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Corrections to the statistical entropy of five dimensional black holes

Alejandra Castro[a] **and Sameer Murthy**[b]

[a] *Department of Physics and Michigan Center for Theoretical Physics, University of Michigan, Ann Arbor, MI 48109-1120, U.S.A.*

[b] *Laboratoire de Physique Théorique et Hautes Energies (LPTHE), Université Pierre et Marie Curie-Paris 6, CNRS UMR 7589, Tour 24-25, 5ème étage, Boite 126, 4 Place Jussieu, 75252 Paris Cedex 05, France*

E-mail: aycastro@umich.edu, smurthy@lpthe.jussieu.fr

Abstract: We compute the statistical entropy of the three charge (D1-D5-p) five dimensional black hole to sub-leading order in a large charge expansion. We find an agreement with the macroscopic calculation of the Wald entropy in R^2 corrected supergravity theory. The two calculations have a overlapping regime of validity which is *not* the Cardy regime in the microscopic conformal field theory. We use this result to clarify the 4d-5d lift for black holes on Taub-NUT space. In particular, we compute sub-leading corrections to the formula $S^{4d} = S^{5d}$. In the microscopic analysis, this correction arises from excitations bound to the Taub-NUT space. In the macroscopic picture, the difference is accounted by a mechanism present in a higher derivative theory wherein the geometry of the Taub-NUT space absorbs some of the electric charge.

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Contents

1 Introduction and summary

In the last few years, there has been significant progress [1–8] in computing the entropy of four-dimensional black holes in string theory beyond the large charge estimate. On the macroscopic side, the dominant contribution to the entropy is given by Bekenstein-Hawking formula and the sub-leading corrections are found by studying higher derivative corrections to the effective action of string theory. In the presence of such higher derivative effects, the definition of the thermodynamic entropy is modified and one has to use the Wald formula [9–11] which generalizes the Bekenstein-Hawking entropy. For extremal black holes, this can be summarized elegantly by the entropy function formalism [12].

What made the higher-derivative problem tractable is the understanding of the off-shell formulation of $\mathcal{N} = 2$ supergravity in four dimensions and the attractor equations of this theory which make it simple to find and analyze black hole solutions. The higher derivative terms analyzed are packaged as corrections to the prepotential [5]. In another analysis [7], a different combination of the higher derivative terms — the Gauss-Bonnet interaction — was studied and found to correctly capture the entropy to sub-leading order. It is still not very clear why only a subset of all possible four derivative corrections correctly captures the sub-leading entropy.

It is natural to ask whether this analysis can be carried over to other dimensions. In five dimensions, there has been work [13] on understanding a certain class of higher derivative corrections, namely the gravitational Chern-Simons term and other terms related to it by supersymmetry. This action was used to find black hole solutions in [14–17] and corrections to the entropy of five dimensional black holes were computed. Further references on sub-leading corrections to the five dimensional solutions in the presence of higher derivative terms include [18–20].

In this paper, we compute the statistical entropy of a 5d black hole with a given set of charges to sub-leading order in a large charge expansion. We find that the sub-leading corrections match those found by the macroscopic analysis. The black hole we analyze is the one in which the first accurate microscopic computation of the leading entropy was done [21], the D1-D5-p black hole in type IIB string theory on $K3$. The theory has 16 real supersymmetries and the black hole preserves four of them.

1.1 Microscopic counting

For *four* dimensional black holes, the exact counting of microstates beyond the large charge estimate has been achieved in $\mathcal{N} = 4$ string theory using the construction of the partition function for 1/4 BPS dyons in terms of an auxilliary mathematical function called the Igusa cusp form, the unique weight 10 modular form of $\mathrm{Sp}(2, \mathbb{Z})$. The degeneracy of states can then be counted by performing an inverse fourier transform of this function using contour and saddle point methods. This counting formula was originally conjectured in [22] and then derived in [23, 24] using a D-brane-monopole setup, and generalized to the counting of all dyons in [25].

The derivation uses the relation of the 4d black holes in question to a three charge spinning black hole in five dimensions which has come to be known as the 4d-5d lift [26]. The 4d black holes carry one extra charge which corresponds to a unit KK monopole at the center of which the 5d black hole is placed. By making the modulus of the KK circle small or large, the authors of [23] then argue that the entropy of the 4d and 5d black holes are related. More precisely [24], the microstates of the 4d system can be counted by putting together the microstates of the 5d system, and the states which are bound to the KK monopole itself.

