RSD-TR-13-84

DYNAMICS OF A CLOSED CHAIN MANIPULATOR[1]

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September 1984

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[1]This work was supported by the Air Force Office Office of Scientific Research/AFSC, United States Air Force under AFOSR contract number F49620-82-C-0089.

TABLE OF CONTENTS

ABSTRACT

In many manipulator configurations, where the end effector of the manipulator is in contact with a fixed object, a complete mathematical model for the manipulator dynamics should include the effects of the resulting contact force between the end effector and the fixed object. Equations for such a closed chain manipulator are developed, where the end effector constraint is defined by a smooth manifold. These equations are shown to be complete in the sense that the direct dynamics problem and the inverse dynamics problem are well-posed. This formulation suggests a new approach to planning and tracking control for closed chain manipulators.

1. INTRODUCTION

There are numerous applications where the end effector of a robot manipulator is in contact with a fixed external object. These applications include the use of manipulators for carrying out assembly and machining tasks. However, the most common models of manipulator dynamics do not take into account the contact forces between the end effector of the manipulator and the fixed object [5,6,11]. If the contact force can be directly sensed, then it is possible to explicitly eliminate the contact force from the model, and this forms the basis for several control approaches [8,9,10,12]. However, direct and accurate sensing of the contact force is not always possible so that a complete dynamic model of the manipulator, including the effect of the contact force, is required. In this paper we develop such a model and indicate two important dynamics problems, each of which we show to be well-posed.

By imposing a contact constraint on the end effector, the manipulator links form a so-called closed chain; hence, we refer to a closed chain manipulator. Closed chain mechanical manipulators have been studied in [1,3,7], but the treatment on which our results are based is in [3].

2. MODELLING A CLOSED CHAIN MANIPULATOR

Let $p \in R^n$ denote the position vector of the end effector of the manipulator, in terms of a fixed workspace coordinate system. Suppose that constraints on the end effector are given as

$$\phi(p) = 0 \tag{1}$$

where $\phi: R^n \rightarrow R^m$ is twice continuously differentiable. A closed chain system is formed through continuous contact of the tip of the manipulator with the manifold defined by the constraints. Let S be a frictionless manifold defined by the constraints

$$S = \{ p \in R^n : \phi_i(p) = 0, \quad i = 1, \ldots, m \} \tag{2}$$

We also assume that the gradient vectors $\nabla\phi_1(p), \ldots, \nabla\phi_m(p)$ are linearly independent for all $p \in S$, so that the constraints define an m dimensional smooth manifold. If p_0 is a point on S, then we can define the normal space of S at p_0 as

$$N(p_0) = \left\{ p : p = \sum_{i=1}^{m} \alpha_i \nabla\phi_i(p_0), \quad i = 1, \ldots, m \right\} \tag{3}$$

and the tangent space of S at p_0 as

$$T(p_0) = \{ p : \langle p, y \rangle = 0, \quad y \in N(p_0) \} \tag{4}$$

$N(p_0)$ and $T(p_0)$ are subspaces of R^n and are orthogonal complements of each other so that

$$R^n = T(p_0) \oplus N(p_0) \tag{5}$$

In order that the manipulator does not lose contact with the constraint manifold, it is required that the velocity of the manipulator end effector lie in the tangent space of S and the contact force lie in the normal space of S. We refer to the constraints on the end effector velocity and force due to contact as the natural constraints.

Let $q \in R^n$ denote the vector of robot joint coordinates [5,6]. The relation between robot coordinates and workspace coordinates can be expressed as

$$p = H(q) \tag{6}$$

where H: $R^n \rightarrow R^n$ is twice continuously differentiable. We also assume that the manipulator is nonredundant, viz., the Jacobian matrix $\frac{\partial H(q)}{\partial q}$ is nonsingular and square.

