THE UNIVERSITY OF MICHIGAN

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# ELECTROMAGNETIC RADIATION FROM SHOCK WAVES
# IN A LOW-DENSITY WEAKLY-IONIZED MEDIUM

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# I

# INTRODUCTION

This interim report presents the results obtained thus far in a program to study the possibility of emission of partially coherent electromangetic radiation during the formation of the shock wave associated with a hypersonic re-entry vehicle. Probstein [1], in a recent article, has considered qualitatively the development of such a shock wave. He concludes that there exists a "transition region" in which the shock wave begins to form. However, in this region, whose altitude bounds are determined by flight conditions and the body configuration, the concepts and equations employed in the free-molecular flow or continum flow regimes cannot be used. Consequently, little is known about the formation of the shock wave except that a highly nonlinear cascading process occurs. In order to treat the problem rigorously, the use of kinetic theory is indicated. If properly applied, this theory can form a bridge between the continuum and free-molecule flow solutions. However, even in simple cases, the resulting equations prove complex. One successful application is that of Lees and Liu who consider non-linear plane Couette flow [2].

Additional complications ensue when one studies the characteristics of the medium surrounding the vehicle in the transition region. For example, if the vehicle has a dimension of the order of one meter, the transition region has a mean altitude of approximately 100 kilometers. At these heights there exists an appreciable number of charged particles (electrons and ions) per cubic centimeter. The neutral particle density is such that the medium must be considered a weakly-ionized gas. As a result, the Coulomb forces existing between the charged components will alter the manner in which the shock wave is formed. Several recent papers have studied the structure of shock waves in fully-ionized gases [3, 4, 5, 6]. In particular, Greenberg, Trève, and Sen [5] and later Greenberg and Trève [6] have discussed charge separation effects that occur; these effects manifest themselves in a spatially dependent electric field that is oscillatory in nature. In the region near the vehicle, the transition from an overall macroscopically neutral configuration to one possessing an oscillatory field structure as a result of shock wave formation could result in the emission of electromagnetic radiation.

The purpose of this study is to investigate the formation of the shock wave in the low-density weakly-ionized transition region with particular emphasis being placed on determining whether or not emission of electromagnetic radiation occurs. To attain this end, kinetic theory is employed to first determine the nonlinear differential equations describing the formation of the shock wave. In Section II, the Grad thirteen-moment procedure [7] as modified by Everett [8] is applied to the Boltzmann equation for a plasma to determine these equations. The Grad procedure is employed because it gives the most general picture of the process, albeit the most complex, in the sense that no ad hoc assumptions need be made concerning the relationships between various physical quantities. Although this may seem needlessly complex, once the equations have been obtained any subsequent assumptions can be easily inserted or, if erroneous, deleted without disturbing the basic framework upon which the theory rests. In Section III, the jump conditions relating the flow field in front of the shock wave to that behind it are developed. And, in Section IV, an application to a plane shock wave is presented.

Due to a premature termination of the program, this study has not been completed. Certain collision terms appearing in the equations have yet to be evaluated; also, no attempt to solve the system of equations has yet been made.

Throughout the report, the Gaussian system of units is used, and the summation convention applies to all the equations that appear. As regards integrals over velocity space, the limits, though not written explicitly, are from $-\infty$ to $+\infty$. And, finally, the tensor notation of Chapman and Cowling [9] is adopted wherein traceless tensors are denoted by a small circle above the tensor symbol and symmetrized tensors by a double bar similarly placed. In symbols:

$$\overset{o}{A}_{ij} = A_{ij} - \frac{1}{3} \delta_{ij} A_{kk} \tag{1}$$

$$\bar{\bar{A}}_{ij} = \frac{1}{2} (A_{ij} + A_{ji}). \qquad (2)$$

For reference, the equation for a traceless symmetrized tensor in terms of its components is given below:

$$\overset{\circ}{\bar{\bar{A}}}_{ij} = \frac{1}{2} (A_{ij} + A_{ji}) - \frac{1}{3} \delta_{ij} A_{kk} \qquad (3)$$

## II

## MOMENT EQUATIONS

The Boltzmann equation for the distribution function characterizing particles of type A in a plasma is [9, 10]

$$\frac{\partial f^A}{\partial t} + V_i \frac{\partial f^A}{\partial x_i} + \frac{e_A}{m_A} \left(E_i' + \frac{1}{c} \epsilon_{ijk} V_j H_k'\right) \frac{\partial f^A}{\partial V_i} = \sum_B I_{AB}. \tag{4}$$

The quantities represented by the symbols appearing in the above equation are as follows:

| | |
|---|---|
| $f^A(\underline{x}, \underline{V}, t)$ | singlet particle density in phase space |
| $V_i$ | $i^{th}$ component of actual particle velocity |
| $e_A$ | charge of type A particle |
| $m_A$ | mass of type A particle |
| c | velocity of light *in vacuo* |
| $\epsilon_{ijk}$ | Levi-Civita density |

