Teaching to the Mathematical Point:
Knowing and Using Mathematics in Teaching

by

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In memory of
Jane and Steve Sleep

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CHAPTER ONE:
THE RESEARCH PROBLEM

Introduction

The ambitious vision of mathematical proficiency for all students demands skillful instruction that is too rarely found in American classrooms. The literature is replete with examples of teaching that fails to engage students in rigorous mathematical work: Teachers who adopt the surface features of new curricula, such as games and manipulatives, but do not use these materials to teach for understanding (Cohen, 1990); efforts to "make math fun" that result in art projects instead of mathematics (Hill et al., 2008); motivating examples and representations that distort the mathematics (Heaton, 1992); and problems intended to engage students in high-level mathematical thinking that deteriorate into lower-level and routine tasks as they are implemented (National Center for Education Statistics, 2003; Stein, Grover, & Henningsen, 1996; Stein, Smith, Henningsen, & Silver, 2000). In each of these examples, teachers seem to miss the mathematical point of the task or materials, or have difficulty maintaining the mathematical focus once the activity is in motion with students.

Teachers are responsible for organizing time and resources so that what students do actually leads to learning. Knowing "the point" of instruction helps in managing this work. Yet what it takes to "teach to the point"—that is, to have a clear understanding of the goals of instruction and to use those goals to manage the work —is not well understood. Though different bodies of research inform the investigation of this problem, a direct focus on this central aspect of teaching practice is missing from the literature. The very notion of goals, as well as what is entailed in identifying, understanding, and utilizing one's goals during instruction, remains underconceptualized. This dissertation aims to address these issues by investigating this core domain of teaching.

Although the task of identifying goals and using them to help students learn arises in teaching any subject, I focus on the teaching of mathematics. My aim is to articulate the practices involved in identifying and using mathematical goals in mathematics teaching and to explore the mathematical knowledge demanded by this work. The significance of this study is framed by the broader problems of mathematics education and efforts to improve it. In particular, my argument rests on three recent developments in education research: (1) evidence that individual teachers have significant effects on student learning; (2) theoretical and practical progress made in specifying the mathematical knowledge needed for teaching; and (3) the need to focus teacher education on practice. I elaborate each below.

Teaching Matters for Student Learning

Improving mathematics education is a matter of broad national concern. The demand for scientific and quantitative literacy has increased, while U.S. students continue to underperform on national and international assessments (Gonzales et al., 2004; Lee, Grigg, & Dion, 2007; Organisation for Economic Cooperation and Development, 2004). The past several decades have seen many efforts to improve students' mathematics achievement. For example, in 1983, the National Commission on Excellence in Education's *A Nation at Risk* spurred the standards movement. Mathematics education led the way with the publication of the *Curriculum and Evaluation Standards for School Mathematics* (1989) by the National Council of Teachers of Mathematics (NCTM), as well as state standards and assessments, now required by *No Child Left Behind. Adding It Up* (2001), a report of the National Research Council, set ambitious goals for the "mathematical proficiency" of all students. In turn, these new standards and goals inspired a host of standards-based curricula.

Noteworthy is that across many of these efforts to improve student achievement, surprisingly little attention was paid to instruction. New standards, assessments, and curricula do not by themselves improve student learning. These resources must be put to effective use by teachers in classrooms with students (Cohen, Raudenbush, & Ball, 2003).

Although it is clear that teaching matters for improving student learning, *how* it matters has been more difficult to document (Hiebert & Grouws, 2007). In their review of research on the effects of teaching on student learning, Hiebert and Grouws (2007) identify opportunity to learn as one of the most important concepts linking teaching and student achievement:

> The emphasis teachers place on different learning goals and different topics, the expectations for learning they set, the time they allocate for particular topics, the kinds of tasks they pose, the kinds of questions they ask and responses they accept, the nature of the discussions they lead—all are part of teaching and all influence the opportunities students have to learn. (p. 379)

In other words, teachers shape "the point" of instruction through the identification and prioritization of learning goals and through the design and implementation of instruction to meet those goals. This, in turn, shapes the mathematics available to students, which impacts their mathematics learning. "Opportunity to learn can be a powerful concept that, if traced carefully through to its implications, provides a useful guide to both explain the effects of particular kinds of teaching on particular kinds of learning and improve the alignment of teaching methods with learning goals" (Hiebert & Grouws, 2007, p. 379). A better understanding of the work of identifying goals and using them to guide instruction would thus enable more nuanced study of the connection between teaching and student learning.

Mathematical Knowledge for Teaching (MKT) as a Resource in Instruction

Efforts to improve student achievement in mathematics have also focused on improving teachers' mathematical knowledge (Ball, Lubienski, & Mewborn, 2001). But teacher knowledge is no different than other resources that can be marshaled to improve instruction: knowledge matters for teaching to the extent that it is used in the work.

Although the idea that teacher knowledge impacts student learning seems obvious, substantial evidence of the influence of teachers' mathematical knowledge on student achievement eluded researchers for many years (Fennema & Franke, 1992). Studies using proxy measures of teachers' mathematical knowledge, such as mathematics courses taken or certification-level, did not find strong relationships between teacher knowledge and student achievement (National Mathematics Advisory Panel, 2007). In the 1980s, sparked in part by Shulman (1986) and colleagues' introduction of *pedagogical content knowledge*, researchers reconceptualized content knowledge for teaching, emphasizing that teachers' knowledge of content must be closely tied to practice. Thus, it is not just any type of mathematical knowledge that matters for teaching; instead, teaching requires knowledge of mathematics that is usable for the work (Ball & Bass, 2003; Ball et al., 2001; Ma, 1999).

Based on this new conceptualization of content knowledge for teaching, a growing number of studies began to examine teachers' knowledge of mathematics. One line of research used open-ended written tests and interviews to explore teachers' knowledge of specific topics in the curriculum (Ball, 1990; Graeber, Tirosh, & Glover, 1989; Simon, 1993; Tirosh & Graeber, 1989). These studies—typically conducted with preservice elementary teachers—uncovered the fragmented and procedural nature of teachers' mathematical knowledge. For example, they found that teachers were often more focused on memorizing rules than understanding concepts, lacked connections between procedural and conceptual knowledge, and did not think flexibly about mathematical ideas.

Other studies investigated how learning about students' mathematical thinking impacts teachers' mathematical knowledge, instructional practices, and student achievement (Fennema et al., 1996; Fennema, Franke, & Carpenter, 1993; Schifter, 1998). For example, in a four-year longitudinal study of 21 teachers, researchers examined the impact of changes in teachers' practices and beliefs on student achievement, finding that a teacher's ability to use students' mathematical thinking to inform instructional decisions corresponded with increased student achievement (Fennema et al., 1996). This research highlights that understanding students' thinking about mathematics is both a component of the mathematical knowledge teachers need, as well as a medium for developing it.

Comparisons of expert and novice teachers were also used to investigate the relationship between teachers' mathematical knowledge and their practice (Leinhardt, 1989; Leinhardt & Smith, 1985). These studies found that expert teachers' knowledge is more hierarchically structured and connected than novices' (Leinhardt & Smith, 1985). Researchers also found differences in teachers' abilities to access their mathematical knowledge in flexible ways while teaching. For example, when explaining new mathematical content, expert teachers better supported students' use of representations, used mathematical language more carefully, and chose better numerical examples than novice teachers (Leinhardt, 1989). Borko et al. (1992) also examined the use of mathematical knowledge in practice in a seminal case study of a preservice elementary teacher. Despite having taken two years of college mathematics courses, the teacher was unable to provide a conceptually based explanation in response to a student's question about the standard division-of-fractions algorithm, and in fact, used a diagram that represented multiplication rather than division.

More recent work, including Liping Ma's (1999) concept of "profound understanding of fundamental mathematics," has made additional progress toward understanding the nature and structure of teachers' mathematical knowledge. Using data from interviews with Chinese teachers, Ma describes how their mathematical knowledge is coherently organized into "knowledge packages," which contain connections among and sequences of the central mathematical ideas, as well as key pieces of knowledge for a particular content area.

Ball, Bass, and colleagues have also investigated the nature and structure of the mathematical knowledge used in teaching by investigating practice to uncover its mathematical demands (Ball & Bass, 2003). This "job analysis" has revealed that elementary teaching is highly mathematical work; even seemingly general pedagogical tasks, such as listening to students or asking questions, require substantial mathematical knowledge and reasoning. In addition, these analyses of teaching have highlighted the centrality of mathematical practices and pointed to essential features of knowing mathematics for teaching, including the need for knowledge to be unpacked and connected. This practice-based approach has led to a conceptualization of *mathematical knowledge for teaching* (MKT)—the mathematical knowledge, skills, and habits of mind used to do the work of teaching (Ball, Thames, & Phelps, 2008).

To test the existence of this professionally specialized knowledge and its relation to student achievement, large-scale measures of MKT have been developed and administered. This empirical work has provided evidence that there is mathematical knowledge specific to the work of teaching and that it can be measured (Hill, Schilling, & Ball, 2004). Furthermore, it has shown that MKT is linked to the mathematical quality of instruction (Blunk, 2007) and is a significant predictor of gains in student achievement (Hill, Rowan, & Ball, 2005).

Through an iterative process of theoretically and empirically based work emerged a framework for categorizing mathematical knowledge for teaching. The current framework for MKT distinguishes four domains of mathematical knowledge for teaching: (1) *common content knowledge (CCK)*; (2) *specialized content knowledge (SCK)*; (3) *knowledge of content and students (KCS)*; and (4) *knowledge of content and teaching (KCT)* (Ball et al., 2008). The first two domains (CCK and SCK) are types of subject matter knowledge; although the knowledge and reasoning in these two domains is used in teaching, knowledge of students or knowledge of pedagogy is not needed. On the other hand, the last two domains (KCS and KCT) are amalgams of subject matter knowledge and pedagogical knowledge, and are thus types of pedagogical content knowledge (Shulman, 1986). In addition to these four domains, two less developed domains have been preliminarily proposed: *horizon content knowledge* and *knowledge of content and curriculum*.

Overall, significant theoretical and practical progress has been made in specifying the mathematical knowledge needed for teaching. This research has demonstrated that teachers' simply knowing the mathematics they teach to their students is not enough: Teachers need mathematical knowledge that is tied to the specific work they do and need to hold this knowledge in ways that enable them to use it in practice. These studies have enabled scholars to develop a more refined conception of the mathematical knowledge needed for teaching, including the nature and structure of that knowledge. Furthermore, teachers' knowledge of mathematics has been linked to both the mathematical quality of instruction and student achievement.

Despite this important progress, there is still much to learn about both the structure of MKT and how teachers' mathematical knowledge influences student learning. Because teachers' knowledge of mathematics impacts student learning through instruction, a better understanding of the relationship between teacher knowledge and student learning requires a better understanding of the relationship between mathematical knowledge for teaching and instruction.

Teaching Teaching to Beginners: The Need to Focus on Practice

Understanding the relationship between teacher knowledge and instruction also has important implications for the preparation and ongoing education of teachers. Teaching requires the integration of knowing and doing, calling for teacher education that links knowledge with its use (Ball & Bass, 2000; Lampert, 2005). Too often, however, teacher education courses are disconnected from practice (Putnam & Borko, 2000), providing little support for preservice teachers to use and extend their course-developed knowledge and skills in real-time teaching. Instead, teaching skill is left to develop independently—and unreliably—from experience.

The ineffectiveness of traditional approaches to teacher education has prompted a growing awareness of the need to focus teacher education on practice—that is, to teach preservice teachers to *do* teaching, rather than simply talk *about* teaching. This shift in focus places new demands on teacher educators, including the need to unpack and articulate the naturally complex and integrated work of teaching so that it can be studied, analyzed, and practiced (Ball, Sleep, Boerst, & Bass, 2009; Franke, Kazemi, & Battey, 2007; Grossman et al., 2009; Grossman & McDonald, 2008; Grossman & Shahan, 2005). This "decomposition of practice" (Grossman et al., 2009) requires frameworks and tools for parsing the work of teaching, as well as language for describing it—both of which are still sorely lacking in the field (Grossman & McDonald, 2008).

Despite its centrality in teaching, the work of identifying and using goals in instruction has yet to be unpacked in such a way that it can be adequately studied or taught to beginners. This is reflected in current teaching and teacher education practice. For example, it is not uncommon for teachers to talk about successful activities and lesson plans without reference to what goals for student learning were achieved, or for standards and goals to be stated in overly general terms (Stigler & Thompson, 2009). When learning to plan lessons, preservice teachers are routinely asked to "state your objectives" or "describe the goals of your lesson" with little guidance as to how to formulate goals or how goals might be used during instruction. Words like "goal," "purpose," and "objective" are used interchangeably and without definition,[1] or are narrowly defined to include only student behaviors that can be directly observed. In addition, little is known about what preservice teachers do to identify and understand the goals of their lessons, or if and how they use those goals in instruction.

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[1] An example of this was seen in work I participated in as part of a study group with Magdalene Lampert, Timothy Boerst, and others at the University of Michigan. The group conducted an informal examination of sample activities and templates used to support preservice teachers in planning lessons. The lack of clarity around the use of purposes, objectives, and goals was so prevalent that our group noted the interchangeability of these terms by referring to them as "POGs."

There is evidence, however, about why a focus on goals would be a strategic site for teacher education. First, the work of identifying and using goals in instruction is difficult for preservice teachers to do well, yet is something that they have inclinations toward. Thus, it provides opportunities for teacher educators to productively build on and scaffold something to which preservice teachers already try to attend. Second, because identifying and using goals demands subject matter knowledge, it provides opportunities for preservice teachers to both develop and practice content knowledge and reasoning in the context of its use in practice. And finally, increased skill with identifying and using goals can be a strategic way to help preservice teachers manage the complexity of in-the-moment instruction because much of the work can be done outside of classroom interactions, using as much time as needed. I discuss each of these reasons in more detail below.

Beginning teachers often struggle with developing and maintaining a focus on long-term goals (Borko & Livingston, 1989; Housner & Griffey, 1985; Kauffman, Johnson, Kardos, Liu, & Peske, 2002) and have difficulty prioritizing content (Borko & Livingston, 1989). For example, when not given a specified curriculum for a given subject, preservice teachers are often uncertain about which details to emphasize or how deeply to go into a topic (Kauffman et al., 2002). They also have difficulty predicting what aspects of the curriculum are likely to be difficult for students (Borko & Livingston, 1989) and sometimes focus on objectives that students have already mastered (Joyce & Harootunian, 1964).

Studies have also shown that preservice teachers often lack clear and coherent goals in their lessons. For instance, Joyce and Harootunian (1964) interviewed 39 preservice teachers before they taught a science lesson and found that most had unclear and vague objectives that often did not relate to the activities in the lesson. In particular, preservice teachers had difficulty matching goals, objectives, activities, and assessments. Novices' lessons tend to be poorly structured and not comprehensive, with few connections between related content (Borko & Livingston, 1989; Leinhardt, 1989; Livingston & Borko, 1989, 1990). For example, Leinhardt (1989) found that novices' lessons had "fragmented lesson structures" and an "ambiguous system of goals that often appear[ed] to be abandoned rather than achieved" (p. 73).

Promising for teacher education, however, was that novices seemed aware of these problems; they just did not yet have the "analytic skills to understand where failures occurred or when goals that were implicit in certain actions were not achieved" (Leinhardt, 1989, p. 73). Thus, even though they often have difficulty developing and using coherent goals (which is not surprising, given that the work is demanding and they are beginners), preservice teachers do recognize that goals are something important to attend to in teaching. Furthermore, preservice teachers sometimes do use goals to inform their work—for example, when analyzing curriculum (Davis, 2006) or planning lessons (Borko, Livingston, McCaleb, & Mauro, 1988). Morris, Hiebert, and Spitzer (2009) found that, although preservice teachers did not spontaneously unpack learning goals in order to evaluate and revise instruction, they could identify subconcepts of learning goals in supportive contexts. Thus, research suggests that work on identifying and using goals could be a site where teacher educators could productively build on and scaffold preservice teachers' inclinations, knowledge, and values (Davis, 2006; Morris et al., 2009).

Not only could work on goals build on what preservice teachers bring to teacher education, it also provides a fruitful site for both practicing and developing content knowledge for teaching. Identifying and using goals requires mathematical knowledge for teaching (Morris et al., 2009). And content knowledge, among other factors, has been found to influence preservice teachers' use of curriculum, planning, and teaching (Behm & Lloyd, 2009; Borko et al., 1988; Kahan, Cooper, & Bethea, 2003). Thus, work on identifying and using goals could provide preservice teachers opportunities to simultaneously practice and develop content knowledge by situating their learning in practice (Ball & Cohen, 1999; Putnam & Borko, 2000). For example, planning and reflecting on mathematics lessons, particularly with a textbook as a guide, is a way to help preservice teachers focus on the particulars of the mathematics (Van Zoest & Bohl, 2002).

Becoming more skilled at identifying and using goals could also help preservice teachers manage interactive aspects of teaching. Teaching requires the simultaneous attention to and management of multiple aspects of instruction. "Novices attempting to solve a problem typically endure high cognitive load because they lack the experience and conceptual framework to make cognitive processing more efficient" (Feldon, 2007, p. 125). For example, during instruction, preservice teachers often have trouble keeping a lesson on track while simultaneously being responsive to students (Borko & Livingston, 1989; Livingston & Borko, 1990). Because identifying goals and determining ways they can be used to steer instruction can be part of planning, it can be done outside of the classroom, with as much time as needed, rather than in the intensity of in-the-moment instruction (Morris et al., 2009). Being clear about the goals of activities and how the activities are designed to meet those goals, therefore, might be a way to help reduce the "cognitive load" of interactive instruction. Furthermore, this would likely have face validity with preservice teachers because they see planning as a way to learn content and a way to solve instructional problems before they are in the classroom with students (Borko, Lalik, & Tomchin, 1987).

At a basic level, a focus on goals could help preservice teachers see teaching as goal-driven work (Stigler & Thompson, 2009) and see content as something that needs to be attended to during instruction. Mewborn (2000) found that during their field placements, preservice teachers first attended to classroom organization and management, then to pedagogy, and then to student thinking; their concern with mathematics content was minimal. Preservice teachers could attend simultaneously to multiple issues, but when confronted with new teaching situations, understandably their attention focused first on context and organization. A more explicit focus on the use of goals in instruction could help preservice teachers see how attending to content can be part of and inform the management of these other concerns. Leinhardt (1989) argues that "beginning teachers need to build more efficient strategies for keeping mental notes about the lessons that they teach—how one lesson connects with others, what the key point of a lesson is, what students need to experience in order to build meanings for themselves, and how long it will take students to do that" (p. 74). A better understanding of the content intended to be taught in a lesson and how the instructional activities are designed to engage students with this content could help preservice teachers develop such strategies.

As expected, preservice teachers rely heavily on textbooks as their source of ideas and decisions (Bush, 1986). When teaching with curriculum materials, being able to "figure out the point" seems particularly important, because the "matching" between the student learning goals and the instructional activity has been done by the curriculum designer (Smith & Ragan, 2005) and is rarely made explicit to the user (Ball & Cohen, 1996). Not understanding this "match" can lead to inadvertent unproductive changes in the activity or its implementation. A focus on mathematical goals and how an activity is designed to meet those goals could help preservice teachers more accurately evaluate the intent of textbook activities rather than rely on their beliefs and familiarity with instructional approach, which is often the case (Lloyd & Behm, 2005). It also could help teachers interpret the orientation of the curriculum materials (Ben-Peretz, 1990).

Work on Identifying and Using Goals as a Strategic Site for Teacher Education

Together, these arguments provide the motivation for this study. My overarching hypothesis is that having a better understanding of the "mathematical point" of an activity and how the activity is designed to engage students with that point will improve the mathematical quality of beginning teachers' instruction, which will, in turn, improve students' learning of mathematics. A better understanding of the mathematical point of an activity helps a teacher steer the activity toward the intended mathematics and manage decisions during instruction. Furthermore, it is a particularly strategic focus for teacher education because much of the work of "figuring out the mathematical point" can be taught and completed outside of the classroom and because it provides an opportunity to both develop and practice mathematical knowledge for teaching in the context of its use (Morris et al., 2009). Testing this hypothesis requires that the work of "figuring out the mathematical point" be taught to beginners. Their ability to "teach to the mathematical point" would then need to be measured in order to study its relationship to MKT, the mathematical quality of instruction, and student achievement. But before any of this work can be done, what is involved in "figuring out the mathematical point" and "using it to steer instruction" must first be investigated. This crucial initial step of examining, identifying, and naming key constituents of this work is the focus of this dissertation.

Overview of Study

This is a study of the mathematical work of teaching. In particular, I seek to probe and conceptualize the work teachers do before and during a lesson to move with students through time toward particular mathematical ideas. To do this, I examine the practices and knowledge demands of identifying mathematical learning goals for students and deliberately designing and implementing instructional activities to move students toward those goals—that is, to be able to "teach to the mathematical point."

In this section, I provide an overview of the study. I begin with some of the premises underlying this work. I then briefly describe the design of the study and present the research questions which frame the investigation. Next, I introduce the conceptualization of "teaching to the mathematical point" that emerged from my analyses and will be discussed in detail throughout the dissertation. I conclude with an overview of the remaining chapters in the dissertation.

Teaching as Purposeful Work

Teaching is, at its heart, being responsible for getting other people to learn something. Doing this requires knowing what that "something" is and deliberately designing and steering instruction so that learners can learn it—in other words, being purposeful.

Hiebert and Grouws (2007) define teaching as consisting of "classroom interactions among teachers and students around content directed toward facilitating students' achievement of learning goals" (p. 372). This conception of instruction as interactions among the teacher, students, and content can be represented by the "instructional triangle" (Cohen et al., 2003; Lampert, 2001), shown in Figure 1. Teachers interact directly with their students and directly with the content; they also mediate the student-content relationship. The "work of teaching" occurs in the dynamic relationships depicted by the arrows emerging from the teacher vertex as students "do the complementary work of making a relationship with the content to learn it"

(Lampert, 2001, p. 423). These interactions occur simultaneously and over time and are influenced by the contexts in which they occur.

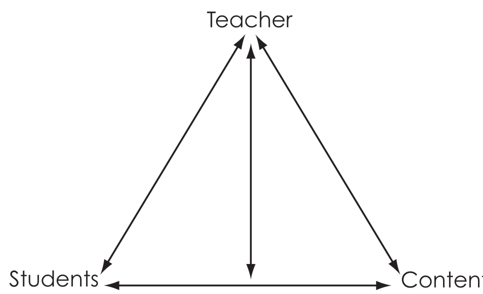

Figure 1: The instructional triangle.

The work of teaching mathematics involves deliberately designing and implementing instruction in order to connect students with particular mathematics. Doing this requires an understanding of where an instructional activity is headed mathematically and how one intends to "get there" with students during the activity's enactment. This involves figuring out the particulars of the content to be taught as well as where that content is located in the larger mathematical terrain. It also involves understanding where students are with respect to that content and what the terrain "looks like" through the eyes of the learner. Furthermore, it requires understanding how the instructional activity is intended to move students toward that content and then, during the activity's enactment, using these understandings to help steer the instruction. It is this work that teachers do before and during instruction to purposefully move with students through time toward particular mathematical ideas that is the focus of this dissertation.

Mathematical Proficiency as the Ultimate Goal for Student Learning

The conceptualization of instruction as interactions between the teacher, students, and content does not assume a particular type of pedagogy or teaching "style," curriculum use or non-use. Nor does my conceptualization of "teaching to the mathematical point." However, I do make assumptions about the nature of the mathematical learning goals for students that are part of "the mathematical point."

The field has been moving toward agreement about the focus of student learning (Franke et al., 2007). This consensus is reflected in the notion of "mathematical proficiency" described in *Adding it Up* (National Research Council, 2001). "Mathematical proficiency" reflects what "is necessary for anyone to learn mathematics successfully" and consists of the following five "interwoven and interdependent" strands:

- *Conceptual understanding*—comprehension of mathematical concepts, operations, and relations;
- *Procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately;
- *Strategic competence*—ability to formulate, represent, and solve mathematical problems;
- *Adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification; and
- *Productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence in one's own efficacy. (National Research Council, 2001, p. 116)

Throughout this dissertation, I am assuming that the ultimate goal of mathematics instruction is to develop the mathematical proficiency of all students.

Study Design

This study uses empirical data—observations of and interviews from 17 preservice teachers' mathematics lessons—to develop a conceptual framework about the nature of mathematical goals and the work of identifying and using mathematical goals in instruction. Although grounded in empirical data, this dissertation is primarily conceptual, using the interview and video data to better understand this central aspect of teaching practice. Three main research questions frame the work in this dissertation:

1. What is the work of determining the mathematical goals of a lesson and using those goals to design instruction?

2. What is the work of using mathematical goals to steer instruction during a lesson's enactment?
    - What problems must be managed in doing this work?
    - What teaching moves can be used to manage these problems?
    - What are some of the issues that arise for beginning teachers when managing these problems?

3. What is the relationship of mathematical knowledge for teaching and the work of determining mathematical goals and using them to design and steer instruction?

The Work of "Teaching to the Mathematical Point"

There is no agreed-upon name in the field for the work of teaching that is the focus of this dissertation (i.e., the work of identifying mathematical learning goals for students and deliberately designing and implementing instructional activities to move students toward those goals). In addition to signaling the underconceptualization of these ideas in the field, the lack of language makes writing clearly about the ideas quite difficult. Therefore, although the language for and conceptualization of what I am calling "teaching to the mathematical point" emerged from my analyses (and, thus, are results of the study), I introduce them here because it is useful (and I hope clearer) to use this language throughout the dissertation.

I struggled with the name "teaching to the mathematical point," in particular whether "mathematical point" conveys my intended meaning. I did not use "teaching to the mathematical purpose" because I name "mathematical purposing" as a subcomponent of teaching to the mathematical point. Nor did I use "teaching to the mathematical goals" because I have tried to reserve "goals" (in particular "mathematical learning goals") to refer to the end goals for student learning and want "mathematical point" to be broader. I am defining "mathematical point" to include the mathematical learning goals for an activity, as well as the connection between the activity and its goals. For example, the mathematical point of an example could be to provoke a common student error in order to develop students' understanding of a particular concept.

I chose the language of "teaching to the mathematical point" to capitalize on various meanings of "point."[2] "Point" can refer to both a particular detail as well as the main subject, conveying both specificity (e.g., the point of a particular choice of numbers in an example) and a connection to the bigger whole (e.g., the ways in which an activity furthers ongoing, broader goals for students' mathematics learning). "Point" also carries a sense of location, which for the mathematical point refers to location both in the mathematical terrain and on the curricular trajectory. I also am trying to harness the everyday, informal way people talk about "the point" of something. For example, asking about "the mathematical point" of an activity provides a familiar and accessible way to ask what an activity is intended to accomplish mathematically and how it is intended to do so.

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[2] I draw on the following definitions for "point," excerpted from the *Oxford English Dictionary (OED)* (2009):

> A separate or single item, article, or element in an extended whole (usually an abstract whole, as a course of action, a subject of thought, a treatise, a discourse, a set of ideas, etc.); an individual part, element, or matter; a detail, a particular.
>
> Something having position in space, time, succession, degree, order, etc.:
> - A position reached in a course or progression of any kind; a definite position in a linear scale (actual or notional); a step or stage in a process; an exact degree or extent of some measurable quality or condition;
> - A place considered in terms of spatial position; a specific location or spot. Also fig., sometimes with modifying word specifying the nature or purpose of the location, as starting point, etc.; or
> - A location along a particular route or in a particular direction.
>
> Something that is the focus of attention, consideration, or purpose:
> - The main subject or focus of a discussion, discourse, etc.; the nub or essence of a matter; the central or pertinent issue; or
> - An objective, aim, or purpose; the thing for which one strives or contends.

There are, of course, a number of problems with the language of "teaching to the mathematical point." One main concern is that "mathematical point" may seem narrower than I intend. Mathematics instruction inherently demands working simultaneously and over time on multiple mathematical goals through activities with multiple mathematical purposes. The mathematical point of an activity is therefore not a single, small idea as the word "point" might imply. I mean "point" to be a package of ideas.[3]

I define the "mathematical point" to be a connected package of mathematical goals and instructional purposes, with depth and weight and time. A particular idea or skill may be the point that is emphasized in a given moment, but always in the background are other concepts, skills, practices, and dispositions to which it is connected. Important in my definition is that the mathematical point is not simply a collection of *mathematical* ideas; it is conceived of and prioritized in relation to the particular students being taught and the activity in which it is being taught through. There is an implicit "of" that accompanies "teaching to the mathematical point": It is the mathematical point *of* something instructional—for example, the mathematical point of a lesson, an activity, a problem, or a teacher question. That the mathematical point is *of* something instructional highlights that "knowing the mathematical point" includes an understanding of the mathematical learning goals as well as how the "something instructional" is intended to move students toward those goals. I view the work of "teaching to the mathematical point" as a subset of "teaching to the instructional point." That is, the mathematical point is one of many instructional points a teacher has at any given time.

I define "teaching to the mathematical point" as comprising three interrelated and mutually informing types of work:

- Articulating the mathematical point (i.e., articulating the intended mathematics and how the instructional activity is designed to engage students with it);
- Orienting the instructional activity (i.e., detailing an instructional activity so it is oriented toward the intended mathematics); and
- Steering the instruction (i.e., deploying teaching moves during instruction in an effort to keep students engaged with the intended mathematics).

[3] My use of "mathematical point" to reflect a connected bundle of ideas is similar to the way Ma (1999) uses "knowledge package" to reflect a set of connected mathematical ideas. However, the mathematical point is not the same as Ma's knowledge package, because in addition to unpacking the mathematical terrain, knowing the mathematical point also includes articulating the specific mathematical ideas a particular instructional activity is intended to focus on, how the activity is intended to develop those ideas with particular students, and where the activity sits in both the mathematical terrain and curricular trajectory.

The first two—articulating the mathematical point and orienting the instructional activity—are what I call "mathematical purposing." (Thus, the work of teaching to the mathematical point can also be seen as having two components: mathematical purposing and steering the instruction.) The result of mathematical purposing is an articulation of the mathematical learning goals for students, an understanding of how the activity is intended to move students toward those goals, and a detailing of the task and likely teacher moves that position the activity so it is more likely to engage students with the intended mathematics. Steering occurs during the interactive phase of instruction as teachers implement the activity with students and deliberately try to keep it heading toward the mathematical point. Although steering happens only in the interactive phase of teaching, mathematical purposing can be part of both the preactive and interactive phases. Even though the work of mathematical purposing and steering do not correspond to the preactive and interactive phases of teaching, there is a temporal, cyclic relationship between the work of mathematical purposing and steering instruction: In order to steer instruction toward the mathematical point, the mathematical point must be known. The work of steering then informs the work of mathematical purposing, for example, the mathematical point might change or the connection between the instructional activity and the intended mathematics become clearer.[4] The components of teaching to the mathematical point and their interdependent and cyclic relationship are depicted in Figure 2.

[4] This cyclic relationship in teaching to the mathematical point is similar to the way Simon (1995) depicts the evolution of hypothetical learning trajectories in his *Mathematics Teaching Cycle*. Simon conceptualizes that teachers have an original goal, which provides the initial direction for the hypothetical learning trajectory. The hypothetical learning trajectory is then continually modified based on the teacher's assessment of students' knowledge during instructional interactions. In his model, Simon depicts this process as being mediated by a number of teacher characteristics such as mathematical knowledge and theories of mathematics learning and teaching. I discuss this further in Chapter 2.

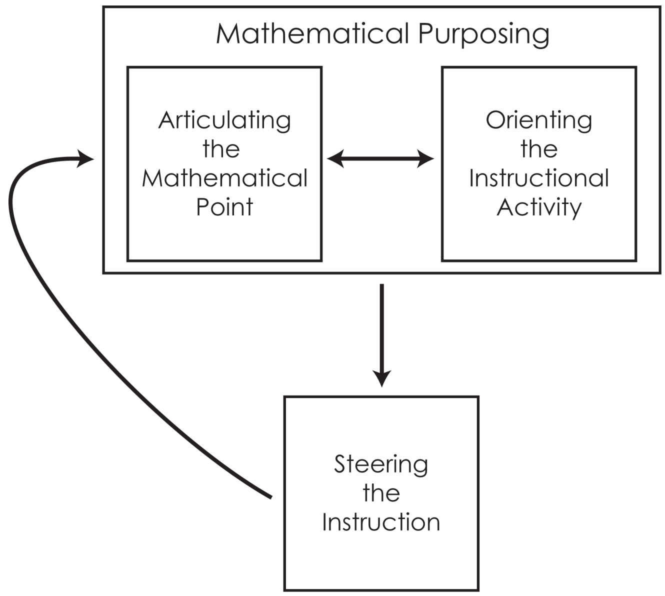

Figure 2. The work of "teaching to the mathematical point."

Contributions of Study

Teacher education must prepare teachers with the knowledge and skills necessary to achieve the bold vision of mathematical proficiency set by current K-12 standards and curricula. Doing so requires a better articulation of the work of teaching and a better understanding of the relationship between teacher knowledge and instruction. The major contribution of this study is a much-needed conceptualization of a central aspect of mathematics teaching practice: the work of articulating mathematical goals and using them to design and steer instruction—what I have named "teaching to the mathematical point." The framework for "mathematical purposing" developed in this study will inform teacher education, as well as research on teaching and teacher education. In addition, the framework can be used in the design and study of curriculum materials, in particular, to identify ways in which these materials can be more "educative" for teachers (Davis & Krajcik, 2005). The results from this dissertation also contribute to furthering the field's understanding of mathematical knowledge for teaching and its relationship to instruction.

Organization of the Dissertation

The dissertation is organized into eight chapters. Chapter 1, the current chapter, frames the research problem, provides an overview of the study, and introduces the language of "teaching to the mathematical point." Chapter 2 describes the theoretical perspective taken toward the design of instruction in this study and reviews the literature basis for the work of mathematical purposing. Chapter 3 describes the data sources and methods of analysis I used. Chapters 4 through 7 present the results of my analyses: Chapter 4 uses two extended examples from the data to examine the complexity of "teaching to the mathematical point." Chapter 5 examines problems in steering instruction toward the mathematical point and issues that can arise when trying to manage them. Chapter 6 presents my conceptual framework for mathematical purposing, and Chapter 7 explores the mathematical knowledge demands of this work. Chapter 8 considers the implications of the study and directions for future research.

CHAPTER TWO:
FOUNDATIONS IN THE LITERATURE
FOR THE WORK OF MATHEMATICAL PURPOSING

Introduction

One of the central aims of this dissertation is to unpack and articulate what is involved in the "mathematical purposing" of instruction by developing a framework that conceptualizes and details this work. This chapter reviews the literature that provides the foundation for this framework. From this review, as well as from my observations during data collection and from my experiences as a teacher and teacher educator, emerged the initial categorization scheme I used to begin analyzing the data. Throughout my analyses, I returned to selected examples from the literature to test and refine my evolving framework.

Developing a coherent framework for mathematical purposing is challenging because there is not a particular moment when, or a prescribed process by which, teachers determine the goals of their lessons and organize their instruction to meet those goals. For example, a teacher might begin planning a lesson by reading the objectives listed in her teacher's guide and then refine her understanding of those goals as she plans the lesson's activities, perhaps modifying the textbook's goals to better meet the perceived needs of her students. A teacher might instead first articulate the content he wants to teach and then create his own instructional activities, drawing from past experience or gathering ideas from a range of curriculum resources. Or, a teacher might decide she wants to use a new manipulative that she learned about in a professional development workshop and then figure out content that would be appropriate to teach with those materials.

Complicating matters further, determining the goals of a lesson does not only happen prior to the lesson's enactment. During instruction, teachers respond to their students and, based on what does or doesn't come up, plans change. For example, a teacher may spend time on an unanticipated topic that arises during discussion, adding new goals to the lesson in the course of instruction; or a teacher might begin a lesson with one set of goals, but then decide to completely change course based on students' reactions.

In order to develop a framework that can incorporate the types of variation reflected above, I make an analytic distinction between *what* the work of mathematical purposing is and *how* that work "gets done." By "the work of mathematical purposing," or simply "mathematical purposing," I mean what is involved in articulating the mathematical point (i.e., articulating the intended mathematics and how the instructional activity is designed to engage students with that mathematics) and orienting the instruction toward the mathematical point (i.e., detailing an instructional activity so it is positioned toward the intended mathematics). By "how that work gets done," I mean describing the ways in which mathematical purposing is accomplished, including the different people and tools involved in doing the work, their context and interactions, and the resources brought to bear. Different instantiations of the "how" can be seen in the scenarios above.

The framework for mathematical purposing I develop in this dissertation aims to describe the *what*. It is grounded, however, in a particular perspective of the *how*. I view mathematical purposing as one aspect of the larger teaching task of designing instruction and consider the design of instruction to be work that can be differently distributed across the teacher and various resources. In the following section, I elaborate this perspective on designing instruction as distributed work—a perspective on the *how*. I then turn in the remaining sections of the chapter to reviewing the literature for insight into the *what*.

Designing Instruction as Distributed Work

In this section, I draw on theories of distributed cognition, in particular Spillane and colleagues' conception of "distributed leadership" (Spillane, Halverson, & Diamond, 2004), and on research that views teachers' use of curriculum as "participating with" the materials (Remillard, 2005) to describe a distributed perspective on designing instruction. Theories of distributed cognition view cognition as residing not solely in the head of an individual actor, but as "distributed—stretched over, not divided among—mind, body, activity, and culturally organized settings (which include other actors)" (Lave, 1988, p. 1). Cognitive processes are seen as distributed across and "in between" the members of a group, internal and external structures and tools, and time (Hutchins, 2000; Salomon, 1993). From a distributed perspective, the unit of analysis for examining "cognition in practice" is "actors in situations working with artifacts, rather than actors abstracted from situations and artifacts" (Spillane et al., 2004). Pea (1993) emphasizes the role of designed artifacts in his notion of "distributed intelligence," which he considers to be manifest in activity, the resources for which are distributed across people, environments, and situations.

The idea that the design of instruction is shared by teachers and tools such as curricula is not new. Research on teachers' curriculum use has long debunked the notion that the curriculum written in textbooks is the curriculum experienced by students. From this research emerged a plethora of models describing how teachers interact with curricula, with different models reflecting the varying analytic foci of this literature. For example, Stein, Remillard, and Smith (2007) capture the temporal phases of curriculum use with the model shown in Figure 3 below.

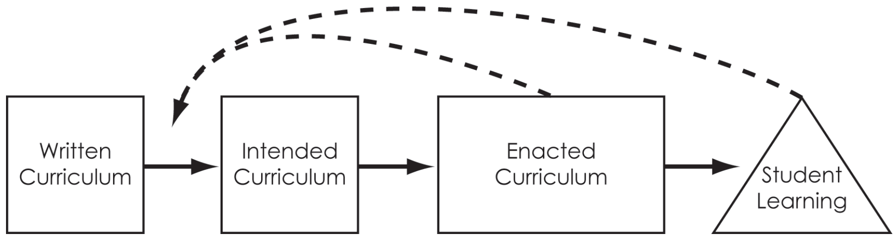

Figure 3. Temporal phases of curriculum use.
Adapted from Stein, Remillard, and Smith (2007).

This model shows the intended curriculum as different from the written curriculum that appears in textbooks or other resources because teachers transform the written curriculum as they plan for instruction. For example, teachers read and interpret the written curriculum, selecting and sequencing the tasks they will use, often making modifications and additions. The curriculum is further transformed during instruction as teachers implement their plans with students in the classroom.

Curriculum materials have also been conceptualized as mediating the work of the teacher. For example, Brown (2002, 2009; Brown & Edelson, 2001, 2003) draws on sociocultural theories to conceptualize the relationship between teachers and curriculum materials. He argues that, as teachers use curriculum resources, they interact with the materials in a variety of ways, for example, selecting which aspects of the materials to use and interpreting the materials during planning and instruction. In turn, curriculum materials afford and constrain teachers' practice, extending teachers' capabilities and mediating their actions. From this perspective, a teacher's use of curriculum is not seen in terms of fidelity or variation, but as a bi-directional relationship between teacher and tool (Brown & Edelson, 2001).

Brown captures this "constructive interplay" between teachers and curriculum materials in his *Design Capacity for Enactment* framework. This framework portrays the teacher-tool relationship as influenced by both the resources the teacher brings to the relationship—subject matter knowledge, pedagogical content knowledge, and goals and beliefs—and the resources brought by the curriculum—physical objects and representations of physical objects, representations of tasks, and representations of concepts. This framework views the varying ways in which teachers interact with curriculum as "different degrees of artifact appropriation" and offers a scale that characterizes the nature of the appropriation by the teacher. The scale positions "offloading" (i.e., a literal use of curriculum materials) at one end of the spectrum and "improvising" (i.e., a teacher's use of her or his own strategies) at the other. Although these labels might seem to impose a value judgment, the offloading-adapting-improvising scale is not meant to imply that one type of interaction is more desirable than another, nor is it meant to be a measure of fidelity of implementation or intent. Instead the scale provides a way to characterize the nature of a teacher's interaction with a particular resource at a particular time. Thus, the *Design Capacity for Enactment* framework is descriptive, not evaluative. In a single lesson, a teacher may, in fact, have multiple instances of different types of interactions. Brown also introduces the construct *pedagogical design capacity* to characterize a teacher's ability to "perceive and mobilize" existing resources in the crafting of instruction (Brown, 2002, 2009; Brown & Edelson, 2003).

A similar, interactive view of the relationship between teachers and curriculum materials underlies a number of models in the literature. Depending on the research focus, the relationship is depicted in different ways: Different aspects of the relationship are foregrounded; different contributing factors identified and elaborated. For example, Silver, Ghousseini, Charalambous, and Mills (2009) and Castro (2006) depict the role of curriculum in mediating a teacher's practice by placing curriculum materials along the practice-arrows of the instructional triangle (Figure 1). Stein and Kim (2009) identify features of curriculum materials that are likely to impact teachers' use (e.g., the nature of tasks and the transparency of design) and examine how these features interact with two key organizational resources (human and social capital).

Yinger and Clark (1982) propose a model to describe how teachers evaluate instructional activities. The model includes the goal of each step and the processes used to attain those goals (the processes are in parentheses):

1. Understand/represent the written description (reading, interpreting, categorizing);
2. Answer the questions, "What would this activity look like in practice?" (mental trying out); "How well would it work?" (evaluating); and "How well do I like it?" (evaluating);
3. Answer the question, "How could I make this activity work?" (editing, justifying); and
4. Make final judgments. (p. 21)

Similarly, Sherin and Drake (2009) examine the work a teacher does inside the teacher-text relationship and propose the "curriculum strategy framework" to characterizes how teachers interact with curriculum materials. Their framework identifies three core interpretive activities in which teachers engage with curriculum materials—*reading, evaluating,* and *adapting*—at three different points in time—*before, during,* or *after instruction*.

In an effort to synthesize across the various models and constructs, Remillard (2009) proposes a "conceptual model of teacher-curriculum interactions and relationships." In this model, the teacher and curriculum materials are depicted in a mediating relationship and the types influences on each are shown. Curriculum resources are shaped by topics and task structure, embedded teacher supports, and pedagogical emphasis; teacher resources by human capital (including pedagogical design capacity), agency and professional status, and social capital. These resources are set in and further shaped by the institutional context. The products of the teacher-text interaction are "instructional outcomes," which include content covered, tasks, and pedagogical emphasis.

All of the above models and frameworks characterize aspects of what I am calling the *how* of designing instruction. Notice that the models themselves do not unpack *what* the work entails. For example, Sherin and Drake's (2009) framework does not describe what teachers <u>do</u> when they "read, evaluate, or adapt" curriculum materials "before, during, or after instruction." Yinger and Clark's (1982) model does not unpack what is involved in "understanding the written description" in a textbook or in "making final judgments." Nor does Remillard's (2009) model describe, for example, the work of determining "pedagogical emphasis" as teachers interact with curriculum resources. This is not to say that the studies from which these models resulted did not also describe aspects of the *what*. My point here is that the models themselves do not unpack and articulate what the work entails; they aim to depict how it is accomplished.

My conceptualization of the *how*—designing instruction as distributed work—uses theories of distributed cognition to build on and broaden the research on teachers' use of curriculum described above. In taking a distributed perspective on designing instruction, I aim to parallel the distributed perspective on leadership taken by Spillane and colleagues in the Distributed Leadership Study.[5] They view leadership *practice* as the product of complex interactions among school leaders, followers, and their situation, which includes tools, routines, and structures. Distributed leadership serves as an analytic frame that can be used, for example, to understand how leadership practice is distributed in different settings and the impacts of those distributions (Harris & Spillane, 2008).

Similarly, I view the work of designing instruction—with or without curriculum materials—as a practice that is distributed across individuals and situations, which include artifacts, environments, and time. In this sense, designing instruction is an activity that is stretched over the teacher, the curriculum materials, state and national standards, established routines and lesson structures, the classroom and school community, yearly planning, grade-level

[5] The Distributed Leadership Study's website is http://www.sesp.northwestern.edu/dls/.

colleagues, and the like. This is not the same as viewing the design of instruction as a distribution of labor; it is instead viewing the design of instruction as occurring in and as a result of the interactions of the "many structural elements that are brought into coordination" during the task (Hutchins, 1995, p. 290).

A few points are worth clarifying. First, unlike the work of the Distributed Leadership Study and the models of curriculum use described above, the focus of this dissertation is not to model the ways in which the design of instruction is or can be distributed, or the impacts of those distributions. Distributed cognitions/practices are necessarily situated (Salomon, 1993). Thus, "precisely how cognition is distributed must be worked out for different kinds of activity, with their different forms of mediation, division of labor, social rules and so on" (Cole & Engestrom, 1993). In the case of designing instruction, depending on the situation (e.g., the nature of the curriculum materials and their use or non-use, the teacher's knowledge and experience, the nature of classroom and school community, etc.) and the instruction being designed (e.g., a unit, lesson, or problem), the work is likely to be differently distributed, and models of the distribution of work could be made for these particular situations. My aim in this chapter (and in this dissertation) is simply to note that my framework for mathematical purposing takes a distributed perspective; I am not yet trying to identify or model that distribution or the factors that influence it.

Second, by taking a distributed perspective on designing instruction, I do not mean to diminish or under-value the role of the teacher or the impact of a teacher's knowledge, experience, and skill. The individual still matters in theories of distributed cognition (Salomon, 1993). This is related to Pea's (1993) use of "distributed intelligence" rather than "distributed cognition": He argues that people, not designed artifacts, "do" cognition (p. 50). This is also why I characterize the *activity* or the *work* of designing instruction as what is distributed. In some sense, then, a teacher can be thought of as "coordinating" or "orchestrating" the design of instruction in his or her classroom. For example, teachers make decisions, both explicit and implicit, about which resources to use and how to use them. Although a variety of factors mediate these decisions, it is the teacher who ultimately assembles the instructional components. Furthermore, some cognitions may not be distributable (Perkins, 1993), which implies that there may be some aspects of the work of designing instruction that can only be done by the teacher. For example, the work of designing instruction involves knowing about the particular students in the classroom. Although the curriculum or educational research can offer information about students in general (e.g., common misconceptions or likely solution methods), and this information may help teachers better "see" their own students, only teachers know about their particular students. And, finally, even though the work of designing instruction can be conceptualized as distributed work, teachers do not necessarily see themselves as active agents in the design process (Silver et al., 2009).

Why Distinguish between the "What" and the "How" of Designing Instruction?

Conceptualizing the design of instruction and its subtask, mathematical purposing, as distributed work has a number of benefits. It offers a way to articulate what work needs to be done to design purposeful instruction, without specifying how, when, or by whom the work is or should be accomplished. This implies, for example, that curriculum use or non-use is not seen as "bad" or "good." Making a distinction between what the work is and how it gets done enables the incorporation of findings from different literatures—for example, teacher planning, curriculum use, and lesson study—into the same framework.

Viewing the design of instruction as distributed work also allows for a consistent framework to be used across the developmental trajectory of teaching, rather than requiring one for beginners and a different one for experienced teachers. What changes with experience is not the work that needs to be done, but the distribution of that work. This has the potential of creating face-validity with preservice teachers who typically do not see their cooperating teachers using the planning processes advocated in their teacher education courses. Distinguishing between the *what* and the *how* of designing instruction can help connect the process of writing lesson plans to the work that is being accomplished. It also supports investigation of questions about how best to distribute the work of designing instruction at different phases of teachers' careers.

Although it is based on a distributed-design perspective, the framework for mathematical purposing presented in this dissertation is not an effort to describe or model the distribution of work. Instead, the framework aims to describe *what* the work of mathematical purposing is. However, this framework could be used in later studies to investigate other questions about mathematical purposing, such as how the work is distributed in particular situations or how teachers learn to do different elements of the work.

Review of the Literature

In this section, I review the literature that provides the foundation for my framework for mathematical purposing. Selecting literature to review was complicated because there is no single body of research that focuses directly on the work of mathematical purposing, yet almost all of the countless studies of teaching or the design of instruction are potentially informing. Thus, my review of the literature is by no means exhaustive. Instead, I have strategically selected studies that seemed most informative for unpacking and articulating the work of mathematical purposing.

Some of this work is prescriptive, specifying what teachers or instructional designers ought to do when they plan for or enact instruction, whereas other studies are descriptive, aiming to portray what teachers actually do during different aspects of their work. Although all of the studies reviewed here are related to practice, only some focus on the work; others investigate teachers' cognitive processes and other influences on their actions. From the latter, I try to extract aspects of the work of mathematical purposing. Throughout the review, I summarize the ways each line of research informs the development of a framework for the work of mathematical purposing.

Instructional Activities and Mathematical Tasks

Mathematics tasks determine the mathematical learning opportunities available to students by shaping both the content students learn and their view of the subject matter (National Research Council, 2001). Doyle (1986) argues that "the curriculum exists in classrooms in the form of academic tasks that teachers assign for students to accomplish with subject matter" (p. 365). Doyle conceptualizes an "academic task" as consisting of four components:

- A goal state or end product to be achieved (e.g., answers to questions, solution to a problem, oral responses in a discussion);
- A problem space or set of conditions and resources available to accomplish the task (e.g., notes, textbook information, models from teacher);
- The operations involved in assembling and using resources to reach the goal state or generate the product (e.g., remembering answers, applying a rule, formulating one's own method); and
- The importance of the task in the overall work system of the class (e.g., a percentage of one's overall or daily grade). (Doyle, 1988, p. 169)

In this work, task is not synonymous with "activity," which refers to how groups of students are organized for working, duration, physical space, type and number of students, the resources used, and the expected behavior of teachers and students (Doyle & Carter, 1984).

Stein and colleagues' work on the QUASAR project focused on the nature of mathematics tasks, how tasks evolve during classroom instruction, and the implications for student learning (Stein et al., 1996; Stein et al., 2000). This extensive body of research built on Doyle's conception of academic task to define "mathematical task" as "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (Stein et al., 1996, p. 460). In their definition, even if there are a number of smaller problems or questions, it is not considered to be a new task unless the "underlying mathematical idea toward which the activity is oriented changes."

Stein et al. (1996) conducted an extensive analysis of mathematical tasks and their enactment. They focused on the following features of tasks that had been identified by mathematics educators as "important considerations for the engagement of student thinking, reasoning, and sense-making": the number of solution strategies; the representations that could be used to solve the problem; and the communication requirements (i.e., the extent to which it demands explanations and/or justifications). A number of significant findings emerged from these analyses. One is that different tasks require different types of thinking. That is, the cognitive demand (i.e., the kinds of thinking needed to a solve task) varies with the task. Second, the cognitive demand of a task can change as is it set up by the teacher and then enacted with students. Third, the researchers found that tasks with high cognitive demands are the most difficult to implement well: High-level tasks often degrade into lower-level tasks during instruction. Finally, their studies have linked maintenance of cognitive demand with student learning, with the greatest gains in student achievement occurring in classrooms where high levels of cognitive demands were consistently maintained (Stein et al., 2000).

Through this work, the researchers distinguished four categories of tasks related to cognitive demand—memorization tasks, procedures without connections, procedures with connections, and doing mathematics—along with features that help differentiate between types of tasks. The types of tasks and their features are captured in their "Task Analysis Guide" (Stein et al., 2000), a tool that has proved useful in teacher education and professional development. Important to note is that this work is not meant to imply that all instruction should focus on cognitively demanding tasks (i.e., procedures with connections or doing mathematics). Stein and colleagues emphasize that tasks should be aligned with instructional goals:

> Since the tasks with which students become engaged in the classroom form the basis of their opportunities for learning mathematics, it is important to be clear about one's goals for student learning. Once learning goals for students have been clearly articulated, tasks can be selected or created to match these goals. Being aware of the cognitive demands of tasks is a central consideration in this matching. For example, if a teacher wants students to learn how to justify or explain their solution processes, she should select a task that is deep and rich enough to afford such opportunities. If, on the other hand, speed and fluency are the primary learning objectives, other types of tasks will be needed. (Stein et al., 2000, p. 11)

Implications for mathematical purposing. The research on mathematical tasks informs the development of a framework for mathematical purposing in two main ways. First, it identifies some of the central features of tasks and instructional activities, on which I based my use of "task" and "instructional activity" in the framework. Loosely, by "tasks," I mean the problems or exercises in which students are asked to engage, including the resources available, operations/methods to be used, and the end products to be achieved. I distinguish "task" from "instructional activity," which I use to include the task, as well as what teachers and students do with the task as it is enacted.[6]

Second, the considerations in the Task Analysis Guide (Stein et al., 2000, p. 16) point to aspects of the work of unpacking a task to determine its mathematical purpose. For example, one of the criteria for "doing mathematics" tasks is that they "require students to access relevant knowledge and experiences and make appropriate use of them in working through the task." Thus, part of the work of mathematical purposing is articulating any relevant student knowledge and experiences and determining how they are connected to the work of the task. One of the criteria for "procedures with connections" tasks is that they "usually are represented in multiple ways." This means that an aspect of the work of mathematical purposing is unpacking the multiple ways in which a procedure can be represented.

I now turn to examples from the instructional design literature. Although this literature focuses mainly on the *how* of designing instruction, examining the components of various models of the instructional design process provides insight into the work of mathematical purposing.

Instructional Design

Smith and Ragan (2005) define *instructional design* as "the systematic and reflective process of translating principles of learning and instruction into plans for instructional materials, activities, information resources, and evaluation" (p. 4). In the instructional design literature, the term "instruction" is used broadly and applies to a range of settings, including business, military, government, vocational, informal adult education, as well as traditional school settings. The instruction is not necessarily "delivered" by a teacher in a classroom, but can refer, for example, to the design of online modules or individualized training programs. The scope of the instruction can vary as well, from large-scale programs to individual lessons. In K-12 settings, the systematic design work of both curriculum developers and teachers can be considered instructional design.

A vast number of models have been created to reflect the instructional design process. These models are prescriptive, detailing the authors' take on the steps that should be implemented to design instruction. Despite the multitude of models, they are more similar than different (Andrews & Goodson, 1980; Tessmer & Wedman, 1990). The core processes included in most models can be described by the generic "ADDIE" process: analyze, design, develop, implement,

[6] Although I further discuss how I am using "instructional activity" in Chapter 6, for purposes of this dissertation, it is not necessary to make clear analytic distinctions between "task" and "instructional activity." What matters for this work is that I am using "instructional activity" to include <u>both</u> the task (e.g., problem statement, given representations, worksheet, available materials, etc.) that students are engaged in, as well as what the students and teachers do as they engage in that task (e.g., work format, language used, questions asked, explanations given, solution methods used, etc.).

and evaluate (Gagné, Wager, Golas, & Keller, 2005; Gustafson & Branch, 2002). And within those processes, most models attend to the following central components: learners, objectives, methods, and evaluation (Morrison, Ross, & Kemp, 2007). The variation tends to be in the number of phases included in the model and its graphic representation (Gagné et al., 2005).

Because my aim in reviewing the instructional design literature is to better understand the *work* of mathematical purposing (not *how* it is done), I do not discuss the processes promoted by the various models. Instead, I look across selected models to describe and compare what they attend to with respect to the four central components (learners, objectives, methods, and evaluation). I chose models to review using a taxonomy proposed by Gustafson and Branch (2002), which classifies instructional design models into three categories: classroom oriented, product oriented, and system oriented. Classroom-oriented models are intended for individual classroom instruction. I examined two classroom-oriented models: the Gerlach and Ely (1980) model and the Reiser and Dick (1996) model. Systems-oriented models are typically used to develop an entire course or curriculum. I examined two systems-oriented models: one from Gagné et al. (2005) and the other from Smith and Ragan (2005). I did not review any product-oriented models because these are typically used to develop short, technologically intensive modules that are not implemented by a teacher.

Learners. Each of the models I reviewed includes a step about assessing the entering characteristics of the learners. For instance, Smith and Ragan (2005) identify four types of learner characteristics that should be analyzed: cognitive (both general characteristics and specific prior knowledge); physiological; affective; and social. Similarly, Reiser and Dick (1996) suggest finding out about students' general ability level; the skills and knowledge they bring to the instructional situation; and the attitudes students have toward both what is being taught and learning in general. Information about the entering characteristics of the learners informs the type and design of instructional strategies. It might impact, for example, the pacing, content, number and difficulty of examples and practice exercises, amount of structure, grouping, or vocabulary used (Smith & Ragan, 2005). Analyzing entering characteristics of the learners is different then identifying prerequisite knowledge or skills needed to achieve a particular learning objective, which is also a part of each model.

Objectives. Each model includes the specification of learning outcomes as a key step in the design process. Outcomes are usually nested from general to specific. With the exception of Gerlach and Ely (1980), the more general outcomes are called "goals" (usually modified, such as "learning goals" or "instructional goals") and the more specific outcomes called "objectives" (e.g., "learning objectives," "performance objectives," "instructional objectives," or "behavioral objectives”). Gerlach and Ely make a deliberate choice to use only one term, “instructional objective,” for all sizes of learning outcomes; however, they do discuss sub-objectives or “en route” objectives. Thus, although the terminology is not consistent across models, the hierarchical, nested, and connected nature of learning outcomes of increasing specificity is seen in each. For example, course goals/objectives are elaborated by unit goals/objectives, which are elaborated by lesson goals/objectives. Even at the lesson level there are usually a number of smaller goals/objectives. Although understanding this relationship between general goals and specific objectives is important, Gagné et al. (2005) notes that the links are often missing:

> One source of complexity in defining educational goals arises form the need to translate goals from the very general to the increasingly specific. Many layers of such goals would be needed to be sure that each topic in the curriculum actually moves the learner a step closer to the distant goal. Probably, this mapping has never been done completely for any curriculum. Thus, there tend to be large gaps from general goals to the specific objectives for courses in the curriculum. A major problem remains—the need to define course objectives in the absence of an entire network of connections between the most general goals and the specific course objectives. (p. 57)

The instructional design models distinguish among different types of learning objectives, most using a taxonomy based on Gagné’s (1985) five categories of learning outcomes: intellectual skill, cognitive strategy, verbal information, attitude, and motor skill. Once a goal is specified, most of the models include a stage of decomposing a larger goal into subcomponents, often by analyzing the possible steps taken to complete a task and/or the perquisite knowledge needed at each step.

Each model stresses the importance of specifying goals and objectives in terms of what <u>students</u> will learn or be able to do as a <u>result</u> of instruction, not what the teacher or students will do during instruction. Although general goals can be less precise, each model specifies the essential components of a well-written objective. The number of components varies with the particular model, but all objectives include a description of: (1) the observable student behavior or action that will demonstrate learning; (2) the conditions under which the behavior/action is to occur; and (3) the standard against which the behavior/action will be evaluated. Thus, when writing objectives, words such as “appreciate,” “know,” and “understand” are considered ambiguous because it is not clear what is meant by these verbs and they cannot be directly observed. Instead, one must determine what it would look like (e.g., what students would be able to <u>do</u>) if they appreciated, knew, or understood.

Smith and Ragan (2005) acknowledge that writing specific, observable objectives has been the subject of controversy:

> Many educators are opposed to writing specific statements of learning outcomes because they believe it leads to lower levels of learning. This inference may have developed because of the common practice of writing objectives that describe declarative knowledge that in no way represents the real goal of instruction, which is often a problem-solving goal. This trivialization of objectives is not the fault of the process of writing goals, but the expertise, creativity, and perseverance of the designer. It is more *difficult* to write good goals for high-level cognitive and affective outcomes, but not impossible. (p. 78)

The proponents of instructional design also argue that the clarity attained from writing learning outcomes as specific statements of observable behaviors helps to articulate the meaning of broader goals and to focus instructional activities and assessments.

Methods and evaluation. Neither the specification of learning objectives nor the identification of prerequisite knowledge dictates an instructional approach or a method of evaluation. In fact, there are usually many different ways to obtain and assess a particular learning outcome. Analyzing the different instructional methods and types of assessments discussed in each of the instructional design approaches is beyond the scope of this dissertation. However, the important idea to take away from this part of the design process is the need for a "match" between the goals, the instructional strategy, and the method of evaluation (Smith & Ragan, 2005).

Implications for mathematical purposing. In many ways, the instructional design literature is far removed from a framework for mathematical purposing. Its main focus is to describe *how* to design instruction through the development of prescriptive models of and techniques for the design process. The instruction being designed is broadly conceived and attempts to apply across content areas, thus the models do not take into account the nuances of mathematics instruction. And, both the method of articulating objectives by decomposing a task into subtasks and the focus on specifying only observable objectives skew the development of learning goals toward procedural fluency over other strands of mathematical proficiency.

Despite these issues, the instructional design literature offers a number of insights into the work of mathematical purposing. One implication for the work of mathematical purposing is the importance of considering the learners—in particular, what they are bringing to instruction and how that influences the nature of the instructional activity. The literature also makes a useful distinction between the prior knowledge of the learners and the content prerequisites of the instructional activity. Anther important idea is that learning goals/objectives are of different types and grain sizes and that it is important to understand the links between broader and more specific goals. There also needs to be a "match" between goals, instructional activities, and assessments.

As mentioned in Chapter 1, understanding this "match" has interesting implications for teaching with curriculum materials. Based on the instructional design literature, it can be assumed that curriculum developers understand the intended match between the activities and student learning goals. The choices developers have made, however, often remain hidden from teachers as they interpret and adapt the materials for use in their classroom (Ball & Cohen, 1996; Ben-Peretz, 1990). Davis and Krajcik (2005) point to this problem in their discussion of "educative curriculum materials"—i.e., materials designed to promote teacher learning (in addition to promoting the learning of the K-12 students). They synthesize from the literature five "high-level guidelines" for the role curriculum materials could play in teacher learning. They also propose a set of design heuristics that suggest kinds of information that could be provided to teachers, how materials could help teachers understand the rationales behind particular decisions, and how teachers could use these ideas in practice. In addition to highlighting the importance of connecting an instructional activity to its instructional purpose, Davis and Krajcik's guidelines and design heuristics point to aspects of the work of mathematical purposing—for example, considering how to relate units across the school year, analyzing how students typically think about a particular topic, and analyzing instructional representations.

I now turn to another line of work that focuses specifically on the design and evaluation of mathematics instruction: criteria and protocols for curriculum analysis and lesson planning.

Criteria and Protocols for Curriculum Analysis and Lesson Planning

Another line of work that helps unpack the work of mathematical purposing is curriculum analysis and lesson planning. As with the instructional design literature, criteria and protocols for curriculum analysis and lesson planning are typically prescriptive[7]; however, descriptions of the work of mathematical purposing can be abstracted from the steps prescribed. I discuss three well-known, research-based protocols below. The first is Project 2061's criteria for curriculum analysis. The others are both lesson planning protocols used in teacher education: the lesson

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[7] A notable exception are the non-prescriptive curriculum analysis tools presented in Ben-Peretz's (1990) *The Teacher-Curriculum Encounter: Freeing Teachers from the Tyranny of Texts*. A main purpose of these tools is to help teachers interpret the intentions/orientations of curriculum materials. Some of the instruments simply provide methods of analysis, and teachers select the "internal" constructs to use for evaluation. Other instruments, including a scheme for analyzing teacher's guides, offer "external" schemes that are meant to be adapted to teachers' interests. Although the main dimensions—subject matter, learners, milieu, and teachers—are useful, because the teacher's guide analysis scheme mainly includes categories that do not help unpack the work of mathematical purposing (e.g., "degree of teacher autonomy" and "consideration of teachers' needs"), I do not discuss this analysis scheme in detail. However, I did find that it suggested two components of the work of mathematical purposing: understanding the nature of the mathematics intended to be taught and considering what the methods of inquiry imply to students about the nature of mathematics and what it means to engage in mathematical work.

planning process presented in Van de Walle's widely used elementary mathematics methods text and Smith et al.'s *Thinking Through a Lesson Protocol*. I first list the components of each protocol and then discuss how this line of work informs the development of a framework for mathematical purposing.

Project 2061's criteria for curriculum analysis. Project 2061, a long-term math and science program of the American Association for the Advancement of Science (AAAS), has developed a curriculum-materials analysis approach that uses research-based criteria to examine both the content and instructional design of textbooks (AAAS, 2006). The first stage in the analysis protocol is the identification of learning goals. Goals are taken from a variety of sources, including NCTM standards documents and state curriculum frameworks. The importance of specific learning goals is emphasized: "These goals must be explicit statements of what knowledge and skills students are expected to learn, and they must be precise. Vague statements such as 'students should understand fractions' are not adequate" (AAAS, 2006). The next stage in the process examines the alignment between learning goals and instruction using the following research-based criteria:

Category I: Identifying a Sense of Purpose I.1 Conveying unit purpose I.2 Conveying lesson purpose I.3 Justifying sequence of activities	**Category V: Promoting Student Thinking about Mathematics** V.1 Encouraging students to explain their reasoning V.2 Guiding interpretation and reasoning V.3 Encouraging students to think about what they've learned
Category II: Building on Student Ideas about Mathematics II.1 Specifying prerequisite knowledge II.2 Alerting teacher to student ideas II.3 Assisting teacher in identifying ideas II.4 Addressing misconceptions	**Category VI: Assessing Student Progress in Mathematics** VI.1 Aligning assessment VI.2 Assessing through applications VI.3 Using embedded assessment
Category III: Engaging Students in Mathematics III.1 Providing variety of contexts III.2 Providing firsthand experiences	**Category VII: Enhancing the Mathematics Learning Environment** VII.1 Providing teacher content support VII.2 Establishing a challenging classroom VII.3 Supporting all students
Category IV: Developing Mathematical Ideas IV.1 Justifying importance of benchmark ideas IV.2 Introducing terms and procedures IV.3 Representing ideas accurately IV.4 Connecting benchmark ideas IV.5 Demonstrating/modeling procedures IV.6 Providing practice	

In mathematics, these criteria have been applied to a range of algebra and middle grades mathematics teacher's guides and student texts. The ratings of various materials that resulted from the analyses are not pertinent to this dissertation, but the analysis process and criteria offer insights into the work of mathematical purposing. For example, Category I suggests that the sequence of activities can convey information about the instructional purpose. This implies that the work of mathematical purposing involves understanding how the sequence of instruction is intended to support and develop students' engagement with the intended mathematics.

Lesson planning protocols. In their teacher education programs, preservice teachers are typically given formats or protocols to follow when planning lessons. The formats and protocols vary widely in detail and structure (Cathcart, Pothier, Vance, & Bezuk, 2000). I focus on two protocols that, rather than specify a particular format, attempt to detail considerations that should be made when planning a mathematics lesson. The first is from a widely used mathematics methods textbook, *Elementary and Middle School Mathematics: Teaching Developmentally* (Van de Walle, 2007), and the second is the *Thinking Through a Lesson Protocol* (TTLP), which was developed by Smith and colleagues at the University of Pittsburgh and is used with both preservice and inservice teachers around the United States (Hughes & Smith, 2004; Smith & Bill, 2004). Both protocols are geared toward a specific problem-based teaching approach in which students are presented with a cognitively demanding task (Stein et al., 2000), are given time to explore the problem independently or in small groups, and then share and discuss their solutions as a whole class. I present each protocol below and then highlight features of the protocols that contribute to an understanding of the work of mathematical purposing.

Van de Walle (2007, p. 62) presents nine steps for planning a lesson, organized into three main categories:

Content and Task Decisions	Teaching Actions	Completed Plan
1. Determine the mathematics. 2. Think about what your students bring to the mathematics. 3. Design or select a task. 4. Predict students' approaches to a solution.	5. Articulate student responsibilities. 6. Plan the BEFORE activities. 7. Plan the DURING hints and extensions. 8. Plan the AFTER discussions.	9. Write out the plan. • Mathematical goals • Tasks and expectations • BEFORE activities • DURING hints/extensions • AFTER format • Assessment notes

"Before," during," and "after" reflect the three phases of the ascribed problem-based teaching approach (i.e., set up and launch the problem, monitor students' independent work, and orchestrate a whole-class discussion of the problem). Van de Walle notes that the first four steps are the most crucial, as "decisions made here will define the content and the task that your students will work on" (p. 61). The tasks used in a lesson can be created by the teacher or selected from a textbook. In addition to the general lesson planning steps above, Van de Walle offers an "activity evaluation and selection guide" to help teachers analyze textbook tasks:

Step 1: How is the Activity Done?

- Actually do the activity. Try to get "inside" the task or activity to see how it is done and what thinking might go on.
- How would *children* do the activity or solve the problem?
    - What materials are needed?
    - What is written down or recorded?
    - What misconceptions may emerge?

Step 2: What is the Purpose of the Activity?

- What *mathematical ideas* will the activity develop?
    - Are the ideas concepts or procedural skills?
    - Will there be connections to other related ideas?

Step 3: Will the Activity Accomplish Its Purpose?

- What is *problematic* about the activity? Is the problematic aspect related to the mathematics you identified in the purpose?
- What *must* children reflect on or think about to complete the activity?
- Is it possible to complete the activity without much reflective thought? If so, can it be modified so that students will be required to think about the mathematics?

Step 4: What Must You Do?

- What will you need to do in the *before* portion of your lesson?
    - How will you activate students' prior knowledge?
    - What will the students be expected to produce?
- What difficulties might you anticipate seeing in the *during* portion of your lesson?
- What will you want to focus on in the *after* portion of your lesson? (Van de Walle, 2007, p. 52)

The *Thinking Through a Lesson Protocol* (TTLP) shares a number of features with the Van de Walle approach. It is designed to move teachers "beyond the structural components of a typical lesson plan and provides the opportunity to focus on specific ways in which the teacher can advance students' mathematical thinking during a lesson" (Hughes, 2006, pp. 65-66). The protocol provides a series of questions for teachers to consider as they plan to implement a high-level mathematics task with their students:

Part 1: Selecting and Setting up a Mathematical Task

- What are your mathematical goals for the lesson (i.e., what is it that you want students to know and understand about mathematics as a result of this lesson)?
- In what ways does the task build on students' previous knowledge? What definitions, concepts, or ideas do students need to know in order to begin to work on the task? What questions will you ask to help students access their prior knowledge?
- What are all the ways the task can be solved?

    - Which of these methods do you think your students will use?
    - What misconceptions might students have?
    - What errors might students make?
- What are your expectations for students as they work on and complete this task?
    - What resources or tools will students have to use in their work?
    - How will the students work—independently, in small groups, or in pairs—to explore this task? How long will they work individually or in small groups/pairs? Will students be partnered in a specific way? If so in what way?
    - How will students record and report their work?
- How will you introduce students to the activity so as not to reduce the demands of the task? What will you hear that lets you know students understand the task?

Part 2: Supporting Students' Exploration of the Task

- As students are working independently or in small groups:
    - What questions will you ask to focus their thinking?
    - What will you see or hear that lets you know how students are thinking about the mathematical ideas?
    - What questions will you ask to assess students' understanding of key mathematical ideas, problem solving strategies, or the representations?
    - What questions will you ask to advance students' understanding of the mathematical ideas?
    - What questions will you ask to encourage students to share their thinking with others or to assess their understanding of their peer's ideas?
- How will you ensure that students remain engaged in the task?
    - What will you do if a student does not know how to begin to solve the task?
    - What will you do if a student finishes the task almost immediately and becomes bored or disruptive?
    - What will you do if students focus on non-mathematical aspects of the activity (e.g., spend most of their time making a beautiful poster of their work)?