It turns out that putting together these two pieces gives a partition function which is given in terms of the above mentioned Igusa cusp form Φ_{10}. This function has modular transformation properties under $\mathrm{Sp}(2, \mathbb{Z})$ which are much more powerful than those of $\mathrm{SL}(2, \mathbb{Z})$ which govern the elliptic genus of a 2d SCFT. Using these modular transformation properties, one can systematically deduce the sub-leading corrections to the 4d black hole entropy [24, 27, 28].

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For the 5d black hole, the analysis in [21] used the related 2d SCFT $Sym^{Q_1Q_5+1}(K3)$ with L_0 eigenvalue equal to the momentum n. The SCFT lives on a circle transverse to the space where the black hole lives. In this 2d SCFT, one can apply the Cardy formula to estimate the density of states at high energies. The Cardy formula is valid for energies much larger than the central charge, i.e. $n \gg Q_1Q_5$. There is a systematic procedure to compute corrections to the Cardy formula [29, 30] in the parameter $\frac{Q_1Q_5}{n} \ll 1$.

On the other hand, in the gravity theory, the configuration looks like a big 5d black hole when the Schwarzschild radius is much larger than the string length. In the type II theory on $K3$, this radius is given by $\frac{R^2_{\text{Sch}}}{l_s^2} = \frac{Q_1Q_5}{n}$. One can now look at finer structures and probe higher derivative corrections to the black hole entropy; these sigma model corrections to supergravity will be governed by the small parameter $\frac{n}{Q_1Q_5}$. This is exactly the opposite regime to the one above where one can compute corrections to the Cardy formula. One cannot therefore, naively compare the macroscopic corrections with the microscopic corrections in the Cardy limit.

One thus needs a new tool to compute the sub-leading expansions of the statistical entropy in the non-Cardy regime.[1] Such a tool can be found by using the above 4d-5d lift in reverse — we can rewrite the 5d partition function in terms of the 4d partition function plus some corrections which physically have to do with the *stripping off* of the modes stuck to the KK monopole. Mathematically, as we shall see in the following, this is expressed as a precise relation between the 5d and the 4d partition functions. Having done this, we can use the powerful mathematical properties of the function Φ_{10} to deduce systematically the corrections to the 5d entropy.

1.2 Five dimensions v/s four dimensions

This new tool allows us to understand certain features of 5d black holes and contrast them against 4d black holes. The first such feature is spacetime duality. The 4d duality group is bigger than the 5d one, and in particular it contains the 4d electric-magnetic duality which is absent in 5d. The manifestation of this duality which exchanges $n \equiv Q^2 \leftrightarrow P^2 \equiv Q_1Q_5$ appears through the prepotential in the 4d gravity theory, to which worldsheet/membrane instantons (depending on the duality frame) wrapping the T^2 contribute in a crucial way. These instanton contributions complete the classical linear prepotential into a transcendental function related to the Jacobi η function which is S-duality invariant. The entropy function which depends on the prepotential is thus also duality invariant.

In five dimensions, one of the circles which these worldsheets/branes wrap becomes large and the five dimensional supergravity does not see their effects, and only the contributions $P^2 \gg Q^2$ are retained. The entropy function as we shall see only contains the

[1] This would not be necessary if one can map the counting problem to that of finding the density of states in the Cardy regime of a different CFT. Indeed, as was observed in [19, 20], the entropy of the 5d black hole which we consider can be expressed to subleading order as a Cardy formula of a putative dual SCFT with $L_0 = Q_1$ and $c = 6Q_5(n+3)$. It would be very interesting to understand the microscopic origin of such a SCFT with these values of charges. We thank the referee for pointing this out.
In the remainder of the paper, the phrase "away from the Cardy limit" should be taken to mean "away from the Cardy limit of any currently understood microscopic SCFT, and in particular the D1-D5-p SCFT".

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residue of the leading linear piece which is *not* duality invariant, which is consistent since 5d supergravity admits no such S-duality.

Our 5d microscopic counting formula matches the 5d gravity calculation in this regime of charges $Q_1 Q_5 \gg n$; it also agrees to sub-leading order with the corrections to the Cardy formula using the Jacobi-Rademacher expansion in the regime $n \gg Q_1 Q_5$. The coefficients of the above two sub-leading corrections are not equal since they are not related by any duality.

