     F1 = F1 + F(2*N-1)
          F2 = 0
          THROUGH ARA, FOR N=1,1, N.G.(J-3)/2
ARA       F2 = F2 + F(2*N)
         R CALCULATION OF THE STRENGTH OF THE SINK
          WHENEVER FURAN.E.1
          NF = NU
          OTHERWISE
          NF = NTH
          END OF CONDITIONAL
          FRI = H/3*(F(0) + 4*F1 + 2*F2 + F(J-1))
          PRINT RESULTS FRI
          PRINT COMMENT $0STRENGTH OF SINK AT HIGHER ENERGY RESONANCES
          SINK = PHIASH*NF*FRI
          PRINT RESULTS SINK
          THROUGH RAJA, FOR M=0,1, M.G.HM
          WHENEVER M.E.0
          AS(M)= 2*SINK*T*(1/PIC)
          OTHERWISE
          AS(M)= 2*SINK*SIN.(M*T)/M*(1/PIC)
          END OF CONDITIONAL
RAJA      CONTINUE
          PRINT RESULTS AS(0)...AS(HM)
         R
         R CALCULATION OF UNINTERFERRED SECOND RESONANCE
         R
          GTS = GNS + GGS
          LAML = (4.55E-10)*(1.0E+12)/SQRT.(ESR)
          CS1 = 4*PI*LAML*LAML*G*GNS/GTS
          CS2 = CS1*SQRT.(ESR)*GGS/GTS
         R
```

```
R ESR IS THE  ENERGY AT THE PEAK OF SECOND RESONANCE
R
 CS3 = CS1*GNS/GTS
 PRINT RESULTS CS1,CS2,CS3
 WHENEVER SURAN .E. 1
 PRINT COMMENT $0 SECOND RESONANCE AT ENERGY ESR IS URANIUM$
 GAMPS = GTS*SQRT.(CS1/SIGPU)
 OTHERWISE
 GAMPS = GTS*SQRT.(CS1/SIGPTH)
 PRINT COMMENT $0 SECOND RESONANCE IS DUE TO THORIUM$
 END OF CONDITIONAL
 SEL = 3.0*GAMPS
 PRINT RESULTS GAMPS,SEL
 HE = ESR + SEL
 LE = ESR-SEL
 H = SEL/NSR
R
R 2SEL IS THE ENERGY INTERVAL FOR THE INTEGRATION OF SECOND
R RESONANCE
R 2NSR IS THE NUMBER OF DIVISIONS USED IN INTEGRATING THE
R SECOND RESONANCE
R
 WHENEVER IMS.E.1
 PRINT COMMENT $0INFINITE MASS APPROX. FOR SECOND RESONANCES$
 OTHERWISE
 PRINT COMMENT $0NARROW RESONANCE APPROX FOR SECOND RESONANCES$
 END OF CONDITIONAL
 J = 0
 THROUGH LOOP2, FOR E=LE,H, E.G.HE
 X = 2*(E-ESR)/GTS
 RT = 1+X*X
 SIGA = CS2/(SQRT.(E)*RT)
 SIGSR = CS3/RT
 WHENEVER  SURAN.E.1
 SIGSI = SQRT.(CS3*G*SIGPU)*X/RT
 SIGT = NU*(SIGA + SIGSR + SIGSI + SIGPU)
 SIGT1 = SIGT + SIGSTH + SIGSO
 TRANSFER TO PR1
 OTHERWISE
 SIGSI = SQRT.(CS3*G*SIGPTH)*X/RT
 SIGT = NTH*(SIGA + SIGSR + SIGSI + SIGPTH)
 SIGT1 = SIGT + SIGSU + SIGSO
 END OF CONDITIONAL
R
R CALCULATION OF COLLISION PROBABILITY
PR1  Y = 0
 THROUGH YURA, FOR N=0,1, N.G.HN
 L1 = (N+1)*TM*SIG2 + N*TF*SIGT1
 L2 = (N+1)*(TM*SIG2 + TF*SIGT1)
 L3 = N*TM*SIG2 + (N+1)*TF*SIGT1
 WHENEVER L1.G.10.0
 Y1 = 0
 OTHERWISE
 Y1 = TAB.(L1, XT,YT,1,1,3,281,SW)
 END OF CONDITIONAL
 WHENEVER SW.E.2.0, TRANSFER TO OUT1
 WHENEVER L2.G.10.0
