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STRUCTURAL INSTABILITY UNDER TIME-DEPENDENT LOADS

E. F. Masur

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ABSTRACT

Possible definitions of time-dependent stability are discussed. The response of simply supported columns to linearly attenuating lateral and axial forces is computed and charted for a broad variety of parameters, including asymptotic cases. The equations governing columns of other boundary conditions are developed and their solution is indicated. Special sections are devoted to multi-story frames and arches.

1. INTRODUCTION

The question of the stability of structures under time-dependent loads has taken on increasing significance in recent years. Its fundamental importance is self-evident. Except under very special circumstances, structures are analyzed and designed with respect to an assumed set of static loads, although the actual loads to be withstood by the structures are almost invariably "time-dependent." This procedure is normally justified by the fact that the dynamic character of the loads affects the response of the building only to a negligible extent. Where such effects can no longer be safely ignored, they are often taken into consideration through the expedient of "impact factors" or "equivalent static forces." Nevertheless, the validity of such procedures is frequently open to question.

When a structure is subjected to dynamic loads caused by blasts, a quasi-static approach to the analysis becomes meaningless. Much research has been conducted during the past ten years to determine the response of beams and frames to blast loads, especially with regard to the influence of permanent plastic deformations. On the other hand, the problem of the elastic stability of such structural elements when subjected to blast loads has received relatively little attention.

It seems likely that one of the reasons for this neglect is to be found in the comparative difficulty which is inherent in such a study. In fact, the very definition of stability under time-dependent loads requires clarification. For this reason, the concept of stability under static loads is reviewed in what follows immediately.

Let a structure, or a structural element such as a column, be subjected to a set of static forces, and let the structure assume a certain equilibrium configuration. For the sake of simplicity, this is frequently assumed to be the original unstressed configuration, although the error thus introduced is not always negligible. When the loads are sufficiently small, the equilibrium of the structure remains stable; it may become unstable, that is, the structure may "buckle," when the loads reach or surpass a certain critical value. To find this value, there are, in general, the following three avenues of approach.

1. The first, and most common, approach is that of investigating the possibility of an equilibrium configuration which is associated with the same set of forces as the unbuckled configuration and which is "adjacent" to the latter. This is the classical approach of Euler and leads to the formulation of an eigenvalue problem, in which the load parameter is the eigenvalue and the buckling mode the eigenfunction. The smallest load parameter for which a nontrivial so-

lution exists separates the stable from the unstable domain.

A slight variation of this line of attack is presented by investigating the uniqueness of the load-deformation relationship. In this case, the smallest load parameter for which a branch point (bifurcation point) may exist constitutes the stability limit. In general, this formulation is identical with the previous one. Differences may develop in the case of irreversible processes; an example for this is furnished by the buckling of a column under axial force when the (prebuckling) stress exceeds the elastic limit.

2. An alternate stability criterion is based on the potential energy. In this case, the potential energy of the unbuckled configuration is compared with that of all geometrically consistent neighboring configurations. So long as these are all associated with an increase in the potential energy, the equilibrium is said to be stable. If there is at least one configuration whose potential energy represents a decrease, the structure is in unstable equilibrium. The two domains are separated by the case in which all configurations show an increase except at least one for which the potential energy remains unchanged; the force parameter associated with this case is the critical one.

This method of attack again leads to an eigenvalue problem, which is generally identical with the one mentioned above. However, the chief significance of this approach seems to lie in its ready adaptability to approximate methods, of which the one due to Rayleigh-Ritz is the best known.

3. The third approach is dynamic. Owing to its comparative mathematical complexity, it is rarely employed, yet it represents the only physically meaningful stability criterion. In essence, it consists in tracing the motion of a structure after its equilibrium has been disturbed by a source of arbitrary character and of arbitrary, but finite, magnitude and duration. If the amplitude of this motion remains finite with time, the equilibrium is called stable. Conversely, if there exists at least one type of disturbance which causes motion of indefinitely increasing amplitude, the structure is in unstable equilibrium.

An obvious difficulty immediately presents itself. Since the concept of instability is linked to "large" amplitudes of motion, the governing equations are almost invariably of nonlinear character; this in turn makes the solution prohibitively difficult. For this reason, the equations of the disturbed motion are usually linearized. As a result, the following contradiction presents itself: the equations of motion are solved on the basis of assumed small amplitudes, and the solution is then investigated relative to the possibility of large amplitudes. Actually, the stability criterion so obtained is usually on the safe side. It can be rationalized by letting the disturbance be very small and by questioning whether the response remains also very small.

In the vast majority of cases, the three stability criteria described above lead to identical results. In particular, this is true in the absence of nonconservative or gyroscopic force systems. An example to the contrary is presented by the case of a cantilever column which is subjected at its free end to a force whose magnitude is given and whose direction is specified to be tangent to the column at the point of application of the force. In this case, the classical eigenvalue approach predicts stability for a force of arbitrarily large magnitude, while the energy approach breaks down since the system is nonconservative, and hence a potential energy cannot be defined. Nevertheless, the dynamic approach leads to a critical force of finite magnitude.[1]

It is almost self-evident that the two static avenues of attack, which serve well in most static stability problems, become vacuous in the event of time-dependent loads. The third, or dynamic, approach alone adapts itself to this problem. In fact, let the response of a structure to dynamic loads be computed; then the motion so determined is considered stable if a slight change in the loads or in the initial conditions corresponds to only a slight change in the response. A few problems of this type have been studied in recent years, notably the case of a column which is subjected to an axial force whose time-dependence is harmonic.[2] Oddly enough, the strict application of the present stability criterion to this case leads to the improbable (and discouraging) result that all columns are unstable under all circumstances. The authors relieve themselves of this difficulty by calling the inevitable presence of friction to the rescue; in addition, they might also have cited the influence of nonlinear factors.

In studying the stability of structures under time-dependent loads, it had originally been planned to apply criteria of the type discussed above. It can be seen readily, however, that such an approach must fail in the case of blast-type loads, whose outstanding characteristic feature is their (more or less rapid) attenuation. This means that, after a sufficiently large time interval has passed, the loads invariably become small enough to make the amplitude of the motion bounded. Hence instability of the nature described above does not occur, unless the kind of reasoning used in Ref. 2 is applied, in which case again all structures are unstable. It is obvious that this is of no practical use in the light of the purpose for which the present study was undertaken.

Rather, to lend realism to this investigation, it was decided to study the response of structures which are subjected not only to dynamic lateral loads, but also to axial forces varying with time. This does not represent a stability problem in the strictest sense of the word; in fact, from a mathematical point of view, the essential feature of the issue of stability—namely, the search for an eigenvalue which renders an essentially homogeneous system singular—has evaporated.

Nevertheless, no remorse need be felt at such an apparent deviation from the straight path. Actually, all true stability problems represent a form of mathematical fiction. For example, in the case of the classical column buckling problem, it is assumed that, prior to buckling, the column is entirely straight. This in turn implies that it is free from imperfections and that the axial load is fully centric; this constitutes an obvious idealization of the actual conditions. The physical significance of the idealized solution lies in the fact that the axial force for which the theoretical column ceases to be stable is also an upper bound to the carrying capacity of the actual column (if nonlinear factors are ignored). In mathematical terms this means that when a homogeneous system becomes singular (that is, it admits nontrivial solutions), the solution of an associated inhomogeneous system in general passes beyond bounds. This limiting condition, however, is the real object of the investigator's search; the associated eigenvalue problem is only a convenient method of finding it.

The method, then, is not available for use in the current investigation, but the object is essentially the same: to determine the behavior of structures and of structural elements when exposed to blast-type loads, of both lateral and axial nature. The presence of these axial forces is an essential feature, and furthermore one which permits the retention of the term "stability" in the title of the report. Indeed, the magnitude of the axial forces may exceed their static stability limit; it will be shown that this is permissible if they are of sufficiently short duration.

A substantial portion of the report (and almost all its numerical data) concerns itself with the case of a simply supported column under both lateral and axial load, both of which attenuate linearly, although not necessarily at the same rate. This may be considered a type of "pilot" problem, whose results, properly modified, are reasonably applicable to all types of structures. Actually it will be seen that the simply supported column occupies a somewhat unique position: the results obtained here are exact (within the limits of the present theory, of course), whereas analogous results become approximate, although technically acceptable, for any other type of structure. This is because, of all possible structural combinations, the simply supported column alone exhibits modes of vibration which are independent of the magnitude of the axial force. The significance of this fact seems to have been pointed out for the first time in Ref. 3.

Some simplifications have been introduced, but these are not considered unduly restrictive. The column is assumed to be initially perfectly straight; if an initial crookedness is present, this can be handled readily through the addition of an equivalent lateral force, as was done, for example, in Ref. 4. The axial force is assumed to be entirely centric; again, the existence of an eccentricity leads to further equivalent lateral loads. The analysis is based on the Bernoulli-Euler beam theory, which appears reasonable in view of the presumed slenderness of the structure. Elastic behavior is postulated throughout; this limits the discussion, but the inclusion of plastic deformations, difficult as it is for the static buckling case, presents insurmountable mathematical obstacles in the case of dynamic instability. Finally, the equations are linearized through the customary use of an approximate expression for the curvature. This implies that the lateral deformations are small compared with the length of the column (not necessarily compared with its thickness!); if this were to be violated, the previous assumption of elasticity would become meaningless, except in the case of unrealistically slender elements.

2. SIMPLY SUPPORTED COLUMNS

Let a simply supported column be subjected to a time-dependent axial force P and a time-dependent lateral load intensity w. Let its deflection be designated by y and its mass per unit length by μ; E and I represent the usual material and cross-sectional constants. If x is measured along the (undeflected) column axis and t represents the time, then, with subscripts designating the appropriate derivatives, the motion of the column is governed by

$$(EIy_{xx})_{xx} + P\, y_{xx} + \mu\, y_{tt} = w \quad , \tag{2.1}$$

where $y = y(x,t)$, $P = P(t)$, $\mu = \mu(x)$, and $w = w(x,t)$.

For the case under consideration, let EI and μ be constants. Also, if the origin of the coordinate system is fixed at one end of the column, whose length is called L, then the boundary conditions are as follows:

$$y(0,t) = y_{xx}(0,t) = y(L,t) = y_{xx}(L,t) = 0 \quad . \tag{2.2}$$

In view of these boundary conditions, it is in general possible to expand the response in the form

$$y(x,t) = \sum_{n=1}^{\infty} Q_n(t) \sin(n\pi x/L) \quad . \tag{2.3a}$$

Similarly, let the loading function w be expressible in the convergent series

$$w(x,t) = \sum_{n=1}^{\infty} p_n(t) \sin(n\pi x/L) \quad , \tag{2.3b}$$

where the coefficients p_n are given, in the usual fashion, by

$$p_n(t) = 2/L \int_0^L w(x,t) \sin(n\pi x/L)\, dx \qquad (n=1,2,\ldots) \tag{2.3c}$$

When Eqs. (2.3) are substituted in Eq. (2.1) and the coefficients of the Fourier expansion are equated, it follows that, for each value of n,

$$(EIn^4\pi^4/L^4 - Pn^2\pi^2/L^2)Q_n + \mu\, Q_n'' = p_n \qquad (n=1,2,\ldots) \quad ,$$

in which primes designate derivatives with respect to t. To simplify this equation, it is convenient to introduce the following definitions:

$$P(t) = \lambda_n(t)P_n = \lambda_n\, n^2\pi^2 EI/L^2$$

$$\omega_n^2 = (n^4\pi^4 EI)/(\mu L^4) \qquad (2.4)$$

$$p_n(t) = g_n(t)\,(n^4\pi^4 EI)/L^4$$

The quantities λ_n and g_n so defined are dimensionless since P_n is the n^{th} eigenvalue for the static case and the coefficient of g_n represents the (sinusoidal) load associated with a unit deflection; ω_n is representative of the n^{th} frequency of vibration in the absence of axial forces.

With these definitions, the n^{th} mode $Q_n(t)$ is governed by

$$Q_n'' + \omega_n^2\,(1 - \lambda_n)Q_n = \omega_n^2\, g_n \quad . \qquad (2.5)$$

For the special case under consideration here, the axial force decreases linearly from its maximum value at t=0 to zero at t=T. It is of course assumed that T is large compared with the time that it takes an axial wave to travel across the column; this assumption is already inherent in the derivation of Eq. (2.1). In other words, let

$$\left.\begin{aligned} \lambda_n(t) &= \alpha_n\,(1 - t/T) & (0 \leqq t \leqq T) \\ \lambda_n(t) &= 0 & (T \leqq t) \end{aligned}\right\} . \qquad (2.6)$$

Then, by Eqs. (2.5) and (2.6):

$$\left.\begin{aligned} Q_n'' + \omega_n^2\,[1 - \alpha_n(1-t/T)]Q_n &= \omega_n^2\, g_n(t) & (0 \leqq t \leqq T) \\ Q_n'' + \omega_n^2\, Q_n &= \omega_n^2\, g_n(t) & (T \leqq t) \end{aligned}\right\} . \qquad (2.7)$$

Furthermore, both Q_n and Q_n' are continuous at t=T.

Consider first the case of $\alpha_n \leqq 1$. Then with the introduction of the dimensionless quantities

$$\left.\begin{aligned} \tau_n &= 1 - \alpha_n(1-t/T) & (1-\alpha_n \leqq \tau_n \leqq 1) \\ k_n &= \omega_n T/\alpha_n & \end{aligned}\right\} . \qquad (2.8)$$

the first of Eqs. (2.7), after dropping the subscript n, assumes the form

$$q_{\tau\tau} + k^2 \tau q = k^2 \gamma \qquad (1-\alpha \leqq \tau \leqq 1) \quad , \tag{2.9}$$

where $q(\tau) \equiv Q(t)$ and $\gamma(\tau) \equiv g(t)$.

The associated homogeneous equation

$$F_{\tau\tau} + k^2 \tau F = 0 \tag{2.10a}$$

is satisfied by the two independent functions

$$\left.\begin{aligned} F_1(\tau) &= \tau^{1/2} J_{1/3}(\theta) \\ F_2(\tau) &= \tau^{1/2} J_{-1/3}(\theta) \\ \text{with} \qquad \theta &= 2/3\, k\, \tau^{3/2} \end{aligned}\right\} . \tag{2.10b}$$

$J_{\pm 1/3}(\theta)$ represents Bessel functions of appropriate order.

If now $q(\tau)$ is assumed to be of the form $C_1(\tau)F_1(\tau) + C_2(\tau)F_2(\tau)$ and the usual "variation of parameter" method is applied, the response can be expressed as follows:

$$q(\tau) = k^2/W \int_{1-\alpha}^{\tau} [F_1(\sigma)\, F_2(\tau) - F_2(\tau)\, F_1(\sigma)]\, \gamma(\sigma)\, d\sigma \quad , \tag{2.11a}$$

where W is the Wronskian of the two functions in Eq. (2.10b) and is determined by

$$\begin{aligned} W = F_1F_{2\tau} - F_2F_{1\tau} &= k\tau^{3/2} [J_{1/3}(\theta)\, J'_{-1/3}(\theta) - J'_{1/3}(\theta)\, J_{-1/3}(\theta)] \\ &= -\, 3^{3/2}/(2\pi) \quad . \end{aligned} \tag{2.11b}$$

In other words, W is a constant; its value is obtained through the use of the well-known relationship

$$J_{\mu}(\theta)\, J'_{-\mu}(\theta) - J'_{\mu}(\theta)\, J_{-\mu}(\theta) = -\, 2 \sin(\mu\pi)/\pi\theta \quad . \tag{2.11c}$$

Equation (2.11a) gives the response for $\tau \leqq 1$, that is, before the axial force vanishes. It can readily be shown that the integral and its derivative with respect to the upper limit vanish when $\tau = 1-\alpha$, or

$$q(1-\alpha) = q_\tau(1-\alpha) = 0 \quad . \tag{2.12}$$

This means that the structure starts from rest. If an initial displacement or velocity is present, this can easily be accounted for through the inclusion of linear combinations of F_1 and F_2.