It is convenient to define the manipulator equations of motion in terms of the joint coordinates [4]. Let $\tau \in R^n$ be the generalized joint torque vector required to maintain satisfaction of the path constraints; then the dynamic equations for a closed chain manipulator, taking into account the contact force, can be written as [2]

$$M(q)\ddot{q} + F(q,\dot{q}) = T + \tau \tag{7}$$

Here $T \in R^n$ is the generalized input joint torque vector. M(q) denotes the inertial matrix which is symmetric and nonsingular. $F(q,\dot{q})$ comprises Coriolis terms, centrifugal terms, and gravitational terms. Since no work is done by the contact torque τ in a virtual displacement δq ,

$$\sum_{l=1}^{n} \tau_l \, \delta q_l = 0$$

or equivalently in workspace coordinates

$$\sum_{l=1}^{n} \overline{\tau}_l \, \delta p_l = 0$$

where $\delta q_1, \ldots, \delta q_n$ ($\delta p_1, \ldots, \delta p_n$) represent scalar virtual displacements of the end effector in robot (workspace) coordinates. $\overline{\tau}$ is the generalized force vector, due to contact, in workspace coordinates. Using (6), we have

$$\sum_{l=1}^{n} \overline{\tau}_l \, \delta p_l = \sum_{i=1}^{n} \sum_{l=1}^{n} \overline{\tau}_l \, \frac{\partial H_l(q)}{\partial q_i} \, \delta q_i = 0$$

Thus

$$\tau_i = \sum_{l=1}^{n} \overline{\tau}_l \, \frac{\partial H_l(q)}{\partial q_i}, \qquad (i=1,\ldots,n)$$

From the constraints we know that the virtual displacements $\delta p_1, \ldots, \delta p_n$ must satisfy

$$\delta \phi_j(p) = \sum_{l=1}^{n} \frac{\partial \phi_j(p)}{\partial p_l} \, \delta p_l = 0, \qquad (j=1,\ldots,m)$$

We now introduce Lagrange multipliers $\lambda_1, \ldots, \lambda_m$; multiply this equation by λ_j to obtain

$$\lambda_j \sum_{l=1}^{n} \frac{\partial \phi_j(p)}{\partial p_l} \delta p_l = 0, \qquad (j=1,...,m)$$

Sum up these m equations; using a previous equation obtain

$$\sum_{l=1}^{n} \left[\overline{\tau}_l - \sum_{j=1}^{m} \lambda_j \frac{\partial \phi_j(p)}{\partial p_l} \right] \delta p_l = 0$$

Since $\dfrac{\partial \phi_1(p)}{\partial p}$,..., $\dfrac{\partial \phi_m(p)}{\partial p}$ are linearly independent vectors, $\lambda_1, \ldots, \lambda_m$ can be chosen such that

$$\overline{\tau}_l = \sum_{j=1}^{m} \lambda_j \frac{\partial \phi_j(p)}{\partial p_l}, \qquad (l=1,...,n)$$

The corresponding contact force components, computed in robot coordinates, are

$$\tau_i = \sum_{l=1}^{n} \sum_{j=1}^{m} \lambda_j \frac{\partial \phi_j(p)}{\partial p_l} \frac{\partial H_l(q)}{\partial q_i}, \qquad (i=1,...,n)$$

Let Jacobian matrices be defined as

$$J(q) \triangleq \frac{\partial H(q)}{\partial q}$$

$$D(p) \triangleq \frac{\partial \phi(p)}{\partial p}$$

then the complete set of equations of motion can be written, using vector notation, as

$$M(q)\ddot{q} + F(q,\dot{q}) = T + \tau \tag{8}$$

$$\tau = J^T(q)D^T(p)\lambda \tag{9}$$

$$p = H(q) \tag{10}$$

$$\phi(p) = 0 \tag{11}$$

or equivalently as

$$M(q)\ddot{q} + F(q,\dot{q}) = T + J^T(q)f \tag{12}$$

$$f = D^T(p)\lambda \tag{13}$$

$$p = H(q) \tag{14}$$

$$\phi(p) = 0 \tag{15}$$

where $\lambda = (\lambda_1, \ldots, \lambda_m)^T$.

