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THE STRENGTH OF VERY SLENDER BEAMS

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By E. F. Masur[2]

SUMMARY

The response of a slender beam to lateral loads and twisting couples is affected by the presence of bending moments in the plane of major stiffness much as the bending of beams may be influenced by the presence of axial forces. If, in addition, the major bending moments are statically indeterminate, and if the beam is sufficiently slender to admit relatively large lateral deformations, these may in turn affect the distribution of the principal bending moments. The resulting nonlinear theory is the subject of this paper.

After the establishment of the basic equations, it is shown that the inclusion of nonlinear terms in the strain-displacement relations corresponds generally to a stiffening of the structure as compared with the familiar linear theory. The (redistributed) major bending moments and reactions are shown to satisfy a minimum principle which represents an extension of the classical Castigliano Theorem. It is demonstrated further that, for increasing lateral loads and torsional moments, a limiting major bending moment distribution is approached asymptotically. For certain singular cases, the corresponding equilibrium configuration may not be unique, in which case the possibility of a snap-through (Durchschlag) phenomenon arises.

The theory presented herein is corroborated experimentally with a fair degree of accuracy. Elastic behavior is assumed throughout.

1. INTRODUCTION AND ESTABLISHMENT OF BASIC EQUATIONS.

In the present paper, a beam will be referred to as being "slender" when its moment of inertia I_y about the (vertical) y-axis is much smaller than the moment of inertia I_x; in addition, its torsion constant K will be

[1]This investigation was conducted under the sponsorship of the Office of Ordnance Research, U. S. Army.

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assumed small in comparison with I_x. An example of a slender beam is furnished by a rectangular beam, whose thickness t is small compared with its depth h.

The response of such a beam to lateral loads and torsional couples in the presence of bending moments about the x-axis has been the subject of numerous investigations, of which [1] and [2][3] may be mentioned here. It is shown in these papers how the bending moments influence, and often aggravate, the displacements of a slender beam. This is analogous to the behavior of beam-columns, whose response to loads in the presence of axial forces is well known.

In all previous publications on the subject, the assumption is made more or less tacitly that the bending moments are either statically determinate or, in the event of statical indeterminacy in the major plane of stiffness, that they may be computed on the basis of the conventional linear theory. This may actually not be the case. In fact, if the lateral displacements u and the rotations β are sufficiently large, the introduction of nonlinear strain displacement relations may serve to modify the predicted moments. This question is explored in detail in what follows. It is shown that a redistribution of bending moments takes place, which serves to stiffen the structure relative to its predicted stiffness according to conventional theory. For example, if there are no vertical loads acting on the beam, the linear theory predicts, of course, vanishing bending moments everywhere in the absence of initial stresses. However, when certain nonlinear terms are included in the analysis, such moments do arise, and approach limiting values as the magnitude of the lateral loads and torques approaches infinity.

[3]Numbers in brackets refer to the Bibliography at the end of the paper. Reference [2] contains a fairly comprehensive list of publications on the subject.

Before proceeding to the analysis, it may be well to point out the limitations of the proposed theory. Actually, the nonlinearity is partial only in that the strain-displacement relations contain terms up to the second order, but ignore those of higher order. Physically this implies that the lateral displacements may be comparable to the thickness of the beam, but are still assumed to be small in relation to its length. Theories of an analogous nature are widely employed in connection with the analysis of structural elements in which at least one dimension is much smaller than the remaining dimensions; the best known example is probably the plate theory of von Kármán [3]. Since, furthermore, the effect of plasticity in this presentation is ignored, it follows that the beam must be very slender to lend physical significance to the proposed theory. In the numerical example treated in a later section, appreciable deviations occur from linear theory at fiber stresses of about 30,000 psi in a beam whose depth-thickness ratio is 16:1.

In what follows, let a beam of the type shown in Fig. 1 be subjected to lateral loads λq and twisting couples λt, in which q and t are given functions of z (measured along the axis of the beam), and λ is a multiplier which is allowed to increase indefinitely. If $\lambda u(z)$ and $\lambda\beta(z)$ are the horizontal displacement and the rotation, respectively, u and β are governed by the equations of equilibrium

$$(EI_y u'')'' + Pu'' + [(M + Py_0)\beta]'' = q(z) \tag{1}$$

$$(E\Gamma\beta'')'' - [(GK - P\rho^2 - 2kM)\beta']' + (M + Py_0)u'' - pa\beta = t(z) \tag{2}$$

where primes denote differentiation with respect to z. In these equations P represents the axial force (measured positive in compression), M(z) the bending moment about the centroidal x-axis, and p(z) the given vertical load applied at a distance a above the shear center S. E and G are the usual elastic constants, Γ is the warping constant, ρ is the radius of gyration about S, and y_0 designates the position of S.[4] Finally, k is a cross-sectional constant and is defined by

$$k = y_0 - \frac{1}{2I_x}\int_A y(x^2+y^2)\,dA \tag{3}$$

which vanishes for sections symmetrical about the x-axis.

Equations (1), (2), and (3) are the familiar equations of lateral bending and torsion and are given, for example, in [4], although slight discrepancies are present, as pointed out in [5]. Through a process which is entirely analogous to the one employed in [5][5], it can be shown further that the vertical displacement v(z) satisfies the relationship

$$v'' = -(M - M^*)/EI_x + \lambda^2(u''\beta - k\beta'^2) \tag{4}$$

in which the second term on the right side represents the effect of the nonlinearity in the strain-displacement relations. M*(z) designates prestressing moments (if any) and has been incorporated for the sake of completeness.

The inclusion of the nonlinear terms in Eq. (4) and the deletion of analogous terms from Eqs. (1) and (2) imply that the vertical displacements are much smaller than the horizontal and rotational displacements. This in turn

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[4]The assumption that S lies on the principal y axis introduces no significant loss of generality.

[5]The writer is indebted to Professor E. Reissner for suggesting, as an alternative, the derivation of these equations from large deflection plate theory. This process can be carried out for the technically most significant case of a thin rectangular beam and, with the exception of the introduction of the term $(1-\mu^2)^{-1}$, leads to relationships which are identical with Eqs. (1) and (2) (properly simplified for this case, of course). However, the form of Eq. (4) is slightly modified by this procedure. This discrepancy, which may affect the results somewhat in the presence of nonuniform bending, is due to a minor variation in the basic assumptions underlying plate theory as compared with the Euler-Bernoulli beam theory on which the present derivation is based.

requires that the beam be slender, as was outlined above. In the development of Eq. (4) it is assumed further that the force P be much smaller than the critical buckling force about the (strong) x-axis; in view of the slenderness of the beam, however, this represents no added restriction.

The bending moments M(z) and M*(z) satisfy the equations of equilibrium

$$M'' = -p(z) \qquad M^{*''} = 0 \tag{5}$$

and a set of appropriate natural boundary conditions. These, together with Eq. (5), determine M and M* uniquely (the latter trivially) if the structure is statically determinate relative to its major bending moments. In that event, for given vertical load p(z) and axial force P, the "calibrated" functions u(z) and β(z) are also uniquely determined by Eq. (1) and (2) and the associated boundary conditions. Hence the total response λu and $\lambda\beta$ increases in proportion to the total lateral load and torque λq and λt, respectively.

A different picture is presented by a structure whose major bending moments are statically redundant. In that case, the actual moment M(z) is distinguished, among all moments satisfying Eq. (5) and boundary conditions, by being associated with a geometrically compatible vertical deflection v(z). Since the latter is related to M by means of Eq. (4), it becomes apparent that increasing lateral loads and torques, as expressed by increasing values of λ, are accompanied by a redistribution of the principal bending moments. The equations governing this redistribution are developed in the remainder of this section.

In a structure of n^{th} degree of indeterminacy, the most general expression for M and M* is

$$M(z) = m_0(z) + \lambda_\alpha m_\alpha(z) \tag{6a}$$

$$M^*(z) = \qquad \lambda^*_\alpha m_\alpha(z) \tag{6b}$$

in which $m_0(z)$ is governed, but not uniquely determined, by

$$m_0'' = -p(z) \tag{7a}$$

and the same boundary conditions which apply to M. The set of self-equilibrated moments $m_r(z)$ $(r=1,2...n)$ satisfies

$$m_r'' = 0 \qquad (r = 1, 2, \cdots n) \tag{7b}$$

and equivalent homogeneous boundary conditions. The set of numbers λ_r $(r=1,2...n)$, as yet unknown, will hereafter be referred to as "redundant parameters," while λ_r^* $(r=1,2...n)$ constitutes a set of "prestressing parameters." Repeated Greek subscripts, as in Eq. (6), represent summation over the range 1 to n. It is finally convenient, and always possible, to select the set of functions $m_r(z)$ in such a way that the "orthonormality condition"

$$\int \frac{m_r m_s dz}{E I_x} = \begin{cases} 1 & (r = s) \\ 0 & (r \neq s) \end{cases} \qquad (r, s = 1, 2, \cdots n) \tag{7c}$$

is satisfied.