A somewhat more interesting case occurs when $\alpha_n > 1$. This means the initial axial force exceeds the value which is associated with instability in the event of static buckling ($T \rightarrow \infty$). As expected, the nature of the dynamic response also changes drastically. In fact, if the definition of τ_n from Eq. (2.8) is used again (and if also the subscript n is deleted), it is seen that τ now assumes both positive and negative values; it increases monotonically, passing through zero at the instant when the axial force equals the static buckling force.

It is convenient to separate the negative and positive domains of τ. As for the former, let a new independent variable $\bar{\tau}$ be defined by means of

$$\tau = -\bar{\tau} \qquad (0 \leq \bar{\tau} \leq \alpha-1) \quad . \tag{2.13}$$

With this definition, the governing Eq. (2.9) reads

$$\bar{q}_{\bar{\tau}\bar{\tau}} - k^2 \bar{\tau} \bar{q} = k^2 \bar{\gamma} \qquad (\alpha-1 \geq \bar{\tau} \geq 0) \quad . \tag{2.14}$$

In Eq. (2.14), $\bar{q}(\bar{\tau})$ is used for $q(\tau)$; similarly $\bar{\gamma}(\bar{\tau})$ takes the place of $\gamma(\tau)$. The associated homogeneous equation

$$\bar{F}_{\bar{\tau}\bar{\tau}} - k^2 \bar{\tau} \bar{F} = 0 \tag{2.15a}$$

has the two independent solutions

$$\left.\begin{aligned} \bar{F}_1(\bar{\tau}) &= -\bar{\tau}^{1/2} I_{1/3}(\bar{\theta}) \\ \bar{F}_2(\bar{\tau}) &= \bar{\tau}^{1/2} I_{-1/3}(\bar{\theta}) \\ \text{with} \quad \bar{\theta} &= 2/3\, k\, \bar{\tau}^{3/2} \end{aligned}\right\} . \tag{2.15b}$$

In Eqs. (2.15b), $I_{\pm 1/3}(\bar{\theta})$ represents modified Bessel functions of appropriate order. The signs were chosen for convenience with a view toward making both F_1 and F_2 and their derivatives continuous at $\tau=\bar{\tau}=0$ since $F_{1\tau}(0) = -\bar{F}_{1\bar{\tau}}(0)$. It may also be worth mentioning that the functions defined in Eqs. (2.15b) represent linear combinations of the well-known Airy functions $Ai(\tau)$ and $Bi(\tau)$.

Through a process which is analogous to the one used in the establishment of Eqs. (2.11), the response is now found to be

$$\bar{q} = k^2/\bar{W} \int_{\alpha-1}^{\bar{\tau}} [\bar{F}_1(\bar{\sigma})\ \bar{F}_2(\bar{\tau}) - \bar{F}_1(\bar{\tau})\ \bar{F}_2(\bar{\sigma})]\ \bar{\gamma}(\bar{\sigma})\ d\bar{\sigma} \quad , \tag{2.16a}$$

where $\bar{W}$ is again the Wronskian and is defined by

$$\bar{W} = \bar{F}_1\ \bar{F}_{2\bar{\tau}} - \bar{F}_2\ \bar{F}_{1\bar{\tau}} = -W = +3^{3/2}/2\pi \quad . \tag{2.16b}$$

It is important to note that $\bar{q}(\bar{\tau})$ as given in Eq. (2.16a) is valid for the range $\alpha\text{-}1 \geqq \bar{\tau} \geqq 0$ only.

When τ exceeds zero ($\bar{\tau} \leqq 0$), the solution is given essentially by Eqs. (2.11). However, at τ=0, neither the response function $q(0)$ nor its derivative necessarily vanish. The initial conditions which led to the establishment of Eqs. (2.11) are therefore violated; hence the term $AF_1(\tau) + BF_2(\tau)$ must be added, with the constants A and B determined from the condition of continuity

$$q(0) = \bar{q}(0) \qquad q_\tau(0) = -\bar{q}_{\bar{\tau}}(0) \quad , \tag{2.17}$$

in which the right sides are computed on the basis of Eqs. (2.16). Since finally, by Eqs. (2.10) and (2.15),

$$F_i(0) = \bar{F}_i(0) \qquad F_{i\tau}(0) = -\bar{F}_{i\bar{\tau}}(0) \qquad (i=1,2) \quad , \tag{2.18}$$

it follows that

$$q(\tau) = k^2/W \left\{ F_2(\tau) \left[\int_0^{\alpha-1} \bar{\gamma}(\bar{\sigma})\ \bar{F}_1(\bar{\sigma})d\bar{\sigma} + \int_0^{\tau} \gamma(\sigma)\ F_1(\sigma)d\sigma \right] - F_1(\tau) \left[\int_0^{\alpha-1} \bar{\gamma}(\bar{\sigma})\ \bar{F}_2(\bar{\sigma})d\bar{\sigma} + \int_0^{\tau} \gamma(\sigma)\ F_2(\sigma)d\sigma \right] \right\} . \tag{2.19}$$

The solution (2.19) applies to the range $0 \leqq \tau \leqq 1$. When τ exceeds unity—that is, when t exceeds T—the second of Eqs. (2.7) becomes valid. Since again the continuity of the response and of its derivative is postulated, the solution may now be written in the form:

$$Q(t) = \omega \int_T^t \sin \omega(t-s)\ g(s)ds + q_\tau(1)/k\ \sin \omega(t-T) + q(1)\ \cos \omega(t-T) \qquad (t \geqq T) \quad . \tag{2.20}$$

This is based on the fact that both the integral and its derivative with respect to the upper limit vanish for t=T; furthermore, $Q(T) = q(1)$ and $Q'(T) = (\alpha/T)q_{\tau}(1)$. Equation (2.20) holds irrespective of whether the value of α does or does not exceed unity.

In concluding this phase of the work which is concerned with the establishment of the general equation of the response function, let the loading function w be expressible in the form

$$w(x,t) = w(x)\ f(t) \ ; \qquad (2.21)$$

then, in view of Eqs. (2.3) and (2.4), the "generalized load" g_n is given by

$$g_n(t) = a_n\ f(t) \qquad \gamma_n(\tau) = a_n\ \phi(\tau) \qquad \bar{\gamma}_n(\bar{\tau}) = a_n\ \bar{\phi}(\bar{\tau}), \qquad (2.22a)$$

where the constant a_n is governed by

$$a_n = L^4/(n^4\pi^4EI)\ (2/L) \int_0^L w(x)\ \sin\ (n\pi x/L)dx \ . \qquad (2.22b)$$

Moreover, if $Q_n(t) \equiv q_n(\tau) \equiv \bar{q}_n(\bar{\tau})$ are defined for the case of $a_n=1$, then, by the principle of superposition,

$$y(x,t) = \sum_{n=1}^{\infty} a_n\ Q_n(t)\ \sin\ (n\pi x/L) \ . \qquad (2.23)$$

Unlike the time-dependence of the axial force, the time variation of the lateral loading function f(t) poses no inherent difficulties. Since the principle of superposition obviously holds in view of the linearity of the problem, different cases may be treated separately through substitution in the general solutions as given in Eqs. (2.11), (2.16), etc.; the integrations may have to be carried out numerically, however.

As an example in the present study, it is assumed that the lateral blasting force attenuates linearly, although it may not necessarily vanish together with the axial force. In other words,

$$f(t) = 1 - t/T_1 \qquad (0 \leqq t \leqq T_1)$$
$$\quad\ = 0 \qquad (T_1 \leqq t) \ . \qquad (2.24a)$$

When the variable τ as defined in Eq. (2.8) is introduced again, f(t) assumes the form:

$$\phi(\tau) = 1-\beta + \beta/\alpha - (\beta/\alpha)\tau \qquad (1-\alpha \leqq \tau \leqq 1-\alpha + \beta/\alpha)$$
$$= 0 \qquad (1-\alpha + \beta/\alpha \leqq \tau) \qquad (2.24b)$$

where $\beta = T/T_1$. When $\tau \leqq 0$, $\bar{\phi}(\bar{\tau})$ can be defined similarly.

It is convenient to consider the cases $\phi=1$ and $\phi=\tau$ separately and then to determine the response to the loading function given in Eqs. (2.24) by superposition. To this end, let $\phi=\tau$ (or $\bar{\phi}=-\bar{\tau}$) be associated with the response function $Q'(t)$ [or $q'(\tau)$]; then Eqs. (2.11) can be integrated directly.

For the case of $\alpha < 1$, this leads to

$$q'(\tau) = 1 + (k/W)(1-\alpha)\ \tau^{1/2}\ [J_{-2/3}(\Theta_0)J_{-1/3}(\Theta) + J_{2/3}(\Theta_0)J_{1/3}(\Theta)] \quad ,$$
$$(2.25)$$
$$\text{with} \quad \Theta = (2/3)k\tau^{3/2} \quad \text{and} \quad \Theta_0 = (2/3)k(1-\alpha)^{3/2} \qquad (1-\alpha \leqq \tau \leqq 1) \ .$$

In the derivation of Eq. (2.25), a number of well-known Bessel identities were utilized, such as $x^{2/3}J_{-1/3}(x) = d/dx\ [x^{2/3}J_{2/3}(x)]$ and $x^{2/3}J_{1/3}(x) = d/dx\ [-x^{2/3}J_{-2/3}(x)]$. It is also recalled that $J_{-2/3}(x) = dJ_{1/3}(x)/dx + (1/3x)J_{1/3}(x$ and $J_{2/3}(x) = -dJ_{-1/3}(x)/dx - (1/3x)J_{-1/3}(x)$. As a consequence it follows that

$$(k/W)\tau^{3/2}\ [J_{-2/3}(x)J_{-1/3}(x) + J_{2/3}(x)J_{1/3}(x)] = -1 \quad .$$

It may also be worth mentioning that Eq. (2.25) can be derived directly by considering that $q'=1$ is a particular integral of Eq. (2.9); the additional terms are obtained in the usual fashion from the complementary solution by satisfying the initial conditions.

For the case of $\alpha > 1$, a similar procedure is employed. This leads to

$$\bar{q}'(\bar{\tau}) = 1 + (k/W)\bar{\tau}^{1/2}(\alpha-1)\ [I_{-2/3}(\bar{\Theta}_0)I_{-1/3}(\bar{\Theta}) - I_{2/3}(\bar{\Theta}_0)I_{1/3}(\bar{\Theta})] \quad ,$$
$$(2.26a)$$
$$\text{with} \quad \bar{\Theta} = (2/3)k\bar{\tau}^{3/2} \qquad \bar{\Theta}_0 = (2/3)k(\alpha-1)^{3/2} \qquad (\alpha-1 \geqq \bar{\tau} \geqq 0) \ .$$

Similarly,

$$q'(\tau) = 1 + (k/W)\tau^{1/2}(\alpha-1)\ [I_{-2/3}(\bar{\Theta}_0)J_{-1/3}(\Theta) + I_{2/3}(\bar{\Theta}_0)J_{1/3}(\Theta)]$$
$$(0 \leqq \tau \leqq 1) \quad . \qquad (2.26b)$$

When $\alpha=1$, the solution can be obtained either from Eq. (2.26a) or (2.26b) through a limiting process. The result is given below:

$$q'(\tau) = 1 - 1.07477\, \theta^{1/3} J_{-1/3}(\theta) \qquad (0 \leqq \tau \leqq 1) \quad , \qquad (2.27)$$

where 1.07477 represents $(\pi/3^{1/2})\, 2^{-2/3}/\Gamma(1/3)$.

For the case $\phi(\tau)=1$, the response is designated by $Q''(t)\equiv q''(\tau)$. In that case, an explicit representation of the integral formula in Eq. (2.11) is not possible. Instead, the response can be written as follows:

a. for $\alpha < 1$,

$$q''(\tau) = (k/W)\tau^{1/2}\left\{[A(\theta) - A(\theta_o)]J_{-1/3}(\theta) - [B(\theta) - B(\theta_o)]J_{1/3}(\theta)\right\} , \qquad (2.28a)$$

in which

$$A(\theta) = \int^{\theta} J_{1/3}(x)dx = 2\,[J_{4/3}(\theta) + J_{10/3}(\theta) + J_{16/3}(\theta) + \ldots]$$
$$B(\theta) = \int^{\theta} J_{-1/3}(x)dx = 2\,[J_{2/3}(\theta) + J_{8/3}(\theta) + J_{14/3}(\theta) + \ldots] \qquad . \ (2.28b)$$

Equations (2.28b) can be derived from the standard differentiation and recursion formulas relating to Bessel functions. Similarly,

b. for $\alpha > 1$,

$$\bar{q}''(\bar{\tau}) = (k/W)\bar{\tau}^{1/2}\left\{[\bar{A}(\bar{\theta}) - \bar{A}(\bar{\theta}_o)]I_{-1/3}(\bar{\theta}) - [\bar{B}(\bar{\theta}) - \bar{B}(\bar{\theta}_o)]I_{1/3}(\bar{\theta})\right\}$$
$$(\alpha-1 \geqq \bar{\tau} \geqq 0) \quad . \qquad (2.29a)$$

$$q''(\tau) = (k/W)\tau^{1/2}\left\{[A(\theta) - \bar{A}(\bar{\theta}_o)]J_{-1/3}(\theta) - [B(\theta) + \bar{B}(\bar{\theta}_o)]J_{1/3}(\theta)\right\}$$
$$(0 \leqq \tau \leqq 1) \quad . \qquad (2.29b)$$

in which

$$\bar{A}(\bar{\theta}) = \int^{\bar{\theta}} I_{1/3}(\bar{x})d\bar{x} = 2\,[I_{4/3}(\bar{\theta}) - I_{10/3}(\bar{\theta}) + I_{16/3}(\bar{\theta}) \pm \ldots]$$
$$\bar{B}(\bar{\theta}) = \int^{\bar{\theta}} I_{-1/3}(\bar{x})d\bar{x} = 2\,[I_{2/3}(\bar{\theta}) - I_{8/3}(\bar{\theta}) + I_{14/3}(\bar{\theta}) \pm \ldots] \qquad . \ (2.29c)$$

Finally,

c. for $\alpha = 1$,

$$q''(\tau) = (k/W)\tau^{1/2} [A(\theta)J_{-1/3}(\theta) - B(\theta)J_{1/3}(\theta)] \qquad (0 \leqq \tau \leqq 1) \quad . \qquad (2.30)$$

With q' and q'' expressed through Eqs. (2.25) through (2.30), and in view of Eqs. (2.24), the actual response to a linearly decreasing lateral load can finally be summarized in the following form:

1. Let $T < T_1$, or $\beta < 1$; then

$$q(\tau) = (1-\beta + \beta/\alpha)q''(\tau) - (\beta/\alpha)q'(\tau) \qquad (0 \leqq t \leqq T) \text{ or } (1-\alpha \leqq \tau \leqq 1)$$

$$Q(t) = (1-\beta t/T) + [q(1) - 1+\beta] \cos \omega(t-T) \qquad (2.31)$$

$$+ [q_\tau(1)/k + \beta/\omega T] \sin \omega(t-T) \qquad (T \leqq t \leqq T_1)$$

$$Q(t) = Q(T_1) \cos \omega(t-T_1) + Q'(T_1)/\omega \sin \omega(t-T_1) \qquad (T_1 \leqq t) \quad .$$

The first of these equations represents the forced vibration under linearly decreasing axial force; the second represents the continued forced vibration after the axial force has vanished; and the third is an expression of the ensuing free vibration after the lateral force has vanished. In all cases the constants were so chosen as to make the response and its derivative continuous.