 Y2 = 0
 OTHERWISE
```

```
          Y2 = TAB.(L2, XT,YT,1,1,3,281,SW)
          END OF CONDITIONAL
          WHENEVER SW.E.2.0, TRANSFER TO OUT1
          WHENEVER L3.G.10.0
          Y3 = 0
          OTHERWISE
          Y3 = TAB.(L3, XT,YT,1,1,3,281,SW)
          END OF CONDITIONAL
          WHENEVER SW.E.2.0, TRANSFER TO OUT1
YURA      Y = Y+Y1+Y3-2*Y2
          P1 = 1-(1-2*Y)/(2*SIGT1*TF)
          PRINT RESULTS E,Y,P1
          WHENEVER IMS.E.1
          F(J) = (SIGT1-(SIGT1-SIGSO)*P1)*SIGA/(E*(SIGT1-P1*(SIGSU+
         1 SIGSTH)))
          J = J+1
          OTHERWISE
          F(J) = SIGA/E-(SIGT1-SIGS1)*P1/(SIGT1*E)*SIGA
          J = J+1
          END OF CONDITIONAL
LOOP2     CONTINUE
          F1 = 0
          THROUGH PIEM, FOR N=1,1, N.G.(J-1)/2
PIEM      F1 = F1+F(2*N-1)
          F2 = 0
          THROUGH SUWAN, FOR N=1,1, N.G.(J-3)/2
SUWAN     F2 = F2+F(2*N)
          PRINT COMMENT $0 UNINTERFERRED SECOND RESONANCE INTEGRAL$
          UNIRI = H/3*(F(0)+4*F1 + 2*F2 + F(J-1))
          PRINT RESULTS UNIRI
         R
         R INTERFERRED SECOND RESONANCE INTEGRAL
         R ES IS THE ENERGY OF SOURCE NEUTRONS
         R
          ALPOX = 15.0*15.0/(17.0*17.0)
          ALPU = 237.0*237.0/(239.0*239.0)
          ALPTH = 231.0*231.0/(233.0*233.0)
          ES = 2.00E6
          US = ELOG.(ES/ESR)
          UF = ELOG.(ES/EFR)
          DELU = US-UF
          PRINT RESULTS US,UF,DELU
          WHENEVER US.G.UF
          STEP = 1
          OTHERWISE
          STEP = 0
          PRINT COMMENT $0FIRST RESONANCE DO NOT HAVE HIGHER ENERGY$
          END OF CONDITIONAL
          CB = STEP*(AS(0)-A(0,1)*AS(1)/A(1,1))/2
          CD = STEP*(S(0)-A(0,1)*S(1)/A(1,1))/2
          PHIZ = CD-CB
          PHIP1 = -STEP*AS(1)*(A(0,1)/(2.0*A(1,1))+ SIN.(T)/T)
          PHIP2 =-STEP*AS(1)*(A(0,1)/(2.0*A(1,1)) - BETA/(PI*(BETA-
         1 ALPH))*SIN.(T))
          AZ = 2.0*PHIZ-STEP*A(0,1)*AS(1)/A(1,1)*EX
          EX = EXP.(-A(1,1)*DELU)
          A1 = -STEP*AS(1)*EX
          PHIB1 = PHIZ+PHIP1*EX
          PHIB2 = PHIZ+PHIP2*EX
```

```
      CON1 = AZ/(2*TM*PHIB2)
      CON2 = A1/(2*TM*PHIB2)*COS.(T-T*D1/D2)
      CON3 = A1/(2*TM*PHIB2)*SIN.(T-T*D1/D2)
      SBP = SIN.(BETAP*(2*SB-SA))
      CBP = COS.(BETAP*(2*SB-SA))
      CBA = COS.(BETAP*SA)
      SBA = SIN.(BETAP*SA)
      ALPHP = PI/(BETA*D1)
      BETAP = PI/(BETA*D2)
      J = 0
      THROUGH LOOP3, FOR E=LE,H, E.G.HE
      X = 2*(E-ESR)/GTS
      RT = 1+X*X