The redundant parameters λ_r may now be determined by multiplying Eq. (7b) by $\bar{v}(z)$ and by integrating over the length of the structure; here $\bar{v}(z)$ represents any geometrically admissible vertical deflection function, that is, one which is sufficiently smooth and satisfies the geometric boundary and continuity conditions pertaining to the vertical deflection. In view of these restrictions, two integrations by parts lead to the relationships

$$\int m_r \bar{v}'' dz = 0 \qquad (r = 1, 2, \cdots n) \tag{8}$$

In particular, let $\bar{v}(z)$ be the actual deflection function $v(z)$ satisfying Eq. (4). Then, by Eqs. (4), (6), and (7c), Eq. (8) is converted into

$$\lambda_r - \lambda_r^* = \lambda^2 \int m_r (u''\beta - k\beta'^2) dz - \int \frac{m_0 m_r dz}{E I_x} \qquad (r = 1, 2 \cdots n) \tag{9}$$

There is no generality lost in letting $m_0(z)$ be the actual bending moment, derived on the basis of linear theory, in the beam in the absence of prestressing and of lateral loads and torques. In other words, let the set of redundant parameters λ_r vanish when all prestressing parameters λ_r^* and the load parameter λ also vanish. When this is substituted in Eq. (9), it means that

$$\int \frac{m_0 m_r dz}{EI_x} = 0 \qquad (r = 1, 2, \cdots n) \tag{10}$$

In view finally of Eq. (10), which constitutes an expression of the well-known and often used principle of virtual work, the "compatibility" conditions (9) become

$$\lambda_r - \lambda_r^* = \lambda^2 \int m_r (u''\beta - K\beta'^2) dz \qquad (r = 1, 2, \cdots n) \tag{11}$$

2. DISCUSSION OF LARGE-DEFLECTION THEORY.

It is seen that there are as many Eqs. (11) as there are redundant parameters. Since the bending moment M(z) is expressed in the form (6a), it is apparent that the nonlinear range is governed by the solution of the differential equations (1) and (2), in which M(z) satisfies simultaneously the compatibility equations derived above. The complexity of the resulting system of equations is therefore evident, and hence the necessity of solving it by inverse or trial-and-error methods.

This is relatively easy for singly redundant beams. In that case, a value for λ_1 may be assumed arbitrarily at the start. With the bending moment M(z) so chosen, Eqs. (1) and (2) are solved and the solution (u,β) is inserted in (the single) Eq. (11), which in turn is solved for the load parameter λ. If the value of λ so obtained is real, and if the assumed moment M(z) is statically admissible in the sense defined later on in this section, then, by virtue of the uniqueness of the solution (proved also below), a point has been established in the load-response diagram. The latter may be completed by repeating this process for different values of λ_1.

For higher degrees of redundancy the process may become prohibitively laborious. However, some insight into the nonlinear behavior of the structure may be gained by means of a number of principles which are developed in what follows. In addition, it is possible that these principles may be instrumental in the reduction of the numerical labor involved.

To this end, let a quadratic form U be defined by

$$2U(M; u, \beta) = \int (EI_y u''^2 + E\Gamma \beta''^2 + GK\beta'^2)\, dz - \int [P(u'^2 + \rho^2 \beta'^2) - 2(M + Py_0)u''\beta + 2KM\beta'^2 + pa\beta^2]\, dz \tag{12}$$

In Eq. (12) the first integral represents the bending and torsional strain energy, while the second integral can be shown to give the work done by the vertical load p(z) and the axial force P. Also, a bending moment M(z) will hereafter be referred to as being "statically admissible" if it satisfies the equations of equilibrium (5) and the stability condition

$$U(M; \bar{u}, \bar{\beta}) \geqq 0 \tag{13}$$

for all non-trivial functions $(\bar{u}, \bar{\beta})$ which are geometrically admissible, that is, which are sufficiently smooth and satisfy the pertinent geometric boundary conditions.

If further the "potential energy" V is defined by

$$V(M; u, \beta) = U(M; u, \beta) - \int (qu + t\beta)\, dz = U(M; u, \beta) - W(u, \beta) \tag{14}$$

in which the integral expression represents the work done by the lateral loads and the torsional moments, it can readily be shown that

$$V(M; \bar{u}, \bar{\beta}) \geqq V(M; u, \beta) \tag{15}$$

if M is statically admissible and if (u,β) satisfy Eqs. (1) and (2).[6] It may be noteworthy that the equality sign in (15) implies trivial equality between $(\bar{u},\bar{\beta})$ and (u,β), provided only the inequality (13) is admitted. Conversely, if U=0 for, say, (u_1,β_1), then the equality (15) applies to any $(\bar{u},\bar{\beta})=(u,\beta)+c(u_1,\beta_1)$, in which c is an arbitrary number. The pair of functions (u_1,β_1) represent the fundamental buckling mode of the beam.

With these definitions, let a bending moment $\bar{M}(z)=m_0+\bar{\lambda}_\alpha m_\alpha$ be statically admissible, relative to a given load, and let the functions $(\bar{u},\bar{\beta})$ be associated with $\bar{M}$ through the solution of Eqs. (1) and (2). Let further M and (u,β) represent, respectively, the correct bending moment and configuration for the same load; this implies, of course, that Eqs. (6a) and (11) are also satisfied. Since (u,β) are certainly geometrically admissible, it follows that

$$V(\bar{M};u,\beta) \geqq V(\bar{M};\bar{u},\bar{\beta}) \tag{15a}$$

similarly to (15). But

$$\begin{aligned} V(\bar{M};u,\beta) &= V(M;u,\beta)+(\bar{\lambda}_\alpha-\lambda_\alpha)\int m_\alpha(u''\beta-k\beta^2)\,dz \\ &= V(M;u,\beta)+\frac{1}{\lambda^2}(\bar{\lambda}_\alpha-\lambda_\alpha)(\lambda_\alpha-\overset{*}{\lambda}_\alpha) \end{aligned}$$

in which the second equality is in consequence of Eq. (11).

Moreover, if Eq. (1) is multiplied by u and Eq. (2) by β, and if several integrations by parts are performed, it is readily demonstrated on account of the boundary conditions that

$$W(u,\beta)=2U(M;u,\beta) \tag{16}$$

[6] Since Eqs. (1) and (2) are the variational Euler equations of V, and since the actual natural boundary conditions are the variational boundary conditions of V, the correct solution (u,β) makes V stationary for all moments satisfying the equations of equilibrium. The additional minimum principle (15) represents an extension to the restricted class of moments which satisfy also the stability condition (13).

An identical relationship applies to the system $(\bar{M};\bar{u},\bar{\beta})$. Thus, in view of the definition of V in Eq. (14), the inequality (15a) now becomes

$$U(\bar{M};\bar{u},\bar{\beta}) \geqq U(M;u,\beta) - \frac{1}{\lambda^2}(\bar{\lambda}_\alpha - \lambda_\alpha)(\lambda_\alpha - \lambda_\alpha^*) \tag{17}$$

or:

$$U(\bar{M};\bar{u},\bar{\beta}) + \frac{1}{2\lambda^2}(\bar{\lambda}_\alpha - \lambda_\alpha^*)(\bar{\lambda}_\alpha - \lambda_\alpha^*) \geqq U(M;u,\beta) + \frac{1}{2\lambda^2}(\lambda_\alpha - \lambda_\alpha^*)(\lambda_\alpha - \lambda_\alpha^*) + \frac{1}{2\lambda^2}(\bar{\lambda}_\alpha - \lambda_\alpha)(\bar{\lambda}_\alpha - \lambda_\alpha) \tag{17a}$$

The last term on the right side of (17a) is positive definite. Hence, in the absence of initial bending moments ($\lambda_r^*=0$), the following inequality is established:

$$U(M;u,\beta) + \frac{1}{\lambda^2} U_b(M) \leqq U(\bar{M};\bar{u},\bar{\beta}) + \frac{1}{\lambda^2} U_b(\bar{M}) \tag{18}$$

with

$$U_b(M) = \frac{1}{2}\int \frac{M^2 dz}{EI_x}$$

U_b represents the familiar expression for the strain energy in bending about the major axis. (18) follows from (17a) through the application of Eqs. (6a), (7c), and (10). An alternate statement of the inequality (18), which takes account of Eq. (16), reads:

$$W(u,\beta) + \frac{2}{\lambda^2} U_b(M) \leqq W(\bar{u},\bar{\beta}) + \frac{2}{\lambda^2} U_b(\bar{M}) \tag{18a}$$

These last two inequalities may be expressed in the following principle:

Of all statically admissible bending moments, the actual one corresponds to the smallest value of λ^2U+U_b (or, alternately, of λ^2W+2U_b) if these values are determined on the basis of a deflected configuration which is related to the bending moments through the solution of Eqs. (1) and (2).

