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ON THE ANALYSIS OF BUCKLED PLATES

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ABSTRACT

A new approach is presented for the analysis of the post-buckling behavior of plates. Based on the large deflection theory of von Kármán, the middle-plane (membrane) stresses are shown to obey a minimum-energy principle. This principle, together with the employment of stress-function space, is utilized to develop a sequence of solutions, each of which can be made to approximate the correct solution to a closer degree than the previous one. An error estimate is provided at each stage through the establishment of upper- and lower-bound principles.

1. INTRODUCTION

Although the basic equations governing the post-buckling behavior of plates were developed by von Kármán [1][1] many years ago, the number of exact solutions that have been found thus far is exceedingly small. A number of approximate solutions have been obtained, notably by Marguerre [2], through an adaptation of the Ritz method. Essentially this consists in selecting from an aggregate of geometrically possible deflection modes a linear combination in such a way as to render the total potential energy stationary. However, as pointed out by von Kármán [3] and by Friedrichs and Stoker [4], [5], this line of attack seems to become increasingly unreliable as the post-buckling domain is penetrated more and more deeply. Furthermore, the success of this method of approach depends heavily on the ingenuity of the computer in guessing the type of deflection mode to be expected; this difficulty is compounded further by the fact that no estimate of the reliability of the results is available.

[1]Numbers in brackets refer to the References at the end of the paper.

In the present paper an alternate avenue of approach is followed. It is shown that the middle-surface (membrane) stresses in the buckled plate are characterized by the satisfaction of a minimum-energy principle. Through the use of a stress function space, this principle is then utilized in the establishment of a sequence of trial solutions, in which each approximation is obtained from the previous one through semi-iterative procedure. Error estimates are obtained at each step through an upper- and lower-bound principle, which "guides" the computer toward a refinement of his trial solution. Moreover, the method is applicable throughout the post-buckling domain; in fact, its use is demonstrated for the limiting case, that is, for the behavior of a buckled plate as its edge tractions approach infinity.

In what follows, let a plate of thickness h (where h is assumed to be small compared with the dimensions of the plane region R of the middle surface) be subjected to prescribed surface tractions λT_i on a portion B' of its boundary B and to displacements λU_i on the remainder B" of its boundary; the subscript i(i=1,2) refers to the coordinates of the middle surface of the plate, while the parameter λ is assumed to take on increasing positive values. When λ is sufficiently small, say $\lambda < \lambda_o$, the equilibrium of the plate is stable. This equilibrium is characterized (within the limits of thin-plate theory) by the plane "membrane stress" field λt^o_{ij} and the plane "membrane displacement" field λu^o_i. If the usual conventions relating to indicial notation are adopted here, the former obeys the field equations of equilibrium

$$t^o_{ij,j} = 0 \qquad \text{in } R \tag{1}$$

and the boundary conditions

$$t^o_{ij} n_j = T_i \qquad \text{on } B' \tag{2}$$

in which n_i is the outer normal vector on the boundary. Moreover, the displacement field λu_i^o satisfies the boundary condition

$$u_i^o = U_i \qquad \text{on } B'' \tag{3}$$

and is related to the strain field λe_{ij}^o by the equations

$$e_{ij}^o = \frac{1}{2}\left(u_{i,j}^o + u_{j,i}^o\right) \tag{4}$$

Finally, the strains are linearly related to the stresses; for the purpose of the present discussion, it is sufficient to postulate only the symmetry and positive definiteness of the matrix of the stress-strain coefficients.

Let the equilibrium so defined become unstable when the parameter λ exceeds λ_o. Then the plate buckles into the configuration $w(x_1,x_2)$ which, again within the confines of thin-plate theory and in the case of isotropy, is governed by

$$D\Delta\Delta w \equiv D w_{,iijj} = h\, t_{ij} w_{,ij} \qquad \text{in } R \tag{5}$$

and a system of linear homogeneous boundary conditions; D represents the stiffness of the plate.

In Eq. (5), the membrane-stress field t_{ij} (and the associated strain field e_{ij} and displacement field u_i) is in general not the same as the one relating to the unbuckled (and unstable) configuration. In fact, let

$$t_{ij} = \lambda t_{ij}^o + t_{ij}' \qquad e_{ij} = \lambda e_{ij}^o + e_{ij}' \qquad u_i = \lambda u_i^o + u_i'. \tag{6}$$

Then the "additional membrane stress" t_{ij}' is readily seen to satisfy the field equations

$$t_{ij,j}' = 0 \qquad \text{in } R \tag{7}$$

and boundary conditions

$$t'_{ij} n_j = 0 \quad \text{on } B' \tag{8}$$

All stress fields satisfying Eqs. (7) and (8), including t'_{ij}, will be referred to as being "statically homogeneous."

The additional strain e'_{ij} is related to t'_{ij} by the same linear law as the one mentioned previously. However, within the limits of the present theory, the strain-displacement relations now become

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i} + w_{,i} w_{,j}\right) \tag{9}$$

or, in view of Eqs. (4) and (6)

$$e'_{ij} = \frac{1}{2}\left(u'_{i,j} + u'_{j,i} + w_{,i} w_{,j}\right) \tag{9a}$$

Finally, the additional displacements satisfy the boundary condition

$$u'_i = 0 \qquad \text{on } B''. \tag{10}$$

2. A MINIMUM-ENERGY PRINCIPLE

It is readily verifiable that the preceding formulation of the problem is equivalent to its formulation in [1]. However, as already pointed out in [5], Eqs. (1) through (10) do not characterize the buckled plate configuration uniquely if λ becomes sufficiently large. For example, if λ exceeds λ_o, there are at least three solutions, of which two differ only by the sign of w, and the third represents the unbuckled state $w = 0$, $t_{ij} = \lambda t^o_{ij}$. When λ increases further, additional solutions (i.e., higher modes) become possible. These, as well as the unbuckled configuration, are unacceptable on the basis of dynamic considerations.

In the present paper, unique characterization of the problem (except for the sign of w) is achieved for any given value of λ by the added requirement that the membrane-stress field t_{ij} be "statically admissible." Any stress field T_{ij} is considered statically admissible if it satisfies the equations of equilibrium in R and B'; in other words, it is expressible in the form

$$T_{ij} = \lambda t^{o}_{ij} + T^{s}_{ij} \tag{11}$$

where T^{s}_{ij} is statically homogeneous. In addition, the inequality

$$V(T;W) \equiv U_b(W) + \frac{h}{2}\int_R T_{ij} W_{,i} W_{,j}\, dA \geqq 0 \tag{12}$$

must be satisfied, in which $U_b(W)$ represents the bending strain energy in the plate and $W(x_1,x_2)$ is any nontrivial deflection configuration which has piecewise continuous second derivatives and satisfies the geometric boundary conditions relating to w on B.