Finally, we use our technique to clear up a slightly confusing point in the literature having to do with the 4d-5d lift. If we put 5d supergravity (+ corrections) on a background Taub-NUT space of everywhere low curvature, the theory should still be valid. On such a space, we can compute the entropy of a black hole sitting at the center which locally looks 5d. It turns out that there is a subtle shift in the definition of charge in the 5d theory having to do with the curvature of the Taub-NUT space, which changes the entropy expressed in terms of the 4d charges. This small change as we shall show, agrees precisely with the change computed by the 4d and 5d microscopic formulas.

The plan of this paper is as follows. In section §2, we present the five dimensional effective theory which arises upon reduction of type IIB string theory on $K3 \times S^1$. At lowest order, this is $\mathcal{N} = 4$ supergravity in five dimensions. We then analyze adding four derivative terms to this action and the corresponding black hole solutions. In section §3, we discuss the Wald entropy formula in the higher derivative theory. We then apply it to the rotating BMPV black hole in five dimensions and present the corresponding corrections to the Bekenstein-Hawking formula. In section §4, we present the microscopic counting formula and compute the first correction to the large charge result. In section §5, we discuss the 4d-5d lift and the slight difference in 4d and 5d black hole entropies. We explain this difference through the different mechanisms in the microscopic and macroscopic understanding. In section §6, we summarize our results and suggest future directions. In the appendix we briefly sketch the evaluation of the contour and saddle point integral, some relevant properties of the Jacobi functions and some details of the Jacobi-Rademacher expansion.

2 Black holes in five dimensional supergravity

In this section we outline the construction of black hole solutions in the presence of higher derivative corrections, which were discussed in great detail in [14, 16]. The framework is $\mathcal{N} = 2$ supergravity in five dimensions coupled to n_V vector multiplets. This theory can be embedded in eleven dimensional supergravity compactified on a Calabi-Yau three fold, where the lower dimensional theory will depend on the topological data of CY_3. In the remaining sections the discussion will focus on D1-D5-p system which corresponds to a 1/4 BPS black hole solutions in $\mathcal{N} = 4$ supergravity. The enhancement of supersymmetry amounts to choosing $\mathrm{CY}_3 = \mathrm{K3} \times \mathrm{T}^2$ and the theory has a dual description in type IIB supergravity on $\mathrm{K3} \times \mathrm{S}^1$.

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At the two-derivative level, the effective action is given by

$$S = \frac{1}{4\pi^2}\int d^5x\sqrt{g}\left(-R - G_{IJ}\partial_a M^I \partial^a M^J - \frac{1}{2}G_{IJ}F^I_{ab}F^{Jab} + \frac{1}{24}c_{IJK}A^I_a F^J_{bc}F^K_{de}\epsilon^{abcde}\right) , \tag{2.1}$$

with $I = 1, \ldots n_V + 1$ and $a, b = 0, \ldots 4$ are tangent space indices. The scalars M^I can be interpreted as volumes of two-cycles and M_I the volume of the dual four-cycle, which are related through the intersection numbers c_{IJK}

$$M_I = \frac{1}{2}c_{IJK}M^J M^K . \tag{2.2}$$

In addition the metric of the scalar moduli space is

$$G_{IJ} = \frac{1}{2}\left(M_I M_J - c_{IJK}M^K\right) . \tag{2.3}$$

The four-derivative corrections of interest are those governed by the mixed gauge-gravitational Chern-Simons term

$$\mathcal{L}_1 = \frac{c_{2I}}{24\cdot 16}\epsilon_{abcde}A^{Ia}R^{bcfg}R^{de}{}_{fg} , \tag{2.4}$$

where c_{2I} is the second Chern class of CY_3. The overall coefficient is determined by the M5-brane anomaly cancelation via anomaly inflow [31]. $\mathcal{L}_1$ by itself is not supersymmetric, but by using the off-shell formulation of the supersymmetry algebra one can construct the supersymmetric completion of (2.4). As discussed in [13], these corrections include all possible terms allowed by the symmetry of the theory which involve the square of the Riemann tensor. This is true under the assumption that the hypers decouple from the theory, and therefore it is consistent to discuss configurations that only involve Weyl and vector multiplets and multiplet.

Taking advantage of the off-shell formalism for the five dimensional theory, the construction of black holes solutions is greatly simplified. The simplest way to obtain the corrected solution is by first imposing BPS conditions, and then utilizing equations of motion for the gauge field and auxiliary fields.