Part 3: Sharing and Discussing the Task

- How will you orchestrate the class discussion so that you accomplish your mathematical goals? Specifically:
    - Which solution paths do you want to have shared during the class discussion? In what order will the solutions be presented? Why?
    - In what ways will the order in which solutions are presented help develop students' understanding of the mathematical ideas that are the focus of your lesson?
    - What specific questions will you ask so that students will:
        - make sense of the mathematical ideas that you want them to learn?
        - expand on, debate, and question the solutions being shared?
        - make connections between the different strategies that are presented?
        - look for patterns?
        - begin to form generalizations?
- What will you see or hear that lets you know that students in the class understand the mathematical ideas that you intended for them to learn?
- What will you do tomorrow that will build on this lesson?

Implications for mathematical purposing. These protocols help identify a number of practices to include in a framework for mathematical purposing. Although the two lesson planning protocols are focused on a particular approach to teaching mathematics, many of the considerations and practices apply to lessons of any type. Like the instructional design literature, there are steps in all three related to identifying goals or purposes, although in the lesson planning protocols there is less attention to having goals of different grain sizes and types. All three stress the importance of aligning enactment with goals, and the lesson planning protocols identify moves teachers can make during a lesson to keep it "on track" (e.g., planning questions that get students to focus on key ideas). Each has steps that involve attending to student thinking (e.g., likely solution strategies, possible misconceptions), their prior knowledge, and the prerequisite knowledge for the task.

The lesson planning protocols include steps that help teachers get "inside" the task (e.g., solving the problem themselves). However, it is not always clear from the protocol why teachers are doing these steps or how they are supposed to use the resulting information in their planning or instruction. Furthermore, the relationship among the steps is masked by the linear nature of the protocols, and therefore, the protocols do not capitalize on the fact that, in practice, all of the steps are interdependent and mutually informing. For example, determining all the different methods that can be used to solve a task or sequencing solution methods for discussion can help determine the mathematical goals of the task. Instead, these protocols simply contain a step that directs teachers to identify the goals, but offer no suggestions for how to figure out what those goals should be.

The instructional design literature and lesson planning protocols offer models for what instructional designers and teachers ought to do when they design instruction. Next, I turn to literature aimed at finding out what classroom teachers actually do when they plan.

Research on Teacher Planning

Empirical studies of teacher planning began in the 1970s. These studies were typically descriptive, seeking to document teachers' planning practices using observations, surveys, interviews, think-alouds, analyses of their written plans, stimulated recall, and ethnography (Clark & Peterson, 1986). The studies found a variety of types of teacher planning, mostly corresponding to different grain sizes of instruction. For example, Yinger (1980) describes five levels of planning in which teachers engage: yearly planning, term planning, unit planning, weekly planning, and daily planning. Like the nested nature of classroom instruction (e.g., lessons in weeks in units in years), the different types of planning are not done in isolation, but are nested and interact with each other (Clark & Peterson, 1986). Early research on planning was usually conducted in isolation from other aspects of teaching practice, but later was included in other studies, such as those about knowledge transformation or in expert-novice comparisons[8] (Clark & Dunn, 1991).

This early research focused mainly on the planning practices of experienced teachers; however, there was typically no attempt to evaluate the practice of these teachers (e.g., whether their planning practices led to higher-quality instruction or student learning). Instead, many of these descriptive studies seemed aimed at disproving the popular linear model of planning most often associated with Tyler (1949). This rational, objectives-based model depicts lesson planning as a four-step process: (1) specifying behavioral objectives; (2) choosing appropriate learning activities; (3) organizing and sequencing the chosen activities; and (4) selecting evaluation procedures.[9]

Studies of teachers' planning practices overwhelmingly found that experienced teachers do not follow this linear model—both the steps teachers take and the order in which they complete the steps are different. Many studies found that objectives are seldom the starting point for teachers during lesson planning; in fact, many experienced teachers do not even write down objectives because they feel the instructional purpose is inherent in the activity itself (Borko & Niles, 1987; McCutcheon, 1980, 1981). Instead, teachers typically begin by identifying the subject matter content and an activity to be used, and then consider aspects such as materials, goals, objectives, and assessments (Borko & Niles, 1987; Clark & Peterson, 1986; Clark & Yinger, 1979; Peterson, Marx, & Clark, 1978; Yinger, 1980). Furthermore, teachers' planning is rarely linear, and alternative models based on empirical studies reflect the cyclic nature of the planning process (Clark & Yinger, 1979; Yinger, 1980). As mentioned above, although these models focus on the *how* of designing instruction, information about the *what* of mathematical purposing can be seen in the components of the models. I describe the components of a few key models below.

Yinger (1979, 1980) observed an elementary teacher for five months in an effort to study planning as it "occurs naturally in the classroom." He characterized two central aspects of the teacher's planning: planning for instructional activities and the use of instructional routines. When planning instructional activities, the teacher made decisions with respect to seven features: location (i.e., the physical spot in the classroom); structure and sequence; duration; participants; acceptable student behavior; teacher's instructional moves; and content and materials. Over time,

[8] I do not review such studies here, but some are included in later sections of this chapter.
[9] This linear model is also reflected in many of the models of instructional design.

many of the decisions for particular activities became routinized through four types of routines: activity routines, instructional routines, management routines, and executive planning routines. The use of routines simplified the teacher's planning work; often planning boiled down to the selection and sequencing of various routines. Decisions about content and materials, however, were aspects of activity planning that did not become routinized (Yinger, 1979).

Shavelson's (1983) model characterizes teacher's plans as "scripts" for carrying out the interactive part of teaching and identifies the "task" as the basic instructional unit of planning. Shavelson synthesized from the literature six elements of a task that teachers plan: (1) the subject matter to be taught; (2) the materials students will work with; (3) the activity (i.e., what the teacher and students will do in the lesson, including sequencing, pacing, and timing); (4) the teacher's goals, or general aims for the task; (5) the students' abilities, needs, and interests; and (6) the social-cultural context, which includes the classroom community and how students will be grouped during the lesson (p. 402). Like Yinger, Shavelson includes a time dimension in his conception of planning. One time-related issue is the nested nature of planning (e.g., yearly, unit, etc.). Another is that planning decisions made early in the year impact the rest of the year's instruction—for example, once beginning-of-the-year, long-term decisions have been made, lesson planning often involves organizing tasks within this framework.

May (1986) proposed a "practical planning model" to reflect how experienced teachers plan. Her model places instructional activities and their flow at the heart of planning, influenced by the simultaneous consideration of various elements such as content, perceived student needs, perceived student interests, and curriculum resources. Her model does not depict planning as a step-by-step process. Instead it is descriptive, showing different types of considerations without proposing an order for their completion.

Studies of teacher planning repeatedly found that teachers' written plans do not reflect the thinking that occurs during their preactive work and that not all of teacher's planning occurs during structured planning times (Clark & Peterson, 1986). To find out more about teachers' "unstated plans," Morine-Dershimer (1978) interviewed teachers at the beginning of the school day about that day's reading lesson. Although the teachers rarely mentioned student ability, specific learning objectives, teaching strategies, or seating arrangement in their initial general response about their plans for the day, they did have ready responses when probed about these specifics. This implies that teachers do think about these aspects of instruction, even if they are not included in their stated plans.

Zahorik (1970) investigated the impact of planning on teachers' sensitivity to students during instruction. He found significant differences between the six teachers who were given a structured lesson plan two-weeks in advance and the six who were told immediately before the lesson what they were supposed to teach. The teachers who planned their lessons were less sensitive to students during their lessons. For example, they asked fewer open-ended questions and more frequently tried to shape student responses to reflect their own views. These findings—though only exploratory—have important (potentially negative) implications for the work of mathematical purposing because they could suggest that "increased mathematical purposing" could make teachers less responsive to students. However, Zahorik offers a more promising (and practical) suggestion in light of his results: that teachers continue to identify goals for student learning, but also plan to be responsive and sensitive to students' contributions:

> Retain the goals-experiences-evaluation type of planning or use any other type of planning that has pupil learning as a basis, but add to it a plan that focuses directly on teacher behaviors. That is, along with the typical plan, which can be described as a plan for pupil learning, develop a teaching plan that identifies types and patterns of teacher behaviors to be used during the lesson. In relation to teacher behavior that is sensitive to pupils, the teacher would identify a group of behaviors such as reflecting pupils' remarks, and deciding when and in what order to use them. The teacher would make teaching behavior that is sensitive to pupils and many other teaching behaviors conscious, purposeful, controlled actions. In short, this suggestion is to broaden the scope of planning to include specific teacher behaviors. (pp. 150-151)

Experience has been found to play a major role in shaping the nature of teachers' planning. Borko and Niles (1987) considered the research on planning in relation to Feiman-Nemser's (1983) developmental model of teaching by asking experienced teachers how their planning practices had changed since they began teaching. The teachers responded that they now had a better understanding of the curriculum and needs of individual students and thus were better able to plan for individual students and groups. In addition, the teachers reported an improvement in their ability to teach to long-range goals. When asked what they tell student teachers about planning, the teachers identified "knowing what you are going to teach, the entering knowledge of those you are going to teach, and where you are headed" as the three most important variables in the planning process (Borko & Niles, 1987, p. 180).

Interestingly, the complicated work of planning with curriculum materials was not recognized in this early planning literature. Researchers thought that there was nothing to study about teacher planning for subjects that rely heavily on textbook materials because "planning is largely eliminated by the publishers and authors of these systems" (Clark & Yinger, 1979, p. 5). For example, Yinger (1979) argued that during textbook teaching, because a teacher relies on the curriculum materials to specify the sequence, duration, and structure of the lesson, "textbook teaching could virtually eliminate the need for preactive planning in some classrooms" (p. 168).

As the research on curriculum use described at the beginning of this chapter has shown, this is clearly not the case.

Implications for mathematical purposing. The research on teacher planning, unlike the literature on instructional design and protocols for lesson planning, is primarily descriptive rather than prescriptive. One of the main findings of this research is the rejection of the linear planning model as a reflection of how teachers plan in practice[10]—although, what teachers *do* do when they plan remains underspecified. The research on teacher planning does, however, point to a number of important considerations and factors to include in a framework for the work of mathematical purposing. For example, the literature identifies central features of instructional activities that need to be considered (e.g., Yinger's (1979, 1980): location; structure and sequence; duration; participants; acceptable student behavior; teacher's instructional moves; and content and materials).[11]

The planning literature also suggests that the work of mathematical purposing includes a time dimension. As discussed in the instructional design section, the nested nature of instruction implies that goals are similarly nested and, therefore, are of different grain sizes and need to be considered over time. In addition, instruction itself occurs over time and builds on past experiences. Thus, once classroom routines have been established, they do not need to be re-planned and, in fact, can be thought of as "doing" some of the design work. Thus, established classroom routines and long-term planning can be viewed as resources over which the design of instruction is distributed, which further elaborates the distributed perspective on the design of instruction. However, what is not discussed in this literature, and seems important for mathematical purposing, is a consideration of if and how established routines support or conflict with the intended mathematical point.

Another central finding from this research is that, in addition to identifying goals for student learning, it is also critical to plan specific teacher moves that support the desired type of instruction. Although Zahorik (1970) focused on planning to be responsive to students, for mathematical purposing, this can be translated into the need to detail specific teacher moves that help students engage with the intended mathematical ideas and practices. In fact, examples of these types of moves were seen in the lesson planning protocols discussed above. For example, Smith et al.'s (2004) *Thinking Through a Lesson Protocol* directs teachers to plan "specific questions" to ask "so that students will: make sense of the mathematical ideas that you want them

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[10] In fact, it is not what instructional designers do in practice either (Tessmer & Wedman, 1990).
[11] These features resonate with Doyle's (1986, 1998; Doyle & Carter, 1984) components of tasks and activities discussed earlier.

to learn; expand on, debate, and question the solutions being shared; make connections between the different strategies that are presented; look for patterns; and begin to form generalizations." Important for mathematical purposing is that the types of moves specified would depend on the mathematics to be taught. That is, different sorts of teacher moves orient students toward different mathematical work.

Next, I review the literuatre on lesson study, a type of professional development based on carefully designing, implementing, and revising lessons. The discussions in which teachers engage and the plans they develop through this process point to key components of the work of mathematical purposing.

Lesson Study

"Lesson study" is a Japanese form of professional development and school improvement in which teachers collaboratively plan, implement, reflect on, and revise a lesson (Fernandez, 2002; Fernandez & Yoshida, 2004; Lewis, 2002; Stigler & Hiebert, 1999). Throughout the process, there is an "unrelenting focus on student learning" (Stigler & Hiebert, 1999). Specific student learning goals are explicitly stated and used in both the design of the lesson and its evaluation. A broad goal or theme is selected (e.g., to develop a love of learning), along with specific, often more content-focused, goals for the particular lesson. Lewis (2002) identifies four levels of goals that teachers simultaneously attend to during lesson study:

- Goals specific to the lesson;
- Goals specific to the unit;
- Broad goals of the subject area; and
- Long-term goals for student development. (p. 61)

Lewis also provides a list of questions to guide planning during lesson study. These questions point to components of the work of mathematical purposing:

1. What do students currently understand about this topic?
2. What do we want them to understand at the end of the lesson?
3. What is the "drama," or sequence of questions and experiences that will propel students from their initial understanding to the desired understanding?
4. How will students respond to the questions and activities in the lesson? What problems and misconceptions will arise? How will the teacher use these ideas and misconceptions to advance the lesson?
5. What will make this lesson motivating and meaningful to students?
6. What evidence about student learning, motivation, and behavior should be gathered in order to discuss the lesson and our larger research them? What data collection forms are needed to do this? (p. 64)

Fernandez and Yoshida (2004) and Lewis (2002) both provide examples of lesson plans from Japanese lesson studies. Although the formats differ, the components included are similar:

The lesson plans contain detailed information about the unit in which the lesson is situated, including a description of the unit's mathematical terrain and main instructional activities. Plans list unit goals that articulate the ideas students will understand, skills and procedures students will be able to do, and/or attitudes and dispositions students will exhibit from engagement in the unit's activities. Plans also describe the particular students in the class, both in general and with respect to the mathematics of the unit. These descriptions include what students already have learned and can do, how they typically solve the types of problems in the unit or lesson, what difficulties they have, and their attitudes toward math. The sequence of lessons for the unit is outlined; each lesson's goals and activities explained.

Descriptions of the unit are followed by details about the particular lesson under study. The specific goals for the lesson are stated. Lesson goals are narrower than unit goals, but like unit goals, can range in content, from concepts to be learned to dispositions to be fostered. For example, the lesson goals for a first-grade mathematics lesson were organized into four types:

1. Interest • Attitude:
   (How well do the students) attempt to progress in calculating subtraction while using concrete objects. (How well do the students) attempt to present their ideas.
2. Way of Thinking:
   Ability to solve problems by using previously learned concepts and/or the idea of breaking numbers into tens.
3. Expression • Processing of Concepts:
   Be able to do the calculation of "12 - 7."
4. Knowledge • Skills:
   Understand the meaning and method of the calculation of "12 - 7." (Fernandez & Yoshida, 2004, p. 78)

The lesson's goals are followed by an elaborate plan for the lesson's progression. The instructional activities and steps to implement them are listed, often with exact wording for the tasks and questions to be posed by the teacher. Possible student responses to tasks and questions are anticipated, along with possible teacher responses. There are notes for the teacher providing information such as the purpose of particular parts of the lesson, what to watch for as students work, things to point out or remind students of, what a particular task or question is designed to bring up, and what order to have students discuss solutions. Also included is how each part of the lesson will be assessed. These evaluation questions are often mapped back to the goals of the lesson.

The plans include explicit comments about the thinking behind the design of both the lesson and unit. These comments reflect the detailed discussions in which teachers engage as they plan the lesson. Discussions focus on issues such as: why the problems and the particular numbers were selected (based on both mathematical and student-focused considerations); and what manipulatives should be provided, including what mathematics each highlights, what solution methods are afforded, what residue is left, as well as managerial things like pieces not getting lost, ease of use, and number available. Teachers consider how students will be encouraged to discuss their work; whether students should work individually or in groups; and how to conclude the lesson. They discuss how to use the board space. All of their decisions relate to the lesson's goals. Although many decisions are reflected in the lesson plan, even with the extent of the elaboration, the written plan cannot reflect all of the detailed design work in which teachers engage while planning the lesson. And, interestingly, routine activities (like bowing at the opening of the lesson) were not discussed or included in the plan (Fernandez & Yoshida, 2004).

When asked why they developed such a detailed plan, the first-grade teachers in Fernandez and Yoshida's (2004) study heralded the benefits of anticipating student solutions and how to respond to them: "Anticipations prepare the teacher for understanding student responses and solutions that occur in the classroom and equip the teacher with appropriate reactions to these….Providing this detail in the lesson plan prepares the teacher to make use of student responses to lead the class to the desired outcome in terms of their thinking and understanding" (p. 46).

Implications for mathematical purposing. The explicit attention to learning goals in the lesson study process make this literature a particularly fruitful site for unpacking aspects of the work of mathematical purposing, and the discussion above identifies many practices and considerations to include in the framework. For example, the lesson study literature unpacks different types of learning goals to attend to (e.g., ideas, procedures and skills, attitudes and dispositions). These different types of learning goals resonate with the goal of mathematical proficiency and show that all of the strands of mathematical proficiency can be worked toward even in a single lesson. Lesson study also attends to student motivation. But unlike many motivation-inspired attempts to "make math fun" that essentially remove the mathematics from the activity (Heaton, 1992; Hill et al., 2008), lesson study ties student motivation directly to the mathematical learning goals.

Lesson study also helps unpack what it means to attend to nested goals of different grain sizes, which the literature reviewed in the sections above also suggested is an important component of the work of mathematical purposing. Even in the plan for an individual lesson, there is explicit attention to the broader unit goals, as well as to the sequence of lessons and the location of the particular lesson in the unit. This points to another important feature: that the lesson (as well as the sequence of lessons) has a "drama." In lesson study, specific attention is paid to how the lesson engages and progresses students from where they enter the lesson to where it is hoped they will be at its conclusion. There is also a plan for how student responses will be used to advance the lesson—not just advance students toward the learning goals, but also to advance the lesson's story.

In addition to attending to broader goals and their progression, in lesson study, each detail of the lesson has a specific purpose. For example, rationales are provided that explain what a particular problem or question is intended to bring up or what mathematics is intended to be highlighted by a representation. This suggests the importance of detailing instruction (e.g., specifying the exact wording of tasks and questions; the numbers used in problems and examples; how ideas will be recorded on the board; etc.) and identifying how these details are designed to engage students with the intended mathematics as components of mathematical purposing.

The last line of research I review is studies of instruction. This literature analyzes and unpacks various aspects of the work of teaching, sometimes from the teacher's perspective and sometimes from the researcher's. Both types of analytic accounts of practice offer important insights into the work of mathematical purposing.

Studies of Instruction

Once I began reading studies of instruction through the lens of articulating the mathematical point and orienting the instruction, I could "see" the work of mathematical purposing in almost every study. Reviewing all studies of instruction is beyond the scope of this dissertation; therefore, I focus on selected research that either specifically aims to unpack the work of mathematics teaching or that has an explicit focus on goals. I begin with the work of three scholars (Magdalene Lampert, Deborah Ball, and Ruth Heaton) who have studied their own practice to unpack both the work of mathematics teaching and its knowledge demands. I then turn to studies of instruction (in general) that also unpack aspects of the work of teaching. These studies include comparisons of expert and novice practice and research on instructional decision-making. Lastly, I review some of the few studies of instruction I found that focus directly on goals (Simon's hypothetical learning trajectories, Mary Kennedy's *Inside Teaching*, and Schoenfeld's Teacher Model Group). As before, after reviewing this collection of research, I step back to discuss the implications for developing a framework for mathematical purposing.

Lampert's problems of practice. Lampert (1986, 1990, 1992, 2001) examines her own practice in order to unpack and articulate the work of teaching. Her work contains detailed analyses of the mathematics she was trying to teach her fifth-grade students, the teaching moves she made, and the rationales for her actions. Two aspects of her research are particularly foundational for the work of mathematical purposing: (1) her characterization of "problems" of practice; and (2) her articulation of the work of preparing for a lesson, in particular, her unpacking of the mathematical terrain.

In her seminal book, *Teaching Problems and the Problems of Teaching*, Lampert (2001) analyzes a year of her own fifth-grade teaching to unpack and analyze the "problems" that teachers routinely encounter in their work:

> To study teaching practice as it is enacted in school classrooms we need an approach to analysis that can focus on the many levels in action at once, integrating the investigation of the problems of practice that a teacher needs to work on in a particular moment with the investigation of problems of practice that are addressed in teaching a lesson or a unit or a year. The study of practical problems a teacher works on to teach each individual student can not be separate from the study of the practical problems of teaching different kinds of groups or teaching the class as a whole, as all of these elements of the work occur simultaneously in the public space of the classroom. The problems are all tackled at once, by the same person. The work aimed toward accomplishing any single goal of teaching needs to be examined in concert with examining concurrent work, perhaps aimed toward other goals, even toward conflicting goals across the temporal, social, and intellectual problem space in which practice occurs. (pp. 2-3)

That teaching involves the simultaneous management of multiple problems across time and space is a theme throughout Lampert's work. Lampert (1985) argues that competing purposes and concerns can sometimes result in "unsolvable problems." She casts this part of the work of teaching as being a "dilemma manager" because, even though there is no "right" answer, a teacher must respond in practice to construct solutions to the dilemma faced.

Three of the problem domains Lampert analyzes in *Teaching Problems and the Problems of Teaching* are particularly informative for describing the work of mathematical purposing: teaching while preparing for a lesson, teaching to deliberately connect content across lessons, and teaching to cover the curriculum. Although a typical lesson in Lampert's classroom was structured around a single, teacher-created mathematics problem, Lampert argues that the work she describes needs to be done by teachers whether or not they are using curriculum materials: "Even with the availability of such resources, one must prepare to use a particular activity with a particular class by investigating the intellectual content of the work entailed in such a way as to be able to support the relationship between that content and a specific group of students" (p. 118).

In her description of the work of preparing for a lesson, Lampert notes that "teaching a lesson begins with figuring out where to set the particular students one is teaching down in the terrain of the subject to be taught and studied" (p. 188). Thus, to determine where to begin a lesson, she needed to "characterize the subject matter to be taught" and "characterize the students to be taught." Lampert intentionally uses "characterize" to reflect "an active, constructive kind of cognition, and to indicate the practitioner's responsibility for the unique content of the characterization" (p. 118). She also notes that this characterization is both tentative and revisable.

Characterizing the students involves thinking about the class as a whole and about particular students in relationship to the mathematics to be taught. To do this work, Lampert anticipated the various strategies students might use to solve the day's problem, as well as where they might get stuck or distracted. She also looked at what methods and procedures particular students had been using in prior work and considered students' dispositions toward mathematics.

To characterize the mathematics, Lampert mapped the mathematical terrain that she expected would be covered in a problem context. Conducting a detailed analysis of the mathematical terrain is seen throughout Lampert's research. For example, Lampert (1986) unpacks what it means to know multi-digit multiplication by analyzing four types of knowledge of multi-digit multiplication and the connections among these types of knowledge: intuitive knowledge (e.g., invented algorithms); computational knowledge (e.g., being able to competently execute standard procedures); concrete knowledge (e.g., knowing how to manipulate objects to get an answer); and principled conceptual knowledge (e.g., place value, commutativity and associativity of addition). Lampert (1992) analyzes the standard long division algorithm to unpack its mathematical content. This analysis includes explaining why the algorithm works and identifying the underlying concepts that justify each step; analyzing what big mathematical ideas it is connected to; and comparing it to other standard procedures for multi-digit arithmetic to identify similarities (e.g., all involve decomposing the numbers to operate on them) and differences (e.g., long division begins on the left rather than the right).

Lampert (2001) displays the mathematical terrain of a problem context as a visual "map," with nodes representing what students would be learning to do (e.g., adding, multiplying, dividing, decomposing and recomposing to work with large numbers, choosing units, judging relative magnitude) and the concepts that support this work (e.g., relationship between addition and multiplication, fractions, place value), along with lines to represent the connections among these concepts and practices. Lampert argues that by elaborating the mathematical terrain in more and more detail, the curriculum can be seen as "engaging students with an organic whole rather than with a set of discrete topics" (p. 258). This enables both "maintaining a holistic perspective on big mathematical ideas" and "teaching conventional topics" (p. 262). Lampert also analyzes how the problem connects to the terrain and how different solution methods engage students with different mathematical ideas.

In addition to specifying big ideas, topics, concepts and procedures, practices, tools, language, symbols, conventions, and the connections among these as learning goals for students, Lampert also wanted students to learn about the nature of mathematics and what it means to know and do mathematics. Lampert (1990) notes that the goal of students' learning new ways of knowing mathematics changes the mathematical focus of a task (e.g., the <u>content</u> to be learned is mathematical argument rather than traditional topics or the finding of answers) and impacts the role of the teacher and students as they engage in those tasks.

These types of mathematical learning goals are not achieved in single lessons but are developed over time. The development of ideas over time is facilitated by what Lampert (2001) calls "teaching to deliberately connect content across lessons." By making connections explicit, teachers help students study "substantial and productive relationships in the content" that are not easily addressed in a single lesson. Lampert illustrates a number of ways teachers can make connections across lessons, for example, by determining a strategic progression of problems from lesson to lesson; by choosing contexts or representations that span lessons; by building a shared language across lessons; and by using student work as a way to build coherence.

In addition to characterizing the subject matter and the students, when preparing for a lesson, teachers make decisions about the activities in which students will engage—that is, what students will *do* during the lesson. To do this, teachers need to figure out what responses a problem or task will elicit from students and specify how those responses support the teaching of the intended subject matter. For Lampert, once she determined the problem she would use, she began the work of "settling on an agenda for classwork," in other words, figuring out the particular moves she planned to make and when she planned to make them. This included determining the precise statement of the task; sequencing the problems to be used; deciding what materials will be available; and making notes about mathematics she wanted to highlight.

Ball's bifocal perspective. Ball (1993a, 1993b) studies her own teaching of third-grade mathematics to unpack the work of mathematics teaching and its knowledge demands. I focus on two key ideas that are emphasized throughout her work: (1) teaching requires managing the tensions inherent in being simultaneously respectful of, responsible for, and responsive to the discipline and to learners; and (2) the goals for students' mathematics learning include developing both students' understanding of specific mathematical ideas and their capacity to engage in mathematical work.

Ball (1993a) characterizes teaching as involving the "insightful consideration of both content and learners, consideration that is at once general and situated" (pp. 158-159). Considering the content involves "careful analysis of the specific content to be learned: the ideas, procedures, and ways of reasoning," and considering the learners involves developing "understandings of students themselves and how they learn the particular content":

> This bifocal perspective—perceiving the mathematics through the mind of the learner while perceiving the mind of the learner through the mathematics—is central to the teacher's role in helping students learn with understanding. (p. 159)

Ball (1993a) examines the work of constructing and using representational contexts by applying this bifocal lens to both the mathematics topic and the representational context. This involves analyzing the mathematics topic itself (e.g., its constructs, its multiple meanings, and its connections to other ideas), as well as what conceptual dimensions of the topic are visible in the representational context (e.g., which ideas are highlighted and which are obscured). It also involves considering how students think about the topic (e.g., their prior knowledge and specific experiences with both the topic and its representations, as well as with related topics, both in and outside of school; how their learning of other topics might support or interfere with their learning of the new topic), as well as the accessibility of the representational context (e.g., what might be unfamiliar or distracting). Getting the representational context up and running during instruction requires considerations about language (e.g., what language will be used and conventions for its use), and, when multiple representations are in play, it involves thinking about how to link them. Throughout her analysis, Ball emphasizes that the student learning goals for mathematics go beyond particular ideas, but also include "ways of seeing, interpreting, thinking, doing, and communicating that are special to the community of those who make and use mathematics" (p. 158). Thus, using representations—as well as engaging in other mathematical practices—are both means of learning mathematics <u>and</u> end learning goals for students.

Ball (1993b) discusses three dilemmas of "developing a practice that respects the integrity both of mathematics as a discipline *and* of children as mathematical thinkers": representing the content, respecting children as mathematical thinkers, and creating and using community (p. 376). Throughout the article, Ball mentions numerous goals for student learning. Some goals are related to understanding specific mathematical ideas (e.g., "that -5 is, in one sense, more than -1 and, in another sense, less than -1" (p. 379)). But many goals focus on developing students' capacity to engage in mathematical work, including:

- engaging in mathematical practices, such as conjecturing, experimenting, and making arguments (p. 374);
- learning mathematical language and ideas that are currently accepted (p. 376);
- developing a sense of mathematical questions and activity; learning how to reason mathematically, including an understanding of the role of stipulation and definition, of representation, and of the different between illustration and proof (p. 376);
- being able to use symbolic conventions and seeing mathematical symbols as powerful ways of communicating mathematical ideas; and

- seeing patterns and conjecturing about their generalizability (p. 384).[12]

As Ball illustrates each dilemma with examples from her teaching, she demonstrates that designing and implementing activities that develop students' capacities with respect to these goals requires unpacking what these mathematical practices look like in the context of particular content and thinking about how to "build bridges between what they already know and what there is to learn" (p. 387).

Heaton's "Relearning the Dance." In *Teaching Mathematics to the New Standards: Relearning the Dance*, Heaton (2000) describes and analyzes her efforts to change her teaching practice to reflect the then new mathematics standards. In contrast to Lampert and Ball, Heaton decided to follow the standards-based textbook adopted by her district because she expected that using the text would make her transition much easier. She quickly realized this was not the case. In fact, reading the teacher's guide often generated more questions than it answered. For example, the meanings of terms—both instructional (e.g., "investigate" or "powerful insight") and mathematical (e.g., "pattern" or "composition of functions")—were often unclear; and she did not know how to decide which parts of a lesson to prioritize or which to skip if pressed for time.

Heaton found, to her surprise, that she could not simply read the teacher's guide and be fully prepared to teach the day's lesson. In order to navigate the interactive work of teaching, she needed to understand the mathematical purposes of both the activities in the textbook and the questions she asked her students. This was particularly evident during a lesson in which she asked students to look for patterns in a table. Uncertain of the point of the lesson, Heaton felt unable to move away from the script in the textbook. She lacked understanding of the mathematical terrain and how to use the task to move students through that terrain.

Heaton concluded that she needed a better understanding of "the nature of the mathematics to be learned, the importance of these particular mathematical ideas in the discipline, the place of these ideas within the K-12 curriculum, and how children make sense of these ideas" (p. 35). Much of this information could not be found in the teacher's guide, and even when the curriculum did offer support, there was still work to do to interpret the resources provided. For example, the teacher's guide presented a sample dialogue as a way to support teachers in understanding the mathematical content. However, the mathematical significance of the sample student responses was not made explicit, and Heaton misunderstood them to be responses she could expect from her students.

[12] I do not list everything that could be counted as a mathematical goal for student learning here. In Chapter 3, I describe how I analyzed the work of Ball (1993a, 1993b) and Lampert (1986, 1992, 2001) to develop my framework for mathematical purposing.

Heaton's experiences highlight the need for teachers to understand "the connections between the problems and activities intended for students and the important mathematical ideas the problems and activities are intended to teach" (pp. 155-156). This includes understanding what students are supposed to notice or "get" out of an activity, and what mathematics a representation is meant to highlight. As Heaton found, students are not going to learn just by doing an activity. Because textbook tasks have been deemed worthwhile by someone else, it is essential for the teacher to figure out how the task is intended to move students through the mathematical terrain: "Part of the intellectual work of teaching becomes trying to see the fundamental mathematical ideas in textbook problems, activities, and representations where the connections to larger mathematical ideas are not necessarily explicit" (p. 150).

I now shift from research by scholars who have analyzed their own mathematics teaching to studies of instruction across subject areas written by researchers who examined classroom practice to articulate aspects of the work.

Comparisons of expert and novice teaching. Leinhardt (1989, 1993) compared the practices of expert and novice teachers across different strands of teaching practice. One strand analyzed in this work is the use of "agendas." An agenda is a teacher's operational plan for teaching a lesson. It includes the goals and subgoals for the different lesson segments, along with the actions that can be used to achieve those goals. Agendas are not necessarily visible in teachers' lesson plans, existing instead in teachers' mental representations of their lessons.

Leinhardt found important differences between the agendas of novice and expert teachers. Novices' agendas tended to include objects and actions, but no goals. In contrast, expert agendas made clear the subject matter topic being taught in the lesson. They contained a sense of what is "conceptually important" in a lesson and where the lesson is situated in the broader continuum of lessons. Experts differentiated between lesson types (e.g., review and introduction), and were aware of the different components of their lessons and the goals for each. Unlike novices, experts simultaneously thought about their lessons along two tracks: their own action-and-goal sequence and their students'. They anticipated possible problems students might have with both the instructional approach and with the content, and their agendas included tests of student understanding. When describing the plans for their lessons, experts began by telling what they had done the previous day, implying that they saw their lessons as connected. Overall, experts had a clear sense of where they were trying to go in their lessons, which helped them deal with the uncertainties of teaching:

> Experts' agendas often provide a sense of the logical flow or, at a minimum, a clear goal. Making the logical flow explicit helps the teachers handle the interrupts that occur in normal classroom situations. If the planned action or sequence must be stopped or

> altered, experts use the basic logic of the lesson to set new goals and substitute comparable moves. Lacking such a general and flexible goal structure, novices are unable to do this because their plans are unintentionally tied to specific actions rather than to the goals of the lesson. (Leinhardt, 1993, p. 24)

Another teaching strand Leinhardt (1993) analyzes is the use of routines. Leinhardt defines routines as “small, socially shared, scripted pieces of behavior,” such as lining-up, question-posing, and turn-taking (p. 16). Routines assist with management, as well as with the development and exchange of ideas. Experts use routines flexibly and across lesson segments, and a particular routine can serve multiple goals. Because routines are automatic, they can be “unpackable” pieces of knowledge and, in most cases, are not mentioned in teachers’ plans and agendas.

Teachers’ use of routines and agendas reduces the information-processing demands of instruction, freeing them to deal with other complex tasks of teaching, such as giving explanations. In fact, Leinhardt argues that novices’ explanations were fragmented and unfocused because they were less skilled at developing agendas and using routines. Novices had a difficult time figuring out the essential ideas and components in their explanations, and their explanations were not connected like those of experts. There were also differences in novices’ and experts’ uses of representations to support their explanations. Experts tended to think carefully before selecting and introducing a representation to determine whether it made salient the aspects of the mathematics they were intending to highlight. Doing this requires knowing what one is using the representation for and how students will likely engage with it. For example, experts typically used “something familiar to teach something new” so that students did not have to learn both the representation and the focal mathematics. In contrast, novices often used “something new to teach something new,” requiring students to learn <u>both</u> the representation and the content, and as a result, students often learned neither (Leinhardt, 1989, p. 66).

Borko and Livingston compared the planning, instruction, and post-lesson reflections of three pairs of student teachers and their cooperating teachers (Borko & Livingston, 1989; Livingston & Borko, 1989, 1990). Like Leinhardt, the researchers found a number of similarities across expert teacher practice. The expert teachers all engaged in yearly planning, unit/chapter planning, and lesson/section planning. The longer-range plans organized the content and general curricular sequence; decisions about details of instruction were made closer to the teaching of a particular lesson. Novices, on the other hand, engaged only in short-term planning, which researchers speculated was likely a reflection of their student-teaching status.

Echoing the findings from the earlier studies on teacher planning, the expert teachers did not prepare written plans for their lessons, but had mental plans that included a general sequence of the lesson's content and instructional activities. Final decisions about specifics such as pacing and exact examples were often determined in response to students during instruction, although experts often had explicit contingency plans for when things did not go as expected. Novice teachers also created mental plans that were flexible with respect to timing, examples, and problems. However, unlike the experts, novices spent much of their planning time thinking about how to represent the content to students. Novices often had difficulty prioritizing aspects of the content to be taught and anticipating student difficulty.

Expert teachers used their lesson agendas to guide their interactive teaching—for example, to make connections between students' contributions and the lesson's objectives. Thus, experts were able to keep their lessons on track, accomplishing their objectives while, at the same time, being responsive to students. Novices had trouble keeping the lesson on track when responding to student comments, particularly when student questions prompted an unplanned explanation or when they needed to generate an example on the spot. In contrast, expert teachers had routines for practices such as explanation, guided problem solving, and summarizing, as well as a repertoire of explanations for concepts and an awareness of common student errors and misconceptions. They could draw on these to maintain the balance between objectives and student questions, enabling them to fill in their general sketch of the lesson using student productions. Experts were also able to generalize across problems to highlight relationships among topics and provide a "big curricular picture" (Livingston & Borko, 1990, p. 380). Novices' explanations, both planned and unplanned, lacked these types of connections.

Next I turn to the literature on teachers' decision-making and thought processes. This research has focused on describing the content and cognitive processes of teacher decision-making; the influences on teacher decisions; and the relationship between teacher decisions, behaviors, and student outcomes (Clark & Peterson, 1986; Shavelson & Stern, 1981). With respect to informing an understanding of the work of mathematical purposing, there is considerable overlap between this research and the research on teacher planning. However, there are a few additional ideas to be taken from this literature that I describe below.

Instructional decision-making and teachers' thought processes. Synthesizing across studies, Clark and Peterson (1986) categorized the content of teachers' thinking during instruction. The largest portion of teachers' reported thoughts were concerned with the learner, followed by instructional procedures; less frequent were thoughts about objectives, content, and materials. Clark and Peterson suggest that the lack of attention during instruction to objectives, content, and materials could be because decisions about these were made during planning. Decision-making during interactive teaching typically arises when a teaching routine is not going as planned. In the cases where changes do occur, they are usually only minor adjustments, not major revisions (Shavelson & Stern, 1981).

Yinger and Clark (1982) studied factors teachers take into account and processes used when evaluating instructional activities. Although the processes identified are more related to *how* the work of mathematical purposing is done, the considerations suggest components of the work. The authors grouped the considerations according to whether they related to the students, teacher, or activity (p. 26):

Students	**Teacher**	**Activity**
Student involvement Student difficulty Students' task-related ability Incidental learning Student interest Cognitive outcomes Affective outcomes Student enjoyment General student outcomes Individual differences Student choice Success Student needs Challenges	Demand Fit with teacher's goals Prerequisite instruction Fit with current practice Fit with past practice "Feel" Enthusiasm	Fit of purpose and description Clarity (of procedures) Appropriateness of instructional strategy Activity type Internal consistency Age level appropriateness Brevity Variety Academic defensibility Terminology Uniqueness Sequence Design/flow Diagnostic opportunity Practicality Expansion potential

Shulman and colleagues (Wilson, Shulman, & Richert, 1987) propose a model of *pedagogical reasoning* that portrays how different forms of knowledge are used and developed in teaching. The model begins with *comprehension,* which involves developing a critical understanding of "a set of ideas, a piece of content, in terms of both its substantive and syntactic structure" (p. 119). Next is the *transformation* process, which is decomposed into four subprocesses: *critical interpretation*, which "involves reviewing instructional materials in light of one's own understanding of the subject matter"; *representing*, or the consideration of different metaphors, analogies, illustrations, activities, examples etc. to "transform the content for instruction"; and lastly, *adaption* and *tailoring*, which involve adjusting the transformations in light of characteristics about students in general and the specific population being taught. The last stages in the model are *instruction*, *evaluation*, and *reflection*. Engaging in the process results in *new comprehension*.

Sherin and Drake's "Curriculum Strategy Framework." Sherin and Drake (2009) try to identify patterns of curriculum use that might be more or less effective for implementing mathematics education reform. Although this is a study of *how* teachers use curriculum to design instruction, the findings suggest aspects of the work of mathematical purposing. For example, the authors found differences in how teachers read curriculum: some read for a general overview, some for details, and some for both. They also found that teachers who focused on their own understanding of the lesson before instruction and on students' responses during instruction were better able to make changes in their lesson in response to students, which suggests that understanding the mathematics and why each of the parts are included in a lesson are a central part of mathematical purposing.

The last set of studies of instruction that I review focus specifically on goals: Simon's hypothetical learning trajectories, Mary Kennedy's *Inside Teaching*, and Schoenfeld's Teacher Model Group.

Hypothetical learning trajectories. Simon (1995) offers the *Mathematics Teaching Cycle* as "a schematic model of the cyclical interrelationship of aspects of teacher knowledge, thinking, decision making, and activity" to address the inherent tension in teaching between direction and responsiveness to students (p. 135). The model emerged from analyses of his own teaching, in which he observed that the goals for and design of his lessons were based on two factors: his mathematical understanding and his hypotheses about students' knowledge. Simon deliberately uses "hypotheses" because, before a lesson, a teacher does not have direct knowledge of students' understandings, but has to infer them. Simon's notion of a teacher hypothesizing about student learning is similar to Lampert's (2001) use of "characterize." The idea of coordinating attention to these two factors builds on Ball (1993a).

One component of Simon's cycle is the construction of a "hypothetical learning trajectory" or "the teacher's prediction as to the path by which learning might proceed" (p. 135). A hypothetical learning trajectory has three components: (1) the learning goal; (2) the learning activities that students will engage in; and (3) the hypothetical learning process of the students (i.e., predictions about the evolution of students' thinking and understanding in the context of the learning activities). In this model, the learning goal points to the direction of the trajectory, and the activities and hypothetical learning process are interrelated and mutually informing. The hypothetical learning trajectory is continually modified throughout the teaching cycle as a result of the teacher's assessment of student learning.

Kennedy's "Inside Teaching." In her book, *Inside Teaching,* Kennedy (2005) reports on a study in which she and her research team observed and interviewed teachers to investigate what teachers actually do in the classroom and how they think about their practice. Each of the 45 participating teachers was observed teaching one lesson. The lessons spanned elementary grade levels and subject areas. After watching a video of their lesson, teachers were interviewed about their understandings of selected teaching episodes and about their rationales for their actions.

During the interviews, teachers described a wide range of intentions for their practices. Kennedy deliberately uses "intentions" rather than "goals" because only some of the rationales mentioned by teachers were things they wanted to accomplish; fears, aspirations, obligations, and personal needs also drove their actions. Teachers' intentions were categorized into the following six "areas of concern":

- Defining learning outcomes;
- Fostering student learning;
- Maintaining lesson momentum;
- Fostering student willingness to participate;
- Establishing the classroom as a community; and
- Attending to [teachers'] personal needs.

Teachers typically mentioned more than one intention for a particular practice, and in many cases, their intentions were competing or even contradictory. Kennedy found that teachers construct their practice by "weighing the momentary importance" of their intentions; lesson momentum usually trumped other concerns.

Kennedy's analysis focused on three central tasks of teaching: (1) developing the day's agenda; (2) managing conversations about content; and (3) establishing a tranquil learning environment. The practice of developing the day's agenda was most informative for articulating the work of mathematical purposing, so I focus on that here.

Kennedy identifies three subtasks of developing the day's agenda. The first is *establishing learning outcomes.* Kennedy found that teachers get ideas about learning outcomes from a variety of sources including curriculum guidelines, textbooks, and student assessments. Those guidelines are then interpreted and translated by teachers based on their own prior ideas. Kennedy points out the difference between content coverage and learning outcomes: "Even when content coverage is held constant, teachers may formulate very different learning outcomes for that content" (p. 44). That is, even though two teachers might be teaching the same topic, what they think is important for students to learn about that topic can be very different.

The second and third subtasks—*portraying content to students* and *constructing learning activities for students*—involve making decisions about how to present and engage students in the content. This includes decisions about the method of portrayal (e.g., telling students about it vs. engaging them in a puzzle), the nature of the content (e.g., procedural vs. conceptual), and what students will actually do to interact with that content. Kennedy found that when teachers used complicated portrayals and activities, it was often confusing or it distracted teachers and students from the content:

> An important substantive idea is harder to keep track of when activities involve multiple props, and it is easy for teachers, as well as students, to forget where they are and where they are going. Instead of enriching the content in the lesson, these activities can distort it. (p. 140)

Another issue Kennedy observed was that teachers sometimes walked students through the steps in various activities without ever directly addressing the content.

Teachers had multiple intentions when developing their agendas. For the subtask of establishing learning outcomes, the majority of teachers' intentions had to do with content; however, all six areas of concern influenced their decisions. For the other two subtasks, non-content related areas of concern, in particular theories about how students learn, became more prevalent.

Teacher Model Group. The Teacher Model Group at the University of California at Berkeley has been developing a comprehensive model of teaching that represents teachers' actions and decisions as representations of their knowledge, goals, and beliefs (Aguirre & Speer, 2000; Schoenfeld, 2000; Schoenfeld, Minstrell, & Van Zee, 2000; Zimmerlin & Nelson, 2000). The beliefs, goals, and knowledge are attributed to the teacher by the researchers based on observations and interviews. Thus, the beliefs, goals, and/or knowledge may not be explicitly held by the teacher and, in some cases, may even contradict those expressed by the teacher. The work of the Teacher Model Group provides insight into the connection between actions and goals, and, for developing a framework for mathematical purposing, the categories and types of goals that drive teachers' actions are particularly informative.

Their model parses teaching into nested action sequences of various grain sizes, with each action sequence corresponding to at least one goal. Goals are defined broadly as "things you want to accomplish," and range in size and scope:

> Goals occur at different grain sizes: "overarching" goals for students over the course of weeks, months, or the year; unit goals; lesson goals; goals for particular parts of a lesson, and "local" goals for particular interactions with students. Goals may be epistemologically oriented ("I want students to understand and experience physics/mathematics as a sense-making discipline"); they may be content-oriented ("students should know the three measures of central tendency and their properties"); they may be socially oriented, at various levels of grain size ("I want the class to function as a community of inquiry," or "I want this student to feel rewarded for having ventured a question"). Goals may be pre-determined (e.g., as part of a lesson image) or they may be emergent (e.g., when the class seems restless, or an interesting issue arises in dialogue with the students). And, of course, multiple goals can be (and usually are) operative at the same time. (Schoenfeld, 2000, p. 250)

Zimmerlin and Nelson (2000) analyze an algebra lesson taught by a student teacher, comparing his lesson image with the model of the enacted lesson. The student teacher had a detailed lesson image that was not reflected in his written lesson plan. His lesson image included nested "action plans" or images of the activities he planned to do, how he thought they would go, and how he thought students were likely to respond to each part. He had three types of goals associated with his action plans; not all goals had equal priority:

1. Overarching goals that span across multiple lessons, or even the year, for example:
   - Help students see where algebraic notations and procedures come from, and why algebraic rules are true
   - Develop a classroom atmosphere in which students contribute to classroom activities (pp. 268-269)
2. Major content and social goals associated with the entire teaching segment:
   - To extend the idea of subtracting exponents, particularly to the idea of zero exponents
   - To engage students in discussion of the problems (p. 271)
3. Local goals that elaborate the major content and social goals associated with each "chunk of the lesson":
   - Build student confidence
   - Confirm procedure and notation
   - Establish variant of the problem
   - Build student understanding of zero exponents (p. 273)

Some aspects of the lesson were left flexible and unplanned, supporting Zimmerlin and Nelson's contention that "teachers plan down to the level of detail at which they feel they can comfortably manage improvisationally using established interactional skills and routines" (p. 267).

For the most part, the lesson played out as the teacher expected. Action plans were implemented, with the details of the actions filled in with routines. Some parts of the lesson, of course, did not match the teacher's lesson image. There were two main causes for differences between the lesson's plan and its enactment. The first was when something took longer than anticipated. In this case, the teacher adjusted based on the priority of his goals. The second type of change was due to unexpected student responses. In this situation, the teacher experienced a "goal shift" and developed a new action plan based on the emergent goal of "dealing with student confusion."

Schoenfeld, Minstrell, and Van Zee (2000) analyze a "non-traditional" physics lesson taught by an expert teacher. The article focuses mainly on the use of the model to capture the lesson's enactment; however, the parsing of the lesson and the types of goals attributed to the teacher's actions inform thinking about the work of mathematical purposing. All actions correspond to one or more high-priority goals that are in effect for an entire segment, as well as goal(s) specific to the particular action. For example, in one "interactive elicitation" segment of the lesson, the authors identify two types of goals that were driving the instruction: goals of content coverage, or a list of topics that the teacher wanted to be sure to cover; and goals of completeness, or how thoroughly the teacher wanted each topic to be discussed.

Implications for mathematical purposing. The above studies of instruction suggest a number of core categories for the work of mathematical purposing, as well as unpack many of the details of that work. I summarize some of the key ideas below.

Chapter 1 characterized mathematics teaching—by definition—as purposeful work. Studies of instruction elaborate this and highlight that mathematics teaching, in fact, requires attention to <u>multiple</u> purposes, both mathematical and non-mathematical, which are often in conflict. These purposes may be explicitly or implicitly held by teachers and, as Kennedy pointed out with her use of "intentions," are not always things teachers want to accomplish, but instead might be things teachers are trying to avoid or feel obligated to do.

Of course, the *intention* of avoiding something could be reframed as the *goal* of wanting that something <u>not</u> to happen; an *obligation* can similarly be reframed as the *goal* of wanting to meet that obligation. However, that is not my point. Instead, I use this to illustrate the lack of common language for goals/purposes/intentions. Not only is there no shared terminology, but the same term gets used in different ways. For example, "goals" are used to denote both the rationales for teacher actions and the aims of student learning (which can of course be reframed as teacher intentions/purposes). Although the multiplicity of meaning creates challenges for articulating the work of mathematical purposing (and precipitates my invention of language in this dissertation), it seems that what is important to take from this issue is not an over-concern about what term is being used. The important idea is that specifying goals/intentions for both student learning <u>and</u> teacher actions is part of the work of mathematical purposing. And, importantly for teaching to the mathematical point, student learning goals and purposes for teacher actions should be linked.

Studies of instruction also help unpack the nature and content of student learning goals. Many of the ideas in this research resonate with the literature discussed earlier in this chapter: Student learning goals are nested and revisable, often emerging and evolving during instruction; they are of different types (e.g., epistemologically oriented, content oriented, practices oriented, etc.) and of different grain sizes. This literature helps expand the notion of different grain sizes. One grain size is related to the grain size of the mathematics (e.g., a goal of learning about a big mathematical idea or developing ways of reasoning vs. learning a particular mathematical fact). A second type of grain size relates to what it is a goal "of" (e.g., a goal for the year vs. a goal of a particular interaction). These types of grain sizes may be correlated—for example, perhaps year-long goals are more likely to be about big ideas. However, this is not always the case. The goal of a particular interaction, for instance, could be about a big idea as well.

Studies of instruction highlight that engaging in mathematical practices and developing ways of reasoning mathematically are both means for and end goals for student learning. This literature also shows that mathematics instruction requires teaching topics and making explicit connections to big ideas—whether lessons are structured to teach with problems or teach by topics just makes this work look different. For example, because Lampert was teaching with problems, her lessons were structured around big ideas. Thus, to show she was covering the conventional school topics she had to locate these topics within her problem contexts. On the other hand, a teacher who is teaching with a curriculum that is organized in a topic-by-topic approach might have to do extra work to make connections across topics and to bigger mathematical ideas.

These studies suggest that mathematical purposing involves understanding the nature of the mathematics being taught through an instructional activity and how that mathematics sits in the larger mathematical terrain. Situating the notion of a hypothetical learning trajectory in the mathematical terrain underlies the "location" meaning of mathematical point mentioned in Chapter 1: the mathematical point of an activity includes its starting point (where students are mathematically); the anticipated path through the terrain (how the activity is intended to engage students with the intended mathematics); the relationship of the path to other mathematical ideas in the terrain; and the intended end point.

In addition, these studies show that, because the mathematics is being characterized for learning, mathematical purposing also involves having a pedagogical view on the mathematical terrain that considers the mathematics from the perspective of the learner. Considering the terrain from the perspective of the students can help with the prioritization of leaning goals, which can inform instructional decisions such as what might be emphasized or skipped. The prioritization of particular learning goals is also reflected by the fact that mathematical purposes/goals often have both a content and a pedagogical aspect. This could be seen, for example, in the goals listed by Zimmerlin and Nelson (e.g., confirm procedure and notation, establish variant of the problem, or build student understanding of zero exponents). This literature also suggests that mathematical purposing includes knowing about learners in general in relation to the mathematics being worked on, as well as about the particular students being taught. Considering what the specific students are bringing to the activity highlights the special role of the teacher in this distributed work.

Echoing some of the research discussed earlier in this chapter, these studies suggest a number of features of instructional activities that need to be planned in detail (e.g., wording of tasks, sequence of problems, materials available, etc.) and show the importance of planning these details in relation to both the mathematics and the students. For example, as seen in Ball's and Leinhart's work, in the mathematical purposing of a representation, it is important to consider both the mathematics the representation is intended to highlight and how accessible the representation is to students. The work of the teacher in establishing and using representations can be similarly applied to other details of the activity. In addition, these studies emphasize the need to understand the connection of the activity to and its intended, logical flow through both the mathematical terrain and the curriculum, as well as the need to develop contingency plans for steering the lesson back on track. Like the research on teacher planning, these studies also demonstrate the role of routines in teaching. This has implications for mathematical purposing—for example, the need to consider if a routine helps steer instruction toward the mathematical point.

One last key idea suggested by these studies is that, during instruction, a particular teaching move is often used to achieve multiple goals. This has obvious implications for steering instruction toward the mathematical point, but also has implications for mathematical purposing. For example, work of mathematical purposing done in the service of learning about students or in the detailing of an activity can also help develop an understanding of the mathematical terrain and how the activity is intended to move students through it. Lampert, for instance, noted that anticipating student responses helped her understand the mathematics students would have opportunities to engage with and helped her learn more about her students in relation to that content. Similarly, planning the details of an explanation or unpacking the features of a representation, as Leinhardt discussed, could help unpack the mathematical terrain and identify what students are bringing to the work.

Summary: Foundations for a Conceptual Framework for the Work of Mathematical Purposing

This dissertation investigates a central task of mathematics teaching: articulating the "mathematical point" and using it to design and steer instruction. One of the results of this investigation is a conceptualization of "teaching to the mathematical point," which includes the development of a framework for the work of "mathematical purposing." This framework aims to describe what is involved in articulating the mathematics intended to be taught through an instructional activity, understanding how that instructional activity is intended to engage students with that mathematics, and orienting the activity so that it is more likely to do so. In this chapter, I reviewed the literature that provided the foundation for the development of that framework.

No single body of research underlies the work of mathematical purposing. Thus, one of the challenges of examining the literature to extract components of the framework was coordinating across the various types of studies. Some of the relevant literature is prescriptive, offering models for instructional design or lesson planning. These models tend to be either too general (e.g., aiming to apply to all types of instruction or to reflect planning for any subject area) or only focus on a particular type of teaching (e.g., problem-based teaching). The work in these models is often displayed as a series of considerations or actions, the purpose of which (i.e., how the results of doing a particular step are supposed to be used to design instruction) may or may not be apparent. Most of the models have a step that involves identifying the mathematical goals, but because this is listed as a separate step (often the initial one), the models do not make explicit how completing the other steps (e.g., solving the problem in multiple ways, anticipating student errors, etc.) actually helps with identifying and understanding the mathematical goals. In addition, the use of the mathematical goals to inform the other steps in the protocol is not usually emphasized. In other lines of research, such as studies of instruction, these tasks of teaching are often integrated. However, because this research typically focuses on some other aspect of teaching, the work of mathematical purposing can be hard to extract from the complex practice portrayed.

Throughout this review, I identified teaching practices and considerations related to the work of mathematical purposing. In most cases, though, the level of detail needed to fully articulate the work remained underspecified. Furthermore, the teaching practices described in the literature were not always tied to their use. For example, a teacher might examine how new vocabulary is used in each part of the lesson (a teaching practice discussed in Sherin and Drake (2009)) to help her better understand the mathematical terrain, to specify her mathematical learning goals, to consider the language demands placed on students in each part of the lesson, or to specify the language she will use when launching the task. Articulating the work of mathematical purposing requires more than listing practices. Attention must also be paid to the purposes for which the practices might be enacted. Moving beyond a list of practices requires a conceptual framework that articulates both particular teaching moves and the purposes they serve.

Reviewing the literature provided a useful first step in developing the framework. The literature suggested three broad, interdependent categories for the work of mathematical purposing: characterizing the mathematics to be taught,[13] characterizing the students to be taught with respect to the intended mathematics, and characterizing the instructional activity:

- *Characterizing the mathematics to be taught*: This involves indentifying mathematical learning goals of different types and grain sizes and situating those in the mathematical terrain. These types and grain sizes are related to the nested nature of instruction, the nature of the intended mathematics (e.g., big vs. specific ideas; different strands of mathematical proficiency; developing concepts, skills, mathematical practices, and connections among them), and the pedagogical activity of which it is the point (e.g., problem, representation, specific teacher question). The mathematical terrain needs to be unpacked and analyzed through both a disciplinary and a pedagogical perspective.
- *Characterizing the students to be taught with respect to the intended mathematics*: This involves considering what students bring to the activity, what they are likely to do as they engage with the mathematics, and how these influence the activity's design and implementation. This includes knowing about learners in general (e.g., likely methods, difficulties, misconceptions, errors, etc.) and about the prior knowledge and experiences of the particular students being taught.
- *Characterizing the instructional activity*: This involves unpacking and planning the details of the activity (the components of the task and what students and teachers will do as they engage in that task) in relation to both the mathematics and the students. It also involves determining the sequence of activity and how that relates to the progression of mathematical ideas.

The heart of mathematical purposing is ensuring that there is a "match" between the intended mathematics and the instructional activity so that the particular students being taught can make progress toward the specified mathematical learning goals. The categories above, though helpful, are far from a framework that adequately conceptualizes and articulates this work. In the next chapter, I explain how I turned to the data collected in this study to look for additional categories, to develop a more productive conceptual organization, and to further decompose the work.

[13] I adapted these "characterizing" phrases from Lampert (2001).

CHAPTER THREE:
METHODS OF DATA COLLECTION AND ANALYSIS

Introduction

This dissertation is a study of the mathematical work of teaching and the relationship between mathematical knowledge for teaching and instruction. It aims to unpack and articulate the practices and knowledge demands of the work of teaching to the mathematical point—in particular, of mathematical purposing. To do this, I analyzed data from mathematics lessons taught by 17 preservice elementary teachers. This chapter describes these data and the methods of analysis used.

I begin by explaining the rationale for the study's design, in particular, why I focused on single lessons taught by preservice elementary teachers and why I selected participants based on their scores on a survey of mathematical knowledge for teaching. Next, I describe the two phases of data collection and the various instruments used. I then turn to the methods of data analysis, organized according to my research questions. I conclude the chapter with a discussion of the limitations of the study.

Rationale for Study Design

Although grounded in empirical data, this dissertation is primarily conceptual. The purpose of the study is to further the field's understanding of the work of teaching mathematics and the nature of the mathematical knowledge that teaching entails. Thus, it is a study of *teaching,* not *teachers*. Even though I analyzed data from lessons taught by particular teachers, I am not trying to make claims about what teachers know or can do in the classroom, or whether they had a mathematical point or taught to it. I am not even trying to make these sorts of claims about the particular preservice teachers in my study. The sample is not representative, and the data reflect only one instance of each preservice teacher's practice.

In addition, although I selected the sample based on MKT scores, I am also not trying to make claims such as "teachers with low MKT do not attend to their goals during instruction" or "teachers with high MKT can determine the mathematical point of a textbook lesson." Instead, I selected participants with a range of MKT scores because subject matter knowledge has been shown to influence teachers' interpretation and use of curriculum and their teaching practice

(Ben-Peretz, 1990; Blunk, 2007; Borko et al., 1992; Borko et al., 1988; Hill et al., 2008; Kahan et al., 2003). I therefore thought differences in MKT would make different aspects of the work of teaching visible. Finally, this dissertation is not a study of fidelity of curriculum implementation. I am investigating what is involved in teaching to the mathematical point of an instructional activity, not how closely a textbook is followed during instruction.

In order to study classroom practice to describe the practices and knowledge demands of teaching to the mathematical point, I had to make choices about the type of data to collect—most importantly, I had to select a sample and a grain size of practice to analyze. Below I explain my choice of preservice teaching as the site for this study and why I decided to collect data related to one lesson per teacher.

Why Use Preservice Teaching as the Site for this Study?

Berliner (1988, 2004) describes five stages of skill development in teaching: (1) novice; (2) advanced beginner; (3) competent teacher; (4) proficient teacher; and (5) expert teacher. It would have been possible to investigate the work of teaching to the mathematical point by studying the practice of teachers at any of these stages. I chose to study novice practice because I hypothesized that it would make visible aspects of the work of teaching that would not be as readily seen at the other stages.

Most studies of experienced teachers do not help to articulate the work of determining and using goals. Experienced teachers do much of their lesson planning mentally, often in informal settings and at random moments in the day—such as in the shower or when driving to school (McCutcheon, 1980, 1981). Minimal information is then recorded in the form of written plans or notes (Clark & Yinger, 1979; McCutcheon, 1980). The goals of experienced teachers are often implicit, intertwined in their instructional activities and routines (Borko & Niles, 1987; Clark & Peterson, 1986). While teaching, experienced teachers make instructional decisions quickly, often without conscious thought or the consideration of alternatives (Clark & Peterson, 1986; Jackson, 1968).

These findings are not surprising given the literature on expertise. Experienced teachers, like experts in other fields, enact much of their practice automatically, and are often unable to articulate—or might not even notice—the routines and scripts being implemented (Berliner, 1986, 1987). If instruction is going well, experienced teachers are unlikely to be consciously deliberate about their actions:

> Experts do things that usually work, and thus, when things are proceeding without a hitch, experts are not solving problems or making decisions in the usual sense of those terms. They go with the flow, as it is sometimes described. When anomalies occur, when

> things do not work out as planned, or when something atypical is noted, deliberate analytic processes are brought to bear on the situation. But when things are going smoothly, experts rarely appear to be reflective about their performance. (Berliner, 2004, p. 206)

In fact, Berliner (1987) further suggests that an understanding of the complexity of their automatic routines might actually be unavailable to the consciousness of many experienced practitioners. Thus, with my aims of unpacking and articulating the work of identifying and using mathematical goals in instruction, it did not seem fruitful to study the practice of experienced teachers who might not be able to articulate how they determined the goals of their lessons and if and how they were using these goals to make decisions during instruction. Furthermore, the literature already contains analytic accounts of expert practice (e.g., the practice of Lampert (1990, 2001) and Ball (1993a, 1993b), and the analyses of expert teaching by Leinhardt (1989, 1993)) that could be examined in light of my research questions.

Nonetheless, preservice teaching might seem a peculiar site for this study because novices are likely to have difficulty teaching to the mathematical point. But it is this difficulty that, in fact, makes preservice teaching an ideal site for this investigation. The "bumps and bruises" of novice practice make visible aspects of the work of teaching and the use of mathematical knowledge in that work that might "go unnoticed in the smoother practice of more experienced teachers" (Heaton, 2000, p. 16). Because novices do not have as many established instructional routines as experienced teachers, when they do use their mathematical goals to design or manage instruction, it is more likely deliberate and thus more readily accessible in an interview. Instances when preservice teachers have difficulty teaching to the mathematical point—for example, not understanding the mathematical point of an activity or making instructional moves that conflict with their expressed mathematical point—help identify problems that must be managed when doing this work. In other words, "the neophyte's stumble becomes the scholar's window" (Shulman, 1987, p. 4).

An additional reason for studying preservice teaching is that it provides information useful for the improvement of teacher education. Even though this dissertation does not make claims about particular preservice teachers, their knowledge, or their practice, the data did surface interesting ideas and questions about preservice teachers' knowledge and instruction. The data also enabled detailed descriptions of preservice teachers' planning and instructional practices. Such descriptions of preservice teaching, though not necessarily generalizable, are valuable because they raise attention to important phenomena (Schoenfeld, 2007). Descriptions such as these can help teacher educators learn more about the knowledge and skill preservice teachers bring to student teaching and the issues they confront when trying to manage the problems of practice. In addition, these types of observations point to productive directions for the design of interventions and future research in teacher education.

Why Observe One Lesson per Teacher?

The mathematics lesson was chosen as the unit of analysis for this study for a number of reasons, both theoretical and practical. The lesson exhibits all of the complex interactions that influence teaching, yet is a small enough unit to enable detailed analysis of the relationships being studied (Hiebert, Morris, & Glass, 2003; Kennedy, 2005). Furthermore, when studying teaching structured by a textbook such as *Everyday Mathematics*, the lesson forms a natural unit of observation: Each lesson in the text typically corresponds to one day's mathematics instruction. Although *Everyday Mathematics* lessons are organized into units (which could also create a natural unit of analysis), because it is a spiral curriculum and even individual lessons touch on multiple topics, it is unclear how much teachers (and preservice teachers in particular) consider connections across lessons when using *Everyday Mathematics*. And, finally, because my observations occurred early in the student teaching semester, many preservice teachers were not teaching math daily and, thus, were only focused on individual lessons in their planning and instruction. As I discuss below, I did ask questions in the interview about if and how the lesson related to goals for the unit and school year, and, if they were teaching the next day, whether and how the observed lesson would impact their next lesson.

Even with the lesson as the unit of analysis, decisions still had to be made about how many teachers and how many lessons per teacher to observe. Constrained by what would be a reasonable size for a dissertation study, the more lessons I observed per teacher, the fewer teachers I could include. I considered the following options: one lesson per teacher; three to four consecutive lessons per teacher; three to four lessons per teacher, spread over the student teaching semester; or two sets of three to four consecutive lessons, spread over the student teaching semester. Because I am not trying to make claims about the practices of individual teachers, nor am I studying change or improvement over time, I did not need the observed lesson to be representative and, therefore, did not need to have multiple lessons per teacher. And, because I am studying *teaching*, I thought that having as many different teachers as possible would better capture the variety of issues that can arise in teaching to the mathematical point.

Data Collection

Data collection occurred in two phases. In the first phase, participating preservice teachers completed a survey measuring mathematical knowledge for teaching. A subset of preservice teachers from phase one were then selected, based on their MKT scores, for the second phase of the study. The second phase of data collection began with interviews about their mathematics and teaching backgrounds, as well as their typical lesson planning practices. Each preservice teacher was then observed teaching one mathematics lesson and was interviewed, that same day, both before and after teaching. Recordings were made of all interviews (audio) and lessons (video), and all recordings transcribed. Any curriculum materials and written lesson plans were collected. In this section, I provide details about each phase of the data collection process.

Teacher Education Context

The participants in this study were undergraduates in their final year of an elementary teacher education program at a large public university in the midwestern United States. Students typically enter this four-semester teacher education program in their junior year, after completing subject matter prerequisites. During the first three semesters of the program, their education coursework includes foundation classes (e.g., educational psychology); subject matter methods courses in literacy, science, social studies, and mathematics; and a field component (six hours per week of classroom-based fieldwork and an accompanying seminar led by their field instructor).

The fourth semester is the student teaching semester. Preservice teachers are in one classroom full time, with most continuing in the classroom in which they completed their third-semester fieldwork. During student teaching, preservice teachers work closely with their cooperating teacher and a university-based field instructor. Their experiences and responsibilities vary widely; for example, at the time of my study, some preservice teachers were already responsible for daily mathematics instruction while others taught mathematics lessons only occasionally. By the end of the student-teaching semester, there is typically a short period (e.g., two weeks) in which they “take over” all instruction for the classroom.

Phase One of Data Collection: MKT Survey

Recruiting. All preservice teachers in the third semester of the program in Fall 2007 (n=50) were recruited to participate in the study. I announced the study during their field seminars and invited anyone who was interested to participate in phase one. I sent a follow-up email that evening with additional details and requested an RVSP so that I could order an appropriate amount of food (a light dinner was provided), but said that they could still participate even if they had not responded. I also sent a reminder message the week of the survey’s administration.

I administered the MKT survey during two evening sessions (November 13 and 14, 2007). Thirty preservice teachers participated: twelve in first session and eighteen in the second. Each participant received $20 for completing the survey.

Selecting the instrument. The Learning Mathematics for Teaching (LMT) project develops survey instruments for measuring mathematical knowledge for teaching. The surveys contain multiple-choice questions reflecting common mathematics problems encountered in elementary school classrooms—for instance, evaluating students' mathematical claims, examining unusual solution methods, and determining how to best represent material or generate examples. The measures have been validated with multiple methods, including cognitive interviews and links to the mathematical quality of instruction and to student achievement (Blunk, 2007; Blunk & Schilling, 2005; Hill et al., 2005; Hill et al., 2004).

The form I administered[14] is composed of three scales: number and operations; geometry; and patterns, functions, and algebra. There are 26 number and operations items; 19 geometry items; and 16 patterns, functions, and algebra items. The form has a reliability of 0.83 (using a two-parameter IRT model) and 0.79 (using a one-parameter IRT model) (Hill, 2007). Preservice teachers had as much time as needed to complete the survey. I did not keep track of the time spent, but everyone finished within an hour.

Scoring the survey. Without viewing the responses, I scanned the completed surveys and stored the electronic copies as back up. I then gave the questionnaires to a research assistant for anonymization. She randomly assigned each participant an identification number from 1 to 30 and then removed the preservice teacher's name from each survey, replacing it with the corresponding identification number. I scored the anonymized surveys and used the LMT conversion tables to determine each participant's IRT score (i.e., the estimate of each participant's ability) for each scale. I averaged the three IRT scores to create an average MKT score.[15]

To select the participants for the second phase of the study, I sorted the average IRT scores in descending order and divided the sorted list into fifths (Table 1). The top six scores became the high-score group, the middle six became the middle-score group, and the bottom six scores became the low-score group. I also identified three alternates in each score group. I then gave the aforementioned research assistant a list of the identification numbers of the six people in each score group. She matched numbers with names and returned to me an alphabetical list of the 18 people selected for phase two (i.e., the six names in each of the high-, medium-, and low-score groups). This selection method ensured that I had six participants in each score group, yet I did

[14] I consulted with LMT Principal Investigator Heather Hill about which of the many LMT forms to use in my study. I followed her suggestion to administer one of the 2004 elementary forms.
[15] I consulted with LMT analysts Geoffrey Phelps and Merrie Blunk about how to interpret and combine the IRT scores from each scale to select high, middle, and low scores across the three scales. They suggested that I average the three IRT scores for each scale.

not know who was in which group. Therefore, my data collection and initial analyses were not influenced by survey scores.