2. When $T > T_1$, or $\beta > 1$, the response becomes a free vibration before the axial force has disappeared. In that case, the three phases may be expressed as follows:

$$q(\tau) = (1-\beta+\beta/\alpha)q''(\tau) - (\beta/\alpha)q'(\tau) \qquad (0 \leqq t \leqq T_1) \text{ or } (1-\alpha \leqq \tau \leqq 1-\alpha+\alpha/\beta = \tau_1)$$

$$q(\tau) = 1/W \Big\{q(\tau_1) [F_{2\tau}(\tau_1)F_1(\tau)-F_{1\tau}(\tau_1)F_2(\tau)] + q_\tau(\tau_1) [F_1(\tau_1)F_2(\tau)-F_2(\tau_1)F_1(\tau)]\Big\} \qquad (\tau_1 \leqq \tau \leqq 1) \quad . \qquad (2.32)$$

$$Q(t) = q(1) \cos \omega(t-T) + q_\tau(1)/k \sin \omega(t-T) \qquad (T \leqq t)$$

It may be noted that for the special case of $T=T_1$, or $\beta=1$, the middle phase in these equations disappears, while Eqs. (2.31) and (2.32) coalesce. Note also that if $\tau_1 < 0$, that is, if the lateral load vanishes before the axial load has been reduced to its Euler value, the functions q, F_1, and F_2 must be replaced by their appropriate counterparts $\bar{q}$, $\bar{F}_1$, $\bar{F}_2$, respectively.

3. ASYMPTOTIC CONDITIONS

In this section, the results of the previous section will be examined in the light of a number of extreme conditions. Such asymptotic analyses may be useful in obtaining the response of the structure, at least from a qualitative point of view, to very severe loading conditions which may otherwise defy exact scrutiny. The following three limiting conditions will be investigated:

A. the lateral load is of impulsive character;
B. the axial force is of impulsive character; and
C. the axial force is of very long duration, without, of course, actually lasting infinitely long. (This would imply a constant axial force, for which the response can be found by elementary methods.)

A. IMPULSIVE LATERAL LOAD

This case is by far the easiest to analyze and involves only a standard limiting technique, which leads, as will be seen, to a conventional and easily predictable result. To this end, let the duration T_1 of the lateral load approach zero while its maximum value becomes infinitely large. In other words, let

$$\begin{aligned} f(t) &= 2/(\omega T_1)\,(1 - t/T_1) \qquad (0 \leqq t \leqq T_1) \\ \text{or} \quad \phi(\tau) &= 2/(\omega T_1)\,(1 - \beta + \beta/\alpha - \beta\tau/\alpha) \qquad (1-\alpha \leqq \tau \leqq \tau_1) \quad , \end{aligned} \tag{3.1}$$

while T_1 approaches zero or, conversely, β approaches infinity. Obviously this implies that the total impulse

$$\int_0^{T_1} f(t)dt = 1/\omega$$

remains constant during the limiting process; the factor $2/\omega$ in Eqs. (3.1) has been added for convenience.

The response can now be determined by means of Eqs. (2.11). In fact, let

$$q(\tau_1) = 2k^2/(\omega W T_1) \int_{1-\alpha}^{\tau_1} [F_1(\sigma)F_2(\tau_1) - F_1(\tau_1)F_2(\sigma)] \left(1-\beta + \frac{\beta}{\alpha} - \frac{\beta}{\alpha}\,\sigma\right) d\sigma \quad .$$

Then it follows from a mean value theorem that

$$\left.\begin{aligned} \lim_{\beta\to\infty} q(\tau_1) &= q(1-\alpha) &= 0 \\ \text{or} \quad \lim_{\beta\to\infty} Q(T_1) &= Q(0) &= 0 \end{aligned}\right\} \quad . \qquad (3.2a)$$

Similarly, the integral can be differentiated with respect to the upper limit. When the same mean value theorem is applied again and the definition of W is recalled, this results in

$$\left.\begin{aligned} \lim_{\beta\to\infty} q_\tau(\tau_1) &= q_\tau(1-\alpha) &= k \\ \text{or} \quad \lim_{\beta\to\infty} Q'(T_1) &= Q'(0) &= \omega \end{aligned}\right\} \quad . \qquad (3.2b)$$

In other words, this case leads to a free vibration with initial values determined by means of Eqs. (3.2). These values can readily be verified by means of the standard impulse-momentum relationship.

B. IMPULSIVE AXIAL FORCE

Of somewhat heightened interest is the case of the axial force becoming very large while its duration shrinks to zero.* In other words, let the axial force be given by

$$\lambda(t) = 2/(\omega T)\ (1 - t/T) \qquad (3.3)$$

while T approaches zero. This case can be handled by expanding the solution of Eq. (2.7) (with $\alpha = 2/\omega T$) in a power series in t/T near t=0. Two different possibilities arise:

a. The bar is initially straight, while both the axial and lateral force are applied simultaneously. With $Q(0) = Q'(0) = 0$, this power series then becomes

$$Q(t) = \omega^2 t^2/2\ [1 - \omega^2 t^2/12\ (1 - 2/\omega T) \pm \ldots] \quad .$$

---

*Actually this contradicts a previous assumption that the duration of the force is large compared with the time a traveling wave requires to traverse the bar. However, while the limiting case is thus outside the exact scope of the present study, the results of the latter seem nevertheless of interest in connection with impulses which are "short but not too short."

In the limit, this leads to

$$\left.\begin{aligned} \lim_{T\to 0} Q(T) &= Q(0) = 0 \\ \lim_{T\to 0} Q'(T) &= Q\ (0) = 0 \end{aligned}\right\} \quad . \qquad (3.4)$$

This result is not entirely self-evident. It means that axial forces of arbitrarily large amplitude are without effect on the response of the structure provided that their duration is sufficiently short. This explains why the accompanying charts show that the motion of the bar becomes increasingly independent of α for decreasing values of $k = \omega T/\alpha$. As a consequence, shock-type axial forces need not be of concern to the designer (at least so far as the phenomena being studied here are involved) if the structure is initially in an unloaded state.

b. A more realistic condition prevails if it is assumed that the lateral load has been applied prior to the application of the axial shock. With the normalizations introduced in this report, this implies that the initial conditions are expressed by $Q(0) = 1$ and $Q'(0) = 0$. The power series expansion then becomes

$$Q(t) = 1 + \omega t^2/T - 1/3\ \omega t^3/T^2 \pm \ldots ,$$

with the result that

$$\left.\begin{aligned} \lim Q(T) &= Q(0) = 1 \\ \lim Q'(T) &= Q'(0) = \omega \end{aligned}\right\} \quad . \qquad (3.5)$$

In other words, the imposition of an axial shock, like that of a lateral impulse, results in the establishment of an initial momentum in conformity with Eqs. (3.5). The ensuing motion is then of the same character as that of an uncompressed bar, differing from the latter only through the imposition of a modified initial condition.

Two points of somewhat academic interest may be mentioned. Firstly, the results of this section could have been obtained by expanding the general solution (involving modified Bessel functions) in a power series near $\bar{\tau} = \alpha - 1$ and by proceeding to the appropriate limits. Secondly, it may be noted that, for the case of the structure being initially not straight, the imposition of an axial impulse results in the establishment of a lateral momentum. This apparent inconsistency is resolved if it is remembered that the lateral end reactions are affected by the axial force; the resulting lateral imbalance accounts for the development of the momentum.

C. AXIAL FORCE OF VERY LONG DURATION

Another interesting problem arises when the duration of the axial force becomes very long (without, of course, going to infinity). In the present discussion this is equivalent to letting $k = \omega T/\alpha$ become very large; in other words, Eq. (2.9) must be integrated asymptotically.

Where the response $q(\tau)$ is given specifically in closed form, this can be achieved by means of the familiar asymptotic expansion formulas of Bessel functions and Airy integrals. When the forcing function $\phi(\tau)$ is a constant, the resulting response $q''(\tau)$ is not given in closed form and the corresponding asymtotic expansion is not at once obvious. In any event, all these expansions break down near $\tau=0$, which occurs in case α exceeds unity. Attention is further directed to the fact that it is desirable to discuss a general method which is applicable even when the axial-force-time relationship is not linear.

There exist several classical methods of asymptotic integration, which are discussed in the literature on the subject. For example, following Ref. 5 (pp. 523 ff.), one can expand the solutions $F_1(\tau)$ and $F_2(\tau)$ of Eqs. (2.10a) in a series of increasing negative powers of k, that is,

$$F(\tau) = h(\tau)\, e^{k\mu(\tau)} \left[1 + f_1(\tau)/k + f_2(\tau)/k^2 + \ldots\right] .$$

When this is substituted in Eq. (2.10a) and terms involving the same powers of k are individually equated to zero, it follows that

$$\left.\begin{aligned} F_1(\tau) &= \tau^{-1/4} \cos\Theta \pm \ldots \\ F_2(\tau) &= \tau^{-1/4} \sin\Theta \pm \ldots \end{aligned}\right\} (\tau > 0) \quad , \qquad (3.6a)$$

if only the first term in the expansion is retained. By the same token, the solution of Eq. (2.15a) is expanded into

$$\left.\begin{aligned} \bar{F}_1(\bar{\tau}) &= \bar{\tau}^{-1/4} e^{\bar{\Theta}} \pm \ldots \\ \bar{F}_2(\bar{\tau}) &= \bar{\tau}^{-1/4} e^{-\bar{\Theta}} \pm \ldots \end{aligned}\right\} (\bar{\tau} > 0) \quad . \qquad (3.6b)$$

Within certain limits, the asymptotic solutions of the homogeneous equations can thus be obtained for arbitrary force-time relationships, even if the direct solutions of the equations themselves are not available in closed form.

It is interesting to note that at $\tau=0$ (or $\bar{\tau}=0$), both types of expansion (3.6a) and (3.6b) become invalid. Moreover, the character of the asymptotic solutions changes drastically as τ changes sign, or, equivalently, as Θ is evaluated along the real or the imaginary axis. This is known as "Stokes'

Phenomenon" and has been the object of extensive studies, notably (in the more recent past) by Langer[6] and Kazarinoff.[7]

As a result of these studies, it has been shown that, in the neighborhood of the singularity (in the present case the origin), the asymptotic solution can be expressed in terms of Bessel functions of suitable order. For any first-order singularity this turns out to be the Airy function; actually, the latter represents the complete solution in the example presently under study. In other words, the special functions given in Eqs. (3.6) are replaced, near the origin, by the general solutions of Eqs. (2.10) and (2.15); moreover, this replacement applies also to any kind of force-time relationship which exhibits a first-order singularity.

The actual response to a lateral load can now be determined, for example, through the method of parameter variation as before. This leads to integral expressions of the type shown in Eq. (2.11), but employing now the asymptotic forms of the pertinent functions. When α is less than unity, no difficulty is encountered in this process; the results are given as follows:

$$\left.\begin{aligned} q'(\tau) &= 1 - \left(\frac{1-\alpha}{\tau}\right)^{1/4} \cos\,(\theta-\theta_0) \pm \ldots \\ q''(\tau) &= (1-\alpha)^{-1}\left[\frac{1-\alpha}{\tau} - \left(\frac{1-\alpha}{\tau}\right)^{1/4} \cos\,(\theta-\theta_0) \pm \ldots\right] \end{aligned}\right\} (1-\alpha \leqq \tau \leqq 1)\;. \tag{3.7}$$

Both of these types of response are similar in form to that of a damped vibration.

When α exceeds unity, a similar simple integration is applicable as long as τ remains negative—that is, before the axial force has decreased to its critical Euler value. This leads to the following type of expansion:

$$\left.\begin{aligned} \bar{q}'(\bar{\tau}) &= 1 - \left(\frac{\alpha-1}{\bar{\tau}}\right)^{1/4} \cosh\,(\bar{\theta}_0-\bar{\theta}) \pm \ldots \\ \bar{q}''(\bar{\tau}) &= (\alpha-1)^{-1}\left[\frac{\alpha-1}{\bar{\tau}} - \left(\frac{\alpha-1}{\bar{\tau}}\right)^{1/4} \cosh\,(\bar{\theta}_0-\bar{\theta}) \pm \ldots\right] \end{aligned}\right\} (\alpha-1 \geqq \bar{\tau} > 0)\;. \tag{3.8a}$$

As pointed out previously, the integration cannot be carried directly across the singularity. However, with the asymptotic expansions replaced by the appropriate exact solutions near the origin, the following solution can be obtained after carrying out a number of limiting procedures:

$$
\left.\begin{aligned}
q'(\tau) &= -\left(\frac{\alpha-1}{\tau}\right)^{1/4} e^{\bar{\Theta}_0} \cos(\Theta - \pi/4) \pm \ldots \\
q''(\tau) &= (\alpha-1)^{-1}\left(\frac{\alpha-1}{\tau}\right)^{1/4} e^{\bar{\Theta}_0} \cos(\Theta-\pi/4) \pm \ldots
\end{aligned}\right\}(0 < \tau \leqq 1) \quad . \qquad (3.8b)
$$

Finally, in the special case of $\alpha=1$, the asymptotic response functions are given by

$$
\left.\begin{aligned}
q'(\tau) &= 1 - 1.07477\,(2/\pi)^{1/2}\,\Theta^{-1/6} \cos(\Theta-\pi/12) \pm \ldots \\
q''(\tau) &= -(1/\tau)\,(2\pi/3)^{1/2}\,\Theta^{1/2} \cos(\Theta+\pi/4) \pm \ldots
\end{aligned}\right\}(0 < \tau \leqq 1) \quad . \qquad (3.9)
$$

The results of the last two sections are plotted in Figs. 1 to 7. The first four figures show the time history of the column response to the lateral load with α assuming the values 1/2, 1, 3/2, and 2 and ωT taking on the values 1, 5, and 25, respectively; also shown, where possible, is the case of ωT approaching infinity. This seems to cover the extremes of a relatively very short shock ($\omega T = 1$) to that of a shock of fairly long duration. The ordinate in these charts is Q_n, which represents (in review) the dynamic magnification over the static case in the absence of an axial force. The abscissa is the dimensionless quantity t/T_p, in which $T_p = 2\pi/\omega$ is the natural period of vibration, measured also in the absence of axial forces. The response curves are divided into solid and dashed sections, the former applying to the time while the axial shock is in effect and the latter to the residual motion after the axial force has disappeared. It is noteworthy, although not unexpected, that the magnitude of the response becomes increasingly sensitive to the duration of the shock with increasing values of α; on the other hand, the effect of the variation of the value of β seems to diminish as α increases.

Figures 5 and 6 show the response of the column to a lateral impulse of magnitude $1/\omega$. This occurs when the value of β is permitted to approach infinity. Actually, these curves may be useful in predicting the behavior of the structure when the duration of the lateral force is very much smaller than that of the axial force; in that case it is reasonable to compute the response in terms of the total impulse imparted to the structure. As pointed out previously, similar considerations apply if the axial force itself is of an impulsive type provided that the structure is already deflected prior to the application of the axial impulse.