Since the actual mode w itself satisfies these conditions, it follows that $V(T;w) \geqq 0$. At the same time, $V(t;w) = 0$ as can be seen in the usual fashion from Eq. (5) through multiplication by w, integration over R, and the application of Green's Theorem. Hence,

$$\frac{h}{2}\int_R (T_{ij} - t_{ij})\, w_{,i} w_{,j}\, dA \geqq 0 \tag{13}$$

or, in view of Eqs. (6) and (11),

$$\frac{h}{2}\int_R (T^{s}_{ij} - t'_{ij})\, w_{,i} w_{,j}\, dA \geqq 0 \tag{13a}$$

If now the expression for e_{ij} in Eq. (9) is considered, then, in view of the symmetry of T_{ij} and t_{ij}, the inequality (13) can be written as

$$h\int_R (T_{ij} - t_{ij})\, e_{ij}\, dA - h\int_R (T_{ij} - t_{ij})\, u_{i,j}\, dA \geqq 0$$

Since both T_{ij} and t_{ij} satisfy the equations of equilibrium in R as well as the same boundary conditions on B', the second integral is converted by Green's Theorem into

$$\lambda h \int_{B''} (T_{ij} - t_{ij})\, n_j\, U_i\, dS$$

Furthermore, in view of the symmetry of the stress-strain matrix,

$$2\, T_{ij}\, e_{ij} = t_{ij}\, e_{ij} + T_{ij}\, E_{ij} - (T_{ij} - t_{ij})(E_{ij} - e_{ij})$$

in which E_{ij} is the strain field associated with T_{ij}. This leads finally to the relationship

$$\begin{aligned} U(T,T) - 2W''(T) &\geqq U(t,t) - 2W''(t) + U(T-t, T-t) \\ &\geqq U(t,t) - 2W''(t) \end{aligned} \qquad (14)$$

in which the second inequality is the result of the positive definiteness of the stress-strain matrix. In (14), U represents twice the strain energy associated with the membrane stresses and is defined by

$$U(T,T) = h\int_R T_{ij}\, E_{ij}\, dA \qquad (14a)$$

while W" is the work done by the boundary tractions over that portion of the boundary on which the displacements are prescribed; it is given by

$$W''(T) = \lambda h \int_{B''} T_{ij}\, n_j\, U_i\, dS. \qquad (14b)$$

The equality sign in (14) is restricted to the trivial case in which $T_{ij} \equiv t_{ij}$. If, as usual, 1/2 U-W" is defined as the complementary energy, the inequality (14) is therefore an expression of the following principle:

The complementary energy corresponding to the actual membrane stresses is smaller than that corresponding to any other statically admissible membrane state of stress.

This principle is, of course, well established for the stable, unbuckled plate (that is, for $\lambda < \lambda_0$), in which case λt^0_{ij} is itself statically admissible. Its applicability to the nonlinear buckled domain, however, represents an extension which will be utilized in the computational procedure to be developed in a later section. Moreover, the variational aspect of this minimum principle can be shown to represent a special case of a broad variational theorem published recently by E. Reissner [6].

It will be more convenient to employ the principle in slightly altered form. To this end, consider the inequality (13a) and substitute Eq. (9a). Then a sequence of steps analogous to the one preceding the establishment of the inequality (14) leads to the following relationship:

$$U(T^s, T^s) \geqq U(t', t') + U(T^s - t', T^s - t') \geqq U(t', t') \tag{15}$$

In other words, the strain energy associated with the additional membrane stresses is as small as possible subject to the requirement of static admissibility.

It is also of some interest to note the similarity between the principle expressed by the inequality (14) and a theorem of Haar and von Kármán [7], which considers the state of stress in an elastic-plastic structure in the absence of unloading. By the same token, the inequality (15) corresponds to the Prager-Hodge version [8] of the same theorem. Furthermore, it is easy to show that the inequality remains valid in the presence of a (statically homogeneous) initial stress field t^*_{ij}, provided the symbols T^S and t' are replaced by $T^S - t^*$ and $t' - t^*$, respectively.

3. GEOMETRIC INTERPRETATION

The discussion in the previous section, as well as in the following sections, lends itself to a geometric interpretation which, although not necessary, may nevertheless be considered useful from the point of view of clarity and simplicity. In this, the present study leans heavily on the contents (though not necessarily notation) of a previous paper by Prager and Synge [9], in which the concept of function space was first utilized in the solution of stress analysis problems.

To this end, let a space be defined (see Fig. 1) in which the vector $\bar{t} \equiv \overline{OP}$ represents the stress field $t_{ij}(x_1,x_2)$. The metric of this space is established by the requirement that if $\bar{t}^1 = \overline{OP}_1$ and $\bar{t}^2 = \overline{OP}_2$ are any two stress fields, then the inner product between the two vectors is defined by

$$\bar{t}^1 \cdot \bar{t}^2 \equiv L(\bar{t}^1)L(\bar{t}^2)\cos\theta = U(t^1, t^2) \equiv h \int_R t^1_{ij}\, e^2_{ij}\, dA \tag{16}$$

where $L(\bar{t})$ represents the length of $\bar{t}$ and θ is the angle between the vectors.

By letting $\bar{t}^1 = \bar{t}^2 = \bar{t}$ and $\theta = 0$, it follows that the length of a vector is given by

$$L(\bar{t}) = {}_{+}\left[U(t,t)\right]^{1/2} \equiv {}_{+}\left[h \int_R t_{ij}\, e_{ij}\, dA\right]^{1/2} \tag{17}$$

which is real in view of the positive definiteness of the stress-strain matrix. The symmetry of the latter results in

$$\bar{t}^1 \cdot \bar{t}^2 = \bar{t}^2 \cdot \bar{t}^1, \tag{18}$$

while $\cos^2\theta \leq 1$ (or θ real) follows from Eqs. (16) and (17) and Schwarz's Inequality.

For the purposes of the present discussion, it is useful to consider the following two subspaces: the one of all statically homogeneous stress fields, and the subspace of all "geometrically homogeneous" stress fields. As for the former, it includes all vectors which designate stress fields satisfying Eqs. (7) and (8). On the other hand, a vector $\overline{t}''$ will be called geometrically homogeneous if its stress field t''_{ij} is associated with a strain field e''_{ij} which is derivable from a displacement field u''_i by means of the relations

$$e''_{ij} = \frac{1}{2}\left(u''_{i,j} + u''_{j,i}\right) \tag{19}$$

and provided further that the boundary condition

$$u''_i = 0 \qquad on \quad B'' \tag{20}$$

is satisfied.

It can readily be shown through Green's Theorem and in view of the boundary conditions that the two subspaces are orthogonal to one another. Furthermore they are complete in the sense that there exists no nonvanishing stress-field vector which is orthogonal to both; in other words, any nonvanishing vector can be resolved completely and uniquely into a statically homogeneous and a geometrically homogeneous "component." This will be demonstrated in the next section.[2]

Only the subspace of statically homogeneous vectors $\overline{T}'$ will be considered in what follows. In that space, for a given value of λ, the inequality (12), in conjunction with Eq. (11), defines a "surface" S (hereafter referred to as "neutral surface") in the sense that it divides the space into two regions: the stable region of stress fields $\overline{T}^s$ in which $V(T;W) > 0$, and the unstable region where V may be negative for at least one function W. On S itself $V(T;W)$ is positive, but not definite; it vanishes when the deflection function W is selected

[2] The proof of the completeness of the two subspaces is essentially the same as the one employed in a recent paper [10] by Budiansky and Pearson. Note, however, that these authors use different subspaces, which are neither homogeneous nor orthogonal in the sense of the present paper.

to be the fundamental buckling mode w which satisfies Eq. (5) and the natural boundary conditions.

For different values of the parameter λ there are separate neutral surfaces. Each surface corresponding to $\lambda \geqq \lambda_o$ contains the actual stress point since, as shown before, $V(t;w) = 0$, and since furthermore t_{ij} is postulated to be statically admissible to rule out higher buckling modes. If $\lambda = \lambda_o$, S passes through the origin in the absence of prestressing. When a prestress field t^*_{ij} is prescribed, the initial neutral surface ($\lambda = \lambda_o$) contains the corresponding point P*.

The inequality (15) can now be interpreted geometrically. In fact, let $\lambda > \lambda_o$ be given, and let point p represent the actual additional stress t'_{ij}, while P is associated with any statically admissible additional stress field T'_{ij}. In view of Eq. (17) it follows from (15) that p is located closer to 0 (or to P*) than any other point P which lies either on S or on the "stable" side of S. Moreover, from the first inequality it is apparent that $(\overline{OP})^2 \geqq (\overline{Op})^2 + (\overline{pP})^2$; in other words, the angle OpP is not acute. Similar considerations apply in the presence of an arbitrary (statically homogeneous) initial stress field t^*_{ij} designated by the point P*. It follows therefore that all surfaces S are convex or plane.