Table 1.
MKT Survey Results

	Number & Operations	Geometry	Patterns, Functions & Algebra	
ID Number	IRT Score	IRT Score	IRT Score	Average IRT Score
		High-Score Group		
21	2.5152	2.2535	1.09	1.9529
3	1.3852	2.2535	1.78	1.8062
30	1.9105	1.1352	1.78	1.6086
6	1.3852	1.6552	1.78	1.6068
8	1.3852	1.6552	1.78	1.6068
13	1.6397	1.1352	1.09	1.2883
2	1.1443	1.1352	1.09	1.1232[HA1]
11	1.3852	1.1352	0.796	1.1055[HA2]
26	1.3852	1.1352	0.796	1.1055[HA3]
27	0.9147	0.2978	1.415	0.8758
9	0.2771	1.1352	1.09	0.8341
1	1.6397	-0.2914	1.09	0.8128[MA2]
		Middle-Score Group		
22	1.3852	0.7957	0.021	0.7340
19	1.3852	-0.2914	1.09	0.7279
29	1.1443	0.5279	0.267	0.6464
20	0.2771	0.7957	0.796	0.6229
25	0.6946	0.0897	0.796	0.5268
23	0.4825	0.5279	0.524	0.5115
18	0.6946	0.7957	0.021	0.5038[MA1]
16	0.6946	0.5279	0.267	0.4965[MA3]
15	0.4825	-0.1049	-0.218	0.0532
24	0.4825	-0.1049	-0.218	0.0532[LA3]
5	0.2771	-0.4737	0.021	-0.0585[LA2]
10	-0.5006	-0.4737	0.267	-0.2358[LA1]
		Low-Score Group		
4	-0.1185	-0.4737	-0.923	-0.5051
14	-0.3107	-0.8378	-0.453	-0.5338
28	-0.1185	-0.6548	-0.923	-0.5654
17	-1.2542	-0.4737	-0.218	-0.6486
7	-1.8406	-0.1049	-0.923	-0.9562
12	-1.0648	-1.2217	-0.923	-1.0698

HA1, HA2, HA3: High-score group alternate 1, 2, 3
MA1, MA2, MA3: Middle-score group alternate 1, 2, 3
LA1, LA2, LA3: Low-score group alternate 1, 2, 3

Phase Two of Data Collection: Lesson Interviews and Observations

Sample size. I contacted the preservice teachers who were selected from phase one, and all 18 agreed to participate in the second part of the study. In early January 2008, one of the preservice teachers had to drop out of the study because her cooperating teacher did not want the classroom to be video recorded. I contacted this person's first alternate but did not get a response, so I contacted the second alternate who agreed to participate. In February, a different preservice teacher (who at the time had already completed a background interview and had scheduled a lesson observation for February 7) canceled her observation with the expressed intention of rescheduling in March. However, despite numerous attempts, I was unable to schedule another observation. Thus, my final sample is 17 lessons.

Background interviews. Background interviews were conducted at preservice teachers' convenience between December 4, 2007 and February 2, 2008. All preservice teachers participated in a background interview before teaching their lesson, although the time between the background interview and lesson observation varied. Because I had only briefly met participating preservice teachers prior to this study (i.e., when I announced the study in their seminar and when they took the survey), the central purpose of the background interview was to establish rapport with each teacher before conducting the lesson observation and interviews.

The 45-minute background interview began with general questions about preservice teachers' experiences with and beliefs about mathematics and mathematics teaching. The second half of the interview asked about their lesson planning practices. The questions about lesson planning were included because planning is a time during which teachers think about the goals of their lessons. The complete protocol is found in Appendix A.

Pre-lesson interviews. Lesson interviews and observations took place between January 10, 2008 and February 7, 2008. The observations were scheduled at preservice teachers' convenience; the only stipulation was that the lesson not be part of a formal observation or course requirement. The pre- and post-lesson interviews were conducted on the same day as the observation at a time that fit the preservice teacher's schedule (e.g., before school, during a prep period, or while eating lunch). Unfortunately, time constraints due to the realities of life in schools sometimes necessitated moving quickly through or even skipping parts of the interviews.

The semi-structured pre-lesson interview often occurred immediately before the lesson was taught. Interviews ranged from 20 to 48 minutes, with an average length of approximately 40 minutes. The interview had four parts. It began with questions about their goals for the lesson. I probed the meaning of any goals stated and then, if not mentioned, asked about goals related to mathematical practices and whether the lesson connected to any broader unit or school-year goals. I also asked how they decided on their goals and whether they thought their goals were the same as the goals of the textbook. I concluded the first part of the interview by asking about the level of understanding they expected for their students with respect to these goals.

During the second part of the interview, I asked preservice teachers to walk me through their plan for the lesson. As they described each activity, I asked questions to probe their mathematical knowledge for teaching. In these probes, I focused on their understanding and choice of: definitions and use of language; explanations; representations; and examples/exercises. In most cases, I had read a copy of the textbook lesson in advance and had prepared specific questions related to each of these categories.[16] After they finished describing each activity or lesson segment, I asked about the main point of the activity. For each activity I also asked whether and why they thought it might be difficult for students and, when appropriate for the activity, inquired about anticipated student answers, solution methods, misconceptions, and errors. I concluded the discussion of each lesson segment by asking whether there was anything in the activity about which they were worried or unsure.

After the walk-through, the third part of the interview returned to global questions about the lesson's main mathematical point/s and what they were most confident and unsure about. The fourth and final part of the pre-lesson interview covered how preservice teachers prepared for the lesson. However, if time was running short, the planning questions were saved for the post-lesson interview. The complete pre-lesson interview protocol can be found in Appendix B.

Lesson observations. Table 2 shows the preservice teacher's name,[17] grade level, and main mathematics topic for each of the 17 lessons. Fifteen of the lessons were based (in varying degrees) on lessons from the *Everyday Mathematics* curriculum. Two lessons (Andrea's and Mia's) were not textbook-based.[18]

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[16] There were a few instances where the lesson being taught was different than the one they had originally told me or where no textbook was used. In these cases, I had not read the curriculum materials in advance of the interview.

[17] All preservice teachers' names are pseudonyms. There was no method for pseudonym selection other than I wanted each pseudonym to begin with a different letter so I could use first initials as a notational shortcut.

[18] Both Andrea's and Mia's cooperating teachers did not regularly use a textbook to teach mathematics. Andrea used a modified version of an *Everyday Mathematics* "Math Message" as a warm-up problem in her lesson; however, the majority of the lesson was spent on practice problems she created. Mia designed the activities in her lesson based on resource materials provided by her cooperating teacher.

Table 2:
Phase-Two Participants: Grade Level and Lesson Topic

Teacher	Grade level	Lesson domain	Lesson topic
Hannah	1	Number and operations	Equivalent names for numbers
Irene	1	Number and operations	Addition and subtraction fact families
Sydney	1	Number and operations	Comparison problems
Beth	2	Data	Graphing
Courtney	2	Number and operations	Comparison number stories
Rachel	2	Number and operations	Introduction to fractions
Erica	3	Number and operations	Using parentheses
Larkin	3	Geometry	Symmetry
Mia	3	Data	Graphing
Nicole	3	Number and operations	Place value (tenths and hundredths)
Tiffany	3	Number and operations	Place value (large numbers)
Gillian	4	Measurement	Metric measures of length (conversion)
Keri	4	Number and operations	Review of fraction concepts
Paige	4	Number and operations	Representing, adding, and subtracting fractions
Andrea	5	Number and operations	Division (4-digit by 2-digit numbers)
Jordan	5	Number and operations	Review of integer addition and subtraction
Zach	5	Number and operations	Adding and subtracting simple fractions

I recorded each lesson using a digital video camera. The camera was stationary, but I tried to situate the tripod at the back or side of the room, where I could be out of the way, yet still clearly capture the board as well as the preservice teacher's movement throughout the room. Preservice teachers wore a high-quality wireless microphone that could, in most cases, also capture the students' talk. I also make a backup digital audio recording, which I used to confirm inaudible student utterances when needed.

As I recorded the lesson, I made brief observational notes in an effort to identify two types of episodes: (1) episodes that could provide insight into the preservice teacher's MKT (e.g., when explaining a mathematical concept, using a mathematical representation, defining or using mathematical terms, selecting or sequencing problems, or interpreting a student comment); and (2) episodes in which the teacher did not seem to act in accordance with her or his stated goals (e.g., choosing an example that did not align with the expressed mathematical point, or not taking up a student comment that was directly related to one of the lesson's mathematical goals). These "goal shifts" have been found to be useful sites for investigating teacher decision-making (Aguirre & Speer, 2000). The collection of episodes became candidates for viewing in the post-lesson interview.

I had one major technical malfunction that resulted in only the last 20 minutes of Erica's lesson being video recorded.[19] In this case, I modified the post-lesson interview: Instead of following the usual protocol, we walked through the lesson and recreated what had happened.

Post-lesson interviews. After each lesson, I conducted a post-lesson interview (Appendix C). Interviews typically began 10 to 20 minutes after the lesson ended; however, some were conducted later in the school day or after school. During this time, I quickly downloaded the video to my computer, burned a copy of the video onto a DVD, and selected the episodes I wanted to watch and/or discuss during the interview. Post-lesson interviews ranged from 18 to 65 minutes, with an average length of 50 minutes.

The interview began by asking preservice teachers to share their reactions to the lesson. I followed up with some general questions, including whether they thought they had accomplished their goals and what they thought students had learned. In most cases, we then watched two or three short video episodes from the lesson. I instructed preservice teachers to interrupt the video at any point to explain what they were thinking or why they made particular decisions. I frequently stopped the video to ask questions as well. The interview concluded with overarching questions about their lesson goals. At the end of the post-lesson interview, preservice teachers received $80 and a copy of their lesson video on DVD.

Written materials. I collected any curriculum materials used, as well as any teacher-made student worksheets, overheads, etc. that were not part of the curriculum materials. I also made a copy of the preservice teacher's lesson plan, if one was made. Their written plans ranged from handwritten notes jotted directly on the teacher's guide to typed plans in more traditional lesson plan formats.

Analytic memos. My data collection schedule was quite intense, but when possible, I wrote informal post-observation memos. In each memo, I recorded emerging ideas related to my research questions, as well as noted anything that stood out to me about the lesson. I was able to write memos after 11 of the lessons.

[19] I did make a backup audio recording of the lesson, which was transcribed.

Data Analysis

This section details how I analyzed the data to investigate each of my research questions. As stated in Chapter 1, my research questions are as follows:

1. What is the work of determining the mathematical goals of a lesson and using those goals to design instruction?

2. What is the work of using mathematical goals to steer instruction during a lesson's enactment?
   - What problems must be managed in doing this work?
   - What teaching moves can be used to manage these problems?
   - What are some of the issues that arise for beginning teachers when managing these problems?

3. What is the relationship of mathematical knowledge for teaching and the work of determining mathematical goals and using them to design and steer instruction?

Investigating Research Question #1: Developing a Framework for Mathematical Purposing

My first question focuses on unpacking the work of determining the mathematical goals of a lesson and using those goals to design instruction—what I am now calling mathematical purposing.[20] I investigated this research question through an iterative analysis of the literature and the data to develop a conceptual framework that decomposes the work. I began by identifying general themes about what was involved in determining mathematical goals and using those goals to design instruction. These ideas were based on my reading of the literature (as summarized at the end of Chapter 2), my own teaching and teacher education experience, and my observations and analytic memos made during data collection (Ryan & Bernard, 2000). I organized these ideas into preliminary categories of what, at the time, I was calling "components of the work of lesson design for mathematics instruction."

Coding the data. My first stage of data analysis was to listen to all of pre-lesson interview audio recordings and watch all of the lesson videos. This served two purposes. One purpose was to "clean" the transcripts. I had received a grant to have the lessons and pre- and post-lesson interviews roughly transcribed, but still needed to review and polish each transcript

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[20] Throughout my discussion of the data analysis process, I use the terms "mathematical purposing" and "teaching to the mathematical point" even though this language was a result of the analyses. When I began analyzing the data, I had sense of the construct I was investigating, but did not have the language to identify it or know exactly what the work entailed (determining these was part of the aim of the study). In my early stages of analysis, I referred to the evolving construct as "the work of X" until I ultimately settled on the language of "mathematical purposing" and "teaching to the mathematical point." Rather than use "X" or cumbersome phrases like "the work of identifying mathematical learning goals for students and deliberately designing and implementing instructional activities to move students toward those goals" in this chapter, I use the language that emerged from the analysis, even though it does not reflect the chronology of the work.

for use in my analysis. The second purpose was to engage in an initial coding of the data in order to revise and further develop my preliminary categorization scheme. I looked for any examples of practices that preservice teachers engaged in that seemed related to identifying and using goals (e.g., types of mathematical goals, details of the instructional activity that were planned to help further those goals, teaching moves that seemed intended to steer the activity toward those goals, etc.). I also looked for non-examples (e.g., when the lesson seemed to get "off track," misunderstanding a key mathematical idea, etc.) and identified—from the literature, my own experience, or from ideas that the preservice teacher brought up in the post-lesson interview—work that might have helped prevent these issues.

Through this process, I reorganized and added categories and subcategories, in particular, to accommodate teaching moves or responses to interview questions that did not fit in one of the categories. In addition, I frequently renamed the categories and subcategories as I gathered more examples and became clearer over time about the work of teaching being described in each. From this initial pass through the data, I developed a set of categories and subcategories that I felt captured the main components of the work, that by this time, I had named "mathematical purposing." I also had an extensive list of example practices (codes) for each subcategory.[21]

The next stage of my analysis had two goals: (1) to ensure that my categories sufficiently captured the data and (2) to further decompose the work of mathematical purposing through a more detailed unpacking of the "insides" of each category. I did this through a process of "focused coding" on a subset of the data:

> Focused coding means using the most significant and/or frequent earlier codes to sift through large amounts of data. One goal is to determine the adequacy of those codes. Focused coding requires decisions about what initial codes make the most analytic sense to categorize your data incisively and completely. (Charmaz, 2006, p. 58)

To develop a procedure for coding (e.g., which aspects of the data to code, whether to code on the computer or on hard copies of the transcript, etc.), I experimented with data from two lessons: Jordan's and Sydney's. I selected these lessons because I thought they were different in ways that would help ensure that my coding procedure would apply across all of my data. For example, Jordan's lesson only loosely relied on a textbook, whereas Sydney's was closely textbook-based. In addition, I thought Jordan's lesson was one of the strongest examples of mathematical purposing in the data set, and Sydney's one of the weakest. Based on this, I guessed

[21] I do not include the categorization schemes in this chapter because I discuss the categories and their components in detail in Chapters 5 and 6.

that Jordan had high MKT and Sydney low MKT.[22] I describe the coding procedure I developed below.

First, I analyzed the pre-lesson interview transcript. I formatted the transcript into a table with each row corresponding to a turn of talk by the teacher or myself and each column corresponding to the first level of subcategory in the coding scheme. Then I analyzed the interview turn by turn, identifying examples of work in each category and making notes about how the example related to the work of mathematical purposing (e.g., labeling the more general practice it exemplified). After coding the entire interview, I went back through the table, column by column, and tried to sort each example I had identified into my categorization scheme. If an example did not fit, I revised or reorganized the subcategories to accommodate it.

Next I coded the lesson transcript. To my surprise, I found it difficult to use some of the categories of the coding scheme to structure my analysis of the lesson transcript.[23] In the interview, the preservice teacher and I were explicitly discussing the mathematical point of the lesson, so the categories and subcategories applied directly. This was not the case during the lesson. Identifying the work of mathematical purposing from the lesson required abstracting from the moves I saw the teacher make (or not make) during the lesson back to the work of mathematical purposing based on my understanding of what the preservice teacher's mathematical point was at the time. Therefore, I used a slightly different process to code the lessons.

As I watched the lesson video and analyzed each turn in the transcript, I made notes in response to the following questions: (1) What is the teacher doing (or could she be doing) at this moment to steer students toward her mathematical learning goals?; and (2) What does this moment have to do with the mathematical point of the lesson? I also wrote descriptive notes to aid with navigation (e.g., work format, which problem was being discussed, what the teacher wrote on the board, etc.). In addition, I recorded any ideas I had related to MKT.[24] Next, I analyzed the post-lesson interview to look for additional ideas and insight related to the two lesson-analysis questions. I then returned to the notes I had made on the lesson transcript, and

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[22] During my first pass through the data, I predicted the MKT-score group of each preservice teacher based on their lesson and interviews. I correctly predicted 12 out of the 17 score groups. I was only off by one level for 4 of the 5 incorrectly guessed scores (e.g., I predicted medium, but the preservice teacher was in the low-score group.). There was one teacher that I predicted was in the low-score group, who surprisingly was in the high-score group.
[23] In retrospect, I now see that this difficulty was the first sign of the analytic distinction I would eventually make between mathematical purposing and steering the instruction. As I describe in the section below on the evolution of the framework, I had originally included codes related to steering in the orienting category.
[24] Coding for MKT was not systematic at this point. I just made notes about anything that struck me as related to MKT as I was I was watching the video.

went, column by column, through a process similar to the one used for the pre-lesson interview of fitting the example moves into the categorization scheme.

After I developed a coding procedure, I needed to select a subset of lessons on which to continue the focused coding. Because I thought variation of MKT would make different aspects of the work of mathematical purposing visible (and this was the basis for the original sample selection), it seemed important that the subset also have variation in MKT scores. At this point, however, I still did not know the scores of the teachers. Therefore, I gave the identification numbers of the three highest- and three lowest-scoring preservice teachers to the research assistant who had anonymized the data and she returned an alphabetized list of their names (Andrea, Beth, Irene, Larkin, Mia, and Paige). This ensured that my subsample had a range of MKT scores, yet I did not know which teacher had which score.[25]

I then utilized the process I had developed with Jordan's and Sydney's data on the data from these six lessons. Over time, my categorization scheme became increasingly stable—that is, I made fewer and fewer new categories, subcategories, and codes. By the time I coded the last of the data in the subsample, I was no longer adding categories and subcategories. This final coding scheme is what was ultimately organized into my framework for mathematical purposing and the problems in steering instruction, which I describe in detail in Chapters 6 and 5, respectively.

The last stage in the coding process was to code the remaining lessons.[26] To code the pre-lesson interviews and lessons, I made coding templates in Excel, again with the main categories and first level of subcategories as the columns and turns of the transcript as the rows. When a turn or sequence of turns related to a subcategory, I noted it in the corresponding cell. Because my framework aims to describe the work of mathematical purposing, not to evaluate how well it is done, I did not distinguish between "good" or "bad" examples when coding the pre-lesson interview. However, when coding the lesson, I did indicate whether or not a particular teaching move or sequence of interactions seemed "mathematically purposeful." I used this information to identify the issues that arise when trying to manage problems in steering instruction toward its mathematical point. This is related to my second research question and is discussed in more detail below.

Although I watched the entire lesson, at this stage of analysis, I coded only the parts of the lesson that were in whole group (i.e., I did not code when students were working independently). I did this for two main reasons. First, when I coded the subsample, I did not seem

[25] However, based on their lessons and interviews, I was able to correctly predict which preservice teachers were in the high-score group and which were in the low-score group for all six preservice teachers in the subsample.
[26] I did not code Erica's lesson in this manner because I did not have a video of the entire lesson.

to be identifying any new types of work in the interactions with individual students that I had not already picked up the whole-group segments. Second, because I did not ask specifically about independent work time in the pre-lesson interview, nor did I typically ask about these interactions in the post-lesson interview (because I could not hear what was being said while I was recording, and therefore did not identify them as episodes to watch together), I did not know the teacher's mathematical point of interactions with individual students, which made them difficult to interpret. I also did not code the post-lesson interview turn by turn; however, I did use the post-lesson interview to inform my analysis of the lesson, in particular, to help me understand the reasons behind the moves made during instruction.

Evolving organization and representation of framework. Through my analysis of the data, what had started as a single-paged document of initial categories and subcategories grew into a 28-page description of the work of mathematical purposing, hierarchically organized into categories and subcategories with brief references to specific examples from the data. Needless to say, using these detailed lists to code the data proved challenging. Not only was the document itself unwieldy, I could not keep track of what work belonged in which category. Furthermore, if I—the person who developed it—found it unmanageable, how could I expect it to be useful to other researchers, teacher educators, or teachers?

The process of reorganizing my lists of categories into a conceptually structured framework occurred over many months, both during and after coding. It involved pages of rejected sketches and countless conversations. Here I recount the three most significant events in the framework's evolution.

One significant change had to do with the structure of mathematical purposing. At the time in the analysis when I decided to name the construct I was studying "mathematical purposing," I conceived of it as having three main (interdependent) categories of work (articulating, orienting, and detailing) instead of two (articulating and orienting). During subsequent analyses, I found that I was making the most progress unpacking the work of articulating and orienting, but was not getting clear about what belonged in the detailing category. I had a list of aspects of instruction that needed to be detailed, but the nature and quality of the detailing for mathematical purposing was captured in the orienting category. Thus, everything that I had listed in the detailing category was also reflected in the orienting category. I eventually realized that the detailing category was redundant and removed it.

A second major change related to my overall conceptualization of (what I eventually named) teaching to the mathematical point. While I was coding, I noticed that there were two types of items that I had been including in the orienting category. The first type was specific teaching moves related to detailing the instructional activity so that it focused instruction on the mathematical point (e.g., selecting numbers from desired cases, using representations that highlighted the intended mathematics, designing strategic questions, etc.). The second type was work of teaching that could be accomplished with any number of moves (e.g., making key ideas explicit, maintaining the mathematical storyline, etc.); in some ways, the second type could be thought of as *reasons* for doing the first type of move. I eventually distinguished between orienting the instructional activity (included as part of mathematical purposing) and steering the instruction, and recognized the second type as problems that must be managed in steering instruction to the mathematical point.[27]

However, even though I separated this second type of code (i.e., problems in steering instruction toward the mathematical point) from my framework for mathematical purposing, there is an important relationship between the work of mathematical purposing and the problems in steering instruction toward the mathematical point. At a basic level, managing the problems in steering instruction toward the mathematical point requires an articulation of the mathematical point, which is part of the work of mathematical purposing. But more than that, in practice, the work of mathematical purposing is intended to help manage the problems in steering instruction. That is, articulating the mathematical point and orienting the instructional activity are conceptualized as work that helps position instruction so that it is more likely to engage students with the intended mathematics—thus, in a way, "doing" some of the work of steering.

This relationship between the work of mathematical purposing and steering the instruction was reflected in my analysis of the data, but in some cases from the other direction. For example, as described above, when analyzing the lesson videos, I asked: *What is the teacher doing (or could she be doing) at this moment to steer students toward her mathematical learning goals?* Working backwards from the answers to this question helped identify aspects of the work of mathematical purposing. For instance, if a poorly worded question seemed to steer the lesson away from the mathematical point because it inadvertently engaged students in unintended mathematical work, a corresponding part of the work of mathematical purposing that could help manage this problem would be to design strategic questions and prompts that focus students on the intended mathematics. This type of analysis is an example of how the "bumps and bruises" of beginning teaching made visible aspects of the work.

[27] While I was coding, I had simply pulled this second type of out of the framework and placed them in their own unnamed category. (In my notes and coding templates, I informally referred to them "emerging themes" or "things you are trying to do when mathematically purposing.") After I made the mathematical purposing and steering the instruction distinction, Deborah Ball suggested using the notion of problems that must be managed.

A third major change had to do with my representation of the work of articulating the mathematical point. I had noticed parallels across the subcategories in the articulating section, which made me think that I might be able to represent the work in a matrix instead of linearly. In addition to highlighting the conceptual connections across categories, a matrix representation helps make the work of articulating the mathematical point easier to track on, because in a matrix you can "see" more of it at one time. The "parallels" I noticed were that the analytic work in each subcategory was being applied to both the mathematical terrain and to the instructional activity. Thus, I reconceptualized the work of articulating the mathematical point as applying three different analytic lenses to the mathematical terrain and to the instructional activity, and the matrix representation that will be presented in Chapter 6 fell into place.

Investigating Research Question #2: Problems in Steering Instruction toward the Mathematical Point

My second research question asks: What is the work of using mathematical goals to steer instruction during a lesson's enactment? In investigating this question, I hoped to unpack some of the problems that must be managed in doing this work and identify examples of teaching moves that can be used to manage these problems. In addition, I aimed to describe beginners engaging in this work and some of the issues they encounter.

As described above, the problems in steering instruction toward the mathematical point that emerged in my analysis began as codes in the orienting category. However, as the framework evolved and I became clearer about the analytic distinctions I was making, these codes were pulled out of mathematical purposing into their own category. In Chapter 5, I present and discuss these codes using detailed examples from the data.

I coded for these problems in the pre-lesson interviews and lessons using the processes described above. In my analyses, I found that the problems were overlapping: Each problem could be managed with a variety of teaching moves, and in many cases, a particular move could be interpreted as addressing multiple problems. In addition to marking instances in which the preservice teacher could be interpreted as managing one of these problems, when applicable, I noted whether the move seemed to steering the instruction toward the mathematical point or if it seemed to be moving things off track. After I had coded all of the data, I compiled the "off track" instances. Looking across these, I identified patterns, which eventually became the issues that arise when trying to manage the problems in steering instruction toward the mathematical point that I discuss in Chapter 5.

Investigating Research Question #3: Exploring the Relationship of MKT and Mathematical Purposing

My third research question asks: What is the relationship of mathematical knowledge for teaching and the work of determining mathematical goals and using them to design and steer instruction? I focused my investigation on the relationship between MKT and mathematical purposing.

To explore the relationship between mathematical knowledge for teaching and the work of mathematical purposing, I applied the MKT categorization scheme developed by Ball et al. (2008) to the framework I had developed for mathematical purposing. As described in Chapter 1, Ball and colleagues currently distinguish six domains of MKT: (1) common content knowledge; (2) specialized content knowledge; (3) knowledge of content and students; and (4) knowledge of content and teaching; along with two preliminary categories: (5) horizon content knowledge; and (6) knowledge of content and curriculum. For each component of my framework, I asked which domain of MKT would be drawn upon to do that aspect the work. The result is a mapping of the domains of MKT onto the framework for mathematical purposing (described in Chapter 7).

To illustrate this mapping in practice, I examined a case from the data. Because “bumps and bruises” make the mathematical knowledge demands visible (Heaton, 2000), at this point in my analysis, I revealed the preservice teachers’ MKT scores in order to select a lesson from the low-score group. Table 3 shows the MKT scores and score groups of the 17 preservice teachers who participated in phase two of the study.

Table 3.
MKT Scores of Phase-Two Participants

	Number & Operations	Geometry	Patterns, Functions & Algebra	
Name	IRT Score	IRT Score	IRT Score	Average IRT Score
		High-Score Group		
Andrea	2.5152	2.2535	1.09	1.9529
Larkin	1.3852	2.2535	1.78	1.8062
Irene	1.9105	1.1352	1.78	1.6086
Nicole	1.3852	1.6552	1.78	1.6068
Rachel	1.3852	1.6552	1.78	1.6068
Jordan	1.6397	1.1352	1.09	1.2883
		Middle-Score Group		
Courtney	1.3852	0.7957	0.021	0.7340
Hannah	1.3852	-0.2914	1.09	0.7279
Gillian	1.1443	0.5279	0.267	0.6464
Keri	0.2771	0.7957	0.796	0.6229
Zach	0.6946	0.0897	0.796	0.5268
Tiffany	0.4825	0.5279	0.524	0.5115
		Low-Score Group		
Erica	0.2771	-0.4737	0.021	-0.0585[LA2]
Beth	-0.3107	-0.8378	-0.453	-0.5338
Paige	-0.1185	-0.6548	-0.923	-0.5654
Mia	-1.8406	-0.1049	-0.923	-0.9562
Sydney	-1.0648	-1.2217	-0.923	-1.0698

LA2: Low-score group alternate 2

Out of the five lessons in the low-score group, I chose to focus on the case of Paige for a number of reasons. First, Paige's lesson was on representing, adding, and subtracting fractions, which is a central topic in elementary school. Second, in her interviews, Paige expressed having a hard time understanding both the mathematics and the connections of the activity to the mathematics, which I thought might make the knowledge demands of the work more visible. Lastly, Paige's case seemed to offer more insight into the knowledge demands of mathematical purposing than the other lessons in the low-score group. Erica was a second alternate and her score was not as low as some of the other scores. Furthermore, Erica's lesson had the technical difficulty and I did not have a full video. Both Beth's and Mia's lessons focused on graphing, which is not a central topic in the elementary curriculum. In addition, Mia's lesson was very short and not from a textbook, and Beth's lesson had a non-mathematical component (food groups). Although Sydney's lesson provides much insight into the work, her interviews do not provide additional information because she often did not try to analyze the mathematics or the activity to make choices about what to teach, but just included the aspects of the lesson she happened to notice.

To analyze Paige's case, I adapted the practice-based approach developed by Ball et al. (Ball & Bass, 2003; Ball et al., 2008). Instead of analyzing the school curriculum or student thinking to gain insight into the mathematical knowledge demands of teaching, this approach studies classroom practice, asking:

- *What* mathematical knowledge is entailed by the work of teaching mathematics?
- *Where* and *how* is mathematical knowledge used in teaching mathematics? How is mathematical knowledge intertwined with other knowledge and sensibilities in the course of that work? (Ball & Bass, 2003, p. 5)

In this approach, teaching is portrayed as a kind of "mathematical problem solving." Analyzing practice has been found to uncover ways that teachers know and use mathematical knowledge in their work that can remain hidden when using other approaches to articulating the subject matter knowledge needed for teaching. Furthermore, asking about the mathematical knowledge entailed in the *work* of teaching "places the emphasis on the use of knowledge in and for teaching rather than on teachers themselves" (Ball et al., 2008, p. 394).

I used this practice-based approach and the domain-mapping described above to illustrate the knowledge demands of mathematical purposing. In particular, I analyzed the interview and lesson transcripts, as well as the curriculum materials, with my framework to identify the aspects of mathematical purposing in which Paige did and did not engage and the domains of MKT on which she seemed to draw or not draw.

Limitations of Study

This study has a number of limitations. First, it is important to acknowledge that I have conceptualized the problem space based on the context of mathematics education in the United States. Thus, the work of teaching to the mathematical point and the problems that must be managed to do so might be different in other countries. But even within the U.S. context, my conceptualization and unpacking of the work of teaching to the mathematical point is limited by the data I used. Although the sample was varied in important ways (e.g., grade level, MKT, lesson topic), there were a number of similarities across the sample that likely influenced my characterization of the work and knowledge demands of mathematical purposing, as well as the problems in steering instruction toward the mathematical point that I identified. For example, all of the preservice teachers were students in the same teacher education program and almost all of their lessons were based on *Everyday Mathematics*. I did not gather data about the children in their classrooms, so I do not know whether their classrooms were ethnically or socioeconomically diverse. Thus, my framework is likely missing aspects of the work of mathematical purposing or problems in steering instruction that were not visible in the settings I observed. Furthermore, there are likely important aspects of teaching to the mathematical point that occur across lessons that I did not attend to in my single-lesson data set. Thus, there may be differences in the work of teaching to the mathematical point in contexts other than those I studied, such as secondary mathematics teaching or even elementary mathematics teaching using a more problem-centered curriculum. However, it is important to note that, because my framework was developed through an iterative analysis of the data and the literature, it does not reflect only the contexts in the data.

Another set of limitations emerges from my methods of analysis. Research like this is necessarily interpretive. Although I have experience studying teaching practice to unpack the work of teaching and its mathematical knowledge demands, my findings reflect only what I was able to "see" in the data. Whether the distinctions being made prove useful or the "right" things are captured in the framework will ultimately be determined by whether and how the ideas can be taken up and used in future research and teacher education.

This points to a third set of limitations resulting from the goals of the study itself. In Chapter 1, I made arguments about the significance of this study that were based on improving students' mathematics learning and teacher education. This dissertation is just a tiny step with respect to these aims. All of the disclaimers made at the beginning of this chapter about what this study is <u>not</u> trying to do (i.e., that I am not making claims about preservice teachers' knowledge or practices, and I am not making claims about whether they had a mathematical point or taught to it) are in fact the very work that needs to be done. This study does not take up questions such as: *How can preservice teachers learn to teach to the mathematical point? Does mathematical purposing help preservice teachers steer instruction toward their mathematical point? Are teachers with stronger MKT more skilled at mathematical purposing? Does skill in mathematical purposing improve the mathematical quality of instruction and lead to increased student learning?* These important questions remain unanswered.

The next four chapters present the results of my analyses. Across these chapters, I unpack and articulate aspects of the work of teaching to the mathematical point and explore the mathematical knowledge demands of this work. I begin in Chapter 4 with a close look at instruction, using detailed examples from two lessons in the data to explore the work of teaching to the mathematical point and to illustrate its complexity. In Chapter 5, I use the data to illustrate problems that must be managed in steering instruction, and, as I discuss each problem, I identify issues that can arise for beginners as they engage in this work. These examples from the data help lay the foundation for the framework for mathematical purposing presented in Chapter 6. Chapter 7 examines the MKT demands of mathematical purposing and illustrates these demands with a case from the data.

CHAPTER FOUR:
THE COMPLEXITY OF TEACHING TO THE MATHEMATICAL POINT

Introduction

As described in Chapter 1, I am conceptualizing "teaching to the mathematical point" as three interrelated and mutually informing types of work: articulating the mathematical point; orienting the instructional activity; and steering the instruction toward the mathematical point. The first two—articulating the mathematical point and orienting the instruction—comprise the work of mathematical purposing. Mathematical purposing results in an articulation of the mathematical learning goals for students and how the activity is intended to move students toward those goals, as well as a detailing of the task and possible teacher moves that positions the activity so it is more likely to engage students with the intended mathematics. Steering the instruction toward the mathematical point is work done during an activity's enactment to try to keep it headed in the intended mathematical direction.

There is a cyclic relationship between mathematical purposing and steering the instruction toward the mathematical point. Steering toward the mathematical point, of course, requires knowing the mathematical point toward which the instruction is being steered; and articulating the mathematical point is part of mathematical purposing. But more than that, the work of mathematical purposing is intended to help steer instruction toward the mathematical point. That is, articulating the mathematical point and orienting the instruction toward it can help manage problems in steering instruction. The work of steering then impacts the work of mathematical purposing, informing both the articulation of the mathematical point and the orienting of the activity. This cycle of work occurs simultaneously at a number of different grain sizes over time. For example, one can consider the mathematical point of an instructional activity as well as the mathematical point of using a particular problem, or even the mathematical point of a particular number in a particular problem. Thus, the work of teaching to the mathematical point involves simultaneous attention to nested mathematical points due to the nested nature of instruction.

Furthermore, as discussed in Chapter 1, the mathematical point of any instructional element (i.e., activity, problem, example) is not a single mathematical idea, nor even a set of only *mathematical* ideas. The mathematical point is a bundled package of ideas: a collection of mathematical learning goals of varying grain sizes, prioritized with depth and weight and time, that are conceived of with respect to the particular students being taught and explicitly connected to the instructional activities being used.

In this chapter, I use two extended examples from the data to explore the work of teaching to the mathematical point. My aim for using these examples is twofold. First, situating the work of teaching to the mathematical point in the context of the lesson highlights the complexity of the work, in particular, the multiple considerations at varying grain sizes that must be simultaneously managed over time and the ways in which different aspects of the work interact with and inform each other. Second, the two lessons are intended to set up the more analytic and abstract decomposition of the work of teaching to the mathematical point presented in Chapters 5 and 6. Consequently, examples from the lessons described in this chapter are reflected in and sometimes explicitly referenced throughout the next two chapters.

To make more visible the aspects of the work of teaching to the mathematical point being illustrated by the lessons in this chapter, I provide an elaborated diagram of the work of teaching to the mathematical point (Figure 4). Because the components of this diagram will be further unpacked and discussed in detail in Chapters 5 and 6, I only highlight here some of the key features of the work of teaching to the mathematical point that the diagram is trying to reflect.

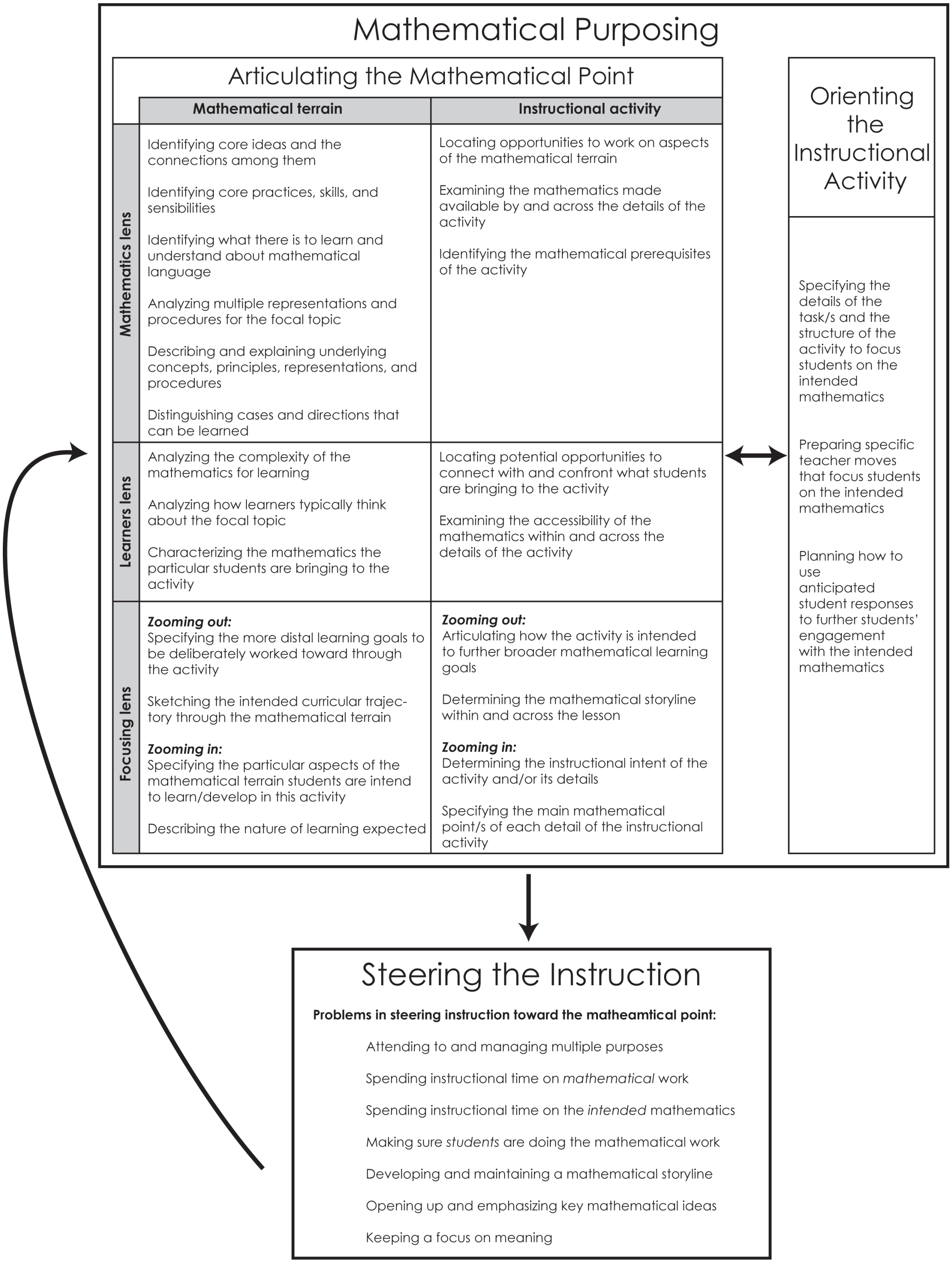

Figure 4. Elaboration of the work of teaching to the mathematical point.

I conceptualize the work of *articulating the mathematical point* as an interactive analysis of both the mathematical terrain and the instructional activity from three different lenses (represented by the matrix in Figure 4). The *mathematics lens* unpacks what there is to learn about the mathematics and how that mathematics is made available for study in the instructional activity. The *learners lens* considers the mathematical terrain from the perspective of the learner and analyzes the accessibility of the mathematics in the instructional activity. The *focusing lens* simultaneously zooms in and out to specify the mathematical learning goals of various grain sizes and their location in and connections within and across both the mathematical terrain and instructional activity. The work of *orienting the instructional activity* involves specifying the details of the activity and preparing specific teacher moves in order to focus students on the intended mathematics. There is no order assumed in the work of mathematical purposing; the various components interact with and inform one another. The details of the work of mathematical purposing will be presented and further unpacked in Chapter 6, but the lessons described in this chapter reflect many aspects of the work of articulating the mathematical point of an instructional activity, as well as ways in which an activity can be oriented to focus students on the intended mathematical work.

One important aspect of the work of mathematical purposing that is not reflected in the diagram, but was discussed in Chapter 2, is that it is based on a conceptualization of the design of instruction as distributed work. Thus, the framework for mathematical purposing developed in this study is not a list of things that any individual teacher is supposed to do for every instructional activity. Instead, it reflects the *work* that is involved in the mathematical purposing of instruction. Depending on the context, this work will be differently distributed across the teacher and various resources, such as curriculum materials, grade level standards, year or unit planning, and established instructional routines. But because the examples in this chapter are taken from the data, they, in fact, reflect the distribution of work in a particular situation. My aim is to abstract from these specific examples a general description of the *work*, not to analyze who or what is doing the work in a particular case. Nonetheless, regardless of the distribution, the teacher does have a special role in the work, as it is the teacher who coordinates the design of instruction in her or his classroom. Furthermore, in order to teach to the mathematical point, a teacher has to "have a handle" on what the mathematical point is and how the instructional activity is intended to engage students with it. However, the ways in which a teacher "gets a handle" on the mathematical point and the difficulty of doing so depends on the particular context, the distribution of work, and the resources brought to bear.

Figure 4 also lists some of the teaching problems[28] that must be managed in steering instruction toward the mathematical point. Although listed separately, the problems are overlapping and occur simultaneously in instruction. Furthermore, the management of a particular problem is not associated with a specific set of teaching practices. In Chapter 5, I examine each problem using examples from across the range of lessons in the data to begin to unpack the work of steering instruction and to identify strategies that can help manage each problem, as well as issues that can arise when trying to do so. A number of the strategies discussed in Chapter 5 are illustrated by the two lessons described in this chapter.

The Work of Teaching to the Mathematical Point: Two Examples from the Data

This section provides detailed images of instruction by two beginning teachers. As described above, I use these two examples to explore the complexity of teaching to the mathematical point and to begin to unpack the work of trying to purposefully engage students with particular mathematical ideas. I intersperse the descriptions of the lessons with excerpts from the interviews, as well as my own commentary. I deliberately use strong examples of beginning teachers' work in this arena to illustrate strategies that can be used to manage problems in steering instruction toward the mathematical point (Chapter 5) and to foreshadow components of my framework for mathematical purposing (Chapter 6).

The first example is Jordan's fifth-grade lesson on adding and subtracting integers. Her lesson is loosely textbook based: the problems in the homework and worksheet come from *Everyday Mathematics*, but the main whole-group work is a review designed by Jordan to meet her articulated learning goals for students. The second example is Courtney's second-grade lesson on comparison number stories that more closely follows the lesson in the *Everyday Mathematics* teacher's guide. Courtney analyzed the representations and problems used in the lesson to articulate her mathematical goals and to determine which aspects of the lesson to implement with her students. In addition to having articulated their mathematical points, both teachers tried to orient their activities to engage students in the intended mathematics and to steer instruction toward their goals.

Before beginning, I want to make a comment about the teaching described here and in the following chapters. In addition to unpacking the complexity of teaching to the mathematical point, these examples are intended to provide images of the range of beginning teaching practice and to recognize and appreciate the difficult and complicated task of student teaching. All of the

[28] I am using Lampert's (2001) notion of "teaching problems" as described in Chapter 2.

teachers in the study are in their first full-time month in the classroom. What they are already able to do so early in their careers and the risks they are willing to take—including sharing their practice with me—is important to acknowledge.

Jordan's Lesson on Adding and Subtracting Integers

"Five, four, three, two, one." Jordan counts backwards to get the attention of her fifth-grade students. As desks close and chatter subsides, she announces, "You need to have your Basic Math out and a checking pen." Students ready themselves for the daily routine of correcting last night's homework. This morning there are two assignments to check: a "Basic Math" and a "Study Link." Basic Math is a mixed computation review that students complete as part of their homework on Mondays and Wednesdays; "Study Link" is *Everyday Mathematics'* name for homework.[29] This Study Link is from the previous math lesson and focuses on integer addition and subtraction, which is the topic of today's lesson.

Jordan's class is nearing the end of Unit 7 from *Everyday Mathematics.* The unit begins with work on exponential notation and order of operations; introduces negative numbers using a variety of models including number lines, counters, and debt; and concludes with addition and subtraction of integers. Today's lesson is a review of their work so far. At the beginning of the pre-lesson interview, Jordan described her goals for the lesson:

> I think for the students, just for them to be able to recognize when you can actually add and when you can actually subtract, because they're still a little hazy on that.…When it's like a positive and a negative then you can subtract and when it's a negative and a negative you can add. And then we've been working on them just being able to give examples of like the different scenarios when it would happen, so if it's like plus a negative, what example could you give, like temperature change or money in an account. They actually have team points up on the board and they can get positive and negative points and they've been doing that all year. So they're really good with that if you set it up, like your team has five points and three negative points, what's your balance? They're really good at that but then not so good at being able to connect it to the actual number sentence. So that's kind of the main goal for this is to get them to be able to set it up and recognize what operation they need to use. (J-Pre, T8-10)[30]

Jordan's response illustrates that teaching to the mathematical point involves simultaneously attending to multiple mathematical goals of various types and grain sizes. For

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[29] At earlier grade levels, it is called a "Home Link." According to the *Everyday Mathematics* teacher's guide: "Each lesson has a Home/Study Link. Home/Study Links include extensions of lessons and ongoing review problems. They show families what students are doing in mathematics."

[30] I use the following notation to reference the interviews: (preservice teacher's initial – data source, location in transcript). Each preservice teacher's pseudonym begins with a different letter of the alphabet. For the data-source identification, "Pre" refers to the pre-lesson interview and "Post" to the post-lesson interview. The location either reflects turn numbers in a Word format of the transcript (e.g., T8-10) or rows (also corresponding to turns in the interview) in an Excel coding table (e.g., R8-10).

example, Jordan wanted students to develop both procedural fluency and conceptual understanding of integer addition and subtraction, and to be able to make connections across representations. Her goals were more than just a list of topics; they were layered with details about what there is to be learned about the focal topic and geared toward her particular students. For example, she had unpacked some of what is involved in attaining procedural fluency with integer addition and subtraction (e.g., recognizing what operation to use) and had a sense of her students' current thinking and experiences with respect to these ideas (e.g., what related ideas they had been working on, what they already seemed to understand, and what they still needed to work on).

When the class is ready, Jordan begins reciting the answers to the Basic Math problems—"Number one: one-half. Two: two-fifths…"—continuing through number eighteen without comment or discussion. At number nineteen, a word problem, she varies her pattern, first reading the problem statement and then asking a student to share both his answer and the operation used. Jordan continues through the remainder of the worksheet, reading answers to the straight computation problems and eliciting students' answers to the word problems. Correcting the 44 Basic Math problems takes approximately six minutes.

Jordan asks a student in each group to collect the Basic Math sheets and, as she sets up the overhead projector at the front of the room, directs the class to get out their Study Links. Her plan for correcting today's Study Link veers from her usual routine. For the other lessons in this unit, she collected and looked through students' Study Links before teaching the lesson, using the students' work to inform her instruction. Today, because she suspected students would have made some errors, she decided to discuss the answers as a whole class. She thought that correcting their papers would provide a form of self-assessment and encourage students to ask questions during the lesson. Jordan calls on a student to explain the first problem:

Jordan	All right. Number one [-25 + (-16) = ____]. Todd, what did you get?
Todd	Negative forty-one.
Jordan	Say it again.
Todd	Negative forty-one.
Jordan	[*Writes "-41" on a projected copy of the Study Link.*] Seth, do you agree?
Seth	Yep.
Jordan	How did you do it?
Seth	I did negative twenty-five plus negative sixteen, so that would mean, say you have in your points, your money, you have negative twenty-five dollars, you owe it to your mom.
Jordan	Okay.
Seth	And then you borrow sixteen to go buy a new Wii game. It adds up, another, adds up more money, so it would equal negative forty-one.
Jordan	Good. I like your example. So you're forty-one dollars in debt to your mom. All right, number two [0 - (-43) = ____], Kelly what did you get?

Kelly	For number two?
Jordan	Yep.
Kelly	I had forty-three.
Jordan	Positive forty-three?
Kelly	Yeah.
Jordan	And how did you get that one?
Kelly	Well, I knew that if there was two, um, like, negative signs…
Jordan	Rusty, close your desk.
Kelly	…it equals to a plus sign.
Jordan	Okay. So right here, these two negative signs means I can add?
Kelly	Yeah.
Jordan	All right. Does anyone disagree?

From the outset, because the Study Link related to her mathematical point, Jordan handled its correction differently than the Basic Math. She recorded the answers to the problems on the overhead and used redundant language to emphasize particular ideas—for example, saying the unnecessary "positive" in front of forty-three—moves she did not make when correcting Basic Math. Although much of the discussion was still at a procedural level, Jordan took more time discussing the Study Link problems, asking students for both answers and explanations. She repeated students' answers to connect them back to the representation that was being used ("So you're forty-one dollars in debt to your mom.") and to emphasize key ideas ("So right here, these two negative signs means I add?"). She spent almost four and half minutes on the first four problems. All of these were moves that helped her engage students with the mathematical point of her lesson.

When planning her lesson, Jordan had not selected which Study Link problems she would discuss in detail. However, she had identified two (overlapping) "cases" of problems that she thought her students were finding more difficult and therefore wanted to focus on in her lesson (J-Pre, T38). One case was subtracting a negative number, which was recently introduced and thus still unclear to some students. The second case was any problem with two negative signs, not necessarily adjacent. Jordan had indentified this case as significant for her learners because, in a previous lesson, a number of students argued that -4 - 4 equals zero because there were two negatives. Jordan had noticed that her cooperating teacher had been telling students that "two negatives make a positive," and she was worried that students were misapplying the rule (J-Post, T184). Like her cooperating teacher, Jordan wanted students to learn to quickly calculate answers to these types of problems and to extract rules for adding and subtracting integers from their observations across problems. However, she also wanted students to use representations to model the problems because she thought that using representations both demonstrated and supported their understanding of the calculations (J-Post, T9).

Thus, from past lessons, Jordan was worried that simply relying on the two-negatives-make-a-positive rule was causing students to add whenever they saw <u>any</u> two negatives in a problem. Addressing this misconception was one of her explicit mathematical points, and she used the next two Study Link problems to raise the issue:

Jordan	Cole, what did you get for number three? [-4 - (-4)]
Cole	I got zero.
Jordan	Zero. Ali, do you agree?
Ali	Mm-hmm.
Jordan	How do you do that one?
Ali	Because, um, because two negatives equals a positive.

Jordan uses Ali's response to address the anticipated two-negatives-make-a-positive misconception by asking which two negatives invoke the rule:

Jordan	So these, which two negatives? This one?
Ali	Yeah, and that one.
Jordan	Which two negatives am I looking at?
Ali	Negative four and negative four.
Jordan	Okay. So I can just change that into a positive? Celia, what do you think?
Celia	[Inaudible]
Jordan	So I can take this negative right here [*pointing to the first -4*] and then this, one of these [*pointing to -(-4)*] and make it a positive? Which two do I need to have? Go ahead.
Celia	You need two, well…you need, you have two negatives next to each other, so you can make that into a positive.
Jordan	So next to each other. But this one out front…
Celia	Right.
Jordan	Doesn't matter, right? So you've got to have minus and then you got to have a negative to make it a plus. Rusty, can you give me an example for that one? Negative four minus negative four?
Rusty	Well, I was just going to say what Celia said.
Jordan	Okay. Can somebody give me an example for that one? It's tricky. Emily? Say it nice and loud.
Emily	Let's say you had negative four points.
Jordan	Okay.
Emily	And then, um, and then, let's just say, instead of adding points, we erased the negative points.
Jordan	Okay.
Emily	And then we, and then you guys erased those four negative points.
Jordan	Right.
Emily	And then that would make zero negative points.
Jordan	Could you guys hear what she said over here? So she said, let's say your team, let's say team two had negative four points and then I took away those negative four points, so I erase them, what would your balance be?
Students	Zero.
Jordan	Zero.

In the discussion of this problem, Jordan restated student comments and used strategic questions to emphasize and open up what she considered to be key ideas. For example, she asked which two negatives were "change[d] into a positive," and then pushed on it further by bringing up a misconception she had hoped to discuss. She continued to use these types of moves to purposely "dwell" on key ideas in her discussion of Study Link problem four:

Jordan	All right, number four [-4 - 4]. Who had number four? Aaron, what did you get?
Aaron	I got negative eight.
Jordan	Negative eight.

Aaron did not make the error that Jordan had hoped to discuss, so Jordan brought up the incorrect answer herself:

Jordan	Why isn't it zero? Matt?
Matt	Well because like, to get, to get like the addition sign, you have to have like, if it was like negative four and then you had like negative four minus negative four and if you had two subtraction signs, then it would be adding.
Jordan	Oh, okay.
Matt	But it's not, it's just two [inaudible].
Jordan	Can you give me an example for that one?
Matt	Not really.
Jordan	Can someone give me an example? Negative four minus four? Jack?
Jack	Okay. It's kind of easy. Um, say I have a bank account and I'm negative four in the account.
Jordan	Okay.
Jack	If I spent another four dollars that I really didn't have, it would make my account four dollars worse.
Jordan	Perfect.
Jack	So I would have negative eight dollars.

Jordan moved through the remaining Study Link problems more quickly. Although there were a few times when she asked for explanations, for the most part, she just called on students to give the answers. In correcting the Study Link, Jordan faced a decision routinely encountered in mathematics teaching: determining which problems to discuss. One of the issues Jordan wanted to address in her lesson was the difficulty students were having with problems that involved subtracting a negative. Both 0 - (-43) and -4 - (-4) provided students with an opportunity to engage with this type of problem. But they were not the only options; two other problems on the Study Link could have served this purpose as well. However, -4 - (-4) had other distinguishing mathematical features that perhaps made it more strategic at this stage in the lesson. For instance, the two problems Jordan discussed used similar numbers (i.e., 4 and -4). Furthermore, -4 - (-4) was also the only problem that had three negative signs, thus making available the opportunity to press students about which two negatives "became" the positive. This type of mathematical analysis of the mathematics made available by different numerical examples and selecting the numbers that are most likely to engage students with the intended mathematics is part of the work of mathematical purposing.

After problems are selected, teachers make many decisions during discussions that impact whether and how students engage with the intended mathematics—for example, what to accept as an explanation, what to press on, and whether to ask for multiple solutions. Jordan handled the two subtracting-a-negative problems differently. With -4 - (-4) she pressed for an example, whereas with 0 - (-43) she accepted an explanation that relied only on the rule. These types of decisions involve weighing multiple—and sometimes conflicting—*mathematical* purposes, as well as *non-mathematical* purposes. Complicating matters further, these purposes must be considered over time. Purposeful decisions reflect what is intended for later in the lesson—that is, they keep the mathematical storyline of the lesson in mind. For example, because Jordan planned to discuss additional problems after she finished correcting the Study Link, she might not have felt the need to spend as much time or raise particular issues at the beginning of the lesson.

Once the Study Links were corrected and collected, Jordan began the review portion of the lesson. Like her cooperating teacher, Jordan usually follows *Everyday Mathematics*, but the lesson that was slotted for this day introduced a slide rule to add and subtract integers, and she and her cooperating teacher decided it would be best not to bring in a new representation at this stage in their work. Instead, Jordan planned to review some of the integer-related concepts the class had been working on and then have students complete a worksheet that she had designed using Math Journal[31] pages from the next two lessons in the textbook. During the pre-lesson interview, Jordan described her purpose for the review:

> [For students to] recall what we did yesterday. And then it gives me a chance if there are misconceptions to kind of address it right there as a group before they start on this. So just kind of get them started, make sure everyone's on the same page before they start working. (J-Pre, T85)

She also described the main mathematical point of this part of the lesson:

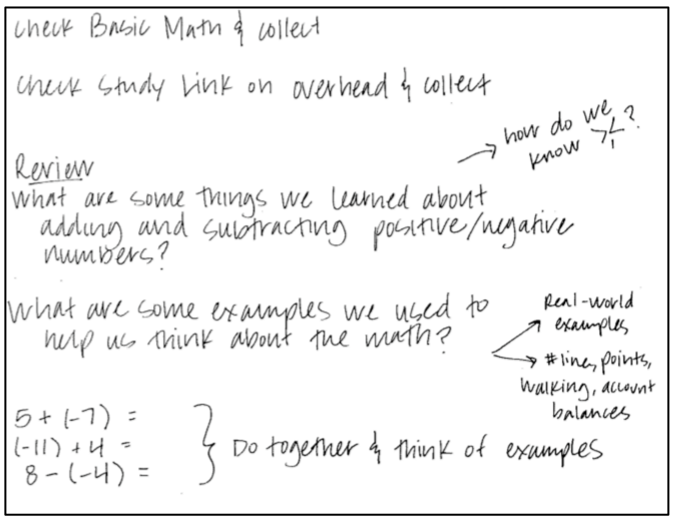
> Probably just being able to explain like how the operations work with negative numbers and being able to verbalize that and explain it to the class. Because a lot, some of the kids can do it correctly and they just see the minus minus and change it to a plus, but I think being able to give an example or put it into words is helpful. (J-Pre, T87)

After taking some general comments from students about what they had learned so far about positive and negative numbers, Jordan shifts the discussion toward comparison:

[31] The "Math Journal" is the student workbook in *Everyday Mathematics*.

Jordan	What about when we, I know this has been on some of your Study Links, Seth, how do we know what number's bigger? If we have two numbers and you have to write a greater than or less than sign, how do you know which one is bigger?
Seth	Um, well, say you're doing like, um…
Jordan	Do you want me to give you an example?
Seth	Yeah.
Jordan	So if I have like negative ten and negative eleven [*writes -10 and -11 on the board*], which one's bigger?
Seth	Um, usually you want to um, say if it's a negative, you want to go to the lowest number, like if it's negative seven against negative eleven, you want to go with the negative seven because it's closer to zero.
Jordan	So the number that's closer to zero is…?
Seth	Bigger.
Jordan	Bigger. Good.

Jordan had planned to ask students about comparing integers, as indicated by the question "how do we know >, <?" jotted on her lesson plan (Figure 5):



check Basic Math & collect

check Study Link on overhead & collect

how do we know >, <?

Review

What are some things we learned about adding and subtracting positive/negative numbers?

What are some examples we used to help us think about the math?

Real-world examples

line, points, walking, account balances

5 + (-7) =
(-11) + 4 =
8 - (-4) =

Do together & think of examples

Figure 5. Excerpt from Jordan's lesson plan.

However, she had not planned to provide specific numbers for students to compare. In the post-lesson interview, Jordan reflected on her choice of -10 and -11:

> I didn't really have that planned out but I just figured that they were, I thought that it might be kind of confusing because they're close and because you think of eleven as being obviously bigger than ten, like you know that it's bigger and…I didn't want to pick like a positive and a negative number, or I didn't want to pick anything that was too close to zero because I think that they're pretty secure in that. But the negative ten and negative eleven I thought might be a little bit more difficult than the examples we had done earlier. And I really was hoping that Seth would be able to explain it, but I saw that he was kind of like, he didn't really know how to put it into words, so I had to give him an example to help him out. (J-Post, T74-76)

Jordan's selection process, which happened quickly and in the moment, distinguished mathematical features of different number choices, including their proximity to zero and to each other and whether they should be both negative, both positive, or one of each sign. In this analysis, Jordan both identified the different numerical cases and compared the mathematics made available for study by each. This type of case analysis is mathematical in nature, yet also takes into account what the example is intended to teach.

This episode also points to another aspect of teaching to the mathematical point: the choice of language and managing the inherent tension between the precision of mathematical language and the use of mathematical language in teaching (Sleep, 2007). Seth's explanation that the number closer to zero is greater is not true in general. However, Seth's comment ("say if it's a negative") could be interpreted as restricting the domain to negatives numbers, in which case, it is true that the number closer to zero is greater. During the interview, when asked if it is always true that the number closer to zero is bigger, Jordan replied, "Well, with negative numbers. Right." I did not ask her whether this was something she noticed at the time (our discussion focused on her interpretation of Seth's explanation), so I do not know if she considered issues of mathematical language in the moment. Perhaps she did not notice the imprecision, or perhaps she did notice it but thought it was clear from the context that the two numbers were negative or did not think this imprecise statement would be incorrectly overgeneralized by her students and therefore did not want to interrupt the flow of the lesson to take time to correct it. Whatever the reason, the episode highlights that, in teaching to the mathematical point, how to respond to students' use of mathematically imprecise language is informed by its impact on students' engagement with the intended mathematics, its interference with the furthering of other purposes, and by the teacher's ability to notice the imprecision, which is informed by MKT.

After the $-10 > -11$ exchange, Jordan took the discussion in a different direction, eliciting from students some of the ways they have been representing integers. Students gave a variety of examples, most of them ones that Jordan had noted on her lesson plan (Figure 5). However, no one mentioned the number line—one of the representations Jordan wanted to be sure to discuss—so she brought it up herself. Throughout this segment of the lesson, Jordan did not deeply probe students' examples nor did she write anything on the board. Her point was to elicit ideas for use later in lesson (J-Post, T96), a move that required determining and keeping track of the lesson's mathematical storyline.

After eliciting representations and before introducing the worksheet, Jordan had planned to ask students to "do together" and "think of examples for" the following problems:

$$5 + (-7)$$

$$(-11) + 4$$
$$8 - (-4).$$

She designed these examples so that "the negative or the subtraction sign was in a different place or set up differently" (J-Pre, T83). She intentionally included different "cases" to help her assess whether students were comfortable across the range of problem types and to reserve all of the worksheet exercises for independent work. Jordan wrote the first problem, 5 + (-7), on the board and called on Missy to give the answer and a corresponding example:

Missy	Wouldn't that be like five minus seven?
Jordan	Okay. So what would I get?
Missy	Um, negative two.
Jordan	Right. And can you think of an example that goes along with that?
Missy	No.
Jordan	You can use team points[32] if you want.
Missy	Okay.
Jordan	You've got to say it, though. So say it, in like a story, a number story.
Missy	Team two had five points, they lost seven, so now they have negative two.
Jordan	So if they lost seven that would kind of be subtraction, but this is addition, so another way to think of it is they have five points and they have seven negative points, right? And then you're adding up their total points.

In the post-lesson interview, Jordan provided a mathematical analysis of the team-points context, explaining that the class' team-points system models integer addition, not subtraction: Instead of reprimanding teams by erasing positive points, negative points are added, and then positive and negative points are reconciled to find the balance. Jordan thought Missy's example more closely corresponded to 5 - 7 (instead of the given problem, 5 + (-7)). It also did not reflect how Jordan wanted the class to think about team points. Jordan wanted students to combine positive and negative points to get the total instead of taking points away. Because Jordan wanted students to understand the distinction between adding -7 and subtracting 7, she thought it was important for the class to hear language that described the addition of a negative number. Furthermore, Jordan had planned to use the examples, team points in particular, later in the lesson when she was helping students with the worksheet, and therefore wanted to make sure that this example was both available and correct during the review (J-Pre, T75-77)—another example of Jordan's awareness of the lesson's mathematical storyline. Jordan decided to make the distinction herself, rather than press on Missy further, because she didn't want to upset Missy and also wanted to acknowledge Missy's connection to 5 - 7 (J-Post, T127-135).

[32] "Team points" refers to the reward system used in Jordan's classroom. Teams (i.e., table groups) are rewarded for positive behavior by placing a tally in the "+" column. When a team loses points, a tally is recorded in the "-" column. The system is designed to model integer addition. Thus, a team's current point balance is determined by adding the number of positive tallies to the number of negative tallies.

Jordan went over the second problem, (-11) + 4, fairly quickly and then posed the third and final problem, 8 - (-4), which involved subtracting a negative and was thus one of the problem types on which Jordan wanted to focus. Although this was not the first problem of this type to be discussed in the lesson, it is one of the more challenging of its kind to model. For example, when a chip or counter model is used to subtract integers, the first number is represented with the corresponding number of positive or negative counters and then the second number is taken away from this initial quantity. To determine the answer, the remaining positive and negative counters are reconciled. For example, to model -5 - (-2), the -5 is represented by five negative counters and then two negative counters are taken away, leaving three negative counters, which correspond to the answer of -3. In the case of -5 - (-2), there were enough negative counters at the outset to take away two negatives. However, when the first number is greater than the negative number being subtracted, a challenge arises because, after modeling the initial quantity, there are not enough negative counters to take away. Jordan's third review problem is an example of this case. To model 8 - (-4), the 8 is represented with eight positive counters, but then there are not any negative counters to take away.

That this third problem was one of the types Jordan most wanted to focus on in her lesson was reflected in both the time spent and the quality of the discussion:

Jordan	All right, last but not least, what if I have two negatives? Dave?
Dave	Um, right here, there's two negatives and that's not possible, so instead of doing minus…
Jordan	Well it's possible, but it's just, we want to think about it a different way.
Dave	Instead of um subtracting you would switch the subtraction sign to an addition sign.
Jordan	Okay.
Dave	So, then it's pretty simple, eight plus four equals twelve.
Jordan	And can you give me an example that goes along with that?
Dave	Um…
Jordan	The minus minus are hard to think of a story. Who can think of a minus minus story, minus a negative number. Sean, what do you think?
Sean	If you had eight, like, points…
Jordan	Okay.
Sean	Or something.
Jordan	Eight points.
Sean	And you minus negative four points
Jordan	Okay.
Sean	You'd really be giving you points because you didn't have any negative.
Jordan	Yeah.
Sean	So…
Jordan	So remember yesterday, we had a problem with like the counters where we needed a balance of eight, but we needed four negative ones too. So how would I draw that? I can't just draw eight positives because then I don't have any negatives to take away.

Represents 8 on the board by drawing eight positive counters:

$$\boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}$$

	So what do I need to do? So if I want to take away four negatives, I don't have any negatives. What do I need to do? Blane, what do I need to do?
Blane	You have four negatives and you need, wouldn't you just add all, um…
Jordan	You're getting there.
Blane	Wouldn't you just add all of them?
Jordan	So I have eight and I want to take away four negatives, but I don't have any negatives, so what do I need to add?
Blane	The negatives.
Jordan	So I just put four negatives?
Blane	You have to add the positives.

Jordan draws four negatives under the positives:

$$\begin{array}{l}\boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\\ \boxed{-}\ \boxed{-}\ \boxed{-}\ \boxed{-}\end{array}$$

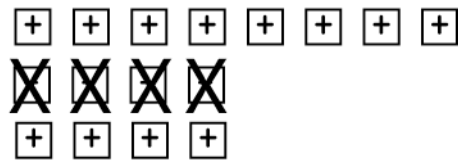
Jordan	Can I do that?
Corey	No.
Jordan	Corey?
Corey	No, don't you like put a, with every negative you put in a positive also and then you just cross off the negatives.
Jordan	Yeah, I need to bring in four more positives so that my balance stays eight. Right?

Draws four more positive counters under the negative counters (note that the balance is 8, the original quantity):

$$\begin{array}{l}\boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\\ \boxed{-}\ \boxed{-}\ \boxed{-}\ \boxed{-}\\ \boxed{+}\ \boxed{+}\ \boxed{+}\ \boxed{+}\end{array}$$

	So what's my balance right now? If that was my bank account and each one, each plus was a dollar and each minus was negative a dollar, what's my balance? Emily?
Emily	Eight.
Jordan	Eight, but I have four negatives that I can take away now, right? So I'm going to take away these four.

Crosses out the four negatives:



So I did eight and then I took away four negatives, so took away four negatives. And what's my balance now?

Emily Twelve.

Jordan spends more time on this third problem than on any of the other examples in her review. She presses students for more complete explanations and, after eliciting the answer and a team-points example, she leads the class in modeling the problem on the board with positive and negative counters. Throughout this explanation, Jordan demonstrates a number of strategies that help emphasize the key ideas and keep the focus on the mathematical meaning of the representation, for example, asking students about the common error of not maintaining the original balance and mapping each step in the modeling back to the problem.

After discussing the three review problems, Jordan shifts the class to the next activity: a 39-problem worksheet of mixed practice with positive and negative numbers. When designing the worksheet, Jordan specified the details of the task and the structure of the activity to orient students toward the intended mathematics: She intentionally mixed operations because, in the previous two lessons, addition and subtraction had been separated, and she thought it was important for students to be able to determine how to solve these problems when they were mixed (J-Post, T21). She also decided to make the worksheet on a separate sheet of paper (rather than have students jump around to the corresponding pages in their *Everyday Mathematics* workbooks), so that students would stay together on the intended problems and not be tempted to work on different tasks.

Jordan introduces the worksheet to the class, pointing out the different types of problems they will be working on. Some of the problems involve a calculator, and Jordan quickly reviews how to use this tool. Familiarizing students with a tool before they begin working is a move that can increase time spent on the intended mathematics because it avoids time wasted with procedural questions later in the lesson. Students work independently for the next 15 minutes. Jordan circulates—answering questions, helping students who are having difficulties, and asking students to explain how they got their answers or to give an example that represents a given problem. As the end of the math period approaches, many students have not yet finished the worksheet. To make sure everyone is prepared to discuss the integer addition and subtraction problems, Jordan gives students more time, but focuses them on the portion of the worksheet they will be discussing: “Make sure you have at least the top part done before you go on to the calculator part so we can go over that in a couple of minutes.” This moved focused students on the part of the worksheet she cared about most—the problems that were most connected to her learning goals for this lesson (J-Post, T53). After a few minutes, Jordan reconvenes students in whole group, outlines the remainder of the class period, and dives into a brief discussion of three problems from the worksheet.

Jordan had not identified the particular problems for discussion in advance of teaching the lesson, but had planned to select some she noticed students were having trouble with when completing the exercises. Then, if there was still time, she planned to read through the rest of the answers so students could check their work; otherwise, she would collect their papers and correct them herself. Because she extended the independent work time, there was only time to discuss a few problems. After the discussion, Jordan posed one last task: an “exit slip” that asked students to write their own addition or subtraction problem with at least one negative number in it and to provide an example or picture that corresponds to the equation. Even though the lesson took longer than she had anticipated and she was feeling pressed for time, Jordan still decided to assign the exit slip. She explained this decision in the post-lesson interview:

> I still wanted to see what they would do for that one and then it gives them a chance to kind of go back to the things we were talking about in the beginning of the lesson. Since this is all computation, it kind of pulls it back in the big picture and let’s them kind of summarize what they’ve learned, I guess. So I thought it kind of pulled everything back together. (J-Post, T57)

Overall, Jordan’s lesson illustrates a number of aspects of teaching to the mathematical point. She had clearly articulated mathematical learning goals for her students and knew how the details of the activities were intended to engage students with particular mathematical ideas. The mathematical point of her activities emerged from analyses of the complexity of the mathematics from the learners’ perspective, as well as what her particular students were bringing to the work. She also analyzed the mathematics made available by the details of the instructional activity, for example, by the numerical examples and representations used. She had a determined a coherent mathematical storyline for her lesson, which she used to make sure that she had reviewed the ideas that students would be using later in the lesson and to build on students’ prior mathematical work. She oriented the various activities toward her mathematical point through the allocation of time and the strategic selection of problems and examples. She also demonstrated a number of strategies for managing problems in steering instruction. For example, she engaged students with the intended mathematics by raising errors and methods that students did not bring up, and she emphasized key ideas through the use of redundant language and "dwelling" on particular cases of problems.

I now turn to the second example from the data: Courtney's second-grade lesson on comparison number stories. As with Jordan's lesson, Courtney's lesson is intended to illustrate the complexity of teaching to the mathematical point and the interaction between the different aspects of the work, as well as provide specific examples of the work of mathematical purposing and strategies for managing the problems in steering instruction toward the mathematical point.

Courtney's Lesson on Comparison Number Stories

Courtney stands in the middle of the large classroom, a copy of the spiral-bound *Everyday Mathematics* teacher's guide in her arms. Her second-grade students sit at their desks, which are clustered in groups of five around the room. Each student already has out an individual whiteboard and marker. Courtney quickly surveys the room to see if the class is ready to begin the morning's mathematics lesson. As usual, the lesson opens with Mental Math. In *Everyday Mathematics*, the Mental Math activity usually reviews skills students have previously worked on and is not necessarily connected to the lesson's main activity. Today's problems (Figure 6) are designed to provide practice with strategies for efficiently adding multiple addends,[33] a topic the class had discussed last week.

Write multiple-addend problems like the following on the board. Encourage children to look for combinations that will make the addition easier.

●○○ 3 + 9 + 7 = ?
14 + 8 + 6 = ?
6 + 8 + 4 = ?