Finally, Fig. 7 is constructed from the previous six figures by considering the maximum response as a function of the various parameters. It can also be used to compute the maximum stresses, which are proportional to the maximum curvature. It is interesting to note that, for short axial shocks ($\omega T=1$), the response is fairly independent of the maximum value of the axial force, which is related to the parameter α. This shows that little damage is to be expected from an axial shock of large magnitude if the duration of the shock is sufficiently short. It is as if the structure "had no time" to buckle before the shock subsides; actually, as pointed out previously, an axial force of impulsive nature has no effect on a previously undeflected structure.

4. STRUCTURES OF ARBITRARY BOUNDARY CONDITIONS

In the preceding two sections, the simply supported column was singled out for analysis because of its inherent simplicity. As will be seen in the present section, the introduction of different boundary conditions, and in particular the analysis of continuous structures, leads to substantial computational difficulties. Briefly, this is due to the fact that, of all possible structural elements, the simply supported column alone exhibits basic modes of vibration which are independent of the magnitude of the axial force.

For the sake of reference, let the basic equation governing the response $y(x,t)$ be repeated here:

$$(EIy_{xx})_{xx} + P\, y_{xx} + \mu\, y_{xx} = p(x,t) \quad . \tag{4.1}$$

It is convenient to introduce the following nondimensional quantities

$$\xi = x/L \qquad \tau = (\Omega_1)^{1/2} t \qquad \lambda(\tau) = P/P_1 \quad , \tag{4.2}$$

so that Eq. (4.1) assumes the form:

$$(EIy'')'' + \lambda\, P_1 L^2 y'' + \mu\, \Omega_1 L^4\, \ddot{y} = p\, L^4 \quad , \tag{4.3}$$

in which primes (') represent derivatives with respect to ξ and dots ($\dot{}$) derivatives with respect to τ. The quantity Ω_1 represents the square of the fundamental frequency of vibration of the structure in the absence of an axial force, while P_1 is the lowest buckling force—that is, the force for which ω_1 (the square of the fundamental frequency) vanishes.

It will now be assumed that both y and p can be expanded in the pair of infinite series

$$\begin{aligned} y(\xi,\tau) &= u_\alpha(\xi,\lambda) f_\alpha(\tau) \\ p(\xi,\tau) &= \mu(\xi) u_\alpha(\xi,\lambda) g_\alpha(\tau) \end{aligned} \quad , \tag{4.4}$$

in which the functions u_i constitute the set of eigenfunctions satisfying, for given value of λ, the homogeneous differential equation

$$(EIu_i'')'' + \lambda P_1 L^2 u_i'' - L^4 \mu \omega_i u_i = 0 \qquad (i=1,2,3\ldots) \qquad (4.5)$$

and associated boundary conditions. In Eq. (4.4) and in what follows, repeated Greek subscripts denote summation from one to infinity; it is presumed that all the series converge uniformly to make the operations meaningful. It may also be noted that, in general, the eigenfunctions u_i depend on the value of the parameter λ.

From Eq. (4.4) it follows that

$$\left.\begin{aligned} \dot{y} &= u_\alpha \mathring{f}_\alpha + v_\alpha f_\alpha \dot{\lambda} \\ \ddot{y} &= u_\alpha \ddot{f}_\alpha + 2v_\alpha \mathring{f}_\alpha \dot{\lambda} + w_\alpha f_\alpha (\dot{\lambda})^2 + v_\alpha f_\alpha \ddot{\lambda} \end{aligned}\right\} , \qquad (4.6)$$

in which $v_i(\xi,\lambda) \equiv \partial u_i/\partial\lambda$ and $w_i(\xi,\lambda) \equiv \partial v_i/\partial\lambda$. When this is substituted in Eq. (4.3), and Eqs. (4.4) and (4.5) are considered, the ensuing equation is of the form

$$\left(\ddot{f}_\alpha + \frac{\omega_\alpha}{\Omega_1} f_\alpha\right) u_\alpha + v_\alpha(2\mathring{f}_\alpha \dot{\lambda} + f_\alpha \ddot{\lambda}) + w_\alpha f_\alpha (\dot{\lambda})^2 = \frac{u_\alpha g_\alpha}{\Omega_1} . \qquad (4.7)$$

It may also be noted here that the usual orthogonality relations are expressed by

$$\left.\begin{aligned} \int_0^1 \mu u_i u_j d\xi &= \delta_{ij} \\ \int_0^1 EIu_i'' u_j'' d\xi - \lambda P_1 L^2 \int_0^1 u_i' u_j' d\xi &= L^4 \omega_i \delta_{ij} \end{aligned}\right\} , \qquad (4.8)$$

where δ_{ij} represents the Kronecker delta.

The "generalized coordinates" f_i appearing in Eq. (4.7) can be separated by multiplying the latter by μu_i and by integrating over the range from zero to one. If the components a_{ij} and b_{ij} are defined by means of

$$a_{ij} = \int_0^1 \mu v_i u_j d\xi \qquad b_{ij} = \int_0^1 \mu w_i u_j d\xi , \qquad (4.9)$$

and if the orthogonality conditions (4.8) are taken into consideration, this operation leads to

$$\ddot{f}_i + (\omega_i/\Omega_1)f_i + a_{\alpha i}(2\dot{f}_\alpha\dot{\lambda} + f_\alpha\ddot{\lambda}) + b_{\alpha i}f_\alpha(\dot{\lambda})^2 = g_i/\Omega_1 \quad . \tag{4.10}$$

$$(i=1,2,3\ldots)$$

In the special case of a linearly attenuating force, let

$$\lambda = \Lambda\,(1 - t/T) \quad , \tag{4.11a}$$

and define the dimensionless constant k by

$$k = (\Omega_1)^{1/2}\, T/\Lambda \quad . \tag{4.11b}$$

Then Eq. (4.10) simplifies to

$$\ddot{f}_i + (\omega_i/\Omega_1)f_i - (2/k)a_{\alpha i}\dot{f}_\alpha + (1/k^2)b_{\alpha i}f_\alpha = g_i/\Omega_1 \quad . \tag{4.12}$$

It may be noted that along with u_i, v_i, w_i, and, consequently, with a_{ij} and b_{ij}, the value of ω_i is also a function of λ and hence of the time t.

To obtain specific formulas for the components a_{ij}, it is recalled that Eq. (4.5) is identically satisfied for all values of λ. If it is therefore differentiated with respect to λ, it follows that v_i is governed by the equation

$$(EIv_i'')'' + \lambda P_1L^2v_i'' - \mu L^4\omega_i v_i = - P_1L^2u_i'' + \mu L^4\overset{*}{\omega}_i u_i \quad , \tag{4.13}$$

in which $\overset{*}{\omega}_i$ is used for $d\omega_i/d\lambda$. Let Eq. (4.13) be multiplied by u_j and integrated with respect to ξ along the length of the structure. By means of several integrations by parts, this can easily be shown to lead to

$$\int_0^1 (EIu_j'')''v_i d\xi + \lambda P_1L^2\int_0^1 u_j''v_i d\xi - \omega_i L^4\int_0^1 \mu u_j v_i d\xi = P_1L^2\int_0^1 u_i'u_j'd\xi + L^4\overset{*}{\omega}_i\int_0^1 \mu u_i u_j d\xi$$

since all the boundary terms vanish regardless of the type of boundary conditions employed. In view of Eqs. (4.5), (4.8), and the definitions (4.9), this reduces to

$$(\omega_j-\omega_i)a_{ij} = P_1/L^2\int_0^1 u_i'u_j'd\xi + \overset{*}{\omega}_i\,\delta_{ij} \quad . \tag{4.14a}$$

In this equation the subscripts i and j refer to a pair of arbitrary integer numbers. In particular, when i=j, this reduces to

$$\overset{*}{\omega}_i \equiv d\omega_i/d\lambda = -P_1/L^2 \int_0^1 (u_i')^2 d\xi \qquad (i=1,2,\ldots) \quad . \qquad (4.14b)$$

It is of interest to note that, as expected, $\overset{*}{\omega}_i$ is negative. Its value can be determined from Eq. (4.14b) for a specific value of λ—that is, without finding the solution of Eq. (4.5) for arbitrary values of λ.

Returning to Eq. (4.14a), let i be different from j; then

$$a_{ij} = -a_{ji} = (P_1/L^2)\,(\omega_j-\omega_i)^{-1} \int_0^1 u_i' u_j' d\xi \qquad (i\neq j) \quad . \qquad (4.14c)$$

Furthermore,

$$a_{ii} = 0 \quad , \qquad (4.14d)$$

as can be seen by differentiating the first of Eqs. (4.8) with respect to λ and by considering again the definitions (4.9). In other words, the coefficients a_{ij} form an infinite antisymmetric matrix.

The determination of the coefficient matrix b_{ij} follows similar lines. In fact, if Eq. (4.13) is differentiated with respect to λ, the set of functions w_i is seen to be governed by

$$(EIw_i'')'' + \lambda P_1 L^2 w_i'' - \mu L^4 \omega_i w_i = -2P_1 L^2 v_i'' + 2\overset{*}{\omega}_i L^4 \mu v_i + \mu \overset{**}{\omega}_i L^4 u_i \quad , \qquad (4.15)$$

in which $\overset{**}{\omega}_i$ stands for $d\overset{*}{\omega}/d\lambda$. As before, let this equation be multiplied by u_j and integrated with respect to ξ over the range of the structure. After some integrations by parts and in view again of Eqs. (4.5), (4.8), and (4.9), this results in the system of equations

$$(\omega_j-\omega_i) b_{ij} = 2P_1/L^2 \int_0^1 u_j' v_i' d\xi + 2\overset{*}{\omega}_i a_{ij} + \overset{**}{\omega}_i \, \delta_{ij} \quad . \qquad (4.16)$$

The integral on the right side of this equation can be determined explicitly, provided it is postulated that the functions $u_i(\xi,\lambda)$ form a complete set. In that case it follows from Eqs. (4.9) and from the first of the orthogonality relations (4.8) that v_i may be written in the form

$$v_i = a_{i\alpha}\, u_\alpha \quad . \qquad (4.17)$$

In other words,

$$2P_1/L^2 \int_0^1 u_j'v_i'd\xi \;=\; (2P_1/L^2)a_{i\alpha}\int_0^1 u_j'u_\alpha'd\xi \;=\; 2(\omega_\alpha-\omega_j)a_{i\alpha}a_{j\alpha} - 2a_{ij}\overset{*}{\omega}_j \;.$$

The second equality follows from Eqs. (4.14), while the last term on the right side is added to account for the case $\alpha=j$ in the summation. Summations and integrals have been freely interchanged, which is presumed legitimate here. Hence:

$$(\omega_j-\omega_i)b_{ij} \;=\; 2(\overset{*}{\omega}_i-\overset{*}{\omega}_j)a_{ij} + 2(\omega_\alpha-\omega_j)a_{i\alpha}a_{j\alpha} + \overset{**}{\omega}_i\,\delta_{ij} \;. \tag{4.18a}$$

Again the cases $i=j$ and $i\neq j$ will be considered separately. In the former case Eq. (4.18a) leads to

$$\overset{**}{\omega}_i \;=\; -\,2(\omega_\alpha-\omega_i)a_{i\alpha}a_{i\alpha} \;, \tag{4.18b}$$

while in the latter case

$$b_{ij} \;=\; -2[(\overset{*}{\omega}_i-\overset{*}{\omega}_j)/(\omega_i-\omega_j)]a_{ij} - 2[(\omega_\alpha-\omega_j)/(\omega_i-\omega_j)]a_{i\alpha}a_{j\alpha} \qquad (i\neq j) \;. \tag{4.18c}$$

Finally, two differentiations of Eq. (4.8) with respect to λ, and consideration of Eqs. (4.8), (4.9), and (4.17) lead to

$$b_{ii} \;=\; -\,a_{i\alpha}a_{i\alpha} \;. \tag{4.18d}$$

Note that b_{ij} is, in general, not antisymmetric; rather

$$b_{ij} + b_{ji} \;=\; -\,2a_{i\alpha}a_{j\alpha} \;,$$

or, equivalently,

$$b_{ij} \;=\; da_{ij}/d\lambda - a_{i\alpha}a_{j\alpha} \;.$$

If the matrix b_{ij} is resolved into a symmetric component b'_{ij} and an antisymmetric component b''_{ij}, then

$$b_{ij} \;=\; b'_{ij} + b''_{ij}$$

$$b'_{ij} \;=\; -\,a_{i\alpha}a_{j\alpha} \;=\; b'_{ji} \tag{4.19}$$

$$b''_{ij} = -2[(\omega_i-\omega_j)/(\omega_i-\omega_j)]a_{ij} - [(2\omega_\alpha-\omega_i-\omega_j)/(\omega_i-\omega_j)]a_{i\alpha}a_{j\alpha} = -\,b''_{ji} \;.$$

For the special case of a single span column of constant mass density μ and stiffness EI, the values of the coefficients a_{ij} can be determined directly from the boundary values of the modes u_i. This is done by considering again Eq. (4.14c). The integral on the right side of this equation can be evaluated through a repeated process of integrations by parts and through several substitutions of Eq. (4.5). Eventually this leads to the formula

$$a_{ij} = (P_1/L^2)\ (\omega_j-\omega_i)^{-2}\ [(\omega_j u_i' u_j - \omega_i u_i u_j') + (EI/\mu L^4)\ (u_i''' u_j'' - u_i'' u_j''')]_0^1 \ . \qquad (4.20)$$

It is of interest to note that, for the simply supported column (and for no other type of support!), all the terms in Eq. (4.20) vanish. In other words, for this case alone the modes u_i are independent of the axial force parameter λ. This is why it is possible to give an explicit solution for the case of the simply supported column (as was done in the last two sections); it can readily be verified that Eqs. (4.10) or (4.12) are reduced to (2.5) or (2.7) if it is also considered that in the case of simple supports the eigenvalues ω_i are linear functions of λ. A discussion of this question can be found in Ref. 3.

The integration of the infinite set of Eqs. (4.10) presents formidable numerical difficulties. It must be remembered that, through the time-dependence of the parameter λ, all the coefficients ω_i, a_{ij}, and b_{ij} are also implicit functions of time. The difficulty is compounded by the fact that the loading function $g_i(\tau)$ is computed from Eqs. (4.4) and (4.8), namely,

$$g_i(\tau) = \int_0^1 p(\xi,\tau)\ u_i(\xi,\lambda)d\xi \qquad (i=1,2,3\ldots) \ . \qquad (4.21)$$

Since u_i itself depends on λ (hence on τ), it follows that even if the lateral load is expressible in terms of a distribution function multiplied by a—say, linear—time function, the generalized load g_i takes on a more complicated form.

Fortunately these effects are all exceedingly small. For example, the constants a_{ij} and b_{ij} (which vanish for the simply supported case) seem to be very small for all other kinds of support. They have been computed, and are shown in Tables I and II, for the case of a column of constant mass density and stiffness, both of whose ends are fixed, for the condition of vanishing axial force. The details of the calculations are omitted here because they are straightforward. They consist in solving Eq. (4.5), with λ=0 and subject to the proper boundary conditions, in normalizing the solutions in the sense of the first of Eqs. (4.8), and in substituting the functions so determined in Eqs. (4.14c) and (4.18c), respectively. The expressions for $\overset{*}{\omega}_i$ and $\overset{**}{\omega}_i$ are obtained similarly.