The quantity $\sum_B I_{AB}$ expresses the change in $f^A$ per unit time owing to binary collisions of A-type particles with B-type particles. The expression for $I_{AB}$ can be written [10]

$$I_{AB} = \int_{\underline{V}_1 \Omega} V_r \sigma_{AB}(V_r, \textcircled{H}) \left[ f^A(\underline{x}, \underline{V}', t) f^B(\underline{x}, \underline{V}_1', t) - f_1^A(\underline{x}, \underline{V}, t) f_1^B(\underline{x}, \underline{V}_1, t) \right] d\Omega \, d^3 V_1 \tag{5}$$

where $\sigma_{AB}(V_r, \Theta)\, d\Omega$ is the differential cross section in the center-of-mass system for elastic scattering of a B-type with an A-type particle, $V_r$ is the relative velocity of the two particles, $\underline{V}'$ $\underline{V}_1'$ and $\underline{V}$, $\underline{V}_1$ are the pre-collision and post-collision velocities, respectively. In the above equations and elsewhere, the symbol $\sum_B$ means a sum over all particle types, type A particles included.

If external electric or magnetic fields are absent, then $E_i'(\underline{x}, t)$ and $H'(\underline{x}, t)$ in the Boltzmann equation are the $i^{th}$ components of the electric and magnetic fields in the plasma induced solely by the charges and currents within the plasma — the internal fields. These fields are given by Maxwell's equations:

$$\frac{\partial E_i'}{\partial x_i} = 4\pi q' \tag{6a}$$

$$\frac{\partial H_i'}{\partial x_i} = 0 \tag{6b}$$

$$\frac{1}{c}\frac{\partial E_i'}{\partial t} - \epsilon_{ijk}\frac{\partial H_k'}{\partial x_j} = -\frac{4\pi}{c} j_i' \tag{6c}$$

$$\frac{1}{c}\frac{\partial H_i'}{\partial t} + \epsilon_{ijk}\frac{\partial E_k'}{\partial x_j} = 0 \tag{6d}$$

where the source terms are

$$q'(\underline{x}, t) = \sum_B \int e_B\, f^B(\underline{x}, \underline{V}, t)\, d^3 V \tag{7a}$$

$$j_i'(\underline{x}, t) = \sum_B \int e_B V_i f^B(\underline{x}, \underline{V}, t)\, d^3 V \qquad (7b)$$

If external fields are present, then the field terms $E'_i$ and $H'_i$ in the Boltzmann equation represent a linear superposition of the internal and external fields. Because $\underline{E}'$ and $\underline{H}'$ are functionals of the particle distribution functions, the Boltzmann equation is nonlinear.

The coordinate system in which the problem is most tractable is one in which a shock wave, moving with a constant velocity U relative to a fixed observer, appears stationary, i.e., a coordinate system moving with the shock wave. Under the transformation to such a system, $f^A(\underline{x}, \underline{V}, t) \rightarrow f^A(\underline{x}, \underline{v}, t)$ where $\underline{v} = \underline{V} - \underline{U}$, the derivatives of $f^A$ with respect to time and space undergo no change and the derivative with respect to velocity becomes $\partial f^A/\partial v_i$. Let the result of the transformation on the field quantities be represented symbolically by $\underline{E}' \rightarrow \underline{E}$, $\underline{H}' \rightarrow \underline{H}$. The specific relations between $\underline{E}'$, $\underline{H}'$ and $\underline{E}$, $\underline{H}$ are given by the usual Lorentz transformation formulas [11]. Thus, in the transformed system, the Boltzmann equation becomes

$$\frac{\partial f^A}{\partial t} + v_i \frac{\partial f^A}{\partial x_i} + \frac{e_A}{m_A}\left(E_i + \frac{1}{c}\,\epsilon_{ijk}\, v_j H_k\right)\frac{\partial f^A}{\partial v_i} = \sum_B I_{AB}\,. \qquad (8)$$

The field quantities satisfy Maxwell's equations with the source terms becoming

$$q(\underline{x}, t) = \sum_B \int e_B f^B (\underline{x}, \underline{v}, t) d^3 v \tag{9a}$$

$$j_i(\underline{x}, t) = \sum_B \int e_B v_i f^B (\underline{x}, \underline{v}, t) d^3 v \tag{9b}$$

as a result of the transformation.

The number density and flow velocity of the type A particles are defined in the usual way, i. e.

$$n^A = \int f^A (\underline{x}, \underline{v}, t) d^3 v \tag{10}$$

$$w_i^A = \frac{1}{n^A} \int v_i f^A (\underline{x}, \underline{v}, t) d^3 v \tag{11}$$