Three representative surfaces, which correspond, respectively, to the case of stable equilibrium ($\lambda < \lambda_o$), initial buckling ($\lambda = \lambda_o$), and postbuckling ($\lambda > \lambda_o$), are shown in Fig. 1 for the unprestressed plate. Also shown is the "stress path" Γ, which is the locus of all actual stress points for increasing values of λ. In the first (stable) case, the origin is seen to lie on the concave side of the corresponding surface of neutral equilibrium; $t'_{ij} \equiv 0$ is therefore statically admissible and represents the actual additional stress field for all prebuckling cases. Conversely, when the initial buckling parameter λ_o is exceeded, the actual stress point p "travels" with the associated surface S in such a way as always to remain closest to the origin. Through an elementary limit process it is easy to deduce that Γ is normal to S at O;[3] however, this orthogonality does not, in general, apply to the other surfaces.

Let now P be a generic stress point which lies on a surface S associated with a loading parameter λ; in other words, let Eq. (5) be satisfied for a stress field t_{ij} which is expressible in terms of Eq. (6). Note, however, that this need not be the actual stress field corresponding to λ since Eq. (9) need not be satisfied. Let it be assumed further that both the stress field t_{ij} and the buckling mode w are differentiable functions of a parameter, relative to which Eq. (5) is identically satisfied. If now a dot ($\dot{}$) above a letter designates differentiation with respect to that parameter,[4] it follows that

$$D\dot{w}_{,iijj} - h(\lambda t^{o}_{ij} + t'_{ij})\dot{w}_{,ij} = h(\dot{\lambda} t^{o}_{ij} + \dot{t}'_{ij})w_{,ij} \tag{21}$$

where $\dot{t}'_{ij}$ is a statically homogeneous stress field.

Let both sides of Eq. (21) be multiplied by w and integrated over the region R, and let further Eq. (5) be multiplied by $\dot{w}$ and also be integrated over R. Then, by comparing the two results, through the repeated application of Green's Theorem, and in view of the boundary conditions, of the symmetry of the stress tensors, and of Eq. (1) and the static homogeneity of t'_{ij} and $\dot{t}'_{ij}$, it follows that

$$h\int_R (\dot{\lambda} t^{o}_{ij} + \dot{t}'_{ij}) w_{,i} w_{,j}\, dA = 0 \tag{22}$$

In Eq. (22) the direction of the path of differentiation is arbitrary. It may therefore be taken normal to the surface S, in which case

[3] It is presumed here that S is sufficiently smooth to make this statement meaningful; this question is dealt with again later on.

[4] This can readily be interpreted geometrically to represent differentiation along a path in stress space.

$$h \int_R \left(\frac{d\lambda}{dn} t^{o}_{ij} + t^{n}_{ij} \right) w_{,i} w_{,j} \, dA = 0 \tag{22a}$$

with the superscript ^{n} denoting the appropriate differentiation.

On the other hand, if Eq. (22) is applied to a direction which is tangent to S, $\dot{\lambda}$ vanishes by definition; hence:

$$h \int_R t^{t}_{ij} w_{,i} w_{,j} \, dA = 0 \tag{22b}$$

in which t^{t}_{ij} represents the totality of all stress fields whose representative vector is tangent to S.

This can again be given a geometric interpretation. In fact, let the strain field E_{ij} be defined by

$$E_{ij} = w_{,i} w_{,j} \tag{23}$$

and let the stress field N_{ij} be associated with E_{ij} through the usual stress-strain relations. Then Eq. (22b) implies that the vector $\overline{N}$ (representing N_{ij}) is normal to all vectors $\overline{t}^{t}$. Now, as pointed out before,

$$\overline{N} = \overline{N}' + \overline{N}'' \tag{24}$$

in which $\overline{N}'$ is statically homogeneous and $\overline{N}''$ is geometrically homogeneous. The latter is the solution of a (generally mixed) plane boundary value problem of linear elasticity and is characterized by

$$N''_{ij,j} = N_{ij,j} \text{ in } R \qquad N''_{ij} n_j = N_{ij} n_j \text{ on } B'' \tag{25}$$

as well as a set of equations of the type of Eqs. (19) and (20). It is itself orthogonal to the (statically homogeneous) vector $\overline{t}^{t}$. It follows therefore that

$$\overline{N}' \cdot \overline{t}^{t} = 0 \tag{26}$$

for all tangent vectors. In other words, the statically homogeneous component of the stress vector which is derived from the buckling mode by means of Eq. (23) is normal to the surface of neutral equilibrium;[5] it can readily be shown to point towards its concave side.

In the foregoing development, the existence of a unique tangent plane and normal to S was presumed. However, S may exhibit "corners" - that is, points of multiple tangent planes. This happens when, for given stress field t_{ij}, the eigenfunction w satisfying Eq. (5) and associated boundary conditions is not unique. Such a case was discussed by Stoker [12] in connection with the effect of prestressing on the buckling parameter of plates; it signifies the discontinuous change from one buckling mode (say, one of circular symmetry) to another (not displaying such symmetry). It implies further that there exists an at least two-dimensional manifold of "normal" vectors which are derivable from linear combinations of buckling modes. It is significant, however, that the minimum principle (14) or (15) as well as the convexity of the neutral surface are unaffected by this type of singularity.

Returning to the case of a smooth surface, it may be desirable, for computational purposes, to determine the curvature of the curve which is formed by the intersection of the surface S with the plane containing the vector $\bar{t}'$ and the normal vector $\bar{N}'$.[6] To this end, consider Eq. (22b), which is identically satisfied on S; it can therefore be differentiated with respect to any tangential direction, and in particular with respect to the direction of $\bar{t}^t$, which yields:

[5]Attention is directed toward the formal similarity between this condition and the orthogonality of the plastic strain with respect to the yield surface [11].

[6]It is, of course, assumed that $\bar{t}'$ and $\bar{N}'$ are linearly independent. If they are not, then the following discussion becomes meaningless. On the other hand, linear dependence between $\bar{t}'$ and $\bar{N}'$ implies, at least in the usual case of open surfaces, that the point P represents the actual stress point. In that event, the need to compute the curvature is obviated.

$$h\int_R t^{tt}_{ij}\, w_{,i} w_{,j}\, dA + 2h\int_R t^t_{ij}\, w^t_{,i} w_{,j}\, dA = 0 \tag{27a}$$

in which $w^t(x_1 x_2)$ is governed by

$$D w^t_{,iijj} - h\left(\lambda t^o_{ij} + t'_{ij}\right) w^t_{,ij} = h\, t^t_{ij}\, w_{,ij} \tag{27b}$$ [7]

The first integral in Eq. (27a) represents $\bar{t}^{tt}.\bar{N}$, or, since $\bar{t}^{tt}$ is statically homogeneous, $\bar{t}^{tt}.\bar{N}'$. But $\bar{t}^{tt}$ has a normal component of magnitude (1/R) x ($\bar{t}^t.\bar{t}^t$), with R representing the radius of curvature. The curvature is therefore given by the formula

$$\frac{1}{R} = -2\,\frac{h\int t^t_{ij}\, w^t_{,i} w_{,j}\, dA}{(\bar{t}^t\cdot\bar{t}^t)(\bar{N}'\cdot\bar{N}')^{1/2}} \tag{28}$$

The vector $\bar{t}^t$, and its associated stress field t^t_{ij}, appears in both the numerator and denominator of Eq. (28). It is obtained by combining $-\bar{t}'$ with a multiple of $\bar{N}'$ in such a way as to make it orthogonal to $\bar{N}'$. Hence

$$\bar{t}^t = -\bar{t}' + \frac{\bar{t}'\cdot\bar{N}'}{\bar{N}'\cdot\bar{N}'}\,\bar{N}' \tag{28a}$$

and the denominator in Eq. (30) becomes

$$(\bar{t}^t\cdot\bar{t}^t)(\bar{N}'\cdot\bar{N}')^{1/2} = (\bar{N}'\cdot\bar{N}')^{-1/2}\left[(\bar{t}'\cdot\bar{t}')(\bar{N}'\cdot\bar{N}') - (\bar{t}'\cdot\bar{N}')^2\right] \tag{28b}$$