      SIGA = CS2/(SQRT.(E)*RT)
      SIGSR = CS3/RT
      WHENEVER SURAN.E.1
      SIGSI =SQRT.(CS3*G*SIGPU)*X/RT
      SIGT = NU*(SIGA+SIGSR+SIGSI+SIGPU)
      SIGT1 = SIGT+SIGSTH+SIGSO
      TRANSFER TO PR2
      OTHERWISE
      SIGSI = SQRT.(CS3*G*SIGPTH)*X/RT
      SIGT = NTH*(SIGA+SIGSR+SIGSI+SIGPTH)
      SIGT1 = SIGT+SIGSU+SIGSO
      END OF CONDITIONAL
     R
     R CALCULATION OF COLLISION PROBABILITIES
     R
PR2   PSI = 0
      PSIP = 0
      CPI = 0
      CPIP = 0
      YM = 0
      YF = 0
      THROUGH KENJI, FOR N=0,1, N.G.HN
      L1 = (N+1)*TM*SIG2 + N*TF*SIGT1
      L2 = (N+1)*(TM*SIG2+TF*SIGT1)
      L3 = N*TM*SIG2+(N+1)*TF*SIGT1
      LP1 = N*TM*SIG2 + (N+1)*TF*SIGT1
      LP2 = (N+1)*(TM*SIG2+TF*SIGT1)
      LP3 = (N+1)*TM*SIG2 + N*TF*SIGT1
      WHENEVER L1 .G. 10.0
      YF1 = 0
      OTHERWISE
      YF1 = TAB.(L1,XT,YT,1,1,3,281,SW)
      END OF CONDITIONAL
      WHENEVER SW.E.2.0, TRANSFER TO OUT1
      WHENEVER L2 .G. 10.0
      YF2 = 0
      OTHERWISE
      YF2 = TAB.(L2,XT,YT,1,1,3,281,SW)
      END OF CONDITIONAL
      WHENEVER SW.E.2.0, TRANSFER TO OUT1
      WHENEVER L3 .G. 10.0
      YF3 = 0
      OTHERWISE
      YF3 = TAB.(L3,XT,YT,1,1,3,281,SW)
      END OF CONDITIONAL
      WHENEVER SW.E.2.0, TRANSFER TO OUT1
```

```
            YF = YF+YF1+YF3-2*YF2
            WHENEVER LP1.G. 10.0
            YM1 = 0
            OTHERWISE
            YM1 =TAB.(LP1,XT,YT,1,1,3,281,SW)
            END OF CONDITIONAL
            WHENEVER SW.E.2.0, TRANSFER TO OUT1
            WHENEVER LP2.G. 10.0
            YM2 = 0
            OTHERWISE
            YM2 =TAB.(LP2,XT,YT,1,1,3,281,SW)
            END OF CONDITIONAL
            WHENEVER SW.E.2.0, TRANSFER TO OUT1
            WHENEVER LP3.G. 10.0
            YM3 = 0
            OTHERWISE
            YM3 =TAB.(LP3,XT,YT,1,1,3,281,SW)
            END OF CONDITIONAL
            WHENEVER SW.E.2.0, TRANSFER TO OUT1
            YM = YM + YM1 + YM3 - 2*YM2
           R
           R CALCULATIONS OF THE INTEGRALS PSI AND PSIPRIME
            T1 = L1/15.0
            WHENEVER T1.L.1.0 .AND. T1.G.0.5
            H1 = (1.0-T1)/20.0
            OR  WHENEVER T1.LE.0.5 .AND. T1.G.0.000001
            H1 = (1.0-T1)/40.0
            OTHERWISE
            PSI1 = 0
            PSIP1= 0
            TRANSFER TO  SI2
            END OF CONDITIONAL
            K = 0
            THROUGH RUSTAM, FOR ST = T1,H1, ST.G.1.0
            IN(K) = ST*EXP.(-L1/ST)/(SIGT1*SIGT1+(ALPHP*ST).P.2.0)