In order to provide a description of the system in terms of macroscopic variables, the "molecular property" function $\phi^A(\underline{x}, \underline{u}, t)$ is introduced. Here, the property function for a particular constituent is defined to be a function of the particle velocity $\underline{u}$ relative to the constituent flow velocity, that is, $\underline{u} = \underline{v} - \underline{w}^A$. Consequently, it will prove convenient to introduce $(\underline{x}, \underline{u}, t)$ as a new set of variables, which replace the independent set $(\underline{x}, \underline{v}, t)$, prior to developing a transport equation for $\phi^A$. In terms of the new variables the Boltzmann equation becomes

$$\left[\frac{\partial}{\partial t} + w_i^A \frac{\partial}{\partial x_i}\right] f^A - \left[\frac{\partial w_i^A}{\partial t} + w_j^A \frac{\partial w_i^A}{\partial x_j} - \frac{e_A}{m_A} (E_i + \frac{1}{c} \epsilon_{ijk} w_j^A H_k)\right] \frac{\partial f^A}{\partial u_i}$$

$$+ u_i \frac{\partial f}{\partial x_i} + \left[(\frac{e_A}{m_A} \frac{1}{c} \epsilon_{ijk} u_j H_k) - \frac{\partial w_i^A}{\partial x_j} u_j\right] \frac{\partial f^A}{\partial u_i}$$

$$= \sum_B I_{AB} \qquad (12)$$

where $f^A = f^A(\underline{x}, \underline{u}, t)$. If the mean value of $\phi^A(\underline{x}, \underline{u}, t)$ — a state variable — is defined as

$$< \phi^A(\underline{x}, t) > = \frac{1}{n^A} \int \phi^A(\underline{x}, \underline{u}, t) f^A(\underline{x}, \underline{u}, t) d^3u, \qquad (13)$$

a transport equation for $< \phi^A(\underline{x}, t)>$ is obtained by evaluating $\frac{\partial}{\partial t}\left\{n^A < \phi^A >\right\}$ with the aid of the transformed Boltzmann equation. Upon carrying out the indicated manipulations and recalling that $f(\underline{x}, \underline{u}, t)$ vanishes at the limits in velocity space, the transport equation for $< \phi^A>$ is found to be

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial x_i} w_i^A\right] n^A < \phi^A > + \frac{\partial}{\partial x_i} \left[n^A < u_i \phi^A >\right]$$

$$+ \left[\frac{\partial w_i^A}{\partial t} + w_j^A \frac{\partial w_i^A}{\partial x_j} - \frac{e_A}{m_A} (E_i + \frac{1}{c} \epsilon_{ijk} w_j^A H_k)\right] n^A < \frac{\partial \phi^A}{\partial u_i} >$$

$$+ \left[\frac{\partial w_i^A}{\partial x_j} - \frac{e_A}{m_A} (\frac{1}{c} \epsilon_{ijk} H_k)\right] n^A < u_j \frac{\partial \phi^A}{\partial u_i} >$$

$$= \sum_B \int \phi^A (\underline{x}, \underline{u}, t) I_{AB} d^3 u \qquad (14)$$

The choice of appropriate values for $\phi^A$ leads to the moment relations.

A general moment equation in which the $n^{th}$ velocity moment is related to the $(n-1)^{th}$ and $(n+1)^{th}$ moments can be obtained. For this purpose, let

$$M_{(n)}^A = M_{i_1 \cdots \cdot i_n}^A = m_A \int (u_{i_1} \cdots u_{i_n}) f^A d^3 u \qquad (15a)$$

$$= n^A < m_A \phi_{(n)}^A > . \qquad (15b)$$

Then the following relations are valid:

$$M_{(n+i)}^A = n^A < m_A u_i \phi_{(n)}^A > \qquad (16)$$

$$\sum_{r=1}^{n} \delta_{i_r i} \, M^{A}_{(n-i_r)} = n^{A} < m_A \frac{\partial \phi^{A}_{(n)}}{\partial u_i} > \tag{17}$$

$$\sum_{r=1}^{n} \delta_{i_r i} \, M^{A}_{(n-i_r+j)} = n^{A} < m_A u_j \frac{\partial \phi^{A}_{(n)}}{\partial u_i} > \tag{18}$$

Substitution of these equations into the transport equation for $< \phi^{A} >$ leads to the generalized moment equation for a particular constituent of the system, namely

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x_i} w^{A}_{i}\right) M^{A}_{(n)} + \frac{\partial}{\partial x_j} M^{A}_{(n+j)}$$

$$+ \sum_{r=1}^{n} \left[ \frac{\partial w^{A}_{i_r}}{\partial t} + w^{A}_{j} \frac{\partial w^{A}_{i_r}}{\partial x_j} - \frac{e_A}{m_A} \left(E_{i_r} + \frac{1}{c} \epsilon_{i_r jk} w^{A}_{j} H_k\right) \right] M^{A}_{(n-i_r)}$$

$$+ \sum_{r=1}^{n} \left[ \frac{\partial w^{A}_{i_r}}{\partial x_j} - \frac{e_A}{m_A} \left(\frac{1}{c} \epsilon_{i_r jk} H_k\right) \right] M^{A}_{(n-i_r+j)}$$

$$= \sum_{B} \int m_A \phi^{A}_{(n)} I_{AB} \, d^3 u \tag{19}$$

Note that $M^{A}_{(0)} = m_A n^{A} = \rho^{A}$, and $M^{A}_{(1)} = n^{A} < m_A u_i > = 0$. The physical significance of the higher moments can be deduced from like quantities appearing in the equations of hydrodynamics. The following table, in which $M^{A}_{(0)}$ is included for completeness, serves not only to exhibit the physical interpretation of some of the moments but also sets forth the notation and definitions adopted for the constituent state variables:

| Quantity | Symbol | $M^A_{(n)}$ |
|---|---|---|
| density | $\rho^A$ | $M^A_{(0)} = m_A n^A$ |
| stress tensor | $\psi^A_{ij}$ | $M^A_{ij} = n^A < m_A u_i u_j >$ |
| temperature | $T^A$ | $\frac{1}{3n^A k} M^A_{ii} = \frac{2}{3k} < \frac{m_A}{2} u_i u_i >$ |
| heat flux | $Q^A_j$ | $\frac{1}{2} M^A_{iij} = n^A < \frac{m_A}{2} u_i u_i u_j >$ |

As a consequence of the above definitions,

$$n^A k T^A = \frac{1}{3} M^A_{ii} = \frac{1}{3} \psi^A_{ii} . \qquad (20)$$

In hydrodynamics, the hydrostatic pressure is expressed as one-third the trace of the stress tensor. Hence, by analogy, let

$$p^A = \frac{1}{3} \psi^A_{ii} = n^A k T^A \qquad (21)$$

be the hydrostatic pressure of the $A^{-th}$ constituent.

In order to compact the notation, the following symbols are introduced for the collision integrals to be encountered subsequently:

$$P_i^A = \sum_B \int (m\, u_i)\, I_{AB}\, d^3 u \tag{22}$$

$$K^A = \sum_B \int (\frac{1}{2}\, m\, u_i\, u_i)\, I_{AB}\, d^3 u \tag{23}$$

$$\overset{o}{X}{}_{ij}^A = \sum_B \int (m\, u_i{}^o u_j)\, I_{AB}\, d^3 u \tag{24}$$

$$F_i^A = \sum_B \int (\frac{1}{2}\, m\, u^2\, u_i)\, I_{AB}\, d^3 u \tag{25}$$

Comparison of the collision integrals with the quantities listed in the above table indicates that the collision transfer of momentum, energy, traceless stress and heat flux relative to the constituent flow velocity are given by $P_i^A$, $K^A$, $\overset{o}{X}{}_{ij}^A$, and $F_i^A$ respectively.

Although equation (19) is as exact as equation (12), it is also as intractable since, as mentioned previously, it expresses the $n^{th}$ moment in terms of the $(n-1)^{th}$ and $(n+1)^{th}$ moments. Its use leads to an infinite set of coupled equations, no finite subset of which forms a determined system. Further progress can be made, however, by employing Grad's procedure [7] as modified by Everett [8]. Basically, the Grad procedure utilizes an approximate expression of the distribution function developed by expanding it into a series of three-dimensional Hermite polynomials. The series is truncated at the appropriate point to form a closed set of moment equations with n, $w_i$, T, $\psi_{ij}$, and $Q_i$, defined relative to the system flow velocity, as the primary variables. Everett modified the procedure by using as the independent variable the particle velocity relative to the flow velocity of the constituent, i.e., the random velocity. His expansion, also a series of appropriate Hermite polynomials, contains terms up to and including a contraction of the third rank polynomial. The first term of the expansion is Maxwellian relative to the constituent flow velocity while the coefficients of the higher order terms are the stresses and heat fluxes relative to the constituent flow velocity.

In the present notation, the approximate distribution function given by Everett is

$$f^A \simeq n^A \left(\frac{\beta_A}{2\pi}\right)^{3/2} \exp\left(-\frac{\beta_A}{2}u^2\right) \left[1 + \frac{\beta_A^2}{2\rho^A} \overset{o}{\psi}{}_{ij}^{A} u_i u_j + \frac{\beta_A^3}{5\rho^A} Q_i^A \left(u^2 - \frac{5}{\beta_A}\right) u_i\right] \tag{26}$$

where $\beta_A = m_A/kT^A$. Insertion of the above expression into equation (19) yields the following determinable set of moment equations:

$$\frac{\partial n^A}{\partial t} = -\frac{\partial}{\partial x_i}\left(n^A w_i^A\right) \tag{27a}$$

$$\frac{\partial w_i^A}{\partial t} = -w_j \frac{\partial w_i^A}{\partial x_j} - \frac{1}{\rho^A}\frac{\partial \psi_{ij}^A}{\partial x_j} + \frac{e_A}{m_A}\left(E_i + \frac{1}{c}\epsilon_{ijk} w_j^A H_k\right) + \frac{1}{\rho^A} P_i^A \tag{27b}$$

$$\frac{\partial T^A}{\partial t} = - w_i^A \frac{\partial T^A}{\partial x_i} - \frac{2}{3} \frac{1}{n^A k} \left[ \frac{\partial w_j^A}{\partial x_i} \psi_{ij}^A + \frac{\partial Q_i^A}{\partial x_i} - K^A \right] \tag{27c}$$