4. PROPOSED ANALYTICAL APPROACH

The remarks of the preceding sections may now serve as a basis for a proposed method of analysis of the postbuckling domain. In fact, let $\lambda > \lambda_o$ be

[7] Although Eq. (27b), together with the boundary conditions, represents a singular nonhomogeneous system in w^t, a solution is assured in view of the orthogonality (22b). This solution is of course not unique, but subject to an arbitrary multiple of w; however, this does not affect the value of the numerator in the expression for the curvature.

given, and let the membrane stress field be governed by Eq. (6), in which t^o_{ij} is presumed to be known. The determination of the additional stress field t'_{ij} and of the buckling configuration can then proceed as follows:

The first step consists in assuming[8] a statically homogeneous stress field t^1_{ij} (corresponding to the vector $\bar{t}^1$) and, letting

$$t'_{ij} = c_1 t^1_{ij} , \tag{29}$$

in finding c_1 and the deflection mode w_1 from the solution of Eq. (5) in conjunction with the first of Eqs. (6). This is a linear eigenvalue problem, which will be considered as solved for the purposes of the present paper.[9] Geometrically this amounts to finding point P_1, which represents the intersection of the line of action of vector $\bar{t}^1$ with the surface S.

The length L_1 of the line OP_1 is readily seen to be given by

$$L_1 = c_1 \left(U_{11} \right)^{1/2} \tag{30}$$

in which U_{11} is used for $U(\bar{t}^1,\bar{t}^1)$, or, in general, $U_{mn} \equiv U(\bar{t}^m,\bar{t}^n) = \bar{t}^m \cdot \bar{t}^n$. In view of the principle discussed in the previous section, L_1 represents an upper bound to the actual length L = Op.

A lower bound to L is established next. To this end, construct the statically homogeneous stress field N^1_{ij} which is normal to S at P_1. This is accomplished as described previously: from a vector $\bar{N}$ based on the strain field of Eq. (23) (with $w \equiv w_1$) a geometrically homogeneous component $\bar{N}''$ [based on Eqs. (25) and on Eqs. (19) and (20)] is subtracted; the remainder $\bar{N}^1$ is statically homogeneous. Since the amplitude of w_1 is arbitrary, the magnitude of $\bar{N}^1$ is also indefinite; otherwise $\bar{N}^1$ is determined uniquely. It can readily be shown that

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[8]A guide for making this initial assumption is described later in this section.

[9]An admittedly optimistic statement. However, the literature is replete with descriptions of exact, approximate, and iterative procedures so that there appears to be no need to pursue this question here.

this process is, in essence, tantamount to the solution of the compatibility equation of von Kármán [1] in the case of an isotropic material.

The angle between $\bar{t}^1$ and $\bar{N}^1$ (see Fig. 1) can now be computed from the relationship

$$\cos^2 \theta_1 = \frac{(\bar{t}^1 \cdot \bar{N}^1)^2}{(\bar{t}^1 \cdot \bar{t}^1)(\bar{N}^1 \cdot \bar{N}^1)} \tag{31}$$

It is a measure of the accuracy of the assumed vector $\bar{t}^1$. Indeed, $L_1 \cos \theta_1$ represents a lower bound to the exact length L, or

$$L_1 \cos \theta_1 \leqq L \leqq L_1 \tag{32}$$

This lower-bound principle is easy to prove by considering that $L_1 \cos \theta_1$ represents the perpendicular (i.e., shortest) distance from O to the plane tangent to S at P_1. That this distance does not exceed the actual distance L follows from the convexity of the surface S.

These results can be generalized somewhat. In fact, let $W(x_1, x_2)$ be an arbitrary geometrically consistent deflection mode, and let the associated stress-field vector $\bar{N}$ [see Eq. (23)] have a statically homogeneous component $\bar{N}'$. Let further

$$\bar{T} = \lambda \bar{t}^o + \bar{T}^k \tag{33a}$$

be "kinematically admissible" if

$$\bar{T}^k = c \bar{N}' \qquad (c \geqq 0) \tag{33b}$$

where c is determined from

$$V(T; W) = 0 \tag{33c}$$

Then

$$U(T^k, T^k) \leqq U(t', t') \tag{34}$$

or:

The strain energy associated with the additional membrane stresses is as large as possible subject to the requirement of kinematic admissibility.

The proof of the inequality (34) follows from the fact that, for a given mode W, the locus of all stress-field vectors $\overline{T}^k$ satisfying Eqs. (33a) and (33c) is a plane which is orthogonal to $\overline{N}'$ and which, in view of the inequality (12), lies "outside," or at most touches the surface S corresponding to λ. Furthermore, Eq. (33b) designates that point on the plane which is closest to the origin; the proof is completed by considering that the actual stress field $\overline{t}$ is itself kinematically admissible.[10]

To return to the proposed method of analysis, an improvement can now be obtained by assuming a new stress field $t^{!}_{ij}$ of the type

$$t^{1}_{ij} = c_2\, t^{2}_{ij} , \tag{35a}$$

where the vector $\overline{t}^2$ is formed by a linear combination of $\overline{t}^1$ and $\overline{N}^1$, or

$$t^{2}_{ij} = \alpha_1\, t^{1}_{ij} + \beta_1\, N^{1}_{ij} . \tag{35b}$$

The process is now repeated; in particular c_2 and the mode w_2 are obtained from the linear eigenvalue problem discussed above, and the new normal vector $\overline{N}^2$ as well as the new angle θ_2 are found as before.

The choice of constants α_1 and β_1 may be left to the judgment of the computer. However, if, in the neighborhood of P_1, the intersection of the $(\overline{t}^1,\overline{N}^1)$ plane with S is approximated by a circle of radius R, then the "best" assumption for $\overline{t}^2$ is clearly that of a vector pointing from 0 to the center of the circle.

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[10]Unlike the upper-bound principle (15), this principle becomes vacuous for the linear problem ($\lambda \leqq \lambda_0$) since, in that case, the region of kinematically admissible stress-field vectors shrinks to zero. It may also be of some theoretical interest to show how the method of Marguerre (see, e.g., [2]) ties in with the present lower-bound principle. In effect, Marguerre can be shown to establish an aggregate of kinematically admissible stress fields from a finite aggregate of deflection functions. By making the energy of the additional membrane stress fields stationary, he therefore arrives at a lower bound to the actual energy.

This is achieved by letting

$$\alpha_1 L(\bar{t}^1) : \beta_1 L(\bar{N}^1) = L_1 : R \tag{35c}$$

in which R is governed by Eqs. (28), (28a), and (28b).

Of course it must not be expected that this procedure automatically and necessarily converges to the correct solution, any more than the conventional Ritz method assures such convergence. In fact, unlike the Ritz method, the present procedure does not even guarantee that a new approximation represents an improvement relative to a previous approximation.

On the other hand, definite advantages seem to be offered. These are embodied primarily in the availability of an error estimate; by determining the angle θ between an assumed stress-field vector and the corresponding gradient vector according to Eq. (31), the computer keeps a check over his progress toward the final solution. Moreover, unlike the Ritz method, the proposed procedure "guides" him in the selection of succeeding approximations.