In view of the smallness of these constants, it appears that an iterative solution of Eqs. (4.10) should converge very rapidly. In other words, a first approximation can be obtained by assuming all the constants a_{ij} and b_{ij} to vanish, that is,

$$\ddot{f}_i^o + (\omega_i/\Omega_1)f_i^o = g_i/\Omega_1 \qquad (i=1,2\ldots) \quad . \tag{4.22a}$$

Successive corrections f_i^n can then be introduced by letting

$$\ddot{f}_i^n + (\omega_i/\Omega_1)f_i^n = -a_{\alpha i}(2\dot{f}_\alpha^{n-1}\dot{\lambda} + f_\alpha^{n-1}\ddot{\lambda}) - b_{\alpha i}f_\alpha^{n-1}(\dot{\lambda})^2 \quad , \tag{4.22b}$$

$$(i=1,2,\ldots; \quad n=1,2,\ldots)$$

so that the final solution appears in the form

$$f_i = f_i^o + f_i^1 + f_i^2 + \ldots \qquad (i=1,2\ldots) \quad . \tag{4.22c}$$

Only a few terms need be included on the right side of Eq. (4.22b) because of the repid convergence of the constants a_{ij} and b_{ij} toward zero. Moreover, the solution of these equations amounts to the setting up of a Duhamel integral of the type of Eq. (2.19)(numerically, if necessary) since the complementary solution of Eqs. (4.22b) is the same for all values of n and is, in fact, equal to that of Eq. (4.22a).

To find this complementary solution, the time-dependence of ω_i must first be determined. As stated before, this is a linear relationship if the column is simply supported; for all other boundary conditions, some variation from linearity is to be expected. Figure 8 shows the solution for the first few modes in the case of a column fixed at both ends; again the details of the computations are omitted here because they are fairly obvious. In any event, it is apparent from the figure that, at least for the case under consideration, this variation from linearity is slight.

This suggests again an iterative approach. Using standard perturbation techniques, let the ω-λ relationship be expressed by a dominant linear term and some additional correction factors. If f_i is similarly expanded and the coefficients of like power are equated, there results a system of equations in which the first equation is similar to Eq. (2.10a) [with solutions similar to Eq. (2.10b)], while the successively higher terms are obtainable from the previous ones by simple quadrature. It is anticipated that the convergence of this process is again extremely rapid.

An inspection of Fig. 8 shows that for the first two modes the deviation from linearity is too small to be appreciable on the chart. Actually, for $\lambda=0$, the calculations show that $\overset{*}{\omega}_1 = -0.970\ \Omega_1$, while at $\lambda=1$, the slope is governed by $\overset{*}{\omega}_1 = -1.038\ \Omega_1$. On the other hand, the nonlinearity becomes noticeable for the third mode. At $\lambda=0$, the slope is given by $\overset{*}{\omega}_3 = -1.068\ \Omega_3\ P_1/P_3$, while for $\lambda=4$ (that is, for $\omega_3=0$), $\overset{*}{\omega}_3 = -0.569\ \Omega_3\ P_1/P_3$. This sharp difference in the slope is startling and seems to presage a much sharper deviation from straightness than is actually shown in Fig. 8. However, the calculations show that the nonlinearity is confined largely to a small region near the buckling value of λ and is accompanied by a rapid variation in the mode u_3 and hence by a comparatively large value of $\overset{**}{\omega}_3$.

5. RIGID FRAMES (MULTIPLE-STORY BENTS)

The results of the preceding section can be applied, at least theoretically, to any type of structural configuration. In practice it appears likely that the numerical difficulties, which are already great for single-span columns (except those with simple supports), become prohibitive for more complicated types of structures. This is especially true in the case of multi-story bents, whose exact analysis presents great computational obstacles even if the axial forces are not time-dependent. It is the purpose of the present section to discuss this problem and to suggest simplifications which, in conjunction with automatic computing equipment, are believed to be capable of rendering the analysis more tractable. Briefly, the essence of the simplifying assumptions is the reduction of the actual system to one of finite degrees of freedom, the number of these degrees being equal to the number of stories.

A typical column tier including the ith and (i+1)st story is shown in Fig. 9. The floors are numbered as shown, while λP_i represents the axial forces in the columns and V_i^u and V_i^ℓ designate, respectively, the horizontal forces in the columns just above and below the ith floor and will be referred to as "shears" in what follows.* In the case of free vibration with frequency $(\omega)^{1/2}/2\pi$, the inertia force associated with the horizontal motion of the ith floor of mass m_i is represented by $\omega m_i x_i$, in which x_i represents the displacement amplitude of that floor. All quantities, including the column moments, are assumed to be positive if they are as shown in the figure.

It follows from the (dynamic) equilibrium of the columns that

$$\begin{aligned} V_i^u &= V_i^{u'} - (1/h_{i+1})\lambda P_{i+1}\,(x_{i+1} - x_i) \\ V_i^\ell &= V_i^{\ell'} - (1/h_i)\lambda P_i\,(x_i - x_{i-1}) \end{aligned} \quad . \qquad (5.1)$$

In these equations the shears $V_i^{u'}$ and $V_i^{\ell'}$ represent the forces which are obtained by considering only the effect of the end moments and of the inertia forces in the column; they are given by expressions of the type $(EIy'')' + Py'$, in which y is the horizontal displacement of the column between the floors and is governed by an equation of the type of Eq. (2.1).

*Strictly speaking, shears are defined as forces which are perpendicular to the deflected column axis. The term is here applied to the horizontal forces for the sake of brevity.

The equilibrium of the ith floor is governed by the equation

$$\Sigma\,(V_i^u - V_i^\ell) + \omega\, m_i x_i \;=\; 0 \qquad (i=1,2\ldots n) \;, \tag{5.2}$$

in which the summation extends over all the column tiers and n is the number of moving floors of the building. Substitution of Eqs. (5.1) in this equation leads to the relationship

$$\Sigma\,(V_i^{u'} - V_i^{\ell'}) - \lambda \frac{x_{i+1}}{h_{i+1}} \;\Sigma\, P_{i+1} + \lambda \left[\frac{x_i}{h_{i+1}} \;\Sigma\, P_{i+1} + \frac{x_i}{h_i} \;\Sigma\, P_i\right] - \lambda \frac{x_{i-1}}{h_i} \,\Sigma\, P_i + \omega m_i x_i = 0$$

$$(i=1,2\ldots n) \tag{5.3}$$

Now it can be shown that the first summation on the left side of Eq. (5.3) can be expressed in the form

$$\Sigma\,(V_i^{u'} - V_i^{\ell'}) \;=\; -\, a_{i\alpha} x_\alpha \tag{5.4}$$

in which the "stiffness coefficients" a_{ij} are functions of the axial force coefficient λ and of the frequency coefficient ω. This has been done in Ref. 8 for the case of a vibrating framework in the absence of axial forces and in Ref. 9 in the static case of axial forces producing buckling (ω=0). The two cases have been combined in Ref. 10. In brief review, the coefficients a_{ij} can be obtained, for an assumed value of λ and ω, by introducing modified "slope-deflection equations" taking into account the effect of the axial forces and of the inertia terms. The term a_{ij} is then found by subjecting the jth floor to a unit lateral displacement (all other floors being held in place) and by balancing the resulting moments until they are in equilibrium. From the shears in the columns above and below the ith floor, it is then possible to compute a_{ij}.

It is convenient to express Eq. (5.3) in matrix form. This leads to

$$K(\lambda,\omega)x - \lambda L x - \omega M x \;=\; 0 \;. \tag{5.5a}$$

In this equation, x is a column vector, i.e.,

$$x \;=\; \{x_i\} \;=\; \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \;, \tag{5.5b}$$

while K represents the stiffness matrix

$$K(\lambda,\omega) = [a_{ij}] = [a_{ji}] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & \text{-------} & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} . \qquad (5.5c)$$

The matrix M is the diagonal mass matrix

$$M = [m_i \ \delta_{ij}] = \begin{bmatrix} m_1 & 0 & \dots & 0 \\ 0 & m_2 & \dots & 0 \\ & \text{------} & & \\ 0 & 0 & \dots & m_n \end{bmatrix} , \qquad (5.5d)$$

while L is almost diagonal and takes the form

$$\begin{bmatrix} \left(\frac{\Sigma P_1}{h_1} + \frac{\Sigma P_2}{h_2}\right) & -\frac{\Sigma P_2}{h_2} & 0 & 0 \dots & 0 & 0 \\ -\frac{\Sigma P_2}{h_2} & \left(\frac{\Sigma P_2}{h_2} + \frac{\Sigma P_3}{h_3}\right) & -\frac{\Sigma P_3}{h_3} & 0 \dots & 0 & 0 \\ 0 & 0 & 0 & 0 \dots & -\frac{\Sigma P_n}{h_n} & \frac{\Sigma P_n}{h_n} \end{bmatrix} . \qquad (5.5e)$$

Physically, the system of Eqs. (5.5) takes into account the distributed mass of the columns as well as the vertical inertia of the floor masses. In other words, it is exact in the sense that it includes the infinite-degree-of-freedom character of the actual structure. Its solution, for given constant value of λ, presents formidable numerical difficulties, however. This is due to the fact that the eigenvalue ω appears not only explicitly, but also implicitly through highly involved transcendental relations in the stiffness matrix K. It is shown in the three references cited above that ω (or, for vanishing ω, the buckling parameter λ) can be bracketed between an upper and lower bound, while the gap between these bounds may be narrowed down through an iterative procedure.

A far-reaching simplification of Eqs. (5.5) is achieved if the stiffness matrix K is computed without regard to the influence of ω and λ, that is, if $K(\lambda,\omega) = K(0,0)$. This reduces the system to one of finite degrees of freedom since, for given value of λ, Eqs. (5.5) exhibit only n eigenvalues ω_i. Physically this means that, by neglecting the column and wall inertia, the structure has been "stiffened." The effect of this stiffening process can be minimized, however, by including in the floor masses half of the column and wall masses above and below each floor. Based on this fictitious finite system of masses, the computed vibration modes and frequencies have been shown[8] to agree well with the exact values, at least for the lowest modes of vibration.

More serious is the neglect of the weakening effect of the axial forces in the columns. This is demonstrated, for example, in the case of a single-story structure, in which the cross beam is rigid. If the bottom of the column is fixed, the exact critical buckling force is well known to be π^2EI/h^2, in which h is the story height. On the other hand, if the "beam-column" effect is neglected, Eq. (5.5) (with $\omega=0$) leads to a buckling force of $12EI/h^2$—an increase of over 20%. The same ratio prevails if the bottom of the column is hinged.

For multi-story structures the effect of this approximation is far less pronounced; the same holds true if the cross beams are not assumed to be infinitely stiff. The reason is that in this case the actual buckling force is considerably smaller than the critical Euler load for between-floor buckling; hence the neglect of its effect is more easily justified. Actual multi-story structures, which are designed to resist wind, exhibit column stiffnesses which are at least of comparable order as the beam stiffnesses. In other words, the approximation described above is not likely to be serious for practical cases.

In view of the foregoing remarks, let the free vibration of the structure be governed by the simplified equation

$$(K - \lambda L - \omega M)x = 0 \quad , \tag{5.6}$$

in which K is now assumed to be the modified stiffness matrix and M the modified mass matrix. For given value of λ, this system of equations has a nontrivial solution if, and only if,

$$f(\lambda,\omega) \equiv |K - \lambda L - \omega M| = 0 \quad . \tag{5.7}$$

This represents the customary characteristic equation and governs the natural frequency $\omega(\lambda)$ as well as the mode $x(\lambda)$.

In particular, let ω_r and ω_s be two different eigenvalues associated, respectively, with the modes x^r and x^s. In other words, let

$$\begin{aligned} (K - \lambda L - \omega_r M)x^r &= 0 \\ (K - \lambda L - \omega_s M)x^s &= 0 \end{aligned} \quad . \tag{5.8}$$

If now, in the usual fashion, the first of Eqs. (5.8) is premultiplied by x^{s*} and the second by x^{r*} (where the asterisk * denotes the transpose), and if one equation is subtracted from the other, then

$$\begin{aligned} x^{r*}\, M\, x^s &= \delta_{rs} \\ x^{r*}\,(K - \lambda L)x^s &= \omega_r\, \delta_{rs} \end{aligned} \quad . \tag{5.9}$$

These orthogonality conditions are based on the symmetry of K, L, and M; $K=K^*$ is an expression of Maxwell's reciprocal relations, while $L=L^*$ and $M=M^*$ is obvious from inspection. The satisfaction of the first of Eqs. (5.9) for r=s is artificial and represents a convenient normalization of the modes.

These relationships are all well known. However, it is important to point out that both the eigenvalues ω_r as well as the eigenvectors x^r depend on the parameter λ and that furthermore Eqs. (5.8) and (5.9) are identically satisfied for all values of λ.

To find the actual response of the structure to time-dependent lateral and axial forces, let $x(t) = \{x_i(t)\}$ be a vector representing the displacements of the floors, and let $p(t) = \{p_i(t)\}$ denote the lateral floor loads. In the spirit of the previous discussion, it is assumed here that these loads include all the contributions of the half-stories above and below each floor. The system of equations governing the motion is then expressed by

$$M\ddot{x} + [K - \lambda(t)L]x = p \quad , \tag{5.10}$$

in which a dot above a letter represents differentiation with respect to t.

It is now convenient to introduce the "rotation" matrix

$$R(\lambda) = [r_{ij}] = [x_i^j] \quad . \tag{5.11}$$

In terms of Eq. (5.11), the orthonormality conditions (5.9) can readily be shown to take the form

$$\begin{aligned} R^* M R &= I = [\delta_{ij}] \\ R^* (K - \lambda L)R &= D = [\omega_i \, \delta_{ij}] \end{aligned} \quad . \tag{5.12}$$

In Eqs. (5.12) I is the unit matrix and D is a diagonal matrix. A new vector $\phi = \{\phi_i\}$ is now introduced by means of

$$\begin{aligned} x &= R\,\phi \\ \dot{x} &= \dot{\lambda}\,R'\phi + R\,\dot{\phi} \\ \ddot{x} &= \ddot{\lambda}\,R'\phi + (\dot{\lambda})^2 R''\phi + 2\dot{\lambda}\,R'\dot{\phi} + R\,\ddot{\phi} \end{aligned} \quad , \tag{5.13}$$

in which the second and third equations are obtained from the previous one by differentiation with respect to t. A prime denotes differentiation with respect to λ, that is, $R' = dR/d\lambda$.

When Eq. (5.13) is substituted in Eq. (5.10) and the latter is premultiplied by R^*, then consideration of Eqs. (5.12) leads to the following system of equations governing the response vector ϕ:

$$\ddot{\phi} + D\phi + R^*M\,(\ddot{\lambda}R' + \dot{\lambda}^2R'')\phi + 2\dot{\lambda}\,R^*MR'\,\dot{\phi} = R^*p \quad . \tag{5.14}$$

The vector on the right side of this equation is usually called the "generalized force." Note that, when λ is not time-dependent, Eq. (5.14) reduces to the customary normal-mode equation, which can be solved by direct quadrature through a Duhamel-type integral.