$$\frac{\partial \overset{o}{\psi}{}_{ij}^A}{\partial t} = - w_k^A \frac{\partial \overset{o}{\psi}{}_{ij}^A}{\partial x_k} - \frac{\partial w_k^A}{\partial x_k} \overset{o}{\psi}{}_{ij}^A + 2 \frac{e_A}{m_A} \frac{1}{c} \epsilon_{i\ell m} H_m \overset{o}{\psi}{}_{\ell j}^A$$

$$- 2 \overset{o}{\overline{\overline{\frac{\partial w_j^A}{\partial x_k} \psi_{ki}^A}}} - \frac{4}{5} \overset{o}{\overline{\overline{\frac{\partial Q_j^A}{\partial x_i}}}} + \overset{o}{X}{}_{ij}^A \tag{27d}$$

$$\frac{\partial Q_i^A}{\partial t} = - w_j^A \frac{\partial Q_i^A}{\partial x_j} - \frac{7}{5} \frac{\partial w_j^A}{\partial x_j} Q_i^A - \frac{\partial w_i^A}{\partial x_j} Q_j^A - \frac{4}{5} \overline{\overline{\frac{\partial w_j^A}{\partial x_i}}} Q_j^A$$

$$+ \frac{e_A}{m_A} \frac{1}{c} \epsilon_{ijk} Q_j^A H_k + \frac{\overset{o}{\psi}{}_{ji}^A}{\rho^A} \frac{\partial \psi_{jk}^A}{\partial x_k} - \frac{kT^A}{m_A} \frac{\partial \overset{o}{\psi}{}_{ij}^A}{\partial x_j} - \frac{7}{2} \frac{k}{m_A} \frac{\partial T^A}{\partial x_j} \overset{o}{\psi}{}_{ij}^A \tag{27e}$$

$$- \frac{5}{2} \frac{kp^A}{m_A} \frac{\partial T^A}{\partial x_i} - \frac{1}{\rho^A} \left( \frac{5}{2} p^A \delta_{ji} + \overset{o}{\psi}{}_{ji}^A \right) P_j^A + F_i^A$$

The derivation of the above equations has made use of the relation

$$\int \phi_{(0)}^A I_{AB} d^3 u = 0 \tag{28}$$

which is merely an expression of the fact that the number of particles remains unchanged for the type of collisions under consideration.

The complexity of the above equations for just one constituent of the medium is evident. Furthermore, coupling of the above set to similar sets of equations for the other constituents occurs through the collision terms — $P_i^A$, $K^A$, $X_{ij}^A$, $F_i^A$ — and the field terms - $E_i$ and $H_i$. Part of the complexity stems from the generality of the equations and some simplification can be expected when an application to a specific problem is made; however, unless only trivial problems are considered, difficulty in obtaining solutions for cases of interest should be expected.

As yet, no explicit consideration has been given to the medium in which the shock wave is moving. Primary interest is directed towards weakly-ionized low-density gases; consequently the following assumptions pertaining to the medium seem feasible. First, assume the medium is comprised of only three constituents — electrons, singly-charged ions, and neutral particles characterized by a single molecular mass value, i.e., the masses of the neutrals, irrespective of species, are taken to be approximately equal. Second, in a weakly-ionized medium, the number density of neutrals will be much greater than either the number density of ions or electrons, and, also, the number of ions will equal the number of electrons. Thus, one can assume that the frequency of close binary encounters of electrons and ions will be negligible in comparison with the collision frequency of either the ions or electrons with the neutrals. As a result, all collision terms that stem from electron-ion interactions can be ignored. The electrostatic influence of the two charged constituents upon each other is not being completely neglected since the field terms account for the long-range Coulomb interactions. The remaining collision terms involve only field-free interactions describable in terms of elastic hard-sphere scattering. The known properties of this scattering and the use of the approximate distribution function, as given by equation (26), permit the evaluation of the collision integrals in terms of the primary variables.

## III

## JUMP CONDITIONS

For a shock wave whose thickness is negligible compared to its extent, the jump conditions are a set of constraints that link the values of appropriate combinations of the macroscopic variables that describe the medium ahead of the shock wave to those in back [12]. They arise as a result of applying the three conservation laws — the conservation of number, momentum and energy — to the moment equations describing the entire system. In most applications the jump conditions prove useful since the structure of the shock wave can be ignored and the shock wave itself can be treated simply as a discontinuity. Moreover, the jump conditions form an "extended" set of boundary conditions relating the values of the variables on the two shock wave surfaces, i. e., given the conditions at one surface the macroscopic variables which describe the shock wave structure must attain certain values at the second surface as prescribed by the jump conditions. Since the jump conditions take no cognizance of the shock wave structure, they are independent of parameters which describe purely internal phenomena such as the collision transfer of momentum or energy.

In order to derive the appropriate jump conditions, the moment equations for the entire system are utilized. These are obtained by summing the constituent equations (equations 27) over the constituents of the system.

The first equation for the system is

$$\frac{\partial}{\partial t}\left[\sum_A m_A n^A\right] = -\frac{\partial}{\partial x_j}\left[\sum_A m_A n^A w_j^A\right] \tag{29}$$

which is left in this form.