●●○ ? = 21 + 5 + 9
34 + 6 + 7 = ?
57 + 10 + 5 = ?

●●● ? = 8 + 5 + 12 + 5
22 + 28 + 7 = ?
63 + 27 + 9 = 99

Figure 6. The Mental Math problems from Courtney's lesson.
Adapted from *Grade 2 Everyday Mathematics Teacher's Lesson Guide*
(Bell et al., 2007b, p. 384).

[33] Strategies that Courtney's class had discussed include "making a 10" (i.e., first adding the two numbers that sum to 10 and then adding 10 to the third number (e.g., 3+9+7→ 3+7=10→9+10=19)); "making a 20 [or 30]" (similar to making a 10, but starting with two addends than sum to 20 [or 30]); and "making a double" (e.g., 5+8+3 → 5+3=8 → 8+8=16).

Before posing the first problem, Courtney introduces the activity by making explicit connections to the class' prior work and reviewing the purpose of the addition strategies (i.e., to make the problems easier). She then asks students to share some of the strategies they remember using:

Courtney	If you can remember way back to Thursday, Ian, we were talking about those problems that have three different numbers that we need to add together, right? We were adding three numbers at a time so we had to think carefully about which order was the easiest to do that in. Raise your hand if you remember one way that we were adding them. We were trying to do certain things when we put them in order. What made those problems easier? Kelsey?
Kelsey	We did from the highest number to the lowest number.
Courtney	Okay. So sometimes it was easier if you started with the bigger numbers and then added the smaller numbers on at the end. Nate, did you have a different way?
Nate	[Inaudible]
Courtney	No? Do you remember sometimes we can make a, make a what? We try to add up until we can make a, Catherine?
Catherine	A number that ends in zero.
Courtney	Make a number that ends in zero. If we could make a ten, or a twenty with two of the numbers, [*student sneezes*] bless you, then adding on that third number was easier, right? So when I read these problems and you write them down, see if you can find the easiest order to add them in. Here's the first one, are we ready? Three plus nine plus seven. Write down the problem so you can look at it. Three plus nine plus seven. And when you get your answer just keep your whiteboard down. You don't need to raise your hands or anything yet, just keep your whiteboard down.

This brief episode depicts one of the main strategies Courtney uses to steer instruction toward her mathematical point: restating a student's answer and then connecting it back to the question, topic, or problem being discussed. For example, in the episode above, each time a student shared a strategy, Courtney not only repeated the strategy but also made a comment about how it made the addition easier—a move that directed attention toward the bigger mathematical idea with which she wanted students to engage. Connecting a response back to the original question is a teaching move that can help keep the work "on point" by maintaining attention to meaning. In addition, repeating students' responses is a way to emphasize and spend more time on key mathematical ideas.[34]

Another steering move Courtney used in the above episode was to provide a mathematical framing that oriented students toward the intended mathematics. In her launch of

[34] In this episode, Courtney was the one who did the restating and connecting. A teacher could also support students to do this work. In either case, the intended mathematics is highlighted.

the Mental Math activity, she did more than simply ask students to solve the problem; she directed them to “find the easiest order to add them in.” This move may have been prompted by the suggestion in the teacher’s guide to “encourage children to look for combinations that will make the addition easier” (Figure 6); however, even if this suggestion did spur her framing, the teacher’s guide does not specify what to *say* to students to do this encouraging. For example, Courtney reviewed some of the strategies before posing the problems—yet another type of orienting move that increased the likelihood that students would understand what it means to “make the addition easier.”

Courtney began with the first problem from the teacher’s guide, an easy one as indicated by the single dot. Although she felt comfortable creating problems, she usually used those provided in the teacher’s guide because she thought they tended to be “pretty good examples.” Furthermore, she felt like the textbook had already done some of the work for her, for example, in this lesson, ensuring that the problems were oriented toward the intended strategies (C-Pre, R58). Interesting, however, is that although the teacher’s guide suggests problems that are appropriate for use with the intended strategies, it does not explicitly state which strategy each problem is designed to elicit. Nor does it provide rationales for the problems’ sequencing and difficulty-ratings or for the ordering of the numbers in a particular example. For instance, it is unlikely coincidental that in the first problem the 3 and the 7 (i.e., the ten to be “made”) are non-adjacent, or that the smallest number, 3, comes before the 9 and the 7. The textbook’s arrangement makes adding the numbers in their given order maximally inefficient, preventing students from accidentally deploying one of the intended strategies by simply adding across. As is often the case, the general purpose of the activity is provided in the teacher’s guide, but to determine the mathematical point of a particular example, the teacher still has to do a mathematical analysis of the activity and the examples provided.

As students solve the first Mental Math problem, Courtney copies it onto the large whiteboard at the front of the classroom, the numbers arranged horizontally. She then quickly circulates around the room, looking over students’ shoulders to gauge their progress. When everyone has an answer, she asks students to show their whiteboards. After checking the raised boards for the correct answer, she engages the class in a brief discussion of the strategies used:

Courtney	Now, one thing that I notice, one thing that I notice is that a lot of you wrote it just like I wrote it on the board, three plus nine plus seven, and I’m wondering, did you really add them in that order, or did you add them in a different order? Raise your hand if you can tell me the order you added them in and why. Chuck?
Chuck	Um, I, um, did the highest to lowest and, well, first I did nine plus seven, which equals sixteen, and then sixteen plus three equals nineteen.

Courtney	Okay, so Chuck did nine plus seven plus three, and I think that's because he knows nine plus seven in his head was what?
Chuck	Sixteen.
Courtney	Sixteen. And then it's easy to just add on three—seventeen, eighteen, nineteen, right? Who did it a different way? Raise your hand. Omar, what order did you add them in?
Omar	Well, I did it the same as Chuck, but I did it like, I did it because it was higher numbers.
Courtney	Okay, so the same reason kind of too, right? You wanted to get the higher number and then only count up three more. Catherine, did you do it a different way?
Catherine	I added, I got seven and added three and that equaled ten, and then I just added nine.
Courtney	So Catherine did seven plus three plus nine because she made a ten, right? Raise your hand if you did it the same as Catherine. Seven plus three, or three plus seven, first so you could make a ten. Only a few of you, huh. Well maybe that will be a good idea for this next problem. See if you can make a ten or a twenty.

In this episode, as before, Courtney framed the discussion so that it was oriented toward the mathematical point—reviewing and practicing strategies for adding multiple addends. Her prompt ("raise your hand if you can tell me the order you added them in and why") made it clear that the focus of the discussion was on the order in which the numbers were added. Her comments after students shared their strategies also kept mathematical meaning in the foreground. For example, after Chuck described how he found the sum, Courtney restated his strategy, pointed out what made the strategy easy ("he knows nine plus seven in his head…and then it's easy to just add on three—seventeen, eighteen, nineteen"), and named the more general strategy it exemplified ("you wanted to get the higher number and then only count up three more").

Even though Mental Math was a routine part of each of her lessons, Courtney tried to be responsive to students (C-Pre, R59); in a sense, she planned to be flexible. In this lesson, Courtney did in fact change her plans in response to students. She had anticipated that the first problem would be a simple review of the "making a ten" strategy and was therefore surprised when students used different strategies (C-Post, T12). Because she wanted students to practice making a ten or twenty, the rest of the Mental Math problems she posed ($14 + 8 + 6$ and $21 + 5 + 9$) were amenable to that strategy. In addition to selecting problems that matched her mathematical point, her framing of the second problem ("Well maybe that will be a good idea for this next problem. See if you can make a ten or a twenty.") oriented students toward the strategy she wanted them to use.

After the third Mental Math problem, Courtney asks students to put their whiteboards away and move to the carpet area in the front corner of the classroom. She designs these types of transitions into her lessons as a management strategy so that students have a chance to move around during the long math period. As students trickle to the rug, Courtney gathers her supplies and sits in a chair next to a whiteboard easel, teacher's guide in her lap. She instructs the class: "I want everyone to read this problem as I write it, but don't say anything. Read it and see if you know how to solve." The class watches as she writes the Math Message on the whiteboard: *Phillip has 17 CDs. Trevor has 8 CDs. How many more CDs does Phillip have than Trevor?* As planned, Courtney used the problem from the teacher's guide; however, she replaced the names in the textbook with names of students in her class because "they get more into it that way" (C-Pre, R84). Writing out the entire problem took some time, but she thought it was a way to "get their attention" and that it might be helpful for "some of them to be able to reread it as much as they wanted" (C-Post, T24).

Here and at almost any point in her lesson, Courtney was simultaneously managing mathematical purposes, as well as non-mathematical purposes such as fostering student engagement. In some cases, non-mathematical purposes may have no impact on teaching to the mathematical point. For example, Courtney's use of student names in the problem was intended to foster student engagement, but it did not impact the mathematics available in the problem. However, if the use of student names had caused a distraction (e.g., Phillip teasing Trevor about having more CDs), then it could have reduced the time spent on mathematical work. Thus, at times, different purposes may be conflict. For instance, Courtney's move to write out the problem as a way of gaining students' attention may have reduced time spent on the intended mathematics. In fact, if the purpose was solely for students to have a written reference, she could have prepared a chart with the problem already written on it. Another possible conflict of purpose is Courtney's frequent management strategy of interspersing students' names to get their attention while she is talking (e.g., as seen with Tori and then Nate in the following episode), which may hinder students' ability to understand the mathematical ideas being conveyed.

After the choral reading of the Math Message, Courtney repeats the problem one more time and then, before having students find the answer, asks them what the problem is asking:

Courtney	Before you give me the answer, I want you to raise your hand if you can tell me, what is this problem asking? What are they trying find here? Don't give me a number but tell me. Think about it. Phillip has seventeen CDs. Trevor has eight. How many more does Phillip have than Trevor? What are we trying to find, Nate?
Nate	Like how, how, more like, like, count up to seventeen and find how much.
Courtney	Count up to seventeen from what?

Nate	Eight.
Courtney	From eight. So we're trying to figure out?
Nate	How much is between seventeen and eight.
Courtney	Ha, listen to this. Nate hit it right on the head. Listen to this. Nate said, we're trying to find out how much more seventeen is than eight. Isn't that what you said Nate?
Nate	Yep.
Courtney	How much is between seventeen and eight. That's what he said. And listen to this, we're going to talk about these kinds of problems all day so, Tori, I want you to make sure that you understand, we're trying to figure out how much bigger, Nate, how much bigger seventeen is than eight.

Asking students what the problem is asking is one of Courtney's routine teaching moves, one she thinks is particularly important when working on number stories because she wants students not only to be able to solve the problem, but also to understand what they are trying to find and what their answer means in the context of the story (C-Post, T165). Courtney's launch of the Math Message demonstrates a number of moves that helped steer the activity toward this mathematical point. Discussing what the problem meant slowed down the pace of the lesson and made explicit what the problem is asking—beyond a simple restatement of the question. Courtney emphasized some of the key mathematical ideas she was trying to develop in her lesson by using the language of comparison ("how many more" and "how much is between") as she built on Nate's response. Another way she helped infuse meaning into the lesson was by repeatedly framing both the problem and the day's work (e.g., "I want you to make sure that you understand we're trying to figure out how much bigger"), which also helped convey the mathematical storyline to the students.

After the discussion of what the problem is asking, Courtney elicits the answer to the problem and asks students for the "full answer." She then once again summarizes the meaning of comparison number stories:

Courtney	Roman, say that again. Here's our full answer.
Roman	Phillip has nine more CDs than Trevor.
Courtney	Phillip has nine more CDs than Trevor, and that would be our answer. Boys and girls, these are called comparison number stories. We're going to talk about all different kinds of comparison number stories, and the reason why they call them comparisons is because we're comparing one number to another and we're figuring out the difference between those numbers, or how much bigger one is than the other one.

Courtney introduces the new diagram students will be using to represent these types of problems:

Courtney	So, we have a brand new diagram we're going to use. Remember those change diagrams and the part-and-total diagrams? Now we're talking about these comparison number story diagrams. So look at this for a minute.

> Everyone look up here. You're going to need to know how to use these, Mira. This is a comparison number story diagram.

As she talks, Courtney erases the problem and pulls from under her chair a large copy of the diagram, laminated on bright yellow paper, which she hangs on the whiteboard with magnets (Figure 7).

Figure 7. Courtney's poster of a comparison number story diagram.

This "comparison number story diagram" is an example of one of the many diagrams used in *Everyday Mathematics* to model word problems. Courtney's class had worked with other diagrams throughout the school year, but this one was new. As Courtney described in the pre-lesson interview, learning to use the diagram to understand and solve comparison number stories was the main focus of her lesson:

> My biggest goal is to get them to see how to use that [the diagram] and how it relates to the number stories and hopefully be able to use those on their own by the end of this.…Also just to be able to think about what a comparison number story is.…So to know that we're comparing these two quantities, and that will be a new word for them too, and trying to find the difference between the smaller one and the bigger one. And since every problem's a little different, that's kind of hard to grasp, too, to be able to pull that out of every single problem. (C-Pre, R9-13)

Courtney's mathematical learning goals were of varying types and grain sizes. She had learning goals related to developing conceptual understanding, to understanding new vocabulary, and to using and making connections across representations. Courtney did not consider these goals in the abstract, but conceived of them with respect to her particular students and what would be new or difficult for them. All of these are key components of the work of articulating the mathematical point of an instructional activity, and thus part of the work of mathematical purposing.

Courtney continues her introduction of the comparison diagram by using it to represent the Math Message problem the class just solved. She begins with the unit box, something that is familiar to students from their work with other diagrams:

Courtney	This is a comparison number story diagram. Up here, [*points to the box at the top of the diagram labeled "unit"*], you've seen this before. What's it called?
Students	Unit/unit box.
Courtney	The unit box. So for this problem, what would our unit be? Raise your hand. What would the unit be for the problem we just did, Blair?
Blair	CDs.
Courtney	Okay, so we're talking about, I get to write right on this and I'll be able to erase it. [*Writes "CDs" in the unit box.*] We're talking about CDs.

Next, Courtney introduces the new vocabulary and explains how the size of the diagram's boxes corresponds to the size of the numbers in the problem:

Courtney	And now, let me read you these words. [*Points to the labels of the other boxes as she reads them.*] This is a quantity, Ian. This is a quantity. And this is a difference. Now these are set up in a very special way. Daniel, can you see it?
Daniel	No.
Courtney	Scooch over that way so you can really see it. These are set up in a special way that help us understand what the problem means. So we have two quantities, and a quantity is how much someone has of something or how much we have of one of those things.
Omar	How much is seven CDs?
Courtney	Omar, do you have a question?
Omar	No.
Courtney	Okay, raise your hand if you ever think of a question, okay. Ellen, you got this? Griffen, I need your eyes up here. We've got two quantities in every problem, or two numbers of something, and we're trying to find the difference between them. Thumbs up if you think you understand what I just said. Nate said it pretty well when he said we're trying to figure out the difference between how many CDs Phillip had and how many CDs Trevor had, right? So, do you think, let's see, we know Phillip had seventeen and Trevor had eight, and those were our two quantities. [*Writes 17 and 8 on the whiteboard below the diagram.*] That's how much Phillip had and how much Trevor had. Those are our two quantities. Raise your hand if you think you know which one goes in the bigger box. Which one do you think goes in the bigger box? Trista, do you know? You looked like you had an idea there? Daniel, which one do you think goes in the bigger quantity box?
Daniel	Seventeen.
Courtney	How come?
Daniel	Because it's bigger.
Courtney	That's our bigger quantity and it's going to go in the bigger box. [*Writes "17" in the bigger quantity box.*] Now see if this makes sense to you. Seventeen is our bigger quantity so, Daniel, what goes in this box? [*Points to the smaller quantity box.*]
Daniel	Eight.
Courtney	Eight is the smaller quantity. [*Writes "8" in the smaller quantity box.*] And we're trying to find the difference. [*Writes "?" in the difference box.*]

This episode demonstrates a number of ways in which Courtney steered instruction toward her mathematical point. She used language that was intentionally redundant to maintain a focus on meaning and to emphasize key ideas. For instance, she overused the new vocabulary words ("quantity" and "difference") and repeatedly said both a term and its definition (e.g., "we've got two quantities in every problem, or two numbers of something"). She narrated what the class was trying to do, referring to both the parts of the diagram and the context of the CDs, a move that could help generalize the use and meaning of the diagram beyond the particular problem.

This deliberate use of language continued as Courtney drew a picture to further explain what it means to compare numbers to find the difference:

Courtney	Now hold on a second, I want you to see this.…Seventeen. [*Counts as she makes seventeen small lines below the diagram*.] One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen, sixteen, seventeen. Just made it. That's how many CDs Phillip has. [*Student makes an inaudible comment about not grouping the tallies by fives.*] I'm not doing tallies I'm just doing little marks. And Trevor has [*draws eight marks directly below the seventeen*] one two three, four, [*students join in the counting*] five, six, seven, eight. Eight CDs. We're trying to find how many more CDs, Ian, Phillip has than Trevor. So, Chuck, can you point to my picture here and show me where that number is, how many more Phillip has than Trevor. This is tricky.
Chuck	Um, right here. [*Points to the nine unmatched lines.*]
Courtney	Do you agree? All of these will show us [*draws a curly bracket to denote the nine marks Chuck pointed to and labels the set with a question mark, shown in* Figure 8.], going to put this [*points to the question mark*] here [*writes "?" in the difference box*] because that what's we're trying to find, how many more seventeen is than eight.

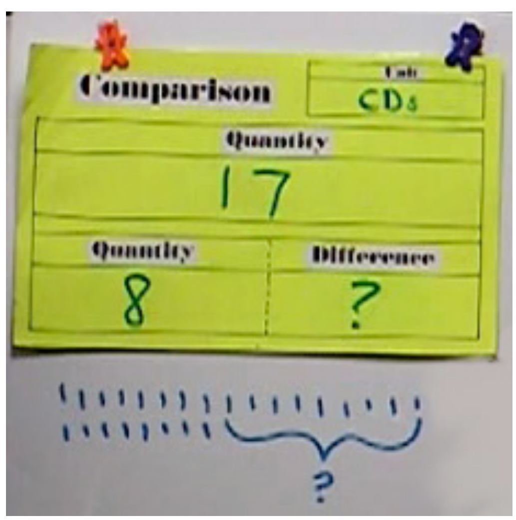


Figure 8. Courtney's use of multiple representations for comparison number stories.

In the above episode, Courtney introduced another representation for comparison problems. She decided to draw a picture so that students "could see these [8 of the 17] are matched and then we're looking for how many aren't matched" (C-Post, T183). She intentionally lined up the marks to match the depiction in the diagram (i.e., bigger quantity on top; smaller quantity below on the left; difference below on the right). Introducing another representation enabled her to dwell on and unpack a key idea—all moves that helped steer the activity toward her mathematical point.

As her explanation continues, Courtney makes explicit the connections between the picture and the diagram, referring back to the problem context and pointing to corresponding parts as she talks:

Courtney	Do you see how this [*points to the picture*] kind up lines up with this [*points to the diagram*]? Here's our big quantity, our number of CDs that Phillip has. Ian, do you need to move? I need your eyes up here. Here's our big quantity up, at the top. Here's our smaller quantity, or how many Trevor had. And we're trying to find the difference. So, you can imagine, Ellen and Griffen, on this diagram, Blair, this number [*points to the smaller quantity box*] plus this number [*points to the difference box*] will always what? Raise your hand. This number, the small quantity, plus the difference will always equal what, Catherine?
Catherine	The, um, bigger.
Courtney	The bigger quantity. Raise your hand if you have a question about that because it's so important. And if you all get that, we're going to have an easy time with math today. Does anyone have a question about that?
Student	Nope.
Courtney	Okay, I'm going to say it one more time. The smaller quantity plus the difference between the two numbers will always equal the bigger quantity. You can see how this big box [*points to the box containing the smaller quantity and the difference boxes*] is the same as this big box [*points to the bigger quantity box*].
Students	Oh.
Courtney	This [*points to the smaller quantity box*] plus this [*points to the difference box*] will equal this [*points to the bigger quantity box*]. So what was our difference? Raise your hand if you remember. Phillip?
Phillip	Nine.
Courtney	Nine. So does eight plus nine equal seventeen?
Students	Yes.
Courtney	So we got it. So the answer was Phillip has nine more CDs than Trevor, right?
Students	Oh, I get it.

This episode provides additional examples of how language can be used to teach to the mathematical point. Once again, Courtney repeated key ideas and vocabulary. She used questions to focus students on the intended mathematics, for example, when she asked "Can you point to my picture here and show me where that number is, how many more Phillip has than Trevor?"

rather than posing a more general question that might spark tangential comments, such as "What do you notice about the picture?" Furthermore, her explanations were not "flat." She used phrases such as "raise your hand if you have a question about that because it's so important" to signal the main point to students.

Although Courtney did elicit responses from students, in this episode and throughout her lesson, Courtney was usually the one who gave explanations and made correspondences across representations. This raises questions about who is doing the mathematical work in the lesson: When teaching to the mathematical point, the goal is for *students* to engage with the intended mathematics, not for the teacher to do all of the thinking.

Courtney's lesson suggests an another tension in the work of teaching to the mathematical point: Not only can mathematical purposes sometimes conflict with non-mathematical purposes, equally valid *mathematical* purposes can also be in conflict. For example, the goal of having students hear a clear and complete explanation can conflict with the goal of having students do the explaining. Or, when trying to manage the problem of *students* doing the intended mathematics, teachers also have to consider whether a representation, problem wording, or worksheet format that is designed to support students' engagement might inadvertently be doing some of the intended mathematics for them. For example, one of Courtney's goals was for students to understand the meaning of comparison. The textbook's diagram could be seen as supporting this understanding, but it might also be enabling students to simply fill in boxes without understanding the very relationships the diagram's geometry is intended to convey.

Making correspondences between the problem context, the diagram, and the picture, as well as explaining how the geometry of the diagram conveys the concept of comparison, required Courtney to have an understanding of the mathematics made available for study by each representation and of the relationships among them—a key part of mathematical purposing. Some of these ideas were discussed in the teacher's guide. In addition to providing suggestions in the lesson narrative about how to introduce the diagram, the teacher's guide included a margin note with a completed comparison diagram for the Math Message and a corresponding picture of matched-up circles (Figure 9), along with the following note: "Point out that the quantity box on the top is as long as the quantity and difference boxes on the bottom. This often provides a good visual for children" (Bell et al., 2007b, p. 385).

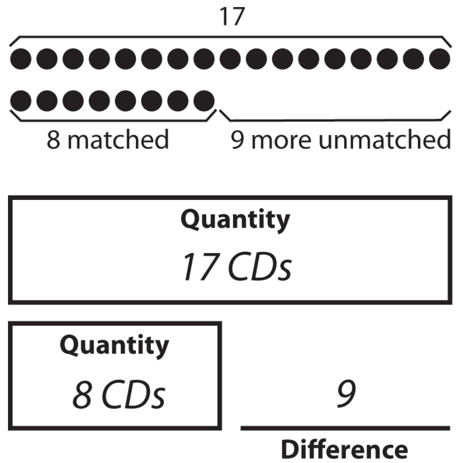


Figure 9. The margin note on comparison diagrams.
Adapted from *Grade 2 Everyday Mathematics Teacher's Lesson Guide*
(Bell et al., 2007b, p. 385).

Of course, information in a teacher's guide does not automatically transfer into an enacted lesson. A teacher would need to closely read and understand the lesson narrative; notice, read, and understand the margin note; evaluate the mathematical importance of the information; determine if it relates to her learning goals for the lesson; decide whether it is something she wants to bring up with students; and then if so, figure out how to best convey it to her particular class. Courtney's understanding of these ideas[35] was evidenced in the pre-lesson interview, for example, in her description of what she found useful about the diagram:

> I like that it's the bigger quantity on top in the bigger box and then the two lines underneath the box and the line should add up to the bigger one, and then you can see on the diagram that they're just as big as the bigger one when you add them up. So I like that about it…because it's a visual. It shows them visually the kind of math we're doing. So these, the smaller quantity plus the difference, is the same as the larger quantity. And also the two quantity boxes, it's not just like we're going to pull the two numbers out of the problems and write them in two quantity boxes, we have to figure out which one's bigger and so then we'll be able to write that in the top box and see that it's the bigger quantity, because of the size of the box. And also it just kind of goes along with the drawing or the picture that I would draw for these problems in my head, which is right above there [referring to the matched-circles picture in the margin note], where you have the longer quantity or the bigger quantity and the smaller quantity and you're looking to match them up. (C-Pre, R96-100)

[35] I do not know whether Courtney read this part of the lesson narrative and/or margin note, and if she did, whether she would have done (or been able to do) this mathematical analysis without having read it. For the purposes of my study, this does not matter. I am using this example to show that the mathematical analysis of the representations used in an activity is part of the work of mathematical purposing, not to make claims about Courtney's understanding of the textbook or about whether she would have been able to do the mathematical analysis without it.

Mathematical purposing also involves analyzing how variations in representations impact the mathematics made available for study. For example, the yellow poster Courtney used was one that her cooperating teacher had made in a previous year and was different than the diagram in the current version of the textbook (Figure 9). Most noticeably, the poster included a unit box and used a box, rather than a line, to denote the difference. Courtney knew the diagrams did not match; in fact, she thought using a line to denote the difference highlighted that it was not the same type of mathematical object as a quantity (C-Post, T127). Courtney noted that if she were to redo poster, she would match the textbook. However, she did not think the discrepancies warranted a new poster or explicitly pointing out the variation to students.[36] She thought the correlation between the two versions would be easy for students to see on their own. Furthermore, she was not sure what she would have said without overly complicating matters (C-Post, T131).

Courtney also introduced the representations in a different order than the textbook: Courtney explained the diagram before drawing the matching picture. Like the selection of problems, the sequencing of representations and solution methods can orient and help steer instruction toward the mathematical point. Thus, specifying a sequence that is likely to focus students on the intended mathematics is part of the work of mathematical purposing. Depending on one's learning goals or the mathematical point of a particular representation or method, different sequences may be more or less productive. For example, although one of the main points of Courtney's lesson was for students to represent comparison stories with number models (e.g., 17 - 8 = 9 or 8 + 9 = 17) and this was the next step in the teacher's guide, Courtney deliberately did not ask students to generate number models that represented the Math Message. She planned to introduce this piece in the discussion of the next problem because thought it might be "beating it to death if [she] kept going and making them do more" with the CDs problem (C-Post, T203).

After the Math Message discussion, Courtney read a new problem aloud, again incorporating the names of her students: *Avery scored 14 points. Ross scored 8 points. How many more points did Avery score than Ross?* In the pre-lesson interview, Courtney explained why she used the numbers provided in the textbook: "They just don't seem too tricky to me. It doesn't seem like that will get in the way of the meaning of the problem" (C-Pre, R86, 148). Thus, Courtney felt that the numbers in the textbook's problems oriented students toward the mathematics with which she intended to engage them. In particular, she wanted to focus on

[36] Another option would have been to use an overhead transparency of the diagram, as suggested in the teacher's guide, but we did not discuss this alternative in the interview.

understanding comparison, not on how to calculate answers, which might have become the focus if "trickier" numbers were used. She also thought it was important for students to work with different contexts, and she felt the context was sufficiently varied in the given examples.

After reading the problem, Courtney elicits from students where to record each quantity and the question mark on the diagram. She then calls on Omar to give the difference and asks him for a number model that reflects how he found his answer:

Courtney	Okay, now, here's the tricky part. We need to write the number model for that. So, Omar, think about the math you did in your head there. Avery scored fourteen points, Mira. Ross scored eight points. How many more points did Avery score than Ross? What math problem do you think you were doing?
Omar	Eight plus six.
Courtney	Eight plus what number equals?
Omar	Six.
Courtney	Eight plus?
Omar	Oh, fourteen.
Courtney	What number equaled fourteen. [*Records 8 + ? = 14.*] That's what he was trying to find. So Omar's strategy, watch up here, Chase, watch up here. This is one of your choices for all these problems. Ross, Mira, watch up here. Omar said, eight plus what equals fourteen, so [*points to the corresponding boxes on the diagram*] my small quantity plus what equals my bigger quantity. And that's what his number model looked like. Did anyone do it a different way? Think about it. Did you do eight plus what equals fourteen? Did anyone do it a different way?

Courtney called on some other students, but their strategies were also modeled by 8 + ? = 14, perhaps because students were unclear about what "a different way" meant in this context or perhaps this was the only method students used. Although Courtney was not planning to require students to use a particular type of number model in their later work, because one of her learning goals was for students to begin to understand comparison in relation to subtraction, Courtney thought it was important for a subtraction model to come out in this discussion. Therefore, she deployed a "contingency plan" in the form of a targeted question to elicit a subtraction number model:

Courtney	Look at this, we're thinking about the small quantity plus what equals fourteen. But how could we use subtraction for a problem like this? That's what I'm looking for. How could we use subtraction with this diagram? Hmm, any new hands? Phillip, how could we use subtraction for a comparison problem?
Phillip	You could start at fourteen and count down to eight.
Courtney	Count <u>down</u> eight or count down <u>to</u> eight?
Phillip	[inaudible]
Courtney	We could start at fourteen and take away a certain number until we get to eight. [*Records 14 - ? = 8.*] Is that what you're saying? Okay. Any other ways we could use subtraction? Omar?

Although Courtney thought it was important that students see at least one subtraction model, it did not matter to her whether all four number models (i.e., two addition and two subtraction) were elicited. In fact, in her analysis of the mathematics made available by the instructional activity, Courtney criticized the suggestion in the teacher's guide to write out and label all four number models. She thought students would then simply rely on their knowledge of fact families rather than map their solution method to the appropriate number model and, as a result, students would be engaging in different mathematical work than she intended (C-Pre, R168).

After discussing the subtraction models, Courtney posed the third and final problem the class discussed at the rug: *A radio costs $47. A watch costs $20. How much more does the radio cost?* The selection of problems and examples plays a central role in orienting and steering instruction toward the mathematical point. During the lesson, Courtney decided to skip one of the problems in the teacher's guide because she thought its numbers were too simple, and its context (points) had already been used. She also wanted to use an example involving money because she thought money was a difficult unit for her students (C-Post, T42-44). She also skipped the following problem: *A radio costs $47. A calculator costs $12 less than the radio. How much does the calculator cost?* This problem was the only example in the teacher's guide in which something other than the difference was unknown.

In the post-lesson interview, Courtney explained that she omitted this problem not because the smaller quantity was unknown, but because she had decided not to do another example and she knew that students would not be completing that type of problem in their Math Journal that day. Of the lesson's seven Math Journal problems, only problems number 5 and 7 had the smaller quantity as the unknown. Because the math period was shorter on Mondays (the day of the lesson), Courtney had planned for students to complete only the first four problems in their Math Journal. They would finish the remaining problems the next day. I asked whether she thought that students' seeing only problems with the difference unknown impacted her ability to assess their understanding of the mathematics in the lesson:

> No, because I think that I can sense, I think it's more important for them to master the skill today of, or to practice the skill today of, plugging the numbers into the diagrams, seeing what the diagram means, being able to write a number model. I think that, that's what we worked on today, so they all got that part, which was important. So tomorrow, if they don't get problems when we already have the difference, then I can be more sure that that's the confusing part, but they do, at least they know those fundamental things that we talked about the day before about the diagram and writing a number model before you get the answer.…If they get, they get what we did today. And I think that they all did and we can say that. So tomorrow, if they struggle, at least I know that they understand

> the diagram…because I can know that the part that's confusing is the fact that we don't have one of the quantities. I know that they get it in this way, and if these are hard, I know what's different about them. Here [today's problems] there are so many new things that they're doing, it's a whole new kind of problem. Here [tomorrow's problems] it's, we're changing one thing about it, so it's almost like if this is much harder than this was I know why. (C-Post, T78-84)

Courtney's comments suggest a number of issues related to teaching to the mathematical point. One is that the mathematical point extends beyond a single lesson. Thus, the work of teaching to the mathematical point needs to take into account the relationship to past and future instruction. Another issue is the role of different types of examples. As seen in Jordan's lesson, different "cases" of problems serve different mathematical purposes. Similarly, in Courtney's lesson, varying what is unknown in the problem generates different cases of comparison problems. What Courtney's lesson raises is the role of cases in developing and assessing student understanding. If students do not experience the entire range of cases, then it can be difficult to say whether they understand a mathematical idea or are just rotely going through the motions without attending to meaning. Teaching to the mathematical point does not mean that teachers need to expose students to examples from all cases in every lesson. In fact, Courtney had reasonable arguments for why she chose to introduce one case at a time. Instead, this example highlights that analyzing cases and their coverage of the mathematical territory in order to strategically select examples is an important component of mathematical purposing.

After the discussion of the third example, Courtney transitions students back to their desks to work on their Math Journal. As students get settled, Courtney erases the yellow diagram and moves it to the large whiteboard at the front of the room. When students are seated with their workbooks open, Courtney reads the first problem aloud and asks students to fill in the diagram on their own, explicitly stating that they will be "talk[ing] about what you put where and why." She circulates as students work and when everyone is ready, asks a student to explain how he filled in the diagram. Once the diagram is filled in, Courtney directs students to find a number model, making explicit what the number model represents and what it should include: "So, let's do a number model now. They might be different from your neighbor. We're trying to figure out what math problem you're doing in your head to get the difference. There should be a question mark in your number model too."

Courtney moves around the room, looking over students' shoulders and prompting them when she notices they have made a mistake or are stuck. After about two minutes, she reconvenes in whole group to share number models. Once again, she tries to keep a focus on meaning by connecting students' number models to the completed diagram:

Courtney	Oh, we have a lot of number models ready to share. Omar, read us your number model.
Omar	Twenty-seven minus ten equals blank.
Courtney	[*Writes 27 - 10 = ? on the whiteboard.*] Equals blank. Twenty-seven minus ten, let's check this. Twenty-seven is our bigger quantity. Ten is our smaller quantity. If we have twenty-seven take away ten, how many will we have left? Thumbs up if you think Omar's number model works.

She then maps the number model back to the problem:

Courtney	If we, Barb had twenty-seven points and Cindy had ten, if we take ten away from twenty-seven, will we have the difference between those two numbers?
Students	Yes.
Courtney	Griffen thinks so. Raise your hand if you had a different number model. I think a lot of people had a different one. Ian, what was your number model?
Ian	Ten plus what equals twenty-seven.
Courtney	Ten plus what equals twenty-seven. [*Writes: 10 + ? = 27.*] Raise your hand if your number model looked like Ian's. Did anyone have the same as Ian? Mason did, Phillip did, Trista did. Okay, so let's look at Ian's. Ten plus what equals twenty-seven. Now that really makes sense to me because I know that on this diagram, this, eyes up here please, Nate and Kelsey and Griffen, this quantity plus this difference need to equal this big quantity. And that's easy for me to remember because these two boxes put together are the same size as this box. I know that this plus this needs to equal this. Any other number models?

No one offers another model, so Courtney asks for the difference and then maps it back to the problem. She then has students fill in the diagram for the second problem on their own. After students complete the diagram, Courtney elicits number models and answers as before. Students then solve Math Journal problems three and four independently. Courtney and her cooperating teacher circulate and check students' answers until it is time for the students' next activity of the day.

Courtney's lesson demonstrates many aspects of the work of teaching to the mathematical point. To help articulate her mathematical point, Courtney analyzed the various problems and representations in the lesson for the mathematics they made available and the relationships among them. She oriented the activity toward the intended mathematics through the selection of problems and sequencing of representations throughout the lesson. During instruction, she steered the activity toward her mathematical point in a number of ways. She provided a mathematical framing for the various activities and narrated the mathematics being worked on throughout the lesson. She was explicit about the meaning of new vocabulary and intentionally overused those terms. She helped keep a focus on meaning by restating students' answers and then connecting them back to the question, topic, or problem being discussed.

Both Courtney's and Jordan's lessons offer a glimpse inside the work of teaching to the mathematical point. The detailed descriptions and extended excerpts illustrate ways that both of these beginning teachers had mathematically purposed their instructional activities and tried to steer those activities toward their mathematical points. In the next two chapters, I provide a more analytic description of what is involved in the work of teaching to the mathematical point. Chapter 5 uses examples from the data to illustrate the problems in steering instruction toward the mathematical point that emerged in my analyses and the issues that can arise when trying to manage these problems. The discussion of these problems helps unpack both the work of steering instruction and the work of mathematical purposing. In Chapter 6, I stand back from the data to present the conceptual framework for mathematical purposing that resulted from my analyses.

CHAPTER FIVE:
PROBLEMS IN STEERING INSTRUCTION TOWARD THE MATHEMATICAL POINT

Introduction

In this chapter, I use examples from the data to explore problems in steering instruction toward the mathematical point. I focus on problems in steering instruction before presenting my framework for mathematical purposing because steering is, in a sense, the "front line" of teaching to the mathematical point. It has a more direct influence on whether and how students engage with the intended mathematics. Mathematical purposing, of course, influences students' engagement with the intended mathematics. However, mathematical purposing "goes through" steering to impact students: The work of articulating the mathematical point and orienting the instructional activity toward the intended mathematics better positions a teacher to steer the activity during instruction. In other words, the purpose of mathematical purposing is to help manage problems in steering instruction. Thus, an examination of problems in steering instruction toward its mathematical point and of issues that can arise when trying to manage these problems lays the foundation for and helps illustrate components of the framework for mathematical purposing presented in the next chapter.

From my analyses, I identified the following problems that must be managed in steering instruction toward the mathematical point:[37]

- Attending to and managing multiple purposes;
- Spending instructional time on *mathematical* work;
- Spending instructional time on the *intended* mathematics;
- Making sure *students* are doing the mathematical work;
- Developing and maintaining a mathematical storyline;
- Opening up and emphasizing key mathematical ideas; and
- Keeping a focus on meaning.

Throughout the descriptions of Jordan's and Courtney's lessons in the previous chapter, I tried to highlight examples of these problems and ways in which I thought the teachers were trying to

[37] As described in Chapter 3, these problems were identified when coding the data, in particular, the lesson videos.

manage them. Situating this discussion in the context of a lesson was intended to demonstrate the complexity of teaching to the mathematical point, in particular, how the problems must be simultaneously attended to and managed during instruction. I now focus on each problem independently using examples from across the set of lessons in my study. Taking the problems out of the context of a particular lesson allows them to be unpacked in more detail; looking across a range of beginning practice illuminates a wider variety of strategies for managing each problem and a broader collection of issues that can arise when trying to do so. Even though I focus on each problem independently, it is important to remember that the problems are overlapping and occur simultaneously in instruction, and that the management of a particular problem is not associated with a specific set of teaching practices. In fact, at any time, there are a variety of teaching moves that could be used to address each one. Similarly, a particular teaching move could be used to manage multiple problems.

For each problem, I provide a general description of its relationship to teaching to the mathematical point and discuss strategies for and issues that can arise when trying to manage it. I illustrate by referring back to examples from Jordan's and Courtney's lessons, as well as to other lessons in the data.[38] Because some of these examples are used to show what is difficult about managing a problem, I sometimes describe an aspect of a lesson that did not go well—for example, an episode that was unclear or unfocused. The purpose of such an example is <u>not</u> to make claims that a preservice teacher's entire lesson was unclear or unfocused, or that a particular teacher was unable to teach to the mathematical point. Here and throughout the dissertation, I use examples from the data to illustrate what is involved in the *work*, not to make claims about the people doing that work. Moreover, a particular teacher's practice and management of these problems most likely varies within and across lessons. Therefore, it would not even be accurate to make those sorts of broad claims about an individual teacher.

Another comment to make at the outset is that some of the issues that arise with respect to the different problems might be attributable to the nature of the curriculum or to the details of an activity that were taken directly from a textbook. For example, issues related to the problem of developing and maintaining a mathematical storyline might be more evident in a spiral curriculum such as *Everyday Mathematics*. Because the aim of this dissertation is to unpack the *work* of teaching to the mathematical point, I am not trying to make claims about or even consider the causes of particular issues at this time. For the purposes here, the distinction is not important; however, in considering the implications for practice, the role of the curriculum materials, as well

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[38] It may be helpful to refer back to Table 2 (Chapter 3), which lists each preservice teacher's name, grade level, and lesson topic.

as the influence of other factors such as teacher knowledge, in helping (or not helping) to manage the problems is of tremendous importance.

In addition to illustrating problems in steering instruction toward the mathematical point, the episodes discussed in this chapter simultaneously illustrate aspects of the work of mathematical purposing that will be discussed in the next chapter. Although I have conceptualized the work of steering instruction and mathematical purposing as distinct types of teaching work, it is not possible to analyze problems in steering instruction toward the mathematical point without articulating what the mathematical point is and how the activity is oriented toward it. Thus, in describing the episodes from preservice teachers' lessons, I rely on excerpts from the interviews and my own analysis to describe the mathematical point and how the activity was set up (or not set up) to engage students with it.

Furthermore, as described in Chapter 3, many aspects of my framework for mathematical purposing emerged in response to problems in steering instruction toward the mathematical point. For example, as will be discussed below, one of the issues that can arise in trying to manage the problem of spending instructional time on the intended mathematics is the non-strategic selection of numbers in problems. There were many instances in the data where the numbers used in a problem resulted in students engaging with different mathematical ideas than the preservice teacher intended. As a way to help manage this problem, the work of mathematical purposing therefore includes unpacking the mathematics made available by different numerical examples (i.e., a component of articulating the mathematical point) and selecting cases that are most related to the intended mathematics (i.e., a component of orienting the instruction). In other words, some aspects of the work of mathematical purposing were conceptualized by working backwards from the strategies and issues identified in the data and being discussed in this chapter. Thus, for each problem in steering instruction toward the mathematical point, there is corresponding work of mathematical purposing. In this way, the examples in this chapter are intended to illustrate both problems in steering instruction toward the mathematical point and the work of mathematical purposing that will be examined in the next chapter.

I begin now with the first problem that emerged in my analysis: attending to and managing multiple purposes.

Attending to and Managing Multiple Purposes

That teachers have to attend to and manage multiple purposes during instruction is certainly not a new observation. A recurrent theme in the literature and one seen throughout the data is that these multiple purposes are often in conflict. Managing multiple purposes and the resulting dilemmas is, in fact, one of the reasons that teaching is so complex (Lampert, 1985). I identify two components of the problem of attending to and managing multiple purposes: attending to multiple mathematical learning goals and managing mathematical and non-mathematical purposes. I discuss each below.

Attending to Multiple Mathematical Learning Goals

The multiplicity of mathematical learning goals in a single lesson stems from a variety of factors. As discussed in Chapter 1, the ultimate goal of mathematics instruction is to develop students' mathematical proficiency. Mathematical proficiency, by definition, is multi-faceted and requires concurrent attention to the development of its five strands[39] (National Research Council, 2001). In instruction, this translates into simultaneously working toward different types of learning goals related to the different strands.

Another reason lessons have multiple mathematical goals is that students' opportunities to learn are nested and occur over time—for example, problems are situated in activities, which are situated in lessons, which are situated in units, which are situated in school years. Teachers have different mathematical learning goals for problems, activities, units, and school years, which are correspondingly nested, and thus, during instruction, concurrently at play. Similarly, mathematical learning goals themselves occur at different grain sizes. For example, in a single lesson related to fractions, a teacher might be trying to help students learn a specific fact (e.g., that $\frac{1}{2}$ is greater than $\frac{1}{3}$), understand more general concepts about fractions (e.g., the need for equal parts and the importance of attending to the whole), and develop the ability to use representations to explain their solutions.

Clearly articulating and understanding the connections between mathematical learning goals of different types and grain sizes is an important aspect of teaching to the mathematical point. Articulating overarching mathematical learning goals broadens the mathematical terrain of a lesson and makes more visible connections across a lesson's activities, problems, examples, and exercises. Attending to overarching goals can provide ways to work on all of the strands of mathematical proficiency in a given lesson. Articulating smaller, more specific goals for particular examples or activities both unpacks the mathematical terrain and clarifies which part of the terrain is intended to be traveled through with students at a particular time. Articulating learning goals of different types and grain sizes, their connection, and how an activity is intended

[39] Recall from Chapter 1 that the five interwoven strands of mathematical proficiency are: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition.

to move students toward these goals is the work of mathematical purposing and will be further unpacked in the next chapter.

Having a clear articulation of these different types and grain sizes of mathematical learning goals helps teachers better understand what they are steering their lessons toward. Many instructional decisions—such as which solutions to discuss, how to respond to a student's comment, or what activity to skip if running late—can be informed by identifying the courses of action that best support the mathematical learning goals. For example, in Jordan's lesson, when the end of the independent work time was approaching, she directed students to focus on the problems that were most related to the intended mathematics rather than on the other sections of the worksheet.

Of course, clearly articulating mathematical learning goals does not ensure that the learning goals will be attained. However, being aware that to develop mathematical proficiency the "point" of a lesson includes a complex web of mathematical learning goals of varying types and grain sizes and understanding the ways in which the mathematical point can be used to inform instructional decisions are important first steps.

Managing Mathematical and Non-Mathematical Purposes

Even with attention to the multidimensionality of the mathematical point, not all instructional decisions will be—or should be—decided for mathematical reasons. However, as discussed in the literature (e.g., Kennedy, 2005; Lampert, 1985, 2001) and seen in the data, decisions made to achieve non-mathematical purposes, such as maintaining lesson momentum or encouraging participation of a normally quiet student who is raising his hand, can impact whether a lesson stays on its intended mathematical course. One way teachers can manage the problem of having non-mathematical purposes while trying to teach to the mathematical point is, when making decisions for non-mathematical purposes, to ask if a decision impacts their mathematical point. This again seems rather simplistic (and of course is not always possible in the moment) and does <u>not</u> mean that a teacher would then decide to abandon a non-mathematical purpose in favor of the mathematical point. Rather, explicitly asking about the impact of non-mathematical decisions on the mathematical point can help a teacher manage this problem by providing more information to weigh.

In my analyses, I observed a few ways that decisions made for non-mathematical purposes can impact a lesson's progress toward its mathematical point. One possibility is that a decision made for non-mathematical purposes has no impact on the mathematics. For example, Courtney's changing the names in the textbook's problems to the names of her students—a move made to foster student engagement—did not impact the mathematics available to be worked on in those problems.

There were many other cases, though, where decisions made for non-mathematical purposes did impact the mathematics available in the lesson. For example, in Mia's third-grade lesson on reading and interpreting bar graphs, she followed her classroom's usual routine of students' completing their work on individual whiteboards, while she displayed the graph and accompanying questions on the overhead projector. Students did not have individual copies of the worksheet at their desks and, as a result, had a hard time answering the questions because they could not determine the heights of the bars from afar. To manage this problem, Mia read and recorded the height of each bar on the projected copy. This move did not take away from her main mathematical point of students' using this information to answer questions about the graph; however, it did take away the opportunity to also use the activity to provide students with practice reading the values of the bars. This is not to say that it was a bad decision to use the whiteboards for this lesson or that she should not have read the graphs for the class. In fact, the impact on the mathematics seems fairly minor because the lesson could still progress toward her main mathematical point. And, Mia may have known that her students were already able to read bar graphs and, therefore, her reading the graphs for them was not taking away needed opportunities to practice that skill.

There are many times, however, when decisions made for non-mathematical purposes have more critical consequences for the mathematical point. I observed a number of these examples in my analyses. Echoing the findings in Kennedy's (2005) study, teachers in my study frequently made moves to motivate students or get them "into" an activity. As seen with the above example from Courtney's lesson, these decisions do not necessarily conflict with teaching to the mathematical point. In fact, decisions made to increase student engagement can enhance the mathematics being worked on in a lesson. For example, two of Mia's non-mathematical purposes were to increase student engagement in her lesson and to gain access to her students' thinking. To accomplish these, she routinely asked students to explain how they got their answers. Thus, Mia did not ask for explanations because she had learning to give mathematical explanations as an explicit goal for student learning. However, because she asked for explanations to further her non-mathematical purposes, her students were given opportunities to develop this aspect of mathematical proficiency.[40]

[40] Davis (2006) observed a similar issue in science instruction: In science lessons, preservice teachers often have their students answer questions and design investigations to foster students' interest, not because engaging in scientific practices is the learning goal.

In other cases, decisions made to foster student engagement distorted or distracted from the mathematical point. For example, Hannah taught a first-grade lesson on equivalence, in which students placed different combinations of seven unifix cubes[41] on each side of a pan balance and the resulting equations (e.g., 1 + 1 + 1 + 1 + 1 + 1 + 1 = 7, or 5 + 1 + 1 = 7) were recorded to demonstrate that there were many ways to name the number 7. In her launch of the activity, Hannah tried to pique her first graders' interest: "Watch me. I have two magical sticks of unifix cubes. Do you know why they're magical?…Because they are equivalent." Throughout the activity, Hannah repeatedly referred to the "magical balance" or commented that it was "like magic" that the two sets of cubes were equal. When asked about the use of magic in the post-lesson interview, Hannah replied: "They think it's awesome. [Magic is] just a word to throw in. Yeah, it's just to keep their engagement, really. There's nothing magical about it" (H-Post, T77-80). Whether Hannah's talk about magic impacted students' ultimate understanding of equivalence is not known. However, it seems that a mathematical, rather than magical, framing would have better supported the development of mathematical proficiency.

I also observed situations in which efforts to manage mathematical and non-mathematical purposes degraded the mathematical learning goals. For example, in Nicole's introductory lesson on place value with decimals, her non-mathematical purpose of reducing students' anxiety led to a reduction in both the quality and quantity of mathematics in her lesson. The textbook's stated objective was "to understand tenths and hundredths; and to exchange between tenths and hundredths" (Bell et al., 2004a, p. 332). In the pre-lesson interview, Nicole explained that she thought this lesson was "developmentally inappropriate" for her third-grade students and stated the following as her goals:

> Just getting them feeling a little more comfortable with knowing parts of a whole…If they walk away still not understanding this is the tenths place, this is the hundredths place, but knowing, okay, it's less than one, I'd be happy. (N-Pre, R21)

Nicole's teaching reflected her worries about the lesson's difficulty. Each time she taught the lesson (three small groups rotated through the activity), she repeatedly told students "not to stress." Although the lesson was designed to develop students' understanding of the relationship between tenths and hundredths, Nicole focused most of the instruction on telling students the names of the places (i.e., tenths, hundredths, and thousandths) and pointing out that when naming

[41] Unifix cubes are colorful interlocking cubes that link in one direction. They can be snapped together to form long "sticks" or "trains." These manipulatives are often used in elementary school to work on concepts related to counting and sorting, number and operations, and patterns.

decimals it was acceptable to say “and.”[42] Her non-mathematical goal of not “stressing” her students spurred her decision not to introduce how to read decimals in her lesson. For example, instead of reading .68 as “sixty-eight hundredths,” she used the language of “sixty-eight out of one hundred.” She explained this decision in the post-lesson interview:

> With the third group, we talked about kind of how you would read that, like the twenty-seven hundredths or thirty-three hundredths. But with the other groups I thought that would just be kind of too much, like, they were kind of at a breaking point like, “I’m over it.” So I didn’t want to push them and be like, “And we read this a special way.”…[It’s] not really important how we read it yet, let’s just understand that they’re there and they have value, as opposed to well, how do we read it. Because I think they just would have been like, “What?” (N-Post, T40-44)

This shift in learning goals is reflected throughout her post-lesson interview in her descriptions of what she was trying to accomplish in her lesson:

> I mean, I wanted them to walk away with the…understanding that this is the tenths, this is the hundredths, just that idea of those places have names and they’re not the same as the places over here on this side of the decimal…Because I was actually, just ran into a colleague in the copy room as I was making copies and I was talking about how it went, and I don’t think any of them really have the understanding that, okay, well, ten hundredths make one tenth. But that wasn’t really one of my goals. I don’t need them to have that understanding yet. I just needed them to have the understanding that there are these places after the decimal that have value and have meaning, they’re not part of a whole, yet, but they’re still there. Just really understanding that [those places] exist. That they’re there. They’re not fake. They’re real. We see them when we use money and other things, so they’re really there. (N-Post, R16-22)

Nicole explained that she did not explicitly use her goals to make decisions in her lesson, but did adjust her goals based what she thought students “could handle” (N-Post, T176-178). As a result, the mathematical point of Nicole’s lesson was reduced to knowing that decimals exist, telling students the names of the places, and noting that it is acceptable to say “and.” Thus, her goal of “not stressing students out” resulted in impoverished mathematical goals and an essentially decimal-free lesson. This relates back to the earlier discussion of managing multiple mathematical learning goals. Because the overarching goal is the ongoing development of students’ mathematical proficiency, implicit in teaching to the mathematical point is that the mathematical point is worthwhile.

Table 4 summarizes the above discussion of the problem of attending to and managing multiple purposes. A similar summary table will be included after the discussion of each problem.

[42] This refers to the fact that “and” is not used when naming large whole numbers (e.g., 237 is read as “two hundred thirty-seven” not “two hundred *and* thirty-seven”); however, “and” is part of decimal names (e.g., 2.86 is read “two and eighty-six hundredths”).

Table 4.
Summary of the Problem of Attending to and Managing Multiple Purposes

Managing multiple mathematical learning goals

- Simultaneously working on mathematical learning goals of different types and grain sizes:
    - Learning goals related to each of the strands of mathematical proficiency
    - Learning goals that are nested and developed over time
    - Learning goals of different grain sizes

Managing mathematical and non-mathematical purposes

- Considering whether a decision made for non-mathematical purposes:
    - Has no impact on the mathematical point
    - Supports or further promotes teaching to the mathematical point
    - Distorts or distracts from the intended mathematics

Spending Instructional Time on *Mathematical* Work

The second problem—spending instructional time on *mathematical* work—is in many ways another obvious part of steering instruction toward the mathematical point: To learn math, students need to spend time on math. Yet, it is worth explicitly naming as a problem to manage when teaching to the mathematical point because the literature is filled with examples of mathematics instruction that is not spent on mathematics (Hill et al., 2008; Kennedy, 2005; Stein et al., 2000). Furthermore, many things compete for time in teaching. Therefore the problem of making sure time is spent adequately and well on the mathematical point is especially important. This problem focuses on spending time on *mathematical* work instead of *non-mathematical* work. The next problem takes the idea of spending time on mathematics one step further to consider, when time is spent on mathematical work, whether it is focused on the intended (rather than unintended) mathematics.

Lessons necessarily involve some non-mathematical work. Activities such as distributing materials, getting out supplies, cutting out or coloring manipulatives, or transitioning between lesson segments occur in almost every lesson. One way to steer instruction toward the mathematical point is to reduce the amount of time spent on such necessary but non-mathematical elements. For instance, Courtney and Jordan, like most of the teachers in my study, organized their materials in advance so they could be easily accessed during the lesson and tried to transition quickly between the lesson's activities.

As described in the literature and echoed in the data, when students are cutting or coloring during a lesson, there is a particular danger that the activity might devolve into non-mathematical work (Stein et al., 2000). I observed a number of moves teachers used—successfully and unsuccessfully—to try to manage this problem. When an activity involves cutting out or coloring a manipulative or tool that will be used later in the lesson, one way teachers can reduce the amount of instructional time spent on non-mathematical work is to do some of the manipulative preparation for students before the lesson. For example, in Rachel's second-grade lesson, students folded paper squares to explore fraction concepts. Although the teacher's guide suggested that students cut their own squares out of large pieces of paper, Rachel decided to cut out the squares in advance, which meant students did not have to do this non-mathematical work during the lesson. Whether to do this type of work for the students might also be weighed in light of (possibly conflicting) non-mathematical goals, such developing fine motor skills.

A similar move can be made during instruction. For example, in Larkin's fifth-grade lesson, her students folded and cut out kites for use in an exploration of symmetry. When most students were ready to move on, there was still one student who had not cut out his kite. Instead of having the class wait, Larkin discreetly cut out his kite while transitioning students to the next part of the lesson. This move reduced class time spent on non-mathematical work, yet because the work Larkin completed for the student was non-mathematical, it did not take away his opportunity to engage with the intended mathematics. Imagine instead that the student had not finished problems in his Math Journal. In that case, if Larkin would have simply answered the questions for him, it would have taken away his opportunity to engage with the intended mathematics.

Thus, in teaching to the mathematical point, it is important to ask if doing some or all of the work for students diminishes their opportunity to engage with the intended mathematics. In Larkin's kite-cutting case, the answer was no. However, in Larkin's lesson, passing out the materials, giving directions, and cutting took approximately six and a half minutes. Thus, even though Larkin found ways to reduce the non-mathematical work time, it is still important to ask whether and how time spent cutting out kites furthered her mathematical goals.

Sometimes a teacher's effort to reduce time spent on non-mathematical work can detract from the mathematical point—yet another example of the problem of managing conflicting purposes. Although Rachel cut out the squares for her fraction lesson in advance, she had not organized them for distribution. When she started passing out squares, she informed students that they would each need six. However, she soon realized she had not cut enough, so she then told students they only needed five and asked them pass any extras to their neighbors. While this was happening, the classroom aide (unbeknownst to Rachel) had cut additional squares, which resulted in reshuffling back to the original six per student. Distributing squares took two and a half minutes. This was instructional time not spent on mathematics that could have been reduced by a move such counting and grouping the pre-cut squares in advance. Perhaps it may even have been quicker to use the textbook's suggestion of giving each student three sheets of paper and having them cut their own squares.

Once students had their squares, Rachel led the class through one of the lesson's main activities—folding squares in halves, fourths, and eighths in multiple ways. She elicited from students different ways to fold and used this part of the lesson to emphasize that fractions required equal parts. Although much instructional time was spent folding squares, this was part of the *mathematical* work of the lesson and Rachel used it to steer students toward her mathematical point. Thus, it would not have served her mathematical purposes to reduce the time spent folding, as this was mathematical.

The next part of Rachel's lesson used the folded squares to introduce naming and writing fractions. The textbook suggested that students color a given number of parts on one of their squares and then name the fraction of the square that is colored and not colored. Rachel decided not to have students color because she thought "it would have been chaotic if they were coloring in" (R-Post, T131). Instead she held up a folded square, covered some of its parts with her hand, and asked students to name the fraction represented by the uncovered region. Although in most cases students could answer her questions correctly, there were times when it was unclear to which part of the square she was referring. Thus, Rachel's decision to reduce non-mathematical work (i.e., coloring) adversely impacted students' engagement with the intended mathematics. Rachel reflected on her decision not to color in her post-lesson interview, and brainstormed alternative moves that might have sill reduced the time spent coloring and thus alleviated her fears of "chaos," but made it easier for her to engage students with the mathematical ideas:

> It might have been a nice idea, like if they colored in one-fourth and then everybody looked at their own thing and tried to figure out how many were not colored and how many were. So maybe the next time I would have them color it. But again, when they were coloring in this [referring to a problem on the Math Journal], some of the kids, I specifically said, like, "Just lightly shade," and some of the kids were like filling in every part and taking forever, and so that was my other fear, that it would just take away from the math if I said color this in, because some kids just get crazy.…Or maybe I should have colored it.…Then, then I could have shown them my colored thing and they wouldn't have had to color anything, but they could have looked at it more concretely. So that would have been a good idea. (R-Post, T131, 135-137)

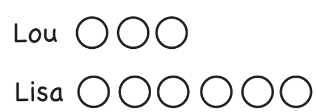
In my analyses, I found that student (and teacher) confusion can also cause time to be spent on non-mathematical rather than mathematical work, thus steering a lesson away from its mathematical point. For example, Sydney used the same names—Lou and Lisa—in the first two problems she posed to her first-grade students:

> Problem 1: Lou saved four cents. Lisa saved six cents. Who saved more money? How much more?
>
> Problem 2: Lou saved three pennies and Lisa saved six pennies. Who saved more pennies and by how much more?

Compounded by the fact that she did not write the second problem on the board and that she read it incorrectly, both Sydney and her first-grade students could not keep track of how many pennies each person had:

Sydney	Okay, let's try another one.…Let's see if you guys can do this one without me writing it out, okay? How about this, Christine? Lisa saved three pennies. Shh. I'll start with Lou again. Lou saved three pennies and Lisa saved six pennies. Who saved more pennies and by how much more? Think about that in your head for a second, okay? Lisa saved three and Lou saved six. Or, sorry, Lou saved three and Lisa saved six. Chloe, who saved more pennies?
Chloe	Lisa.
Sydney	Lisa. How many did Lisa save?
Chloe	Two?
Sydney	Nope. Who, you said Lisa saved more pennies, right? Okay. She saved six pennies and Lou had saved three pennies. Now what I wonder is if anyone can find out the difference of how many more pennies Lisa saved. Joel, do you know?
Joel	Six.
Sydney	No. That's how many pennies she saved. Let's write it out. So here's Lou's pennies, okay? And here's Lisa's pennies.

Sydney displays and labels each person's money on the whiteboard using large magnetic pennies:



Sydney	Amanda, are you looking up here?
Amanda	Yeah.
Sydney	Okay. Boys and girls, Lou is on the top and Lisa is on the bottom. Can everyone see that? Lou has three and Lisa has six. And Chloe said that, who saved more?

Thus, in this example, confusion due to poorly worded problems and muddled language resulted in wasted instructional time and detracted from the mathematical point.

Situating a mathematics activity in a non-mathematical context can also cut into the amount of time spent on mathematics and make it harder to steer the lesson toward the mathematical point. This can happen when an elaborate context needs to be introduced, set up, and explained or, as in the case of Beth's second-grade lesson, when a non-mathematical context is used to gather data for the subsequent mathematical activity. Beth's lesson was based on an *Everyday Mathematics* lesson called "Data Day: The Four Food Groups." The teacher's guide stated the following as the lesson's objective: "To provide experiences with collecting, sorting, tallying, and graphing data" (Bell et al., 2007b, p. 390). The textbook's version of the lesson began with a discussion about "good nutrition" followed by an introduction to the "basic food groups." After this discussion, each student was to name their favorite food and then, as a class, assign it to the appropriate food group, recording the decision in a tally table. Once all of the tallies were recorded, the class was to discuss the completed table and then students were to make a bar graph in their Math Journals that represented the data in the table. The next activity in the lesson was an "ongoing learning & practice" worksheet involving comparison problems.

In her pre-lesson interview, Beth explained that her lesson goals focused on students' being able to "talk about their graphing or the graph and setting up the graph and the data table" (B-Pre, T13). She expressed concern about the non-mathematical nature of the lesson, calling the focus on food groups "kind of random" (B-Pre, T19). Beth decided not to skip the food-group portion of the lesson, but, encouraged by a margin note in the textbook, made a number of modifications aimed at reducing the amount of time spent on non-mathematical work. For example, she had students work in table groups to categorize their favorite foods instead of doing this as a whole class. She also cut out pictures that represented the different food groups in advance to help students remember the types of foods that belonged in each category and to facilitate data collection. Her allocation of time during the lesson also reflected her mathematical goals. She moved quickly through the food group segment, only having a few students share ideas about healthy foods. She did not probe students' food-related comments nor take up whether their suggestions were actually healthy.

Surprisingly, it was later in the lesson where Beth spent more time than anticipated on non-mathematical issues. Students were unsure what to do with the food group cut-outs she had prepared, and although she made an effort to familiarize students with the bar-graph template in the Math Journal, students were confused by the representation. Both she and her cooperating teacher spent time answering logistical questions and eventually Beth interrupted the lesson to explain in more detail where the bars were supposed to be drawn. Beth brought up both of these instances in her post-lesson interview, reflecting that it would have been preferable to spend more time before students began working to talk explicitly about how to use the representation and how she wanted them to work (B-Post, T29-39). She also noticed that the explicit discussion of how to use the representation enabled her to point out key ideas about working with bar graphs:

> It had never really occurred to me but it was a good point to talk about, like how to organize your data if you're, which is the discussion we had afterwards, like if you're talking about bread and cereal is it going to be over the fruits and vegetables? No, it's going to be over that category, so, it was a happy accident. (B-Post, T33)

Thus, not only can familiarizing students with representations increase time spent on mathematical work by reducing later confusion, it can further students' engagement with the intended mathematics by creating an opportunity to explicitly discuss key mathematical ideas.

As these examples show, a number of issues can arise when trying to manage the problem of spending time on mathematical rather than non-mathematical work. However, teaching to the mathematical point does not imply that non-mathematical work should always be minimized. For example, Courtney intentionally *added* time to the transitions between her lesson's activities by teaching different activities in different parts of the classroom. She argued that spending this non-mathematical time helped her teach to the mathematical point because it provided students with an opportunity to move during the lesson, which enabled them to better focus on the mathematical work.

Table 5 summarizes the discussion of the problem of spending instructional time on mathematical work.

Table 5.
Summary of the Problem of Spending Instructional Time on Mathematical Work

Strategies for reducing time spent on non-mathematical work:

- Organizing teacher materials in advance
- Transitioning quickly between lesson activities
- Doing non-mathematical manipulative preparation for students (before or during the lesson)
- Not taking up or not deeply probing students' non-mathematical comments

Issues that can arise:

- Disorganized preparation and/or distribution of materials
- Omitting non-mathematical work that actually helps students engage with the intended mathematics
- Unnecessary and unproductive confusion due to unclear directions, poorly worded problems, or muddled language
- Problems or activities situated in elaborate non-mathematical contexts that take time to explain and establish
- Unfamiliarity with representations and tools leading to interruption of work and re-explanation

Spending Instructional Time on the *Intended* Mathematics

Managing time in teaching is challenging. Many of the preservice teachers in my study expressed concerns related to the timing and pacing of their lessons, for example, commenting that their lessons usually ran long and/or they never got to everything in the teacher's guide. As discussed above, some timing-related issues might be addressed by reducing time spent on non-mathematical activities. However, spending more instructional time on mathematics does not guarantee that work will be focused on the mathematical point. Even when students are engaged in mathematical work, they may not be moving toward the intended mathematical learning goals (Hill et al., 2008; Kennedy, 2005; Stein et al., 2000). This section discusses strategies that can be used and issues that can arise when managing the problem of spending time on the *intended* rather than *unintended* mathematics.

Jordan and Courtney demonstrated many ways to try to manage the problem of steering the lesson toward the intended mathematics. One was the strategic selection of problems, examples, and exercises. Recall, for example, Jordan's focus on problems with two negative signs in order to address her concern that students had overgeneralized the two-negatives-make-a-positive rule, or her choice of -10 and -11 for the comparison discussion. Jordan also focused students on the portion of the worksheet most related to the intended mathematics when she was running short on time. Jordan and Courtney both had strategies for raising a mathematical idea or method that did not come up as expected from students. For example, Courtney used targeted questions to elicit a subtraction number model when it was not offered by students, and in Jordan's lesson, when an anticipated error did not surface, she brought it up herself.

Many factors can cause an activity to shift toward unintended mathematics. One cause is when a reduction in cognitive demand during an activity's set up or enactment results in students doing less mathematically substantial work than intended (Stein et al., 2000). There were numerous examples of this in the data. In many cases, the use of leading questions or overly supportive representations inadvertently changed the content of the mathematical work in which students were engaged. Both of these issues can be seen in the following example from Nicole's third-grade lesson.

As described above, Nicole's lesson was designed to use base ten blocks[43] to introduce decimals. After reviewing the names and relationships of the blocks, Nicole asked her students to name the value of a long if the flat were one whole. Students wrote $\frac{10}{100}$ and $\frac{1}{10}$ on their whiteboards. After a brief discussion, Nicole posed the next example, leading students to the answer in her launch of the problem:

Nicole	Let's see, what if I had one long and six cubes? One long and six cubes? How many do I have?
Paul	Want us to do it both ways?
Nicole	Do what you can. Show me, if I have, so it's, if I have one long, how many cubes are in one long?
Students	Ten.
Nicole	Ten. Plus another six cubes, how many cubes total is that?
Students	Sixteen.
Nicole	Sixteen. So if we have sixteen cubes out of one hundred, what is that going to look like?

By rephrasing the question during her launch, Nicole shifted the problem from "What if I had one long and six cubes? How many do I have?" to "So if we have sixteen cubes out of one hundred, what is that going to look like?" Because Nicole's questions gave away the answer, students did not need to engage with fraction concepts to write $\frac{16}{100}$. Nicole next tried to elicit writing $\frac{16}{100}$ as a decimal:

Nicole:	I'm wondering though, because you've been writing them all as fractions, I'm wondering if there is another way to write it. What's another way that we could write it? Do we have any ideas? Judy?

Judy suggested 100 - 16. Nicole pointed out that 100 - 16 was too big to be the same as "sixteen out of one hundred" and then made a second attempt to elicit a decimal representation, this time using money:

Nicole	This is really tricky guys, and I know it's new, so we're going to work on a different way to write it today. I want you to think about money. If this [the flat] is a dollar and these little cubes are pennies.
Mike	Pennies.
Nicole	How many pennies would we have?
Students	Six.
Nicole	Six pennies, so this [the rod] would be what? Ten pennies make what?

[43] Base ten blocks are manipulative materials designed to represent and teach place value concepts. A typical set contains four types of blocks: a 1 cm^3 "little cube"; a "long" or "rod" composed of ten little cubes; a "flat" made up of ten longs (i.e., a ten by ten array of little cubes); and a "block" or "big cube" built from ten stacked flats. By assigning the blocks different values, a wide range of numbers can be represented. For example, if the little cube is assigned the value of 1, then a rod has a value of 10 and a flat equals 100; however, if the flat is assigned the value of 1—as in Nicole's lesson—then the rod is $^{1}/_{10}$ and the little cube $^{1}/_{100}$.

Students A dime.
Nicole A dime. So if I have one dime and six pennies, how much money do I have?
Paul A dime and a nickel and…
Vicki Sixteen cents.
Nicole Sixteen cents. Thank you, Vicki. Well, what if I had one dollar and sixteen cents, how would we write that? Show me on your slates, how we would write one dollar and sixteen cents?

Base ten blocks can be a powerful representation for place value concepts. However, here and throughout her lesson, Nicole relied on money to name and describe the numbers represented by the blocks. Thus, because students are familiar with money, they could record the value represented by the blocks without having to think about tenths and hundredths. Money <u>can</u> be used to engage students with decimals. However, Nicole did not prompt students to think about a dime as $\frac{1}{10}$ of a dollar, a penny as $\frac{1}{100}$ of a dollar, or 16 cents as $\frac{16}{100}$ of a dollar. As a result, the cognitive demand was reduced and the mathematical work shifted away from place value and decimals to practice with dollars and cents.

In other cases, time is spent on *unintended* mathematics not because the cognitive demand is reduced, but simply because the work goes in a different direction than intended. This can happen when a student solves a problem using a method that draws upon different mathematical ideas than the intended focus of the lesson. A minor instance of this was seen in the Mental Math portion of Courtney's lesson: Courtney had intended for students to practice "making a ten" but a student used a different, unanticipated strategy. This is not inherently problematic, but could become an issue when a teacher does not recognize that using a particular method engages students with different mathematics than the intended point.

Student questions can also shift the mathematical course of a lesson. This possibility arose during Mental Math in Rachel's second-grade lesson. In this activity, Rachel read a number and students wrote down its double or half. In her launch, she offered students a suggestion for recording their work:

Rachel Okay, so we're going to practice our halving and doubling skills to get ready for math today. So I'm going to give you a number and ask you to double it. So, just write it down on your paper and then we'll share as a class. So I'm going to give you the number six and write down the double. Shh, don't tell me. Write it down. If you want to write six first and then like an arrow or a line and then draw what it doubles to, that's a good idea.

When the class reached the halving portion of the activity, one of her second graders asked an unanticipated question about the "halving sign":

Chester	Um, I have a question. How would you write the halving sign? Because, because if you do six minus three, you might not know the answer. It's six minus the answer equals the answer.

Rachel's initial reaction was to tell Chester to use division, but she decided to try to steer him toward something more familiar (R-Post, T292):

Rachel	Um, you could do, it's more of like a thing that you do in your head, or you could do in your head what plus what is equal to eighteen so you could think what plus what equals eighteen and you know your doubles facts so well that you should know it's nine. Or we talked about multiplication a little bit with the arrays and you could think what times two is equal to eighteen. Does that help?