Nor is he entirely without guidance in making his first selection. As pointed out in the previous section, the stress path Γ is normal to the surface S which passes through the origin ($\lambda = \lambda_0$). This would suggest that in the early postbuckling range, that is, when λ exceeds λ_0 by only a small amount, a useful first approximation for $\bar{t}'$ is furnished by a vector which is based on the (generally known) initial buckling mode. Also, once the actual point p corresponding to a given value of λ has been obtained, an estimate for the distance from p to the surface S corresponding to the next higher value of λ is furnished by Eq. (22a). This can be seen by letting $\bar{t}' = c\bar{N}'$, whence $\bar{t}^n = (dc/dn)\bar{N}'$. Along the normal to S at p, Eq. (22a) can therefore be written in the form

$$\frac{dc}{d\lambda} = - \bar{t}^o \cdot \bar{N} / \bar{N}' \cdot \bar{N}' \tag{36}$$

After the stress field t^1_{ij} and the mode $w(x_1,x_2)$ associated with a certain value of λ have been found, there remains the problem of determining the magnitude of the actual deflection function. To this end let

$$w(x_1, x_2) = k\, w'(x_1, x_2) \tag{37}$$

in which w' has an arbitrary amplitude.[11] To find the multiplier k, consider that

$$\bar{t}' \cdot \bar{t}' = h \int_R t'_{ij}\, e'_{ij}\, dA \tag{38}$$

in which the strain field e'_{ij} satisfies Eq. (9a). When this is substituted in Eq. (38) and Green's Theorem is applied, then, by virtue of Eq. (7), of the boundary conditions (8) and (10), and of the symmetry of t'_{ij}, it follows that

$$\bar{t}' \cdot \bar{t}' = \frac{k^2}{2}\, \bar{t}' \cdot \bar{N}' \tag{39}$$

The left side of Eq. (39) represents $L^2(\bar{t}')$. Since, for the actual stress point p, $\bar{t}'$, and $\bar{N}'$ point in the same direction, the inner product on the right side of Eq. (39) equals $L(\bar{t}')L(\bar{N}')$. The amplification factor k is therefore given by the relationship

$$k^2 = 2\, L(\bar{t}') / L(\bar{N}') \tag{40}$$

5. NUMERICAL EXAMPLE

The analytical procedure proposed in the previous section is now applied to the case of a simply supported circular plate which is subjected to uniform radial pressure and which buckles into a mode of circular symmetry. This example appears to be almost the only one for which an exact solution exists

[11] The elegance of many of the equations can be enhanced by suitable normalizations; however, no computational advantage seems to result from such a step.

throughout the postbuckling range. It was treated exhaustively by Friedrichs and Stoker in [4] and [5]; hence it provides a good check on the accuracy and efficacy of the proposed method.

Let a be the radius of the plate, and let $\rho = r/a$ be the dimensionless radial coordinate of a generic point. If h is the thickness of the plate and the external pressure p is expressed in the form

$$p = \lambda_0 \alpha D/a^2 h \tag{41a}$$

then the most general radially symmetric, statically consistent stress field can be written in the form:

$$\begin{aligned} t_{rr} &= (D/a^2 h)\lambda_0(-\alpha + \beta f) \\ t_{\theta\theta} &= (D/a^2 h)\lambda_0\left[-\alpha + \beta(\rho f)_{,\rho}\right] \end{aligned} \qquad t_{r\theta} = 0 \qquad (0 \leq \rho \leq 1) \tag{41b}$$

where $f(\rho)$ satisfies the boundary conditions

$$f_{,\rho}(0) = f(1) = 0 \tag{41c}$$

For the case of an isotropic plate of material constants E and μ, respectively, the equation of equilibrium perpendicular to the plane of the plate [corresponding to Eq. (5)] takes the form

$$\left[\rho^{-1}(\rho\phi)_{,\rho}\right]_{,\rho} + \lambda_0(\alpha - \beta f)\phi = 0 \qquad (0 \leq \rho \leq 1) \tag{42a}$$

while the boundary conditions relating to regularity at the center and simple support at the edge are

$$\phi(0) = \phi_{,\rho}(1) + \mu\phi(1) = 0 \tag{42b}$$

In Eqs. (42), ϕ represents the slope $w_{,r}$. In the case of incipient buckling ($\alpha = 1$, $\beta = 0$), Eq. (42a) simplifies to a Bessel equation which, in conjunction with the boundary conditions (42b), exhibits a nontrivial solution when λ_0 is equal to 4.20 for $\mu = 0.30$.

Consider now the postbuckling range, for which $\alpha > 1$. An initial assumption for $f(\rho)$ could be obtained from the buckling mode for $\alpha = 1$ (see previous section). To demonstrate the rapidity of the convergence, this was not done here. Instead, the additional stress field was selected on the basis of simplicity alone, i.e.,

$$f_1(\rho) = 1 - \rho^2$$

When this is substituted in Eq. (42a), the resulting equation is solved in the usual manner by means of a power series approach. For a given value of α and an assumed value of β, this leads to recursion formulas among the coefficients. The correct value of β is then obtained by trial and error (and interpolation) from the boundary conditions (42b). For example, let $\alpha = 1.25$; this results in $\beta_1 = 0.631$, while the mode is given, subject to a multiplicative constant, by

$$\phi_1 = \rho - 0.325\rho^3 - 0.075\rho^5 + 0.022\rho^7 + 0.002\rho^9 \pm \cdots$$

With the stress-field vector $\bar{t}^1$ and the mode ϕ_1 thus found, the next step consists in determining the gradient vector $\bar{N}^1$. As outlined before, this is done by assuming a strain field on the basis of Eq. (23) and by subtracting from the associated stress-field vector $\bar{N}$ its geometrically homogeneous component. For the case under consideration, this amounts to letting

$$\begin{aligned} \phi_1^2 &= e'_{rr} + \frac{1}{a} u''_{,\rho} \\ 0 &= e'_{\theta\theta} + \frac{1}{a} u''/\rho \end{aligned} \qquad e'_{r\theta} = 0 \tag{43}$$

from which, by eliminating u", it follows that

$$e'_{rr} - (\rho e'_{\theta\theta})_{,\rho} = \phi_1^2 \tag{43a}$$

With the usual stress-strain relations for isotropic materials

$$Ee'_{rr} = F_1 - \mu(\rho F_1)_{,\rho} \qquad Ee'_{\theta\theta} = (\rho F_1)_{,\rho} - \mu F_1 \tag{44}$$

the radial stress F_1 and circumferential stress $(\rho F_1)_{,\rho}$ are governed by the equation

$$\rho^2 F_{1,\rho\rho} + 3\rho F_{1,\rho} \equiv \rho^{-1}(\rho^3 F_{1,\rho})_{,\rho} = -E\phi_1^2 \qquad (45)$$

Both the complementary and particular solutions to Eq. (45) are readily obtained by elementary methods; the two constants of integration are determined from the boundary conditions (41c). For the example under discussion, the gradient vector $\overline{N}^1$, when suitably "normalized," is governed by

$$F_1 = 1 - 1.275\rho^2 + 0.276\rho^4 + 0.009\rho^6 \pm \cdots$$

The angle Θ_1 between the vectors $\overline{t}^1$ and $\overline{N}^1$ is found next from Eq. (31). For isotropic materials and radially symmetric stress fields, the bilinear forms $U_{k\ell}$ are given by

$$U_{k\ell} = \overline{t}^k \cdot \overline{t}^\ell = \frac{2\pi a^2 h}{E} \int_0^1 \left[t_{rr}^k t_{rr}^\ell + t_{\theta\theta}^k t_{\theta\theta}^\ell - \mu \left(t_{rr}^k t_{\theta\theta}^\ell + t_{rr}^\ell t_{\theta\theta}^k \right) \right] \rho \, d\rho$$

For statically homogeneous stress fields this simplifies somewhat; in particular, the term involving Poisson's Ratio can be shown to vanish in view of the boundary conditions. This leads to:

$$U_{k\ell} = \frac{2\pi a^2 h}{E} \int_0^1 f_{k,\rho} f_{\ell,\rho} \rho^3 d\rho \qquad (46)$$

Applied to the vectors $\overline{t}^1$ and $\overline{N}^1$, this results in $\cos\Theta_1 = 0.99308$, or $\sin\Theta_1 = 0.117$.