This minimum principle, it may be well to emphasize, relates to the distribution of the redundant moments rather than to the deflection configuration itself; that the latter is itself governed by minimum principles is well-known and was, in fact, utilized in the derivation of the present principle. It represents an extension of the classical Castigliano "Theorem of Least Work" into the nonlinear range. Indeed, the Theorem of Castigliano represents a special case, which can be obtained from either (18) or (18a) by setting λ equal to zero.[7]

To carry the discussion further, let the inequality (15a) be subtracted from (15). This results in the relationship

$$(\lambda_\alpha-\bar{\lambda}_\alpha)\int m_\alpha(\bar{u}''\bar{\beta}-k\bar{\beta}'^2)\,dz \geqq (\lambda_\alpha-\bar{\lambda}_\alpha)\int m_\alpha(u''\beta-k\beta'^2)\,dz$$

If now both $(M;u,\beta)$ and $(\bar{M};\bar{u},\bar{\beta})$ are assumed to satisfy the conditions of compatibility (11), it follows that

$$(\lambda_\alpha-\bar{\lambda}_\alpha)(\bar{\lambda}_\alpha-\overset{*}{\lambda}_\alpha) \geqq (\lambda_\alpha-\bar{\lambda}_\alpha)(\lambda_\alpha-\overset{*}{\lambda}_\alpha)$$

or, after some rearrangement of terms,

$$0 \geqq (\lambda_\alpha-\bar{\lambda}_\alpha)(\lambda_\alpha-\bar{\lambda}_\alpha)$$

Obviously, the inequality above is impossible for real values of the redundant parameters. On the other hand, the equality is possible only if $M(z)$ and $\bar{M}(z)$ are trivially identical. This in turn implies the following principle:

If there exists a statically admissible bending moment which satisfies the conditions of compatibility (11) relative to a configuration (u,β) which is a solution of Eqs. (1) and (2), that bending moment is the actual bending moment and is unique.

In general, it follows from the uniqueness of the $M(z)$ that $u(z)$ and $\beta(z)$

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[7]Actually, the customary statement of the Theorem of Castigliano implies not a minimum, but only a stationary property of the strain energy in bending. The present principle can be broadened in a similar fashion. In fact, if M satisfies the equation of equilibrium (5), but not necessarily the stability condition (13), the quantities dealt with in (18) and (18a) can be shown to be stationary. In other words, if (u,β) as solved from Eqs. (1) and (2) are differentiable functions of λ_r, then the satisfaction of Eq. (11) ensures that the derivative of (λ^2U+U_b) or of (λ^2W+2U_b) with respect to λ_r vanishes.

are also unique, as discussed before. However, singular cases may occur in which, owing to a condition of "neutral equilibrium," two equilibrium configurations become possible. This was already hinted at following the establishment of the inequality (15). A detailed discussion of this question is contained in the Appendix as well as in later sections in connection with the numerical example and the report on the experiments.

Returning to the minimum principle established above, it is clear then that any statically admissible bending moment furnishes an upper bound to the two functions with which the principle deals. In what follows it will be the object to derive also a lower bound and to discuss a possible application of these bounds toward the simplification of the computational labor.

To this end, let a "kinematically admissible" bending moment $\bar{M}(z)$ be defined as one which satisfies the equation of equilibrium (5) [and which, therefore, can be expressed in terms of Eq. (6a) subject to Eq. (7)], which is compatible in the sense of satisfying Eq. (11) for some geometrically consistent configuration $(\bar{u},\bar{\beta})$, and which finally does not violate the inequality[8]

$$U(\bar{M};\bar{u},\bar{\beta}) - \frac{1}{2}W(\bar{u},\bar{\beta}) \leqq 0 \tag{19}$$

If this inequality is now subtracted from the inequality (15), in which the actual state $(M;u,\beta)$ is compared with the state $(M;\bar{u},\bar{\beta})$, then, in view of Eq. (16),

$$2(\lambda_\alpha - \bar{\lambda}_\alpha)\int m_\alpha(\bar{u}''\bar{\beta} - k\bar{\beta}'^2)dz - W(\bar{u},\bar{\beta}) \geqq -W(u,\beta) \tag{20}$$

where $\bar{\lambda}_r$ represents the set of redundant parameters designating $\bar{M}(z)$. The integrals in (20) may be replaced in line with Eq. (11); after some rearrange-

[8]The concept of "kinematic" admissibility follows from the fact that if the equality sign in (19) is satisfied the external work equals the "internal work." Note, however, that the equations of equilibrium (1) and (2) need not be satisfied.

ment of terms, this leads to

$$W(u,\beta)+\frac{1}{\lambda^2}(\lambda_\alpha-\overset{*}{\lambda}_\alpha)(\lambda_\alpha-\overset{*}{\lambda}_\alpha) \geqq W(\bar{u},\bar{\beta})+\frac{1}{\lambda^2}(\bar{\lambda}_\alpha-\overset{*}{\lambda}_\alpha)(\bar{\lambda}_\alpha-\overset{*}{\lambda}_\alpha)+\frac{1}{\lambda^2}(\bar{\lambda}_\alpha-\lambda_\alpha)(\bar{\lambda}_\alpha-\lambda_\alpha)$$

in which the last term, as before, is positive definite. Hence, by the same argument as the one employed previously

$$W(\bar{u},\bar{\beta}) + \frac{2}{\lambda^2} U_b(\bar{M}) \leqq W(u,\beta) + \frac{2}{\lambda^2} U_b(M) \tag{21}$$

or, in view of the relations (16) and (19)

$$U(\bar{M};\bar{u},\bar{\beta}) + \frac{1}{\lambda^2} U_b(\bar{M}) \leqq U(M;u,\beta) + \frac{1}{\lambda^2} U_b(M) \tag{21a}$$

Since the actual bending moment M(z) is also kinematically admissible, it follows then that:

Of all kinematically admissible bending moments, the actual one corresponds to the smallest possible value of the functions $\lambda^2 W + 2U_b$ or $\lambda^2 U + U_b$.

It appears that this lower-bound principle may be useful in estimating the nonlinear response of the structure. In fact, let an upper and a lower bound be found for a given value of the load parameter λ; an error estimate has then been established. Since the bounds can be narrowed down arbitrarily, a correct solution may presumably be approached in this fashion.

ASYMPTOTIC BEHAVIOR

As the load parameter λ grows beyond bounds, the bending moment M(z) and with it the deflection mode (u,β) usually approaches a limiting condition. This asymptotic behavior is discussed in this section in regard to both the governing equations and the appropriate energy principles.

For given load $p(z)$, it was shown in [5] that the stable domain [that is, the one in which the inequality (13) is satisfied] is generally a closed region in a space in which the coordinates of a point are given by the redundant parameters λ_r associated with $M(z)$ through Eq. (6). As p approaches an ultimate value p_u, this region shrinks to a point; for $p > p_u$, no statically admissible bending moments are available any more. In the present paper, p is assumed to be fixed and less than p_u. Because of the boundedness of the associated stable domain, it follows therefore that all the redundant parameters λ_r remain finite as the load parameter λ goes to infinity.

This determines the form of the governing equations of the limiting state. In fact, if λ approaches infinity in Eq. (11) while the left side remains bounded, the asymptotic deflection mode (u_ℓ, β_ℓ) is seen to satisfy the set of equations

$$\int m_r (u_\ell'' \beta_\ell - K \beta_\ell'^2)\, dZ = 0 \qquad (r = 1, 2, \cdots n) \tag{22}$$

together with the equations of equilibrium (1) and (2). It may be of some interest to note that this system of equations does not contain the prestressing parameters λ_r^*. In other words, the "final" state is independent of whatever initial stresses may exist in the structure (due to settlement of support, temperature gradients, etc.), although the "history" of the structure does display such dependence.

By the same token, an asymptotic minimum principle can be derived. For if λ is permitted to approach infinity in the inequality (18), it follows (for $\lambda_r^*=0$) that

$$U(M_\ell; u_\ell, \beta_\ell) = U(m_0; u_\ell, \beta_\ell) \leqq \mathrm{U}(\bar{M}; \bar{u}, \bar{\beta}) \tag{23}$$

or

$$W(u_\ell, \beta_\ell) \leqq W(\bar{u}, \bar{\beta}) \tag{23a}$$

in which $(\bar{M};\bar{u},\bar{\beta})$ represents, as before, any statically admissible bending moment and associated deflection mode. The first equality in (23) follows from Eq. (22). Accordingly:

The limiting bending moment is characterized by making U (or W) smaller than does any other statically admissible bending moment.

In general, this minimum is in the interior of the region of stability. It is approached asymptotically, and the corresponding configuration is unique. The existence of such an interior minimum is proved in the Appendix; also explored are certain singular cases, in which U may assume a minimum on the boundary of the stable domain. Suffice it to state here that in that event the limiting bending moment may be reached for a finite value of the load parameter λ. Also, the deflection mode need not be unique in that case, as would appear reasonable in view of the limitations on the uniqueness principle established in the preceding section.