But Chester pressed for division:

Chester	Mmm, I want, I'm looking for dividing.
Rachel	Dividing, or you could do eighteen divided by two equals.
Chester	Okay.
Rachel	But we're not too familiar with divided by, so I don't expect everyone to do the divided by. But if that's something you're comfortable with you could do eighteen divided by two equals nine.

In the post-lesson interview, Rachel explained her response and why she added in the disclaimer about not being "too familiar with divided by":

> I said that because I didn't want some of the kids to be like, wait, what's divided, and like get all confused. Also I wanted them to know like, that's what you do, but don't worry about it, you don't have to know that, like. Because that was, I mean, I wanted to answer his question, so I figured I'd answer his question but let the rest of the class know they didn't have to be on the same page as him. (R-Post, T300)

One can imagine any number of ways that Rachel might have responded to Chester's question: She could have switched gears and used halving to launch a lesson on division, or she could have picked up on Chester's observation that "six minus the answer equals the answer" to introduce algebraic notation. Instead, Rachel's move of quickly acknowledging that division could be used to record halving and saying to the rest of the class that it was okay if they were not familiar with division, served to both signal to students that the connection to division was not the mathematical point of the Mental Math activity and keep her lesson on her intended course.

My intention here is not to make claims about whether Rachel responded correctly in this situation or whether she should have seized one of the mathematical opportunities presented. Teachers do intentionally change the mathematical course of their lessons—perhaps deciding to take up an unanticipated method or introduce a new concept in response to a student's question. In these situations, students are still engaged in the intended mathematics; the intended mathematics has just changed. Thus, teaching to the mathematical point does not require teachers to stick with their original plans at all costs. It does, however, require having a mathematical point and trying to steer the lesson toward it—which, in turn, necessitates an awareness of whether the mathematics students are working on furthers the intended mathematical learning goals—regardless of whether these are the original goals or ones that emerged during the lesson.

Another issue that can inadvertently cause time to be spent on unintended mathematics is the non-strategic selection of numbers in examples and exercises. This can happen when the numbers used lend themselves to different methods than those most related to the intended mathematics (Rowland, 2008; Rowland, Thwaites, & Huckstep, 2003). I observed many instances in which a teacher's chosen examples did not set up the desired mathematical discussion, for example, in Andrea's fifth-grade lesson on division.

One of the goals of Andrea's lesson was for students to learn to use rounding to estimate how many times a two-digit divisor "goes in" at each step in the long division algorithm. For example, when using long division to divide 8760 by 18, the first step is to divide 87 (hundreds) by 18. Andrea wanted students to complete this step by rounding 18 to 20, estimating that 87 divided by 20 is about 4, and therefore concluding that 87 divided by 18 must also be about 4. She planned to introduce rounding the divisor using the following warm-up problem: *A rope measuring 87.6 meters long is cut into 12 equal pieces. Estimate the length of each piece. Be prepared to explain your estimation strategy.* She selected this problem from *Everyday Mathematics* because it asked students to "estimate," which she saw as related to her goal of rounding. However, during the problem's enactment, because many students knew their multiples of 12, the problem did not lend itself to rounding the divisor to 10, as Andrea had hoped. Instead, many students solved the problem by rounding 87.6 to 84 and using the known fact 12 x 7 = 84. In the post-lesson interview, Andrea commented that if she were teaching the lesson again, she would use a divisor such as 22 so that students would be more inclined to round it to a multiple of 10. Thus, although students were engaged in productive mathematical work during this warm-up activity, it did not initially steer the lesson toward the mathematics Andrea had intended.

Another way the choice of numbers can steer a lesson toward unintended mathematics is when they are generated randomly (Rowland, 2008; Rowland et al., 2003). This was seen in Sydney's first-grade lesson on comparison. In the second half of her lesson, Sydney introduced "The Difference Game," a partner activity in which each player draws a number card and counts out the corresponding number of pennies. The players then compare their pennies and the player with more pennies keeps the difference. The player with more pennies at the end of the game wins. To demonstrate the game, Sydney randomly drew two cards—3 and 0—from the deck. The selection of 0 as one of the numbers distracted from her mathematical point:

Sydney	So boys and girls, if we each had a deck of cards, we would each flip the top card over, okay? [*Draws a card from each deck she is holding.*] So if Michelle was playing, the first card she did was a zero. And if Joel was playing, he got a three. So we got a zero and a three. And once you flip the cards over, you're going to represent those cards in pennies. So how many pennies would I use to represent zero? Raise your hand if you know. Julia?
Julia	None.
Sydney	No pennies, because it's zero. Sandra, how many pennies would I use to represent this card three?
Sandra	Three.
Sydney	Three pennies. So boys and girls, that's what I want you to do. Now you might think this is a little funny because it's only zero for this side, but what I think is a good idea, boys and girls, is you divide, pretend you have an imaginary line on your desk, and you divide it in half, okay? So you'd put the zero pennies on one side and the three pennies on the other side and that's what I'm going to do up at the board, okay?

Sydney arranged three magnetic pennies horizontally on the whiteboard and then walked around helping students set up their pennies on their desk. When everyone was ready, she explained the rest of the game:

Sydney	So everyone should have three pennies, that's really all I should see, is three pennies and then a corner of your pennies, okay? So remember, I drew a zero and I drew a three. So we need to match up, um, well first we need to decide who had more pennies, the person who drew a zero or the person who drew a three? Erica, who drew more pennies? Which player, the one with the zero card or the one with the three card?
Erica	[inaudible]
Sydney	The person with three had a higher card, right? Does everyone agree the person with three had a higher card? Sandra, you agree? Okay if, so remember we had, [*points to the pennies on the board*] this is the person who had three and this would be the person that had zero. Is there anything to match up? Are there any pennies up here for us to match up?
Students	No.
Sydney	No. So what we would do is the person, if we were playing this in pairs, the person who drew the three, they would get to keep the difference. The difference in this problem is three. Marina, can you sit down please? Okay. So that, this, the person who had the three card would be the winner and they would get to keep all three pennies, because the difference in that problem would be three, okay? That was kind of a funny example because it was a zero. So let's try it again.

The random generation of 0 was problematic for the initial example because it did not support students' learning how to set up and do the matching. This is not to say that 0 would not have been an interesting example to discuss—for a different purpose—later in the lesson. However, the fact that 3 was both one of the original numbers and the difference between them was potentially confusing for students just learning what "difference" means. Obscuring the intended mathematics through numbers playing dual roles may have been an issue in one of

Sydney's other examples as well (i.e., 6 and 3 pennies have a difference of 3). In general, Sydney did not pick her examples or activities by strategically choosing among alternatives in order to orient students toward the intended mathematics. Her haphazard selection process instead seemed based on random chance or what she happened to notice in the teacher's guide. For example, when asked how she decided to have students play a game instead of complete the worksheet that was also included in the textbook's lesson, she replied:

> Honestly, this is the first time I'm really looking at this worksheet, to tell you the truth.…I picked the game because it was the biggest thing on the page, and I usually follow like, you know, it's kind of laid out one, two, three, or whatever. This is what I was drawn to first. (S-Pre, T93-95)

Another issue that can shift an activity toward unintended mathematical work is when it gets bogged down in a complicated mathematical idea or procedure that is not its focus. The strategic selection of numbers in problems and examples can also help manage this issue. For example, Courtney used small numbers in her comparison problems so that students would not have difficulty with the computation and thus could focus on the diagram and number models. She did not, for example, use numbers that required regrouping, which would have likely been challenging for her second-grade students.

An example of an activity getting mired in unintended mathematical work was seen in Gillian's fourth-grade lesson on metric measurement. The class discussed the names and relationships among metric units of length, and students solved a few basic conversion problems, such as: *How many centimeters are there in 3 meters?* Gillian ran out of time in her lesson and decided to skip an activity that had students measuring objects around the room. She instead asked students to complete a problem in their Math Journal, which asked them to measure two line segments (with lengths of 9 cm and 12 cm) to the nearest centimeter. As students began working, she spontaneously added an extra component to their task:

> Gillian If you're done, what we're going to do is convert your measurement in centimeters, convert it to decimeters, to meters, and to millimeters. Convert your measurement, convert answers. You have nine centimeters; so convert it to how many decimeters, how many meters, how many millimeters.

Students immediately expressed confusion about how to convert nine centimeters to decimeters. A number of students said the answer was zero. Gillian tried to explain, but had trouble:

> Gillian So, could everyone pause for a moment and have a seat. What are we doing here? Okay, let's try it one more time. Could everyone please have a seat? I'm going to explain what I mean, okay? So for the first one, let's say the answer is nine centimeters. I'm thinking, in fact, let's just do the first one if you haven't done it yet, okay? So I'm thinking, how many decimeters are in

> nine centimeters? It's just like our slate problems. And you might think zero, but zero would mean nothing and nine centimeters is not equal to nothing. Actually, I have centimeters and I know that one decimeter equals ten centimeters. So if I'm thinking okay, problem, I have less then ten, I'm thinking okay, how many do I have? In one whole decimeter is ten centimeters, that means ten out of ten centimeters equals one decimeter. But if I don't have ten out of ten centimeters, then I have nine centimeters so I have nine out of ten centimeters, er, decimeters, because, okay. Ten out of ten is a whole, so ten out of ten decimeters equals one decimeter. This is a little bit confusing isn't it?

She then asked a student to help out. The student came up to the board and explained: "Since we only have nine centimeters and we need ten, ten centimeters to be one decimeter, then we have to write it in millimeters instead of decimeters." Another student volunteered after that, explaining that the answer was .9 because it is nine out of ten. At this point it was time for the afternoon assembly. In closing, Gillian attempted to explain the conversion one more time and then told students not to worry because they would talk about it tomorrow.

Gillian's decision to ask students to write their centimeter measures in decimeters, meters, and millimeters was certainly within the mathematical terrain of the lesson and is a task that could provide a rich context for mathematical work. However, this was not a task that could easily be completed by her students in the last two minutes of class. Gillian did not seem to recognize the significance for learners of switching the direction of the conversion and, because this was not the intended mathematical focus of her lesson, was not prepared to explain the calculation and thus got bogged down in confusion related to unintended mathematical work.

Table 6 summarizes the discussion of the problem of spending instructional time on the intended mathematics.

Table 6.
Summary of the Problem of Spending Instructional Time on the Intended Mathematics

Strategies for steering the lesson toward the intended (rather than unintended) mathematics:

- Strategically selecting problems for discussion
- Strategically selecting numbers for examples and exercises
- Focusing students on the most relevant problems
- Cutting the least relevant work if running short on time
- Raising key mathematical ideas that do not get brought up by students

Issues that can arise:

- Reducing the cognitive demand through leading questions or overly supportive representations
- Using methods that draw upon different mathematical ideas than the planned focus of the lesson
- Student question taking the lesson in an unintended direction
- Non-strategic selection of numbers in examples or exercises:
    - Using numbers that better lend themselves to different methods
    - Using numbers that are generated randomly
    - Obscuring meaning through numbers playing dual roles
- Getting bogged down in complicated mathematical ideas that are not the intended focus

Making Sure *Students* are Doing the Mathematical Work

If instructional time is not spent on the intended mathematics, then students will certainly not be engaged with it. However, even if instructional time is spent on the intended mathematics, the *students* still might not be doing the majority of the work. This fourth problem—making sure students are doing the mathematical work—foregrounds the important idea that mathematics instruction is about student learning (Hiebert & Grouws, 2007): Teachers engage students in mathematical work so that *students* can learn the intended mathematics.

The problem of making sure students are doing the mathematical work overlaps with managing the problem of spending time on the intended mathematics. For example, when the mathematical knowledge needed to answer a teacher's question is changed by a teacher's asking easier and easier questions, the students are not doing the intended mathematical work (Brousseau, 1997). In the data, I indentified three strategies for managing the problem of making sure students are doing the mathematical work: asking questions that engage students in mathematical reasoning, getting students into the work without doing it for them, and distributing the mathematical talk. Although these strategies overlap, I discuss them separately below. I also describe some of the issues that arose for beginners that resulted in either the teacher or the details of the task "doing the work" for the students.

Asking Questions that Engage Students in Mathematical Reasoning

One issue that can arise is that the presentation of a task can do some of the mathematical work for the students—for example, when it is set up in such a way that students can get a correct answer by means other than engaging with the intended mathematics. Asking questions that engage students in mathematical reasoning can be a useful strategy for addressing this issue. An example of this strategy can be seen in Keri's introduction of a Math Journal page during her fourth-grade lesson on fractions. Keri introduced the worksheet with a whole-group discussion of its first problem (Figure 10).

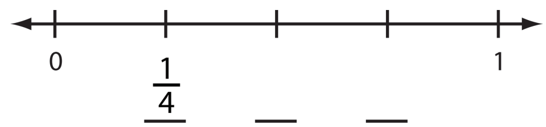


Figure 10. Number line problem from Keri's worksheet.
Adapted from *Grade 4 Everyday Mathematics Teacher's Lesson Guide*
(Bell et al., 2007c, p. 573).

In this problem, students are to identify a fraction that corresponds to each marked location on the number line. Because the $\frac{1}{4}$ was already filled in as an example, students could generate correct fractions for the other blanks simply by counting by fourths, rather than by thinking about the number of equal parts into which the unit interval is divided.[44] This is exactly what happened when Keri asked Eliza what she thought the problem was asking:

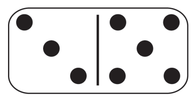
Eliza	Figure out the next one, it's one-fourth, count the next spaces to know…
Keri	Okay.
Eliza	And see how many fourths there are.
Keri	So how did you know it was divided up into fourths?
Eliza	Because the first problem is done for you.
Keri	Okay, so they gave you one-fourth here. What if they didn't give you this number at all and they just gave you these blank spaces, how might you know it's in fourths still? Victor, how would you know it's in fourths even if the one-fourth, let's say they didn't give you the one-fourth, how might you know it still?

In this episode, when Eliza responded that she knew the number line was divided into fourths because "the first problem is done for you," Keri asked a strategic "what if" question to engage students in reasoning about the fraction concepts that were the mathematical point of her

[44] Not all of the number line problems had fractions filled in, so I am not claiming that the entire worksheet did the mathematical work for the students. I only use this example to illustrate this issue and Keri's use of questioning to address it.

lesson. Asking questions that require students to reason can help make sure that it is the students—not the task—doing the mathematical work.

As discussed above, another issue that can arise is that the teacher's questions give away the answer, making it so students do not have to engage with the intended mathematics to answer correctly. Examples of leading questions doing the work for students could be seen throughout Irene's first-grade lesson. The activities in her lesson used dominoes to generate addition-subtraction fact families. For example, the 3|5 domino corresponds to the 3-5-8 fact family shown Figure 11:



$$3 + 5 = 8$$
$$5 + 3 = 8$$
$$8 - 3 = 5$$
$$8 - 5 = 3$$

Figure 11. The addition and subtraction fact family corresponding to the 3|5 domino. Adapted from *Grade 1 Everyday Mathematics Teacher's Lesson Guide* (Bell et al., 2007a).

In her pre-lesson interview, Irene expressed multiple mathematical goals for her lesson. She had overarching goals of developing students' understanding, as well as a conceptual learning goal for students (i.e., to understand the relationship between addition and subtraction), which she distinguished from the procedural goal of being able to compute answers to addition and subtraction facts. Irene seemed to have done a thoughtful analysis of the mathematics made available by the domino and had decided that the representation worked well for her purposes. During the lesson, however, instead of engaging students in thinking about the relationships between the numbers and operations, she asked questions that simply required naming the number of dots she was pointing to. This could be seen in the first example she posed to her class:

Irene	So I'm going to draw a domino up on the board, okay? Let's see. [*Draws a 3|5 domino on the whiteboard*.] So, let's see here. How many dots are on this side?
Students	Three.
Irene	Three, okay. [*Writes 3 +*] And then how many am I adding to this?
Students	Five.
Irene	Five. So how many dots do I have on my domino here?
Students	Eight.
Irene	Ryan, how many do I have?
Ryan	Eight.
Irene	Eight. [*Writes 3 + 5 = 8*]

Irene's next question is a little less leading, at least requiring students do the mathematical work of generating the "turn around fact":

Irene	Now, do you remember doing turn around facts?
Students	Yeah.
Irene	So, how else can we write this? Tina, how else can we write this?
Tina	Five plus three equals eight.
Irene	Five plus three equals eight. Good.

But the remaining facts were elicited in the same manner as the first, with Irene doing all of the mathematical work for the students:

Irene	Now, I want to show you something a little different today. So I need a sheet of paper. [*Covers the three dots so only the five are showing.*] So how many do I have here all together?
Students	Eight.
Irene	Eight. Now let's see, if I cover three, if I take away three, how many do I have here?
Students	Five.
Irene	Okay, so how many did I start with?
Students	Eight.
Irene	How many do I have all here. How many do I start with here? Eight.
Students	Eight.
Irene	Minus, oh, and how many did I take away?
Students	Three.
Irene	Three.
Students	Three.
Irene	So equals, what do I have left?
Students	Five.
Irene	Five. Okay, that's one way we can do it now. Okay, now let's do it this over. So how many do I start up with up here?
Students	Three.
Irene	No, how many do I have all together as a whole?
Students	Eight.
Irene	Now, instead of taking away the three, instead of taking it away from here.
Students	Minus five equals five.
Irene	Minus five?
Students	Equals three.
Irene	So I took away five, I'm covering up five. So how many do I have left here?
Students	Three.
Irene	Three.

In this episode, even though students correctly answered Irene's questions and the 3-5-8 fact family was recorded on the board, students were not necessarily engaging with the intended mathematics. Because Irene's questions could be answered by counting dots, students did not need to do the mathematical work of thinking about the relationships between the numbers and operations, which was the mathematical point of the activity. Of course, it is possible that students were noticing these relationships themselves; however, the leading questions Irene asked did not help steer attention toward this mathematical point.

What counts as "doing the work for the students" depends on the mathematical point. Because Irene's goal was for students to be able to generate fact families and to understand the relationship between addition and subtraction, her leading questions did the intended mathematical work for the students. However, what can be seen as doing the work for the students in one situation, can be seen as students' engaging in the intended mathematics in another. For example, when Andrea posed her warm-up problem to her fifth-grade students, she got students into the work as follows:

Andrea	The first problem I want you to think about is this, if I have a rope that is eighty-seven and six-tenths long, meters long, so it's a really long rope, and I want to cut it into twelve pieces, approximately how much, how long is each piece? So estimate how long each piece is. Do it on your paper. The rope is eighty-seven and six-tenths meters long. I want to estimate how long each of the twelve pieces is. So don't solve eighty-seven and six-tenths divided by twelve, I want you to estimate. Estimate how long each one is.
Student	Do we write our answer right here?
Andrea	Mmhm. Be able to explain how you thought about it in a minute. Approximately how long is each piece? [*Students begin working as Andrea continues to talk to the whole class.*]
Student	Oh, wait, oh.
Andrea	So I have a rope that's really long and I'm going to cut it into twelve pieces, approximately how long is each piece? I'll give you a hint. You should round the numbers. Use easy numbers. Jacob, you need, did you figure it out?
Jacob	I don't get it. Are we adding or subtracting?
Andrea	We're dividing. We're dividing.
Jacob	Oh, that explains a lot.
Andrea	All right, you should be writing not talking.

Andrea "gives away" that the rope problem can be solved using division, both in her launch ("so don't solve eighty-seven and six-tenths divided by twelve, I want you to estimate") and again in her response to the student's question ("we're dividing"). However, Andrea's mathematical point in using this problem was to introduce rounding the divisor and estimating as a strategy for long division with two-digit divisors, not, for example, to discuss when division is an appropriate operation. As discussed earlier, the choice of numbers was not ideal for her purposes, nor is it clear why she used a contextualized problem (i.e., she did not use the context to help teach to her mathematical point). And certainly, this episode can be seen as a missed opportunity to simultaneously work on multiple mathematical learning goals (e.g., developing students' understanding of the meaning of division). But given Andrea's purposes, telling students to use division did not do the intended mathematical work for them.

Getting Students into the Work without Doing it for Them

A common teaching strategy to get students into the work is to do some examples together as a class. I observed many variations of "doing some examples together first." Jordan conducted her review with examples other than those on the worksheet and did not discuss additional problems from the worksheet in whole group before students worked independently, thus leaving the entire worksheet for students to complete on their own. Courtney did some initial examples as well as the first two problems from the Math Journal in whole group. However, the two discussions were structured differently: On the rug, the problems were completed entirely as a group; back at their desks, students worked on one step of the problem independently, Courtney went over that step in whole group, and then students continued to the next step independently. Thus, students completed the steps of the problems on their own, but had fewer problems available to complete entirely independently.

Doing examples and problems with students is an important and productive strategy for getting students into mathematical work. The common practice of "doing some of the problems together" to introduce a worksheet meant for independent practice has interesting implications for teaching to the mathematical point. As seen in the following example from Zach's fifth-grade lesson on fractions, doing some problems together can be a way to help engage students in doing the mathematical work and focus their attention on the intended mathematics.

The Math Journal page in Zach's lesson focused on modeling addition and subtraction of fractions. Shaded fraction bars were given, along with one of the fractions and the operation. Students were to use the fraction bars to determine the other fraction that was being added or subtracted and then calculate the answer. The first problem on the worksheet is shown in Figure 12. In this problem, students would first determine that the $\frac{5}{12}$ corresponds to the top fraction bar. They would write $\frac{4}{12}$ (or $\frac{1}{3}$) in the second blank (because it corresponds to the bottom fraction bar) and then add to get an answer of $\frac{9}{12}$ (or $\frac{3}{4}$).

$$\frac{5}{12} + \_\_\_\_ = \_\_\_\_$$

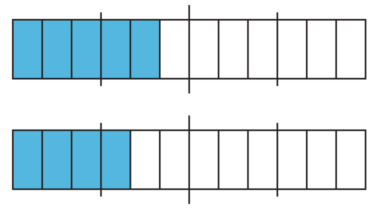


Figure 12. The first problem on Zach's fraction worksheet.
Adapted from *Grade 5 Everyday Mathematics Teacher's Lesson Guide*
(Bell et al., 2007c, p. 425).

Zach got his students into the Math Journal page by first asking a student to read the directions:

Zach	Can you read the directions to us, Matt?
Matt	Write the missing number, fraction for each pair of fraction sticks. Then write the sum and difference of the fractions.
Zach	Okay, sum and difference. That means we're going to be doing adding and subtracting, so make sure you pay attention to your operation.

Zach repeated "sum and difference" and then clarified that this is "adding and subtracting." Making sure students understand the language being used is an important part of getting students into the work. Zach then led the class through the first problem:

Zach	So, for the first one, it gives us five-twelfths. Matt, which one of these fraction sticks gives us five-twelfths and why?
Matt	Five-twelfths. Um, the first.
Zach	How can you tell?
Matt	Because, um, it has five blue, like sticks.
Zach	So five of these sticks are colored in?
Matt	Yeah.
Zach	And how many sticks, or how many boxes are there total inside this box?
Matt	Twelve.
Zach	Twelve. All right. Now Matt, this is, might be a little bit of a tough question, what is this box? What does this represent?
Matt	The whole.
Zach	The whole. Very good. So this whole box equals one.
Matt	Mm-hmm.
Zach	But only five-twelfths of that box is filled.
Matt	Yeah.
Zach	So that's our first number, five-twelfths. What about our second number here? What do we have?
Matt	Um, four-twelfths.
Zach	Four. [*Counts the shaded rectangles.*] One, two, three, four. So we have four-twelfths here. Great. Do guys see how that works? Four-twelfths. Now, Max, what's five-twelfths plus four-twelfths?
Matt	Um, do you times or plus?

Zach	What's our operation here?
Matt	Plus.
Zach	Plus.
Matt	Um, so five plus four is nine-twelfths, so it's just, you just keep the twelve. Nine-twelfths.
Zach	Okay, why do you keep the twelve?
Matt	Because the denominator is the same so it's already a whole.
Zach	Okay, so you're saying the twelve just tells us how many pieces go in a whole.

In this episode, Zach tried to engage students (or at least one student) in doing the mathematical work, while explaining the directions for the Math Journal page. He asked questions aimed at unpacking the correspondences between the written fractions and the fractions bars. For example, he did not just accept the student's answer that the $\frac{5}{12}$ is represented by the first fraction stick; he pressed for an explanation and used this opportunity to emphasize the whole, a key fraction concept. After discussing this first problem, students completed the remainder of the worksheet on their own.

The mathematical point of doing some of the problems together to introduce a worksheet or activity is not always clear when the class has already completed other similar examples. For instance, in Hannah's first-grade lesson, she engaged her class in two example problems at the board, and then when introducing a Math Journal page composed of similar problems, completed most of those problems in whole group as well. In some cases, the introduction can leave no mathematical work for students to do on their own. This issue was seen in Beth's second-grade lesson, which, as described above, involved categorizing students' favorite foods and then graphing the results. When Beth explained how to make a tally table from the data represented in the picture graph, she filled in a table at the overhead, having kids "draw tallies in the air" as she recorded. Once Beth had recorded the tallies on the projected table, she asked students to complete an identical table in their Math Journal, which then became an exercise in copying off the overhead.

Distributing the Mathematical Talk

The distribution of mathematical talk in a lesson also has implications for who is doing the mathematical work. This does not mean that students should always talk more and teachers talk less. Rather, when teaching to the mathematical point, the distribution of talk should be informed by the intended mathematics. For example, in Jordan's lesson, when the student gave an example for 5 + (-7) that actually corresponded to 5 - 7, Jordan decided to clarify the distinction herself because she thought it was important that students heard a clear example for modeling the addition of a negative number.

However, because learning to reason mathematically is a component of mathematical proficiency, making mathematical arguments, giving mathematical explanations, making connections, and using mathematical language is part of the *content* of mathematics instruction. Therefore, if teachers are the only ones doing the explaining or making connections, as was often the case in the data, then *students* are not the ones engaging in that mathematical work. For example, Courtney explained and represented mathematical ideas and made connections across multiple representations, all moves that, I argue, helped engage students with the main content of the lesson. However, it did not engage students in practicing giving explanations. Greater sharing of the mathematical talk, and the kinds of mathematical talk, in her lesson could have been a way to attend to and manage these multiple mathematical goals.

Table 7 summarizes the above the discussion of the problem of making sure students are doing the mathematical work.

Table 7.
Summary of the Problem of Making Sure <u>Students</u> are Doing the Mathematical Work

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Important: What counts as "doing the work for students" depends on the mathematical point.

Strategies for making sure students are doing the mathematical work:

- Asking questions that engage students in mathematical reasoning
- Getting students into the work without doing it for them
- Distributing the mathematical talk and the kinds of mathematical talk

Issues that can arise:

- Doing the mathematical work for the students through leading questions, a task structure that gives away the answer, or overly supportive representations
- "Doing some together first" reducing the amount of work students are left to do on their own
- Not having a clear point in doing more examples together
- Doing all of the mathematical work in the activity's introduction so that all students are left with is copying the answer
- Students not engaging in mathematical practices

Developing and Maintaining a Mathematical Storyline

A fourth problem in steering instruction toward the mathematical point is developing and maintaining a mathematical storyline. Mathematics lessons need to be coherent (Fernandez, Yoshida, & Stigler, 1992; Leinhardt, 1989). One way to develop coherence is to design and maintain a "mathematical storyline"—a deliberate progression of the mathematical ideas. This involves making connections across mathematical work, both within a lesson (e.g., between activities and between parts of an activity) and across lessons. In the data, there were examples of instruction where a storyline was developed and conveyed to students, yet the mathematics involved was superficial. Thus, implicit in this problem is that mathematical storyline being developed is worth maintaining.

Developing a Within-Lesson Mathematical Storyline

One way to think about a within-lesson storyline is that it creates a coherent structure with a clear beginning, middle, and end (Mathematics Methods Planning Group, 2005). Both Jordan and Courtney had determined coherent mathematical storylines for their lessons, which helped them steer their lessons toward their mathematical points. For instance, because Jordan knew where her lesson was headed mathematically, she was able to review ideas that students would need in their later work. She also concluded her lesson with an exit slip that was designed to "pull everything back together" and allowed students to "summarize what they've learned."

One issue that can arise when developing a mathematical storyline is that it can be difficult to identify mathematical connections across the activities in a lesson. Furthermore, even if mathematical connections are identified, the activities then need to be sequenced and implemented in ways that highlight the coherence of the mathematics. Both issues were seen in Larkin's fifth-grade lesson on symmetry.

Larkin's lesson engaged students in a number of different activities. First, the class discussed the previous day's homework, which involved measuring angles. Larkin then described the homework students would complete that night on symmetry—the topic the lesson was intended to introduce. Next, each student folded and cut out a symmetrical kite. Larkin elicited their observations about the kite and their initial ideas about symmetry. There were a number of pre-made points on one side of the kite, which students reflected over the kite's line of symmetry using a mirror. After marking the points, students measured the distance between the line of symmetry and selected points to conclude that corresponding points were equidistant from the line of symmetry. The students then completed and discussed a Math Journal page in which they drew the missing half of symmetrical shapes. At the end of the lesson, Larkin revisited the homework and introduced the Math Boxes.[45] If students finished their Math Boxes, they could design their own symmetrical kite.

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[45] "Math Boxes" is the name of the mixed-review worksheet included in most *Everyday Mathematics* lessons.

Not only was the order of the activities confusing (e.g., introducing the new homework before the lesson and then going back to it at the end), Larkin did not make many *mathematical* connections across the activities. We discussed the sequence of activities and their connections in the post-lesson interview:

Larkin	I don't think they [the activities] were necessarily sequential. I just sort of think that they are different activities that are around the same concept, but I don't think it was like you had to do the first one to understand the second one, or that you even really had to do it in that order. You could have started with the shape and measured some points from a line. And I don't know, I kind of was surprised that like the first thing was making that kite. I didn't really like that.
Interviewer	How come?
Larkin	I think when you have them do something, like they didn't really know why there were doing it at first, and I felt like, okay, we're going to talk about this, but for now, just fold it in half and cut along here and then trace these points, like I kind of just wanted to do a lesson about symmetry first.
Interviewer	Did you consider like saying anything about symmetry in front, like before doing that?
Larkin	Well, we said a little, and I just, I mean I knew that they [the students] would come along with it, and it would come together, so I just didn't want to totally redo the lesson. I mean, it wasn't, I didn't think it was bad, I just didn't get why they [*Everyday Mathematics*] started with that. So, I mean, we did talk about it a little bit first.
Interviewer	So you feel like it gets repetitive if you would have said more stuff first and then done it…
Larkin	Yeah. I sort of felt like I had already given away some of the stuff they were going to do later, but they kind of needed it to make any sense out of the kite thing, so I just felt that was kind of backwards. (L-Post, T189-195)

Certainly, all of the activities in Larkin's lesson were related to symmetry. However, a coherent mathematical storyline that helps steer students toward the mathematical point requires more than topical connections. For example, when students were discussing the missing-parts-of-symmetrical-shapes worksheet, Larkin could have asked students to explain why their answers were correct. This would have created an opportunity for students to use the properties of symmetry (in particular, the equal-distance property they had investigated with the kite) to justify the placement of their reflected points. In fact, the missing-parts problems were on grid paper, so distance could have been easily measured by counting dots or spaces from the line of symmetry.

Teaching to the mathematical point does not require everything in a lesson to be connected. For example, as seen in Larkin's lesson, as well as in both Courtney's and Jordan's, there are often routines, such as correcting homework or practicing Mental Math, that may not relate to the main body of the lesson. There are also cases when it is a stretch to make mathematical connections across all of the activities included in a textbook. This issue was seen in Rachel's second-grade lesson on fractions.

Rachel's lesson began with Mental Math. As mentioned above, the problems focused on doubling and halving, which Rachel did not connect to the later fraction work (although halving numbers could have been connected to the next activity, which was folding squares in half). However, after the fraction work, the teacher's guide included an additional Math Journal page (under the heading "Ongoing Learning & Practice"), which asked students to "find all possible combinations of three pairs of pants and three shirts." In this case, developing mathematical coherence across all of the textbook's activities would have been difficult, as there is not an obvious mathematical storyline connecting the "outfits" page to the earlier work on fractions. Finding all the ways to fold a square into eighths could have been connected to finding all of the possible pants-shirts combinations; however, this is not a mathematical storyline that would have helped Rachel teach to her mathematical point, which was introducing fraction concepts. Thus, teaching to the mathematical point does not mean that a mathematical storyline should be developed for the sake of having a storyline. The storyline should be developed in service of better engaging students with the intended mathematics.

In the pre-lesson interview, Rachel complained about the lesson's lack of coherence:

> And then they're going to do this [the outfits worksheet], which I still don't really understand how it goes very much with the lesson, and I think it's just going to be a disaster. It talks about how a boy has three different colored shirts and three different colored pants. How many outfits can he wear? And I just feel like them organizing it is going to be a disaster, so I'm going to give them a suggestion to start with like one color shirt and match it up with all the different types of pants, and then move on to a different colored shirt. Because they want you to set it up where you go like, all red, and then blue, yellow, orange. (R-Pre, R91)

Rachel's above comments reveal another issue that can arise from incoherent lessons: Because the outfits worksheet was not connected to the mathematical point of her lesson and Rachel did not think her students would know how to complete it, she planned to give them a hint—one that would significantly reduce the cognitive demand of the activity. I asked why she decided not to skip the Math Journal page. She explained that even though she thought it did not "fit," she felt obligated to complete it because her cooperating teacher always had students complete all of the pages in the book (R-Pre, T94-99). As a result, her point for this activity became "just to get it done."

Clearly, one option would have been to omit the outfits worksheet from the lesson. However, there are reasons (besides feeling pressure to complete all of the Math Journal pages) that a teacher might choose to keep it in the lesson despite its disconnect from the mathematical storyline—for example, to use as an assessment or to lay the foundation for future work. In such case, the mathematical storyline of the lesson might be better maintained by concluding the fractions work before introducing the unrelated activity. Rachel did not do this. Instead, she introduced both Math Journal pages at the same time, which resulted in students working on a range of mathematical topics for the reminder of the lesson without a clear conclusion to the fraction work.

Developing an Across-Lesson Mathematical Storyline

Because mathematical proficiency develops over time, the mathematical point necessarily extends beyond an individual lesson. So too does the mathematical storyline. Across-lesson mathematical storylines were evident in both Jordan's and Courtney's lessons. For example, Jordan selected problems for her review based on her observations of students' misconceptions in prior lessons. She also made connections to students' prior knowledge and experiences, for example, through her use of the class' team-point system to represent integers. Courtney saw her lesson on comparison number stories as part of her students' year-long subtraction-learning trajectory.

However, many preservice teachers in my study did not have well developed mathematical storylines across their lessons. Perhaps this is because they were novices or, as student teachers, they only felt responsible for the lesson they were teaching that day. The spiral nature of *Everyday Mathematics* was also often cited as a reason they did not look for or find connections across lessons. Because mathematical coherence and progression across lessons is central to teaching to the mathematical point, even when connections across lessons are less obvious, it is important to look for ways that the lesson and its activities fit with the others in the unit. That is, part of teaching to the mathematical point involves asking: What mathematical storyline *could* make the day-to-day instruction mathematically coherent?

Beth was very concerned about the within-lesson storyline for her food-group graphing lesson. She tried to design a coherent transition from the healthy-foods discussion to graphing and even skipped the textbook's Mental Math activity because she thought "it'd be too confusing to go from a math problem to talking about food and then back to math again" (B-Pre, T201). Beth did not, however, try to develop an across-lesson storyline. At first glance, Beth's food-group graphing lesson might seem disconnected from the other lessons in the *Everyday Mathematics* unit, which are about subtraction. When asked in the pre-lesson interview if her lesson related to any unit goals, she mentioned that the Math Journal pages on comparison problems—which she planned to skip—were related to the other lessons in the unit, but that the food-group graphing activity was not. From her interview, she did not seem to have sought ways that the graphing activity *could* have been connected. For example, one way to connect the graphing work to the rest of the unit would be to ask comparison questions about the graph (e.g., How many more students' favorite foods were in the dairy group than the bread group?). Another mathematical storyline that can almost always be developed across lessons is work on mathematical practices.

Developing an across-lesson mathematical storyline can support students' engagement with the intended mathematics by making more visible connections to and ways to build upon students' prior knowledge and experiences. One issue that can interfere with the development of an across-lesson storyline is not understanding how topics are connected—both mathematically and in the trajectory of students' learning. For example, Nicole's disconnected view of the mathematics in her lesson prevented her from building on what students already knew as a way to engage them with the intended mathematics. Nicole did not capitalize on the connection between fractions and decimals to create the storyline for her lesson. Instead, she treated fractions and decimals as separate topics and developed a storyline using money. And, as described above, this storyline resulted in students not engaging with much decimal content.

Progressing the Mathematical Storyline

A "story" implies not only coherence, but also a plot—something "happens." Similarly, a mathematical storyline *progresses* students' mathematical proficiency. This might involve learning something new, as in Courtney's lesson, or it might mean solidifying or deepening understanding of previously worked on ideas, as in Jordan's. In each case, one move used to progress the mathematics was to select problems and examples that they thought would be challenging for their students.

In my analyses, one issue that arose in relation to progressing the mathematical storyline was when activities or examples in a lesson seemed redundant. This often happened when discussing problems as a whole class before students worked independently. There were cases in the data when the point of doing a problem together was unclear, as the additional example did not seem to raise anything new. A similar issue arose when teachers "went over" problems students had already completed. As with any of a lesson's activities, to steer it toward the intended mathematics, it helps to know how the activity is designed to progress the mathematical storyline. This seems particularly important when the mathematical work involved is not new. In these cases, it can be helpful to think about shifting the mathematical point of the activity to developing mathematical reasoning and practices—for example, by raising the level of expectation for students' use of mathematical language, pressing on students for more complete explanations, or asking students to make connections across different mathematical ideas and solutions. Furthermore, developing a mathematical storyline can help manage the problem of timing and pacing. Asking "how does this activity progress the storyline?" or "what new mathematical points can be made with this activity?" can help teachers make decisions about what can be skipped or gone over more quickly. These kinds of questions can also help teachers develop strategic questions that help steer the work toward the intended mathematics.

The most pronounced version of storyline non-progression occurs when an entire lesson neither teaches students anything new nor develops more sophisticated understanding or skill. For example, Beth thought completing the tally chart and bar graph would be "pretty straightforward" for her students because it was something that they had done before. She expected a "high level" of mastery in the lesson and thought all of her students would "get it." She decided not to introduce the word "axis" because she did not know if it had been taught before and did not think the focus of the lesson was to "get into that kind of language" (B-Pre, T51). Of course, there is nothing inherently wrong with providing additional practice or review to maintain students' current level of understanding or skill. However, because the ultimate goal is developing mathematical proficiency over time, it seems that at least part of every lesson should aim to help students make some mathematical *progress*.

Conveying the Mathematical Storyline to Students

Fernandez, Yoshida, and Stigler (1992) present findings from preliminary studies showing that "coherent lessons lead to more coherent representations [for students], which in turn lead to greater learning" (p. 363). One way to help students develop more coherent representations is to make the mathematical storyline explicit. This does not mean simply announcing the main topic at the beginning of a lesson or writing "today's objective" on the board. Making the structure of a lesson visible involves deploying moves *throughout* a lesson that frame the mathematical work, summarize where the class has been, and narrate where things are headed.

One of the main issues that can arise when there is no mathematical framing or narration is the feeling that students are being "dragged" through the steps of an activity. In these instances, students may appear to be doing mathematics, but are not engaging with the intended mathematical point. Examples of this can be seen throughout Irene's first-grade lesson on using dominos to generate addition-subtraction fact families. As discussed above, although Irene had articulated conceptual learning goals for her students and seemed to have done a thoughtful analysis of the mathematics made available by the domino, during the lesson, Irene had a difficult time engaging students in doing the work. These difficulties seemed related to the fact that she did not clearly convey the mathematical storyline to students, as seen in her introduction to the lesson:

> Irene We'll be working with dominos today. Do you remember, do you remember dominos? I know we've been using a lot of dominos in our lessons. So I'm going to draw a domino up on the board, okay? Let's see. [*Draws a 3|5 domino on the whiteboard*.] So, let's see here. How many, what's the number, what's the, let's see, how many dots are on this side?

Although she did tell students they would be using dominos, there was no mention of the *mathematics* on which they would be working. Furthermore, her initial question did not provide students with a mathematical reason for counting the number of dots on one side of the domino. The teacher's guide suggested the following sequence of questions to launch the activity:

- What four number models go with this domino?
- What two number models show how to find the total number of dots?
- What two subtraction number models can you make up that use 3, 5, and 8? (Bell et al., 2007a, p. 550)

While not ideal, these questions at least hint at the mathematical point of the activity. Another alternative would have been to ask a more open-ended question (e.g., Can anyone think of a number model that shows the relationship between the dots on this domino?) and then have targeted questions prepared, like those in the teacher's guide, in case students do not understand the open-ended question or it does not elicit all four facts. Instead, Irene asked fill-in-the-blank questions that simply engaged students in naming the number of dots to which she was pointing.

After going through the 3|5 domino, Irene immediately began the next example. There was no summary, nor an explanation of why they were writing number sentences. After the second example (a 4|6 domino), Irene tried to introduce the idea of fact families by eliciting patterns from students:

> Irene Okay now. I want you to take a look at this. This, this one and this one. [*Pointing to the two sets of fact families.*] So this problem and this problem, so what, do you notice a pattern? Owen, do you notice a pattern here? Look at the numbers that we used. What pattern do you notice?

Students started naming random "patterns" they noticed across the problems (e.g., that the numbers counted by twos), a common response to an unclear request for patterns (Heaton, 2000). Throughout her lesson, Irene continued to have difficulty asking focused questions that framed the work and helped convey the mathematical point to students. Instead her questions were at the extremes: general questions that elicited tangential responses (e.g., do you notice any patterns?)

or leading questions that dragged them through the work. In either case, the mathematical storyline is not visible to students.

Table 8 summarizes the above discussion of the problem of developing and maintaining a mathematical storyline.

Table 8.
Summary of the Problem of Developing and Maintaining a Mathematical Storyline

Strategies for developing and maintaining a mathematical storyline:
- Developing a coherent within-lesson storyline (e.g., having a beginning, middle, and end) by making mathematical connections across a lesson's activities
- Developing an across-lesson mathematical storyline by looking for mathematical coherence across students' prior and future work
- Progressing the mathematical storyline by engaging with new ideas/practices or engaging with ideas/practices in new (more challenging) ways
- Conveying the mathematical storyline to students by framing, narrating, and summarizing the mathematical work

Issues that can arise:
- Difficulty identifying mathematical connections across activities in a lesson
- Not sequencing activities within the lesson in ways that promote connections or the progression of mathematical ideas
- Not summarizing or closing mathematical work before moving onto a different activity that is focused on a new topic
- Not looking for broader ideas that connect lessons across the unit
- Not understanding how mathematical ideas are connected (both mathematically and in the curricular trajectory) resulting in missed opportunities to build on students' prior knowledge
- Engaging students in mathematically redundant activities
- Not teaching any new mathematics in the lesson
- Lack of mathematical framing and narration:
    - Too general questions that elicit tangents
    - Too narrow questions that result in "dragging" students through the mathematics

Opening Up and Emphasizing Key Mathematical Ideas

Another problem in steering instruction toward the mathematical point is trying to deliberately open up and emphasize key mathematical ideas (e.g., concepts, terms and notation, explanations) during instruction. Hiebert and Grouws (2007) identify attending explicitly to concepts as a key feature of teaching that promotes conceptual development. Opening up and emphasizing key mathematical ideas, of course, requires an understanding and articulation of what the key ideas are, which is a component of mathematical purposing. Once key ideas are identified, there are many teaching moves that can help open up and emphasize them during instruction. In my analyses, I identified three main types of strategies: intentional redundancy, being explicit, and "dwelling" in strategic places. I describe these strategies and some of the issues that can arise below.

Intentional Redundancy

"Intentional redundancy" is the deliberate repetition or unnecessary use of mathematical language—e.g., vocabulary, definitions, explanations—to emphasize key ideas (Sleep, 2007). One type of redundancy is using unnecessary language to emphasize a mathematical point, for example, Jordan's referring to 43 as "positive forty-three" to emphasize the number's sign. Also seen throughout the data was the deliberate overuse of new mathematical vocabulary. For instance, Courtney repeated "quantity" and "difference" throughout her introduction of the comparison diagram. Another type of intentional redundancy is saying a mathematical term immediately followed by its definition, seen for instance in Courtney's "two quantities in every problem, or two numbers of something."

A major problem that can arise with respect to intentional redundancy is when the language that is repeated is mathematically imprecise or in some way problematic. For example, in Mia's lesson, she emphasized repeatedly that "'how many more'…always, always means subtraction." A more common issue is that it can be difficult to maintain repetitive language use throughout an entire lesson. One reason is that it is unnatural (Ball, 2007) to say the same words over and over. It also takes more time to say the "full mathematical name" and, therefore, is much easier and quicker to take shortcuts with language, an issue I return to in the next section. In addition, intentional redundancy can be difficult to manage when there are multiple key ideas a teacher wants to emphasize. This issue was seen in Rachel's second-grade fraction lesson.

Rachel had identified a number of key terms and ideas she wanted to emphasize throughout her lesson (e.g., half, fourths, eighths, equal parts, whole, numerator, and denominator), and during the lesson, Rachel's talk was filled with fraction language. For example, in the introduction to the folding squares activity, Rachel mentioned both "half" and "two equal parts" (as shown in bold):

Rachel	Okay, so will everybody please put your papers in the corner of your desk, except grab one sheet. And I'm going to ask you to fold it in **half**, so that there's **two equal parts**. And think about it, there's different ways, maybe you want to fold it one way, maybe your friend next to you wants to fold it a different way, but we're going to talk about both the ways.

As students folded their first square and Rachel circulated, she continued to repeat this language:

Rachel	Ryan, did you fold yours in **half**?
Ryan	He folded it the same way.
Rachel	Just in **half** so that there's **two equal parts**.

She then quickly reconvened in whole group and asked students to share their methods of folding, using both “half” and “two equal parts” throughout the discussion:

Rachel Okay, someone with a raised hand, will you share with the class the way you folded it in **half**? Natalie.

Natalie I folded it into a triangle.

Rachel Into a triangle. Open up the whole piece. Raise your hand without making a comment if you folded yours that way. Okay, so are those **two equal parts**?

Students Yeah.

Rachel Okay. Someone with a raised hand, did you fold it a different way, Victor?

Victor I folded it the hotdog way.

Rachel So did I. Hold it up. Is that **two equal parts**?

Students Yeah.

Rachel Raise your hand if you folded it this way. Okay, now everybody please use your pencil and draw a line down your fold. It doesn’t matter if you have the fold like a triangle like Natalie’s or like a hotdog like Victor’s. You have **two equal parts**.

Learning that halves are two equal parts was one of the main mathematical points of Rachel’s activity. Her redundant language can be seen as a way that she helped steer students’ attention toward these ideas.

Rachel’s overuse of “equal parts” at the start of the lesson can be contrasted with her use of language as the lesson progressed. Later in the lesson, Rachel rarely used “equal” when discussing “parts.” For example, in her introduction of how to label halves and fourths, although she repeatedly defined the “number on the bottom” as “the number of parts we have” and the “number on the top” as “the number of parts that we’re talking about,” and repeatedly said the fraction names, she did not even once mention that the parts need to be equal:

Rachel So when we’re talking about our one-half, the number two on the bottom is representing **the number of parts we have**, and the number one on the top is representing **the number of parts that we’re talking about**. [*Holds up a square that had been folded in half.*] So if we’re talking about just **half** of this, we’re only talking about one of these pieces, so it’s **one-half**. So everybody, um, on your half pieces of paper, write **one-half** and **one-half**… [*Students label their squares.*]…Okay, so now, if we’re dealing with the piece that you have four pieces of, what do you think we would call one of those sections? Celine?

Celine One-fourth.

Rachel **One-fourth**. And this is how we would write **one-fourth**. *[Writes ¼ on the overhead*.] Because the number four on the bottom is **how many total pieces we have**, and the number one on the top is **how many pieces we’re talking about**.…[*Holds up a square that had been folded in fourths.*] Okay, so one of these pieces is called **one-fourth**, like this. Everyone can see this? What if I covered up this one piece and we were talking about these three pieces here. What would you call that fraction? Lisa?

Lisa One-third.

Rachel	**One-third**, that's a good guess, but **how many total pieces** do we still have in this **whole**? Everyone can shout out **how many total pieces**.
Students	Four.
Rachel	And **how many pieces are we talking about**?
Students	Three/One-third.
Rachel	How many pieces, just the number?
Students	Three.
Rachel	How many are showing? Three. Okay, so remember what I said, **the number on the bottom is how many total pieces**. So **how many total pieces**? Say it louder, who said that?
Students	Four.
Rachel	Four. Okay, so there's **four total pieces**. [*Writes a four in the denominator.*] And **the number on top is the number of pieces we're talking about**. So **how many pieces are we talking about** when I cover up this, Tyler?
Tyler	Three.
Rachel	Three. So what do we write above the four?
Tyler	Three-fourths.
Rachel	Three. So when I go like this [*covers up one of the fourths of her folded square*], we're talking about **three-fourths**. Because, I'm going to review it one more time, because **the total number of pieces** is four. So the number, **the total number of pieces** always goes on the bottom. And **we're talking about three pieces**, so we put the three on the top.

Thus, when Rachel switched to emphasizing the definition of numerator and denominator, she lost track of one of the other key concepts—that the parts need to be equal. This episode shows the complexity of using intentionally redundant language to emphasize and open up key mathematical ideas when there are a number of important ideas.

Being Explicit and "Dwelling" in Strategic Places

Two often-overlapping strategies for managing the problem of emphasizing and opening up key ideas are to make mathematical ideas explicit and to "dwell" in strategic places during the lesson. Being explicit about key ideas can occur, for example, by pointing out the use of a focal skill during an explanation or by providing a definition for a new term. "Dwelling" occurs when the instruction lingers on a key idea, for example, by giving or asking for more detailed explanations, revoicing a student's comment or asking another student to revoice, or asking multiple students versions of the same question.

In my analyses, there were many instances when teachers did not dwell on or make explicit a mathematical idea they had identified as central. In Larkin's lesson, this occurred because she thought many of the mathematical ideas seemed obvious, and "dwelling" therefore felt redundant and unnecessary. In the post-lesson interview, Larkin explained why she did not ask students for explanations during her lesson on symmetry:

> There's some kids that, they're just waiting to go on and if I went to ask them to like explain or tell me something about it, they might have gotten upset…Yeah, like, I just, with time and with trying to get some people caught up, I just didn't. And sometimes I do feel like I'm being really redundant, like, okay, so where's the line of symmetry? Why is that the line, blah, blah, blah…Well, duh, it's the dotted line…I knew they were going to know, especially when there was only one. Like it's <u>the</u> one dotted line that you use to draw the other half. (L-Post, T203-209)

In cases such as Larkin's, a more detailed unpacking of the mathematical terrain, a clearer articulation of what there is to learn about the key ideas, and a better understanding of how learners conceive of these ideas—all aspects of mathematical purposing—might make more visible what there is to emphasize or open up, and therefore make dwelling on key ideas seem useful rather than redundant.

Another issue observed in my analyses was that, when a teacher tried to be explicit or dwell to emphasize a key idea, what was said or done was in some way mathematically problematic. For example, the language used may have been imprecise or unclear and therefore did not serve to steer the instruction toward the mathematical point. For example, in her lesson on fractions, Keri dwelled on the meaning of "the whole," but her language and examples were problematic:

Keri	Um, one thing I wanted to add is that when you're talking about fractions, it's really important to think about the whole. So what is the whole unit that I'm measuring? Pizza, I had one slice of pizza. That means that the whole is going to be what?
Student	Eight slices.
Keri	Eight slices or the whole piece of pizza. If I had, um, let's say I'm working with some geometric numbers, oh I'm sorry, geometric figures, and I have half of a hexagon, well what is the whole that we're measuring? Yeah?
Student	The hexagon.
Keri	The hexagon is the whole. So this concept of what is the whole, what is the whole thing we're measuring, is important.

In this episode, Keri tried to emphasize the importance of attending to the whole, a key fraction concept. She dwelled on the idea by explicitly naming it as important ("when you're talking about fractions, it's really important to think about the whole"), giving two examples (pizzas and hexagons), and then concluding by restating its importance ("this concept of what is the whole, what is the whole thing we're measuring, is important"). However, her language and examples were confusing throughout. For instance, in responding to the student's answer with "eight slices or the whole piece of pizza" she accepts that the whole in her pizza example is eight slices, even though she did not say that one slice was one-eighth of a pizza. Her effort to clarify this by saying "the whole piece of pizza" is also mathematically problematic because she seems to be using "whole" in the everyday sense rather than with its technical meaning in the context of fractions. Thus, it sounds like she is just talking about the original slice.

Another issue I observed was dwelling with language that was not geared toward the learners. This issue arose repeatedly in Gillian's fourth-grade lesson on metric conversion. She spoke of "conversion factors" and "doing things to both sides," almost as if she was talking herself through her own solution methods, not giving explanations to her students. This can be seen in the following discussion of the number of millimeters in one meter:

Gillian	And then, what do we think, how many millimeters are there in this, in one meter? And how would we figure that out? Let's think for a moment. Diane?
Diane	One hundred times ten.
Gillian	You would do, what would you do? A hundred times ten? And why would you do a hundred times ten?
Diane	Because there are a hundred centimeters in a meter.
Gillian	Right.
Diane	And there are ten millimeters in a centimeter.
Gillian	Right, exactly. So, what Diane said was, to figure out how many millimeters in this whole long meter stick, I know there's a hundred centimeters equals one meter and that one centimeter equals ten millimeters. [*Writes on board: 100 cm = 1 m and then below that 1 cm = 10 mm.]* So, to get from millimeters all the way to meter, I would have, for each one centimeter over here, I have ten millimeters. So that's why I know, so really first, we're converting from the centimeters to the millimeters, I think, okay, I need to multiply by ten to get from one centimeter to ten millimeters [*writes a x10 arrow*], so then, here, one, well, that's how many, there's ten millimeters in a centimeter, right, so the number value is multiplied by ten. So then here, a hundred centimeters in one meter, if I want to figure out how many millimeters it is, then I would also do one hundred times ten equals one thousand millimeters in a meter. [*Writes 100 x 10 = 1000.* Figure 13 shows what was displayed on the board at the end of her explanation.]

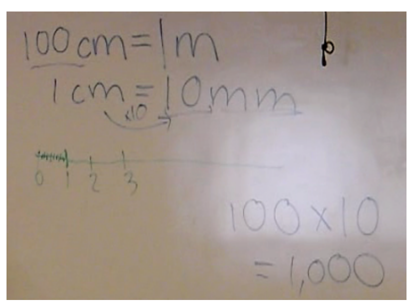

Figure 13. Gillian's explanation for the number of millimeters in a meter.

This episode shows an example of a teacher dwelling on an important idea. The student had already given a correct answer, but Gillian spent additional time on the problem, trying to give a more detailed explanation and making a record on the board. Although some parts of

Gillian's explanation seemed within reach of her fourth-grade students (e.g., "for each one centimeter over here, I have ten millimeters"), much of her reasoning did not. For example, she did not explain why the fact that you "multiply by ten to get from one centimeter to ten millimeters" means you would "also do one hundred times ten." Thus, although a teacher "thinking aloud" and making records on the board can be useful strategies for emphasizing and opening up key mathematical ideas, in order to engage students with those ideas, the language and reasoning needs to be accessible to the learners.

Being explicit about or dwelling on key ideas to steer a lesson toward its mathematical point does not occur only through teacher talk. Teachers can also ask questions or elicit explanations from students as a way to open up or emphasize important ideas. However, as seen in previous examples, asking questions that are targeted enough to steer students toward the intended mathematics, yet not so narrow as to drag them through the work, can be difficult. In the following episode, Tiffany used interactions with students to dwell on a key idea about place value:

Tiffany	So we know that each place value is ten times greater than that to the right, which is why we're allowed to keep adding those extra zeros. Remember when we did those multiplying, those large numbers where you just, let's say a hundred times three thousand, you do one times a three and that's why we're allowed to add on those zeros on the end? Okay? That's why. Because of the relationship, because we have something called a base ten number system. And that is why in our, different places have different number systems. We work with a base ten number system, and because our place values are ten times that of the number to its right…So, so, let's just recap. What is the relationship between these two numbers? [*Pointing to the 4,000,000 and 40,000,000 that were written on the projected place value table.*] Aaron?
Aaron	Um, they all…
Tiffany	Use the number in your answer. Use the numbers.
Aaron	They're both, like, in the millions column?
Tiffany	Okay, good, I hadn't thought of that. You're right. Use the numbers in your explanation in the relation of those two. Kylie, how are they related? Think about our discussion that we just had between how place values are related. Who sees where I'm going with this? Jeff?
Jeff	Um, I think they're like the same because, um.
Tiffany	Are they the same number?
Jeff	No, but.
Tiffany	No.
Jeff	Um, they're kind of the same because, like, but, like but, kind of the zero difference, but they're like the same.
Tiffany	So what does that mean?
Jeff	Like it's um, ten, um, it gets ten times less.
Tiffany	What's ten times less?
Jeff	Four million is ten times less than the forty.
Tiffany	Forty?

Jeff	Million.
Tiffany	Correct. That's what I'm looking for. This number, four million, is ten times less than forty million. And Jeff knew that because the forty million had one more place value with the zero filled in.

In this episode, Tiffany used a combination of teacher and student talk to dwell on a central idea about place value. First, she offered a recap of the key idea that each place is ten times the value of the place to its right. She continued to dwell on this idea by asking students about the relationship between two numbers that differed by one place. She began with a general question ("What is the relationship between these two numbers?"), but had trouble prompting her third-grade students to explicate the intended mathematics—in particular, that four million is ten times less than forty million. She managed this problem by using gradually more focused questions. Finally, she concluded the segment by restating the key idea.

Table 9 summarizes the above discussion of the problem of opening up and emphasizing key mathematical ideas.

Table 9.
Summary of the Problem of Opening Up and Emphasizing Key Mathematical Ideas

Strategies for opening up and emphasizing key mathematical ideas:

- Intentional redundancy
    - Purposefully saying unnecessary mathematical language
    - Deliberately overusing new vocabulary
    - Saying a mathematical term immediately followed by its definition
- Being explicit and "dwelling" in strategic places
    - Pointing out the use of a focal concept or skill
    - Providing definitions
    - Spending more time on key ideas (e.g., by revoicing, giving more detailed explanations, "thinking aloud," asking multiple students versions of the same question, making records, etc.)
    - Using a combination of teacher and student talk

Issues that can arise:

- Repeating imprecise or unclear language
- Can be difficult and feel unnatural to maintain repetitive language and use "full mathematical names"
- Losing track of some of the key ideas and omitting them from repetitive language
- Thinking ideas are obvious and therefore not explaining or spending time on them
- Giving imprecise or confusing explanations
- Using language that is not geared toward the learner
- Making confusing records of mathematical work
- Difficulty eliciting key ideas from students

Keeping a Focus on Meaning

Keeping a focus on meaning is the last problem that I discuss in this chapter. Because the ultimate goal for student learning is the development of mathematical proficiency, which intertwines conceptual understanding and reasoning with procedural fluency, mathematics instruction needs to help students attend to mathematical meaning (Hiebert & Grouws, 2007; National Research Council, 2001). Many of the moves that are useful for emphasizing key ideas and for developing and maintaining the mathematical storyline are also useful for keeping a focus on meaning. For example, being explicit about and dwelling on key mathematical ideas, and framing and narrating the mathematical work, can also help focus on meaning. In addition to the meaning-focusing moves discussed in other sections, I also observed other strategies that can help manage this problem: using meaning-focused language, explicitly connecting the activity to the intended mathematics, and deploying representations in ways that highlight intended meaning. I discuss these strategies and some of the issues that can arise below.

Using Meaning-Focused Language

Teachers can deliberately use mathematical language to help keep an activity infused with mathematical meaning. One type of meaning-focused language is saying "full names"—that is, not taking shortcuts with language when naming mathematical objects or definitions. For example, Courtney repeatedly used full names when referring to the components of the comparison diagram (e.g., "bigger quantity box"), rather than shorter phrases such as "the big box," "the bigger one," "that box," or simply pointing. In addition to keeping a focus on meaning, using full names can simultaneously help manage the problem of emphasizing and opening up these key ideas.

When the mathematical point of a lesson is to develop procedural fluency, using meaning-focused language can be a way to keep concepts in the background, thus enabling simultaneous work on multiple strands of mathematical proficiency. For example, the mathematical point of Andrea's fifth-grade lesson was to develop procedural fluency with the long division algorithm. Although her lesson was not focused on decimals, she still read 87.6 as "eighty-seven and six-tenths" rather than "eighty-seven point six," language that makes more visible the decimal's underlying place value structure and its connection to fractions. She did not, however, use place value language when describing the steps in the division algorithm. Even though developing conceptual understanding was not her intent, using meaning-focused language to describe the steps of the procedure could have been a way to maintain some connection to the underlying foundational concepts, even when it was not the main mathematical point.

In my analyses, I observed many missed opportunities to use meaning-focused language to engage students with the intended mathematics, and in some cases, not using meaning-focused language interfered with steering an activity toward its mathematical point. For example, in her third-grade lesson on decimals, Nicole rarely read fractions and decimals using the language of "tenths" and "hundredths" (e.g., she read $\frac{16}{100}$ and .16 as "sixteen out of a hundred" rather than as "sixteen-hundredths"). In fact, with one of the small groups, there was not a single instance of reading a fraction or decimal using place value language in the entire 20-minute lesson. This was particularly problematic because place value with decimals was the lesson's focal topic (contrast this with Andrea's lesson on division that happened to have decimals in it). However, as described in her post-lesson interview, Nicole intentionally used the "out of" language both to be consistent with her cooperating teacher and because she thought it supported students' understanding of the part-whole relationship:

> I was trying to mostly stay consistent with what they've been learning with my CT [cooperating teacher], with especially with like the parts of a whole. So if this is one whole, this is only sixteen parts of it. So it's sixteen out of the one hundred total. And just understanding that it's not a full one yet, and I think that gets tricky because they're used to base ten blocks being the cube is one, like one whole, and then ten, and then the flat is one hundred. And so making that shift that, okay, this flat is now one, and this is only part of it, so just really just kind of stressing that language. (N-Post, T38)

Thus, she was intentionally not using meaning-focused language to further her (impoverished) goals of having kids learn that decimals exist.

Another issue that can arise when managing the problem of keeping a focus on meaning is using language that distorts or detracts from the mathematical point. For example, in her second-grade graphing lesson, Beth repeatedly characterized the activity as "changing our information from our table to our graph." While such statements may not have interfered with student learning, it is not true that the data changed; they were just represented differently.

When students use mathematically imprecise language, the problem of keeping a focus on meaning becomes even more complicated to manage. One strategy I observed is briefly correcting the student without taking up the imprecision. For example, when a student in Jordan's class was explaining how to calculate 8 - (-4) and said "there's two negatives and that's not possible," Jordan quickly interrupted him with the comment, "Well it's possible, but it's just, we want to think about it a different way," and the student continued his explanation. In this case, Jordan did not dwell, but she did not allow the student's mathematically imprecise statement to stand untroubled. During the discussion of comparing -10 and -11, however, Jordan did not correct a student's imprecise statement that the number closer to zero was the bigger number.

Making decisions about how to handle students' use of mathematically imprecise language is part of managing the problem of keeping a focus on meaning and can be informed by the impact of the language on students' engagement with the mathematical point.

Explicitly Connecting the Activity to the Intended Mathematics

Another strategy for managing the problem of keeping a focus on meaning is to explicitly connect students' activity back to the *mathematics* they are intended to be learning from engaging in that work. For example, a teacher can make explicit the connection between student responses and the problem being solved. Courtney made this type of move throughout her discussion of the comparison diagram. For instance, when a student correctly answered "eight" for the number that went in the smaller quantity box, she connected this back to what the problem was asking: "Eight is the smaller quantity. And we're trying to find the difference." Making connections back to the problem narrates the mathematical work and thus also helps manage the problem of conveying the mathematical storyline to students. In Courtney's example, the teacher did the connecting. However, teachers can also support students in making these connection themselves, for example, with a follow-up question that asks students how their response connects back to the problem being solved.

One issue that can arise when these types of connections are not made is that the mathematical reasons for doing an activity are unclear. This could be seen in Sydney's first-grade lesson on comparison problems in which students matched pennies to find the difference between two numbers. In her introduction of the matching strategy, Sydney directed students to match pennies, but offered few explicit connections between the activity of matching Lou's four pennies with Lisa's six pennies and the intended mathematics:

Sydney	So we said that Lou saved four cents, is that right? [*Displays four magnetic pennies horizontally on the board.*] Am I remembering it correctly?
Students	Yeah.
Sydney	Jill, did I remember it correctly?
Jill	Yeah.
Sydney	Okay. And how many cents did we say that Lisa saved?
Jill	Six.
Sydney	Six, okay. So I'm going to put out six. Is everyone being quiet and looking up here, please? Great. [*Places six pennies directly below the other four.*] Okay so I'm going to match them up.
Student	Ten cents.
Sydney	It's really loud in here boys and girls. Okay, so I put Lou's pennies up here. I don't know where the best place to stand is. I put Lou's pennies up here. He had four cents. And Lisa had six cents, right? Does everyone agree with what I did? Edward, did I do it right?
Edward	Yeah.

Sydney	Okay. So I'm going to match them up. [*Counts as she slides the matched pennies off to the side*] Two, two, two, two. What am I left with? Michelle, how many pennies am I left with?
Michelle	Two.
Sydney	Two. The pennies that I didn't pair together [*points to the two unmatched pennies*] these pennies, that represents how many more Lisa had than Lou, because Lisa we said had more pennies, right?
Students	Yeah.
Sydney	Okay, does anybody know what these pennies are called? It's a word that starts with a D. I don't know if you've heard it before.
Student	Dime.
Sydney	No. Does anyone know? Shana, do you know?
Shana	Dollar.
Sydney	Nope. Okay, I was thinking that you might not know. These two pennies are called the difference. The difference. They're the result of what's left over when we matched up the pennies, okay? You guys want to try another one?

Although there was some explicit talk about what the class was trying to find out by matching pennies ("these pennies, that represents how many more Lisa had than Lou"), no explanation was given for *why* matching could be used to determine how much more one number was than another. Instead of focusing on the meaning of matching and its connection to comparison, Sydney announced that matching gave the answer and defined "difference" in terms of matching ("These two pennies are called the difference. The difference. They're the result of what's left over when we matched up the pennies, okay?"), not in terms of the amount between the original two numbers.

Deploying Representations and Making Records in Ways that Highlight Meaning

Written records and representations can also be used to manage the problem of keeping a focus on meaning. This can be seen, for example, in the strategic use of the blackboard in lesson study (Fernandez & Yoshida, 2004). Courtney's lesson provided many examples of deploying representations in ways that highlight meaning. She used a picture to explain the meaning of comparison and to demonstrate how comparison was visible in the geometry of the diagram. She arranged the marks in ways that kept a focus on the mathematical meaning, intentionally matching the arrangement of the marks to the layout of the comparison diagram to better illustrate the relationship between the two representations. And to help students attend to this, she explicitly pointed out how the marks corresponded to the diagram.

One basic issue that can arise when using records and representations to focus on meaning is that students have difficulty seeing or reading what is being displayed. This happened in Mia's third-grade lesson on graphing. Mia wanted to emphasize the importance of reading the labels on a graph, so she designed an activity in which students were to compare two bar graphs that had identical shapes but different labels. However, because she used the overhead projector, she was unable to display the graphs simultaneously, which made it difficult for students to compare the graphs and thus detracted from her mathematical point.

Another issue that can occur is when what is recorded inadvertently obscures or confuses the idea it was intended to emphasize. For example, in her third-grade lesson on decimals, Nicole tried to emphasize the names of the places by recording them on the whiteboard. She labeled the tens place with a “T” and then, because she wanted to emphasize the –th sound in “tenths,” she labeled the tenths column with a “Th,” as seen in Figure 14:

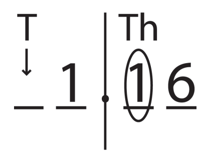

Figure 14. Nicole’s labeling of the tens and tenths places.

We discussed her labels in the post-lesson interview:

Interviewer	And how did you decide what to record, like the T or the Th?
Nicole	They’re used to T for ten, and then I wanted to stress the –th part of it, so Th.
Interviewer	And were, had that been something, you had done that in another group, I think, you did that in the group before?
Nicole	I just kind of made it up on the, made it up as I went.
Interviewer	And what do you think about that?
Nicole	I’m thinking I probably could have used something better than Th.
Interviewer	Like what would have you used that was better?
Nicole	I don’t know what I could’ve, there’s got to be something that would be better because Th they think thousands because that’s generally how we record the thousands place. So I probably could have found a better way. I hope that doesn’t confuse them later, to be like, “Well, it was Th, that’s what we call thousands.”
Interviewer	So you were trying to emphasize the –th part of it, but now you’re seeing that it maybe can…
Nicole	Now that it could potentially be problematic for some of them. (N-Post, T115-124)

In this example, Nicole’s use of recording to emphasize the –th sound may have, in fact, confused students’ learning of the names of the places. Thus, when making records to emphasize a particular idea, it is important to consider the potential impact of the recording on its conveyance of mathematical meaning.

Another issue that arises is inherent in using representations: Students might attend to mathematically irrelevant features of the representation (Nolder, 1991; Sierpinska, 1994), which can detract from students’ engagement with the intended mathematics. One way teachers can help manage the problem of keeping a focus on the intended meaning is by being explicit about what students are supposed to attend to (and not attend to) in a particular representation. For example, when Courtney was drawing little lines to model comparison, one of her students made a comment about how she did not use a slash to make a group of five tallies. Courtney acknowledged this potential confusion by clarifying, “I’m not doing tallies I’m just doing little marks.” Whether this cleared up the issue and steered the student toward the intended mathematical focus of the marks is unknown, and it could be argued that in this in case, it might have been clearer to use another type of diagram (such as small dots). But in either case, this example illustrates the importance of not assuming that students are attending to the intended mathematical meaning.

A related issue that can arise when using representations to focus on meaning is that the referent is unclear to students. Unclear referents create the potential for confusion and obfuscation of intended meaning (Back, 2000; Rowland, 1999; Sierpinska, 1994). For instance, as described above, in Rachel’s fraction lesson, when she was asking students to name fractions of a folded square, it was often unclear to which portion of the square she was referring because she decided not to spend time coloring. Similarly, in Irene’s lesson on fact families, she covered a side of the domino and then asked students what was taken away, hiding the number to which she was referring.

Table 10 summarizes the above discussion of the problem of keeping a focus on meaning.

Table 10.
Summary of the Problem of Keeping a Focus on Meaning

Strategies for keeping a focus on meaning:

- Using meaning-focused language
    - Saying "full names"
    - Using conceptual language (even when the focus is on procedures)
- Explicitly connecting the activity to the intended mathematics
    - Connecting the answer back to what the problem is asking
- Deploying representations in ways that highlight intended meaning
    - Being explicit about which features of a representation students are supposed to attend to (and not attend to)
    - Displaying representations in ways that make correspondences more visible
    - Making correspondences between representations explicit

  In addition, many strategies that are useful for emphasizing and opening up key mathematical ideas and for developing and maintaining a mathematical storyline also can help keep a focus on meaning (e.g., being explicit and "dwelling" in strategic places; framing, narrating, and summarizing the mathematical work; etc.)

