Dominance and Nash Equilibrium

Professor Yan Chen
Fall 2008

Some material in this lecture drawn from http://gametheory.net/lectures/level.pl
Agenda

– Dominance and best response
  » Dominance
  » Best response
  » Dominant strategy equilibrium
– Rationalizability and iterated dominance
  » Dominance-solvable equilibrium
– Nash equilibrium
  » Pure strategy Nash equilibrium
  » Mixed strategy Nash equilibrium
Dominance and Best Response

(Watson Chapter 6)
## Example: Prisoners’ Dilemma

Tchaikovsky

<table>
<thead>
<tr>
<th>Conductor</th>
<th>Confess</th>
<th>Not Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Confess</strong></td>
<td>-5, -5</td>
<td>0, -15</td>
</tr>
<tr>
<td><strong>Not Confess</strong></td>
<td>-15, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>
• A strategy is a *best response (reply)* to a particular strategy of another player, if it gives the highest payoff against that particular strategy

• How to find best responses
  – Discrete strategy space: for each of opponent’s strategy, find strategy yielding best payoff
  – Continuous strategy space: use calculus
### Best Response: Prisoners’ Dilemma

Tchaikovsky

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<td><strong>-1, -1</strong></td>
</tr>
</tbody>
</table>
Best response: Battle of Sexes

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1, 3</td>
<td>0, 0</td>
</tr>
<tr>
<td>B</td>
<td>0, 0</td>
<td>3, 1</td>
</tr>
</tbody>
</table>
Solving strategic form games: Dominance

- Confess is a best reply regardless of what the other player chooses.
- Strategy $s_1$ strictly dominates another strategy $s_2$, if the payoff to $s_1$ is strictly greater than the payoff to $s_2$, regardless of which strategy is chosen by the other player(s). Or $u_i(s_1, t) > u_i(s_2, t)$, for all $t$.
- Strategy $s_2$ : strictly dominated strategy
Examples of dominance:

For player 1, U strictly dominates D.
**Weak Dominance**

- Strategy $s_1$ *weakly dominates* another strategy $s_2$, if the payoff to $s_1$ is at least as good as the payoff to $s_2$, regardless of which strategy is chosen by the other player(s). Or

$$u_i(s_1, t) \geq u_i(s_2, t) \text{ for all } t, \text{ and }$$

$$u_i(s_1, t') > u_i(s_2, t'), \text{ for some } t'$$

- In this case, strategy $s_2$ is called a *weakly dominated strategy*. 
Example of strict and weak dominance:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>8, 3</td>
<td>0, 4</td>
<td>4, 4</td>
</tr>
<tr>
<td>M</td>
<td>4, 2</td>
<td>1, 5</td>
<td>5, 3</td>
</tr>
<tr>
<td>D</td>
<td>3, 7</td>
<td>0, 1</td>
<td>2, 0</td>
</tr>
</tbody>
</table>

For player 1, M strictly dominates D, U weakly dominates D.
Player 2: C weakly dominates R.
Example of dominance:

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>4, 1</td>
<td>0, 2</td>
</tr>
<tr>
<td>M</td>
<td>0, 0</td>
<td>4, 0</td>
</tr>
<tr>
<td>D</td>
<td>1, 3</td>
<td>1, 2</td>
</tr>
</tbody>
</table>

Randomize between U and M dominates D, or D is dominated by the mixed strategy \((\frac{1}{2}, \frac{1}{2}, 0)\).
Dominant strategy equilibrium

• If every player has a dominant strategy, the game has a dominant strategy equilibrium (solution).

• Dominant strategy axiom: if a player has a dominant strategy, she will use it.

• Problem with dominant strategy equilibrium: in many games there does not exist one
**DSE: Prisoners’ Dilemma**

Tchaikovsky

<table>
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<th>Conductor</th>
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<th>Not Confess</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Confess</strong></td>
<td>-5, -5</td>
<td>0, -15</td>
</tr>
<tr>
<td><strong>Not Confess</strong></td>
<td>-15, 0</td>
<td>-1, -1</td>
</tr>
</tbody>
</table>

(Confess, Confess) is a dominant strategy equilibrium.
So, both confess

Question: They both would be better off with (Not, Not). So why don’t they play “Not Confess”?
Efficiency and equilibrium

• Game equilibrium is a characterization of the outcome of individually rational behavior
  – Because of strategic interactions, rational behavior does not always lead to outcomes that are mutually the best

• Dominant strategy equilibrium in Prisoner’s Dilemma: (Confess, Confess)

• But this is not socially efficient: both players are better off with (Not Confess, Not Confess)

• Many applications
  – Arms race
  – Tragedy of commons
A solution is **Pareto optimal** if and only if there is no other solution that is

1. Better for at least one agent
2. No worse for everyone else

A mild (weak) criterion for social efficiency

The Prisoner’s Dilemma solution is *not* Pareto optimal
**Example: (Low, Low) is DSE**

<table>
<thead>
<tr>
<th></th>
<th>Low price</th>
<th>High price</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm A</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low price</td>
<td>0, 0</td>
<td>50, -10</td>
</tr>
<tr>
<td>High price</td>
<td>-10, 50</td>
<td>10, 10</td>
</tr>
</tbody>
</table>

Firm B
What to do when equilibrium is inefficient?