As before, the upper-bound principle expressed through (23) has a counterpart in the form of a limiting lower-bound principle. This is derived readily by considering (21), with λ going to infinity. If then (u_c,β_c) represents any pair of functions which, in addition to being otherwise acceptable, satisfies the set of equations (22), and if this "collapse mode" furthermore satisfies the condition[9]

$$2U(m_0; u_c, \beta_c) - W(u_c, \beta_c) = 0 \tag{24}$$

then

$$U(m_0; u_c, \beta_c) \leqq U(m_0; u_\ell, \beta_\ell) \tag{25}$$

or

$$W(u_c, \beta_c) \leqq W(u_\ell, \beta_\ell) \tag{25a}$$

[9]Unless one of the terms in Eq. (24) vanishes, this condition can always be satisfied by selecting the amplitude and sign of the assumed collapse mode properly. This follows from the fact that U is quadratic, but W linear in u and β.

In other words:

Of all collapse modes, the actual one corresponds to the largest value of U (or W).

NUMERICAL EXAMPLE

In this section, the principles and equations of the preceding sections are applied to an illustrative example. For this purpose, consider a beam of length L which is of thin rectangular cross section (Γ=0) and of second degree of redundancy relative to bending in its major plane by virtue of being elastically restrained at both ends; however, only one degree of indeterminacy need be considered here owing to the complete symmetry of the problem. Simple supports are provided at both ends, so far as lateral movement is concerned; a single lateral force of magnitude λ is applied halfway between supports and at an eccentricity e above the center line. If, as before, the total response is denoted by (λu,$\lambda\beta$), and if m is the restraint moment, then, in the absence of the vertical load p(z), Eqs. (1) and (2) and the boundary conditions take the following form:

$$\left.\begin{aligned} EI_y u^{iv} - m\beta'' &= 0 \\ GK\beta'' + m u'' &= 0 \end{aligned}\right\} \quad (0 \leqq z \leqq L/2)$$

$$u(0) = u''(0) = \beta(0) = 0 \tag{26}$$

$$u'(L/2) = EI_y u'''(L/2) - m\beta'(L/2) + 1/2 = GK\beta'(L/2) - e/2 = 0$$

In Eq. (26) the second set of boundary conditions follows from the symmetry of the problem; hence the solution for (u,β) need only be considered for half the beam, that is, from the left support (z=0) to the center (z=L/2).

For the case under consideration, the compatibility condition (11), in the absence of prestressing, becomes

$$m = -2\lambda^2 (EI_x/L)\,\gamma \int u''\beta\, dz \tag{27}$$

where $\gamma = \dfrac{C}{C+\dfrac{2EI_x}{L}}$

represents the degree of end restraint and varies from zero (no restraint) to unity (full fixity). C denotes the "spring constant" and is measured in in.-lb per radian.

For the range between 0 and L/2, the solution of Eq. (26) is given by

$$u(z) = A(1-\phi\alpha)\left(\frac{\sin \alpha z/L}{\cos \alpha/2} - \frac{\alpha z}{L}\right)$$
$$e\beta(z) = A\phi\alpha\left[-(1-\phi\alpha)\frac{\sin \alpha z/L}{\cos \alpha/2} + \frac{\alpha z}{L}\right] \qquad (28)$$

in which

$$A = \frac{L^3}{2EI_y\alpha^3} \qquad \phi = \frac{e}{L}\sqrt{\frac{EI_y}{GK}}$$

and

$$0 \leq \alpha^2 = \frac{m^2L^2}{EI_yGK} = \pi^2\left(\frac{m}{m_1}\right)^2 < \pi^2$$

where m_1 identifies the moment associated with neutral equilibrium, and ϕ is a dimensionless constant.

If Eq. (28) is substituted in Eq. (27), this can be shown to lead to the relationship

$$\alpha^5 = \frac{I_x}{2I_y}\frac{L^5}{EI_yGK}\gamma\lambda^2F(\alpha) \qquad (29)$$

with $F(\alpha) = (1-\phi\alpha)\left[\left(\tan\frac{\alpha}{2} - \frac{\alpha}{2}\right) + \frac{1}{2}(1-\phi\alpha)\left(\tan\frac{\alpha}{2} - \frac{\alpha}{2} - \frac{\alpha}{2}\tan^2\frac{\alpha}{2}\right)\right]$

This equation can be made dimensionless. In fact, let the dimensionless ratio ω be defined by

$$\omega = s_y/s_1 \qquad (30)$$

where $s_y = \dfrac{\lambda L/4}{S_y} \qquad s_1 = \dfrac{m_1}{S_x} = \dfrac{\pi}{S_xL}\sqrt{EI_yGK}$

with S_y and S_x designating, respectively, the section moduli about the y and x axes. It is readily apparent that s_y represents the maximum fiber stress in lateral bending, while s_1 is the maximum fiber stress associated with the buckling moment m_1. With these definitions, and in consideration of the relationship between the moments of inertia and section moduli for rectangular sections, Eq. (29) now takes the simple form:

$$(8\gamma\pi^2)\,\omega^2 = \alpha^5 / F(\alpha) \tag{31}$$

As ω (and hence the lateral load λ) approach zero, both numerator and denominator on the right side of Eq. (31) also approach zero. For very small values of ω, Eq. (31) can therefore be replaced, through the usual limit procedure, by

$$(\gamma\phi\pi^2)\,\omega^2 = 3\alpha \qquad (\alpha << \pi) \tag{31a}$$

Conversely, as the lateral load becomes increasingly large, it follows from Eq. (31) that

$$\lim_{\omega \to \infty} F(\alpha) = 0 \tag{31b}$$

The best way to establish a functional relationship between α and ω is probably to assume a value for the former and to solve for the latter by means of Eq. (31), although, for small values of α, the inverse procedure may be followed through the use of Eq. (31a). The asymptotic magnitude of α (for very large lateral loads) may be obtained from Eq. (31b); it can readily be verified that the smallest positive root of Eq. (31b) is given by $\alpha=1/\phi$ for ϕ exceeding π, while, for $\phi < \pi$, the smallest root must be computed through a trial-and-error process.

With the ω-α relationship thus established, the maximum lateral deflection, which occurs at the midpoint of the beam, is found to be

$$u_{max} = A(1-\phi\alpha)(\tan\alpha/2 - \alpha/2) = K_u \frac{L^3}{48EI_y}$$
$$K_u = 24/\alpha^3\,(1-\phi\alpha)(\tan\alpha/2 - \alpha/2) \tag{32}$$

In Eq. (32), the quantity A was defined in Eq. (28), while κ_u represents the ratio between the computed maximum deflection and the equivalent value determined on the basis of the linear theory, which does not predict the development of restraining moments m (that is, α=0). For very small values of α, Eq. (32) can be approximated by

$$\kappa_u = 1 - \phi\alpha = 1 - (\pi^2/3)\,\gamma\phi^2\omega^2 \qquad (\alpha \ll \pi) \tag{32a}$$

in which the second equality follows from Eq. (31a).

Similary, the maximum rotation occurs also at the middle of the beam and is governed by the equation

$$e\beta_{max} = A\phi\alpha\left[\alpha/2 - (1-\phi\alpha)\tan\alpha/2\right] = \kappa_\beta\,\frac{e^2L}{4GK}$$
$$\kappa_\beta = (2/\phi\alpha^2)\left[\alpha/2 - (1-\phi\alpha)\tan\alpha/2\right] \tag{33}$$

in which, similarly to the preceding equation, κ_β is defined as the ratio between the computed maximum rotation and the corresponding rotation computed on the basis of the linear theory. As before, for very small values of α, the second equation of Eq. (33) can be approximated by

$$\kappa_\beta = 1 - (\alpha/12\phi) = 1 - (\pi^2/36)\,\gamma\omega^2 \qquad (\alpha \ll \pi) \tag{33a}$$

Of some interest, finally, is the lateral deflection of the point of application of the force itself. This is found by adding Eqs. (32) and (33), or

$$(u+e\beta)_{max} = A\left[(1-\phi\alpha)^2\tan\alpha/2 - (1-2\phi\alpha)\alpha/2\right] = \kappa\,\frac{L^3}{48EI_y}(1+12\phi^2)$$
$$\kappa = (24/\alpha^3)(1+12\phi^2)^{-1}\left[(1-\phi\alpha)^2\tan\alpha/2 - (1-2\phi\alpha)\alpha/2\right] \tag{34}$$

Here, κ is defined analogously to κ_u and κ_β. For small values of α, this goes over into

$$\kappa = 1 - (2\phi\alpha)(1+12\phi^2)^{-1} = 1 - (2/3)\,\pi^2\phi^2(1+12\phi^2)^{-1}\gamma\omega^2 \qquad (\alpha \ll \pi) \tag{34a}$$

It may be worth noting that, from Eq. (34), $d\kappa/d\alpha$ can be shown to be proportional to $F(\alpha)$. In other words, for very large values of ω, $d\kappa/d\alpha$

vanishes. Since furthermore the second derivative is positive, it is seen that the limiting value of κ is an absolute minimum if only stable values of α (lying in the closed region between $-\pi$ and $+\pi$) are admitted in comparison. This, however, is not surprising. Since κ is a measure of the total work done by the external force (all other terms being independent of α), this property of κ is a natural concomitant of the minimum principle expressed in the inequality (23a).