To find the matrices R' and R", differentiate the two orthogonality relations (5.12) with respect to λ. This leads to the two conditions

$$\begin{aligned} R'^*M\,R + R^*M\,R' &= 0 \\ R'^*\,(K - \lambda L)R + R^*\,(K - \lambda L)R' &= D' + R^*L\,R \end{aligned} \quad . \tag{5.15}$$

On account of the completeness of the vectors x^r, it is possible to express the matrix $R' = [r'_{ij}] = [y^j_i]$ in the form

$$R' = R\,A^* \quad , \tag{5.16a}$$

that is, in vector form,

$$y^r = a_{r\alpha}\,x^\alpha \tag{5.16b}$$

in which $A = [a_{ij}]$. When this is substituted in Eqs. (5.15) and Eqs. (5.12) are considered, the first leads to

$$A + A^* = 0 \quad , \tag{5.17a}$$

while the second, in view of Eq. (5.17a), becomes

$$A\,D - D\,A = D' + R^*L\,R \quad , \tag{5.17b}$$

According to Eq. (5.17a), the matrix A is antisymmetric; hence

$$a_{rr} = 0 \quad . \tag{5.17c}$$

On the other hand, for $r \neq s$, Eq. (5.17b) implies that

$$a_{rs} = -\,a_{sr} = (x^{s*}L\,x^r)/(\omega_s - \omega_r) \quad , \tag{5.17d}$$

while, from the same equation and for r=s,

$$\omega'_r = -\,x^{r*}L\,x^r \quad . \tag{5.17e}$$

An interesting conclusion can be drawn from Eqs. (5.17). In fact, consider the second of Eqs. (5.9) and let K and L be linearly dependent; then the numerator on the right side of Eq. (5.17d) vanishes. In other words, the modes x^r are not functions of λ if such linear dependence exists; in that case the eigenvalues ω_r are linear functions of λ by Eq. (5.17e), and the system of Eqs. (5.14) becomes uncoupled in much the same way as in the case of a simply supported column in the previous sections.

The matrix R" is obtained by differentiating Eqs. (5.15) with respect to λ. This leads to the relationships.

$$\begin{aligned} R''^{*}M\,R + 2\,R'^{*}M\,R' + R^{*}M\,R'' &= 0 \\ R''^{*}(K-\lambda L)R + 2\,R'^{*}(K-\lambda L)R' + R^{*}(K-\lambda L)R'' & \\ = D'' + 2\,R'^{*}LR + 2\,R^{*}LR' & \end{aligned} \quad . \tag{5.18}$$

As before, let the matrix $R'' = [r''_{ij}] = [z^j_i]$ be of the form

$$R'' = R\,B^{*} \quad , \tag{5.19a}$$

which, in vector notation, means that

$$z^r = b_{r\alpha}\,x^{\alpha} \quad , \tag{5.19b}$$

where $B = [b_{ij}]$. In view of Eqs. (5.19), (5.12), and (5.17), the two relationships (5.18) now become, respectively,

$$B + B^{*} = -\,2\,A\,A^{*} = 2\,A\,A \tag{5.20a}$$

$$BD - DB = D'' - 2AD' + 2D'A + 2A(AD-DA) \quad . \tag{5.20b}$$

Equation (5.20a) implies that

$$b_{rr} = -\,a_{r\alpha}\,a_{r\alpha} \quad , \tag{5.20c}$$

while, for $r \neq s$, it follows from Eqs. (5.20b) that

$$b_{rs}(\omega_r-\omega_s) = -\,2(\omega'_r-\omega'_s)a_{rs} - 2(\omega_{\alpha}-\omega_s)a_{r\alpha}a_{s\alpha} \tag{5.20d}$$

As a further consequence of Eq. (5.20b), the condition r=s implies that

$$\omega''_r = -\,2(\omega_{\alpha}-\omega_r)a_{r\alpha}\,a_{r\alpha} \quad . \tag{5.20e}$$

Finally, by differentiating Eq. (5.17b) and after several substitutions of previously found results, it can be shown that

$$B = A' + A\,A \quad , \tag{5.20f}$$

that is,

$$b_{rs} = a'_{rs} - a_{r\alpha} a_{s\alpha} \quad . \qquad (5.20g)$$

In other words, the matrix A may be thought of as representing a "rotation" of the vector space for increasing values of λ.

In view of Eqs. (5.16a) and (5.19a), the basic equation of motion (5.14) governing the vector ϕ can be simplified, by considering the first of Eqs. (5.12), to

$$\ddot{\phi} + D\,\phi + (\ddot{\lambda}A^* + \dot{\lambda}^2 B^*)\phi + 2\dot{\lambda}A^*\,\dot{\phi} = R^*\,p \quad . \qquad (5.21)$$

A further simplification takes place if $\lambda(t)$ varies linearly with time—that is, if

$$\lambda(t) = \alpha(1-t/T) \quad . \qquad (5.22a)$$

In that case, the governing equation reduces to

$$\ddot{\phi} + (D/\Omega_1)\phi - 2(A^*/k)\dot{\phi} + (1/k^2)B^*\phi = (R^*/\Omega_1)p \quad , \qquad (5.22b)$$

in which a dot ($\dot{}$) signifies differentiation with respect to τ. In this equation, the parameters τ, Ω_1, and k are the same as used in the previous chapter and are defined in Eqs. (4.2) and (4.11), respectively. Note that Eq. (5.22b) is in essence the same as Eq. (4.12); its solution should therefore proceed along similar lines.

6. ARCHES

In the present section, the investigation of the prefious sections is extended to include the stability of arches. Obviously it is beyond the scope of this study to make a thorough and exhaustive analysis of so broad a subject; only an exploratory investigation is therefore attempted here. For example, only circular arches will be considered in what follows, although other types, e.g., parabolic ones, could at least theoretically be analyzed in the same manner. Also, only motion within the plane of the arch is to be taken into consideration; buckling out of the plane is governed by similar equations, although the effect of torsion is normally to be taken into account.

In what follows, let $u(\theta)$ represent the radial component and $v(\theta)$ the tangential component of the displacement of a generic point on the circular centroidal axis of the arch; u is considered positive outward, and v is positive if it is in the direction of increasing argument θ. Let N be the axial force (in tension) and M the bending moment, which will be positive if associated with a compressive stress on the outside. If R is the radius of the undeformed centroidal axis and μ (as before) denotes the mass per unit length, then the linearized equations of motion relative to the tangential and normal directions of the deformed element are as follows:

$$[1 - (u/R) - (v'/R)]N' - [1 - 2(u/R) - (u''/R) - (v'/R)](M'/R) - \mu R\ddot{v} = -p_t R$$
$$[1 - (u/R) - (u''/R)]N - (u'+v'')(M'/R) + [1 - 2(u/R) - 2(v'/R)](M''/R) + \mu R\ddot{u} = -p_n R \quad . (6.1)$$

In Eqs. (6.1), primes (') represent partial derivatives with respect to Θ while dots ($^\circ$) are time derivatives as before. The external pressure is included in the form of its normal and tangential components p_n and p_t, respectively.

Only "thin" arches are to be considered here—that is, arches whose thickness h is much smaller than R. In view of this restriction, the force-displacement relationships are given by

$$N = (EA/R)\,(u+v')$$
$$M = (EI/R^2)\,(u+u'') \quad . \quad (6.2)$$

Let Eqs. (6.2) be substituted in Eqs. (6.1) and consider the case of p_t=0 (calling, for convenience, the normal pressure component p without subscript). If furthermore the inertia term $\mu\ddot{v}$ is neglected (which is plausible on physical grounds) and if the mean axial strain N/EA is neglected in comparison with unity, then the equations of motion reduce to

$$N' - (EI/R^3)\,(u'+u''') = 0$$
$$N(1-u/R-u''/R) + EI/R^3\,(u''+u^{iv}) + \mu R\ddot{u} = -pR \quad . \quad (6.3)$$

In the present discussion, the stability of the "unbuckled" motion is to be investigated. This is achieved by letting

$$N = N_o + N_1$$
$$u = u_o + u_1 \quad , \quad (6.4)$$

in which N_o and u_o represent the axial force and radial displacement, respectively, prior to buckling. All quantities shown in Eqs. (6.4) are in general functions of Θ and t. To simplify the present study, let the pressure p be independent of Θ and let the structure be represented by a closed circular ring; in that case N_o and u_o are also independent of Θ and are governed by the equations

$$N_o(1-u_o/R) + \mu R\,\ddot{u}_o = -pR$$
$$N_o = (EA/R)u_o \quad , \quad (6.5)$$

or, equivalently, and subject to the same approximation made previously,

$$(\mu R^2/EA)\ddot{N}_o + N_o = -pR \quad . \tag{6.6}$$

Equations (6.4), (6.5), and (6.6) are now substituted in Eq. (6.3), with the subscript "1" dropped for convenience. After linearization with respect to N and u, this becomes

$$\begin{aligned} N' - (EI/R^3)\,(u'+u''') &= 0 \\ N - (u+u'')N_o/R + (EI/R^3)\,(u''+u^{iv}) + \mu R\ddot{u} &= 0 \end{aligned} \quad . \tag{6.7}$$

This can be solved by setting

$$u(\Theta,t) = f(t)\sin n\Theta \qquad (n=2,3\ldots) \quad , \tag{6.8a}$$

and, in view of the first of Eqs. (6.7),

$$N(\Theta,t) = -(n^2-1)\,(EI/R^3)\,f(t)\sin n\Theta \quad . \tag{6.8b}$$

Substitution of Eqs. (6.8) in the second of Eqs. (6.7) leads to

$$\ddot{f} + (n^2-1)/(\mu R^2)\,[(n^2-1)\,(EI/R^2) + N_o]f = 0 \quad , \tag{6.9}$$

in which $N_o(t)$ is governed by Eq. (6.6)

In the static case this leads to well-known results. In fact let p (and hence N_o) be a constant; then f(t) is bounded if the coefficient of f in Eq. (6.9) is positive—that is, if $N_o > -(n^2-1)EI/R^2$ or, in view of the first of Eqs. (6.5), if $p < (n^2-1)EI/R^3$. The critical (static) pressure is obtained by setting n=2.

For the general dynamic case no universally applicable solution to Eqs. (6.6) and (6.9) can be given, of course. However, it is interesting, especially in view of the aims of the present study, to consider the effect of a short shock-type pressure impulse, of the kind of time-dependence that has been investigated in the previous sections. In that case, after the shock has subsided, there will remain a residual free vibration of the form

$$N_o(t) = A\sin(\omega t-\tau) \quad , \tag{6.10a}$$

where ω is given by

$$\omega^2 = EA/\mu R^2 \quad . \tag{6.10b}$$

When this is substituted in Eq. (6.9), the solution of the latter can be written explicitly in terms of Mathieu functions.

The question of the boundedness of these functions has been the object of numerous studies, of which Ref. 2 was mentioned earlier. Hence it appears that, for certain combinations of parameters, the type of instability mentioned in the Introduction may actually take place, at least as applied to the case of a circular ring. Of course the linearization of the relevant equations rules out anything but "small" deformations; expressed in physical terms, the buckling amplitudes must remain bounded on account of the boundedness of the energy input. In other words, the additional energy must originate from the "unbuckled" motion. However, since the latter is associated essentially with axial-stress energy (which is relatively large), while the former involves primarily bending energy, it appears reasonable that even a nonlinear investigation may disclose inadmissibly large buckling amplitudes.

7. CONCLUSION

The foregoing study of the dynamic stability of structures is obviously far from exhaustive. Its purpose has been exploratory, in the main; the subject matter is too broad, and the knowledge gathered thus far too scanty, to permit anything more. In fact, many questions that are vital from a practical point of view have been entirely ignored, prominent among them the issue of the effect of yielding on the performance of structures under time-dependent buckling conditions.

Chief emphasis has been placed on the derivation of the relevant equations and on proposed methods of solving them. In general, these equations cannot reasonably be solved without the aid of elaborate computational equipment; with such equipment, on the other hand, no great difficulties are anticipated. If iterative schemes are employed, they should converge quickly. Only in the case of a single-span simply supported column has it been possible to obtain explicit solutions, although even these are not necessarily in closed form. As pointed out in Section 5, similar possibilities exist for multi-story bents under severely restricted conditions.

In general, time-dependent stability is difficult to define, let alone to analyze. In view of the potential practical significance of the problem, a testing program may be set up in which experimental results are obtained to corroborate, if possible, the analytical predictions. It appears that this type of program may be most fertile in connection with arches or shell-type structures, whose resistance to blast loads may become a focal point of interest.

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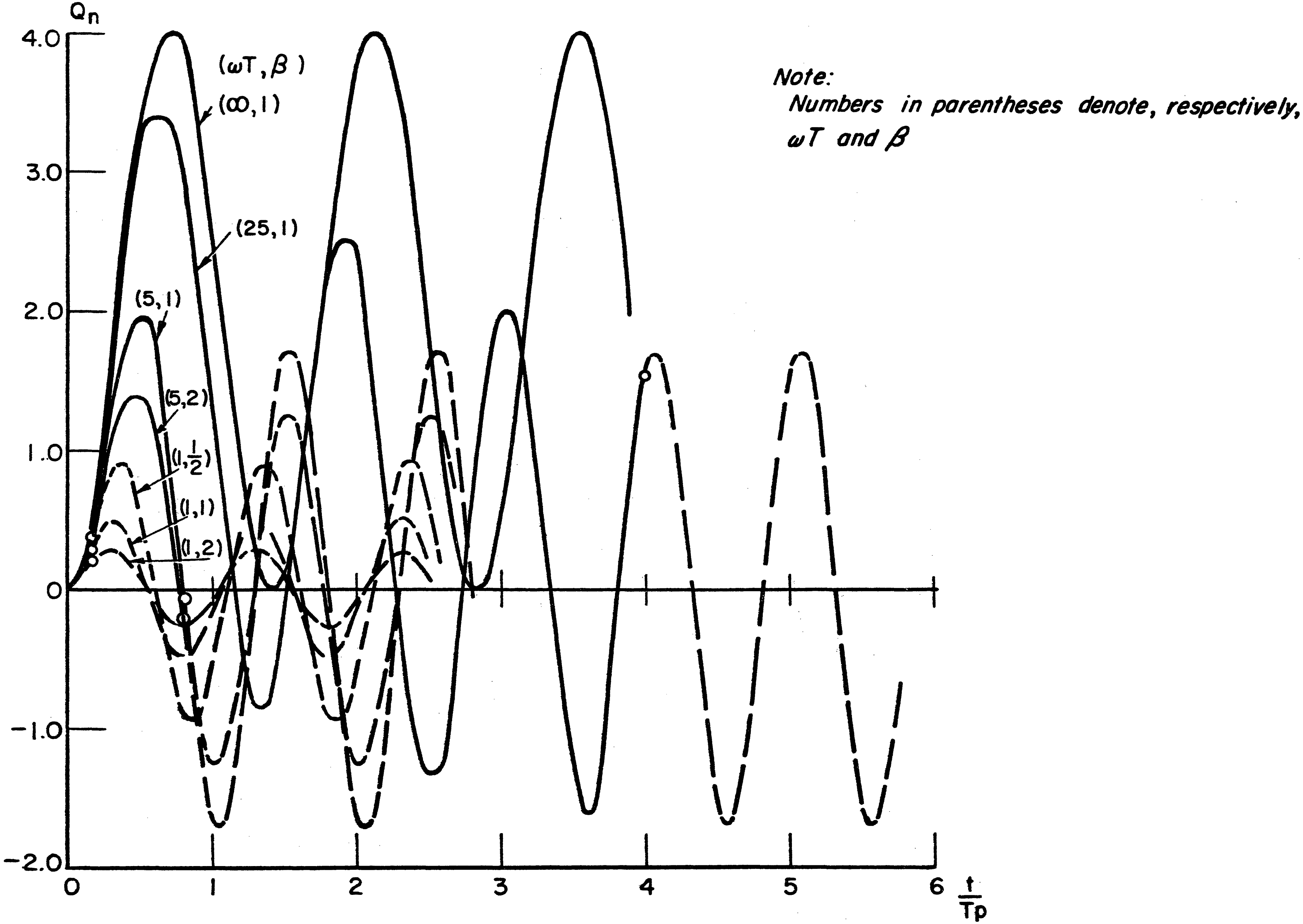

Fig. 1. Column response for $\alpha = 1/2$.