The second equation for the system can be written as

$$\frac{\partial}{\partial t}\left[\sum_A m_A n^A w_i^A\right] = -\frac{\partial}{\partial x_j}\left[\sum_A (m_A n^A w_i^A w_j^A + \psi_{ij}^A)\right]$$

$$+ \sum_A\left[e_A n^A (E_i + \frac{1}{c}\epsilon_{ijk} w_j^A H_k) + P_i^A\right] \tag{30}$$

But

$$\sum_A\left[e_A n^A (E_i + \frac{1}{c}\epsilon_{ijk} w_j^A H_k)\right] = q E_i + \frac{1}{c}\epsilon_{ijk} j_j H_k \tag{31}$$

from equations (9), (10) and (11). Subsequent use of Maxwell's equations yields

$$q E_i + \frac{1}{c} \epsilon_{ijk} j_j H_k = - \frac{\partial}{\partial x_j} T_{ij} - \frac{1}{c^2} \frac{\partial}{\partial t} S_i \tag{32}$$

where $T_{ij}$ is the Maxwell stress tensor given by

$$T_{ij} = - \frac{1}{4\pi} \left[ E_i E_j + H_i H_j - \frac{1}{2} \delta_{ij} (E^2 + H^2) \right] \tag{33}$$

and $S_i$ is the Poynting vector defined as

$$S_i = \frac{c}{4\pi} \epsilon_{ijk} E_j H_k \tag{34}$$

Inserting these results in equation (30) yields

$$\frac{\partial}{\partial t} \left[ \sum_A m_A n^A w_i^A + \frac{1}{c^2} S_i \right] = - \frac{\partial}{\partial x_j} \left[ \sum_A (m_A n^A w_i^A w_j^A + \psi_{ij}^A) + T_{ij} \right] + \sum_A P_i^A \tag{35}$$

The third moment equation for the system can be written as

$$\frac{\partial}{\partial t} \left[ \sum_A (\frac{3}{2} n^A kT^A) \right] = \sum_A \left[ - \frac{\partial}{\partial x_j} (\frac{3}{2} n^A kT^A w_j^A) - \frac{\partial w_i^A}{\partial x_j} \psi_{ij}^A - \frac{\partial Q_j^A}{\partial x_j} + K^A \right] \tag{36}$$

If one forms the product of $w_i^A$ and the second moment equation for a particular constituent, i.e., equation (27b), sums over all constituents, and adds the result to equation (36), one obtains

$$\frac{\partial}{\partial t}\left[\sum_A \left(\frac{1}{2}\, m_A n^A w_i^A w_i^A + \frac{3}{2}\, n^A kT^A\right)\right] =$$

$$-\frac{\partial}{\partial x_j}\left[\sum_A \left(\frac{1}{2}\, m_A n^A w_i^A w_i^A w_j^A + w_i^A \psi_{ij}^A + \frac{3}{2}\, n^A kT^A w_j^A\right.\right.$$

$$\left.\left. + Q_j^A\right)\right] + \sum_A \left(e_A n^A w_i^A E_i + w_i^A P_i^A + K^A\right) \tag{37}$$

Now, again from equations (9), (10), and (11),

$$\sum_A e_A n^A w_i^A E_i = j_i E_i \tag{38}$$

which can be written

$$j_i E_i = -\frac{\partial}{\partial x_i} S_i - \frac{1}{8\pi}\frac{\partial}{\partial t}\left(E_i^2 + H_i^2\right) \tag{39}$$

Consequently, equation (37) becomes

$$\frac{\partial}{\partial t}\left[\sum_A \left(\frac{1}{2}\; m_A n^A w_i^A w_i^A + \frac{3}{2}\; n^A kT^A\right) + \frac{1}{8\pi}\;(E_i^2 + H_i^2)\right] =$$

$$- \frac{\partial}{\partial x_j}\left[\sum_A \left(\frac{1}{2}\; m_A n^A\; w_i^A\; w_i^A\; w_j^A + \frac{3}{2}\; n^A\; kT^A\; w_j^A\; + w_i^A\; \psi_{ij}^A\right.\right.$$

$$\left.\left. + Q_j^A\right) + S_j\right] + \sum_A (w_i^A\; P_i^A + K^A) \tag{40}$$

The conservation of momentum and energy in the collision process is expressed by the equations

$$\sum_{A,B} \int (m\,v_i)\; I_{AB}\; d^3\, v = 0 \tag{41}$$

$$\sum_{A.B} \int \left(\frac{mv_i^2}{2}\right) I_{AB}\; d^3\, v = 0 \tag{42}$$

which state that the net collision transfer of momentum or energy from the $A^{th}$ constituent to the remaining constituents when summed over all constituents is zero. The conservation of number has been already used implicitly since equation (28), which expresses this fact, was employed in deriving the moment relations for the individual constituents. The integrals in equation (41) and (42) have $v_i$ as the variable of integration but their form is similar to those occurring in equations (22) and (23). In fact, it can be shown that the following relations exist between the two pairs of integrals:

$$\sum_B \int (mv_i)\, I_{AB}\, d^3 v = P_i^A \tag{43}$$

$$\sum_B \int \left(\frac{mv_i^2}{2}\right) I_{AB}\, d^3 v = K^A + w_i^A\, P_i^A \tag{44}$$