To obtain the next trial stress field $\overline{t}^2$ it is useful to compute the curvature $1/R$. This is done by means of Eq. (28). For the case under consideration, this can be shown to lead to a problem of simple quadrature, so far as the numerator in formula (28) is concerned. In fact, if the tangent vector $\overline{t}^t$ [see Eq. (28a)] designates a stress field in which the radial stress is $f_t(\rho)$ and the circumferential stress $(\rho f_t)_{,\rho}$, then differentiation of Eq. (42a) along the tangent to S leads to

$$(D/a^2h)\left[\rho^{-1}(\rho\phi^t)_{,\rho}\right]_{,\rho} + (D/a^2h)\lambda_o(\alpha-\beta f)\phi^t = f_t\phi \tag{47a}$$

with the boundary conditions

$$\phi^t(0) = \phi^t_{,\rho}(1) + \mu\phi^t(1) = 0 \tag{47b}$$

Equations (47) can be solved by letting

$$\phi^t(\rho) = y(\rho)\,\phi(\rho) \tag{48}$$

which, when substituted in Eqs. (47), and in view of Eqs. (42), results in

$$(D/a^2h)(y_{,\rho\rho}\phi + 2y_{,\rho}\phi_{,\rho} + \rho^{-1}y_{,\rho}\phi) = f_t\phi \tag{49a}$$

and the boundary condition

$$y_{,\rho}(1) = 0 \tag{49b}$$

Equation (49a) can be rewritten by multiplying both sides by $\rho\phi$. It then takes the simple form

$$(D/a^2h)(\rho\phi^2 y_{,\rho})_{,\rho} = f_t\rho\phi^2 \tag{49c}$$

On the other hand, the numerator in Eq. (28), for the case under consideration, equals

$$2\pi a^2h\int_0^1 f_t\phi\phi^t\rho\,d\rho = 2\pi a^2h\int_0^1 f_t\phi^2 y\,\rho\,d\rho$$

in which the equality sign follows from Eq. (48). When Eq. (49c) is substituted in this expression and one integration by parts is performed, then, by virtue of Eq. (49b), and by letting

$$z(\rho) \equiv \rho\phi^2 y_{,\rho} \tag{50a}$$

the numerator can finally be written in the form

$$h\int_R t^t_{ij}\, w^t_{,i}\, w_{,j}\, dA = -2\pi D\int_0^1 \frac{z^2}{\rho\phi^2}\,d\rho \tag{50b}$$

where $z(\rho)$, in consequence of Eq. (49c), is obtained from

$$z(\rho) = (a^2 h/D) \int_0^\rho f_t(\rho)\, \phi^2 \rho\, d\rho \tag{50c}$$

and where, in line with Eq. (28a), f_t is given by

$$f_t(\rho) = -(D/a^2 h) \lambda_o \beta_1 \left(f_1 - \frac{\bar{t}^1 \cdot \bar{N}^1}{\bar{N}^1 \cdot \bar{N}^1} F_1 \right). \tag{50d}$$

When the functions f_1 and F_1, as found previously, are substituted in these expressions as well as in Eq. (28b), and when furthermore the normalizing factor in arriving at F_1 is taken into consideration, the radius of curvature is found to be $R_1 = 71.81\, D(2\pi/Ea^2h)^{1/2}$. On the other hand, the length L_1 of the trial vector $(D/a^2h)\lambda_o\beta_1\bar{t}^1$ is $2.17\, D(2\pi/Ea^2h)^{1/2}$. In line with Eqs. (35), a new stress field of the type of Eq. (41b) is now assumed, in which, of course, α retains its value of 1.25, and in which, after convenient normalization,

$$f_2 = 1 - 1.268\rho^2 + 0.269\rho^4 + 0.009\rho^6 \pm \cdots$$

This results in a new eigenvalue problem, which is solved, as before, by expanding the solution $\phi \equiv \phi_2$ of Eq. (42a) in a power series and by determining $\beta = \beta_2$ on the basis of the boundary conditions (42b). This leads to $\beta_2 = 0.7166$ and a mode

$$\phi_2 = \rho - 0.280\rho^3 - 0.133\rho^5 + 0.045\rho^7 + 0.003\rho^9 \pm \cdots$$

A new gradient vector $\bar{N}^2$ (identified by F_2) is now determined in the same manner as before - that is, by solving an equation of the type of Eq. (45). The normalized result is given below:

$$F_2 = 1 - 1.254\rho^2 + 0.234\rho^4 + 0.039\rho^6 - 0.021\rho^8 \pm \cdots$$

while the angle between $\bar{t}^2$ and $\bar{N}^2$ is governed by $\sin\theta_2 = 0.0015$. In other words, the new angle (which is a measure of the error involved) is little more than 1% of the previous angle.

This process could, of course, be continued further. However, the result of one iteration, which is given by the stress field $F_2(\rho)$ and the mode $\phi_2(\rho)$, already represents a very good approximation to the correct solution. In fact, since $\phi = (1/a)\, w_{,\rho}$ and since $w(1) = 0$, the deflection mode $w(\rho)$ is found directly by integrating the expression for ϕ_2. In this matter, the maximum "relative" deflection $w(0) = 0.414a$ is obtained. With the multiplier k determined on the basis of Eq. (40), this leads to an "absolute" maximum deflection of $3.26h \times (12-12\mu^2)^{-1/2}$, which is in excellent agreement with the exact solution in [4]. Furthermore, the ratio between the radial membrane stresses at the center and at the edge is given by $(\alpha-\beta_2)/\alpha = 0.425$, which again agrees well with the exact solution.

Of special interest is the case of α approaching infinity. The ensuing boundary-layer problem, which has been discussed in great detail in [4] and [5], can be solved readily by the present method after the introduction of some modifications.[12] To this end it is convenient to replace the term $\alpha\lambda_0$ in Eqs. (41) by λ^2 and the expression $\beta\lambda_0$ by $\beta^2\lambda^2$ and to let λ go to infinity. When this limiting process is carried out, it follows from Eq. (42a) that, outside of the boundary-layer strip, ϕ is everywhere zero; this in turn implies by Eq. (45) that the radial stress F_1 is a constant, although the value of that constant cannot be determined directly since the second of the boundary conditions (41c) refers to the boundary layer.

Within the boundary layer itself, the scale is enlarged by introducing the new independent variable

$$\tau = \lambda(1-\rho) \qquad (0 \leqq \tau \leqq \lambda \to \infty) \tag{51}$$

With the external pressure p [see Eqs. (41)] given by

[12]The writer is of course under no illusion as to the extent to which his work has been simplified by the above-cited references.

$$p = \lambda^2 D/a^2 h \tag{52a}$$

the limiting stress field is generally expressed by

$$\begin{aligned} t_{rr} &= (D/a^2h)\lambda^2(-1+\beta^2 f) \\ t_{\theta\theta} &= -(D/a^2h)\lambda^3\beta^2 f_{,\tau} \end{aligned} \qquad t_{r\theta} = 0 \tag{52b}$$

where $f(\tau)$ has to satisfy the boundary conditions

$$f(0) = f_{,\tau}(\infty) = 0 \tag{52c}$$

In view of these modifications, the slope ϕ is now governed by [see Eqs. (42)]:

$$\phi_{,\tau\tau} + (1-\beta^2 f)\phi = 0 \tag{53a}$$

and by the boundary conditions

$$\phi_{,\tau}(0) = \phi(\infty) = 0 \tag{53b}$$

To solve Eqs. (53) by means of a power series, it is useful to introduce a further change in coordinates through

$$x = e^{-\omega\tau} \qquad (0 \leq x \leq 1) \tag{54}$$

in which case Eqs. (53) take the form

$$x^2\phi_{,xx} + x\phi_{,x} + \omega^{-2}(1-\beta^2 f)\phi = 0 \tag{55a}$$

$$\phi(0) = \phi_{,x}(1) = 0 \tag{55b}$$

On the other hand, the stress field $f(x)$ now has to satisfy the boundary conditions

$$f_{,x}(0) = f(1) = 0 \tag{55c}$$

In these equations, the constant ω is as yet arbitrary and will be governed by considerations of convenience.