Issues that can arise:

- Not capitalizing on opportunities to use meaning-focused language
- Using language that distorts or detracts from the mathematical point
- Not noticing students' use of mathematically imprecise language
- Mathematical reasons for engaging in an activity are unclear
- Deploying representations and records that students cannot see
- What is recorded distorts, obscures, or confuses the intended mathematics
- Students attending to irrelevant features of representations
- Unclear referents when deploying representations

Toward a Framework for Mathematical Purposing

This chapter used the data to explore problems in steering instruction toward the mathematical point. I identified strategies that can help manage these problems, as well as issues that can arise when trying to do so. Although these problems must be managed during interactive instruction, an activity can be positioned in ways that make it easier to steer toward its mathematical point, thus increasing the likelihood of engaging students with the intended mathematics. This is the work of mathematical purposing. Mathematical purposing is intended to help manage problems in steering instruction toward the mathematical point. Thus, the problems discussed above help lay the foundation for the framework for mathematical purposing described in the next chapter.

CHAPTER SIX:
A FRAMEWORK FOR THE WORK OF MATHEMATICAL PURPOSING

Introduction

As mentioned throughout the dissertation, two of my overarching assumptions are that the ultimate goal of mathematics instruction is to develop students' mathematical proficiency over time and that purposeful mathematics instruction aims to move with students toward specified mathematical learning goals. Through the analysis in this study, I have conceptualized "teaching to the mathematical point" as being composed of three different types of work: articulating the mathematical point; orienting the instructional activity; and steering the instruction toward the mathematical point. In Chapter 4, I provided two detailed examples of preservice teachers teaching to the mathematical point. Through my commentary and the excerpts from their interviews, I tried to highlight ways in which these teachers had articulated their mathematical points and the strategies they used to orient and steer the instruction to engage students with the intended mathematics. In Chapter 5, I identified some problems in steering instruction toward the mathematical point, various strategies that can be used to help manage these problems, and some of the issues that arise for beginners when doing this work. In this chapter, I focus on the precursor to the work of steering instruction toward the mathematical point: determining the mathematical learning goals and their connection to the instructional activity, and setting up the instructional activity so that it is more likely to move students toward those goals—what I am calling "mathematical purposing."

The aim of this chapter is to explain the components of what ended up being a rather elaborate framework for the work of mathematical purposing. The chapter begins with a description of the general structure of the framework. I then walk through each of its sections to further decompose the work. After presenting the entire framework, I step back and discuss some of its significant features, potential uses, and limitations.

As described in Chapter 3, the categories in the framework, as well as their components and subcomponents emerged from the conceptual analytic work of this study. The framework is the final version of the "codes" that emerged from my review of the literature and were elaborated, organized, and refined through my analysis of the data. Unlike the previous two chapters, which were grounded in examples from the data, my discussion of the framework in this chapter is more abstract. I do this for two reasons. First, because the framework was developed through an iterative analysis of the literature and the data, the literature review in Chapter 2 and the examples from the data in Chapters 4 and 5 foreshadow the components of the framework. Thus, the components of the framework have, in a sense, been pre-illustrated in these other chapters. Second, it is useful to have the framework represented completely and in a generalized form for use in future research and to develop tools for teacher education.

Architecture of Framework

The framework presented in this chapter aims to articulate the work of mathematical purposing a given instructional activity to teach a given focal mathematics topic. As discussed in Chapter 2, I consider mathematical purposing to be a component of the work of designing instruction and base this framework on a distributed perspective of the design of instruction. This framework is only an effort to describe *what* the work of mathematical purposing entails, not *how* (who/what does it, in what order, etc.) it is done in particular situations. Thus, in saying that the instructional activity and focal topic are both given, I am not specifying how they are given. For example, the activity and/or topic may have been created and selected by the teacher, the curriculum, the district, or some combination of sources.

By "instructional activity," I mean a mathematics task (as described in Chapter 2) and what the teacher and students do as they engage in that mathematics task during instruction. The reason I focus on an instructional activity rather than on a lesson is because, in some cases, lessons are composed of multiple instructional activities that may or may not relate to the same focal topic, or in other cases, an instructional activity might stretch across multiple lessons. What I am calling a "focal topic" is not a large domain (e.g., geometry, number, or algebra), but more of a "lesson-sized" topic.[46] For example, in Jordan's lesson (described in Chapter 4), I would consider the focal topic to be adding and subtracting integers. Courtney's lesson had two focal topics, one for each of the main instructional activities: addition strategies for the Mental Math activity, and comparison problems for the rest of the lesson. A given instructional activity could have more than one focal topic.

Although I try to provide some notion of what I mean by instructional activity and focal topic, my intent here is not to spur debates about the boundaries of an instructional activity or what to call its focal topic. I am not using these terms in an analytically strict sense. The reason I say that the activity and focal topic are given is that, in most cases, a particular activity can be

[46] The "focal topic" could be a traditional school mathematics topic or it could be part of a "conceptual field" as Lampert (2001) describes in Chapter 7 of *Teaching Problems and the Problems of Teaching*.

used to teach a variety of topics, and a particular topic can be taught through many different activities (Ben-Peretz, 1990). Mathematical purposing is not about creating instructional activities or about examining all of the possible uses for a problem or activity; it is about specifying both the intended mathematics and the details of an activity so that there is a match between them. I say that the instructional activity and focal topic are given in order to assume that some general focus for the activity has already been determined. Therefore, precise definitions of instructional activity and focal topic are not important for my framework.

Mathematical purposing has two main components: *articulating the mathematical point* and *orienting the instructional activity*. One way to characterize the difference between the two is that the former is focused on analysis and the latter on setting up enactment. Articulating the mathematical point is analytic work. It involves unpacking and analyzing both the mathematical terrain and the instructional activity to develop a nuanced understanding of the mathematics intended to be taught through the activity. Orienting the instructional activity maps out details of the intended enactment in ways that better position the activity toward the mathematics to be taught.

A key feature of my framework is that these two types of work are mutually informing and often occur simultaneously. For example, as the mathematics of the activity is analyzed, decisions might be made about which examples or representations to use. And making decisions about the details of the activity can help unpack the connection between the activity and the mathematical terrain. Furthermore, because they are interdependent, there is no specified order for doing the work. This is reflected in the basic architecture of the framework shown in Figure 15 below.

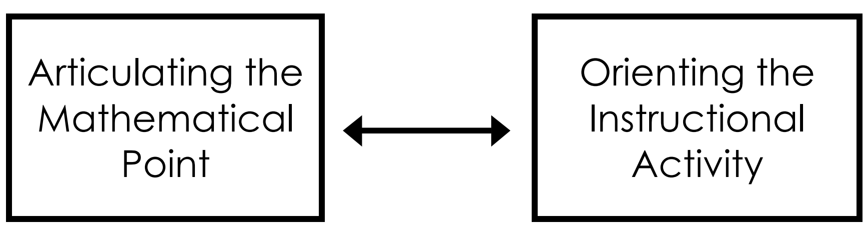

Figure 15: The basic structure of mathematical purposing.

In the next two sections, I unpack the work of articulating the mathematical point and orienting the instructional activity. My intention is to provide a more detailed decomposition of the work of mathematical purposing. However, I am not implying that teachers do the detailed design work depicted in the framework for every activity they teach. Because the design of instruction is distributed, it is likely that much of the work is stretched across the curriculum, state and district standards, unit or yearly planning, established classroom routines, etc. Furthermore, the work will be differently distributed in different contexts. After detailing the insides of the framework, I summarize with an elaborated version of Figure 15 that incorporates the main ideas of from each component of the work.[47]

Articulating the Mathematical Point

The work of *articulating the mathematical point* results in a detailed description of the mathematics students are intended to learn from engaging in the instructional activity and how the activity is intended to engage students with that mathematics. This description emerges from an analysis of the focal topic and a close examination of how the details of the instructional activity and what the particular students are bringing to it shape the opportunities to learn. The work of articulating the mathematical point of an instructional activity is not only about specifying particular topic-focused mathematical goals for student learning (i.e., proximal learning goals). Because the ultimate goal is the development of mathematical proficiency, mathematics instruction needs to simultaneously have an eye on ongoing, larger grain-sized goals for student learning. Thus, the mathematical point includes both proximal and more distal mathematical learning goals, which—as seen in the literature and discussed in Chapters 4 and 5—are nested and of different grain sizes. Furthermore, articulating the mathematical point involves more than simply listing these different types of goals. It also requires understanding the connections between the details of the activity, the specific topic goals, and the ongoing development of mathematical proficiency; prioritizing the mathematical goals for a given activity; and having a sense of where the activity sits in the overall trajectory of students' mathematics learning.

Because articulating the mathematical point means understanding the relationship between the mathematics and the activity, it requires an in-depth analysis of both the mathematical terrain and the instructional activity. My framework accomplishes this detailed analysis by applying three different analytic lenses to both the mathematical terrain and the instructional activity. The first analytic lens is the *mathematics*. Applying a "mathematics lens" to the mathematical terrain and the instructional activity results in an unpacking of the mathematical terrain around the focal topic and an analysis of which aspects of the terrain could be worked on through the instructional activity. The second analytic lens is the *learners*. Applying a "learners

[47] The elaborated version of Figure 15 mentioned here was introduced at the beginning of Chapter 4 (i.e., the mathematical purposing component of Figure 4).

lens" to the mathematical terrain results in a characterization of how students think about the focal topic. Applying a learners lens to the instructional activity analyzes the accessibility of the activity for the particular students. The third analytic lens provides the *focusing*. The "focusing lens" simultaneously zooms in and zooms out[48] on the mathematical terrain and the instructional activity to specify which of the many mathematical possibilities—both ongoing goals and particular aspects of the mathematical terrain—will be the main foci of instruction and how the activity is intended to support their development.

In summary, I conceive of the work of articulating the mathematical point as conducting a trifocal analysis—with the lenses of mathematics, learners, and focusing—on both the mathematical terrain and the instructional activity. The work of articulating the mathematical point can be represented using a 3 x 2 matrix, as shown in Figure 16.

	Mathematical terrain	**Instructional activity**
A. Mathematics lens	A.1	A.2
B. Learners lens	B.1	B.2
C. Focusing lens	C.1	C.2

Figure 16. Articulating the mathematical point:
A trifocal analysis of the mathematical terrain and the instructional activity.

Next, I describe the work involved in applying each of these analytic lenses. To facilitate this discussion, I assigned letters to the lenses (i.e., rows) and corresponding labels to the cells in Figure 16. However, like the overall architecture of the framework, there is not an order in which the analyses occur. Furthermore, the analysis in one cell informs the others, both within and across rows. The depiction of this work in a matrix, unfortunately, does not reflect this interaction.

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[48] In using the notion of both zooming out and zooming in when applying a focusing lens, I am drawing on the "photographic metaphor" used by Lampert (2001), in particular, that zooming in and out occur simultaneously. Therefore, the narrow is always embedded in the bigger context:

> To do the work of teaching, the teacher in the classroom also needs to do something akin to zooming in and zooming out, acting simultaneously in both "the big picture," across time and relationships, and in the moment-by-moment interactions with individual students.…These actions must be, at the same time, both narrowly convergent and widely panoramic, and everything in between. And, they must often converge on more than one focal point. (p. 430)

Applying a Mathematics Lens to the Mathematical Terrain and the Instructional Activity

The first analytic lens I discuss is the *mathematics lens* (Row A). This analytic lens is intended to open up and examine the mathematical learning possibilities of the instructional activity.[49] Although I call it a "mathematics lens," the analysis is not mathematical in a strictly disciplinary sense; it is a mathematics lens with an eye toward teaching. Thus, although it does not yet consider the mathematical terrain from the perspective of the learner, it does analyze the mathematics of the terrain and activity in ways that matter for instruction.

Applying a mathematics lens to the mathematical terrain unpacks the focal topic and related mathematical ideas to determine what *could* be taught about the focal topic—from specific facts to broad foundational ideas. Applying a mathematics lens to the instructional activity analyzes what aspects of the unpacked mathematical terrain *could* be worked on and in what ways through the activity. The interaction of these analyses results in an articulation of the range of mathematics that could be taught through the particular activity.

As mentioned above, just as the analyses of the terrain and the activity are unordered and interdependent, the other two analytic lenses—learners and focusing—as well as the work of orienting the instructional activity, shape the mathematics-lens analysis of the terrain and the activity. For example, once a particular mathematical idea has been zoomed in on as something that is intended to be worked on in the activity or the numbers for an example have been selected, the degree to which the terrain and activity need to be unpacked is narrowed. Thus, analyzing the mathematical terrain and the instructional activity with a mathematics lens does not imply that <u>all</u> of the mathematical possibilities are mapped out—this would be both unrealistic and undesirable for every instructional activity. The analysis is informed and directed by the other aspects of the work of mathematical purposing.

I now describe each of the mathematics-lens analyses: a mathematics lens on the mathematical terrain (cell A.1) and a mathematics lens on the instructional activity (cell A.2).

***Cell A.1: A* mathematics lens *on the* mathematical terrain.** The work of mathematical purposing located in this cell analyzes the mathematical terrain, asking: *What is there to learn about the given focal topic?* This analysis helps articulate what *could* be the mathematical learning goals for students. Although the aim of this analysis is to unpack the mathematics related to the given focal topic, it does not require unpacking the entire mathematical terrain or detailing

[49] Other research has discussed ideas related to unpacking and examining the mathematical learning possibilities of an instructional activity. For example, Li, Knudsen, and Empson (2005) discuss the "*possible curriculum*—the range of valid mathematical goals that can be addressed through the same material, and different routes teachers and students could move along to fulfill the goals."

every possible aspect of what there is to learn about a topic. As mentioned above, not only would this be unrealistic, but what is unpacked through this analysis of the terrain is informed and bounded by the analysis of the instructional activity and the other aspects of mathematical purposing.

I identified six components of the work of applying a mathematics lens to the mathematical terrain (each is in relation to the given focal topic):

- Identifying core ideas and the connections among them;
- Identifying core practices, skills, and sensibilities;
- Identifying what there is to learn and understand about mathematical language;
- Analyzing multiple representations of or multiple procedures for the focal topic;
- Describing and explaining underlying concepts, principles, representations, and procedures; and
- Distinguishing cases and directions that can be learned.

I elaborate each below and then summarize the work of applying a mathematics lens to the mathematical terrain in Table 12.

Identifying core ideas and their connections refers to both key ideas "inside" the focal topic (e.g., the main quantities and objects related to the focal topic, their attributes, and how they can be acted upon) and related "outside" ideas (e.g., the concepts, ideas, principles, and properties that underlie the focal topic; the concepts for which the focal topic is foundational; and cross-topic "big ideas"). In addition to naming these core ideas, unpacking the terrain involves understanding how they are connected and build upon each other.

Identifying core practices, skills, and sensibilities involves identifying the central skills, types of reasoning, and procedures related to the focal topic. It also involves articulating the types of problems that are typically solved or modeled, and the representations and tools that are typically used. And, finally, it involves identifying the important dispositions, sensibilities, and aesthetics related to the focal topic.

Identifying what there is to learn and understand about mathematical language focuses on unpacking the vocabulary and notation related to the focal topic. This includes identifying key mathematical terms and articulating their precise definitions; as well as identifying key symbols, their relevant systems of notation, and connections between symbols and their referents. It also involves being able to describe when and how mathematical language is used in relation to the focal topic.

Analyzing multiple representations of or multiple procedures for the focal topic involves identifying, describing, and explaining the different ways core concepts can be represented (e.g., models, problems, equations, contexts, etc.) and identifying, describing, and explaining the different procedures related to the focal topic. In addition, it involves evaluating and comparing the various representations and procedures for their breadth or expandability (e.g., usefulness across the terrain), generality (e.g., numbers for which it works), and efficiency; as well as being able to map the correspondences among them.

Describing and explaining underlying concepts, principles, representations, and procedures involves identifying what there is to explain or describe, and then being able to describe and explain it in multiple ways. Detailing a description involves articulating what it is and how it is done (e.g., the steps involved in a procedure, how to use a representation, etc.), whereas detailing an explanation provides meaning (e.g., the underlying mathematical structure, what concepts a procedure is an instantiation of, what big ideas it is connected to or relies on, what is the mathematical reason for each step, etc.). Because the detailing is for teaching, part of the work is figuring out what are the key steps and/or ideas that need to be touched on in any description or explanation of the concept or procedure. In addition, the analysis involves articulating the conditions under which the concept/procedure/representation is used/true/holds/applies; generating examples/contexts of when it is used/applied; and articulating why a particular model or procedure would be used. Finally, it involves identifying the logical implications and consequences of the concept, principle, procedure, or representation.

Distinguishing cases and directions that can be learned involves articulating and analyzing the features of a problem or ways of engaging with a concept that matter for instruction. For example, this includes distinguishing and analyzing the different "directions" that can be taken in demonstrating a concept or executing a skill (e.g., for the skill of identifying fractions some different directions are: given a picture, write the fraction that is represented; given the fraction, draw a picture; etc.); distinguishing the features in a problem that might impact the learning of a procedure (e.g., for learning long division: the number of digits in divisor; the number of digits in the dividend; whether there is a zero in the quotient; whether the divisor and/or dividend has a decimal; whether there is a remainder; etc.); or distinguishing the "cases" of numbers that can be used in a problem (e.g., when comparing integers x and y, $x \neq y$: (a) $x, y < 0$; (b) $x, y > 0$; (c) $x > 0, y < 0, |x| > |y|$; (d) $x > 0, y < 0, |x| < |y|$; (e) $x > 0, y = 0$; (f) $x < 0, y = 0$; and (g) $x = -y$).

***Cell A.2: A* mathematics lens *on the* instructional activity.** The work of mathematical purposing located in this cell applies a mathematics lens to the instructional activity to examine what about the focal topic and the related mathematical terrain could be taught using the given activity. In other words, it asks: *What mathematics (related to the focal topic) could this activity make available for study?*

Of course, not everything about a topic can be worked on in one activity, and not every activity is equally suited for teaching particular concepts, skills, or practices. The analyses in this cell help match an activity with the mathematics to be taught by examining whether and how the details of the activity (e.g., the representations, numbers, language, etc.) make different aspects of the focal topic available for study. The analyses also consider how the activity can be used to further all of the strands of mathematical proficiency, as well as any broader, ongoing mathematical learning goals. As with all other aspects of the framework, no direction is implied. That is, the mathematics of a topic does not need to be unpacked (i.e., cell A.1) before the activity is analyzed. Instead, the analyses are meant to be mutually informing. For example, examining the details of an activity to determine what can be taught from it is a way to unpack the mathematical terrain.

I identified three types of analyses involved in applying a mathematics lens to the instructional activity:

- Locating opportunities to work on aspects of the mathematical terrain;
- Examining the mathematics made available by and across the details of the activity; and
- Identifying mathematical prerequisites of the instructional activity.

The first type of analysis examines the activity to locate opportunities to work on aspects of the mathematical terrain and to develop each of the strands of mathematical proficiency. This involves identifying opportunities in the activity to engage students with different aspects of the focal topic; to elicit and connect across multiple solutions and representations; to work on overarching mathematical goals; to work on mathematical practices; and to keep "background" strands of mathematical proficiency "in the air." For example, if the focal topic is something procedural, part of analyzing the activity with a mathematics lens is considering how the activity could be simultaneously used to develop conceptual understanding, adaptive reasoning, etc.

The second type of work in this cell—examining the mathematics made available by and across the details of the activity—analyzes the details of the instructional activity. In my analyses of the literature and the data, the following details emerged as important for teaching to the mathematical point: representations, manipulatives, tools, and contexts; procedures and solution methods (including those anticipated to be generated by students); numbers and figures used in problems, examples, and exercises; explanations and examples; language (including technical vocabulary and symbolic notation, wording of task/explanations, etc.); what counts as an answer;

and the structure of the activity (including work format and location; expected duration of task and its components; materials available; how the task is presented to students; etc.). To understand how an activity could be used to engage students with the focal topic, each detail is examined to unpack the mathematics it makes available for study. This mathematical analysis considers what aspects of the focal topic can be taught through the activity (e.g., what mathematical ideas the activity "makes" students confront; what ideas it highlights; and what ideas could be pulled out during its enactment), as well as what ideas are obscured or cannot be as easily worked on in the activity. Applying a mathematics lens to the details of the instructional activity is similar to the *critical interpretation* stage of the model for pedagogical reasoning described in Wilson et al. (1987). Examples of questions that can be asked to support the analysis of the details of the instructional activity with a mathematics lens can be found in Table 11 below.

Table 11.

Example Questions to Guide Analysis of the Details of the Instructional Activity with a Mathematics Lens: What Mathematics is Made Available for Study?

Representations, manipulatives, tools, contexts

- Does/could its use support understanding of key concepts? Which ones? How?
- Does/could it help give meaning to a procedure? How?
- Does/could its use reveal/draw attention to the underlying structure/meaning/properties; or foreshadow important mathematical ideas? If so, which ones?
- Does/could its use help reinforce connections between it and what it is representing?
- What numbers numbers/problems/concepts is/can it be used with? Which cases/directions are/can be worked on?
- Are all of the quantities being operated/acted upon visible? What interpretation of the operation/action is shown? Where is the answer?
- How is the mathematics available impacted by whether it is explained during or after its construction?
- Does it distort the math in any way?
- Does its use make mathematics available for study that wouldn't be available without it?
- Does it create opportunities to address/raise likely misconceptions, errors, or other difficulties?
- What mathematical thinking is being done by it instead of students?

Procedures and/or solution methods *(including those anticipated to be generated by students)*

- What numbers and/or types of problems is/can this procedure/method be used with?
- If students used this procedure/method, would they be engaging with the focal topic? If so, with what aspects?
- Does/could its use reveal/draw attention to underlying concepts/structure/meaning/properties? If so, which ones and how? Which are obscured?
- Is it a case of or foundational for other mathematical ideas?

Numbers and/or figures used in problems, examples, and exercises

- Do the numbers/figures necessitate/encourage the mathematical idea/skill being taught?
- Where in the terrain do the numbers lead? Might the numbers/figures bring you into unwanted mathematical territory?
- Do they create opportunities to address/raise likely misconceptions, errors, or other difficulties?
- If numbers/figures are being generated randomly or by students, might something unwanted come up, or something wanted not come up?

Explanations and examples

- What is being explained or illustrated? What is not being explained and why?
- Does it include all of the key steps/concepts that need to be included?
- Does it support understanding of key concepts? Which ones? How?
- Does it reveal/draw attention to the underlying structure/meaning/properties; or foreshadow important mathematical ideas? Which ones? How?
- Is the explanation/example mathematically accurate? Does it distort the math in any way?
- Does it create opportunities to address/raise likely misconceptions, errors, or other difficulties?

Language *(including technical vocabulary and symbolic notation, wording of task/explanations, etc.)*

- Is the language mathematically precise?
- Does the language used convey meaning/connections? What meanings/connections are hidden through language?
- Does the wording "give away" what students are supposed to do?
- Is there casual or intended-to-be-helpful language that distorts or obscures the mathematics?

What counts as an answer

- How does what students are being asked to do relate to the focal topic (e.g., does it draw on skills, concepts, etc.)? What kinds of reasoning does it engage them in?
- Does what students are being asked to do engage them in mathematical practices (e.g., provide explanations, use representations, etc.)?
- If students are giving an explanation, what are the key concepts that must be mentioned?
- Is it possible to get a correct answer without engaging with the intended mathematics?

Structure of the activity

- Does the work format impact the mathematics?
- Which problems are students left to do on their own and what mathematical work does that leave them?
- How does the use of any established routines impact the mathematics being worked on?

In addition to analyzing each detail, applying a mathematics lens to the instructional activity involves looking across the details to: compare and distinguish among them (e.g., which example makes a particular mathematical idea more or less salient); examine whether and how slight variations change the mathematics made available; consider the mathematics made available by the whole collection (e.g., which directions/cases are included, coverage of terrain, sequence, variety, etc.); and analyze what mathematics can be learned from making connections across the details.

The third type of mathematics-lens analysis of the instructional activity is identifying its mathematical prerequisites. This includes indentifying any ideas, concepts, definitions, procedures, tools, or directions and cases that need to be understood or familiar in order to engage in the activity toward various mathematical goals. Some prerequisites may be independent of the mathematics the activity is being used to teach. But other prerequisites might depend on the mathematical focus of the activity and thus shape the mathematics available for study. Note that this is not yet an analysis of whether or not the particular students have the prerequisite knowledge and skills. That is an important consideration, but is part of analyzing the terrain with a learners lens.

Summary. The work of mathematical purposing described in this section (Row A of Figure 16) uses a mathematics lens to analyze the mathematical terrain and the instructional activity. As described above, these analyses are interdependent and overlapping, and their interaction results in an articulation of the range of mathematics that could be taught through the particular activity. The main components of the analyses are summarized in Table 12 below.

Table 12.
*Applying a **Mathematics Lens** to the Mathematical Terrain and the Instructional Activity*

Mathematical terrain	Instructional activity
Identifying core ideas and the connections among them • Quantities and objects, their attributes, and how they can be acted upon • Underlying concepts, ideas, principles, and properties • Concepts for which the topic is foundational • Important relationships within the topic/terrain • Cross-topic/terrain "big ideas" **Identifying core practices, skills, and sensibilities** • Central skills, mathematical practices, ways of reasoning, and procedures related to topic • Types of problems to solve or model; situations to analyze; representations & tools to use • Important dispositions, sensibilities, aesthetics **Identifying what there is to learn and understand about mathematical language** • Key terms and precise definition/s • Key symbols and relevant systems of notation; symbol rules; connections between symbols and meaningful referents • When and how language/symbols/notation system is used **Analyzing multiple representations or multiple procedures** • Different ways core concepts can be represented (models, problems, equation, contexts, etc.) • Different procedures that can be used • Describe/explain each representation or procedure • Evaluate and compare features of the representations or procedures **Describing and explaining the underlying concepts, principles, representations, and procedures** • Identifying what there is to explain or describe about a concept or procedure • Detailing the description, explanation, and/or justification • Articulating alternative descriptions and explanations • Articulating conditions under which it applies • Generating examples/contexts of when it is used/applied • Articulating why a particular model or procedure would be used • Identifying logical implications and consequences **Distinguishing cases and directions that can be learned** • Describing different "directions" that can be taken in demonstrating a concept or executing a skill • Identifying the features in a problem that might impact learning the topic • Distinguishing all of the different "cases" and how they are the same, yet different	**Locating opportunities to work on aspects of the mathematical terrain** • Identifying opportunities to work on the different aspects of the mathematical terrain around the focal topic • Identifying opportunities for eliciting and connecting across multiple solutions / representations • Identifying opportunities to work on overarching mathematical goals • Identifying opportunities to work on mathematical practices / process standards • Identifying opportunities to keep "background" strands of mathematical proficiency present **Examining the mathematics made available by and across the details** *Analyzing the details to unpack the mathematics available to be worked on through activity (e.g., what is highlighted, obscured, etc.):* • Representations, manipulatives, tools, contexts • Procedures and/or solution methods (including those anticipated to be generated by students) • Numbers and/or figures used in problems, examples, and exercises • Explanations and examples • Language (including technical vocabulary and symbolic notation, wording of task/explanations, etc.) • What counts as an answer • Structure of the activity *Looking across the details:* • Comparing and distinguishing among different details in the activity • Examining if/how slight variations change the mathematics made available • Considering the mathematics of the collection • Analyzing what mathematics can be learned from making connections across representations, solutions, problems **Identifying mathematical prerequisites of the instructional activity** • Identifying ideas, concepts, definitions, procedures, tools, or directions and cases that are needed to engage in activity, use a representation, etc. for a particular mathematical purpose

Applying a Learners Lens to the Mathematical Terrain and the Instructional Activity

In this section, I describe another component of articulating the mathematical point: applying a learners lens to both the mathematical terrain and the instructional activity (Row B of Figure 16). I use the term "learners lens" to denote a lens that takes the perspective of the learner—both of general learners of the intended mathematics and of the particular students being taught. Applying a learners lens to the mathematical terrain examines the mathematical complexity of the topic for learning it, analyzes how learners typically think about the focal topic, and characterizes the mathematics the particular students are bringing to the activity. Applying a learners lens to the instructional activity analyzes ways in which the instructional activity could be used to connect with the mathematics students are bringing to the activity and examines the details of the activity to evaluate the accessibility of the mathematics. As before, although these analyses are interdependent and overlapping, I describe them separately below.

***Cell B.1: A* learners lens *on the* mathematical terrain.** The work of mathematical purposing in this cell aims to unpack what the mathematical terrain looks like, not from the perspective of the discipline, but from the perspective of someone who is first learning the concepts, practices, and skills related to the focal topic. Whereas a mathematics lens provides a detailed road map of the mathematical terrain, pointing out what there is to learn about a topic with the connections between ideas and their "distances" informed by the discipline, a learners lens provides a "topographical map" of the mathematical terrain, "annotated by a guide for an expedition with novices":

> These maps include less territory but magnify the particular features of the landscape for those who intend to explore it. Topographical maps indicate the difficulty or ease of particular routes by the rise in elevation and potential obstacles ahead. The guide's annotations might include routes that help inexperienced hikers learn to ford a treacherous river and suggestions for worthwhile sidetrips for panoramic views or spectacular wildflowers. (Grossman, 1991, p. 211)[50]

Grossman also points out that the annotated map is only useful for a particular terrain—there is no generic topographic map. Thus, the learners lens creates a view of the mathematical terrain from the perspective of particular students learning particular content. Applying a learners lens is similar to Dewey's (1910/1997) notion of psychologizing the subject matter and to Ball's (1993a) bifocal perspective.

[50] I am appropriating Grossman's analogy for my purposes here. Grossman (1991) uses the map analogy to distinguish subject matter knowledge and pedagogical content knowledge. Although applying the lenses certainly demands subject matter knowledge and pedagogical content knowledge, the mathematics and learners lenses do not map solely onto a single knowledge domain. I discuss this in Chapter 7.

Because it is an analysis of the mathematical terrain, this cell is not about anticipating what students will do as they engage in the activity; however, anticipating student responses, of course, informs and influences the analysis. I have identified three main types of learners-focused analyses of the mathematical terrain:

- Analyzing the complexity of the mathematics for learning;
- Analyzing how learners typically think about the focal topic; and
- Characterizing the mathematics the particular students are bringing to the activity.

I describe each in more detail below and provide a summary at the end of the section (Table 14).

Analyzing the complexity of the mathematics for learning involves examining relationships to other mathematical topics that might support or interfere with learning—for example, the similarities and differences in a concept or procedure due to a change in number domain; different uses of a term or concept across mathematical contexts; what could be misinterpreted or misapplied from what students have previously learned; and whether the new content will impact or confuse students' understanding of the content they have already been learning.[51] It also involves analyzing the ways in which meaning is masked by compression, and indentifying what could be overgeneralized. Lastly, it involves examining the differences in mathematical complexity of different cases and directions—for instance, whether one direction of a representation is more mathematically complicated to explain than another; or whether a particular case of numbers complicates the use of a procedure.

A second aspect of applying a learners lens to the mathematical terrain is analyzing how learners typically think about the focal topic. This includes describing the general difficulties and typical approaches that learners have related to the topic, such as what tends to be easily understood or intuitive for learners; common solution methods; common misconceptions or common errors related to topic; what the hardest part is about learning or understanding the topic; and the relative difficulty of particular cases and directions. Another aspect of applying a learners lens to the mathematical terrain is analyzing the connections and "cognitive distance" between ideas for the learner across the terrain. By this, I mean analyzing not how ideas are related in the discipline, but how mathematical ideas are related and how "far apart" they seem to learners. For example, there are ideas that are mathematically equivalent (e.g., multiplying by ½ and dividing by 2) and therefore seen as closely connected by people who already know the content, but these same ideas might seem completely different and unrelated to learners. Thus, although the mathematical distance is close, the cognitive distance is far for those learners.

[51] Hiebert (1992) has a nice example of this type of analysis for the topic of decimal fractions.

The third type of analysis included in applying a learners lens to the mathematical terrain is characterizing the mathematics the particular students are bringing to the activity. This involves indentifying students' prior knowledge and experiences relevant to the topic. For example: What do they already know and can do? What have students already studied or experienced about the topic: this year, in prior grades, in other subjects? What has the class been working on lately with respect to the terrain, and how has it been going? In addition to indentifying the connected prior knowledge, this analysis also involves evaluating students' current level of understanding of the focal topic, including identifying aspects of the focal topic that most students seem to understand; what they are having trouble with and the fragility of their understandings; particular types of problems or cases that are hard; and common errors or misconceptions that students have been demonstrating. Finally, the analysis includes considering the mathematical prerequisites of the activity with respect to the particular students to be taught and articulating what will be new for students about the focal topic or the way it is being worked on.

Cell B.2: A* learners lens *on the* instructional activity*: Applying a learners lens to the instructional activity analyzes the opportunities to learn the intended mathematics through the instructional activity from the perspective of these students. It involves two main types of analyses:

- Locating potential opportunities to connect with and confront what students are bringing to the activity; and
- Examining the accessibility of the mathematics within and across the details of the instructional activity.

These two analyses parallel the mathematics-lens analysis of the instructional activity described above (cell A.2). The first—locating potential opportunities to connect with and confront what students are bringing to the activity—involves finding opportunities in the activity to build on students' prior knowledge and experiences and to address common difficulties or misconceptions.

The second—examining the accessibility of the mathematics within and across the details—involves looking closely at the same details discussed in the mathematics-lens analysis, but the overarching focus of the analysis is on the accessibility of the details for learners (e.g., familiarity, difficulty, etc.). Examples of questions that can be asked to support this analysis are shown in Table 13 below. The questions parallel the analysis of the details of the instructional activity using a mathematics lens discussed above (Table 11). Appendix D combines Table 11 and Table 13 to highlight the complementary analyses.

Table 13.
Example Questions to Guide Analysis of the Details of the Instructional Activity with a Learners Lens: How Accessible is the Mathematics?

Representations, manipulatives, tools, contexts

- Are all students familiar with how to use it? Is it used differently than students might have used in the past? If so, how does that impact understanding?
- Does the way it is used to explain concepts/procedures build on what students already know and can do?
- Does the accompanying notation or language facilitate its use (e.g., support understanding, remembering, etc.)?
- Do everyday uses (if any) support or interfere with its mathematical use?
- Does it make assumptions about students' experiences or background that might interfere with understanding?
- What might be difficult or tricky about using it? Is one of the directions more complex than another?
- What mathematical elements might be confusing or distracting?
- Are there non-mathematical elements that could be potentially confusing or distracting?
- How intricate is it to use/teach/get into play? Does the number of steps involved or the complexity of teaching it detract focus from learning the intended mathematics? What residue is left from its use?
- How prone to errors is it?
- What mathematical ideas could be incorrectly overgeneralized from it?

Procedures and/or solution methods *(including those anticipated to be generated by students)*

- What might be difficult about using the procedure/method? Are there numbers or cases for which it is more mathematically complex?
- How is it similar or different to what students have done before? Do these similarities/differences support or interfere with understanding/use?
- How does any accompanying notation or language facilitate its use?
- How intricate is it to use/teach/get into play? Does the number of steps involved or the complexity of teaching it detract focus from learning the intended mathematics?
- How likely would students be able to devise the procedure/method on their own?

Numbers and/or figures used in problems, examples, and exercises

- How "friendly" or familiar are the numbers to students?
- Is there anything that might be masked or left implicit by the familiarity of the numbers?
- How do the numbers/figures impact the difficulty?
- Are any of the cases more complex than another?
- How visible to students is the idea in the example?
- Does the same number serve multiple roles, and might that make things less visible, cause unnecessary confusion, or hinder explanation?
- What mathematical ideas can be incorrectly overgeneralized from numbers/figure?

Explanations and examples

- Will the explanation/example be understood by students?
- Is the way that the necessary key steps/concepts are explained accessible to these students? Does it build on what they already know and can do?
- What might be difficult or tricky about using it?
- What mathematical ideas can be incorrectly overgeneralized from the explanation/example?
- What might be confusing or distracting? Are there non-mathematical elements that could be potentially distracting?
- How intricate is it to use/teach/get into play? Does the number of steps involved or the complexity of teaching it detract focus from learning the intended mathematics?
- Does the accompanying notation or language facilitate its use?

Language *(including technical vocabulary and symbolic notation, wording of task/explanations, etc.)*

- Are students familiar with any terms and symbols?
- Does compression mask meaning? Is this likely to cause difficulty for students?
- Are there potential conflicts or confusions with the everyday use of language? Or with how language or symbols have been used in previous topics?
- Will students understand the wording of the task?

What counts as an answer • Will what students are likely to do engage them with the intended mathematics? • Will students understand what counts as "different"?
Structure of the activity • Are students familiar with the structure of the activity? • What might be confusing or distracting? Are there non-mathematical elements that could be potentially distracting?

In addition to analyzing each detail independently, a learners lens on the instructional activity also looks across the details to consider the impact on the accessibility of the mathematics—for example, examining their relative difficultly and error-proneness; distinguishing difficulty of cases; considering their progression and order; considering what mathematical ideas could be incorrectly overgeneralized because certain cases aren't included; and considering the marginal benefit/cost of students' learning another method or representation.

Summary. The work of mathematical purposing described in this section (Row B of Figure 16) uses a learners lens to analyze the mathematical terrain and the instructional activity. The main components of this analysis are summarized below in Table 14.

Table 14.
*Applying a **Learners Lens** to the Mathematical Terrain and the Instructional Activity*

Mathematical terrain	Instructional activity
Analyzing the complexity of the mathematics for learning • Examining relationships to other mathematical topics that might support or interfere with learning • Identifying the ways in which meaning is masked (e.g., by compression, language, or surface features) or ideas could be overgeneralized • Analyzing the differences in mathematical complexity of different cases and directions **Analyzing how learners typically think about the focal topic** • Describing the general difficulties that learners have related to the topic • Describing learners' typical ways of thinking about and/or approaches to the topic • Analyzing the connections and "cognitive distance" for the learner across the terrain **Characterizing the mathematics the particular students are bringing to the activity** • Indentifying students' prior knowledge and experiences relevant to the topic/activity • Evaluating their current level of understanding of the focal topic • Considering prerequisite mathematics with respect to <u>these</u> students. • Identifying what will be new for students about the focal topic/s and/or the way it is being worked on	**Locating potential opportunities to connect with and confront what students are bringing to the activity** • Identifying opportunities to connect to prior knowledge and experiences • Identifying opportunities to address common misconceptions • Identifying opportunities to bring up things that are difficult **Examining the accessibility of the mathematics within and across the details** *Analyzing the accessibility of the details:* • Representations, manipulatives, tools, contexts • Procedures and/or solution methods (including those anticipated to be generated by students) • Numbers and/or figures used in problems, examples, and exercises • Explanations and examples • Language (including technical vocabulary and symbolic notation, wording of task/explanations, etc.) • What counts as an answer • Structure of the activity *Looking across the details:* • Examining their relative difficultly • Considering their progression/order • Considering what could be overgeneralized because cases are not included • Analyzing marginal cost/benefit of introducing another method or representation

Applying a Focusing Lens to the Mathematical Terrain and the Instructional Activity

This section discusses the work of applying a focusing lens to the mathematical terrain and the instructional activity (Row C of Figure 16). I named this lens "focusing" for two reasons. First, I wanted to pick up on the following meaning of focus: "to draw to a focus; to cause to converge to or as to a focus" (OED, 2009). This meaning helps distinguish the focusing lens from the mathematics and learners lenses. Whereas the other two lenses unpack the mathematics and examine the learning opportunities available in the instructional activity, the focusing lens hones in to specify the intended mathematics (i.e., the particular mathematics it is hoped that students will learn) and the purpose of the details of the activity.

However, this is not to imply that the intended mathematics is narrowly defined. Hence my second reason for using "focusing." Another meaning of focus is "to adjust the focus of (the eye, a lens, etc.)" (OED, 2009). This definition captures the need to articulate the intended mathematics at different grain sizes. Focusing on the mathematical terrain involves simultaneously zooming in and out on the mathematical terrain to determine the mathematical learning goals at various grain sizes and to understand how they are nested and connected. Focusing on the instructional activity involves simultaneously zooming in and out on the instructional activity to determine the point of the particular details of the activity (e.g., the representations or numbers used), as well as how the activity relates to broader learning goals and can be used to develop all of the strands of mathematical proficiency.

As with the other lenses, I describe separately what is involved in applying a focusing lens to the mathematical terrain and to the instructional activity. In each case, I further separate (a) zooming out and (b) zooming in.

***Cell C.1(a):* Zooming out *on the* mathematical terrain**. Zooming out on the mathematical terrain uses a wide lens to determine the bigger mathematical learning goals toward which the activity is intended to help students make progress. There are two main parts of this analysis:

- Specifying the more distal mathematical learning goals to be deliberately worked toward through the activity; and
- Sketching the intended curricular trajectory through the mathematical terrain.

The first—specifying distal mathematical learning goals to be deliberately worked toward through the activity—involves articulating the bigger grain-sized goals that are intended to be developed in the activity (even if just in the background). Bigger grain-sized goals include ongoing foundational goals and unit goals. The work of articulating these goals might not be done while planning daily lessons (again, this framework is not about *how* the work gets done), but it is important that even at the level of a particular activity the big picture of students' mathematical development is kept in mind. This is especially important for learning goals related to the development of mathematical practices or understanding the nature of mathematics. For example, when an activity is focused on teaching a particular concept or skill, it can be easy to forget about ongoing goals like developing students' capacity to make reasoned arguments, understanding the nature of mathematical work, or developing a productive disposition toward math.

The second part of zooming out on the mathematical terrain is sketching the intended curricular trajectory through the mathematical terrain. This includes identifying the main concepts, skills, vocabulary, representations, types of problems, cases, etc. that students will learn related to the focal topic. Important here is that all of the strands of mathematical proficiency are reflected in the trajectory. In addition to naming the specific content, specifying the curricular trajectory also involves describing connections among the content, the order in which it will be developed, and what will be assessed, as well as what students already have or will be learning about the topic at different grade levels. This is not a detailed description (hence the use of "sketching"), but more like knowing the major landmarks in the terrain and the order in which students are going to work on them. Again, the curricular trajectory is not likely analyzed anew with each activity, but including it in the framework emphasizes that teaching to the mathematical point requires knowing where the intended mathematics is located in the bigger trajectory of student learning.

Cell C.1(b): **Zooming in *on the* mathematical terrain**. The other side of the focusing lens zooms in on the mathematical terrain to pinpoint the particulars of the mathematics intended to be taught through the activity. This involves two types of analyses:

- Specifying the particular aspects of the mathematical terrain students are intended to learn from engaging in the activity; and
- Describing the nature of learning expected.

The first type of analysis—specifying the particular aspects of the mathematical terrain students are intended to learn from engaging in the activity—articulates which of the ideas, skills, representations, language, and types of reasoning related to the focal topic are intended to be taught through the activity. The aim of this analysis is to not only specify which of these are intended to be developed, but also to articulate exactly what about them is going to be developed. This analysis also clarifies which things are end learning goals and which are means to engage students with other mathematical ideas (e.g., using a representation to understand a key concept). In these cases, zooming in involves specifying how the representation or explanation is intended to support learning this concept. To illustrate, Table 15 provides examples of questions that could be used to help articulate the particulars about the intended mathematics.

Table 15.
Zooming In on the Intended Mathematics

Core ideas and connections

- Which underlying foundational concepts, ideas, principles, and/or properties are students to develop an understanding of in this activity? What, in particular, about each will students learn? In relation to which numbers, quantities, objects, actions, etc.?
- How are students supposed to think about the concept?
- What kinds of reasoning (e.g., explanation, description, etc.) are students to do with these ideas? And in what direction? (e.g., Do students need to be able to generate explanations, understand them, apply them, etc.)? How do the kinds of reasoning support their understanding of the intended key ideas?
- Do students need to understand any connections among ideas (e.g., relationships between/within concepts; distinctions between/within concepts; links between the concrete and abstract, etc.)?

Procedures and skills

- With what numbers or objects are students to be able use the procedure or skill in this activity?
- What kinds of reasoning will students be expected to do (e.g., explanation, description, etc.) related to the procedure or skill? Do students need to explain *why* in addition to *how*?
- Is the focus on meaning and/or fluency/memorization? If it's on meaning, meaning of what?
- Is learning the procedure or skill an end learning goal and/or is it a means for teaching something else (e.g., Is a procedure being used to teach a concept, about the underlying mathematical structure, about the nature of mathematical reasoning, etc.?) How is the procedure/skill intended to support students' understanding of this?

Representations, solution methods, and tools

- With what numbers or objects do students need to be able to use the representation, method, or tool in this activity?
- What kinds of reasoning do students need to do with these ideas? How do the kinds of reasoning support their understanding of the intended key ideas?
- Is learning to use the representation, method, or tool an end learning goal, and/or is the main reason for its use to support the learning of some other idea or skill? How is the representation/method/tool intended to support their understanding of this?
- Will there be a focus on multiple representations or methods? If so: Does each student need to know how to use multiple methods? Are the multiple methods being used to raise some other concept, relationship, or practice? Do students need to understand how the different representations/solutions map onto each other?

Mathematical language (including symbols) and conventions

- What terms and notation are students expected to learn and use in the activity?
- Is learning to use this language an end learning goal, and/or is the main reason for its use to support the learning of some other idea or skill? How is the use of language intended to support students' understanding of the intended ideas/skills?
- Do students need to be able to use the language themselves? What are the expectations of this use (e.g., being able to define? being able to use, but not define? recognizing instances of it, etc.?)?
- What are the expectations for precision?

Engagement in mathematical practices and discourse

- What kinds of reasoning are students expected to do in the activity? (e.g., Will students describe their solutions? Do students need to be able to generate explanations? evaluate others' explanations?)
- What are the key concepts that need to be included in an explanation? What is the level of detail, precision, etc. expected?
- Is engaging in this practice/reasoning the end learning goal, and/or is the main reason for its use to support the learning of some other idea or skill?
    - If the practice/reasoning is to support learning some other idea or skill, what type of explanation best supports that understanding?
    - If the practice/reasoning is an end goal, are students just to engage *in* the practices or also learn *about* the practices?
- What do students need to learn about the nature of solutions?

Even if the particulars of the intended mathematics are specified, this does not yet reflect the level of understanding expected or prioritize these goals for learning. This is the second aspect of zooming in on the mathematical terrain: describing the nature of student learning expected. Describing the nature of student learning expected includes determining the expected level of understanding in this activity (e.g., how "far" students are to move in this activity with respect to the intended mathematics), as well as the ultimate depth of treatment. It also includes prioritizing the intended mathematics for this activity. Prioritizing includes evaluating the importance of the intended mathematics (e.g., what is significant with respect to the discipline; weighing the (relative) mathematical significance of ideas, skills, etc.; determining the degree of usefulness to future mathematics learning, etc.). Prioritizing also includes determining the "weight" of particular mathematical ideas for this activity (e.g., which concepts, representations, or methods will be emphasized; which ideas are important for all students to understand and which are fine to have mixed progress; which ideas or skills students are supposed to have "mastered" by the end of the activity; which ideas are of less concern for this activity because either they are not foundational or students will have additional opportunities to work on them; etc.). Finally, analyzing the nature of the learning expected includes identifying the "mathematical boundary" of the activity (i.e., related mathematics that could arise during the activity, that may or may not be taken up) and deciding how these elements will be handled (e.g., which ideas will not intentionally be brought up, but would be taken up if they arise; which ideas will be tabled if they arise; which ideas will be mentioned in passing, but it does not matter if all students pick up on them; which ideas will be pressed on or extended to if students are having an easy time with the mathematics originally intended; etc.).

***Cell C.2(a):* Zooming out *on the* instructional activity.** In this cell, the focusing lens turns to the instructional activity, zooming out to see the big picture and zooming in on the details. Zooming out on the instructional activity involves two types of analyses:

- Articulating how the activity is intended to further the broader mathematical learning goals; and
- Determining the mathematical storyline within and across the lesson.

The first—articulating how the activity is intended to further the broader mathematical learning goals—connects the focusing of the terrain to the activity. Zooming out on the mathematical terrain specifies what the more distal goals are; zooming out on the instructional activity indentifies how those goals are intended to be furthered through the activity, making visible the match between the activity and the broader mathematical learning goals.

The second aspect of zooming out on the instructional activity is determining the mathematical storyline within and across the lesson in which the activity is being taught. As discussed in Chapter 5, coherence matters for student learning. One way to develop coherence is to look for ways to make explicit connections within and across the activities. There are a number of different types of mathematical connections that can be made: connections within the same lesson (e.g., between activities and homework; across types of problems; across procedures and methods); connections to other lessons and topics in the unit; and connections to "real life" and/or other subjects. In addition to noting the connections, this analysis also examines the similarities and differences with respect to the treatment of the mathematics. It also involves considering how the mathematical ideas progress through the lesson and across related activities.

***Cell C.2(b):* Zooming in *on the* instructional activity**. Just as zooming out on the instructional activity makes visible the connections between the activity and the broader mathematical learning goals, zooming in on the instructional activity makes visible the connections between the details of the activity and the particular mathematics they are intended to develop. This is done through two related analyses:

- Determining the instructional intent of the activity and/or its details; and
- Specifying the main mathematical point/s of each detail of the instructional activity.

Determining the instructional intent of the activity and its details uses what is known about learners and their intended mathematical trajectory to articulate the general pedagogical purpose of the activity and its details. For example, an activity might be intended to help students review a particular idea; to explain or clarify a procedure; to introduce, explore, or expose students to a new topic; to deepen or assess students' understanding; or to provoke a common misconception. This analysis informs and is informed by the determination of the nature of the learning expected that is part of zooming in on the mathematical terrain. A related, second type of analysis is specifying the main mathematical point/s of the details, for example, by identifying what general idea an example is intended to illustrate or how the problem is supposed to further students' learning of a particular aspect of the terrain. This analysis connects the details of the activity to the zoomed-in-on mathematical terrain and is informed by the examinations of the details of the instructional activity from the mathematics and learners lenses (Table 11 and Table 13).

Summary. The work of mathematical purposing described in this section (Row C of Figure 16) uses a focusing lens to analyze the mathematical terrain and the instructional activity. This lens is different than the other two analytic lenses, which are intended to unpack and examine the instructional possibilities of the terrain and activity. Focusing on the mathematical terrain and the instructional activity identifies the mathematical priorities for the activity and connects the particulars of the activity to the particulars of the intended mathematics by specifying what mathematics it is intended to develop and how it is intended to do so. The main components of the focusing lens are summarized in Table 16.

Table 16.
Applying a ***Focusing Lens*** *to the Mathematical Terrain and the Instructional Activity*

Mathematical terrain	Instructional activity
Zooming out: **Specifying the more distal learning goals to be deliberately worked toward through the activity** • Articulating ongoing foundational mathematical learning goals to be worked on in this activity • Specifying unit learning goals **Sketching the intended curricular trajectory through the mathematical terrain** • Identifying the main mathematics content for all strands of mathematical proficiency (knowledge & skills; representations, tools, contexts; types of problems; etc.) • Describing the connections that will be emphasized across this content • Describing the intended order it will be taught in • Knowing what is going to be assessed • Having a general sense of the curricular trajectory through this terrain in earlier/later grades	***Zooming out:*** **Articulating how the activity is intended to further the broader mathematical learning goals** • Identifying how the activity is intended to contribute to developing each strand of mathematical proficiency • Identifying how the activity is intended to further students' progress on the intended curricular trajectory for the focal topic • Identifying how this activity is intended to further students' development of the ongoing foundational learning goals **Determining the mathematical storyline within and across the lesson** • Making connections to other activities (including homework) in the same lesson • Making connections to other lessons and topics in the unit • Making connections to "real life" and other subjects • Considering the storyline's progression
Zooming in: **Specifying the particular aspects of the mathematical terrain students are intend to learn/develop in this activity** • Core ideas and connections • Procedures and skills • Representations, solution methods, and tools • Mathematical language (including symbols) and conventions • Engagement in mathematical practices, reasoning, and discourse **Describing the nature of learning expected** • Determining the (ultimate) depth of treatment of the intended mathematics • Determining the expected level of understanding in this activity • Prioritizing the learning of the intended mathematics for this activity • Identifying the "mathematical boundary" of the activity	***Zooming in:*** ***Determining the instructional intent of the activity and/or its details*** • To review/practice/reinforce • To explain/clarify/make something explicit • To introduce • To explore • To expose • To assess • To deepen understanding • To familiarize • To provoke **Specifying the main mathematical point/s of each detail of the instructional activity** • Identifying what more general idea it is going to serve as an instance of • Identifying how it is intended to further students' learning of a particular aspect of the terrain

In sum, the work of articulating the mathematical point, described above, results in a detailed understanding of the intended mathematics and how the activity is intended to engage students with that mathematics. I now turn to the other main component of mathematical purposing (see Figure 15): orienting the instructional activity.

Orienting the Instructional Activity

Whereas articulating the mathematical point is analytic work, the work of orienting the instructional activity focuses on enactment. Though enactment focused, orienting the instruction is not yet about implementing the activity with students. Instead, the work of orienting aims to set up or position an activity so that when it <u>is</u> implemented, it is more likely to head toward its mathematical point. As discussed previously, I distinguish *orienting* (i.e., specifying the details of an activity so that it is positioned toward the intended mathematics) from *steering* (i.e., the in-the-moment, interactive work of navigating the activity's enactment toward the intended mathematics). That being said, the distinction between orienting and steering can sometimes be tricky to maintain because orienting often involves specifying teacher moves that are intended to occur in the enactment phase. Furthermore, when orienting is done during instruction, a particular move can be seen as simultaneously orienting and steering.

Although I use the language of "orienting instruction toward the intended mathematics," this is not meant to imply that the first step in mathematical purposing is to articulate the intended mathematics and the second to orient instruction toward it. As with all aspects of my framework, the direction in which the work occurs (the *how*) is not specified. It is possible, for example, to first detail an activity's enactment and then analyze what mathematics the activity seems best oriented toward.

My framework decomposes orienting the instructional activity into three interdependent types of work:

- Specifying the details of the task and the structure of the activity to focus students on the intended mathematics;
- Preparing specific teacher moves that focus students on the intended mathematics; and
- Planning how to use anticipated student responses to further students' engagement with the intended mathematics.

I describe each in detail below. At the conclusion of the section, I summarize the work of orienting the instruction in Table 17.

Specifying the Details of the Activity to Focus Students on the Intended Mathematics

Chapters 4 and 5 showed that both the details of the task (e.g., numbers used in examples, wording of problems, representations presented, etc.) and the structure of the activity (e.g., work format, materials available, etc.) shape the mathematics with which students engage during an activity. The first type of orienting work is to specify the details of the task itself and the activity structure so that when the task is enacted, it is more likely to result in students engaging with the intended mathematics. I identify six orienting moves related to the specifying the details of the task and the activity structure:

- Selecting examples, problems, and exercises most related to the intended mathematics;
- Selecting numbers and/or figures from desired cases;
- Choosing representations, manipulatives, contexts, and tools that highlight the mathematical ideas to be made salient;
- Matching the wording of the task and what counts as an answer to the intended mathematical work;
- Choosing an activity structure that helps students focus on the intended mathematics; and
- Allocating time within and across activities to focus on the intended math.

The first three—selecting examples, problems, and exercises most related to the intended mathematics; selecting numbers and/or figures from desired cases; and choosing representations, manipulatives, contexts, and tools that highlight the mathematical ideas to be made salient—have much in common. Each involves trying to encourage students' engagement with the intended mathematical ideas and to prevent bogging down the activity in unintended mathematical work. Encouraging engagement with the intended mathematics involves specifying examples, representations, numbers, etc. in order to direct students' attention and work toward the intended mathematics. In addition, it involves sequencing and prioritizing these details, as well as ensuring that the selected examples, numbers, representations, etc. cover the intended mathematical territory. In addition, it involves specifying details that orient *away* from *unintended* mathematics, for example, by avoiding or modifying examples, numbers, representations, etc. that have distracting features or are likely to surface unwanted difficulties. Specific moves include using the least complex numbers that still provide access to the intended mathematics in order to avoid unwanted computational difficulty, or asking students to use a tool that will help them avoid making distracting errors.

The fourth orienting move in this category—matching the wording of the task and what counts as an answer to the intended mathematical work—has two parts. The first is specifying what will be accepted as an answer and/or solution to the task so that, when students give an acceptable answer, they have most likely engaged with the intended mathematics.[52] Orienting what counts as an answer to the intended mathematics might involve making small adjustments to the task's wording or directions—for example, adding the word "explain" in the written form of the task if part of the goal is for students to develop explanations; adding sub-questions to help make particular concepts or problem-solving steps more explicit; or instructing students to use a particular method (e.g., draw a picture) or representation (e.g., use a number line to solve) if the goal is for students to develop their understanding of or skill with a particular concept, procedure, or representation. A second, related part of this orienting move is making sure the intended mathematical work is not being done for students though the presentation of the task, perhaps by leading questions or a representation that can be mindlessly manipulated.

Another orienting move in this category is choosing an activity structure that helps focus students on the intended mathematics. For example, if one of the learning goals of the activity is for students to develop their skills with mathematical explanation, then a partner format might be selected so that students have to explain their solutions to another person. Or, if the focus of an activity is not on computation, then calculators might be made available so that incorrect calculations do not obscure the mathematics the task is meant to make visible. However, if this same task were intended to provide students with computation practice, then calculators would likely not be provided.

A last orienting move in this category is allocating time in relation to the intended mathematics. One way to increase the likelihood that students engage with the intended mathematics is to distribute the time spent in a lesson so that (if things go according to plan) more time will be spent on activities or parts of activities that best engage students with the intended mathematical ideas.

Preparing Specific Teacher Moves that Focus Students on the Intended Mathematics

The second category of orienting work is preparing specific moves for the teacher to use during the activity's enactment that are designed to focus students on the intended mathematics. I have indentified the following practices as part of this category:

- Designing strategic questions and prompts;

[52] By "acceptable answer" I do not mean "correct answer." I mean that students acceptably engaged in the task—for example, they tried to solve the problem, answer the question, complete all of the exercises, etc.

- Determining what, when, and how to make something explicit;
- Determining when and how to give, ask for, and press on explanations;
- Planning to "dwell" in strategic places;
- Identifying ways of using mathematical language to focus on the intended mathematics;
- Determining how to deploy representations and/or make records in ways that help students engage with the intended mathematics; and
- Deciding what mathematical ideas need to surface during the activity's enactment and making contingency plans to raise them in case they do not otherwise come up.

The moves themselves are nothing out of the ordinary, nor are they unique to this framework; teachers routinely plan questions and explanations, and choose problems for discussion. What is important here is that when these moves are used to orient an activity toward the mathematical point, they are designed with the specific intent of strategically focusing students on the intended mathematics. There is also some overlap in the list. For example, questions can be used to dwell or to make something explicit, or can be included as part of a contingency plan. I do not consider this problematic, but it does highlight once again the need to keep purpose in mind: When designing teacher moves to orient instruction, it is important to detail both the move and the intent of that move.

As mentioned earlier, in many cases, the moves identified here are strategies for managing the problems discussed in Chapter 5. For example, because moves such as dwelling and intentional redundancy can be used to emphasize key ideas and thus help focus students on the intended mathematics, or using meaning-focused language can help keep attention on meaning, one way to make it more likely that instruction will focus students on the intended mathematics is to *plan* to make these types of moves at strategic moments in the activity's enactment. Planning specifically how and when to make these types of moves can orient instruction toward the intended mathematics, helping to avoid the issues that can arise when trying to steer instruction toward its mathematical point. For instance, detailing what will be said in an explanation can help prevent the use of imprecise mathematical language; scripting targeted questions that focus on the main mathematical idea can help prevent asking questions that are either too general or too specific. I discuss each of the moves in more detail below.

The first move—designing strategic questions and prompts to orient instruction—requires determining both the purpose of the question and scripting the actual question or prompt. The purpose of the question reflects how it is intended to be used to steer instruction. There are a number of ways questions can be used to steer instruction, for example: to frame mathematical work; to emphasize and/or probe core mathematical ideas; to launch a task without doing the mathematical work for the students; to focus student attention on a key idea during a discussion; to intentionally raise a mathematical issue or make a mathematical observation (rather than "hoping" something will come up); to make the steps of a problem explicit; or to connect student responses back to what they are trying to solve. And there are different ways questions can be designed. One strategy for question design is to script a single question or prompt targeted at the intended mathematics. Another strategy is to develop a "question package" that elicits some of the key mathematical ideas related to a particular problem type. For example, a question package around a fraction representation could be: What's the whole? How many equal parts are there? How many of the equal parts are shaded in? What part of the fraction indicates the number of equal parts that we are referring to? How do you read that fraction? A question package such as this could be used to highlight important ideas about fractions (e.g., attending to the whole, the need for equal parts); make correspondences across representations (e.g., between symbols and pictures); or model for students the components of a complete explanation. Question packages can also be developed to scaffold students' presentations of their solution methods (e.g., What is your answer?; What did you do to get it?; What does ____ mean in the context of the problem?; etc.) Another question-design strategy is to develop "cases" of questions, similar to the distinguishing of cases of numbers discussed earlier. For example, in an primary-grade activity focused on interpreting bar graphs, different cases of question might be: most/least; compare two categories; combine categories; total. Developing cases of questions is a useful strategy for orienting instruction because the cases are general and thus one set of question-cases applies to a whole class of activities, which is easier to remember than many different sets of specific questions, yet it provides some structure (e.g., compare with planning to "ask students a variety of questions about the graph").

The next orienting move is determining what, when, and how to make something explicit. At any point during instruction, there are many things about the mathematics that can be made explicit, for example: a mathematical idea (e.g., an instantiation of a concept; steps in a procedure; why a method works; how to use notation); connections to students' prior knowledge/experiences (e.g., what is similar and different about what they are learning now to what they have learned before); connections between what they are doing and what mathematics they are working on (e.g., connections between a student response and concept; pointing out when a student is using a new skill); and correspondences and connections (e.g., between symbols and representations; between problems). Although all of these can be made be explicit, it is certainly not desirable to be explicit about everything. First, too much explicitness would unnecessarily bog down instruction. Second, what is useful to be made explicit depends on the students and the intended mathematics. And third, there can be a tension between being explicit as a way to focus students' attention on the intended mathematics and doing the mathematical work for students. In addition to detailing what to make explicit, part of the orienting move is determining when and how to make it explicit—for example, by the teacher narrating or asking a targeted question, perhaps after an idea is brought up by a student or before the class begins the activity. Furthermore, the degree of explicitness can also vary (e.g., simply pointing out in passing versus explaining with meaning). This too, of course, depends on the intended mathematics and on what the particular students will likely be able to understand.

The next orienting move in this category—determining when and how to give, ask for, and press on mathematical explanations—involves matching what is explained and the nature of that explanation to the intended mathematics. This includes determining (based on the mathematical point of the activity) when and why it might be strategic for the teacher to give an explanation, when it would be useful to ask students to explain, or when a combination of the two seems appropriate. In addition, it involves figuring out which aspects of the mathematics need explanation. Like explicitness, not everything warrants a detailed explanation. Depending on the intended mathematics or the students being taught, for example, it could be useful to focus only on the "tricky" part of an explanation. Or, explanations could be used to orient an activity toward multiple strands of mathematical proficiency. For example, if an activity involves review problems designed mainly to help students practice a previously learned skill, then pushing students for more complete explanations could be a way to also work on conceptual understanding or reasoning. In addition, it is important to determine the degree of detail and precision desired—for example, when and why it is important to elicit a mathematically complete explanation from the student, when a sketchy explanation would suffice, and when it would be preferable for the teacher to fill in an incomplete argument.

Planning to "dwell" in strategic places is the next orienting move in this category. This move overlaps with those described above. There are many reasons to dwell during an activity, for example: to emphasize, unpack, or practice a core idea; or to spend time on an aspect of the mathematics that is likely to be difficult for or confusing to students. And, as shown in Chapters 4 and 5 and discussed above, there are also many ways to dwell, for example, by the teacher repeating explanations or asking multiple students to explain the same idea; by asking "why" about some parts of a procedure and not others; by writing things out on the board; or by bringing up an error not made by students. Again, there is not time to dwell on all aspects of the intended mathematics. In fact, this would be counterproductive, as part of the power of dwelling lies in the fact that more time is being spent on a particular aspect of the mathematics relative to others. Because dwelling is being used to orient instruction toward the intended mathematics, as with all of the moves discussed here, the intended math should guide decisions about when and on what to dwell. For example, it would not likely be strategic to plan to dwell on ideas that students have already mastered or on a peripheral idea.

The next orienting move is identifying ways of using mathematical language to focus on the intended mathematics. As shown in Chapters 4 and 5, there are a number of ways teachers can use language to help steer instruction toward the mathematical point, for example: intentionally using consistent language in explanations; intentionally repeating key terms; and using meaning-focused language. This orienting move details such language use—both the purpose of the move and how to implement it.

The next move on the list—determining how to deploy representations and/or make records in ways that help students engage with the intended mathematics—involves moves such as: selecting media to use so that all of the relevant mathematical information can be seen and connections across them can be made; planning what to record and why (e.g., showing steps in a procedure so that a confusing residue isn't left, or making "think-in-your-head steps" visible); and deciding what will be written and why when launching problems (e.g., writing out the question along with the key information so students can keep track of what they are trying to find). Planning the deployment of representations also involves making decisions about who is going to do the deploying, and if it will be students, considering additional factors such as how to support them, and what degree of precision and clarity is needed.

The last orienting move in this category is deciding which mathematical ideas must surface during the activity's enactment and making contingency plans to raise them in case they do not otherwise come up. This move involves first identifying the main points, concepts, and/or methods that are imperative to have surface and then developing a way to get these on the table if they are not elicited from students. There are many strategies that can be invoked in a contingency plan, for example, using targeted questions to refocus students' work; asking students to try a particular method; or demonstrating the unelicited method. This orienting move can be in tension with the aim of not doing the mathematical work for the students. Therefore, it is also important to consider when it might be better to wait for an idea to come from students rather than have the teacher bring it up.

Planning How to Use Anticipated Student Responses to Further Students' Engagement with the Intended Mathematics

The moves in this final orienting category are particularly challenging to design and implement because they involve specifying moves in response to what students *might* do as they engage in an activity. Despite this challenge, it is worth including this category of orienting moves in the framework for a number of reasons. First, some student responses are predictable. It may not be possible to predict which student or students will give a particular response or how the response will be worded, but in many cases, it can be confidently said that a particular response will be given by some student in the classroom. Second, planning responses to likely student productions can help teachers manage very complicated in-the-moment work. Responding to students during instruction requires eliciting student thinking, hearing (or seeing) the mathematics in their talk (or written work), assessing how it relates to their engagement in the intended mathematics, determining an appropriate response, and then giving that response in the intended manner—all while simultaneously engaging the whole class and trying to manage multiple other purposes. Designing some general types of responses in advance can help navigate this complex instructional space and keep the activity more on-track during instruction (Fernandez & Yoshida, 2004). However, planning responses to students could cause mathematical purposing to conflict with being responsive to students (Zahorik, 1970). Therefore, it is important that the planned moves are designed with responsiveness in mind and are not forged ahead with at all costs.

I have identified the following four orienting moves in this category:

- Determining how to strategically discuss solutions or "go over" problems in ways that focus on the intended mathematics;
- Planning which aspects of student responses to take up;
- Determining how to handle likely errors; and
- Determining how to scaffold and/or help students if they get stuck.

The first move in this category—determining how to strategically discuss solutions or "go over" problems in ways that focus on the intended mathematics—involves selecting and ordering for discussion problems or solution methods that are most related to the intended mathematics, as well as determining the intended nature of the discussion. The importance of planning for a discussion of student solutions to a complex problem is seen in the literature (Chapin, O'Connor, & Anderson, 2003; Smith & Bill, 2004; Smith, Hughes, Engle, & Stein, 2009). For example, Smith et al.'s *Thinking Through A Lesson Protocol* asks teachers to think about how their orchestration of a class discussion will accomplish their goals, in particular, how the solution paths selected for discussion and their order will "help develop students' understanding of the mathematical ideas that are the focus of [the] lesson" (Smith & Bill, 2004). Similar types of thinking (e.g., strategically selecting and ordering the exercises to discuss) can orient the "going over" of a worksheet that students have completed. One strategy is to select, in advance, a subset of problems that is most focused on the intended mathematical ideas. Because students will have already completed the problems on their own, an important aspect of the selection process is to consider what learning opportunities are made available to students by going over the problems—for example, reinforcing key concepts, making connections across solution methods, or giving complete explanations. In addition, it is important to plan the nature of the discussion, calibrating the nature of explanation required to the mathematical point.

The next orienting move in this category is planning which aspects of student responses to try to take up. This can involve planning in advance how to handle particular responses or generating more general guidelines to use during the activity. For example, a teacher might plan to revoice the parts of student responses that use key vocabulary. Or, a teacher might plan to probe student methods that highlight an important aspect of the intended mathematics, and to acknowledge but not probe student solution methods that are likely to take the activity into the mathematical boundary. To help hear students' ideas during the lesson, a teacher might also determine where in the activity will require listening especially carefully to students in order to know exactly how or where to steer. Careful listening around key mathematical ideas can help teachers identify kernels of the intended mathematics in student talk or avoid getting bogged down in unintended mathematical work.

The next move—determining how to handle likely errors—is closely related to planning which aspects of student responses to take up. However, it is worth listing separately because teachers often handle errors differently than other student responses (Ball, 1997). As with any orienting move, when using the handling of errors to orient an instructional activity, decisions are made in order to further students' engagement with the intended mathematics. For example, a teacher might plan to simply correct computational errors that are unrelated to the intended mathematics, but take up and discuss errors that reflect a possible misconception of an idea central to the intended mathematics. However, like all instructional decisions, determining how to handle errors is influenced by a number of factors, including who the student is and his or her relationship to the teacher and to other students.

The final orienting move in this category is determining how to scaffold students and/or help students who are stuck. This move is related to the problem discussed in Chapter 5 of ensuring that it is the *students* who are engaged in the intended mathematics. It is very common for teachers to inadvertently reduce the cognitive demand of a task in their efforts to help students when they are having difficulties (Stein et al., 2000). One strategy that can be helpful here is to plan hints or strategic questions that point students in a productive direction, but do not do the work for them.

The orienting moves are summarized in Table 17 below.

Table 17.
Orienting the Instructional Activity

Specifying the details of the task/s and the structure of the activity to focus students on the intended mathematics • Selecting examples, problems, and exercises most related to the intended mathematics • Choosing representations, manipulatives, contexts, and tools that highlight the mathematical ideas to be made salient • Selecting numbers and/or figures from desired cases • Matching the wording of the task and what counts as an answer to the intended mathematical work • Choosing an activity structure that helps students focus on the intended mathematics • Allocating time within and across activities to focus on the intended math
Preparing specific teacher moves that focus students on the intended mathematics • Designing strategic questions and prompts • Determining what, when, and how to make something explicit • Determining when and how to give, ask for, and press on explanations • Planning to "dwell" in strategic places • Identifying ways of using mathematical language to focus on the intended mathematics • Determining how to deploy representations and/or make records in ways that help students engage with the intended mathematics • Deciding what mathematical ideas need to surface during the activity's enactment and making contingency plans to raise them in case they do not otherwise come up
Planning how to use anticipated student responses to further students' engagement with the intended mathematics • Determining how to strategically discuss solutions or "go over" problems in ways that focus on the intended mathematics • Planning which aspects of student responses to take up • Determining how to handle likely errors • Determining how to scaffold and/or help students if they get stuck

Putting It All Together: A More Detailed Picture of Mathematical Purposing

I have conceptualized mathematical purposing as the interaction of two main types of work: articulating the mathematical point and orienting the instructional activity. Articulating the mathematical point is analytic work, which involves applying a mathematics, a learners, and a focusing lens to both the mathematical terrain and the instructional activity. Through this trifocal analysis, the mathematical terrain is unpacked and mathematical goals for student learning—both distal and proximal—are specified; the instructional activity is examined and the mathematical point of its details determined; and the connection between the activity and the terrain made explicit. Orienting the instructional activity is more enactment focused, specifying the details of instruction so that the activity is better positioned toward the intended mathematics. Articulating the mathematical point and orienting the instructional activity are interdependent and mutually informing and, together, result in a deep and nuanced understanding of the mathematical learning goals of a given activity, an understanding of how the activity is intended to engage students with that mathematics, and a detailing of the activity that matches its design to the intended mathematical point.