• Can’t always be improved (arms race not an easy problem!)

• Opportunities:
  – Collude / cooperate (sometimes illegal!)
    » OPEC
    » marriage
    » Might involve side payments if not win-win
  – Design systems to increase trust
  – Repeated interactions
    » Build trust
    » Or create opportunities for punishment!
Rationalizability and Iterated Dominance

(Watson Chapter 7)
Dominance Solvability

• In some games, there might not be a dominant strategy, but there are dominated strategies (i.e., bad)

• If we can reach a unique strategy vector by iterated elimination of dominated strategies, the game is said to be dominance solvable.
Example:
Playing mind games

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3.3</td>
<td>0.5</td>
<td>0.4</td>
</tr>
<tr>
<td>B</td>
<td>0.0</td>
<td>3.1</td>
<td>1.2</td>
</tr>
</tbody>
</table>

If you are player A, which strategy should you play?
FIGURE 7.2 (a)
Iterative removal of strictly dominated strategies.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3.3</td>
<td>0.5</td>
<td>0.4</td>
</tr>
<tr>
<td>B</td>
<td>0.0</td>
<td>3.1</td>
<td>1.2</td>
</tr>
</tbody>
</table>
**FIGURE 7.2 (b)**
Iterative removal of strictly dominated strategies.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3.3</td>
<td>0.5</td>
<td>0.4</td>
</tr>
<tr>
<td>B</td>
<td>0.0</td>
<td>3.1</td>
<td>1.2</td>
</tr>
</tbody>
</table>
FIGURE 7.2 (c)
Iterative removal of strictly dominated strategies.

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>3,3</td>
<td>0,5</td>
<td>0,4</td>
</tr>
<tr>
<td>B</td>
<td>0,0</td>
<td>3,1</td>
<td>1,2</td>
</tr>
</tbody>
</table>

1, 2
Rationalizable Strategies

• The set of strategies that survive iterated dominance is called the *rationalizable strategies*

• Logic of rationalizability depends on
  – Common knowledge of rationality
  – Common knowledge of the game
Example: rationalizability/iterated dominance

<table>
<thead>
<tr>
<th></th>
<th>L</th>
<th>C</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>5,1</td>
<td>0,4</td>
<td>1,0</td>
</tr>
<tr>
<td>M</td>
<td>3,1</td>
<td>0,0</td>
<td>3,5</td>
</tr>
<tr>
<td>D</td>
<td>3,3</td>
<td>4,4</td>
<td>2,5</td>
</tr>
</tbody>
</table>

L is strictly dominated by (0, ½, ½), etc. Set of rationalizable strategies is {(M, R)}. 
**Strategic Uncertainty**

- Rationalizability requires players’ beliefs and behavior be consistent with common knowledge of rationality.
- It does not require that their beliefs be correct.
- It does not help solve the strategic uncertainty in coordination games.
Coordination game: want to go to an event together, with slightly different preferences

Any dominant strategies?
Any dominated strategies?

<table>
<thead>
<tr>
<th></th>
<th>Opera</th>
<th>Movie</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opera</td>
<td>2, 1</td>
<td>0, 0</td>
</tr>
<tr>
<td>Movie</td>
<td>0, 0</td>
<td>1, 2</td>
</tr>
</tbody>
</table>
Example: Stag hunt

<table>
<thead>
<tr>
<th></th>
<th>Stag</th>
<th>Hare</th>
</tr>
</thead>
<tbody>
<tr>
<td>Stag</td>
<td>5,5</td>
<td>0,4</td>
</tr>
<tr>
<td>Hare</td>
<td>4,0</td>
<td>4,4</td>
</tr>
</tbody>
</table>

Any dominant strategies?  
Any dominated strategies?  
Pareto optimal outcomes?
Facilitate Coordination

• Focal point
  – Schelling: *The strategy of conflict*
  – Rome

• Institutions, rules, norms

• Communication
Nash Equilibrium

(Watson Chapters 9, 11)
• A set of strategies forms a Nash equilibrium if the strategies are best replies to each other

• Recall: A strategy is a best reply to a particular strategy of another player, if it gives the highest payoff against that particular strategy
Hawk-Dove

• In this situation, the players can either choose aggressive (hawk) or accommodating strategies.