Consequently, the quantities $\sum_A P_i^A$ and $\sum_A (K^A + w_i^A\, P_i^A)$ which occur in equations (35) and (40) vanish. As a result, equations (29), (35) and (40) all have the same form which may be written symbolically as

$$\frac{\partial}{\partial t}\left[\mathcal{A}\right] = - \frac{\partial}{\partial x_j}\left[\mathcal{B}_j\right] \tag{45}$$

If we integrate the three equations over the volume of the shock wave, employ the general divergence theorem, which states

$$\int_{\text{volume}} \frac{\partial}{\partial x_j} A\, d\tau = \int_{\text{surface}} A\, d S_j\,, \tag{46}$$

assume the shock wave has a negligible thickness in comparison to its extent, and finally assume that a stationary state exists, the jump conditions are:

$$\left[\left[\sum_A n^A\, w_j^A\right]_j\right] = 0 \tag{47a}$$

$$\left[\!\left[\sum_A (m_A n^A w_i^A w_j^A + \psi_{ij}^A) + T_{ij}\right]_j\right] = 0 \tag{47b}$$

$$\left[\!\left[\sum_A (\frac{1}{2} m_A n^A w_i^A w_i^A w_j^A + \frac{3}{2} n^A kT^A w_j^A + w_i^A \psi_{ij}^A + Q_j^A) + S_j\right]_j\right] = 0 \tag{47c}$$

where the double bracket notation represents the difference in the normal components of the enclosed quantity at the two surfaces. In effect, equations (47) state that the normal components of the mass flow, momentum flow, and energy flow are each conserved.

IV

# APPLICATION TO A PLANE SHOCK WAVE

As an application of the foregoing, the moment equations for a particular constituent will be obtained for the case of a plane shock wave moving through a field-free quiescent medium with a constant velocity U in the z-direction. Only the stationary state is considered, i. e. all derivatives with respect to time are set equal to zero. One-dimensional flow is assumed so that the spatial dependence of the various factors in the equations is a function of z only. Symmetry about the flow axis — the only preferred axis in space — exists; therefore the distribution function must be invariant with respect to rotations about this axis. As a consequence, it can be shown that

$$w_x^A = w_y^A = 0 \tag{48}$$

$$\psi_{xx}^A = \psi_{yy}^A \tag{49}$$

$$\psi_{ij}^A = 0 \text{ for all } i \neq j \tag{50}$$

$$Q_x^A = Q_y^A = 0 \tag{51}$$

i.e., there exists no mass or heat flow in the radial direction and the off-diagonal elements of the stress tensor vanish while the first two diagonal elements are equal. The flow velocity in the z-direction is given by

$$w_z^A(z) = W_z^A(z) - U \tag{52}$$

where $W_z^A(z)$ is the flow velocity relative to a fixed observer. Finally, the assumptions of time independence, one-coordinate dependence, and the absence of external fields, when applied to Maxwell's equations for the internal fields, imply that no internal magnetic field exists, the internal electric field has a component only in the z-direction, and non-trivially,

$$j_z = 0 \tag{53}$$

With these introductory remarks and results, we return to equations (27) and exhibit the explicit moment relations for the $A^{th}$ constituent. From equation (27a), we obtain

$$\frac{d}{dz} (n^A w_z^A) = 0 \tag{54a}$$

from (27b),

$$P_x = P_y = 0 \tag{54b}$$

$$w_z^A \frac{dw_z^A}{dz} = -\frac{1}{\rho^A} \frac{d\psi_{zz}^A}{dz} + \frac{e_A}{m_A} E_z + \frac{1}{\rho^A} P_z^A \tag{54c}$$

from (27c),

$$w_z^A \frac{dT^A}{dz} = -\frac{2}{3} \frac{1}{n^A k} \left(\frac{dw_z^A}{dz} \psi_{zz}^A + \frac{dQ_z^A}{dz} - K^A\right) \tag{54d}$$

from (27d)

$$\mathbf{X}_{ij} = 0 \text{ for } i \neq j \tag{54e}$$

$$w_z^A \frac{d\overset{o}{\psi}{}_{xx}^A}{dz} = -\frac{dw_z^A}{dz} \overset{o}{\psi}{}_{xx}^A + \frac{2}{3} \frac{dw_z^A}{dz} \psi_{zz}^A + \frac{4}{15} \frac{dQ_z^A}{dz} + \overset{o}{\mathbf{X}}{}_{xx}^A \tag{54f}$$

$$w_z^A \frac{d\overset{o}{\psi}{}_{yy}^A}{dz} = -\frac{dw_z^A}{dz} \overset{o}{\psi}{}_{yy}^A + \frac{2}{3} \frac{dw_z^A}{dz} \psi_{zz}^A + \frac{4}{15} \frac{dQ_z^A}{dz} + \overset{o}{\mathbf{X}}{}_{yy}^A \tag{54g}$$