The framework presented in this chapter begins to decompose what is involved in doing the work of mathematical purposing. The main categories of work included in the framework are summarized in Figure 17. Figure 17 elaborates the basic architecture described at the beginning of the chapter (Figure 15).

Articulating the Mathematical Point

	Mathematical terrain	Instructional activity
Mathematics lens	Identifying core ideas and the connections among them Identifying core practices, skills, and sensibilities Identifying what there is to learn and understand about mathematical language Analyzing multiple representations and procedures for the focal topic Describing and explaining underlying concepts, principles, representations, and procedures Distinguishing cases and directions that can be learned	Locating opportunities to work on aspects of the mathematical terrain Examining the mathematics made available by and across the details of the activity Identifying the mathematical prerequisites of the activity
Learners lens	Analyzing the complexity of the mathematics for learning Analyzing how learners typically think about the focal topic Characterizing the mathematics the particular students are bringing to the activity	Locating potential opportunities to connect with and confront what students are bringing to the activity Examining the accessibility of the mathematics within and across the details of the activity
Focusing lens	***Zooming out:*** Specifying the more distal learning goals to be deliberately worked toward through the activity Sketching the intended curricular trajectory through the mathematical terrain ***Zooming in:*** Specifying the particular aspects of the mathematical terrain students are intend to learn/develop in this activity Describing the nature of learning expected	***Zooming out:*** Articulating how the activity is intended to further broader mathematical learning goals Determining the mathematical storyline within and across the lesson ***Zooming in:*** Determining the instructional intent of the activity and/or its details Specifying the main mathematical point/s of each detail of the instructional activity

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Orienting the Instructional Activity
Specifying the details of the task/s and the structure of the activity to focus students on the intended mathematics Preparing specific teacher moves that focus students on the intended mathematics Planning how to use anticipated student responses to further students' engagement with the intended mathematics

Figure 17. The main components of the work of mathematical purposing.

Discussion of Framework

Before turning to the relationship of MKT and the work of mathematical purposing in the next chapter, I step back to make some general comments about the framework. I first recap some of its significant features and briefly describe some potential contributions and uses.[53] I then discuss the limitations of this work.

Significant Features

One of the significant features of this framework is that, by taking a distributed perspective toward the design of instruction, it accommodates the range of lesson-design processes. For example, the framework applies when teachers create their own instructional activities as well as when teachers closely follow a textbook lesson. As a result, the framework does not privilege the classic direction depicted in the liner model of planning (i.e., (1) specifying behavioral objectives; (2) choosing appropriate learning activities; (3) organizing and sequencing the chosen activities; and (4) selecting evaluation procedures). Instead, the framework names the work without specifying an order in which it must occur or by whom (or what) it should be done. This means that it is “legal” for a teacher to first select an activity from a textbook and then analyze the activity to articulate the intended mathematics. In fact, this was the direction that was often taken by experienced teachers in the planning literature, which means the framework can be mapped onto what many teachers do in practice.

Another significant feature of the framework is its explicit attention *at the activity level* to all of the strands of mathematical proficiency and to the development of mathematical practices. That mathematical learning goals are nested and of different grain sizes was a major theme in the literature. As described in Chapter 5, attending to goals of different types and grain sizes can be challenging. Most teachers in my study specified topic-related mathematical learning goals and did not explicitly mention goals such as developing students’ skills with explanations or using representations, even if they may have engaged students with these in their lesson. The framework’s explicit attention to broader, ongoing mathematical goals at the activity level aims to bring these important ongoing goals to the foreground and to encourage the development of all strands of mathematical proficiency in every activity.

This points to another significant feature: The framework is mathematics focused, yet not topic specific. Mathematics standards documents are typically organized by content domain (e.g., number and operations, geometry, algebra, etc.), with perhaps some standards that cut across

[53] In Chapter 8, I explore potential uses of the framework in more detail, in particular, as a tool that helps mediate connections between research, practice, and problems in mathematics education scholarship (Silver & Herbst, 2007).

topics (e.g., NCTM's (2000) Process Standards). Similarly, analyses of students' mathematical thinking and of the mathematical terrain in the research literature (e.g., Leinhardt, Putnam, & Hattrup, 1992; Lester, 2007) also tend to focus on specific topics. At the other end of the spectrum is the instructional design literature. The instructional design frameworks for articulating student learning goals are not topic specific (they are not even mathematics specific) and are often organized by type-of-learning hierarchies (e.g., Bloom's (1956) taxonomy or Gagné's (2005) domains of learning). Because these hierarchies are not mathematics specific, it can be difficult to extract a focus on mathematical meaning. In many cases, student learning objectives are identified through a "learning-task analysis" or an "information processing analysis," which involves working backwards from the end task (e.g., being able to subtract multi-digit whole numbers) to name the essential or supporting prerequisite tasks (Gagné et al., 2005). Although this method is intended to be applicable to developing concepts, dispositions, and practices, learning outcomes not focused on skills are more difficult to specify and can become proceduralized in the process (Smith & Ragan, 2005).

Although the overall architecture of my framework may be applicable to the teaching of different subjects, its insides are clearly focused on mathematics. However, it is not a pure content analysis: The content is unpacked and analyzed in order to *teach* it to learners. This is reflected, for example, in the inclusion of "distinguishing cases and directions that can be learned" as part of the work of applying a mathematics lens to the mathematical terrain. Naming this as part of the work of applying a mathematics lens to the mathematical terrain signals that the mathematical analysis of the terrain is not strictly from a disciplinary perspective, but is an analysis of the mathematics *for teaching*. The framework also reflects ways of knowing and working on mathematics that the mathematics education literature has identified as important.

And, unlike standards documents or the instructional-design hierarchies, the central idea that is highlighted through my framework's architecture is the connection between the instructional activity and the mathematics it is intended to teach. For example, the details of the task are bundled with their specific purposes that link them to specified mathematical learning goals. This is true for teacher moves as well; teacher moves are tied to their mathematical point. Highlighting this connection might be useful for those beginners who are so concerned about what they are going do and say when teaching that explicit attention to student learning slips into the background.

The framework also takes a slightly different cut on the work of teaching. A typical division in teacher education is between preactive and interactive phases (Jackson, 1968); or between planning, enactment, and reflection. My framework offers the following (noncomprehensive) divisions of the work of teaching: mathematical purposing (which is further divided into articulating the mathematical point and orienting the instructional activity) and steering the instruction toward the mathematical point. Steering coincides with the interactive/enactment phase of teaching; but mathematical purposing could happen at any time. Within mathematical purposing, the division is between analysis and setting up for enactment. Thus, the distinctions in my framework reflect different types of work, not different times when it happens. However, because much of the work of mathematical purposing can be done during the preactive phase of teaching, it may be a strategic site for teacher education.

As discussed in Chapter 1, another way to slice the work of teaching is to parse it into instructional routines and practices such as leading a discussion, going over homework, or posing a problem (Ball et al., 2009; Franke & Chan, 2007; Franke et al., 2007; Grossman, Hammerness, & McDonald, in press; Kazemi, Lampert, & Ghousseini, 2007; Lampert & Graziani, 2009; Smith et al., 2009). Such "discrete general practices" can be contrasted with "cross-cutting practices" such as eliciting student thinking, giving students feedback, and cultivating norms (Teacher Education Initiative, 2009). Cross-cutting practices do not occur in only a particular instructional format. Teaching to the mathematical point is an example of a cross-cutting practice, as is mathematical purposing. However, mathematical purposing can be productively worked on as part of a discrete general practice: planning a lesson.

Potential Contributions and Uses

The major contribution of the framework is the conceptualization of a central aspect of mathematics teaching practice: identifying the mathematical goals of an activity and orienting instruction toward those goals. It is not the idea that teaching should have instructional goals or that there should be a match between an activity and the intended mathematics that is the contribution of the framework. These are not new ideas; they are essentially the definition of teaching. In many cases, however, the work of determining the mathematical goals of an activity and how the activity is designed to move students toward those goals is left implicit. Thus, what this framework does is bring the work of mathematical purposing into the foreground. The overall conceptualization of the work of mathematical purposing (Figure 17) may, in fact, be applicable to the "content purposing" of instruction in other subject areas. For example, instead of applying a mathematics lens to the mathematical terrain and the instructional activity, the lens and terrain would reflect the respective content area.

In addition to providing an overall conceptualization of mathematical purposing, the framework begins to further parse what is involved in doing this work. Such decompositions of practice are critical for both teacher education and future research (Grossman & McDonald, 2008). The framework for mathematical purposing presented here also provides much needed language for teaching practice. Even if the language used in the framework does not live beyond this dissertation, it has helped identify aspects of teaching practice that warrant naming and further research. Just as mathematical terms and symbols compress mathematics concepts into objects that can more easily be manipulated and operated upon, naming the work of mathematical purposing compresses a set of important ideas and practices into an object that can be more easily studied.

The types of decompositions of practice found in this framework are also needed to teach beginners to do the work of teaching. In addition to articulating practices that beginning teachers need to learn to do, the framework can be used as a lens for studying or reflecting on teaching. It could be used to develop teachers' skill at interpreting curriculum materials and for seeing "curriculum potential" (Ben-Peretz, 1990). The framework could also inform the design of educative curriculum materials (Davis & Krajcik, 2005) by providing additional insight into the type of supports curriculum designers might include to help teachers engage in mathematical purposing.

Limitations

One major limitation is the framework's manageability. The decomposition of practice is, by definition, detailed and intricate. However, extensive detailing can interfere with usefulness for and usability in practice (Ball et al., 2009). As I parsed each section of the framework, it became long lists of things to do and consider, and it is unclear how useful that format is for teacher education or research. In addition, despite the length of the lists, the framework is still not exhaustive. Thus, part of the challenge of this kind of work is determining the strategic aspects to unpack and the right grain size of detail so that it is meaningful and can be acted upon in practice, yet does not become an unwieldy laundry list of things to remember. To help mediate this issue, I tried to make visible an overarching architecture for the framework (Figure 17), in hopes that some coherency would support understanding and remembering the details. Whether this helps remains to be seen. And, even if it helps, the architecture has its own limitations. For example, the matrix representation does not depict the interactions across cells that are central to the conceptualization.

Another limitation is that, although the framework aims to describe the work of mathematical purposing through decomposition, it does not attempt to characterize the quality of that work. That is, it does not describe how one would know if mathematical purposing is being done well. For example, simply doing everything in the framework is not enough: the mathematical terrain and the activity can be unpacked with different degrees of sophistication and understanding; specified learning goals can be more or less central; connections across activities can be differently compelling; and orienting questions can be more or less strategically designed. Certainly, an articulation of the work is an important first step, but there is much more to be to done to describe what it means to do this work well. Being able to describe the quality of mathematical purposing has implications for future research that might, for example, try to measure mathematical purposing to study whether there is a relationship between mathematical purposing and other aspects of mathematics teaching and learning, such as student achievement or the mathematical quality of instruction.

Another limitation arises from my adoption of a distributed perspective on the design of instruction. While this perspective does enable the development of a framework for describing what the work is separate from how that work gets done, in practice the *how* is important. The work of mathematical purposing will be differently distributed in different teaching contexts. But as mentioned in Chapter 2, the individual matters in theories of distributed cognition (Salomon, 1993), and, in any context, the teacher is ultimately responsible for coordinating the available resources and for steering the instruction toward the mathematical point. Thus, regardless of the distribution of work, to steer instruction toward the mathematical point, the teacher must have some understanding of the mathematical point. The specific aspects of the work that have to be done by the teacher to develop that understanding and the knowledge required will reflect of the distribution of work in the particular context. Furthermore, I hypothesize that there are likely some aspects of mathematical purposing that, no matter what the context, have to be done by the teacher and not distributed to other resources. Although important to the questions at hand, these issues are beyond the scope of the present study.

A final limitation is the applicability of the framework beyond this study. Although a product of iterative analyses of the literature and the data, it is likely that I missed key aspects of the work and that the framework is too narrow. As mentioned in Chapter 3, it might too heavily reflect *Everyday Mathematics* or the teacher education program in which the preservice teachers participated. For example, maybe the lack of explicit attention to mathematical practices as learning goals was a reflection of the spiral curriculum (or teachers' perceptions of it as disconnected), or perhaps the teacher education program did not emphasize teaching toward goals of a larger grain size. In other words, although the work of mathematical purposing will certainly be differently distributed in different teaching contexts (i.e., the how), there may be also be components of the work (i.e., the what) that vary across contexts. Studying teaching in different contexts could make these aspects of the work visible and further develop the framework presented here.

In the next chapter, I turn to the relationship between mathematical knowledge for teaching (MKT) and mathematical purposing. To explore this relationship, I apply the domains of MKT (Ball et al., 2008) to the framework for mathematical purposing described above. I then present a case from the data to further illustrate the knowledge demands of mathematical purposing.

CHAPTER SEVEN:
THE RELATIONSHIP BETWEEN MATHEMATICAL KNOWLEDGE FOR TEACHING AND THE WORK OF MATHEMATICAL PURPOSING

Introduction

This chapter explores the relationship between mathematical knowledge for teaching (MKT) and the work of mathematical purposing. The analysis presented here provides two types of insight. First, it illustrates the ways in which MKT is drawn upon in the work of mathematical purposing and the different types of knowledge each component of the framework entails. Second, the analysis contributes to the development of the theory of MKT by further elaborating the structure and content of MKT and illustrating the use of MKT in practice. I begin by describing in more detail the domains of mathematical knowledge for teaching proposed by Ball et al. (2008). I next map the domains onto the framework for mathematical purposing by analyzing which domains of MKT each aspect of the work draws upon. I then illustrate the MKT demands of mathematical purposing with a case from the data.

Domains of Mathematical Knowledge for Teaching

As described in Chapter 1, the current framework for MKT distinguishes six domains of mathematical knowledge for teaching (Ball et al., 2008). Four of the domains have been empirically tested and are better conceptualized: common content knowledge (CCK); specialized content knowledge (SCK); knowledge of content and students (KCS); and knowledge of content and teaching (KCT). Two provisional domains are at earlier stages of conceptualization: horizon content knowledge (HCK) and knowledge of content and curriculum (KCC). CCK, SCK, and HCK are types of subject matter knowledge (requiring no knowledge of students or pedagogy); whereas KCS, KCT, and KCC are amalgams of subject matter knowledge and pedagogical knowledge and are thus types of pedagogical content knowledge (Shulman, 1986). I describe each domain in more detail below.

Common content knowledge (CCK) captures mathematical knowledge that is used in the work of teaching in ways that are in common with how mathematics is used in other professions. This category includes, for example, being able to do arithmetic, identify geometric shapes, and solve algebraic equations. It is knowledge that teachers use, but is not specific to the work of teaching.

Specialized content knowledge (SCK) is content knowledge that is tailored for the specialized uses of mathematics that come up in the work of teaching, and is thus not commonly used in those ways by most other professions or occupations. Examples in this category include explaining why common arithmetic procedures work, representing mathematical concepts in multiple ways, and analyzing nonstandard solution methods.

Horizon content knowledge (HCK) is "a kind of elementary perspective on advanced knowledge" (Ball & Bass, 2009). It includes understanding which ideas being taught now are foundational for later topics, considering whether the way an idea is currently portrayed will maintain its mathematical integrity as more sophisticated mathematical ideas are introduced, and recognizing how a current topic is an instantiation of something that was taught before or will be taught later.

Knowledge of content and students (KCS) is content knowledge intertwined with knowledge of how students think about, know, or learn the particular content. It is used in tasks of teaching that involve attending to both the specific content and something particular about learners. Examples include identifying common student misconceptions and determining a problem's difficulty relative to a particular stage in students' mathematical development.

Knowledge of content and teaching (KCT) is content knowledge intertwined with knowledge of how to teach particular content. It is used for tasks of teaching that involve attending to both the specific content and specific methods of teaching it to others. This category includes knowing how to choose and sequence examples and how to guide student discussions toward accurate mathematical ideas.

Knowledge of content and curriculum (KCC) is provisionally classified as a form of pedagogical content knowledge; however, Ball et al. do not offer much by way of definition for the types of knowledge and reasoning the domain includes. They instead defer to Shulman and colleagues' work, in particular, Grossman (1990).

In Shulman and colleagues' early conceptualizations of forms of teacher knowledge, knowledge of curriculum was often separated from pedagogical content knowledge (Shulman, 1986, 1987; Wilson et al., 1987). Shulman (1986) defines "curricular knowledge" as an understanding of the "curricular alternatives available for instruction" (p. 10). In addition to knowledge of the range of programs and materials for teaching a subject, Shulman includes "lateral curriculum knowledge" and "vertical curriculum knowledge," which are, respectively, familiarity with content taught simultaneously in other subject areas and with topics in the same subject taught at earlier and later grade levels. Grossman and Richert (1988) also include "awareness of the prerequisite knowledge for studying particular content" as part of curricular knowledge. In later publications, curricular knowledge was included as part of pedagogical content knowledge (Grossman, 1990, 1991; Grossman & Richert, 1988).

Grossman (1990, 1991) identifies four components of pedagogical content knowledge: conceptions of the purposes for teaching particular subject areas; knowledge and beliefs regarding student understanding; curricular knowledge; and knowledge of instructional strategies and representations for teaching particular topics. Comparing Grossman's components with the domains of MKT, "knowledge and beliefs regarding student understanding" would be part of knowledge of content and students (KCS) and "knowledge of instructional strategies and representations for teaching particular topics" part of knowledge of content and teaching (KCT). Grossman's "curricular knowledge" reflects Shulman's description above, and "conceptions of the purposes for teaching a particular subject" includes "teachers' beliefs about what is most important for students to know, understand, and appreciate about specific content, and their understanding of the interrelationship of topics within a subject" (Grossman, 1991, p. 209). Thus, it seems that both of these components—conceptions of the purposes for teaching particular subject areas and curricular knowledge—would be considered part of Ball et al.'s knowledge of content and curriculum (KCC).

The domains of MKT and their relationship to Shulman et al.'s subject matter knowledge and pedagogical content knowledge are shown in Figure 18.

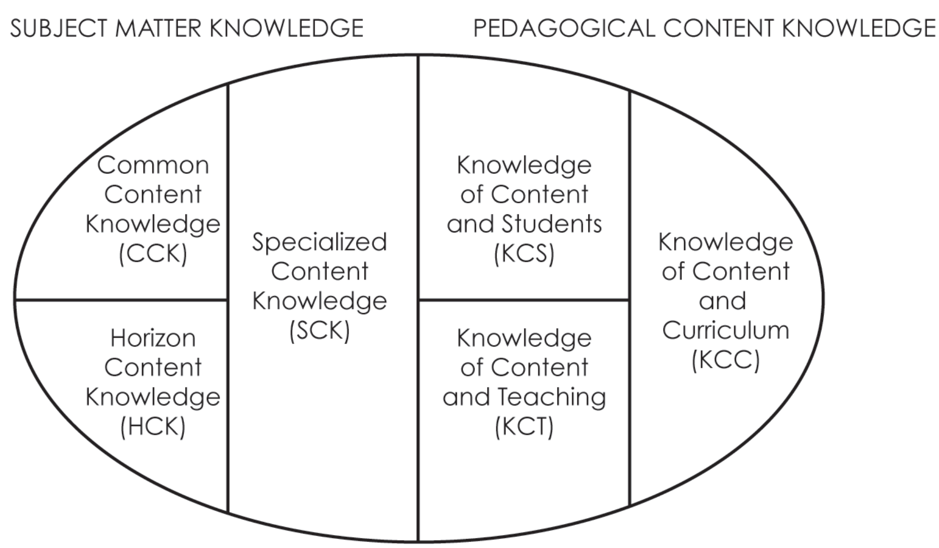


Figure 18. Domains of mathematical knowledge for teaching.
Adapted from Ball et al. (2008).

MKT Demands of Mathematical Purposing

As described in Chapter 3, the approach to studying mathematical knowledge for teaching utilized by Ball and colleagues first studies the work of mathematics teaching and then analyzes the mathematical knowledge demands of that work (Ball & Bass, 2003; Ball et al., 2008). Therefore, to use this same practice-based approach to investigate the mathematical knowledge demands of identifying mathematical goals and using them to design instruction, I began by identifying the work of what I came to call "mathematical purposing." I then analyzed the knowledge or kinds of reasoning entailed in that work. In this case, the analysis of the work yielded the framework discussed in Chapter 6.

To analyze the mathematical knowledge entailed by the work of mathematical purposing, I analyzed this framework using the MKT domains described above. To do this, I mapped each component of the framework to the domain of MKT upon which it most heavily draws. Of course, because the components of the mathematical purposing framework are interdependent, each domain of MKT can be seen as being utilized in every aspect of the work. However, this amounts to saying that the work of mathematical purposing draws upon all of the domains of MKT, which is true, but not particularly useful. Therefore, for purposes of analysis, I considered each component, and the decomposed practices within each component, separately. Figure 19 shows the results of this analysis. I discuss these findings in more detail below.

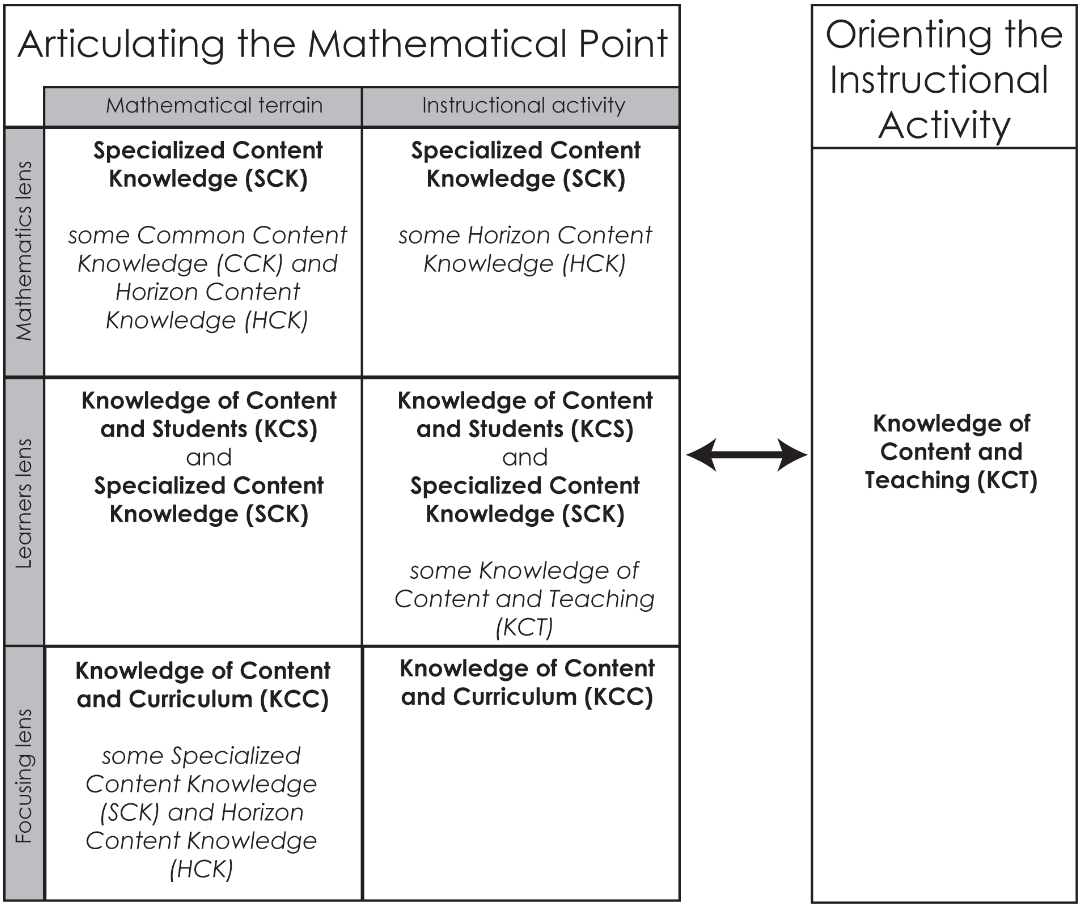


Figure 19. Domains of MKT drawn upon in the work of mathematical purposing.

MKT Demands of Applying a Mathematics Lens

Applying a mathematics lens to the mathematical terrain and to the instructional activity does not require knowledge of students or pedagogy. Therefore, applying a mathematics lens draws upon knowledge in the subject-matter-knowledge half of Figure 18. A closer analysis of the components of mathematical purposing shows that specialized content knowledge (SCK) is the prominent domain in use. When applying a mathematics lens to the mathematical terrain, much of the work involves identifying and explaining core concepts, procedures, and representations—all work that relies heavily on SCK. Because a mathematics lens is being applied to the mathematical terrain in the service of mathematical purposing, the work goes beyond simply recognizing ideas that are related to a topic (which might draw only on common content knowledge).

A few aspects of the work of applying a mathematics lens to the mathematical terrain also seem to draw upon common content knowledge (CCK) and horizon content knowledge (HCK). CCK might be all that is used to describe the steps of procedures, how to use a representation or tool, and recognize or generate situations when a particular skill or procedure would be used. HCK might be needed (along with SCK) to recognize concepts for which a topic is foundational and to unpack the relationship to big mathematical ideas.

Similar arguments can be made about the predominant use of specialized content knowledge in the work of applying a mathematics lens to the instructional activity. Looking for potential opportunities to work on different aspects of the terrain in the instructional activity draws upon SCK to do the mathematical analysis of unpacking the activity to match it to various mathematical options. The detailed analysis of the mathematics made available in representations, procedures, explanations, and examples also draws upon SCK because it is not yet analyzing how students will most likely interpret them or reasoning about which would be the best fit for a particular instructional purpose. Instead, the mathematics-lens analysis of the details of the instructional activity examines aspects such as the coverage of the terrain by a set of exercises, which mathematical ideas are most visible in a particular representation, the precision of the mathematical language in an explanation, whether the use of a procedure draws attention to its underlying concepts, or to what parts of the terrain the numbers in a problem might lead. Applying a mathematics lens opens up the mathematical options (i.e., what *could* be worked on) and evaluates the mathematical features of these options. There are also a few types of analyses for which horizon content knowledge might also be drawn upon when analyzing the details of an instructional activity with a mathematics lens: analyzing precision and distortion, and analyzing whether a particular detail foreshadows upcoming ideas.

MKT Demands of Applying a Learners Lens

Applying a learners lens to the mathematical terrain and to the instructional activity, by definition, requires knowing about students and knowing about content. Thus, it is not surprising that knowledge of content and students (KCS) is one of the domains of MKT most drawn upon in the learners-lens analysis. Knowledge of students' common misconceptions, typical solutions methods, and likely difficulties are all part of KCS and are needed when unpacking the terrain and the instructional activity from the perspective of the learner. In fact, I originally thought KCS would be the only domain utilized in the learners-lens analysis. However, mapping the MKT domains onto the mathematical purposing framework revealed that specialized content knowledge also plays an important role in developing a learners perspective on both the mathematical terrain and the instructional activity. In particular, SCK is drawn upon when analyzing the complexity of the mathematics for learning—for example, examining how other mathematical topics might interfere with or support learning about the focal topic. Analyzing the similarities and differences across mathematical contexts does not require knowledge of students; knowing which of these differences actually impact student learning is what requires KCS. A similar use of SCK to unpack what *could* matter from the learners' perspective along with KCS to determine which of those things *do* matter can also be seen in the analysis of the details of the instructional activity.

MKT Demands of Applying a Focusing Lens

Applying a focusing lens involves zooming out on the mathematical terrain to specify bigger grain-sized learning goals and to sketch the intended curricular trajectory and zooming out on the instructional activity to match those goals to the activity and to determine the lesson's mathematical storyline. Zooming in, applying a focusing lens specifies the learning goals related to particular aspects of the focal topic and determines the mathematical point of the activity's details. Doing these analyses requires knowledge of what is taught at particular grade levels, as well as the ability to construct and prioritize coherent learning goals of appropriate size and importance in light of the unpacking of the mathematical terrain and instructional activity through the mathematics and learners lenses. Applying a focusing lens thus draws upon vertical curricular knowledge (Shulman, 1986) and conceptions of the purposes for teaching particular subject areas (Grossman, 1988, 1990). I therefore have categorized the work of applying a focusing lens to the mathematical terrain and instructional activity as demanding knowledge of content and curriculum (KCC). I did find two examples of work that draw upon SCK and possibly HCK: Prioritizing the intended mathematics involves articulating mathematical significance (with respect to the discipline), which only demands subject matter knowledge (not pedagogical content knowledge).

MKT Demands of Orienting the Instructional Activity

Finally, the work of orienting the instructional activity toward the intended mathematics draws primarily upon knowledge of content and teaching (KCT). Strategically selecting examples, developing strategic questions, determining when to press on student explanations, and selecting student responses to take up all involve the design of teaching moves to match specific instructional purposes. Doing this requires knowing about teaching and knowing about the content.

Contributions to the Developing Theory of MKT

As discussed above, analyzing the framework for mathematical purposing in light of the domains of MKT provides insight into the types of mathematical knowledge and reasoning required to do this work. At the same time, this analysis contributes to better understanding mathematical knowledge for teaching by proposing elaborations of some of the domains and suggesting ideas about how MKT might be developed in practice. I discuss some of the contributions to the ongoing development of the practice-based theory of MKT below.

Identifying KCT and KCC

One of the contributions of this analysis is an elaboration of the domains of knowledge of content and teaching and knowledge of content and curriculum. With respect to knowledge of content and teaching, the three main categories of work in orienting the instructional activity—specifying the details of the task and structure of activity to focus students on the intended mathematics; preparing specific teacher moves that focus students on the intended mathematics; and planning how to use anticipated student responses to further students' engagement with the intended mathematics—are already reflected in Ball et al.'s work. However, the more detailed unpacking of these categories provided in the framework for mathematical purposing suggests more specific ways that KCT is used in practice.

Knowledge of content and curriculum, on the other hand, is underspecified in Ball et al.'s work. Thus, analyzing the MKT demands of mathematical purposing contributes to the development of the theory of MKT by identifying possible components of KCC. Building on Shulman's and Grossman's work, my analysis suggests the following as types of knowledge, reasoning, and dispositions included in the domain of KCC:

- Knowledge of foundational mathematical learning goals (including knowledge of the strands of mathematical proficiency) and the instantiation of those goals at particular grade levels; and the ability to determine how they can be worked toward in particular activities;
- Knowledge of productive curricular trajectories through the mathematical terrain for different topics [including the main content taught (e.g., concepts, skills, procedures; representations and tools; types of problems); connections across this content; the typical order in which it is taught; what is assessed; etc.]—detailed for the grade being taught, more general for grades above and below;
- The ability to develop a coherent mathematical storyline across instructional activities and lessons;

- The ability to specify coherent mathematical learning goals of different types and grain sizes appropriate for a particular instructional activity and to understand how the details of the instructional activity are intended to move students toward those goals;
- The ability to prioritize mathematical learning goals and determine the appropriate depth of treatment and expected level of understanding for a given instructional activity;
- The ability to identify the "mathematical boundary" of an instructional activity; and
- The disposition that the ultimate goal of mathematics instruction is to develop mathematical proficiency in all students and that mathematics instruction should aim to deliberately move students toward specified mathematical learning goals.

Using SCK to Develop a Learners' Perspective

The analysis of the MKT demands of mathematical purposing also has implications for how mathematical knowledge for teaching might be developed in practice. One of the interesting results of this analysis is that applying a learners lens to the mathematical terrain and the instructional activity does not depend only on knowledge of content and students; some aspects of the work draw more heavily on specialized content knowledge. This means that it is possible to begin to develop an understanding of the learners' perspective of the mathematical terrain and the instructional activity without having extensive knowledge of students' mathematical thinking. In fact, engaging in these types of analyses is likely to help *develop* knowledge of content and students, suggesting a way that subject matter knowledge might be "transformed" into pedagogical content knowledge (Wilson et al., 1987).

In the next section, I illustrate the knowledge demands of mathematical purposing using a case from the data.

A Case of Knowing and Using Mathematics in Mathematical Purposing: *What is the Mathematical Point of the Clock?*

Paige taught a fourth-grade lesson on representing, adding, and subtracting fractions on a clock face. She had a difficult time teaching the lesson, as evidenced by her self-reflection in the interviews and by the mathematical issues that arose during instruction. At the heart of the difficulties seemed to be a mismatch between her instructional activities and her goals for student learning—that is, she was unclear about her mathematical point, in particular, the mathematical point of the clock.

I begin by describing Paige's lesson. Throughout this description, I make some comments about the mathematics made available by different problems and representations; however, I do not discuss Paige's purposes or analyze the quality of the instruction. This analysis occurs in the subsequent section where I use the framework for mathematical purposing to analyze Paige's articulation of the mathematical point of the clock.

Description of Paige's Fourth-Grade Lesson on Clock Fractions

Paige begins her mathematics lesson with her classroom's usual warm-up routine: a Math Message followed by Mental Math problems. She selected both the Math Message problem (*How many minutes are in a half an hour, a fourth of an hour, and a third of an hour?*) and the Mental Math problems ("fraction of" problems—e.g., $\frac{1}{2}$ of 12; $\frac{1}{6}$ of 12; $\frac{1}{4}$ of 60) to ready her fourth-grade students for the main work of the day, which involves using a clock face to represent, add, and subtract fractions. Paige makes this connection across the lesson's activities explicit to her students as she transitions from Mental Math to the first clock activity:

Paige	All right you guys, let's start something. We're going to be working with fractions on the clock today. Okay, and if you were thinking about it, the numbers that I asked you [during Mental Math], I asked you numbers with twelve in the denominator and sixty in the denominator. Why do you think I did that? Why'd I do that? Lyle, do you know?
Lyle	Because when the big hand is on the twelve that means an hour has passed and that's sixty minutes.
Paige	Okay, so Lyle said that we're looking at the clock and if the, the hand's on the twelve or an hour has passed or the clock is broken off into twelve parts, right? And then the little parts in between, okay. And he said that there's sixty minutes in an hour. Okay, so we're thinking in time here. And that's why we did those fractions because we're going to be working with time today.

Paige's lesson is based on the fifth lesson in a 13-lesson unit on fractions from the second edition of *Everyday Mathematics*. The textbook's stated objectives for the lesson, entitled *Clock Fractions*, are "to model fractions on a clock face; and to use a clock face to help add and subtract fractions" (Bell et al., 2004b, p. 535). In addition, three "main objectives" are listed in the front matter of the unit: (1) to provide reminders, review, and practice of fraction ideas introduced earlier; (2) to develop good understanding of equivalent fractions; and (3) to provide informal activities related to chance and probability (p. 498). Also listed in the front matter are "learning goals" for the unit, two of which are specifically associated with the clock fractions lesson. The first—add and subtract fractions—is a "beginning goal," and the second—identify fractional parts of regions—is a "secure goal" (p. 500).

The first of the textbook's clock fractions activities is called *Representing Fractions on a Clock Face*. The teacher's guide suggests displaying shaded sectors of a clock face (Figure 20) on the overhead and asking students to write the corresponding fractions on their slates. The given examples include both unit fractions ($\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{6}$) and non-unit fractions ($\frac{2}{3}, \frac{5}{6}$, and $\frac{5}{12}$), with the expressed direction to the teacher to "include examples of sectors that do not begin at 12:00" (p. 536).

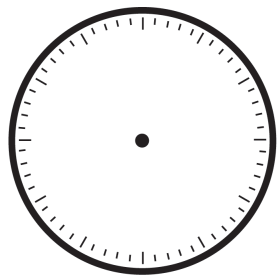


Figure 20. Blank clock face for clock fractions.
Adapted from *Grade 4 Everyday Mathematics Teacher's Lesson Guide*
(Bell et al., 2004b).

Based on a suggestion from her cooperating teacher, Paige modifies the activity. Instead of displaying shaded sectors and asking students to name the fraction, Paige distributes small manipulative clocks to pairs of students and asks them to display various fractions using the moveable hands, beginning with $\frac{1}{2}$. Students hold up their clocks, Paige checks their answers, and then asks them to show $\frac{1}{2}$ a different way. She projects blank clock faces (Figure 20) on the SMART Board and records their answers.

The next fraction Paige poses is $\frac{1}{4}$, which she again asks students to show in two ways. She checks students' displayed clocks and then records the answer on the board, first connecting the fraction to minutes:

Paige	So one-fourth on the clock, how many minutes is that?
Students	Fifteen.
Paige	Fifteen minutes, okay. So we had a hand here and can go like this. [*Draws hands at 12 and 3.*[54]] How many spaces are in between? Or, how many, yeah, how many spaces?
Students	Three.

[54] Although the hours on the clock faces are not numbered, for ease of description, I indicate the position of the hands or partitions by the number of the hour they point to.

Paige	Three. Does everybody see that? Okay. So there's three spaces in between here to the three. And when I asked if you could show other ways I saw some people had, let's just erase this, so some people had their answers here [*erases, and redraws hands at 3 and 6*] or they'd have this one gone and your answers, oops, would be here [*erases, and draws hands at 6 and 9*]. So your answer, your hands could actually be anywhere as long as there are like, let's say, two dots in between. So it could have been here to here [*erases, and draws hands at 1 and 4*], right? Is that still one-fourth of your clock?
Students	Yeah.
Paige	How many minutes are in between here? How many minutes are in between?
Students	Three/Fifteen.
Paige	Fifteen minutes, right? Because each spot is worth five. So five, ten, fifteen minutes.

On the surface, the textbook's version and Paige's modification seem similar, as both activities have students representing fractions with clocks. But there are important differences that impact the mathematics made available by each. One difference is that the two activities go in different "directions": In the textbook activity, the shaded sector is given, and students identify the fraction it represents; in Paige's version, the fraction is given, and students represent it on the clock. Both directions are needed for the upcoming work on addition and subtraction; however, identifying the shaded region is more mathematically complex than representing a given fraction. Representing a given fraction requires partitioning the clock face into the denominator-number of equal parts and then indicating the numerator-number of those parts. In contrast, identifying the fraction that represents a given shaded sector requires determining a way to partition the clock into equal parts so that some subset of those parts measures the shaded sector. This is easier when the entire clock face can be partitioned into equal pieces that are each the size of the shaded sector (e.g., when the shaded sector is $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$, or $\frac{1}{12}$), but is more difficult when the shaded sector does not partition the clock face, and thus smaller-sized pieces have to be used (e.g., when the shaded sector is $\frac{7}{12}$ or $\frac{5}{6}$). This complexity is perhaps why the textbook chose to focus on this direction in the activity.

Second, there are mathematical differences between how fractions can be represented with a drawn clock face and a small clock with moveable hands. The small clock has only two hands, which means they can be positioned to divide the face into two regions. Thus, for fractions other than $\frac{1}{2}$, it is not possible to show all of the equal parts into which the whole is divided. For example, when representing $\frac{1}{4}$, the hands can be placed at, say, 12 and 3, but it is not possible to depict the whole as partitioned into four equal pieces. Furthermore, once the hands are positioned, it is not clear which of the two sectors corresponds to the given fraction. A drawing, on the other hand, can be partitioned into as many sectors as desired and can use shading to identify the sector that corresponds to the fraction being discussed. This issue of which region corresponds to the given fraction arose in the discussion of $\frac{2}{3}$, the last fraction Paige asked students to represent:

After students hold up their clocks to display $\frac{2}{3}$, Paige asks a student to record her answer on the board:

Paige	Could somebody come up here and show me where two-thirds is on our clock? Krista, you want to come up and show?

Krista draws hands at 12 and 4.

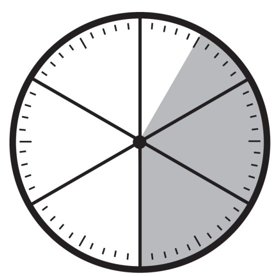
Paige	Okay, and if you're going to shade something, what part would you shade?
Krista	Um, this part right there [*draws a squiggly line in the smaller sector*].
Paige	And how much is that worth?
Krista	Well, like in times or?
Paige	No, in fractions.
Krista	Mmm, two-thirds?
Paige	Is that part worth two-thirds?
Krista	No?
Paige	No. What part is worth two-thirds?
Krista	This part? [*Points to the larger sector.*]
Paige	Exactly, the bigger part, right? So this part right here, okay, you can have a seat, thanks. This part right here [*points to the smaller sector*] this would be one-third, right? And let's say we draw our mark over [*draws another hand, pointing to the 8, so that the clock face is now partitioned into three equal sectors*], we just broke our clock into three thirds, right? So here's one third, two third, three third, so we could either shade in this part right here [*as talks, points to different pairings of the sectors*], or we could shade in this part, or this part right here, okay? So how many minutes is two-thirds? How many minutes is two-thirds? So let's say we got rid of this [*erases the hand pointing to the 4 and shades in the two-thirds of the clock from 12 to 8*], how many minutes is this piece right here? Monica?
Monica	Forty.
Paige	Forty minutes. So two-thirds on the clock is forty minutes.

After this example, Paige has students work independently on a page in their Math Journal. The worksheet has problems in both directions—identifying the fraction that is shaded and shading the clock face to represent the given fraction. Before having students work independently, Paige displays a projected copy of the worksheet and completes the first two problems in whole group. Students complete the next three problems on their own while Paige circulates, answering questions. As students finish, Paige tells them to move onto the bottom half of the worksheet where they shade clock faces to represent given fractions. After about 12 minutes, Paige reconvenes the class to discuss the problems they just completed. Of the three problems discussed, identifying $\frac{5}{12}$ (Figure 21) is the most difficult.

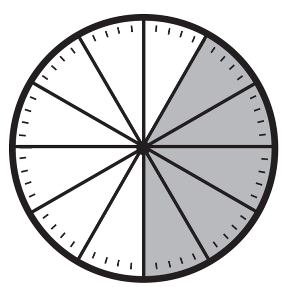
Figure 21. Five-twelfths on the clock face.

Paige calls on Emma to share her answer:

Paige	Emma, did you have this answer? What did you have?
Emma	Five-sixths.
Paige	Five-sixths? Okay, how did you get five-sixths? Could you break, could you break this clock into six different parts? Okay, if we broke the clock into six different parts it would look like this. [*Partitions the clock face into sixths.*] One, two, three, four:

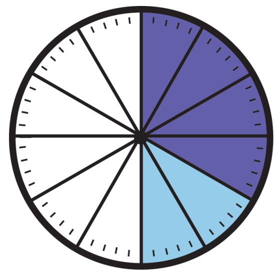


Six parts. Now would five parts out of the six be shaded? No. Okay. But we can break the clock into twelve parts, right? [*Divides each sixth in half to make twelfths.]* One, two, three:



Okay? Then how many parts are shaded? Five. So then our answer would be five-twelfths, okay? Does everybody have five-twelfths?

The class does not discuss the rest of the Math Journal page. Instead, Paige asks students to turn to the next page so that she can introduce adding fractions on a clock. She begins by reviewing the example problem at the top of the page (Figure 22).



$$\frac{1}{3}+\frac{1}{6}=\frac{1}{2}$$

Figure 22. Adding fractions on a clock face.

Paige	Our example says one-third plus one-sixth equals twelve [*sic*]. What color is the one-third? What represents the one-third on here? I'll give you a second to look. What color represents the one-third? Lily?
Lily	Um, the dark blue.
Paige	The dark blue, okay. So this part right here is the one-third. So then this leaves this [*points to the light blue*] to be the one-sixth, okay? Does everybody see that?
Student	Mm-hmm.
Paige	Okay, great. So in here [*points to the given equation*] it says one-third plus one-sixth equals one-half. Okay so they added [*points to each region*] one-third plus a sixth and they got the one-half. Do we see how that works? But let's try the math. Okay, we've been working on adding fractions, right? That's what we did yesterday. Who can, who can come up and show me what one-third plus one-sixth is if we're doing math, if we're showing math. Kevin, why don't you come up and show. If you think you can write small. Maybe write underneath. And maybe tell what you have to do.
Kevin	Okay, we're trying to get this [the 3 in the denominator of 1/3] to be a six.
Paige	Right, so.
Kevin	And then so you would times two. [*Multiplies the numerator and denominator of 1/3 each by 2 to get 2/6, recording the steps throughout his description* (see Figure 23)]
Paige	Great.
Kevin	And then, then it would be two-sixths.
Paige	Okay.
Kevin	And one-sixth…
Paige	Okay, so you add the…
Kevin	And it would just be three-sixths.
Paige	Okay.
Kevin	Which is the same as one-half.
Paige	Perfect. Would you all have been able to, is that how you would do it?
Students	Yeah.

Paige Yeah? So we know because we're super smart and we know this about adding and subtracting fractions, we know that our bottom numbers have to be the same. So Kevin did a really nice job of showing what he would do to the one-third. You times it by two, times by two, the top and the bottom. And then you get two times one is two, two times three is six. You don't have to do anything to one-sixth because it already has six on the bottom. And then he just added straight across, right? And you don't do anything to the bottom. Good.

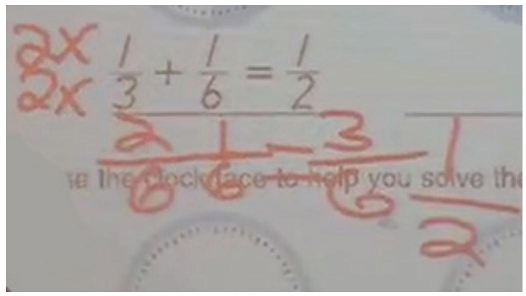


Figure 23. Kevin's "doing the math" for the problem $\frac{1}{3}+\frac{1}{6}$.

Paige completes one more problem together in a similar manner. This problem gives the two shaded regions and students write the number model. After the class identifies the corresponding number model, Paige calls up another student to "do the math" at the board. She then has students work independently on the four remaining fraction addition problems. After about eight minutes, Paige brings the class back together to discuss their work. She starts with problem two (Figure 24).

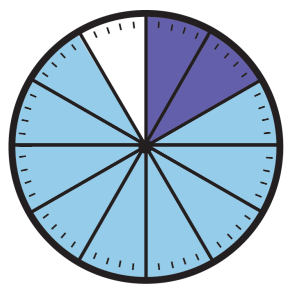


Figure 24. Problem two: $\frac{1}{6}+\frac{3}{4}$.

Paige All right you guys, let's talk about number two. Let's talk about number two. What fraction is our dark part? What fraction is our dark part? Evan?

Evan One-sixth.

Paige One-sixth. Okay, is everybody paying attention up here? [*Labels the dark sector as 1/6.]* Okay, one-sixth. Because we have two parts out of twelve. Two parts out of twelve equals one-sixth. Because two goes into twelve six times, two goes into two one time, so it reduces to one-sixth. [*Records 2/12 = 1/6.]* Okay, how about our light blue part? What fraction is that?...Monica?

Monica Um, nine-twelfths.

Paige Nine-twelfths, okay. [*Records 9/12.*] So we have nine-twelfths and what does that break down into? Evan?

Evan Three-fourths.

Paige Three-fourths. [*Records 9/12 =3/4.*] Okay, because we know that three goes into three, or three goes into nine three times. Three goes into twelve four times, okay? And if we're just looking at our twelfths, though, which we can do, that's just fine. What's two-twelfths plus nine-twelfths? Two-twelfths plus nine-twelfths? Lyle?

Lyle Eleven-twelfths.

Paige Okay, eleven-twelfths. [*Records 11/12.*] So if we just look at our twelfths that's fine, or we could do one-six plus three-fourths and we know we need to change our numbers so that twelve's on the bottom, um, and then, so nine plus two equals eleven.

The remaining problems provide the addition expression and students have to shade the corresponding regions on the clock face and find the sum. Paige asks a student to come to the board and show how she shaded the clock. At this point, Paige has been teaching for over an hour and is running late, so instead of finishing the addition problems, she decides to introduce subtraction. But because she is almost out of time, she does not explain the convert-to-minutes method that she had planned to introduce, and students do not have time to complete any subtraction problems independently. Instead, Paige reviews the example and completes the first two problems in whole group, showing students how to shade the clock face to model subtraction and also "doing the math" to find the answer. She then concludes the lesson:

Paige So do you guys understand how that works? Yeah? And if you want to check to make sure, because I know you guys know how to do your addition and subtraction of fractions, you can do your math first and then figure out your clock. Because you're not always going to have a clock on you, right? Yeah, I mean you might be wearing a watch, but I don't think you're going to sit there and really try to figure it out on your wrist, right? That might be a little hard. So it's good to know the math behind it and figure out what it means.

Paige's Articulation of her Mathematical Point for the Clock Activities

The above section provides a glimpse into Paige's lesson. Before discussing the difficulties she had, it is important to note that there are many aspects of the lesson that went well. For example, Paige chose problems that provided students with opportunities to engage with fraction ideas, such as having students represent $\frac{1}{6}$ and $\frac{2}{12}$ so that she could discuss equivalent fractions; students came up to the board to describe their work; she transitioned smoothly between the numerous activities in her lesson; and she coordinated the use of the manipulative clocks, Math Journal pages, and the SMART board. Mathematically, however, the lesson had a number of problems. For example, Paige's explanations and language were, for the most part, unclear and confusing, which was not surprising given her difficulty explaining the procedures and concepts during the interviews. For example, in the pre-lesson interview I asked how she had explained "changing" fractions in a previous lesson (in her response she uses the example $\frac{1}{3}+\frac{1}{6}$):

> I just said that when we're adding fractions or subtracting fractions they need to be of equal parts. So a third of something you, as it is, as a third, it doesn't, it's just hard, or it doesn't make sense to add it to a sixth, but it makes sense to have, like, a set of sixths, or whatever, added to another set of sixths, so. And I just said that you just have to have the bottom numbers equal. It's just how the math works. Because then it's the, I guess, then it's part of the same whole. It's not like, a third would be part of a different whole than, you know, a sixth, which might not be the right explanation, but. And then I just tell them that whatever you do to the bottom you have to do to the top, too….Like if you multiply the bottom by two, or three, or four, whatever, you have to do the same thing to the top, because you're increasing that fraction by that much, so, but it stays the same fraction. (P-Pre, T126-128)

In fact, Paige was aware of the difficulties she had understanding and explaining the concepts she was teaching. For example, in the post-lesson interview when asked why multiplying the "top and bottom by the same thing" generates an equivalent fraction, she replied: "I honestly don't know, and that's why I'm glad the kids don't ask why. And it's something that I need to figure out and I probably should know better" (P-Post, T237). She also said that she did not "feel very well" about her lesson when asked for her general thoughts at the beginning of the post-lesson interview:

> I just don't think, well, I just don't think, I think I'm kind of a little scattered. Like, I went down their sheets and like through like what the book was, but it just doesn't, I don't know, I don't know whether just I don't like the book, and that's why I have a problem with it. I mean, I haven't seen anything else, but I just, I don't know. I'm afraid that they're not getting the fractions, and I think maybe just for me because, like I said before, I'm very much just the math person, like the strict, just okay, here's our equation, let's solve it and move on. And just, I think all the other stuff just seems to be getting in the way of teaching that. 'Cause I think they get lost in are we learning about time and the clock, or are we learning about fractions? (P-Post, T21)

She elaborated what she meant by being "scattered":

> I just felt, you know, because when you start something, you introduce it to them and then you give time for them to work on it, and then you're all over the place in the room, and kids are, you know, moving on to different parts, and then you have to try and get back to where you were, but the kids are now working on parts on the far and don't want to talk about it, so then I'm trying to move forward with the lesson and like get to another part, but I may not think that they still get the one part, and it's just like, so, so much back and forth. Okay, here, I'm going to show you this, like let's do one together, let's you

guys try some, let's try and get back, let's keep moving on, like back and forth, and I just don't, I don't know whether I lost the like whole point of the lesson. (P-Post, T25)

I pressed on this issue further:

Interviewer	And so what do you think the whole point of the lesson was?
Paige	I don't even know anymore.
Interviewer	Well what was <u>your</u> point?
Paige	I don't even know. I guess, just I, with, just showing fractions or adding fractions because that, and I think that's what like, I guess, I would want them to be able to do, is add and subtract fractions because I, maybe that's just because the part, the lesson before, was adding and subtracting fractions, but, I don't, I don't know.
Interviewer	What makes you say that you feel like you lost the point?
Paige	Because I just feel like I was, so just, I just seemed, like I said, all over the place, or just, like I wasn't being clear enough, or I don't know, I think I maybe wasn't even thinking at all and that's why I just, I don't know. (Post, T26-31)

It would be easy to attribute the mathematical problems in Paige's lesson solely to her inability to explain the procedures she was teaching. But mathematical purposing and the knowledge demands of this work provide another lens for gaining insight into the difficulties she had. Paige's comment that she felt like she "lost the whole point of the lesson" is particularly telling; however, it is not clear that Paige ever had a clear sense of the mathematical point of her lesson—especially of the mathematical point of the clock representation and how it connected to the ideas about fractions that she was trying to teach. Overall, there seemed to be a mismatch between her use of the clock in the various activities and her mathematical learning goals for students.

I used the framework for mathematical purposing and its MKT demands to explore this issue. By applying a mathematics, learners, and focusing lens (i.e., the articulating component of the framework for mathematical purposing (Figure 17)) to Paige's pre-lesson interview, I created a portrait of Paige's articulation of the mathematical point of the clock. From my analysis of the interview, I made a map of how Paige viewed the mathematical terrain and how she situated the clock fraction activities in it (Figure 25). Second, I analyzed this map and the pre-lesson interview using the framework for mathematical purposing to determine in what aspects of the work Paige did and did not engage and what aspects of MKT she did not seem to know or be able to draw upon in practice.

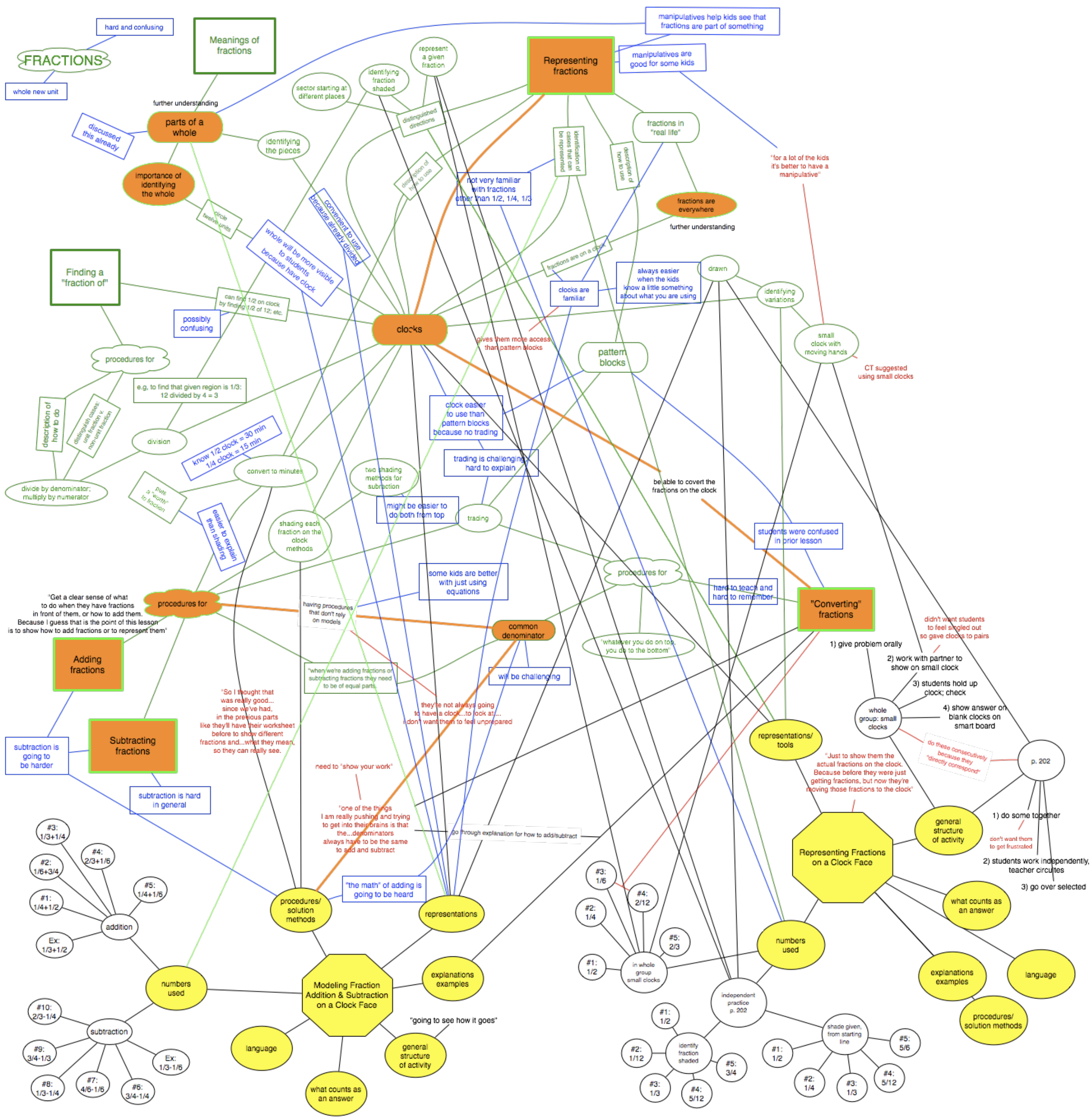


Figure 25. Paige's trifocal view of the mathematical terrain related to the two clock fractions activities.[55,56]

[55]Key: mathematics lens = green; learners lens = blue; focusing lens (specified learning goals) = orange; focusing lens ("points" of details) = red; activity = yellow/black

[56] Unfortunately, printing the map on a single page makes it difficult to read. I include it here only to illustrate this type of analysis and resulting product.

Figure 25 shows the aspects of the mathematical terrain (related to the clock fractions activities) that Paige mentioned in her pre-lesson interview. The colors indicate the different lenses: green for mathematics, blue for learners, and orange for focusing. Thus, it is an example of the type of annotated map of the mathematical terrain that Grossman (1991) analogized in her discussion of pedagogical content knowledge that I mentioned in Chapter 6. Such a map is useful for a number of reasons. It reduces the long, text-heavy interview to a one-page snapshot of Paige's view of the mathematical terrain and the connection to the instructional activities. The map also makes visible the ways in which the three lenses inform one another and how analyzing the terrain or the activity can serve to unpack the mathematics of a lesson.

As with any representation, there are also a number of things that are not depicted. For instance, the map does not indicate distance. That is, although the lines show connections, they do not convey how closely connected Paige viewed particular mathematical ideas, either for herself or for her learners. The map does not indicate how Paige unpacked the terrain (e.g., whether an aspect of the terrain was identified from thinking about fractions in the abstract or from examining the activities). The map also does not reflect aspects of the terrain that Paige did not mention or know (e.g., it does not note that Paige could not explain the procedure for generating equivalent fractions). Therefore, to analyze Paige's understanding of the mathematical point of the clock, I compared both the map and the pre-lesson interview to the framework for mathematical purposing. I noted which aspects of the work Paige seemed to have done and which aspects she did not or was unable to do. The results of this analysis are shown in Table 18.

Table 18.
Paige's Articulation of the Mathematical Point of the Clock in the Pre-Lesson Interview

	Mathematical Terrain	Instructional Activity
Mathematics Lens	Identified some central fraction concepts (attending to whole; fractions as parts of whole; equivalence (though used language of "converting"); no mention of equal parts. Described and compared representations for fractions (clocks and pattern blocks) and identified multiple methods for how to use them (partitioning region; converting to minutes; finding "fraction of"). Described different procedures for adding/subtracting fractions (combining/taking away areas (multiple ways); converting to minutes; standard algorithm); and procedure for "converting fractions" (multiply "top and bottom" by same number).	Noted that clock makes whole visible; noted that clocks are already partitioned, but did not examine impact on the mathematics available. Recognized which fractions can be represented on a clock; distinguished directions of representing (identify shaded versus represent given; sector starting a different places). Could map the quantities and answer in the combining/taking away areas methods onto the clock representation, but did not examine whether/how it gives meaning to procedure; did not connect standard algorithms for adding/subtraction or generating equivalent fractions.

	Could not explain why standard procedures work.	Did not unpack all of the steps for both directions (e.g., given 2/3 shaded, how do you determine it is 2/3?). Did not examine the mathematics differently available through small clocks and drawn clock faces (e.g., did not relate this to directions; or that cannot show all of the partitions on the small clock that only has two hands). Did not examine how the different procedures engage students differently with fraction concepts; did not unpack key aspects of explanation. Identified being able to name the minute equivalents for fractions as a prerequisite for the convert-to-minutes method. Language imprecise and confusing; did not examine language for distortion.
Learners Lens	Knows that students are familiar with ½, $^{1}/_{3}$, and ¼ ; but less familiar with other fractions. Knows that students are familiar with clocks, and that students know minute equivalents for ½ hour and ¼ hour, but will be less likely to think about hours in thirds. Thinks that manipulatives are good for some students because they help them see that fractions are part of something. But thinks that some students are better at just using equations. Identified some general student difficulties: fractions are confusing for students; subtraction is harder than addition. Identified current experiences and level of understanding (at a surface level): new unit for students on fractions; had talked in prior lesson about parts of a whole and "converting fractions." Students were confused about this with pattern blocks[57]; found pattern blocks challenging because of trading.	Evaluated accessibility of clock representation: whole will be more visible than with pattern blocks; convenient that clocks are already partitioned. Considered difficulty/accessibility of methods for adding/subtraction (thought converting to minutes more accessible and easier – "shows the worth"); thought "the math" (i.e., standard algorithm) would be difficult. Did not consider the difference in difficulty of the two directions for representing fractions on the clocks. Did not unpack how students would figure out what fraction a shaded region represented.

[57] Pattern blocks are colored shapes that can be used to teach a variety of concepts related to geometry and number. In the prior lessons Paige mentioned in her interviews, students used the following blocks to represent, add, and subtract fractions: the yellow regular hexagon, the red trapezoid (1/2 of the hexagon), the blue rhombus (1/3 of the hexagon); and the green equilateral triangle (1/6 of the hexagon).

	Did not consider “cognitive distance” or mathematical complexity of ideas from perspective of learner; seemed to think ideas were closer and more connected for students than they likely are (e.g., thought students would figure out that a given sector was 1/3 by doing 12 divided by 4).	
Focusing Lens	Ongoing goals: • Seemed to have overarching goal of students’ having procedures that do not rely on models. • No mention of mathematical practices, reasoning, or explaining. • Did not look at unit goals in textbook. Trajectory (for adding and subtracting fractions): • Had a mathematical end point: learning the algorithm (finding common denominators by algorithm for generating equivalent fractions; add/subtract across numerators). • Did not examine or have a sense of the curricular trajectory to get students to that mathematical end point or how the ideas would unfold and build on each other; rushed the end to the beginning. Identified some learning goals for students: • To develop further understanding of fractions as parts of a whole and that fractions are everywhere (very underspecified) • To be able to represent fractions on a clock • To be able to “convert” fractions (e.g., to simplify using the equivalent fractions procedure) • To be able to add and subtract fractions using common denominators	Mathematical storyline: • Made connections across activities in lesson (e.g., foreshadowed minutes because was planning to teach the convert-to-minutes procedure; designed math message problems to give practice with fractions/minutes). • Ordered the activities so that like work was adjacent in the lesson. • Did not look at the rest of the unit before teaching it. Main mathematical point of the representing fractions activity: • To “show them…the actual fractions on the clock” (P-Pre, T100) Main mathematical point of the adding/subtracting activity: • To “show them the addition on the clock” • To “show subtracting fractions and actually seeing it in front of them” • To “go through the conversion” • To “get into their brains…that the bottom numbers need to be the same, the denominators always have to be the same to add and subtract” (P-Pre, T124, 192) “Point” of some of the details of the activity: • To make things accessible • To focus on “the math” (i.e., the standard procedures without models); to show your work • To not have students be frustrated • To do related activities in close proximity in lesson

Discussion: Knowledge Demands of Articulating the Mathematical Point

Based on the analysis described above, I identified three aspects of the work of articulating the mathematical point where Paige’s lack of MKT, in particular her lack of specialized content knowledge (SCK) and knowledge of content and curriculum (KCC), seemed to play a central role in the difficulties she had in her lesson: distinguishing the mathematical and curricular trajectory; distinguishing means and ends; and coordinating the availability and accessibility of the mathematics in the details of the activity.

Distinguishing the mathematical and curricular trajectory. Throughout the interviews, Paige repeatedly complained that the textbook was trying to do too much in one lesson:

> I hate how fast it is….Not only are we looking at what fractions mean on, or like showing fractions on a clock, like taking a piece and showing what that means out of a whole, but then you're taking that and they're having them draw them, they're having them label them, then you're having to show addition of fractions. So that's a whole, that's like a whole lesson right there, I think, is showing the addition of fractions with this. But then you have to move on to subtraction of fractions. And not only are you just doing the math part of it, but you're showing, you know, on the clock how to take the part and then shade another part of it, and like the part you have left. Like there's just so much to explain, and I just feel like a day, like, or fifty minutes isn't, I don't know. (P-Post, T35)

It is true that *Everyday Mathematics* includes many different activities in each lesson, which can certainly make it feel like there is too much to do in one math period. However, Paige seemed to compound this issue because she was unable to distinguish the *mathematical* trajectory related to adding and subtracting fractions from the *curricular* trajectory. For example, Paige knew that learning to add and subtract fractions involves being able to generate equivalent fractions and being able to use the standard algorithm; however, she did not have an understanding of this trajectory from the learners' perspective, in particular, how ideas need to be spaced out and developed for learning. In the pre-lesson interview, Paige described why she thought discussing equivalent fractions was important to include in her lesson:

> Because they're not always going to have a clock in front of them and they're not going to always be able to see that, hey, like, you know, this and this. It doesn't make sense to just show one-third plus one-sixth equals one-half, without showing your work. At least that's how I was always taught math, like, you need to have your work down. So, I mean, for everything that we pretty much do I always have them, I'll always show them the work that goes along with it. Because it does, they're not always going to have a clock or pattern blocks, or you know, something that is in front of them to look at. They may just have one-half plus one-sixth equals what? And if they don't have something in front of them that shows it to them, I don't want them to feel unprepared. Plus, I just don't, I get the using examples and stuff, but I'm very much like, and I know that there's kids in my class who are better with just equations or with just, you know, the straight math of it, so and that's how I always was, so. I mean, you'll probably see me, I'll probably struggle explaining some of this stuff, like I am right now to you just because it just, this is easier than the lesson before it, but it still is just, I'd much rather just be doing the math part of it and saying, "All right! Here we go!" (P-Pre, T130)

Wanting students to be able to add and subtract fractions without manipulatives is a legitimate goal. However, Paige does not seem to understand that other understandings need to be developed *before* students can get there. Lacking this aspect of KCC, Paige's lesson goals began at the mathematical end point, collapsing the entire mathematical trajectory into each lesson rather than developing it over time.

As a beginning teacher, it is not surprising that Paige does not have knowledge of the curricular trajectory. But she also did not seek to gather this information, for example, by reading the unit goals or the other lessons in the unit—in fact, she said that looking ahead "frightens" her (P-Post, T43). Ironically, in this case, looking ahead, or at least looking more closely at the notes in the teacher's guide, might have helped her feel less pressure to teach the entire mathematical trajectory in a single lesson. There are notes throughout the unit and in the clock fractions lesson emphasizing that, in fourth grade, the goal is not to teach the standard algorithm for adding and subtracting fractions, but "to provide initial exposure to these kinds of problems in a concrete context" (Bell et al., 2004b, p. 539). The teacher's guide also expressly states in a margin note that: "Equivalent fractions are addressed later in this unit. For now, simply note that a given sector can often have more than one fraction name, so many of these problems can have more than one correct answer" (p. 537). Important to note, however, is that even if she had read these remarks, understanding what they mean demands MKT, in particular SCK and KCC.

Knowledge of the curricular trajectory also could have also helped her determine the mathematical point of particular problems in her lesson. For example, I followed up on Paige's comment that the textbook did not teach "converting fractions…but they do it all the time" (P-Post, T35) by asking where in the lesson she thought the textbook was "converting." She indicated a problem that showed $\frac{3}{4}$ of a clock face shaded and asked students to identify the fraction:

> The three-fourths, a lot of them found that as nine-twelfths…So if they have nine-twelfths, like then I'm trying to get them to say three-fourths, but they don't know it. They like, they don't know how to do it. But that's, that's what it has in the book. It has three-fourths. And if they're looking at it another way and don't know how to reduce, you're just like, all right, so how does that work. So they haven't learned. (P-Post, T39)

Because the teacher's guide had $\frac{3}{4}$ as the suggested answer, when her students instead answered $\frac{9}{12}$, Paige thought they needed to "reduce." She was frustrated by this because the procedure for simplifying fractions had not yet been introduced in the textbook. Paige's response to this issue was to teach her students to divide the numerator and denominator by the same number. However, if she would have understood the curricular trajectory and known that this procedure was taught later, she might have been prompted to consider how students could see the answer as both $\frac{3}{4}$ and $\frac{9}{12}$ *without* using this procedure. For example, by partitioning the clock face so that both four equal pieces and twelve equal pieces are visible (Figure 26), the same shaded region can be described as three out of four equal pieces or as nine out of twelve equal pieces. Such an analysis could have helped Paige unpack some of the foundational fractional concepts along the curricular trajectory rather than jumping to the end of the mathematical trajectory.

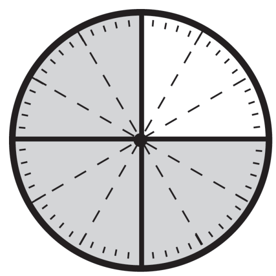


Figure 26. Showing that three-fourths and nine-twelfths are equivalent.

Distinguishing means and ends. Paige considered representing fractions on the clock, as well as learning the standard procedures for generating equivalent fractions and for adding and subtracting fractions, to be the end learning goals for students in her lesson. This was evident in the description of her main point for the addition segment of her lesson:

> I think just to show them the addition on the clock and to show how the fractions, like how, for example…for the one on the bottom to show how the one-third plus the one-sixth equal one-half. And then I'll probably just, I'll go through the conversion of one-third to have the six on the bottoms, because that was one of the things I'm really pushing and trying to get into their brains is that the bottom numbers need to be the same, the denominators always have to be the same to add and subtract. So we'll go through how, you know, the one-third will change to two-sixths, and then the two plus one equals three, and then the six on the bottom and how that's one-half, and then it will be right there on their clock. (P-Pre, T124)

The textbook, on the other hand, does not have these same concepts and skills as end goals for the lesson, or even for the unit. Instead, the textbook uses representing, adding, and subtracting fractions on the clock as a means for working on other fraction concepts. As mentioned earlier, throughout the teacher's guide are notes saying that teaching standard procedures for equivalent fractions and for adding and subtracting fractions is not the goal of the lesson. Certainly, the spiral nature of the curriculum makes this a little confusing—for example, even though the procedure is not taught until fifth grade, adding and subtracting fractions is listed as a "beginning goal" for the unit. However, the nature of the lesson's activities suggests that addition and subtraction is not the end goal, but is serving as a context for work on more foundational fraction concepts. That is, for the textbook, addition and subtraction on clocks is the means rather than the end.

I am not implying that Paige needed to adhere to the textbook and therefore not introduce the standard procedure for adding and subtracting fractions. However, if teaching the addition and subtraction algorithm was Paige's main mathematical point, then her use of the clock representation should have supported that goal. Because Paige was unable to distinguish that the textbook was using both the clock and addition and subtraction as means for learning fraction concepts rather than as end learning goals (a distinction that requires MKT to make), there was a mismatch between the textbook's activities and the mathematics she was trying to teach.