Some of the results of the preceding discussion become invalid for the singular case of $\phi=1/\pi$. Physically this means that the force is applied at a point which would not move if the beam were to buckle laterally under the critical end moment m_1 ($\alpha=\pi$). This special case, which was briefly alluded to in the previous section, is treated in a general fashion in the Appendix. In its application to the illustrative example being discussed here, it is analyzed fully in what follows; a comparison with the Appendix shows that the general principles developed there are confirmed for the case under consideration.

It can be verified easily that $F(\alpha)$, which is defined in Eq. (29), has no root in the range $0 < \alpha < \pi$ for the singular case of $\phi=1/\pi$. $F(\pi)$ is indeterminate, but the customary limit procedure leads to a value of $1/\pi$; hence, by Eq. (31), the boundary of the stable range is reached when ω assumes the finite value governed by

$$\omega_0^2 = \pi^4/8\gamma \tag{35}$$

while, by Eqs. (32), (33), and (34) and similar limit procedures

$$\left.\begin{aligned} K_U(\pi) &= 48/\pi^4 \\ K_B(\pi) &= 1 - 4/\pi^2 \\ K(\pi) &= (1 + \pi^2/12)^{-1} \end{aligned}\right\} \tag{36}$$

Similarly, the deflection mode approaches the limiting functions

$$\left.\begin{aligned} u_o(z) &= \lim_{\alpha\to\pi} u(z) = \frac{L^3}{\pi^4 EI_y} \sin \pi z/L \\ e\beta_o(z) &= \lim_{\alpha\to\pi} e\beta(z) = \frac{L^3}{\pi^4 EI_y}(\pi^2 z/2L - \sin \pi z/L) \end{aligned}\right\} \quad (0 \leqq z \leqq \tfrac{L}{2}) \tag{37}$$

For values of $\omega > \omega_0$, the restraint moment m retains its critical value m_1, or $\alpha \equiv \pi$ throughout. However, the solution (37) of Eq. (26) is no longer unique; it can in general be expressed by

$$\left.\begin{aligned} u(z) &= u_o(z) + c\,\frac{L^{7/2}}{\pi^4 EI_y}\, u_1(z) \\ e\beta(z) &= e\beta_o(z) + c\,\frac{L^{7/2}}{\pi^4 EI_y}\, e\beta_1(z) \end{aligned}\right\} \tag{38}$$

in which c is an as yet arbitrary multiplier; the factor associated with c has been added for convenience, and (u_1, β_1) is the normalized buckling mode given by

$$u_1(z) = (L)^{-1/2} \sin \pi z/L = -e\beta_1(z) \tag{39}$$

The value of c is now determined from the compatibility condition (27) which, in view of Eqs. (37), (38), and (39) and of the definitions of α and ω, becomes

$$c^2 = 1 - (\pi^4/8\gamma\omega^2) \tag{40}$$

Let a factor $\tau \leqq 1$ be defined by

$$\tau = \omega_0/\omega \tag{41}$$

where ω_0 is given in Eq. (35) and represents (in review) the value of ω as $\alpha=\pi$ is reached initially. Then c is related to τ by the equation of the unit circle

$$c^2 + \tau^2 = 1 \tag{42}$$

It is noted, and indeed expected from the discussion in the Appendix, that for any value of $\tau < 1$ there are two possible values of c; in particular, for $\tau=0$ (i.e., as the force λ goes to infinity), $c = \pm 1$. In other words, two distinct equilibrium configurations are now possible, which are not adjacent to one another; thus, a "snap-through" (Durchschlag) phenomenon is to be expected.

The reduction factors κ are obtained by considering Eqs. (37), (38), (39), (41), and (42). This leads to the following set of relationships:

$$\left.\begin{aligned} &\kappa_u = (48/\pi^4)(1 \pm \sqrt{1-\tau^2}) && \lim_{\tau\to 0} \kappa_u = (96/\pi^4),\ 0 \\ &\kappa_\beta = 1 - (4/\pi^2)(1 \pm \sqrt{1-\tau^2}) && \lim_{\tau\to 0} \kappa_\beta = 1 - 8/\pi^2,\ 1 \\ &\kappa = (1 + \pi^2/12)^{-1} && \end{aligned}\right\} \qquad (43)$$

In other words, the limiting configurations represent either predominant bending with little twisting, or else pure twist without any bending. Only one value of κ appears, however, since the transition from one configuration to the others constitutes a rotation about the point of application of the force. Moreover, this value of κ remains constant once the condition $\alpha=\pi$, or $\omega=\omega_0$, is reached. For $\omega > \omega_0$ $(\tau < 1)$, the force-displacement relationship is therefore linear, but the apparent stiffness of the structure is almost double that predicted by the linear theory.

In concluding this section, it may be noted that, for sufficiently large values of the applied force, the response of the structure is a discontinuous function of the eccentricity e of the force. In fact, for "small" eccentricities $(\phi < 1/\pi)$, the configuration approaches one of mostly bending and little twisting, the ratio being nearly independent of ϕ. More surprisingly, perhaps, the response for "large" eccentricities $(\phi > 1/\pi)$ approaches one of pure twist without any bending, the reduction factors κ_u and κ_β being zero and unity irrespective of the value of ϕ. For the singular case $(\phi=1/\pi)$, the two possible configurations discussed above represent limiting cases as the value of ϕ approaches $1/\pi$ from below or from above, respectively.

This type of discontinuity seems to be one of the salient features which distinguishes the present theory from the conventional linear approach. Other examples have been investigated and have been found to lead to similar results. For example, if the same beam is subjected to two forces applied at equal distances from the ends and at equal (nonvanishing) eccentricities, but pulling in opposite directions, then the asymptotic response is found to be a discontinuous function of the ratio of the magnitudes of the forces. In fact, the larger of the two forces dominates the behavior entirely; singularity, that is, snap-through, occurs when they are equal.

EXPERIMENTAL RESULTS

To obtain an experimental check on the results of the preceding section, a test arrangement similar to the one described in [5] was employed. The beam specimen used was a strap, one inch high and 1/16 inch thick, which was made of heat-treated steel with a yield point of about 180,000 psi. Spanning a distance of 20 in., it was elastically restrained at both ends in the vertical plane, with the degree of end fixity γ [see Eq. (27)] computed at 0.74. However, by loosening the clamps this end restraint could be removed entirely; in this fashion, the beam was made statically determinate to provide a check for the linear theory. The values for the elastic constants E and μ (Poisson's Ratio) were known to be 30,000,000 psi and 0.3, respectively; this establishes the relationship $\lambda = 1.48\omega$, in which λ is the applied force measured in pounds, and ω is defined in Eq. (30).

A relatively rigid vertical bar was attached to the beam at midspan; this bar was then subjected to a horizontal lateral force of increasing magnitude and varying eccentricity. The results corresponding to eccentricities e of 4 in., 8 in., and 12 in. are discussed in what follows; these eccentricities are associated with values of ϕ equaling, very nearly, $1/2\pi$, $1/\pi$, and $3/2\pi$, respectively, where the second value represents the singular case. The lateral deflections λu and rotations $\lambda\beta$, as well as the total displacement $\lambda(u+e\beta)$ of the applied force, were measured in the usual manner by means of scales and mirrors.

A comparison between the predicted and measured results is given in Figs. 2, 3, and 4, which correspond to the three types of eccentricites (small, singular, and large) mentioned above. In each case, the curves give the computed values of the reduction ratios on the basis of Eqs. (32), (33), and (34) as functions of the dimensionless loading parameter ω [defined in Eq. (30)]. For the singular case of e=8", or $\phi=1/\pi$, the boundary of the stable domain is reached for $\omega_0=4.06$, as given in Eq. (35), with the associated reduction coefficients computed in accordance with Eq. (36). For values of ω in excess of ω_0, these coefficients are governed by Eq. (43).

The experimental points shown in the figures represent average values based on several test sequences. It is seen that reasonable agreement was obtained for small and for large eccentricities. Such quantitative discrepancies as do occur seem to be due to the effects of initial imperfections and of prestressing moments; for large values of ω, other non-linear factors which were ignored in the present analysis cause further discrepancies. In fact, since both the measured deflections and rotations were somewhat in excess of the limitations laid down in the Introduction, a more accurate analysis, which involves, for example, powers of u and β above the second, yields more compatible results. Nevertheless, there is excellent qualitative agreement between the theory and experiment; in particular, the discontinuous character of the asymptotic solution is fully corroborated.

For the singular case (e=8", $\phi=1/\pi$) the agreement is confined to values of ω below ω_0. For larger values of ω, two possible solutions (Durchschlag) did indeed occur, but the quantitative agreement between the theory and the experimental values is poor. A possible explanation for this can be found in the effect of initial imperfections. These imperfections, such as initial curvatures, etc., become increasingly important as the boundary of the stable domain is approached, i.e., as the value of α approaches π. In fact, if Eq. (26) is properly modified to include the effect of an initial deviation from perfect shape, it can readily be verified that, in the general case, the solution "blows up" as the support moments approach their critical values. In effect, the orthogonality condition (A7) (see Appendix) is violated, which means that neutral equilibrium is approached only as the load goes toward infinity, instead of the finite value of ω_0 predicted by the idealized theory. This may explain why the double-valued equilibrium configuration was "delayed" beyond its theoretical value.