Backgrounds with unbroken supersymmetry allows for stationary solutions of the form

$$ds^2 = e^{4U(x)}(dt+\omega)^2 - e^{-2U(x)}h_{mn}dx^m dx^n , \tag{2.5}$$

where h_{mn} is a 4D hyper-Kahler base space, and the particular case of Taub-NUT is given by

$$h_{mn}dx^m dx^n = \frac{1}{H^0(\rho)}(dx^5 + p^0\cos\theta d\phi)^2 + H^0(\rho)\left(d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)\right) , \tag{2.6}$$

with $x^5 \cong x^5 + 4\pi$ and $H^0(\rho) = 1 + \frac{p^0}{\rho}$. The rotation is described by $\omega = \omega(x^m)dx^m$, and since we are interested in corrections to the BMPV black holes we will restrict the discussion to self dual rotation $d\omega = \star_4 d\omega$. For the Taub-NUT base space this fixes the one-form

$$\omega = \frac{J}{8\rho}(dx^5 + p^0\cos\theta d\phi) . \tag{2.7}$$

The last piece of information from supersymmetry is given by the variation of the gaugino in the vector multiplet. This results in a condition between the gauge field to the corresponding scalar field

$$F^I = d(M^I e^{2U}(dt+\omega)) \ . \tag{2.8}$$

After exhausting the supersymmetry conditions, the equations of motion for the explicit action will further determine the full solution. The variation of the action with respect to the gauge field, i.e. Maxwell's equation, results in an exact harmonic equation

$$\nabla^2 \left[M_I e^{-2U} - \frac{c_{2I}}{8}\left((\nabla U)^2 - \frac{1}{12}e^{6U}(d\omega)^2 \right) \right] = \frac{c_{2I}}{24\cdot 8}\nabla^2\left[2\frac{(\nabla H^0)^2}{(H^0)^2} - \frac{2}{\rho}\right] \ . \tag{2.9}$$

The term to the right of the equality arises from the from curvature of the base space coupled to the gauge field through $A^I \wedge \mathrm{Tr}(R^2)$, which behaves as a charge density governed by the curvature of the base space. Solving (2.9) determines the scalar fields as

$$M_I(\rho) = e^{2U}\left[M_I^\infty + \frac{q_I}{4\rho} + \frac{c_{2I}}{8}\left((\nabla U)^2 - \frac{1}{12}e^{6U}(d\omega)^2\right)\right] + e^{-2U}\frac{c_{2I}}{24\cdot 4}\left[\frac{(\nabla H^0)^2}{(H^0)^2} - \frac{1}{\rho}\right] , \tag{2.10}$$

with M_I^∞ the value of the moduli at infinity. The constants q_I are identified with conserved 5d charges by Gauss's law, i.e. the integral of the conserved current associated with the variation of the action with respect to A^I. Writing (2.9) as the divergence of the current one can identify the conserved current. In the absence of dipole charges, this is equivalent to

$$q_I(\Sigma) = -\frac{1}{4\pi^2}\int_{\partial\Sigma} \star_5 \frac{\partial \mathcal{L}}{\partial F^I} \, , \tag{2.11}$$

with Σ a spacelike surface and $\partial\Sigma$ is the asymptotic boundary. As shown in [16] this integral is sensitive to the geometry of the base space. For example, the Taub-NUT geometry interpolates between $\mathbb{R}^4$ at the origin and $\mathbb{R}^3 \times S^1$ at infinity, and dialing the size of the circle interpolates between a 4d and 5d black hole. Naively one might expect that (2.11) is independent of the location of Σ, but the delocalized source in (2.9) amounts for the shift

$$q_I(\Sigma_\infty) - q_I(\Sigma_0) = -\frac{c_{2I}}{24} \, , \tag{2.12}$$

where the 5d electric charge is $q_I(\Sigma_\infty) = q_I$ as defined in (2.10), and $q_I(\Sigma_0)$ corresponds to the 4d electric charge. Notice that this discrepancy appears after the inclusion of higher derivatives; for the two-derivative theory the four and five dimensional charge are equal. We will return to this shift in section §5 when discussing the 4d-5d lift.