A reasonable and simple initial stress function $f(x)$, which satisfies Eqs. (55c), is given by

$$f_1(x) = 1 - x^2$$

When this is substituted in Eq. (55a), the solution can be written directly in the form

$$\phi_1 = A\,J_\nu(kx) + B\,Y_\nu(kx)$$

in which J and Y are Bessel Functions of the first and second kind, and the constants ν and k are given, respectively, by

$$\nu^2 = (\beta^2 - 1)/\omega^2 \qquad\qquad k = \beta/\omega$$

The first of the boundary conditions (55b) requires that B vanish, while the second, for an assumed value of ν, determines the constant k. Although a better initial value for ν could be assumed on the basis of energy considerations (see next section), the rapid convergence of the present method becomes apparent when ν is set arbitrarily equal to unity. This leads to k = 1.841, which in turn implies that $\beta^2 = 1.417$ and $\omega^2 = 0.417$.

With the first eigenvalue problem thus solved, the next step consists in determining the gradient vector $\overline{N}'$ which represents the radial stress field $\lambda^2 F_1(x)$ and the circumferential stress field $\lambda^3 \omega x F_{1,x}$. Through a procedure which is analogous to the one used in the development of Eq. (45), it can be shown that, with λ going to infinity, $\overline{N}'$ is governed by

$$x^2 F_{1,xx} + x F_{1,x} = -(E/\omega^2)\phi_1^2 \qquad (0 \leq x \leq 1) \tag{56}$$

and the boundary conditions (55c). When the Bessel Function representing ϕ_1 is expressed in the conventional power-series expansion, the integration of Eq. (56), in view of the boundary conditions and after the usual normalization, leads to

$$F_1 = 1 - 1.222x^2 + 0.259x^4 - 0.041x^6 + 0.005x^8 \pm \cdots$$

The angle Θ_1 between the initial vector $\overline{t}^1$ and the gradient vector $\overline{N}^1$ is next obtained by means of Eq. (31); however, the bilinear forms $U_{k\ell}$ as given by

Eq. (46) now take the following limiting form:

$$U_{k\ell} = (2\pi a^2 h/E)\,\lambda^5 \omega \int_0^1 x\, f_{k,x}\, f_{\ell,x}\, dx \tag{57}$$

It may be interesting to note that this represents only the membrane energy associated with the circumferential stresses in the boundary layer; the remaining stresses (i.e., the circumferential stresses outside of the boundary-layer strip as well as the radial stresses throughout the region) contribute to energy expressions which approach infinity with a power of λ less than five. As applied to the present problem, the angle between the two vectors is found to be governed by $\sin \theta_1 = 0.099$.

The next trial stress field f_2 is now determined, as before, through a linear combination of f_1 and F_1. This requires the calculation of the curvature $1/R$ by means of Eq. (28), as adapted to the limiting stress-field condition. With the curvature computed in this manner, the next trial stress field is determined on the basis of Eqs. (35). This leads to

$$f_2 = 1 - 1.165x^2 + 0.192x^4 - 0.030x^6 + 0.003x^8 \pm \ldots$$

which, when substituted in Eqs. (55), leads to a new mode $\phi_2(x)$ through the usual power-series approach. As before, the recursion relations depend on the assumed value of β_2, whose final value is determined by satisfying the second of boundary conditions (55b). A first guess for β_2 is easily obtainable by considering the surface S (see Fig. 1) to be a circle of radius R between points P_1 and P_2. In the present example such a first estimate leads to $\beta_2^2 = 1.484$, while the solution of the eigenvalue problem (55) is given by $\beta_2^2 = 1.486$ and

$$\phi_2 = x^{1.079}\left(1 - 0.499x^2 + 0.112x^4 - 0.019x^6 + 0.003x^8 \pm \ldots\right)$$

Finally, the new gradient vector $\overline{N}^2$ is governed by the stress field

$$F_2 = 1 + x^{2.158}\left(-1.282 + 0.345x^2 - 0.074x^4 + 0.013x^6 \pm \ldots\right)$$

These results of the first iteration are already in fairly good agreement with the exact results. In particular, the ratio of the (tensile) radial stress at the center to the (compressive) stress at the edge is $1 - \beta^2 = - 0.486$, as compared with the exact value of -0.473. Also, the maximum deflection w_{max} at the center of the plate, computed in line with Eqs. (37) and (40) equals 6.66 x $h\alpha^{1/2}(12-12\mu^2)^{-1/2}$, which is in good agreement with the chart shown in Fig. 8 of [4]. Similar agreement holds for the value of the maximum slope at the edge. The angle Θ_2 between the vectors $\overline{t}^2$ and $\overline{N}^2$ is given by $\sin \Theta_2 = 0.014$, which represents a sizable reduction from the previously established angle $\sin \Theta_1 = 0.099$.

A further increase in accuracy, although at mounting expenditure of labor, can of course now be obtained through a repetition of the previous process. This is not attempted here. However, the next value for the ratio $1-\beta^2$ between the radial stresses at the center to those at the edge can be estimated "cheaply" in the same manner in which an initial guess (1.484) was obtained for ${\beta_2}^2$ during the previous iteration. A function $f_3(x)$ is constructed on the basis of Eqs. (35); for this purpose the same radius R is used as found before. This leads to $1-\beta^2 = -0.473$, which is indistinguishable from the exact value given in [5].

6. FURTHER REMARKS AND CONCLUSION

The analytical approach which was described and demonstrated in the last two sections consists in alternately determining the intersection of a line with the surface of instability (an eigenvalue problem) and in constructing the gradient vector to that surface (a problem of plane elasticity). Since the governing equations - say, Eqs. (42) and (45) for the problem under discussion - are essentially the same as the equations of equilibrium and compatibility of the orginal problem {see, for example, Eqs. (8) and (9) in [4]}, it is clear that the present procedure is in essence a geometric interpretation of the solution of the two von Kármán equations [1] through an alternate iterative process.

Whether such an interpretation is of more than academic interest depends, of course, on whether it facilitates the solution of problems which were heretofore considered beyond the reach of analysis. Offhand it appears that this may conceivably be so. At least so far as the specific problem discussed in the previous section is concerned, there is no doubt that the convergence of the process is speeded up considerably through various devices which were developed on the basis of geometric reasoning. In fact, without these devices, a straightforward iteration (using, e.g., the gradient vector itself as the basis for the next eigenvalue problem) may lead to rapid divergence.

Nor were all the shortcuts fully utilized in the preceding problem. For example, the solution of Eqs. (55) in the boundary-layer problem (with $f = f_1 = 1-x^2$) is given by Bessel Functions of order ν, the latter being set arbitrarily equal to unity. If, instead, various values of ν are considered and if the associated membrane-strain energy is computed, it can readily be shown that this energy assumes a minimum when ν is equal to 1.25 (corresponding to $k = 2.155$, $\omega = 0.570$, and $\beta^2 = 1.507$). The angle Θ_1 between the initially assumed stress field vector and the gradient vector is now given by $\sin \Theta_1 = 0.041$, which is less than half the value corresponding to $\nu = 1$.

Of some theoretical interest may be the question of singularities of the surface S. Such singularities occur when, for a given stress field, the eigenvalue problem (5) has more than one independent solution - say, w_1 and w_2. It is obvious that in that case $w = c_1 w_1 + c_2 w_2$ satisfied Eq. (5), where c_1 and c_2 are arbitrary constants. Geometrically this means that if $\overline{N}^{11}$ and $\overline{N}^{22}$ are the gradient vectors of the two branches of S meeting at the singular point, then the totality of all possible gradient vectors at that point is represented by a linear combination (with positive coefficients) of $\overline{N}^{11}$ and $\overline{N}^{22}$.

This statement, which again establishes an analogy with the direction of the plastic strain vector for singular yield surfaces [11], can be proved without difficulty. In fact, the most general gradient vector can be expressed in the form $\bar{N}' = c_1{}^2\bar{N}^{11} + 2c_1c_2\bar{N}^{12} + c_2{}^2\bar{N}^{22}$, where $\bar{N}^{k\ell}$ is the statically homogeneous component of the stress-field vector which is associated with the strain field $1/2(w_{k,i}w_{\ell,j} + w_{k,j}w_{\ell,i})$ $(k,\ell = 1,2)$. However, it can be shown similarly to the establishment of Eq. (26) that $\bar{N}^{12}$ is perpendicular not only to all the tangential directions relative to which S is smooth, but also to the two discontinuous directions formed by the intersection of the $(\bar{N}^{11},\bar{N}^{22})$ plane with S. Since this encompasses the entire statically homogeneous space, it follows that $\bar{N}^{12}$ vanishes, which completes the proof.