• From each player’s perspective, preferences can be ordered from best to worst:
  – Hawk – Dove
  – Dove – Dove
  – Dove – Hawk
  – Hawk – Hawk

• The argument here is that two aggressive players wipe out all surplus.
Hawk-Dove Analysis

• We can draw the game table as:

<table>
<thead>
<tr>
<th></th>
<th>Hawk</th>
<th>Dove</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hawk</td>
<td>0, 0</td>
<td>4, 1</td>
</tr>
<tr>
<td>Dove</td>
<td>1, 4</td>
<td>2, 2</td>
</tr>
</tbody>
</table>

• Best Responses:
  – Reply Dove to Hawk
  – Reply Hawk to Dove

• Equilibrium
  – There are two equilibria
    – (Hawk, Dove)
    – (Dove, Hawk)
FIGURE 9.2 (1)
Equilibrium and rationalizability in the classic normal forms

Matching Pennies

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1,-1</td>
<td>-1,1</td>
</tr>
<tr>
<td>T</td>
<td>-1,1</td>
<td>1,-1</td>
</tr>
</tbody>
</table>
FIGURE 9.2 (2)
Equilibrium and rationalizability in the classic normal forms

Prisoners’ Dilemma

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</thead>
<tbody>
<tr>
<td>Confess</td>
<td>3,0</td>
<td>1,1</td>
</tr>
<tr>
<td>Not Confess</td>
<td>0,3</td>
<td>2,2</td>
</tr>
</tbody>
</table>
FIGURE 9.2 (3)
Equilibrium and rationalizability in the classic normal forms

<table>
<thead>
<tr>
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<tbody>
<tr>
<td>Opera</td>
<td>2,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Movie</td>
<td>0,0</td>
<td>1,2</td>
</tr>
</tbody>
</table>

Battle of the Sexes
FIGURE 9.2 (4)
Equilibrium and rationalizability in the classic normal forms

Hawk-Dove/Chicken
Equilibrium and rationalizability in the classic normal forms
FIGURE 9.2 (6)
Equilibrium and rationalizability in the classic normal forms

Pareto Coordination
FIGURE 9.2 (7)
Equilibrium and rationalizability in the classic normal forms

\[
\begin{array}{c|cc}
 & P & D \\
\hline
P & 4,2 & 2,3 \\
D & 6,-1 & 0,0 \\
\end{array}
\]
FIGURE 9.3 (a)
Determining Nash equilibria.

<table>
<thead>
<tr>
<th></th>
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<th>Y</th>
<th>Z</th>
</tr>
</thead>
<tbody>
<tr>
<td>J</td>
<td>5,6</td>
<td>3,7</td>
<td>0,4</td>
</tr>
<tr>
<td>K</td>
<td>8,3</td>
<td>3,1</td>
<td>5,2</td>
</tr>
<tr>
<td>L</td>
<td>7,5</td>
<td>4,4</td>
<td>5,6</td>
</tr>
<tr>
<td>M</td>
<td>3,5</td>
<td>7,5</td>
<td>3,3</td>
</tr>
</tbody>
</table>
**FIGURE 9.3 (b)**
Determining Nash equilibria.

<table>
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<td>3,1</td>
<td>5,2</td>
</tr>
<tr>
<td>L</td>
<td>7,5</td>
<td>4,4</td>
<td>5,6</td>
</tr>
<tr>
<td>M</td>
<td>3,5</td>
<td>7,5</td>
<td>3,3</td>
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</table>
Mixed Strategies

(Watson Chapter 11)
Tennis Anyone?

Source: John Morgan, gametheory.net
Serving

Source: John Morgan, gametheory.net
The Game of Tennis

• Server chooses to serve either left or right
• Receiver defends either left or right
• Better chance to get a good return if you defend in the area the server is serving to

Source: John Morgan, gametheory.net
**Game Table**

<table>
<thead>
<tr>
<th>Server</th>
<th>Receiver</th>
<th>Left</th>
<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Left</td>
<td>Left</td>
<td>$\frac{1}{4}, \frac{3}{4}$</td>
<td>$\frac{3}{4}, \frac{1}{4}$</td>
</tr>
<tr>
<td>Right</td>
<td>Left</td>
<td>$\frac{3}{4}, \frac{1}{4}$</td>
<td>$\frac{1}{4}, \frac{3}{4}$</td>
</tr>
</tbody>
</table>

Source: John Morgan, gametheory.net
# Game Table

<table>
<thead>
<tr>
<th>Server</th>
<th>Receiver</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Left</td>
<td>Right</td>
<td></td>
</tr>
<tr>
<td>Left</td>
<td>(\frac{1}{4}, \frac{3}{4})</td>
<td>(\frac{3}{4}, \frac{1}{4})</td>
<td></td>
</tr>
<tr>
<td>Right</td>
<td>(\frac{3}{4}, \frac{1}{4})</td>
<td>(\frac{1}{4}, \frac{3}{4})</td>
<td></td>
</tr>
</tbody>
</table>

For server: Best response to defend left is to serve right
Best response to defend right is to serve left

For receiver: Just the opposite

Source: John Morgan, gametheory.net
• Notice that there are *no* mutual best responses in this game.
• This means there are no Nash equilibria in pure strategies
• But games like this always have at least one Nash equilibrium
• What are we missing?