3. DIRECT DYNAMICS AND INVERSE DYNAMICS

Given a specified motion of the manipulator, satisfying the imposed constraints, can we compute the input joint torque vector which would generate this specified motion and the corresponding contact force? This problem is usually referred to as the direct dynamics problem. Similarly, the inverse dynamics problem considers the following: given the input joint torque vector, what is the manipulator motion, satisfying the imposed constraints, and the corresponding contact force necessary to maintain satisfaction of the constraints? Calculation of the contact forces plays an important role in these problems. The existence (and uniqueness) of the manipulator motion and joint torques is also addressed. Throughout this section, we assume that the constraint manifold is frictionless.

The desired motion of a manipulator is often specified in workspace coordinates instead of robot coordinates. Using the coordinate transformation given by (6) define

$$\dot{J}(q,\dot{q}) \triangleq \frac{d}{dt}\,[J(q(t))] \qquad ;$$

then velocity and acceleration of the end effector in workspace coordinates are

$$\dot{p} = J(q)\dot{q}$$

$$\ddot{p} = \dot{J}(q,\dot{q})\dot{q} + J(q)\ddot{q}$$

Assume the motion $p(t) \in R^n$, $t_0 \leq t \leq t_f$, given in workspace coordinates, satisfies (15). The corresponding contact forces and the system dynamics are implicitly defined by (12),(13),(14). Note that $f(t)$ is defined in workspace coordinates. We now make the following assumptions:

(A1) The inertial matrix M: $R^n \rightarrow R^n$ is nonsingular. F: $R^n \times R^n \rightarrow R^n$ is bounded and Lipschitz continuous.
(A2) $\phi : R^n \rightarrow R^m$ is twice continuously differentiable.
(A3) J(q) is nonsingular for all $q \in R^n$. D(p) has rank m for all $p \in R^n$.
(A4) The $m \times m$ matrix

$$A(q) = D(H(q))J(q)M^{-1}(q)J^T(q)\,D^T(H(q))$$

is nonsingular.

Two basic propositions are derived below.

Proposition 1:

Suppose that the torque vector $T(t) \in R^n$ *is piecewise continuous on* $t_0 \leq t \leq t_f$. *Given T(t) and initial values* $q(t_0) = q_0$, $\dot{q}(t_0) = \dot{q}_0$, *satisfying* $H(q_0) \in S$, $J(q_0)\dot{q}_0 \in T(H(q_0))$. *Then with assumptions (A1)~(A4) there exists a unique contact force f(t) such that the solution of (12) satisfies the path constraints (15), assuming there is no finite escape time, for* $t_0 \leq t \leq t_f$.

Proof:

Our objective is to obtain an expression for the contact force which guarantees satisfaction of the path constraints. To this end, suppose that $\phi(p)=0$ so that

$$\frac{d\phi(p)}{dt} = D(p)\dot{p} = 0$$

$$\frac{d^2\phi(p)}{dt^2} = D(p)\ddot{p} + \dot{D}(p,\dot{p})\dot{p} = 0$$

where $\dot{D}(p,\dot{p}) \triangleq \frac{d}{dt} D(p(t))$. Since M(q) is nonsingular, $\ddot{q}$ is obtained as

$$\ddot{q} = M^{-1}(q)[\ T - F(q,\dot{q})\ ] + M^{-1}(q)J^T(q)f$$

From (13) obtain

$$A(q)\lambda = D(H(q))J(q)M^{-1}(q)[\ F(q,\dot{q}) - T\ ] - [\ D(H(q))\dot{J}(q,\dot{q}) + \dot{D}(H(q),J(q)\dot{q})J(q)\ ]\ \dot{q}$$