            JN(K) = IN(K)*ST
RUSTAM      K = K+1
            JJ = 0
            II = 0
            THROUGH ALI, FOR M=1,1, M.G.(K-1)/2
            II = II+IN(2*M-1)
ALI         JJ = JJ+JN(2*M-1)
            IK = 0
            JK =0
            THROUGH GOLAM, FOR M=1,1, M.G.(K-3)/2
            IK = IK+ IN(2*M)
GOLAM       JK = JK + JN(2*M)
            PSI1 = H1/3.0*(IN(0)+4*II+2*IK+IN(K-1))
            PSIP1 = H1/3.0*(JN(0)+4*JJ+2*JK+JN(K-1))
SI2         T2 = L2/15.0
            WHENEVER T2.L.1.0 .AND. T2.G.0.5
            H2 = (1.0-T2)/20.0
            OR  WHENEVER T2.LE.0.5 .AND. T2.G.0.000001
            H2 = (1.0-T2)/40.0
            OTHERWISE
            PSI2 = 0
            TRANSFER TO  SI3
            END OF CONDITIONAL
            K = 0
```

```
          THROUGH FAROOQ, FOR ST =T2,H2, ST.G.1.0
          IN(K) = ST*EXP.(-L2/ST)/(SIGT1*SIGT1+(ALPHP*ST).P.2.0)
FAROOQ    K = K+1
          II = 0
          JJ = 0
          THROUGH ANO, FOR M=1,1, M.G.(K-3)/2
ANO       II = II+IN(2*M-1)
          IK = 0
          JK =0
          THROUGH WARA, FOR M=1,1, M.G.(K-3)/2
WARA      IK = IK+ IN(2*M)
          PSI2 = H2/3.0*(IN(0)+4*II+2*IK+IN(K-1))
SI3       T3 = L3/15.0
          WHENEVER T3.L.1.0 .AND. T3.G.0.5
          H3 = (1.0-T3)/20.0
          OR  WHENEVER T3.LE.0.5 .AND. T3.G.0.000001
          H3 = (1.0-T3)/40.0
          OTHERWISE
          PSI3 = 0
          PSIP3= 0
          TRANSFER TO NAHID
          END OF CONDITIONAL
          K = 0
          THROUGH BEGUM, FOR ST=T3,H3,ST.G.1.0
          IN(K) = ST*EXP.(-L3/ST)/(SIGT1*SIGT1+(ALPHP*ST).P.2.0)
          JN(K) = IN(K)*ST
BEGUM     K = K+1
          II = 0
          JJ = 0
          THROUGH RABEYA, FOR M=1,1, M.G.(K-1)/2
          II = II + IN(2*M-1)
RABEYA    JJ = JJ+JN(2*M-1)
          IK = 0
          JK =0
          THROUGH MALEKA, FOR M=1,1, M.G.(K-3)/2
          IK = IK+ IN(2*M)
MALEKA    JK = JK + JN(2*M)
          PSI3 = H3/3.0*(IN(0)+4*II+2*IK+IN(K-1))
          PSIP3 = H3/3.0*(JN(0)+4*JJ+2*JK+JN(K-1))
NAHID     PSI = PSI + PSI1 + PSI3 - 2*PSI2
          PSIP = PSIP+PSIP1-PSIP3
         R
         R CALCULATIONS OF INTEGRALS CAP-PI AND CAP-PI PRIME
         R
          TP1 = LP1/15.0
          WHENEVER TP1.L.1.0 .AND. TP1.G.0.5
          HP1 = (1.0-TP1)/20.0
          OR  WHENEVER TP1.LE.0.5 .AND. TP1.G.0.00001
          HP1 = (1.0-TP1)/40.0
          OTHERWISE
          CPI1 = 0
          CPIP1 = 0
          TRANSFER TO PI2
          END OF CONDITIONAL
          K = 0