CONCLUSION

It may be stated that the theory presented herein constitutes a third approximation to the problem of a beam subjected to a combination of bending and torsion. In the first approximation, the strains are assumed to be linear functions of the displacements, while the equations of equilibrium refer to the undistorted configuration. This leads to an entirely linear theory, to which the principle of superposition applies. Such a theory, which is one commonly used in ordinary design, is valid in case the lateral stiffness of the beam is comparable in magnitude to its major stiffness.

When this condition is violated, the need for a second approximation arises. In fact, the presence of large lateral deflections and rotations makes it imperative that the equations of equilibrium be referred to the distorted geometry. The resulting theory is still linear with respect to the lateral loads and twisting couples; however, the effects of the vertical loads can no longer be added linearly.

The present, or third, approximation further abandons the assumption of linear strain-displacement relations. This is a natural result of the presence of large lateral displacements, in consequence of which the second approximation appears to be illogical. This is not quite the case, however. Actually, at least so far as the lateral displacements and rotations are concerned, the results of the last two approximations are identical for beams which are statically determinate relative to moments in their major plane of stiffness. This may provide an explanation as to why this point has been ignored heretofore. In any event, it appears that, at least in the case of statically redundant structures, the introduction of nonlinear strain-displacement relations leads to substantial modifications in their predicted behavior.

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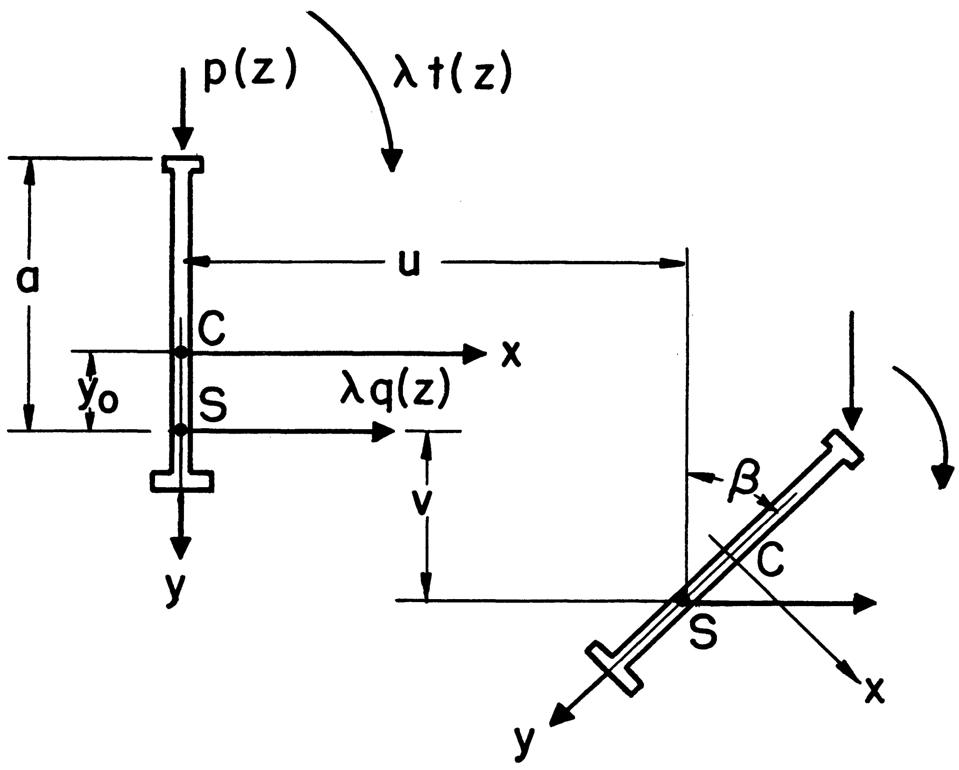

Fig. 1. Typical cross section.

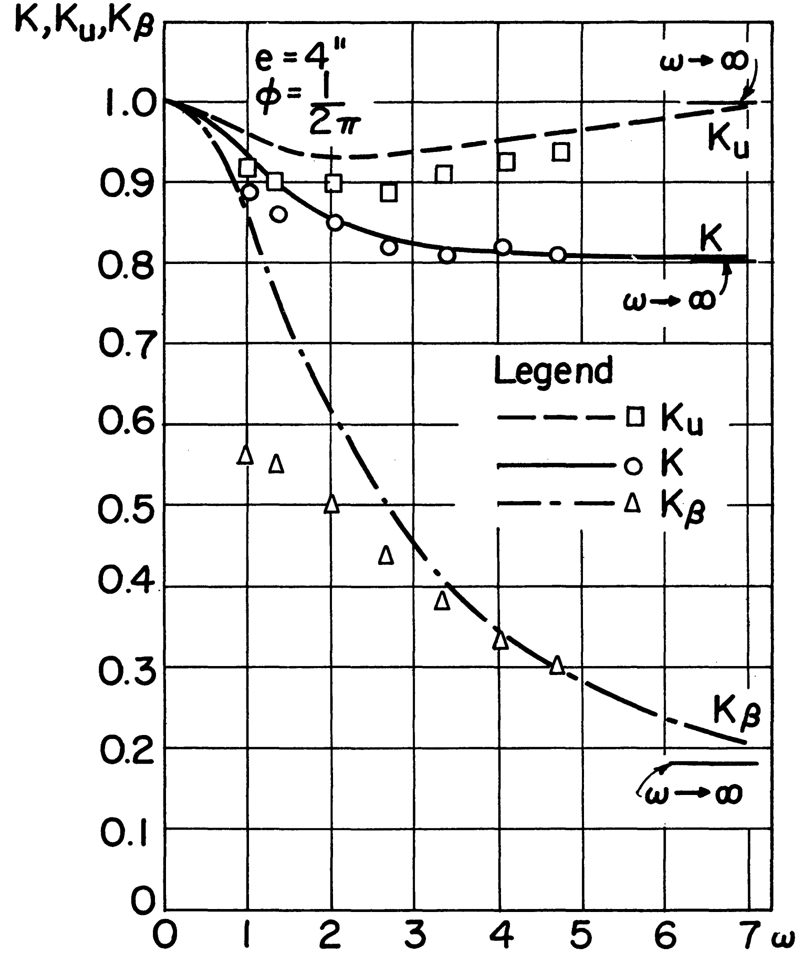

Fig. 2. Reduction factors for small eccentricity.