Coordinating the availability and accessibility of mathematics in the details of the activity. Paige made many instructional decisions based on what she thought would make the activities more accessible to her students rather than on what mathematics was made available for study. When planning her lesson, Paige did consider some aspects of the mathematics made available by the clock representation. In particular, she thought that using a clock made the whole visible, thus helping students see that fractions are "part of something" (P-Pre, T4). However, her main reasons for using clocks in the lesson were based on accessibility to the learner (e.g., "they need to have something in front of them to look at" (P-Pre, T132)). Similarly, although her comparison of clocks to pattern blocks (which were used in the previous lesson) noted some of the mathematical differences between the materials (e.g., pattern blocks require trading (P-Pre, T32)), her preference for clocks was based on her perceived difficulty of pattern blocks both for students to understand and for her to explain (P-Pre, T134), and on students' familiarity with clocks:

> I hadn't thought of using a clock before until I, you know, turned to my next lesson, and I said, hey, that makes a little more sense to me. Because they see clocks, like, there's, you know, clocks in their houses, clocks at school, and you know, in stores or wherever. And I think it's just something that I think that they'd have more access to if they're looking at something than pattern blocks. (P-Pre, T136)

Throughout the interviews, Paige did notice details of the representations that impacted the mathematics. However, in making instructional decisions, she did not consider how these features impacted the mathematics made available for study. Instead, Paige made choices based on what she thought would be more accessible to her students. For example, she noticed that the clock had "built-in divisions." She liked this feature because it was convenient for students (P-Pre, T138), but did not consider the impact of "built-in divisions" on the mathematics made available for study. For instance, although having the minute and hour marks facilitates more accurate drawings of fractional regions, the fact that clocks are pre-marked can mask the central idea that fractions require equal parts. In fact, Paige never once mentioned "equal parts" during any of the whole-group portions of her lesson.

Another mathematical issue that was not considered is that clocks can, in some sense, be interpreted as an area, linear, or discrete model for fractions: As an area model, the area of the clock face can be seen as the whole. If just considering the edge of the clock, the clock can be seen as a portion of the number line, for example, as the unit interval or as the interval from 0 to 12. But unlike the number line, it is uncommon to partition the clock into a number of pieces that is not a factor of 60. Finally, when considering the whole to be 12 hours or 60 minutes, a clock functions like a discrete (or set) model in which the whole is 12 or 60 objects. This use can be seen in Paige's connection between the "fraction of" problems (e.g., $\frac{1}{6}$ of 12) and the clock. Such mathematical features of a representation are not inherently good or bad. In this case, depending on the implementation, the fact that clocks can be interpreted as different types of models could provide an opportunity to make connections across fraction models, or it could obscure distinctions and cause confusion, especially when there is an implicit (or inadvertent) shift in model type. In Paige's lesson, because she did not analyze the impact of these mathematical features on the mathematics made available, her accessibility-based decisions did not always further students' engagement in the intended mathematics.

Paige's consideration of accessibility over the mathematics made available was also seen in the addition/subtraction activity. Paige was worried because she thought subtraction would be harder for students than addition, but was somewhat relieved by the method for subtraction on the clock that was described in the teacher's guide: convert the shaded region to minutes, subtract the minutes, and then convert back to fractions. She thought this method would be easier for her to explain and for her students to understand (P-Pre, T168). She did not, however, consider whether this method engaged students in the fraction concepts she was trying to teach. Paige's main mathematical point was to teach students the standard algorithm. Although converting to minutes can be seen as making a common denominator of 60, this is very subtle and masked by thinking of the whole as 60 minutes rather than as one hour. Thus, although converting to minutes might be a more accessible method for students, it obscures some of the key mathematical ideas Paige intended to teach.

Considering the accessibility of the details of a task for students is, of course, an extremely important part of mathematical purposing. However, mathematical purposing involves considering <u>both</u> the mathematics made available for study and the accessibility of that mathematics to the learners. As Paige's case shows, coordinating these to determine how the details of a task can best be used to teach to the mathematical point depends on MKT.

In sum, Paige's case illustrates some of the mathematical knowledge demands of an aspect of mathematical purposing: articulating the mathematical point of a representation. Using the lens of mathematical purposing provides insight into the difficulties Paige had in her lesson beyond noting that she was unable to explain the concepts she was teaching. In addition to illustrating some of the MKT demands of mathematical purposing, this case also demonstrates the use of the framework as a tool in data analysis. I continue the discussion of possible uses of the framework in research and teacher education in the next chapter.

CHAPTER EIGHT:
CONCLUSIONS AND DIRECTIONS FOR FUTURE RESEARCH

Summary of Dissertation

Even though it is fundamental to the work of teaching, what is involved in determining the mathematical goals of an activity and using them to design and steer instruction has not been well-specified in the research literature or in teacher education. A better articulation of the work of determining and using mathematical goals in teaching is crucial both for studying the relationship between instruction and student achievement and for teaching novices how to do this complicated work. Many different bodies of literature inform an understanding of this task of teaching, but unlike this dissertation, none foreground it as the central object of study.

In this dissertation study, I analyzed both the literature and data from preservice teachers' mathematics lessons to unpack the practices and knowledge demands of determining the mathematical goals of an activity and using those goals to design and steer instruction. This led to a conceptualization of "teaching to the mathematical point" as three interrelated types of work: (1) articulating the mathematical point; (2) orienting the instructional activity; and (3) steering the instruction toward the mathematical point. I identified some of the problems teachers have to manage when trying to steer instruction toward its mathematical point, as well as some of the issues that arise for beginners. From these analyses, I developed a conceptual framework that parses the work of articulating the mathematical point and orienting the instructional activity—what I have named "mathematical purposing." In addition to decomposing this work, I explored the relationship of mathematical purposing to mathematical knowledge for teaching by identifying the MKT demands of different components of mathematical purposing. These results contribute to the ongoing development of the practice-based theory of MKT.

What is the Point of this Work?
Potential Contributions to Research in Education

On one level, this dissertation can be seen as an application and extension of some of the current ideas in education research. In particular, I took up Grossman and McDonald's (2008) call for the decomposition of teaching practice and the development of a language for teaching by applying these ideas to the work of determining and using mathematical goals in instruction. I also utilized Ball et al.'s (2008) practice-based theory of mathematical knowledge for teaching to analyze the mathematical knowledge demands of this work. This analysis resulted in hypotheses about the nature of knowledge of content and curriculum (KCC).

But what about what I consider to be the central findings of this dissertation: the conceptualization of teaching to the mathematical point and the framework for the work of mathematical purposing? Unlike other products of research that have "industry standards" for evaluation, evaluating this type of conceptual analytic work is not so straightforward. However, one way to evaluate the results of this dissertation is by examining its usefulness for and potential contributions to mathematics education scholarship, and perhaps even education scholarship more broadly.

Silver and Herbst (2007) characterize mathematics education scholarship as a set of relationships between *research*, *practices*, and *problems*, which they depict in a "scholarship triangle" (Figure 27a). The bidirectional arrows imply that each aspect of scholarship informs the other. For example, new practices in mathematics education can arise in response to research or to perceived problems; in the other direction, existing practices can pose problems or be the subject of research.

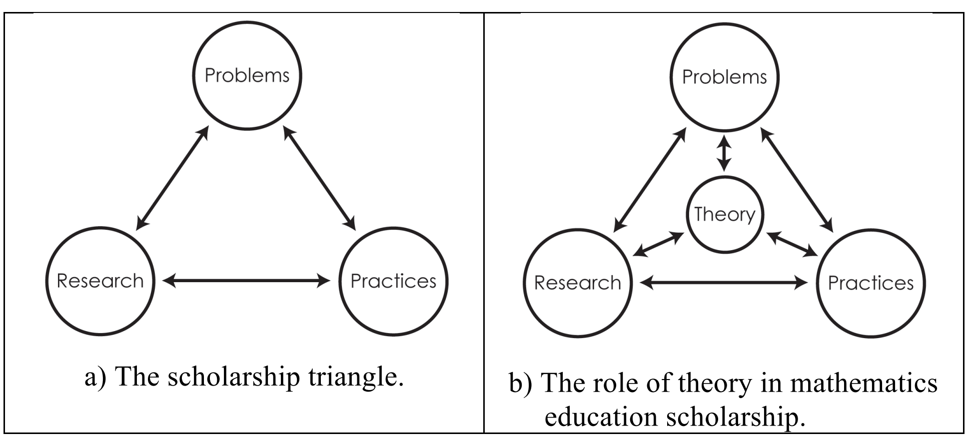


Figure 27. The role of theory in mediating the research, problems, and practices in mathematics education. Adapted from Silver and Herbst (2007).

Silver and Herbst conceptualize the role of *theory* in mathematics education scholarship as mediating the interactions in the scholarship triangle (Figure 27b). They elaborate this frame by identifying specific ways in which theory can mediate connections between the three pairs of vertices in the triangle, illustrating each with examples from the literature. Because this dissertation is conceptual work, it has the potential to contribute to the development of "local theories" that "help mediate specific connections among the three vertices of the scholarship triangle" (Silver & Herbst, 2007, p. 60). In this section, I use Silver and Herbst's frame to explore this study's potential contributions to mathematics education scholarship. I then consider possible applications to education scholarship more broadly. I conclude the chapter with ideas for specific next steps in this line of work.

Mediating Connections between Research and Problems

One way theory can mediate the connection between research and problems is "to provide ways of examining and transforming a problem that had initially been formulated through common sense, turning the problem into a researchable problem" (Silver & Herbst, 2007, p. 48). The example Silver and Herbst provide to describe this use of theory in mathematics education scholarship is Ball et al.'s efforts to study the relationship between teachers' knowledge of mathematics and instruction. Theory enabled a reconceptualization of the nature of the mathematical knowledge needed in teaching, which led to the development of different kinds of measurement instruments to research the problem of teacher knowledge.

Similarly, the results of this dissertation have the potential to contribute to theory that could connect commonsense problems and research. The conceptualization of teaching to the mathematical point and, in particular, the conceptualization of its subcomponent, mathematical purposing, has turned the important, yet underspecified, notion that mathematics teaching involves determining mathematical learning goals and using them to design and implement instruction into an object that can be studied. Naming and defining the work of mathematical purposing helps mediate research on a variety of commonsensical problems, such as investigating the relationship between purposeful instruction and student learning in mathematics. For example, now that mathematical purposing is an "object," instruments can be designed to try to measure it. Such measures could then be used to study its relationship to mathematical knowledge for teaching, the mathematical quality of instruction, and/or student achievement.

Mediating Connections Between Research and Practice

According to Silver and Herbst, one way theory can mediate connections between research and practice is to help research understand practice by providing a "language of description of an educational practice" (p. 56). The conceptualization of teaching to the mathematical point and mathematical purposing developed in this dissertation could help make connections between research and practice in this manner.

The conceptualization of teaching to the mathematical point in this dissertation and the resulting framework for mathematical purposing provide a different lens for viewing teaching. The focus of the framework is not on the mathematics task, the mathematical terrain, the student learning goals, or how the task is planned to be implemented; the focus is on the *connection* among these. This has a number of implications for helping research understand practice. For example, the framework for mathematical purposing shows that, in practice, analyzing the connection between the details of the task and the student learning goals could simultaneously unpack the mathematical terrain. In addition, the detailed description of the work of teaching represented in the framework makes visible aspects of practice that might be missed with other lenses, thus helping research have a more nuanced view of teaching. For instance, the framework enabled me to notice productive instincts and sensibilities in preservice teaching, even when the enactment seemed to steer an activity in an unproductive mathematical direction. A number of examples of this were seen in Chapter 5. For instance, although Nicole's recording of "Th" for "tenth" in her decimal lesson was potentially confusing mathematically, her use of recording to emphasize a key mathematical idea can be a productive teaching move. By naming these types of moves, the framework helps observers of practice distinguish between the intent of a move and the way it is implemented.

Silver and Herbst note that there is a fine line between theory as a tool for understanding practice and theory as a prescription for practice. If the work of mathematical purposing were further researched and connected to improved quality of mathematics instruction, then a framework for mathematical purposing could ultimately reflect a prescription for practice. Such a framework could inform the evaluation of teaching or instructional materials. With additional research into the ways the work of mathematical purposing can be distributed, suggestions could be made about how the work is best distributed in particular contexts or inform the design of educative curriculum materials.

Another way theory can connect research and practice is through "an organization of by-products of research that practitioners might use" (Silver & Herbst, 2007, p. 55). Silver and Herbst illustrate this use of theory with an example from Stein and colleagues' work on mathematics tasks (Stein et al., 1996; Stein et al., 2000), in particular, their development of the *Mathematical Tasks Framework* (MTF). The MTF theorizes about the influence of mathematics tasks on student learning by depicting the role of the teacher in shaping the nature of the task through the stages of its implementation. Silver and Herbst explain how the MTF has been used to develop narrative cases for teacher education and professional development, and argue that "this example raises to visibility the potentially important role that tools, such as frameworks, can play in the work of theory mediating connections between research and practice" (p. 56).

The work in this dissertation could similarly be used to develop tools for practice, particularly for use in teacher education. For example, Hiebert and colleagues (2007) argue for the design of teacher education programs that prepare teachers to learn from teaching, and propose "analyzing teaching in terms of student learning" as one of the core practices needed for teachers to continue learning from their teaching once they are in the classroom. They identify four skills that need to be developed in order to learn to analyze teaching in terms of student learning, the first of which is "specifying learning goals." The four skills would become the learning goals for teacher education programs. Hiebert et al. argue that the first step in realizing this type of teacher education program is to specify the *teacher education* learning goals more precisely, which involves "unpacking the four skills into more detailed and precise component skills and examining how the component skills can be developed and recomposed" (p. 59). The work of this dissertation could help translate Hiebert et al.'s proposal into teacher education practice by specifying what might be component skills of the teacher education learning goal of learning to specify learning goals.

The results of this dissertation could also lead to the development of tools that teacher educators could use to help preservice teachers learn to specify student learning goals or to develop their skills with other aspects of teaching to the mathematical point. For example, the framework could be used to create tools for analyzing or reflecting on practice. Such tools could be used in teacher education courses to study records of practice, to support preservice teachers' reflection on their own teaching, or to assist field instructors when debriefing observed lessons. I discuss more specific ideas for teacher education tools at the end of the chapter.

One interesting thing to note is how language plays a role in mediating connections between both research and problems and research and practice. With respect to this dissertation, the role language plays in each is slightly different. In connecting research and problems, language helps compress a complicated set of ideas into an object that can be studied. In connecting research and practice, language opens up and names the work of teaching to enable more nuanced lenses for interpreting practice and the development of tools for teaching beginners to do the work. Thus, the language of the dissertation might be used differently and be differently useful across contexts. For example, although the language of "mathematical purposing" might provide useful shorthand for research, in teacher education it might be preferable to talk about articulating the mathematical point of an instructional activity and orienting the details of the activity toward it. In fact, the everydayness of "mathematical point" and "orienting" is part of what might make this language accessible for and useful in teacher education.

Mediating Connections Between Practice and Problems

Silver and Herbst propose that theory can mediate the connection between practice and problems by being "a proposed solution to a problem of practice" (p. 59). In this role, theory provides a way of understanding a problem so that a response can be developed in practice. The work in this dissertation could contribute to the development of proposed solutions to the problem of beginners learning to navigate the complexities of in-the-moment teaching. A hypothesis of this study is that having a deeper and more nuanced understanding of the mathematical point of an instructional activity and its details would help beginners manage the interactive aspects of mathematics instruction. This way of conceptualizing the problem of navigating in-the-moment teaching could lead to the development of practices that support beginners in doing this work. For example, one of the challenges of managing instruction is remembering the many different things that need to be tracked on. One idea for practice that emerges from viewing the problem through the lens of needing a better understanding of the mathematical point of an activity is the development of "bundled" teaching moves—that is, sets of teaching moves that are bundled together with a particular mathematical purpose. Another example is the idea of developing question packages, which was discussed in Chapter 6.

Applications to Education Scholarship More Broadly

The conceptualization of teaching to the mathematical point and the framework for mathematical purposing developed in this study could also be extended to and then used to mediate education scholarship in subjects other than mathematics. For example, in history instruction, one could conceive of "teaching to the historical point" or "historical purposing." Like mathematical purposing, "purposing" in other subjects would focus on the connection between the instructional activity, the content it is intended to teach, and how the details of the instructional activity are oriented so that it is more likely to engage students with that intended content. Although the overall conceptualization of "teaching to the point" seems readily extendable to other subject areas, further research would be needed to examine more closely the analogous work and to develop a framework for the work of "purposing" in other subject areas, as well as to identify the problems in and strategies for steering instruction. Once unpacked and articulated, frameworks in other subjects could be similarly used to mediate research, practice, and problems in education scholarship.

But unlike this dissertation, these new studies would not need to start from scratch. For example, it seems like the basic structure of the work of articulating the mathematical point as a trifocal analysis of the mathematical terrain and the instructional activity would also be applicable to "articulating the [other subject] point," except that the work would involve a trifocal analysis of the terrain in another content domain and the instructional activity using lenses of the other subject, learners, and focusing. In that sense, the framework developed in this study provides a kind of graphic organizer for decomposing the work of purposing in relation to teaching other subjects. Furthermore, the decomposition of work in this dissertation might help make visible parallel aspects of the work in other subject areas and provide language for naming this work.

In turn, decomposing the work of purposing in other-subject-area instruction would inform an understanding of mathematical purposing. First, if the framework for mathematical purposing was usefully extended to teaching other subjects, this would provide information about the robustness of the framework and evidence of its usefulness beyond this study. Second, if there are components of the framework that are similar across the teaching of different subject areas, this would be a way to help identify which components of mathematical purposing are more high-leverage (Ball et al., 2009) for pre-service elementary teachers. In other words, overlapping components would help identify aspects of the work of teaching that apply across subjects, which can be strategic sites for elementary teacher education because of the need to prepare beginners to teach all subjects in a limited amount of time.

The End is Really the Beginning: Next Steps in this Line of Work

In many ways, this dissertation is about setting the stage for future work. Conceptualizing teaching to the mathematical point and beginning to unpack the work of mathematical purposing enables these aspects of teaching to be studied. The above discussion of this study's potential contributions to education scholarship points to a number of concrete next steps in this line of work. I briefly discuss some of these below.

Use the framework to analyze preservice teachers' mathematical purposing. One next step would be to reanalyze the data from preservice teachers' lessons using the framework for mathematical purposing. For this dissertation, the data were analyzed to *develop* the framework,[58] which meant that instances of preservice teachers' practice were used to spur ideas about what could be the work of mathematical purposing, not to determine whether an episode was an example of mathematical purposing being done or being done well. In many cases, it was the

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[58] The case from Paige's lesson is an exception. In this case, I used the developed framework to analyze her articulation of the mathematical point of a representation. Thus, Chapter 7 shows an example of an application of the framework for data analysis.

absence of a move that was coded. Therefore, a possible next step would be to develop a way to use the framework to code whether a particular aspect of mathematical purposing was present and whether it was done in a productive way.[59]

The results of this analysis could be quantified and then examined for trends in preservice teacher practice. Such descriptions of preservice teaching, while not generalizable, could help teacher educators learn more about what preservice teachers bring to student teaching and suggest productive directions for future research. Preservice teachers' mathematical purposing could also be compared with their MKT scores to see whether there are differences in mathematical purposing related to mathematical knowledge for teaching. I could analyze the mathematical quality of the lessons and compare this to the mathematical purposing analysis. Finally, I could develop a framework for coding the steering of instruction toward its mathematical point and look for relationships between mathematical purposing (as seen in the interviews) and steering.

Develop measures for knowledge of content and curriculum (KCC). Another next step would be to try to develop measures for KCC. These could be multiple-choice items like those on the survey used in this study or open-ended items, perhaps involving the examination of curriculum materials or the description of possible curricular trajectories for teaching a particular topic. Looking at the list of proposed knowledge, reasoning, and dispositions to be included in KCC (from Chapter 7) gives ideas about the content of items that might be developed. This list also signals a likely challenge of developing such measures: Although some aspects of KCC may be general (i.e., apply across contexts), much of the knowledge and reasoning used in teaching to the mathematical point involves matching intended mathematics to particular instructional activities to be engaged in with particular students. For example, what is taught at different grade levels depends on the particular district or state; what makes a good learning goal for an activity depends on factors such as students' prior knowledge and the amount of time that will be allotted. I do think it would be possible though, with adequate framing, to develop scenarios that sufficiently contextualize KCC items. Developing items would be useful not only to try to measure teachers' knowledge of KCC, but also to learn more about the nature of this knowledge and its development. For example, piloting items could provide information about whether KCC is distinct from other domains of MKT. Thus, item development would help clarify and refine hypotheses about what is included in KCC.

[59] The ideas for this type of coding scheme are based on the Learning Mathematics for Teaching (LMT) project's video coding rubric for the mathematical quality of instruction (MQI). In this rubric, for each feature of mathematics instruction in a five-minute episode, coders determine whether the feature is "present" or "not present" and then evaluate whether it is "appropriate" or "inappropriate." This enables the coding scheme to capture when an element is there but is mathematically problematic and to reflect when an element's absence is mathematically inappropriate.

Study the work of mathematical purposing and its distribution in different contexts. Another important next step would be to study the work of mathematical purposing in different contexts. This line of research would investigate two types of information: (1) whether the *work* of mathematical purposing is different in different contexts (e.g., with different curricula, experts instead of novices, middle or high school instead of elementary, different countries, etc.); and (2) how the work is differently distributed in different contexts. With respect to the first type, I hypothesize that, like with purposing in other subject areas, the general structure of the framework presented in this dissertation would be applicable across different contexts of mathematics instruction in the United States. Even though the lesson data were from preservice teaching, the framework also reflects the literature I reviewed, which includes a range of practice (including experts). However, it is likely that there would be new things to add to the framework that were not visible in the data or in the literature I reviewed.

With respect to the second type, even in similar contexts, the work will be differently distributed in each particular situation. Across more diverse contexts, I imagine that differences in distribution would be even more pronounced. For example, if a teacher is using a curriculum that provides detailed information about the mathematics and student thinking related to an instructional activity, the work of the *teacher* in the mathematical purposing of that activity would include interpreting the information provided and using it to analyze the details of the instructional activity. When that kind of background information is not provided by the curriculum, then the work of the teacher includes either abstracting that information from an analysis of the details of the activity or drawing on other resources. In either case, the work of mathematical purposing is the same, but the teacher's role in that work, as well as the kinds of mathematical knowledge and reasoning required by the teacher, are different.

Understanding more about how the work of mathematical purposing is distributed in different contexts would provide important information for teacher education. For example, it could help identify aspects of the work of mathematical purposing that would be high-leverage to teach novices. If looking across contexts reveals that there are certain aspects of the work of mathematical purposing that the teacher has to do in most settings, or that the distribution of work for beginning teachers tends to be similar across contexts, then those more common practices and the requisite knowledge might be strategic foci in teacher education.

Design tools for teacher education. As discussed above, the framework developed in this dissertation could be translated into tools for teacher education. I think two types of tools would be particularly interesting to try to develop and experiment with at different points in preservice teacher education. The first would be a tool that helps preservice teachers learn to analyze textbook activities to determine their (possible) mathematical point(s). For example, the details of the activity could be examined using questions like those in Appendix D (Table 11 and Table 13 from Chapter 6). What would be different about this tool from similar tools already being used in teacher education is that there would be an expressed "teacher education point" of engaging preservice teachers in the textbook analysis in order to unpack the mathematical terrain and to focus on the connection between the activity and the intended mathematics. This focus is not necessarily different than other tools currently used in teacher education; however, as discussed in Chapter 2, lesson analysis/planning protocols often have determining the goals of an activity as a discrete (often initial) step in the analysis/planning process. Thus, it is not made explicit that the work of analyzing the lesson and detailing the plans for implementation could, in fact, inform an understanding of what the activity is trying to accomplish. The tool I would like to develop would engage in the analysis of an instructional activity in order to determine the mathematical point and how the activity is set up to engage students with that point. Through this analysis, preservice teachers would have opportunities to practice this central task of teaching and to develop MKT, in particular, specialized content knowledge (SCK) and knowledge of content and students (KCS).

A second tool I would like to develop is one that could be used with preservice teachers before and after they teach lessons in the field. Such a tool could build off the interview protocols used in this study. During data collection, I was surprised by how many preservice teachers said that they learned from the interviews—even though I was only asking detailed questions about their lessons, not offering suggestions. It seemed that for many preservice teachers the questions themselves signaled important things to consider, and just my asking a question put them in the position to do the analysis themselves. For example, as described in Chapter 7, going into her lesson, Paige had preferred the convert-to-minutes method for adding and subtracting fractions on a clock. When I pressed on this during the post-lesson interview, Paige engaged in an analysis of the mathematics made available for study in the convert-to-minutes method and concluded that, although it might be an accessible way for students to find the answer, they may not be using fraction concepts in the process:

Interviewer	And why did you decide to ask them how many minutes?
Paige	Because I thought that might be easier for them. When you have twenty minutes, if they're looking and thinking in minutes, if they see that twenty minutes are shaded, like fifteen minutes—whether you're starting from the bottom or the top—will leave the same amount left. I guess that's what I was thinking. Because I, I mean, even for me it's easier to think in minutes here, and I wish I would've been able to use the example of minutes, like they did in the book, because for me that makes so much sense. "Okay, well, you know, thirty minutes minus ten minutes is twenty minutes" and that's how much is [inaudible].

Interviewer	And how do you think using the example of minutes helps them understand subtracting fractions?
Paige	Well, I, understand subtracting fractions?
Interviewer	Right, because isn't that what you were trying to get at was subtracting fractions? So how does the minutes help you see fractions?
Paige	Well, the minutes, I guess, doesn't with, if we're looking strictly, if we're doing minutes and fractions, maybe it doesn't. Or if you're thinking of minutes as part of sixty, in an hour.
Interviewer	Do you think that when they think of thirty minutes they're thinking out of sixty?
Paige	No, I think they're probably just thinking thirty minutes. So that might not be a good example, because if they're not thinking as part of a whole, then it kind of defeats the purpose of fractions, which is thinking of something as part of a whole, so that half—that thirty minutes—would be part of your sixty minutes which is your whole.
Interviewer	But if they aren't thinking as part of a whole, then they're just thinking in minutes, then you're not really working on fractions?
Paige	No. (P-Post, T252-261)

I hypothesize that talking closely about the connection of the details of an activity to the intended mathematics is an important site for teacher learning. Before teaching a lesson, the tool would engage preservice teachers in a discussion about the details of the activity, with the driving question being "what is the mathematical point of ___?" After the lesson, the focus of the discussion would be on what they were trying to do to steer the instruction toward the mathematical point.

In conclusion, this dissertation set out to investigate one of the central aspects of teaching: that instruction is about moving with students over time toward particular learning goals. Because this is so fundamental to teaching, when working on this dissertation, I often felt that the ideas were obvious and not new. For example, once "teaching to the mathematical point" became my lens for viewing the literature, the work of determining goals and using them to design and steer instruction could be seen in everything I read. But maybe that is the point of this dissertation: This study brings to the foreground a foundational aspect of teaching that is often taken for granted and assumed to be in the background. That teaching involves determining mathematical goals and using them to design and steer instruction may be obvious, but this dissertation shows that what is involved in doing this work is certainly not.

APPENDICES

Appendix A: Background Interview

Introduction

I first want to thank you again for participating in this study. As you know, the purpose of the study is to find out more about how beginning teachers think about and prepare for teaching textbook-based math lessons. This part of the study has two main components: (1) is today's background interview; and (2) is observing and interviewing you about one of your math lessons. As you know, I plan to record the interviews and your lesson. And to show appreciation for the time that this requires of you, at the end of the study you will receive an honorarium of $80.

Before beginning the interview, I wanted to see if you have any questions about the study or what you will be doing. Is there anything else before we get started?

The purpose of today's interview is to find out about your background, in particular about your experiences with mathematics and teaching mathematics. So why don't we start with your math background?

Mathematics background

1. What math classes did you take in high school? What math classes did you take in college?

2. What was learning math like for you in elementary school? How do you feel now about doing math?

3. What do you think it means to be "good" at math? What do you think it takes for someone to be "good" at math?

4. What do you think is the main goal of teaching math in elementary school?

Teaching background

5. What is your teaching major and minor?

6. What types of teaching experiences have you had other than in the teacher education program? Did you teach math during any of those experiences?

7. Describe the field placements you've had in the program. For their current placement, ask:

    How would you describe your CT's approach to teaching math? What is a typical math lesson like in your CT's classroom?

    How would you describe the students in your class? How do you know this about your students?

8. What experiences teaching math have you had in your placements?

9. How do you feel about teaching math? What do you feel comes pretty naturally for you? What aspects of teaching math have you found challenging?

10. What do you think makes a "good" math teacher? What types of things do good math teachers do?

11. What do you think makes a "good" math lesson?

Planning math lessons

12. What math curriculum is used in your placement?

13. Suppose your CT asks you to teach a lesson tomorrow from [math curriculum]. What would you do to prepare for teaching the lesson?

    If not mentioned, ask about:

    Do you read the stated objective? How do you use it?

    How do you prepare to use the student problems in the textbook?

    What do you write down to use while you are teaching? What are the types of things you include on your plan? (e.g., goals, scripted opening, examples to use, summary)

    What other parts of the curriculum materials do you use? Do you look at other lessons in the unit?

    What role does your knowledge of your students play in your planning? Can you give an example? How do you know that about your students?

14. Is what you do to prepare for a math lesson different when you are doing it for a teacher ed course or observation by your field instructor, and when you are just teaching a math lesson that your CT asked you to teach? If so, how?

    Can you give me an example of something you've had to prepare for a TE course? How would you have done it differently if you were just preparing to teach?

Teaching math lessons

15. When you're teaching a math lesson, how do you use the lesson plan that you made?

16. Do you ever veer off or change your plan? How do you decide when to make this kind of change?

17. How do you decide how your math lesson went? How do you determine what students learned in a lesson?

18. What have you learned about lesson planning in the program? What kind of feedback or support have you received from course or field instructors? From your CT?

19. Anything else about teaching math lessons that you'd like to add?

Appendix B: Pre-Lesson Interview

<u>Introduction</u>

Thank you again for participating in this study and for letting me observe and record your lesson today. The purpose of this pre-lesson interview is to find out more about your plans for your lesson, what you hope to accomplish with your students, and what you did to prepare.

I'm usually going to ask you a general question, and then some related follow-up questions. Some of the questions might feel a little repetitive and you definitely aren't expected to have answers to all of them. It's just that it can be hard to remember everything you are thinking about for a lesson, so I want to follow-up to make sure I don't miss anything.

Do you have any questions before we begin?

<u>Lesson goals</u>

1. Why don't you start by telling me your goals for your lesson?

   For each goal mentioned, first ask for elaboration and clarification:

   e.g., What do you mean by _______?
   Can you give an example of _______?
   How do you define ______?

   If they don't mention any of the following goal-types, ask:

   - Do you have any mathematics content goals?
   - Do you have any social goals or goals related to student behavior?
   - You mentioned a number of goals about mathematical topics. Do you have any goals related to *doing* math? (mathematical practices)

2. How did you decide on these goals? Do you think your goals are the same or different from the goals stated in the textbook version of the lesson?

3. Do the goals that you've mentioned for this lesson relate to any goals for the unit? In what ways?

4. Do the goals that you've mentioned relate to any goals for the school year? In what ways?

5. What kinds of understanding or skill are you expecting kids to have with the different goals you mentioned?

<u>Walking through the lesson</u>

Next I thought you could walk me through your lesson. For each activity or lesson segment, I'm going to ask you the some of the same questions, so sorry if it feels a little repetitive…

6. Why don't you start by telling me how you plan to begin your lesson?

As they describe <u>each</u> major activity/segment of the lesson:

a) Probe MKT

Ask about definitions and use of <u>mathematical terms and language</u>:

- How would you define ___? Is that how you would define it with your students?
- Why did you decide to use the language of _______?

Ask for <u>explanations</u> of main mathematical concepts and procedures:

- Can you explain why ____ works?
- What would you say if a student asked why ____?

Ask for explanations of <u>representations</u>:

- How does that representation show ___? Why does the representation have/use ___?
- Why did you decide to use that representation for ______?
- Does the representation highlight any particular mathematical ideas?

Ask about <u>examples/numbers</u>:

- Why did you decide to use those examples/numbers in the problem?
- Is there a particular mathematical idea you are trying highlight with the example?
- Did you consider using other examples/numbers?

b) What do you think the point of this activity is? Is there a specific mathematical point?

How is that point related to your mathematical goals for the lesson?

c) Do you think this part of the lesson will be difficult for your students? Why?

If say difficult:

- What about it will be hard?
- What will you do if they have that difficulty?
- How will you know if they are having that difficulty?

d) *If it is a problem-solving activity:*

Anticipated answers & solution methods:

- How do you think your students will solve that problem?
- Are there any solutions that you particularly want to highlight? Why?

Anticipated misconceptions & errors:

- Can you think of any errors students might make when solving this problem? Why do you think that is a likely error?
- What information would that error give you about the student's understanding?

e) Is there anything in this activity that you are worried or unsure about?

Overall comments about the lesson

7. What do you think is the main mathematical point or points of the lesson? How important do you think that is mathematically? Why?

8. Is there any part of the lesson that you aren't sure about? Why are you unsure?

Is there any part of the math that you are worried about or aren't sure you understand?

Is there anything you are worried will come up when you teach the lesson? What will you do if that comes up?

9. Is there a part of the lesson that you feel most confident about? Why are you confident about that?

10. Is there anything else you'd like to add about your thinking about this lesson?

Planning this lesson

11. Why don't you walk me through your planning process. What did you do first? What did you do next?

If not mentioned, ask about:

- How did you figure out what the lesson was about mathematically?
  - Did you do the problems yourself?
  - Did you think about what your students would do with the problems?
- Did you read other components of the Everyday Math curriculum?
  - other lessons in the unit, front part of unit, other EM components
- Did you consult with other resource/reference materials?
- Did you talk about the lesson with anyone else to help you prepare?
  - CT, field instructor, other colleagues
- How did you use what you know about your students when you were planning?

11. When did you start preparing for this lesson?

How much time do you think you spent planning this lesson?

How did you decide when you were ready to stop planning?

12. What did you write down when you were planning? Why do you find it helpful to write that down?

13. Is how you prepared for this lesson typical of how you prepare for your other math lessons?

    What was the same? What was different?

    Did you do anything different or special because it was being video taped?

14. Anything else you'd like to add about your lesson planning and what you did to prepare for this lesson?

Appendix C: Post-Lesson Interview

Introduction

I thought we'd start by having you share your thoughts about the lesson, and I'll ask some follow-up questions as you talk. After that we'll watch a few short clips from your lesson together and talk about your thinking and decisions at that point in the lesson. Does that sound okay?

General reactions to lesson

1. Why don't you start by sharing your thoughts about the lesson?

   Probe their comments as they talk:
   - What do you mean by ____?
   - Can you give a specific example of that from the lesson ___?
   - How do you know that students learned ___?
   - Why would you have done that differently?

2. Can you say again what you were hoping to accomplish in this lesson? How well do you think you did that?

3. Did anything go differently than you had planned? Why did you decide to make that change? Can you give an example? Was there anything else that went differently?

Student learning

4. What do you think students learned in this lesson? How do you know?

During my observation of the lesson, I identified episodes where:
- *Teacher seems to show use or not use of mathematical goals in instructional decision-making; or where I was surprised by his/her move or response to student.*
- *I want to find out more about teacher's MKT (e.g., Did the teacher understand a student comment or method? What do they think of their explanation of a mathematical idea or procedure?)*

5-9. At one point in the lesson… [*show clip from video or briefly describe episode*].

For instructional decision-making episodes, ask:

Can you describe what was happening at this point in the lesson?
Why did you decide to ___? What were you trying to accomplish?

If not mentioned, ask:

- What did you think about when you decided to ___?
- Did you consider other options?
- How did this relate to the point of this activity or your goals for the lesson?

Do you think it accomplished what you had hoped? How do you know?

What would you do in that situation if you were teaching the lesson again?

For MKT episodes, ask them to explain the mathematics in the episode. For example:

Can you describe what was happening at this point in the lesson?

Can you explain the explanation/representation you gave/used ___?

What do you think about [student's] solution? Is it mathematically valid? Would it work in general?

What do you think [student] meant when he/she said ____?

Why did you decide to record it on the board in that way?

How did this relate to the point of this activity or your goals for the lesson?

10. Were there any other specific parts of the lesson that you wanted to talk about, or any parts of the video you'd like to see?

Mathematical goals of the lesson

11. Now that you've taught the lesson, do you think your goals are different from your original goals when you planned your lesson? In what ways?

If they don't mention any of the following goal-types, ask about:

- What about your mathematical goals?
- What about social goals or goals related to student behavior?
- Did you have any goals related to doing math?

If they say their goals changed: Why did your goals change during the lesson? Were you aware when you were teaching that you had changed your goals?

12. Do you think you explicitly used your goals to help you make decisions while you were teaching?

Can you think of an example when you thought about your goals during the lesson? What did you decide to do as a result? Can you think of another example of using your goals?

If not mentioned, ask:

- Did you use your goals to decide how to respond to a student (e.g., who to call on, what to say in response to their answer)?

- Did you use your goals to pick examples or problems?

- Did you use your goals to manage the time (e.g., decide when to move on or what part of the lesson to skip)?

- Did you use your goals to decide how well the lesson went?

13. Anything else about your goals for this lesson?

<u>Planning for tomorrow</u>

Ask if they are teaching math tomorrow. If so, ask the following questions.

14. Will today's lesson impact what are you thinking about for tomorrow's lesson? In what ways? How does that relate to what happened today?

15. Does today's lesson impact your goals for tomorrow's lessons?

16. Anything else about today's lesson or what you are planning for tomorrow?

Appendix D: Example Questions to Guide Analysis of the Details of the Instructional Activity

<table>
<tr><th colspan="2">Examining the Details of the Instructional Activity</th></tr>
<tr><th>Mathematics Lens
What mathematics is available?</th><th>Learners Lens
How accessible is the mathematics?</th></tr>
<tr><td colspan="2">Representations, manipulatives, tools, contexts</td></tr>
<tr><td>
• Does/could its use support understanding of key concepts? Which ones? How?
• Does/could it help give meaning to a procedure? How?
• Does/could its use reveal/draw attention to the underlying structure/meaning/properties; or foreshadow important mathematical ideas? If so, which ones?
• Does/could its use help reinforce connections between it and what it is representing?
• What numbers numbers/problems/concepts is/can it be used with? Which cases/directions are/can be worked on?
• Are all of the quantities being operated/acted upon visible? What interpretation of the operation/action is shown? Where is the answer?
• How is the mathematics available impacted by whether it is explained during or after its construction?
• Does it distort the math in any way?
• Does its use make mathematics available for study that wouldn’t be available without it?
• Does it create opportunities to address/raise likely misconceptions, errors, or other difficulties?
• What mathematical thinking is being done by it instead of students?
</td><td>
• Are all students familiar with how to use it? Is it used differently than students might have used in the past? If so, how does that impact understanding?
• Does the way it is used to explain concepts/procedures build on what students already know and can do?
• Does the accompanying notation or language facilitate its use (e.g., support understanding, remembering, etc.)?
• Do everyday uses (if any) support or interfere with its mathematical use?
• Does it make assumptions about students’ experiences or background that might interfere with understanding?
• What might be difficult or tricky about using it? Is one of the directions more complex than another?
• What mathematical elements might be confusing or distracting?
• Are there non-mathematical elements that could be potentially confusing or distracting?
• How intricate is it to use/teach/get into play? Does the number of steps involved or the complexity of teaching it detract focus from learning the intended mathematics? What residue is left from its use?
• How prone to errors is it?
• What mathematical ideas could be incorrectly overgeneralized from it?
</td></tr>
<tr><td colspan="2">Procedures and/or solution methods
(including those anticipated to be generated by students)</td></tr>
<tr><td>
• What numbers and/or types of problems is/can this procedure/method be used with?
• If students used this procedure/method, would they be engaging with the focal topic? If so, with what aspects?
• Does/could its use reveal/draw attention to underlying concepts/structure/meaning/properties? If so, which ones and how? Which are obscured?
• Is it a case of or foundational for other mathematical ideas?
</td><td>
• What might be difficult about using the procedure/method? Are there numbers or cases for which it is more mathematically complex?
• How is it similar or different to what students have done before? Do these similarities/differences support or interfere with understanding/use?
• How does any accompanying notation or language facilitate its use?
• How intricate is it to use/teach/get into play? Does the number of steps involved or the complexity of teaching it detract focus from learning the intended mathematics?
• How likely would students be able to devise the procedure/method on their own?
</td></tr>
</table>

Numbers and/or figures used in problems, examples, and exercises

- Do the numbers/figures necessitate/encourage the mathematical idea/skill being taught?
- Where in the terrain do the numbers lead? Might the numbers/figures bring you into unwanted mathematical territory?
- Do they create opportunities to address/raise likely misconceptions, errors, or other difficulties?
- If numbers/figures are being generated randomly or by students, might something unwanted come up, or something wanted not come up?
- How "friendly" or familiar are the numbers to students?
- Is there anything that might be masked or left implicit by the familiarity of the numbers?
- How do the numbers/figures impact the difficulty?
- Are any of the cases more complex than another?
- How visible to students is the idea in the example?
- Does the same number serve multiple roles, and might that make things less visible, cause unnecessary confusion, or hinder explanation?
- What mathematical ideas can be incorrectly overgeneralized from numbers/figures?

Explanations and examples

- <u>What</u> is being explained or illustrated? What is not being explained and why?
- Does it include all of the key steps/concepts that need to be included?
- Does it support understanding of key concepts? Which ones? How?
- Does it reveal/draw attention to the underlying structure/meaning/properties; or foreshadow important mathematical ideas? Which ones? How?
- Is the explanation/example mathematically accurate? Does it distort the math in any way?
- Does it create opportunities to address/raise likely misconceptions, errors, or other difficulties?
- Will the explanation/example be understood by students?
- Is the way that the necessary key steps/concepts are explained accessible to these students? Does it build on what they already know and can do?
- What might be difficult or tricky about using it?
- What mathematical ideas can be incorrectly overgeneralized from the explanation/example?
- What might be confusing or distracting? Are there non-mathematical elements that could be potentially distracting?
- How intricate is it to use/teach/get into play? Does the number of steps involved or the complexity of teaching it detract focus from learning the intended mathematics?
- Does the accompanying notation or language facilitate its use?

Language

(including technical vocabulary and symbolic notation, wording of task/explanations, etc.)

- Is the language mathematically precise?
- Does the language used convey meaning/connections? What meanings/connections are hidden through language?
- Does the wording "give away" what students are supposed to do?
- Is there casual or intended-to-be-helpful language that distorts or obscures the mathematics?
- Are students familiar with any terms and symbols?
- Does compression mask meaning? Is this likely to cause difficulty for students?
- Are there potential conflicts or confusions with the everyday use of language? Or with how language or symbols have been used in previous topics?
- Will students understand the wording of the task?

What counts as an answer

- How does what students are being asked to do relate to the focal topic (e.g., does it draw on skills, concepts, etc.)? What kinds of reasoning does it engage them in?
- Does what students are being asked to do engage them in mathematical practices (e.g., provide explanations, use representations, etc.)?
- If students are giving an explanation, what are the key concepts that must be mentioned?
- Is it possible to get a correct answer without engaging with the intended mathematics?
- Will what students are likely to do engage them with the intended mathematics?
- Will students understand what counts as "different"?

Structure of the activity

• Does the work format impact the mathematics? • Which problems are students left to do on their own and what mathematical work does that leave them? • How does the use of any established routines impact the mathematics being worked on?	• Are students familiar with the structure of the activity? • What might be confusing or distracting? Are there non-mathematical elements that could be potentially distracting?

REFERENCES

Aguirre, J., & Speer, N. (2000). Examining the relationship between beliefs and goals in teacher practice. *Journal of Mathematical Behavior, 18*(3), 327-356.

American Association for the Advancement of Science. (2006). *Project 2061 textbook evaluations.* Retrieved March 4, 2009, from http://www.project2061.org/publications/textbook/default.htm.

Andrews, D. H., & Goodson, L. (1980). A comparative analysis of models of instructional design. *Journal of Instructional Development, 3*(4), 2-16.

Back, J. (2000). Inducting pupils into mathematical discourse. In T. Rowland & C. Morgan (Eds.), *Research in mathematics education* (Vol. 2, pp. 33-44). London: British Society for Research into Learning Mathematics.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*(2), 132-144.

Ball, D. L. (1993a). Halves, pieces, and twoths: Constructing and using representational contexts in teaching fractions. In T. P. Carpenter, E. Fennema & T. A. Romberg (Eds.), *Rational numbers: An integration of research* (pp. 157-195). Hillsdale, NJ: Lawrence Erlbaum.

Ball, D. L. (1993b). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *Elementary School Journal, 93*(4), 373-398.

Ball, D. L. (1997). What do students know? Facing challenges of distance, context, and desire in trying to hear children. In B. Biddle, T. Good & I. Goodson (Eds.), *International handbook on teachers and teaching* (Vol. II, pp. 679-718). Dordrecht, Netherlands: Kluwer Press.

Ball, D. L. (2007). *Teaching mathematics as an unnatural activity.* Presentation made at the annual meeting of the National Council of Supervisors of Mathematics, Atlanta, GA, March 20, 2007.

Ball, D. L., & Bass, H. (2000). Interweaving content and pedagogy in teaching and learning to teach: Knowing and using mathematics. In J. Boaler (Ed.), *Multiple perspectives on the teaching and learning of mathematics* (pp. 83-104). Westport, CT: Ablex.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian Mathematics Education Study Group*. Edmonton, AB: CMESG/GCEDM.

Ball, D. L., & Bass, H. (2009). *With an eye on the mathematical horizon: Knowing mathematics for teaching to learners' mathematical futures.* Unpublished manuscript.

Ball, D. L., & Cohen, D. (1996). Reform by the book: What is--or might be--the role of curriculum materials in teacher learning and instructional reform? *Educational Researcher, 25*(9), 6-8, 14.

Ball, D. L., & Cohen, D. (1999). Developing practice, developing practitioners: Toward a practice-bases theory of professional education. In G. Sykes & Darling-Hammond (Eds.), *Handbook of policy and practice* (pp. 3-32). San Francisco: Jossey Bass.

Ball, D. L., Lubienski, S., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), *Handbook of research on teaching* (4th ed., pp. 433-456). New York: Macmillan.

Ball, D. L., Sleep, L., Boerst, T. A., & Bass, H. (2009). Combining the development of practice and the practice of development in teacher education. *The Elementary School Journal, 109*(5), 458-474.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal for Teacher Education, 59*(5), 389-407.

Behm, S. L., & Lloyd, G. M. (2009). Factors influencing student teachers' use of mathematics curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 205-222). New York: Routledge.

Bell et al. (2004a). *Grade 3 Everyday Mathematics teacher's lesson guide*. Chicago: Wright Group/McGraw-Hill.
Bell et al. (2004b). *Grade 4 Everyday Mathematics teacher's lesson guide*. Chicago: Wright Group/McGraw-Hill.
Bell et al. (2007a). *Grade 1 Everyday Mathematics teacher's lesson guide*. Chicago: Wright Group/McGraw-Hill.
Bell et al. (2007b). *Grade 2 Everyday Mathematics teacher's lesson guide*. Chicago: Wright Group/McGraw-Hill.
Bell et al. (2007c). *Grade 5 Everyday Mathematics teacher's lesson guide*. Chicago: Wright Group/McGraw-Hill.
Ben-Peretz, M. (1990). *The teacher-curriculum encounter: Freeing teachers from the tyranny of texts*. Albany, NY: State University of New York Press.
Berliner, D. C. (1986). In pursuit of the expert pedagogue. *Educational Researcher, 15*(7), 5-13.
Berliner, D. C. (1987). Ways of thinking about students and classrooms by more and less experienced teachers *Exploring teachers' thinking* (pp. 60-83). London: Cassell.
Berliner, D. C. (1988). *The development of expertise in pedagogy.* Washington D.C.: American Association of Colleges for Teacher Education.
Berliner, D. C. (2004). Describing the behavior and documenting the accomplishments of expert teachers. *Bulletin of Science, Technology & Society, 24*(3), 200-212.
Bloom, B. S., Englehart, M. D., Furst, E. J., Hill, W. H., & Krathwohl, D. R. (1956). *Taxonoy of educational objectives: Handbook I, cognitive domain*. New York: McKay.
Blunk, M. L. (2007). *The QMI: Results from validation and scale-building.* Paper presented at the annual conference of the American Educational Research Association.
Blunk, M. L., & Schilling, S. G. (2005). *Validating measures of content knowledge for teaching mathematics: A validity argument approach.* Paper presented at the annual meeting of the American Educational Research Association.
Borko, H., Eisenhart, M., Brown, C., Underhill, R., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education, 23*(3), 194-222.
Borko, H., Lalik, R., & Tomchin, E. (1987). Student teachers' understandings of successful and unsuccessful teaching. *Teaching and Teacher Education, 3*(2), 77-90.
Borko, H., & Livingston, C. (1989). Cognition and improvisation: Differences in mathematics instruction by expert and novice teachers. *American Educational Research Journal, 26*(4), 473-498.
Borko, H., Livingston, C., McCaleb, J., & Mauro, L. (1988). Student teachers' planning and post-lesson reflections: Patterns and implications for teacher preparation. In J. Calderhead (Ed.), *Teachers' professional learning* (pp. 65-83). London: Falmer Press.
Borko, H., & Niles, J. (1987). Descriptions of teacher planning: Ideas for teachers and researchers. In V. Richardson-Koehler (Ed.), *Educators' handbook: A research perspective* (pp. 167-187). New York: Longman, Inc.
Brousseau, G. (1997). *Theory of didactical situations in mathematics: Didactique des mathematiques, 1970-1990* (N. Balacheff, M. Cooper, R. Sutherland & V. Warfield, Trans.). Dordrecht: Kluwer Academic Publishers.
Brown, M. W. (2002). *Teaching by design: Understanding the intersection between teacher practice and the design of curriculum innovations*. Unpublished doctoral dissertation, Northwestern University, Evanston, IL.
Brown, M. W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. Remillard, G. M. Lloyd & B. Herbel-Eisenmann (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 3-36). New York: Routledge.

Brown, M. W., & Edelson, D. C. (2001). *Teaching by design: Curriculum design as a lens on instructional practice.* Paper presented at the American Educational Research Association.

Brown, M. W., & Edelson, D. C. (2003). *Teaching as design: Can we better understand the way in which teachers use materials so we can better design materials to support changes in practice?* Evanston, IL: Center for Learning Technologies in Urban Schools, Northwestern University.

Bush, W. S. (1986). Preservice teachers' sources of decisions in teaching secondary mathematics. *Journal for Research in Mathematics Education, 17*(1), 21-30.

Castro, A. M. (2006). *Planning for mathematics instruction: A study of the teacher guide as a resource*. Unpublished doctoral dissertation, University of Michigan, Ann Arbor, MI.

Cathcart, W. G., Pothier, Y. M., Vance, J. H., & Bezuk, N. S. (2000). *Learning mathematics in elementary and middle schools*. Upper Saddle River, NJ: Prentice-Hall, Inc.

Chapin, S. H., O'Connor, C., & Anderson, N. C. (2003). *Classroom discussions: Using math talk to help students learn*. Sausalito, CA: Math Solutions Publications.

Charmaz, K. (2006). *Constructing grounded theory*. London: Sage Publications.

Clark, C. M., & Dunn, S. (1991). Second-generation research on teachers' planning, intentions, and routines. In H. C. Waxman & H. J. Walberg (Eds.), *Effective teaching: Current research* (pp. 183-201). Berkeley, CA: McCutchan Publishing Corporation.

Clark, C. M., & Peterson, P. L. (1986). Teachers' thought processes. In M. Witrock (Ed.), *Handbook of research on teaching* (3rd ed., pp. 255-296). New York: Macmillan.

Clark, C. M., & Yinger, R. J. (1979). *Three studies of teacher planning* (Research Series No. 55). East Lansing: Michigan State University.

Cohen, D. (1990). A revolution in one classroom: The case of Mrs. Oublier. *Educational Evaluation and Policy Analysis, 12*(3), 311-329.

Cohen, D., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis, 25*(2), 119-142.

Cole, M., & Engestrom, Y. (1993). A cultural-historical approach to distributed cognition. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 1-46). New York: Cambridge University Press.

Davis, E. A. (2006). Preservice elementary teachers' critique of instructinoal materials for science. *Science Teacher Education, 90*(2), 348-375.

Davis, E. A., & Krajcik, J. S. (2005). Designing educative curriculum materials to promote teacher learning. *Educational Researcher, 34*(3), 3-14.

Dewey, J. (1910/1997). *How we think*. Mineola, NY: Dover Publications, Inc.

Doyle, W. (1986). Content representations in teachers' definitions of academic work. *Journal of Curriculum Studies, 18*(4), 365-379.

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. *Educational Psychologist, 23*(2), 167-180.

Doyle, W., & Carter, K. (1984). Academic tasks in classrooms. *Curriculum Inquiry, 14*(2), 129-149.

Feiman-Nemser, S. (1983). Learning to teach. In L. S. Shulman & G. Sykes (Eds.), *Handbook of teaching and policy*. White Plains, NY: Longman.

Feldon, D. F. (2007). Cognitive load and classroom teaching: The double-edged sword of automaticity. *Educational Psychologist, 42*(3), 123-137.

Fennema, E., Carpenter, T. P., Franke, M. L., Levi, L., Jacobs, V., & Empson, S. (1996). A longitudinal study of learning to use children's thinking in mathematics instruction. *Journal for Research in Mathematics Education, 27*, 403-434.

Fennema, E., & Franke, M. L. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 147-164). New York: MacMillan Publishing Co.

Fennema, E., Franke, M. L., & Carpenter, T. P. (1993). Using children's mathematical knowledge in instruction. *American Educational Research Journal, 30*(3), 555-583.
Fernandez, C. (2002). Learning from Japanese approaches to professional development: The case of lesson study. *Journal for Teacher Education, 53*(5), 393-405.
Fernandez, C., & Yoshida, M. (2004). *Lesson study: A Japanese approach to improving mathematics teaching and learning*. Mahwah, NJ: Lawrence Erlbaum Associates.
Fernandez, C., Yoshida, M., & Stigler, J. W. (1992). Learning mathematics from classroom instruction: On relating lessons to pupils' interpretations. *The Journal of the Learning Sciences, 2*(4), 333-365.
Franke, M. L., & Chan, A. (2007). *Learning about and from focusing on routines of practice.* Paper presented at the annual meeting of the America Educational Research Association, Chicago.
Franke, M. L., Kazemi, E., & Battey, D. (2007). Mathematics teaching and classroom practice. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*. Charlotte, NC: Information Age Publishing.
Gagné, R. M. (1985). *The conditions of learning* (4th ed.). New York: Holt, Rinehart & Winston.
Gagné, R. M., Wager, W. W., Golas, K. C., & Keller, J. M. (2005). *Principles of instructional design* (5th ed.). Belmont, CA: Wadsworth.
Gerlach, V. S., & Ely, D. P. (1980). *Teaching and media: A systemic approach* (2nd ed.). Englewood Cliffs, NJ: Prentice-Hall, Inc.
Graeber, A. O., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education, 20*(1), 95-102.
Grossman, P. (1990). *The making of a teacher: Teacher knowledge and teacher education*. New York: Teachers College Press.
Grossman, P. (1991). Mapping the terrain: Knowledge growth in teaching. In H. C. Waxman & H. J. Walberg (Eds.), *Effective teaching: Current research* (pp. 203-215). Berkeley, CA: McCutchan.
Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record, 111*(9).
Grossman, P., Hammerness, K., & McDonald, M. (in press). Redefining teaching, reimagining teacher education. *Teachers and Teaching: Theory and Practice*.
Grossman, P., & McDonald, M. (2008). Back to the future: Directions for research in teaching and teacher education. *American Educational Research Journal, 45*(1), 184-205.
Grossman, P., & Richert, A. E. (1988). Unacknowledged knowledge growth: A Re-examination of the effects of teacher education. *Teaching and Teacher Education, 4*(1), 53-62.
Grossman, P., & Shahan, E. (2005). T*he anatomy of practice: The use of decomposition.* Paper presented at the annual meeting of the American Educational Research Association.
Gustafson, K. L., & Branch, R. M. (2002). *Survey of instructional development models* (4th ed.). Syracuse, NY: ERIC Clearinghouse on information and technology.
Harris, A., & Spillane, J. P. (2008). Distributed leadership through the looking glass. *Management in Education, 22*(1), 31-34.
Heaton, R. (1992). Who is minding the mathematics content? A case study of a fifth-grade teacher. *The Elementary School Journal, 93*(2), 153-162.
Heaton, R. (2000). *Teaching Mathematics to the new standards: Relearning the dance*. New York: Teachers College Press.
Hiebert, J. (1992). Mathematical, cognitive, and instructional analyses of decimal fractions. In G. Leinhardt, R. Putnam & R. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching*. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371-404). Charlotte, NC: Information Age Publishing.
Hiebert, J., Morris, A. K., Berk, D., & Jansen, A. (2007). Preparing teachers to learn from teaching. *Journal of Teacher Education, 58*(1), 47-61.
Hiebert, J., Morris, A. K., & Glass, B. (2003). Learning to learn to teach: An "experiment" model for teaching and teacher preparation in mathematics. *Journal of Mathematics Teacher Education, 6*, 201-222.
Hill, H. C., Blunk, M. L., Charalambous, C., Lewis, J. M., Phelps, G., Sleep, L., et al. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction, 26*(4), 430-511.
Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal, 42*(2), 371-406.
Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematical knowledge for teaching. *Elementary School Journal, 105*(1), 11-48.
Housner, L. D., & Griffey, D. C. (1985). Teacher cognition: Differences in planning and interactive decision-making between experienced and inexperienced teachers. *Research Quarterly for Exercise and Sport, 56*(45-53).
Hughes, E. K. (2006). *Lesson planning as a vehicle for developing pre-service secondary teachers' capacity to focus on students' mathematical thinking*. Unpublished doctoral dissertation, University of Pittsburgh, Pittsburgh, PA.
Hughes, E. K., & Smith, M. S. (2004). *Thinking through a lesson: Lesson planning as evidence of and a vehicle for teaching learning*. Poster presented as part of a symposium, "Developing a Knowledge Base for Teaching: Learning Content and Pedagogy in a Course on Patterns and Functions" at the annual meeting of the American Educational Reseach Association, San Diego, CA.
Hutchins, E. (1995). *Cognition in the wild*. Cambridge: The MIT Press.
Hutchins, E. (2000). Distributed cognition. The International Encyclopedia of the Social and Behavioral Sciences. Retrieved from http://eclectic.ss.uci.edu/~drwhite/Anthro179a/DistributedCognition.pdf.
Jackson, P. W. (1968). *Life in classrooms*. New York: Holt, Rinehart and Winston, Inc.
Joyce, B. R., & Harootunian, B. (1964). Teaching as problem solving. *Journal of Teacher Education, 15*, 420-427.
Kahan, J. A., Cooper, D. A., & Bethea, K. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for reserach applied to a study of student teachers. *Journal of Mathematics Teacher Education, 6*, 223-252.
Kauffman, D., Johnson, S. M., Kardos, S. M., Liu, E., & Peske, H. (2002). "Lost at sea": New teachers' experiences with curriculum and assessment. *Teachers College Record, 104*(2), 273-300.
Kazemi, E., Lampert, M., & Ghousseini, H. (2007). *Conceptualizing and using routines of practice in mathematics teaching to advance professional education.* Report to the Spencer Foundation, Chicago.
Kennedy, M. M. (2005). *Inside teaching: How classroom life undermines reform*. Cambridge, MA: Harvard University Press.
Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. *Harvard Educational Review, 55*(2), 178-194.
Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal, 27*(1), 29-63.

Lampert, M. (1992). Teaching and learning long division for understanding in school. In G. Leinhardt, R. Putnam & R. Hattrup (Eds.), *Analysis of arithmetic for mathematics teaching*. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc.
Lampert, M. (2001). *Teaching problems and the problems of teaching*. New Haven, CT: Yale University Press.
Lampert, M. (2005). *Preparing teachers for ambitious instructional practice: Learning to listen and to construct an appropriate response.* Paper presented at the annual meeting of the American Educational Research Association.
Lampert, M., & Graziani, F. (2009). Instructional activities as a tool for teachers' and teacher educators' learning. *The Elementary School Journal, 109*(5), 491-507.
Lave, J. (1988). *Cognition in practice: Mind, mathematics, and culture in everyday life*. Cambridge: Cambridge University Press.
Leinhardt, G. (1989). Math lessons: A contrast of novice and expert competence. *Journal for Research in Mathematics Education, 20*(1), 52-75.
Leinhardt, G. (1993). On teaching. In R. Glaser (Ed.), *Advances in instructional psychology* (Vol. 4, pp. 1-54). Hillsdale, NJ: Lawrence Erlbaum Associates.
Leinhardt, G., Putnam, R., & Hattrup, R. (Eds.). (1992). *Analysis of arithmetic for mathematics teaching*. Hillsdale, NJ: Lawrence Erlbaum Associates.
Leinhardt, G., & Smith, D. A. (1985). Expertise in mathematics instruction: Subject matter knowledge. *Journal of Educational Psychology, 77*(3), 247-271.
Lester, F. K. (Ed.). (2007). *Second handbook of reserach on mathematics teaching and learning*. Charlotte, NC: Information Age Publishing.
Lewis, C. C. (2002). *Lesson study: A handbook of teacher-led instructional change*. Philadelphia, PA: Research for Better Schools, Inc.
Li, X., Knudsen, J., & Empson, S. (2005). *The possible curriculum: Encompassing intended and enacted versions of mathematics curriculum.* Poster presented at the 27th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education. Eugene, OR.
Livingston, C., & Borko, H. (1989). Expert-novice differences in teaching: A cognitive analysis and implications for teacher education. *Journal of Teacher Education, 40*, 36-42.
Livingston, C., & Borko, H. (1990). High school mathematics review lessons: Expert-novice distinctions. *Journal for Research in Mathematics Education, 21*(5), 372-387.
Lloyd, G. M., & Behm, S. L. (2005). Preservice elementary teachers' analysis of mathematics instructional materials. *Action in Teacher Education, 26*(4), 48-62.
Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum Associates, Inc.
Mathematics Methods Planning Group. (2005). *Teaching children mathematics (ED 411): Class 11 slides*. University of Michigan, Ann Arbor.
May, W. T. (1986). Teaching students how to plan: The dominant model and alternatives. *Journal of Teacher Education, 36*(6), 6-12.
McCutcheon, G. (1980). How do elementary school teachers plan? The nature of planning and influences on it. *Elementary School Journal, 81*(1), 4-23.
McCutcheon, G. (1981). Elementary school teachers' planning for social studies and other subjects. *Theory and Research in Social Education, 9*(1), 45-66.
Mewborn, D. S. (2000). Learning to teach elementary mathematics: Ecological elements of a field experience. *Journal of Mathematics Teacher Education, 3*, 27-46.
Morine-Dershimer, G. (1978). Planning in classroom reality: An in-depth look. *Educational Research Quarterly, 3*(4), 83-99.

Morris, A. K., Hiebert, J., & Spitzer, S. M. (2009). Mathematical knowledge for teaching in planning and evaluating instruction: What can preservice teachers learn? *Journal for Research in Mathematics Education, 40*(5), 491-529.

Morrison, G. R., Ross, S. M., & Kemp, J. E. (2007). *Designing effective instruction* (5th ed.). New York: John Wiley & Sons, Inc.

National Center for Education Statistics. (2003). *Teaching mathematics in seven countries: Results from the TIMSS video study*. Washington DC: U.S. Department of Education.

National Commission on Excellence in Education. (1983). *A nation at risk: The imperative for educational reform*.

National Council of Teachers of Mathematics. (1989). *Curriculum and evaluation standards for school mathematics*. Reston, VA: NCTM.

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

National Mathematics Advisory Panel (2007). *Teachers task group progress report.* St. Louis, MO.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford & B. Findell (Eds). Mathematics Learning Study Committee, Center for Education, Division of Behavioral and Social Science and Education. Washington, DC: National Academy Press.

Nolder, R. (1991). Mixing metaphor and mathematics in the secondary classroom. In K. Durkin & B. Shire (Eds.), *Language in mathematical education: Research and practice* (pp. 105-113). Philadelphia: Open University Press.

Pea, R. D. (1993). Practices of distributed intelligence and designs for education. In G. Salomon (Ed.), *Distributed cognition: Psychologoical and educational considerations* (pp. 47-87). New York: Cambridge University Press.

Perkins, D. N. (1993). Person-plus: A distributed view of thinking and learning. In G. Salomon (Ed.), *Distributed cognition: Psychologoical and educational considerations* (pp. 88-110). New York: Cambridge University Press.

Peterson, P. L., Marx, R. W., & Clark, C. M. (1978). Teacher planning, teacher behavior, and student achievement. *American Educational Research Journal, 15*(3), 417-432.

Putnam, R. T., & Borko, H. (2000). What do new views of knowledge and thinking have to say about research on teacher learning? *Educational Researcher, 29*(1), 4-15.

Reiser, R. A., & Dick, W. (1996). *Instructional planning: A guide for teachers* (2nd ed.). Boston: Allyn and Bacon.

Remillard, J. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research, 75*(2), 211-246.

Remillard, J. (2009). Considering what we know about the relationship between teachers and curriculum materials. In J. T. Remillard, B. A. Herbel-Eisenmann & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 85-92). New York: Routledge.

Rowland, T. (1999). Pronouns in mathematics talk: Power, vagueness and generalisation. *For the Learning of Mathematics, 19*(2), 19-26.

Rowland, T. (2008). The purpose, design and use of examples in the teaching of elementary mathematics. *Educational Studies in Mathematics, 69*, 149-163.

Rowland, T., Thwaites, A., & Huckstep, P. (2003). The choice of examples in the teaching of mathematics: What do we tell the trainees? *Proceedings of the British Society for Research into Learning Mathematics, 23*(2), 85-90.

Ryan, G. W., & Bernard, H. R. (2000). Data management and analysis methods. In N. K. Denzin & Y. S. Lincoln (Eds.), *Handbook of qualitative research* (2nd ed., pp. 769-802). Thousand Oaks, CA: Sage Publications, Inc.

Salomon, G. (1993). No distribution without individuals' cognition: A dynamic interactional view. In G. Salomon (Ed.), *Distributed cognitions: Psychological and educational considerations* (pp. 111-138). New York: Cambridge University Press.
Schifter, D. (1998). Learning mathematics for teaching: From a teachers' seminar to the classroom. *Journal of Mathematics Teacher Education, 1*, 55-87.
Schoenfeld, A. H. (2000). Models of the teaching process. *Journal of Mathematical Behavior, 18*(3), 243-261.
Schoenfeld, A. H. (2007). Method. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 69-107). Charlotte, NC: Information Age Publishing.
Schoenfeld, A. H., Minstrell, J., & Van Zee, E. (2000). The detailed analysis of an established teacher's non-traditional lesson. *Journal of Mathematical Behavior, 18*(3), 281-325.
Shavelson, R. J. (1983). Review of research on teachers' pedagogical judgments, plans, and decisions. *The Elementary School Journal, 83*(4), 392-413.
Shavelson, R. J., & Stern, P. (1981). Research on teachers' pedagogical thoughts, judgments, decisions, and behavior. *Review of Educational Research, 51*(4), 455-498.
Sherin, M. G., & Drake, C. (2009). Curriculum strategy framework: Identifying patterns in teachers' use of a reform-based elementary mathematics curriculum. *Journal of Curriculum Studies, 41*(4), 467-500.
Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*(2), 4-14.
Shulman, L. S. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review, 57*(1), 1-22.
Sierpinska, A. (1994). *Understanding in mathematics*. London: The Falmer Press.
Silver, E. A., & Herbst, P. G. (2007). Theory in mathematics education scholoarship. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 111-155). Charlotte, NC: Information Age Publishing.
Silver, E. A., Mills, V., Ghousseini, H., & Charalambous, C. Y. (2009). Exploring the curriculum implementation plateau: An instructional perspective. In J. Remillard, G. M. Lloyd & B. Herbel-Eisenmann (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 245-273): Routledge.
Simon, M. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education, 24*(3), 233-254.
Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.
Simpson et al. (Ed.). (2009). *Oxford English dictionary* (Online 3rd ed.). London: Oxford University Press.
Sleep, L. (2007). *Investigating the knowledge demands and practices of carefully attending to and using mathematical language in teaching.* Paper presented at the the annual meeting of the American Educational Research Association, Chicago, IL.
Smith, M. S., & Bill, V. (2004). *Thinking through a lesson: Collaborative lesson planning as a means for improving the quality of teaching.* Paper presented at the Annual Meeting of the Association of Mathematics Teacher Educators.
Smith, M. S., Hughes, E. K., Engle, R. A., & Stein, M. K. (2009). Orchestrating discussions. *Mathematics Teaching in the Middle School, 14*(9), 548-556.
Smith, P. L., & Ragan, T. J. (2005). *Instructional design* (3rd ed.). Hoboken, NJ: John Wiley & Sons, Inc.
Spillane, J. P., Halverson, R., & Diamond, J. B. (2004). Towards a theory of leadership practice: A distributed perspective. *Curriculum Studies, 36*(1), 3-34.

Stein, M. K., Grover, B. W., & Henningsen, M. (1996). Building student capacity for mathematical thinking and reasoning: An analysis of mathematical tasks used in reform classrooms. *American Educational Research Journal, 33*(2), 455-488.
Stein, M. K., & Kim, G. (2009). The role of mathematics curriculum materials in large-scale urban reform: An analysis of demands and opportunities for teacher learning. In J. T. Remillard, B. A. Herbel-Eisenmann & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 37-55). New York: Routledge.
Stein, M. K., Remillard, J., & Smith, M. S. (2007). How curriculum influences student learning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 319-369). Charlotte, NC: Information Age Publishing.
Stein, M. K., Smith, M. S., Henningsen, M. A., & Silver, E. A. (2000). *Implementing standards-based mathematics instruction: A casebook for professional development*. New York: Teachers College Press.
Stigler, J. W., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom*. New York: The Free Press.
Stigler, J. W., & Thompson, B. J. (2009). Thoughts on creating, accumulating, and utilitizing shareable knowledge to improve teaching. *The Elementary School Journal, 109*(5), 442-457.
Teacher Education Initiative. (2009). *High-leverage teaching practices (report revised 3/27/09)*. Ann Arbor: University of Michigan, Curriculum Group.
Tessmer, M., & Wedman, J. F. (1990). A layers-of-necessity instructional development model. *Educational Technology Research and Development, 38*(2), 77-86.
Tirosh, D., & Graeber, A. O. (1989). Preservice elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics, 20*, 79-96.
Tyler, R. W. (1949). *Basics principles of curriculum and instruction*. Chicago: University of Chicago Press.
Van de Walle, J. A. (2007). *Elementary and middle school mathematics: Teaching developmentally* (6th ed.). Boston: Pearson Education, Inc.
Van Zoest, L. R., & Bohl, J. V. (2002). The role of reform curricular materials in an internship: The case of Alice and Gregory. *Journal of Mathematics Teacher Education, 5*, 265-288.
Wilson, S. M., Shulman, L. S., & Richert, A. E. (1987). '150 different ways' of knowing: Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teachers' thinking*. Sussex: Hold, Reinhart, & Winston.
Yinger, R. J. (1979). Routines in teacher planning. *Theory into Practice, 18*(3), 163-169.
Yinger, R. J. (1980). A study of teacher planning. *The Elementary School Journal, 80*(3), 107-127.
Yinger, R. J., & Clark, C. M. (1982). Understanding teachers' judgments about instruction: The task, the method, and the meaning.
Zahorik, J. A. (1970). The effect of planning on teaching. *The Elementary School Journal, 71*(3), 143-151.
Zimmerlin, D., & Nelson, M. (2000). The detailed analysis of a beginning teacher carrying out a traditional lesson. *Journal of Mathematical Behavior, 18*(3), 263-279.