$$w_z^A \frac{d\overset{o}{\psi}{}_{zz}^A}{dz} = -\frac{dw_z^A}{dz} \overset{o}{\psi}{}_{zz}^A - \frac{4}{3} \frac{dw_z^A}{dz} \psi_{zz}^A - \frac{8}{15} \frac{dQ_z^A}{dz} + \overset{o}{\mathbf{X}}{}_{zz}^A \tag{54h}$$

and from (27e)

$$F_x^A = F_y^A = 0 \tag{54i}$$

$$w_z^A \frac{dQ_z^A}{dz} = - \frac{16}{5} \frac{dw_z^A}{dz} Q_z^A + \frac{\overset{o}{\psi}{}_{zz}^A}{\rho^A} \frac{d\psi_{zz}^A}{dz} - \frac{kT^A}{m_A} \frac{d\overset{o}{\psi}{}_{zz}^A}{dz}$$

$$- \frac{5}{2} \frac{k}{m_A} \frac{dT^A}{dz} \overset{o}{\psi}{}_{zz}^A - \frac{5}{2} \frac{kp^A}{m_A} \frac{dT^A}{dz} - \frac{1}{\rho^A} (\frac{5}{2} p^A + \overset{o}{\psi}{}_{zz}^A) P_z^A \tag{54j}$$

$$+ F_z^A .$$

The jump conditions, (equations 47), become

$$\left[\left[ \sum_A n^A w_z^A \right]\right] = 0 \tag{55a}$$

$$\left[\left[ \sum_A (m_A n^A (w_z^A)^2 + \psi_{zz}^A) - \frac{1}{8\pi} E_z^2 \right]\right] = 0 \tag{55b}$$

$$\left[\left[ \sum_A (\frac{1}{2} m_A n^A (w_z^A)^3 + \frac{3}{2} n^A kT^A w_z^A + \psi_{zz} w_z^A + Q_z^A) \right]\right] = 0 \tag{55c}$$

The boundary conditions at the shock wave limits are such that the ambient conditions are realized on the appropriate surfaces. Recall, also that the properties of the medium, as discussed in Section II, restrict the form of the collision terms appearing in equations (54).

# V

# DISCUSSION

By starting with the Boltzmann equation and utilizing the Grad thirteen-moment approximation with a modification by Everett, the nonlinear differential equations governing the formation and structure of the shock wave associated with a hypersonic vehicle moving through a weakly-ionized low-density medium have been obtained; see equations (27) and the following discussion. The corresponding jump conditions are also set forth; see equations (47). The shock wave equations have been applied to the case of a plane shock wave and, after some simplification, are exhibited explicitly; see equations (54). No definite conclusions about either the structure of the shock wave or the possibility of the emission of electromagnetic radiation during its formation can as yet be drawn since the study is incomplete and has been terminated. Equations (54b, 54e, 54i) are, in a sense, solutions for the transverse collision transfer of momentum, traceless stress, and heat flow, respectively, in a plane shock wave. The fact that all these quantities vanish is to be expected; however, these results do serve as a check on the validity of the method and the assumptions employed.

In order to proceed further, the collision terms appearing in equations (54) must be evaluated subject to the restrictions imposed by the properties of the medium. As a consequence of this calculation, further simplification of equations

(54) is to be expected. Various approximation procedures can be used to solve the final set of equations since it is very likely that they will not be amenable to formal methods of solution. To study the stationary state, either numerical or iterative procedures, or a combination of both, can be adopted. To study the approach to the stationary state, the time dependent equations may be linearized by means of a perturbation technique.

## REFERENCES


[1] Probstein, R. E., ARS Journal 31 No. 2, 185 (1961).

[2] Lees, L. and Liu, C. Y., Proc. Second Int. Symposium on Rarefied Gas Dynamics (New York: Academic Press, 1961).

[3] Shafranov, V. D., JETP 32, 1453, (1957).

[4] Jukes, J. D., J. of Fluid Mechanics 3 No. 3, 275 (1957).

[5] Greenberg, O.W., et.al., Phys. of Fluids 3 No. 3, 379 (1960).

[6] Greenberg, O. W. and Trève, Y. M., Phys. of Fluids 3 No. 5, 769 (1960)

[7] Grad, H., Comm. on Pure and Appl. Math. 2 331 (1949).

[8] Everett, W., "Generalized Magnetohydrodynamic Equations for Plasma Systems with Large Currents", Doctoral dissertation, University of Michigan (1961).

[9] Chapman, S. and Cowling, T.G., The Mathematical Theory of Non-Uniform Gases (Cambridge: University Press, 1960).

[10] Osborn, R.K., "Notes on Plasma Physics", Jet Propulsion Laboratory Technical Report No. 32-4 (1960).

[11] Panofsky, W., and Phillips, M., Classical Electricity and Magnetism (Cambridge: Addison-Wesley, 1955).

[12] Courant, R. and Friedrichs, K. O., Supersonic Flow and Shock Waves (London: Interscience Publishers, 1948).