Note that D(p) is not a square matrix in general. These are m linear equations in m unknowns $\lambda \in R^m$; Since A(q) is nonsingular, λ can be uniquely determined as a linear function of $q, \dot{q}, T$

$$\lambda = g(q, \dot{q}, T)$$

where

$$\begin{aligned} g(q,\dot{q},T) = A^{-1}(q)D(H(q))J(q)M^{-1}(q)[\ F(q,\dot{q}) - T\ ] \\ - A^{-1}(q)[\ D(H(q))\dot{J}(q,\dot{q}) + \dot{D}(H(q),J(q)\dot{q})J(q)\ ]\dot{q} \end{aligned}$$

Hence, the resulting contact force is

$$f = D^T(H(q))g(q,\dot{q},T)$$

Eqns. (12), (13) reduce to the following initial value problem.

$$\begin{aligned} & M(q)\ddot{q} + F(q,\dot{q}) = T + J^T(q)D^T(H(q))g(q,\dot{q},T) \\ & q(t_0) = q_0, \quad \dot{q}(t_0) = \dot{q}_0 \end{aligned}$$

This initial value problem has a unique solution q(t) defined for $t_0 \leq t \leq t_0 + \delta$ for some $\delta > 0$. Assuming there is no finite escape time, the solution is defined for $t_0 \leq t \leq t_f$.

We now show that the constraints are satisfied throughout the motion. For the contact force $\lambda = g(q,\dot{q},T)$, $t_0 \leq t \leq t_f$, it can be verified that p(t)=H(q(t)) satisfies

$$\frac{d^2\phi(p)}{dt^2} = 0, \quad t_0 \leq t \leq t_f$$

But, since $H(q_0) \in S$, $J(q_0)\dot{q}_0 \in T(H(q_0))$ it follows that

$$\phi(p(t_0)) = 0, \quad \frac{d\phi(p(t_0))}{dt} = 0$$

Thus necessarily

$$\phi(p(t)) = 0, \quad t_0 \leq t \leq t_f$$

□

Proposition 2:

Given motion q(t), satisfying $\phi(H(q(t))) = 0$, $t_0 \leq t \leq t_f$ *, which is twice continuously differentiable. Under assumptions* (A1)~(A3) *, there exist input joint torque vector T(t) and corresponding contact force vector* $f(t) \in N(H(q(t)))$ *satisfying (12) for* $t_0 \leq t \leq t_f$.

Proof:

From (12), (13), we have

$$M(q)\ddot{q} + F(q,\dot{q}) = T + J^T(q)\, D^T(H(q))\lambda$$

Given q(t), $t_0 \leq t \leq t_f$, the left hand side of the equation is determined. Clearly there are many T(t), $\lambda(t)$, $t_0 \leq t \leq t_f$, satisfying the equation.

□

As indicated in the proof, there are many input joint torque vector functions and contact force function which generate the same given motion. For a specified contact force function, the torque T(t) is uniquely defined. The case of an open chain manipulator corresponds to the assumption that the contact force $f(t)=0$, $t_0 \leq t \leq t_f$, so that the torque T(t) is uniquely determined.

4. CONCLUSIONS

We have carefully developed a general model for a closed chain manipulator, and we have shown that the direct and inverse dynamics problems are well-posed. We believe that the closed chain manipulator model is the appropriate model to use in control design and analysis, where the manipulation task involves contact between the manipulator end effector and a fixed object.

Numerous approaches to control of manipulators, not including a closed chain constraint, have been developed [5,6,11]. It is likely that those control approaches can be suitably modified to apply to the closed chain case, but the modifications and extensions have not yet been developed.

It is possible to eliminate the path constraint and the contact force from the dynamic model [3], by a suitable elimination of variables. However, the resulting equations are usually extremely complicated; moreover, elimination of the contact force from the model may not be most desirable if the contact force represents a variable to be controlled. Thus an interesting challenge is to develop suitable control approaches, based on the complete equations (12)$\sim$(15)

developed here.

In our development, a number of specific assumptions have been made. Important extensions would include incorporation of friction effects on the constraint set, allowance of nonsmooth constraint sets, and consideration of constraint sets defined in terms of inequalities so that contact between the end effector and the fixed object need not be continuously maintained.

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