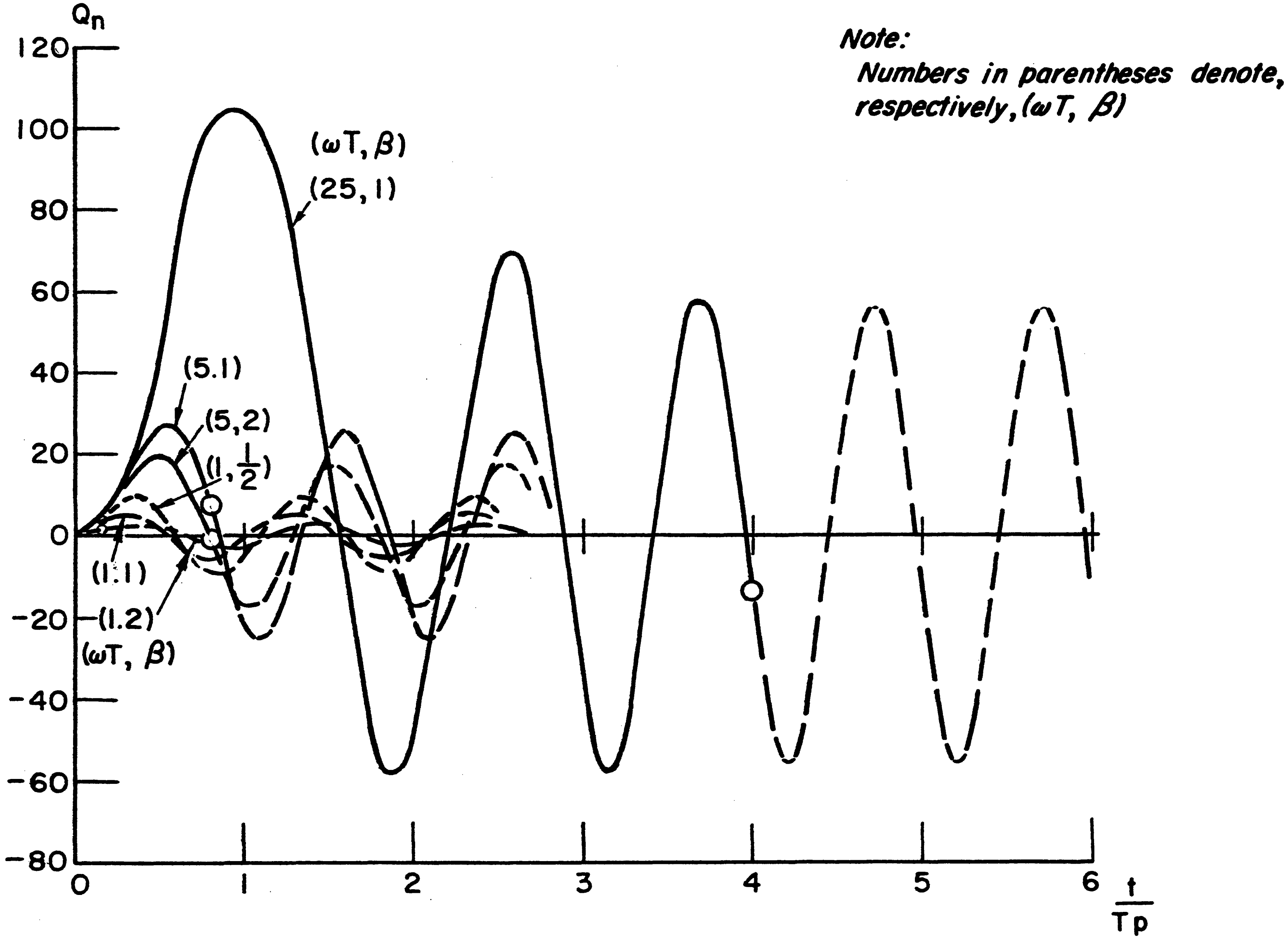

Fig. 2. Column response for $\alpha = 1$.

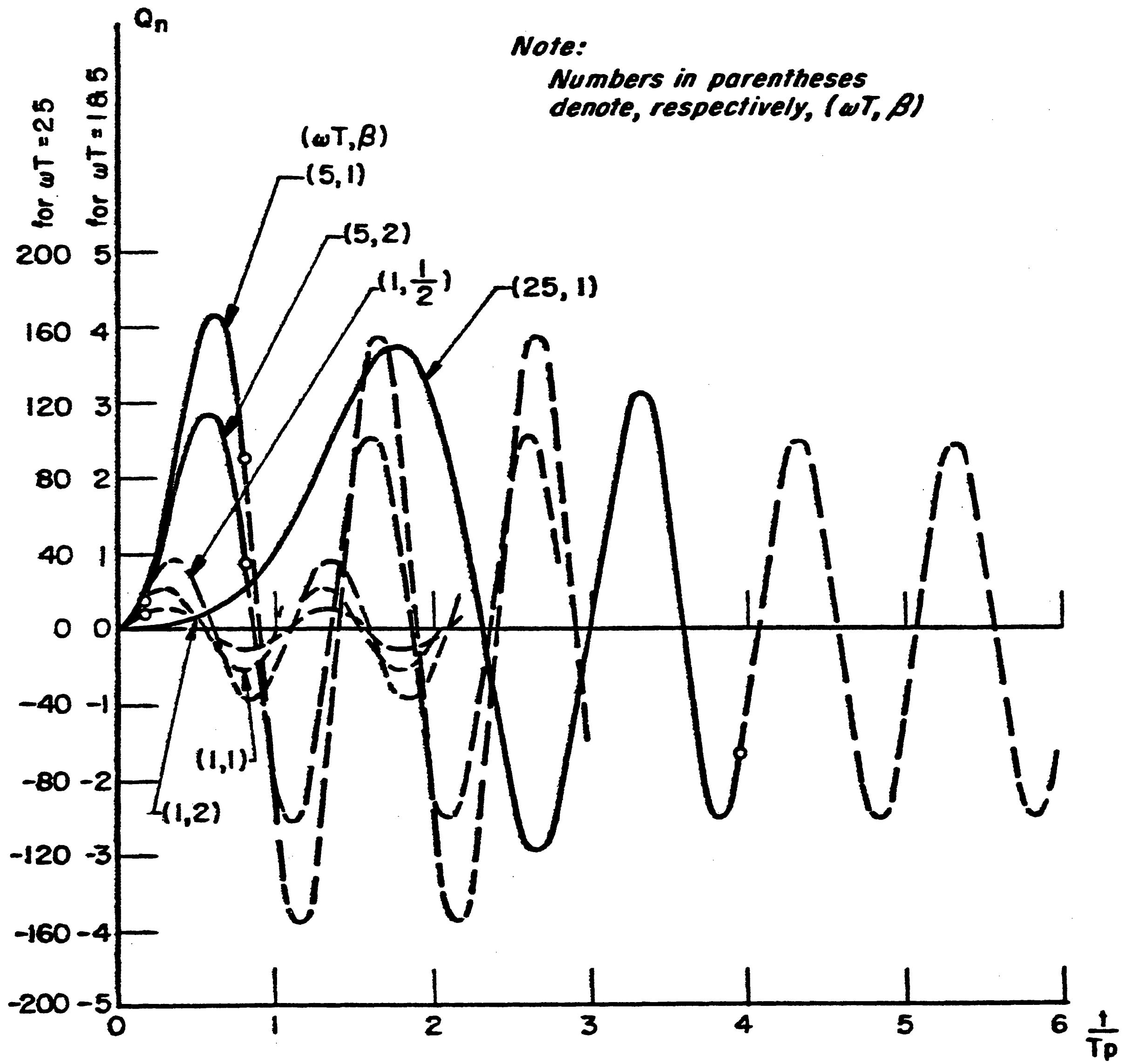

Fig. 3. Column response for $\alpha = 3/2$.

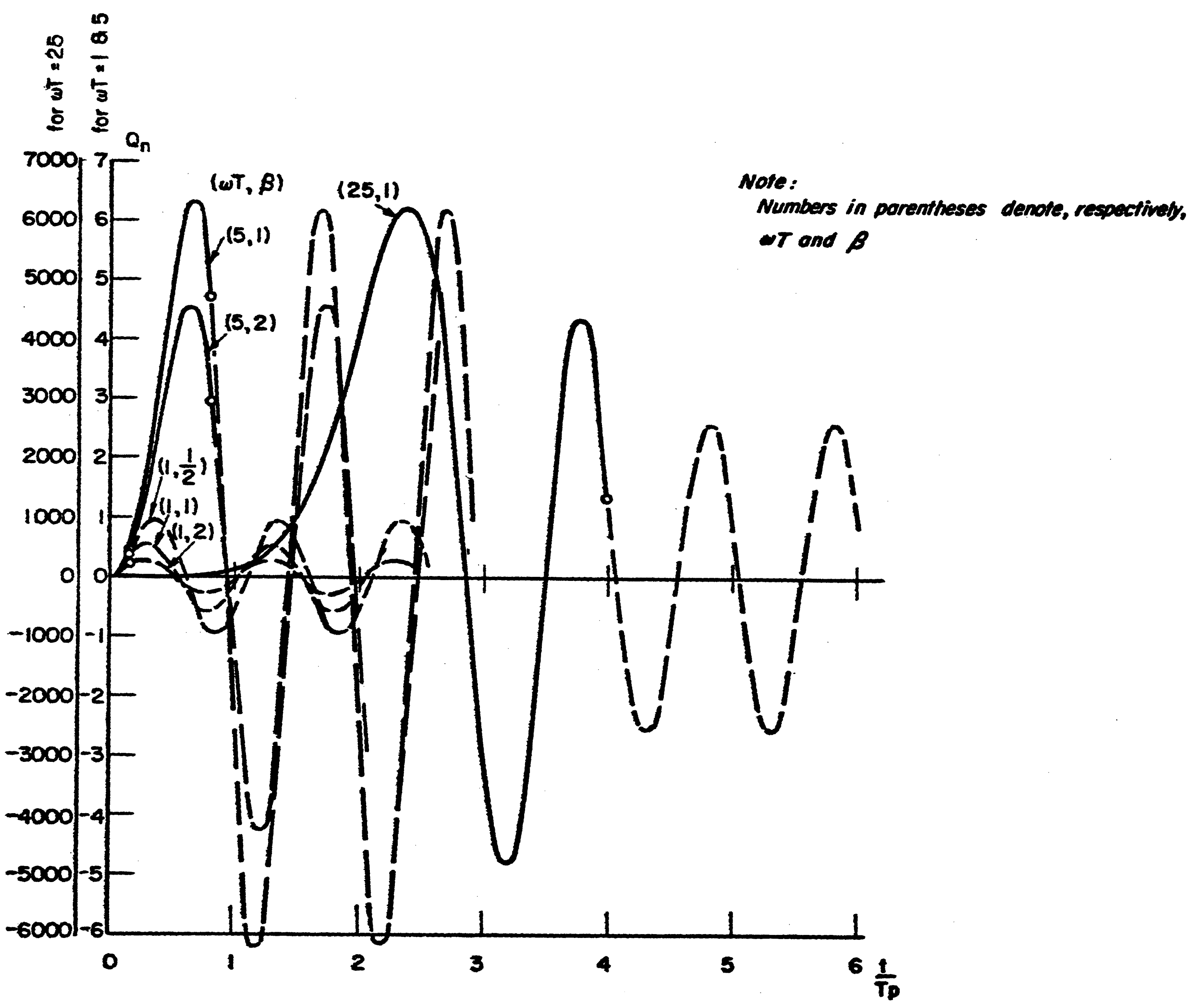

Fig. 4. Column response for $\alpha = 2$.

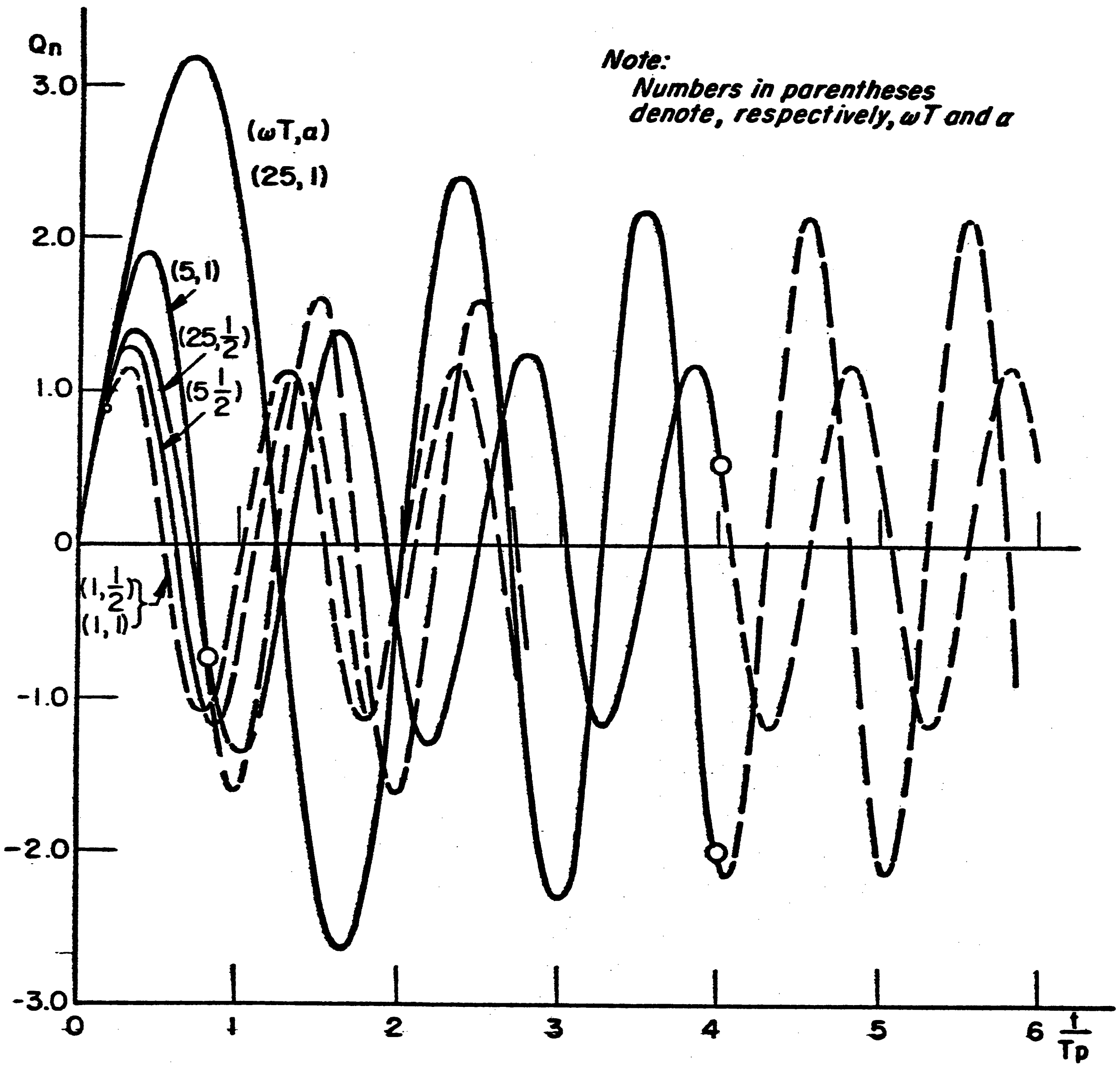

Fig. 5. Column response to lateral impulse $1/\omega$ ($\alpha = 1/2$ and 1).