The only function we haven't specified so far is $U(\rho)$. In the off-shell formalism, the variation of the scalar auxiliary field modifies the special geometry constraint and for the solution in question the equation reads

$$\frac{1}{6}c_{IJK}M^I M^J M^K = 1 - \frac{c_{2I}}{24}\left[e^{2U}M^I\left(\nabla^2 U - 4(\nabla U)^2 + \frac{1}{4}e^{6U}(d\omega)^2\right) + e^{2U}\nabla^i M^I \nabla_i U\right] . \tag{2.13}$$

By specifying the internal CY_3 manifold and the charge vector q_I, one can iteratively solve non-linear differential equation for the metric function $U(\rho)$ and fully specify the geometry.

3 Macroscopic derivation of black hole entropy

Our discussion focuses on supersymmetric rotating black holes and the sub-leading corrections to the entropy found in [16]. The corrections where found by exploiting the consequences of the attractor mechanism [32–35] and utilizing the entropy function formalism. Here we will briefly outline the key features of the procedure.

For a semi-classical theory of gravity described by a local action, the black hole entropy can be obtained as a Noether charge associated to the diffeomorphism invariance of the theory. This is the well known Wald's entropy formula

$$S = -\frac{1}{8\pi G_D}\int_\Sigma d^{d-2}x\sqrt{h}\frac{\partial\mathcal{L}}{\partial R_{\mu\nu\rho\sigma}}\epsilon^{\mu\nu}\epsilon^{\rho\sigma}\ . \tag{3.1}$$

For two-derivative gravitational theories this gives Bekenstein-Hawking area law, i.e. $S = \frac{A}{4G}$. In practice (3.1) is somewhat complicated to manipulate, specially if we are interested in actions which contain higher powers of the curvature tensor.

As discussed in [12] one can reformulate (3.1) as a Legendre transformation of the action for extremal black holes. The near horizon geometry of these black holes contain an AdS_2 factor which allows one to rewrite (3.1) as a functional of the on-shell Lagrangian. Define the Lagrangian density

$$f = \frac{1}{4\pi^2}\int dx^{D-2}\sqrt{g}\mathcal{L}\ . \tag{3.2}$$

The black hole entropy is given by[2]

$$S = 2\pi\left(e^I\frac{\partial f}{\partial e^I} + e^0\frac{\partial f}{\partial e^I} - f\right) \tag{3.3}$$

Here e^I and e^0 are potentials associated to electric charge and rotation, respectively. To evaluate explicitly (3.3) we need the on-shell values of the potentials as a function of the charges, which is greatly simplified by the attractor mechanism

3.1 Attractor solution and black hole entropy

The near horizon geometry of an extremal black hole is governed by the attractor mechanism. For supersymmetric configurations this highly constrains the geometry independently of the precise action one uses. One feature of the attractor is that the values of the scalar fields at the horizon are fixed by charges carried by the black hole, independent of initial conditions at infinity. Additionally, the attractor enhances the solution to be maximally supersymmetric at the horizon.

The rotating attractor solution in five dimensions is described by a circle fibered over $\mathrm{AdS}_2\times\mathrm{S}^2$,

$$ds^2 = -\ell^2(1-\hat{J}^2)(dx_5 + \cos\theta d\phi + e^0\rho dt)^2 + \ell^2\left(\rho^2 dt^2 - \frac{d\rho^2}{\rho^2}\right) - \ell^2 d\Omega_2^2\,, \tag{3.4}$$

[2] The derivation of (3.3) assumes gauge invariance of the action, which is not true for Chern-Simons terms in discussion. The resolution is well understood and we refer the reader to [36] for a detailed discussion.

where ℓ is the AdS$_2$ radius and $\hat{J}$ is the potential associated with rotation. For simplicity we set $p^0 = 1$ in (3.4). The configuration also holds a 2-form flux carrying electric charges and the corresponding gauge field is

$$A^I = e^I \rho d\tau - \frac{e^0 e^I}{(1+(e^0)^2)}(dx_5 + \cos\theta d\phi + e^0 \rho dt) \ . \tag{3.5}$$

Here the potentials e^I and e^0 are related to the near horizon fields by

$$e^0 = -\frac{\hat{J}}{\sqrt{1-\hat{J}^2}} \ , \quad e^I = \frac{\hat{M}^I}{2\sqrt{1-\hat{J}^2}} \ . \tag{3.6}$$

Both (3.4) and (3.5) are solely determined by solving the off-shell BPS conditions, which assures that the background is an exact solution even after including higher derivative corrections. The next step is to relate the charges (q_I, J) with the potentials $(\hat{M}^I, \hat{J})$ and the geometry governed by the scale ℓ. The modified special geometry constraint (2.13) relates ℓ with the potentials,

$$\ell^3 = \frac{1}{8}\left(\frac{1}{6}c_{IJK}\hat{M}^I\hat{M}^J\hat{M}^K - \frac{1}{12}c_{2I}\hat{M}^I(1-2\hat{J}^2)\right) \ . \tag{3.7}$$