Obviously such a singularity is eliminated by artificially excluding one of the two buckling modes through the imposition of geometric constraints. This seems to be the case for the example treated in the previous section. As pointed out in [4] and elsewhere, the assumption of circular symmetry constitutes a constraint which, according to experimental evidence, is violated for sufficiently large loads. In the Authors' Closure [4] this kind of "second buckling" is ascribed to a weakening of the edge constraints during the tests since, according to the authors, nonsymmetric configurations of lower potential energy {based on Eq. (31) of [4]} could not be ascertained.

Actually, if the potential energy of Eq. (12) of the present paper is used as the basis, the symmetric membrane state of stress does become statically inadmissible relative to unsymmetric modes for sufficiently large values of the load parameter λ. For example, consider the limiting (boundary layer) stress field, and assume a deflection pattern $W(\rho,\Theta)$ consisting of circumferential waves whose length is of the same order of magnitude as the width of the boundary-layer strip. Then it is readily apparent that the second integral in Eq. (12) approaches $(-\infty)$ to a higher power of λ than the first approaches $(+\infty)$. In other words, the very nature of the boundary-layer stress pattern becomes inadmissible if nonsymmetrical wave patterns are not ruled out. In fact, fairly cursory calculations tend to indicate that the actual (nonsymmetrical) stress field exhibits radial, circumferential, and shearing stresses which all approach infinity to the second power of λ.

The following and concluding discussion concerns itself with the special condition that, for increasing amplitudes, the load parameter λ may not increase indefinitely, but may instead approach a limiting (ultimate) value λ^u. Concurrently, the additional stress field t'_{ij} approaches a (generally finite) limiting form t^u_{ij}, while the deflection function w approaches kw^c, where k grows beyond bounds and where the "collapse mode" w^c is governed by

$$\int_R T'_{ij}\, w^c_{,i}\, w^c_{,j}\, dA = 0 \tag{58}$$

for all statically homogeneous stress fields T'_{ij}.

To prove Eq. (58), multiply Eq. (9a) by T'_{ij} and integrate over the region R. For the special case under consideration, the integral on the left side remains finite,[13] as postulated. On the right side of the equation, the first two integrals (involving the displacements u'_i) vanish following the usual reasoning. Hence, with k going to infinity, Eq. (58) is approached in the limit.

Moreover, the existence of a mode w^c satsifying Eq. (58) for all statically homogeneous stress fields is a sufficiency condition for the existence of a limiting load parameter λ^u, provided further that the parameter λ^c computed by means of the equation

$$U_b(w^c) + \frac{1}{2}\lambda^c h \int_R t^0_{ij}\, w^c_{,i}\, w^c_{,j}\, dA = 0 \tag{59}$$

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[13]Under special circumstances, especially in the case of the development of a boundary layer, the stress field t^u_{ij} may not remain finite. However, this does not invalidate the proof of Eq. (58) provided that the integral in question approaches infinity to a power of k less than two.

is positive. Indeed such a "kinematically admissible" parameter represents an upper bound to λ^u. For if λ were to exceed λ^c, then by Eqs. (58) and (59) and by the positive definiteness of U_b, the inequality (12) would be violated at least for $W = w^c$. "Geometrically" this special condition means that the neutral surfaces S are closed and approach a limiting point as λ approaches λ^u.

Equation (58) implies further that the strain field $e^c_{ij} = w^c_{,i}w^c_{,j}$ is associated with a geometrically homogeneous stress field. It is therefore derivable from a displacement field u^c_i by means of Eq. (19) subject to the boundary condition (20). As is well known from linear elasticity theory, the necessary and sufficient condition for the existence of such a displacement field is the satisfaction of the compatibility condition, which in the present case takes the form

$$w^c_{,11}\,w^c_{,22} - \left(w^c_{,12}\right)^2 = 0 \tag{60}$$

The left side of Eq. (60) represents the Gaussian curvature. In other words, by Eq. (60) and subject to the restrictions named above, an ultimate load parameter λ^u exists if the geometry of the plate is such as to admit the formation of a developable surface.

An elementary application of this principle is furnished by a rectangular plate which is free along the edges $x_2 = 0,b$ and simply supported along the edges $x_1 = 0,a$; also along the latter edges it is subjected to compressive tractions of given distribution. An obvious possible collapse mode is given by $w^c = \sin \pi x_1/a$. This satisfies Eq. (60) identically; Eq. (59) can easily be shown to be satisfied if the total force applied at either edge equals the Euler force $\pi^2 Db/a^2$, irrespective of the distribution of the edge tractions (subject, of course, to the conditions of equilibrium). In other words, the Euler force represents an upper bound to the collapse load regardless of its actual mode of application.

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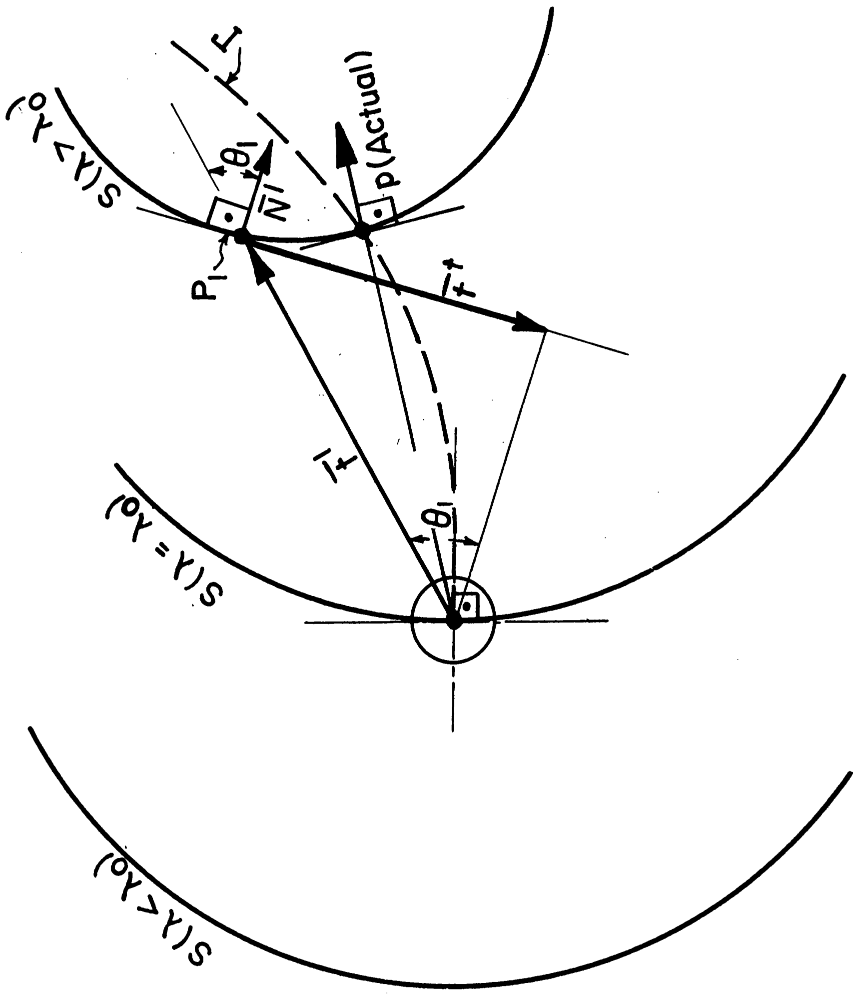

Fig. 1. Statically homogeneous stress space.