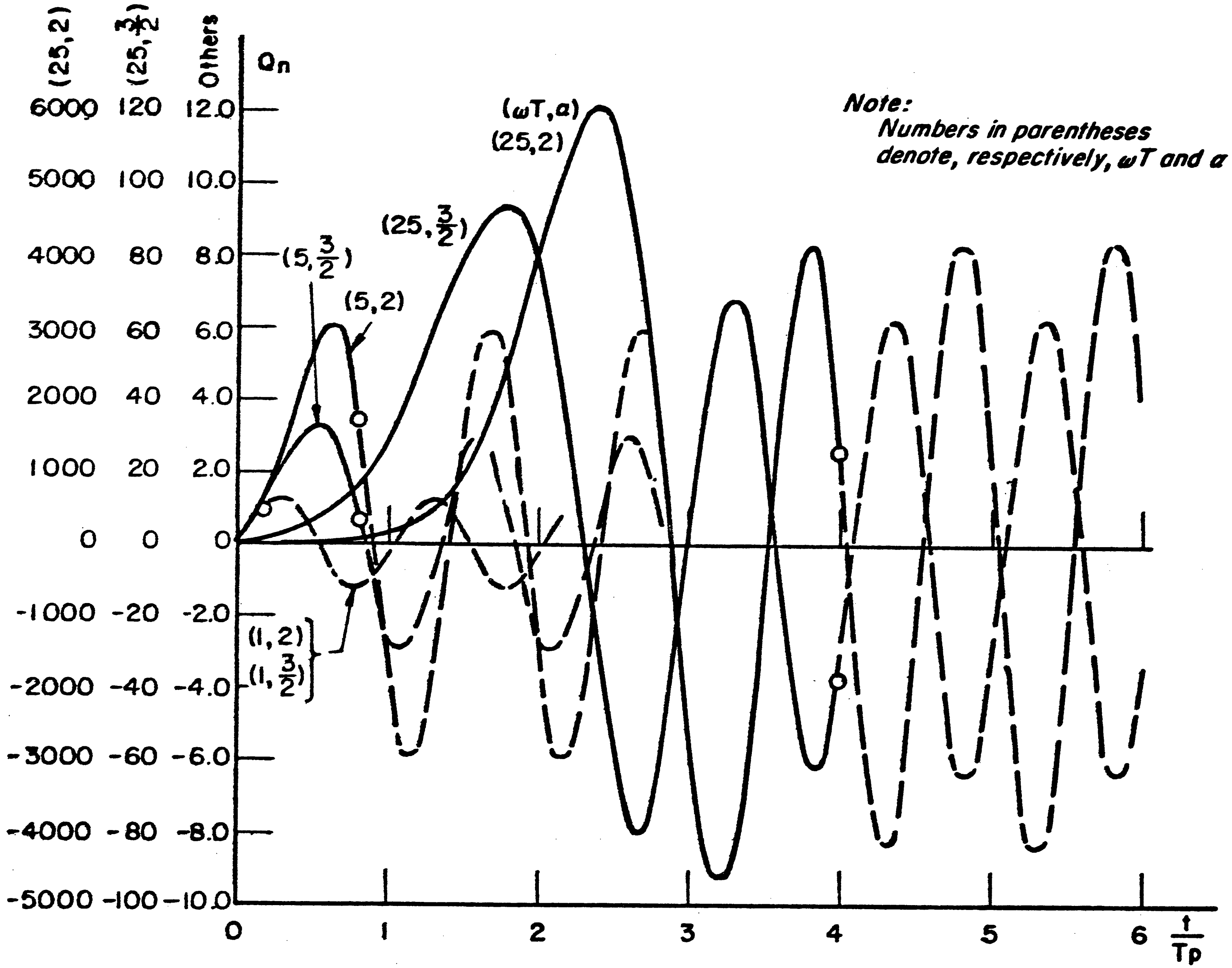

Fig. 6. Column response to lateral impulse $1/\omega$ ($\alpha = 3/2$ and 2).

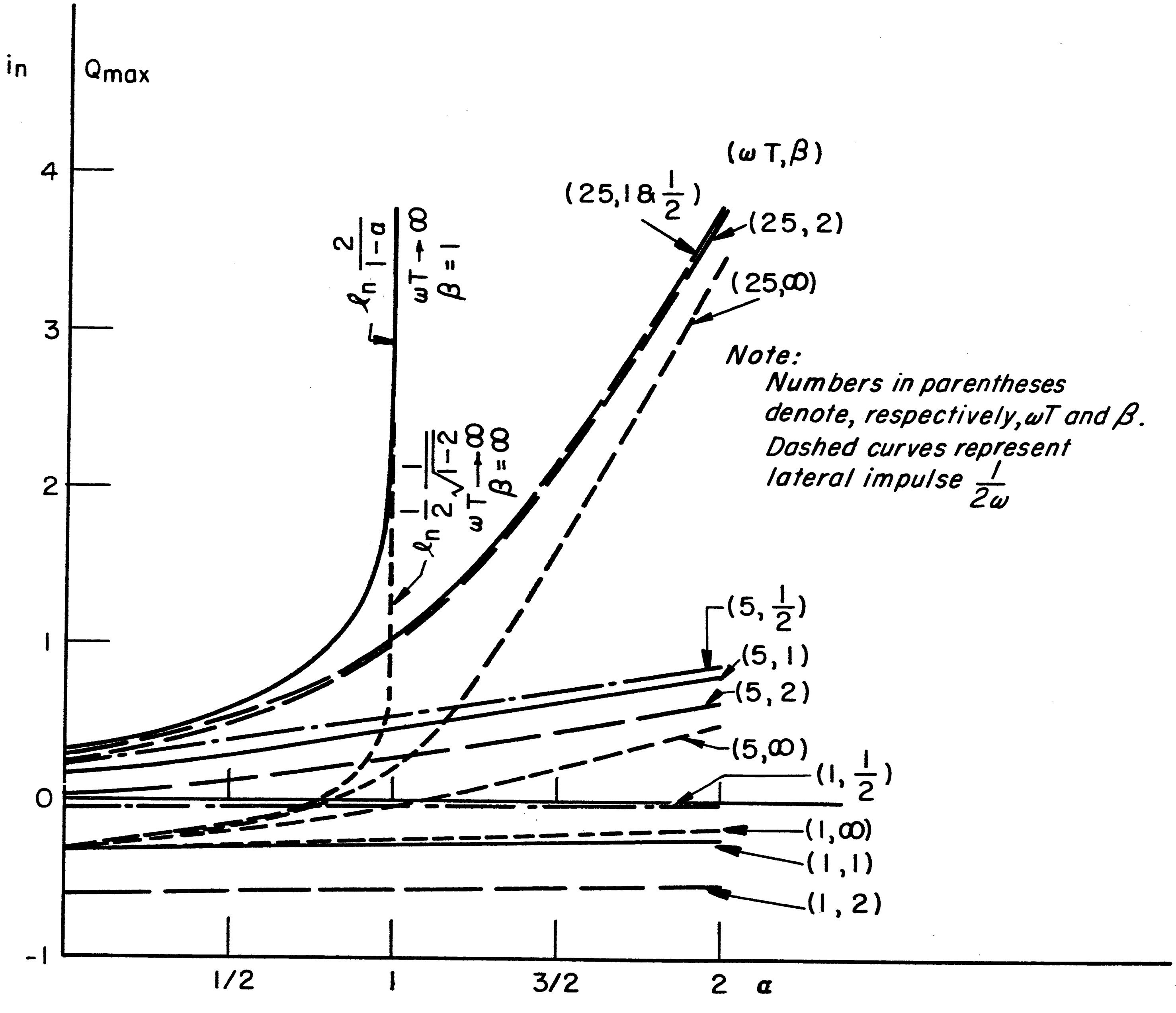

Fig. 7. Maximum column response.

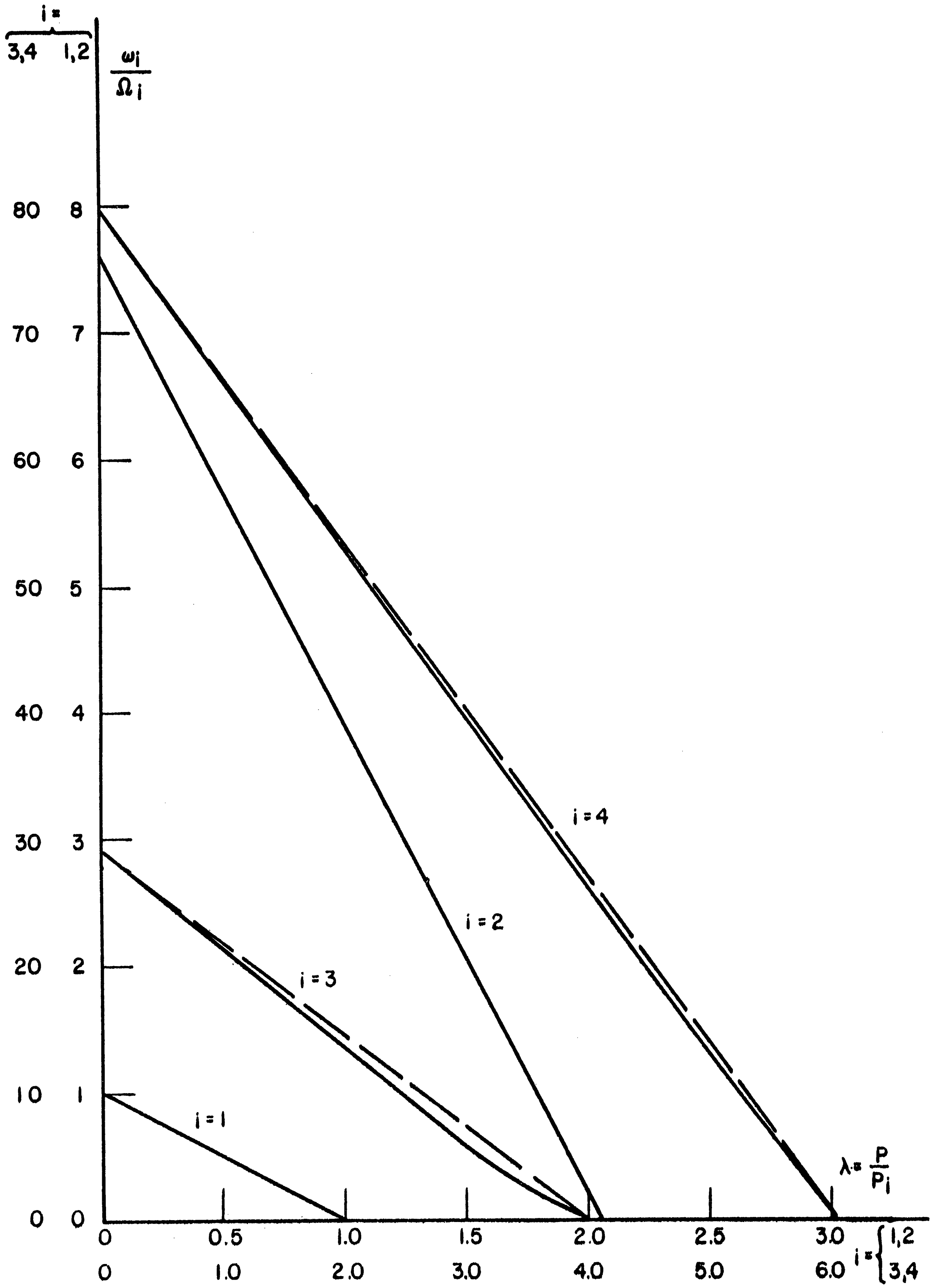

Fig. 8. Frequency vs. axial force.

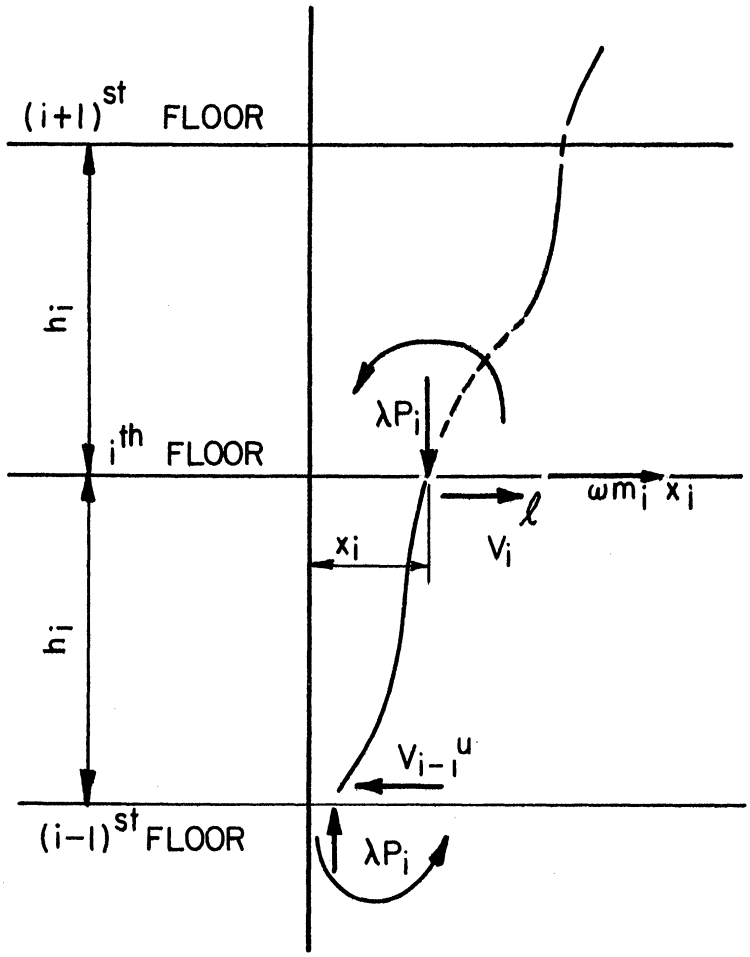

Fig. 9. Typical column pier.

TABLE I

$(4000\pi^2)(a_{ij})$ for i, j = 1, 2,....9

(λ = 0)

i \ j	1	2	3	4	5	6	7	8	9
1	0	0	-0.689	0	-0.086	0	-0.020	0	-0.006
2	0	0	0	-0.474	0	-0.089	0	-0.026	0
3	+0.689	0	0	0	-0.327	0	-0.078	0	-0.027
4	0	+0.474	0	0	0	-0.233	0	-0.065	0
5	+0.086	0	+0.327	0	0	0	-0.174	0	-0.054
6	0	+0.089	0	+0.233	0	0	0	-0.133	0
7	+0.020	0	+0.078	0	+0.174	0	0	0	
8	0	+0.026	0	+0.065	0	+0.133	0		
9	+0.006	0	+0.027	0	+0.054	0			

TABLE II

$(4000\pi^2)^2(b_{ij})$ for i, j = 1, 2,....5

(λ = 0)

i \ j	1	2	3	4	5
1	-0.482	0	-8.002	0	-0.090
2	0	-0.234	0	-3.073	0
3	+7.944	0	-0.588	0	-1.463
4	0	+3.028	0	-0.284	0
5	+0.138	0	+1.315	0	-0.148