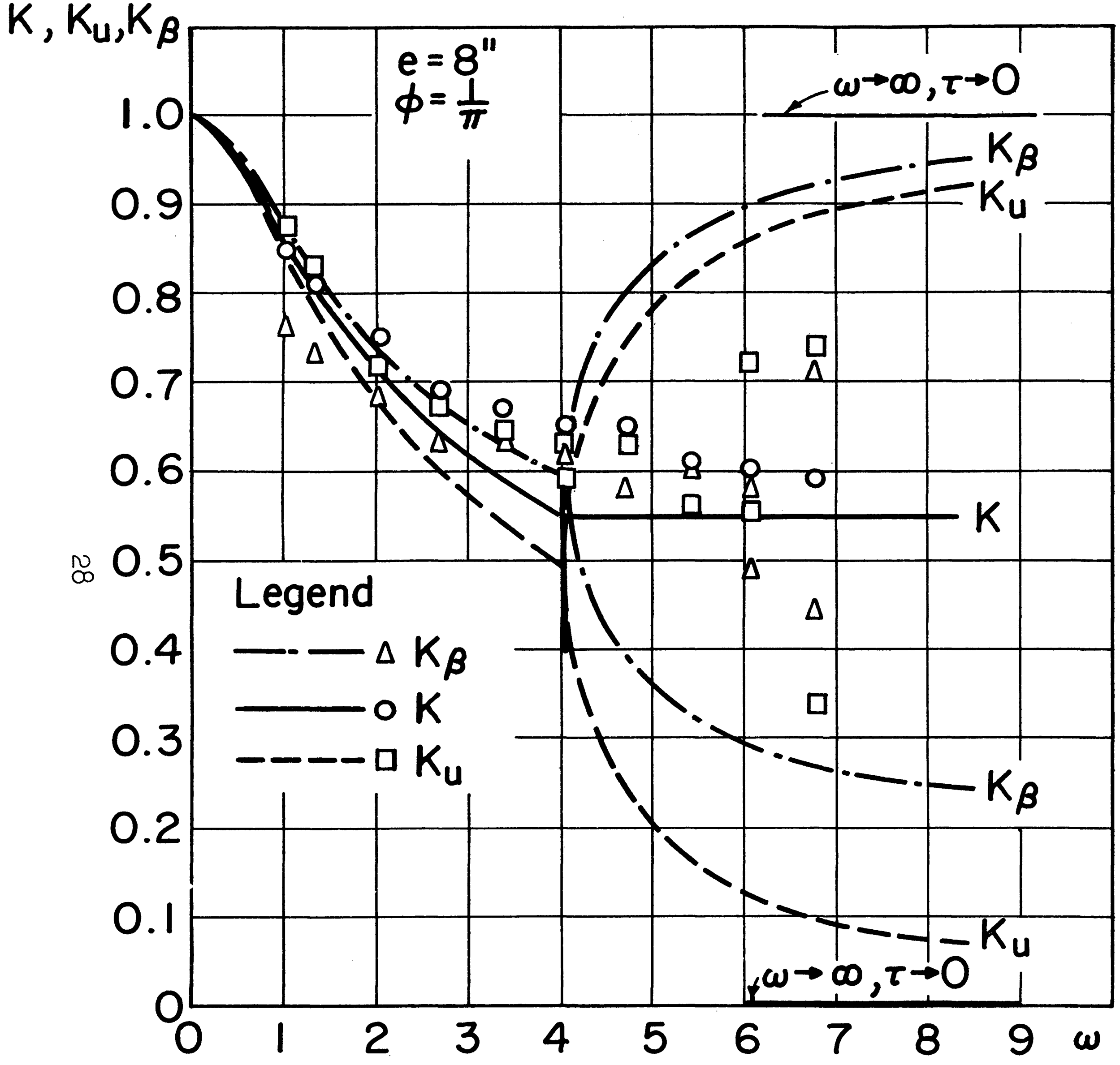

Fig. 3. Reduction factors for singular eccentricity.

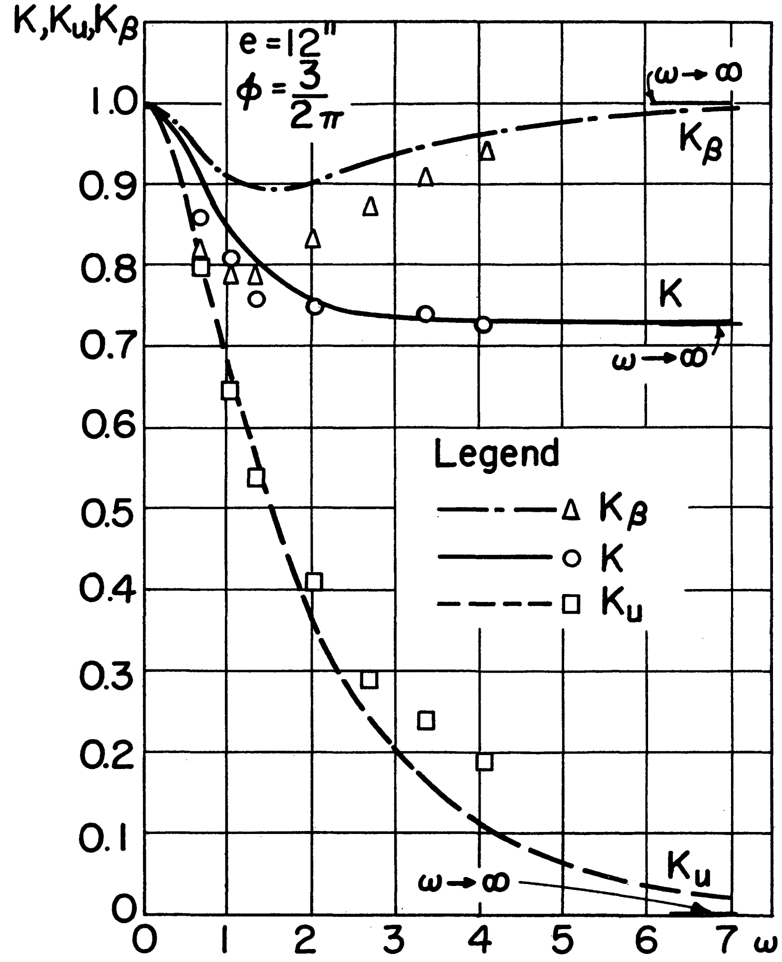

Fig. 4. Reduction factors for large eccentricity.

APPENDIX

In what follows it will be shown that, in general, the function $U(M;u,\beta) \equiv U(\lambda_r)$ [or, equivalently, $W(\lambda_r)$] has an interior minimum, which is approached asymptotically as the load parameter λ goes beyond bounds. In addition, the singular case, in which such an interior minimum may not exist, is explored further.

To this end, consider again the region of stability, which will be assumed to be bounded, as before. In the λ_r-space this region R is enclosed by the boundary B, on which the homogeneous equations (1) and (2) admit a nontrivial solution (u_1,β_1); on B the equilibrium is therefore "neutral," with (u_1,β_1) representing the buckling mode. Moreover, in the general case, the inhomogeneous equations (1) and (2) admit no solution on B, while U grows beyond bounds as the stress point P (whose coordinates are the redundant parameters λ_r) approaches B. If now U can be shown to be convex, the existence of a unique interior minimum can be postulated.

To show this, it is recalled, in line with footnote 7, that if u and β are differentiable functions of λ_r, then, by the inequality (18) or by an independent calculation,

$$\frac{\partial U}{\partial \lambda_r} = -\frac{1}{\lambda^2}\frac{\partial U_b}{\partial \lambda_r} = -\frac{1}{\lambda^2}(\lambda_r - \lambda_r^*) \tag{A1}$$

Consider now a plane which is tangent to U at some point (λ_r); its equation is given by

$$U_t(\bar{\lambda}_r) = U(\lambda_r) + \frac{\partial U}{\partial \lambda_\alpha}(\bar{\lambda}_\alpha - \lambda_\alpha)$$

or, in view of Eq. (A1), by

$$U_t(\bar{\lambda}_r) = U(\lambda_r) - \frac{1}{\lambda^2}(\bar{\lambda}_\alpha - \lambda_\alpha)(\lambda_\alpha - \lambda_\alpha^*) \tag{A2}$$

It follows therefore as a consequence of the inequality (17) that

$$U(\bar{\lambda}_r) \geqq U_t(\bar{\lambda}_r) \tag{A3}$$

In other words, the surface U does not cross any of its tangent planes; it is therefore convex, and the existence of a unique interior minimum is proved, provided all the other previous assumptions are found to hold also.

The exceptional (singular) case occurs when the inhomogeneous equations (1) and (2) do admit a finite solution. For the sake of simplicity, let Eqs. (1) and (2) be rewritten in the symbolic form

$$L_u(M;u,\beta) = q \qquad L_\beta(M;u,\beta) = t \qquad \text{(in R)} \tag{A4}$$

and let, in accordance with the previous discussion,

$$L_u(M_1;u_1,\beta_1) = L_\beta(M_1;u_1,\beta_1) = 0 \qquad \text{(on B)} \tag{A5}$$

Then the system of equations

$$L_u(M_1;u,\beta) = q \qquad L_\beta(M_1;u,\beta) = t \qquad \text{(on B)} \tag{A6}$$

has a finite solution on that portion B' of B on which the "orthogonality condition"

$$\int (qu_1 + t\beta_1)\,dz = 0 \qquad \text{(on B')} \tag{A7}$$

holds. Moreover, since U is finite on B', the existence of an interior minimum can no longer be postulated.

For the purpose of studying singular behavior, let it be assumed that an interior minimum does not exist; this means in turn that U has an exterior minimum, which lies on B'. In that event, as the load factor λ increases, the stress point P reaches some point on B', from which it proceeds along B' toward the minimum. If B' is represented by a single point, as in the case of the illustrative example, the stress point stays fixed after reaching B' and the structure exhibits the quasi-linear behavior discussed earlier in the paper.

After P has reached the boundary B', the most general solution of Eq. (A6) subject to the condition (A7) can be expressed in the form

$$(u, \beta) = (u_o, \beta_o) + c\,(u_1, \beta_1) \tag{A8}$$

in which (u_o, β_o) represents some solution of Eqs. (A6) and (u_1,β_1) is governed, subject to an arbitrary multiplicative constant, by Eqs. (A5). It is convenient to identify these functions further by the orthonormality conditions

$$\int (u_o u_1 + b^2 \beta_o \beta_1)\, dz = 0$$
$$\int (u_1^2 + b^2 \beta_1^2)\, dz = 1 \tag{A9}$$

in which b is an arbitrary number having the dimension of a length. Except for the sign of (u_1,β_1), conditions (A9) determine all the functions uniquely for given value of b.