Source: John Morgan, gametheory.net
Extended Game

• Suppose we allow each player to choose randomizing strategies

• For example, the server might serve left half the time and right half the time.

• In general, suppose the server serves left a fraction $p$ of the time

• What is the receiver’s best response?

Source: John Morgan, gametheory.net
Calculating Best Responses

• Clearly if \( p = 1 \), then the receiver should defend to the left

• If \( p = 0 \), the receiver should defend to the right.

• The expected payoff to the receiver is:
  
  \[
  \begin{align*}
  &p \cdot \frac{3}{4} + (1 - p) \cdot \frac{1}{4} \quad \text{if defending left} \\
  &p \cdot \frac{1}{4} + (1 - p) \cdot \frac{3}{4} \quad \text{if defending right}
  \end{align*}
  \]

• Therefore, she should defend left if

  \[
  p \cdot \frac{3}{4} + (1 - p) \cdot \frac{1}{4} > p \cdot \frac{1}{4} + (1 - p) \cdot \frac{3}{4}
  \]

Source: John Morgan, gametheory.net
When to Defend Left

• We said to defend left whenever:
\[ p \times \frac{3}{4} + (1 - p) \times \frac{1}{4} > p \times \frac{1}{4} + (1 - p) \times \frac{3}{4} \]

• Rewriting
\[ p > 1 - p \]

• Or
\[ p > \frac{1}{2} \]

Source: John Morgan, gametheory.net
Receiver’s Best Response

Source: John Morgan, gametheory.net
Server’s Best Response

• Suppose that the receiver goes left with probability $q$.

• Clearly, if $q = 1$, the server should serve right

• If $q = 0$, the server should serve left.

• More generally, serve left if
  
  \[ \frac{1}{4} q + \frac{3}{4} (1 - q) > \frac{3}{4} q + \frac{1}{4} (1 - q) \]

• Simplifying, he should serve left if
  
  \[ q < \frac{1}{2} \]

Source: John Morgan, gametheory.net
Server’s Best Response

\[ q \]

\[ \frac{1}{2} \]

Right \hspace{1cm} Left

Source: John Morgan, gametheory.net
Putting Things Together

Source: John Morgan, gametheory.net
Equilibrium

Source: John Morgan, gametheory.net
• A mixed strategy equilibrium is a pair of mixed strategies that are mutual best responses

• In the tennis example, this occurred when each player chose a 50-50 mixture of left and right.

Source: John Morgan, gametheory.net
General Properties of Mixed Strategy Equilibria

- A player chooses his strategy so as to make his rival indifferent.
- A player earns the same expected payoff for each pure strategy chosen with positive probability.
- Funny property: When a player’s own payoff from a pure strategy goes up (or down), his mixture does not change.

Source: John Morgan, gametheory.net
Does Game Theory Work?

• Walker and Wooders (2002)
  – Ten grand slam tennis finals
  – Coded serves as left or right
  – Determined who won each point

• Tests:
  – Equal probability of winning
    » Pass
  – Serial independence of choices
    » Fail

Source: John Morgan, gametheory.net
## Find all NE: Battle of the Sexes

<table>
<thead>
<tr>
<th></th>
<th>Opera</th>
<th>Boxing</th>
</tr>
</thead>
<tbody>
<tr>
<td>Op</td>
<td>3,1</td>
<td>0,0</td>
</tr>
<tr>
<td>Bo</td>
<td>0,0</td>
<td>1,3</td>
</tr>
</tbody>
</table>

Source: John Morgan, gametheory.net
# Find all NE: Hawk-Dove

<table>
<thead>
<tr>
<th></th>
<th>Krushchev</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Kennedy</strong></td>
<td><strong>Hawk</strong></td>
</tr>
<tr>
<td>Hawk</td>
<td>0, 0</td>
</tr>
<tr>
<td>Dove</td>
<td>1, 4</td>
</tr>
</tbody>
</table>

Source: John Morgan, gametheory.net
Highlights

• Dominance

• Rationalizability and iterated dominance

• Nash equilibrium
  – Pure strategy NE
  – Mixed strategy NE
Homework Assignment

• Chapter 6: #1
• Chapter 7: #1, 2, 3
• Chapter 11: #4, 6