          THROUGH MATIOR, FOR ST=TP1,HP1, ST.G.1.0
          IN(K) = ST*EXP.(-LP1/ST)/(SIG2*SIG2+(BETAP*ST).P.2)
          JN(K) = IN(K)*ST
MATIOR    K = K+1
```

```
            THROUGH RAHMAN, FOR M=1,1, M.G.(K-1)/2
            II = II + IN(2*M-1)
RAHMAN      JJ = JJ + JN(2*M-1)
            IK = 0
            JK =0
            THROUGH KHALEQ, FOR M=1,1, M.G.(K-3)/2
            IK = IK+ IN(2*M)
KHALEQ      JK = JK + JN(2*M)
            CPI1 = HP1/3.0*(IN(0)+4*II+2*IK+IN(K-1))
            CPIP1 = HP1/3.0*(JN(0)+4*JJ+2*JK+JN(K-1))
PI2         TP2 = LP2/15.0
            WHENEVER TP2.L.1.0 .AND. TP2.G.0.5
            HP2 = (1.0-TP2)/20.0
            OR  WHENEVER TP2.LE.0.5 .AND. TP2.G.0.00001
            HP2 = (1.0-TP2)/40.0
            OTHERWISE
            CPI2 = 0
            TRANSFER TO PI3
            END OF CONDITIONAL
            K = 0
            THROUGH KERR, FOR ST=TP2,HP2, ST.G.1.0
            IN(K) = ST*EXP.(-LP2/ST)/(SIG2*SIG2+(BETAP*ST).P.2)
KERR        K = K+1
            II = 0
            THROUGH BOHR, FOR M=1,1, M.G.(K-1)/2
BOHR        II = II + IN(2*M-1)
            IK = 0
            THROUGH PAUL, FOR M=1,1, M.G.(K-3)/2
PAUL        IK = IK+ IN(2*M)
            CPI2 = HP2/3.0*(IN(0)+4*II+2*IK+IN(K-1))
PI3         TP3 = LP3/15.0
            WHENEVER TP3.L.1.0 .AND. TP3.G.0.5
            HP3 = (1.0-TP3)/20.0
            OR  WHENEVER TP3.LE.0.5 .AND. TP3.G.0.00001
            HP3 = (1.0-TP3)/40.0
            OTHERWISE
            CPI3 = 0
            CPIP3 = 0
            TRANSFER TO YIP
            END OF CONDITIONAL
            K = 0
            THROUGH ZWE, FOR ST=TP3,HP3, ST.G.1.0
            IN(K) = ST*EXP.(-LP3/ST)/(SIG2*SIG2+(BETAP*ST).P.2)
            JN(K) = ST*IN(K)
ZWE         K = K+1
            II = 0
            JJ = 0
            THROUGH IFEL, FOR M=1,1, M.G.(K-1)/2
            II = II + IN(2*M-1)
IFEL        JJ = JJ+JN(2*M-1)
            IK = 0
            JK =0
            THROUGH OSBORN,FOR M=1,1, M.G.(K-3)/2
            IK = IK+ IN(2*M)
OSBORN      JK = JK + JN(2*M)
            CPI3 = HP3/3.0*(IN(0)+4*II+2*IK+IN(K-1))
            CPIP3 = HP3/3.0*(JN(0)+4*JJ+2*JK+JN(K-1))
YIP         CPI = CPI+CPI1+CPI3-2*CPI2
KENJI       CPIP = CPIP+CPIP1-CPIP3
```

```
          P1 = 1-(AZ/SIGT1*(0.5-YF)/2+A1*(SIN.(T)/ALPHP*(1-SIGT1/ALPHP*
         1 ATAN.(ALPHP/SIGT1))+SIGT1*COS.(T)*ELOG.(1+(ALPHP/SIGT1).P.2)
         2 /(2*(ALPHP).P.2)) -A1*(SIGT1*COS.(T)*PSI+ALPHP*SIN.(T)*PSIP)
         3)/(TF*PHIB1)