Let the point at which the stress path "first" reaches the boundary B' be designated by P', and let P' be associated with a load parameter λ' and a parameter $c=c'$ [see (A8)]. Then the value of c' can be determined by a limit process as follows:

While the stress point P is in the interior of the region of stability, the solution of Eq. (A4) can be represented by Eq. (A8), subject to the orthonormality conditions (A9), if (u_1,β_1) satisfies the equations

$$L_u(M; u_1, \beta_1) = \mu u_1 \qquad L_\beta = (M; u_1, \beta_1) = \mu b^2 \beta_1 \tag{A10}$$

in which μ is the eigenvalue of the system (A10). This eigenvalue satisfies the relationship

$$\int [u_1 L_u(M; u_1, \beta_1) + \beta_1 L_\beta(M; u_1, \beta_1)]\, dz = \mu \tag{A11a}$$

which is obtained by multiplying both sides of Eq. (A10) by u_1 and β_1, respectively, by integrating, and by considering the second equation of Eq. (A9). Similarly, the condition

$$\int [u_0 L_u(M; u_1, \beta_1) + \beta_0 L_\beta(M; u_1, \beta_1)] dz = \int [u_1 L_u(M; u_0, \beta_0) + \beta_1 L_\beta(M; u_0, \beta_0)] dz = 0 \tag{A11b}$$

is obtained, in which the second integral follows from the first through integrations by parts. Finally, from Eqs. (A4), (A8), (A11), and the linearity of the differential operators, the parameter c is seen to be given by the equation

$$c = \left(\frac{1}{\mu}\right) \int (q u_1 + t \beta_1) dz \tag{A12}$$

As the point P approaches the boundary B, the eigenvalue μ approaches zero and the factor c increases generally beyond bounds. However, on B' the numerator in Eq. (A12) also vanishes, as postulated in Eq. (A7); hence the fraction becomes indeterminate, and the following relationship is established by means of the usual limit considerations:

$$c' = \lim_{\mu \to 0} c = \frac{1}{\dot{\mu}} \int (q \dot{u}_1 + t \dot{\beta}_1) dz \tag{A13}$$

In Eq. (A13) and in what follows, a dot (such as in $\dot{\mu}$) signifies differentiation along the path, that is, $\frac{\alpha}{\alpha \lambda_s}$.

The denominator in Eq. (A13) is obtained from Eq. (A11a) through differentiation. This leads to

$$\begin{aligned} \dot{\mu} = & \int [\dot{u}_1 L_u(M; u_1, \beta_1) + \dot{\beta}_1 L_\beta(M; u_1, \beta_1)] dz \\ & + \int [u_1 L_u(M; \dot{u}_1, \dot{\beta}_1) + \beta_1 L_\beta(M; \dot{u}_1, \dot{\beta}_1)] dz \\ & + \int \{ u_1 (m_s \beta_1)'' + \beta_1 [m_s u_1'' + 2(k m_s \beta_1')] \} dz \end{aligned} \tag{A14}$$

The last integral on the right side of (A14) follows from the definitions of L_u and L_β and from the fact that $\dot{M}=m_s$ by Eq. (6a).

The first integral in (A14) vanishes; this can be seen by substituting Eq. (A10) and by differentiating the second equation of Eq. (A9) with respect to λ_s. The second integral vanishes similarly, since it can be converted into the first integral (and hence zero) through four integrations by parts and in view of the boundary conditions. Two more integrations by parts finally convert the third integral into

$$\dot{\mu} = 2\int m_s(u_1''\beta_1 - k\beta_1'^2)\,dz \tag{A15}$$

The establishment of the numerator in Eq. (A13) proceeds along analogous lines. In fact, since (u_0,β_0) satisfies Eq. (A4) at P' (for $M=M_1$), it follows that the relationship

$$\int(q\dot{u}_1 + t\dot{\beta}_1)\,dz = \int\left[\dot{u}_1 L_u(M; u_0, \beta_0) + \dot{\beta}_1 L_\beta(M; u_0, \beta_0)\right]dz$$
$$= \int\left[u_0 L_u(M; \dot{u}_1, \dot{\beta}_1) + \beta_0 L_\beta(M; \dot{u}_1, \dot{\beta}_1)\right]dz$$

holds. The second equality in the preceding equation is the result of a number of integrations by parts. However, if Eq. (A10) is differentiated with respect to λ_s, and if μ is set equal to zero, it follows that

$$\dot{L}_u(M; u_1, \beta_1) = L_u(M; \dot{u}_1, \dot{\beta}_1) + (m_s\beta_1)'' = \dot{\mu}\,u_1$$
$$\dot{L}_\beta(M; u_1, \beta_1) = L_\beta(M; \dot{u}_1, \dot{\beta}_1) + m_s u_1'' + 2(k m_s \beta_1')' = \dot{\mu}\,b^2\beta_1$$

on B'. When these relations are substituted in the last integral and the first equation of Eq. (A9) is considered, then, after some more integrations by parts, the relationship

$$\int(q\dot{u}_1 + t\dot{\beta}_1)\,dz = -\int m_s(u_0''\beta_1 + u_1''\beta_0 - 2k\beta_0'\beta_1')\,dz \tag{A16}$$

is established. Hence, by Eq. (A13), c' is expressed as the quotient of the two integrals appearing on the right side of Eqs. (A15) and (A16).

Offhand, this quotient seems to depend on the direction of the approach path. That this is not the case can be seen from the following considerations:

In the linear space in which the stress point P is defined, the moment component m_s can in general be expressed in the form $m_s=\gamma_\alpha m_\alpha$, in which the set of γ_r represents the direction cosines of the approach path relative to the coordinate axes. Similarly, the integrals on the right side of Eqs. (A15) and (A16) are representable as linear combinations involving these direction cosines.

Now both denominator and numerator components vanish for all directions which are tangent to the boundary surface B'. Indeed, since μ vanishes identically throughout B, $\mathring{\mu}=0$ for all directions other than the one normal to B. Similarly, the numerator is different from zero only in the normal direction, which can be verified by considering that the orthogonality condition (A7) is satisfied identically on B'. Hence the direction of the approach path is immaterial, and the limiting coefficient c' can finally be expressed by

$$c' = -\frac{\int m_n(u_0''\beta_1 + u_1''\beta_0 - 2k\beta_0'\beta_1')\,dz}{2\int m_n(u_1''\beta_1 - k\beta_1'^2)\,dz} \tag{A17}$$

in which m_n represents the moment component normal to B'.

After the bounding surface of instability B' has been reached (for $\lambda = \lambda'$), the stress point P travels on B' as the load parameter λ is increased further. The response $\lambda(u,\beta)$ continues to satisfy Eq. (A8); when this is substituted in the compatibility conditions (11), the following system of equations is obtained:

$$\lambda_n - \lambda_n^* = \lambda^2\int m_n(u_0''\beta_0 - k\beta_0'^2)\,dz + c\lambda^2\int m_n(u_0''\beta_1 + u_1''\beta_0 - 2k\beta_0'\beta_1')\,dz$$
$$+ c^2\lambda^2\int m_n(u_1''\beta_1 - k\beta_1'^2)\,dz \tag{A18n}$$

$$\lambda_t - \lambda_t^* = \lambda^2 \int m_t (u_o'' \beta_o - k\beta_o'^2)\, dz \tag{A18t}$$

The subscript n denotes the direction of the outer normal to B', while the subscript t designates the totality of all directions which are tangent to B' and for which the condition (A7) is identically satisfied. In the set of equations (A18t) only one integral appears on the right side; two more integrals, similar in form to the second and third integrals in (A18n), can be shown to vanish.

In general, P moves on B' in such a way as to minimize further the value of $U+U_b/\lambda^2$. With λ going to infinity, a limiting point P_ℓ is approached, which corresponds to the smallest value of U on B'. This point is found by setting the integrals appearing on the right side of the system of equations (A18t) equal to zero; a limiting value of c is then obtained by substituting the functions (u_o, β_o) so determined in Eq. (A18n). (Note that there are two solutions.)

In special cases the stress point P may remain at P'. This happens when either P' is the only point on B' (such as in the illustrative example), or else when P' happens to be the point associated with the lowest value of U on B'; this latter case occurs when the vector $\overline{P^*P'}$ is normal to the instability surface. In either case, the final state of stress designated by P' is generally reached when the load parameter is finite ($\lambda = \lambda'$). For $\lambda > \lambda'$, the response functions $\lambda(u_o, \beta_o)$ increase linearly with λ similarly to a statically determinate structure. However, as before, two solutions for c are possible according to Eq. (A18n).[10] In other words, two distinct equilibrium configurations are possible such that

[10] In the $(\lambda, c\lambda)$ plane, Eq. (A18n) defines a hyperbola. This becomes apparent from the fact that, if multiplied by two, the right side can be converted into the form $-\lambda^2\dot{W} + c\lambda^2 B + c^2\lambda^2\dot{\mu}$, in which the dot represents differentiation in the n-direction. It is clear that $\dot{\mu}$ is negative, since μ is positive inside the region of stability and negative outside of it. Similarly, $\dot{W}$ is seen to be negative; indeed if it were positive then W would exhibit an interior minimum, in which case P would not reach (or stay) on the boundary. And finally, if $\dot{W}$ were zero, P' would be reached only in the limit. The quadratic form has therefore a negative discriminant; hence it designates a hyperbola.

one may be reached from the other only through a snap-through (Durchschlag) process. As before, a limiting value of c is approached as the parameter λ grows beyond bounds.

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