          P2 = 1-CON1/SIG2 *(0.5-YM )-CON2*(SIG2/(2*BETAP*BETAP)*
         1 ELOG.(1+(BETAP/SIG2).P.2)*(CBP+CBA) + (1-SIG2/BETAP*ATAN.
         2 (BETAP/SIG2))*(SBP-SBA)/BETAP-SIG2*(CBP+CBA)*CPI-BETAP*
         3 (SBP-SBA)*CPIP) + CON3*(SIG2/(2*BETAP*BETAP)*ELOG.(1+(BETAP
         4 /SIG2).P.2)*(SBP+SBA) + (CBP+CBA)*(1-SIG2/BETAP*ATAN.(BETAP
         5 /SIG2))/BETAP - SIG2*(SBP+SBA)*CPI+BETAP*(CBP-CBA)*CPIP)
          PRINT RESULTS E, P1, P2
          WHENEVER IMS.E.1
          F(J) = SIGA/((SIGT1-P1*(SIGSTH+SIGSU))*E)*P1
          FN(J) = (1-P2)*SIGA/(SIGT1-P1*(SIGSTH+SIGSU))/E
          J = J+1
          OTHERWISE
          F(J) = SIGA*P1/(E*SIGT1)
          FN(J) = SIGA*(1-P2)/(E*SIGT1)
          J = J+1
          END OF CONDITIONAL
LOOP3     CONTINUE
          F1 = 0
          FN1 = 0
          THROUGH KILL, FOR M=1,1, M.G.(J-1)/2
          F1 = F1+F(2*M-1)
KILL      FN1 = FN1+FN(2*M-1)
          F2 = 0
          FN2 = 0
          THROUGH EEN , FOR M=1,1, M.G.(J-3)/2
          F2 = F2+F(2*M)
EEN       FN2 = FN2+FN(2*M)
          RI1 = H/3.0*(F(0)+4*F1+2*F2+F(J-1))
          RI2 = H/3.0*(FN(0)+4*FN1+2*FN2+F(J-1))
          PRINT RESULTS RI1,RI2,PHIZ,PHIB1,PHIP1,PHIP2
         R INTERFERED RESONANCE INTEGRAL
          AMU = (1-ALPU)*(A(1,1)-1)
          AMTH = (1-ALPTH)*(A(1,1)-1)
          AMOX = (1-ALPOX)*(A(1,1)-1)
          AMM = (1-ALPM)*(A(1,1)-1)
          LNU =  ELOG.(1/ALPU)
          LNTH = ELOG.(1/ALPTH)
          LNOX = ELOG.(1/ALPOX)
          LNM = ELOG.(1/ALPM)
          EXDU = EXP.(-DELU)
          WHENEVER LNU .G.DELU
          IU = CD/(1-ALPU)*(EXDU-EXP.(-LNU)) + PHIZ/(1-ALPU)*(1-EXDU)
         1 +PHIP1*(EXDU-EX)/AMU
          OTHERWISE
          IU = PHIZ + PHIP1*(EXP.((A(1,1)-1)*LNU-A(1,1)*DELU)-EX)/AMU
          END OF CONDITIONAL
          WHENEVER LNTH.G.DELU
          ITH = CD/(1-ALPTH)*(EXDU-EXP.(-LNTH)) + PHIZ*(1-EXDU)/
         1 (1-ALPTH) + PHIP1*(EXDU - EX)/AMTH
          OTHERWISE
          ITH = PHIZ + PHIP1* (EXP.((A(1,1)-1)*LNTH-A(1,1)*DELU)-EX)/
         1 AMTH
          END OF CONDITIONAL
          WHENEVER LNOX.G.DELU
          IOX = CD/(1-ALPOX)*(EXDU-EXP.(-LNOX)) + PHIZ*(1-EXDU)/(1-
```

```
          1 ALPOX) + PHIP1*(EXDU-EX)/AMOX
            OTHERWISE
            IOX = PHIZ + PHIP1*(EXP.((A(1,1)-1)*LNOX-A(1,1)*DELU)-EX)/
          1 AMOX
            END OF CONDITIONAL
            WHENEVER LNM .G.DELU
            IMOD = CD/(1-ALPM)*(EXDU-EXP.(-LNM)) + PHIZ*(1-EXDU)/(1-ALPM)
          1+PHIP2*(EXDU-EX)/AMM
            OTHERWISE
            IMOD = PHIZ +PHIP2*(EXP.((A(1,1)-1)*LNM-A(1,1)*DELU)-EX)/AMM
            END OF CONDITIONAL
            WHENEVER IMS.E.1
            RI = (SIGSO*IOX*RI1 + (SB-SA)/SA*SIG2*IMOD*RI2)/PHIB1
            OTHERWISE
            RI = ((SIGSO*IOX + SIGSU*IU +SIGSTH*ITH)*RI1 +(SB-SA)/SA*
          1 SIG2*IMOD*RI2)/PHIB1
            END OF CONDITIONAL
            DELRI = RI-UNIRI
            PCERR = DELRI*100.0/RI
            PRINT COMMENT $0INTERFERED RESONANCE INTEGRAL$
            PRINT RESULTS RI,UNIRI,DELRI ,PCERR
            PCINA =(RI*PHIB1 - UNIRI*CD)/(UNIRI*CD)
            PRINT RESULTS PHIB2,PCINA
            TRANSFER TO START
 OUT1       PRINT COMMENT $0ERROR IN TAB SUBROUTINE$
            TRANSFER TO START
            INTEGER NFR,NSR,HM,HN,M,N,J,K
            END OF PROGRAM
$DATA
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6.0927E-6,   5.4677E-6,   4.9071E-6,   4.4044E-6,   3.9533E-6,3.5488E-6*
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