By construction, the five dimensional rotation (2.7) is defined as

$$\hat{J} = \frac{1}{8\ell^3}J \ , \tag{3.8}$$

and after using (3.7) we have

$$J = \left(\frac{1}{6}c_{IJK}\hat{M}^I\hat{M}^J\hat{M}^K - \frac{1}{12}c_{2I}\hat{M}^I(1-2\hat{J}^2)\right)\hat{J} \ . \tag{3.9}$$

Electric charges are defined as a conserved quantity associated to the variation of action with respect to the corresponding gauge field. Evaluating (2.10) at the horizon, the electric charge q_I is related to the potentials by

$$q_I = \frac{1}{2}c_{IJK}\hat{M}^J\hat{M}^K - \frac{1}{8}c_{2I}\left(1-\frac{4}{3}\hat{J}^2\right) \ . \tag{3.10}$$

Both (3.9) and (3.10) are obtained by taking the near horizon limit of the equations of motion.

Given the attractor geometry (3.4)–(3.5), one can evaluate the full action including higher derivative corrections to compute the entropy function (3.3). After some effort, the semi-classical black hole entropy reads

$$S^{5d} = 2\pi\sqrt{1-\hat{J}^2}\left(\frac{1}{6}c_{IJK}\hat{M}^I\hat{M}^J\hat{M}^K + \frac{1}{6}c_{2I}\hat{M}^I\hat{J}^2\right) \tag{3.11}$$

The next step would be to write the potentials $\hat{M}^I$ and $\hat{J}$ as a function of charges and rotation by solving (3.9)–(3.10), which would allow us to write $S^{5d} = S^{5d}(q_I, J)$. For generic intersection numbers c_{IJK} this can only be done perturbatively, but as we will discuss below the equations are invertible for specific CY$_3$ manifolds.

3.2 Black holes on K3$\times$T^2

We are interested in corrections to the entropy of 1/4 BPS black holes in $\mathcal{N}=4$ with internal manifold CY$_3$=K3$\times$T^2. In the eleven dimensional language, the electric charges q^I correspond to M2-branes wrapping two-cycles. Equivalently, we can consider type IIB string theory on $K3 \times T2$ and D1-D5-P charges. The D1-D5 system is extended in the $K3$ and the effective string extends along one of circles S^1 of the T^2. The momentum P is excited along the circle S^1.

For CY$_3$=K3$\times$T^2, $\hat{M}^1$ denotes the modulus on the torus and $\hat{M}^i$ the moduli on K3, with $i = 2, \ldots 23$. The non-trivial intersection numbers and second Chern class are

$$c_{1ij} = c_{ij} \,, \quad c_{2,1} = c_2(K3) = 24 \,. \tag{3.12}$$

For this specific manifold, equations (3.9) and (3.10) are invertible allowing to write $(\hat{M}^I, \hat{J})$ in terms of (q_I, J)

$$\hat{M}^1 = \sqrt{\tfrac{1}{2}q_i q_j c^{ij} + \tfrac{4J^2}{(q_1+\frac{c_2}{24})^2}}\,(q_1 + \tfrac{c_2}{8})\,, \tag{3.13}$$

$$\hat{M}^i = c^{ij} q_j \sqrt{\frac{(q_1+\frac{c_2}{8})}{\frac{1}{2}q_i q_j c^{ij} + \frac{4J^2}{(q_1+\frac{c_2}{24})^2}}}\,, \tag{3.14}$$

$$\hat{J} = \frac{J}{q_1+\frac{c_2}{24}} \sqrt{\frac{(q_1+\frac{c_2}{8})}{\frac{1}{2}q_i q_j c^{ij} + \frac{4J^2}{(q_1+\frac{c_2}{24})^2}}}\,, \tag{3.15}$$

where we define c^{ij} as the inverse of c_{ij}. Inserting (3.13)–(3.15) in (3.11), the entropy as function of charges becomes

$$S = 2\pi \sqrt{\frac{1}{2}q_i q_j c^{ij}\left(q_1 + \frac{c_2}{8}\right) - \frac{\left(q_1 - \frac{c_2}{24}\right)\left(q_1 + \frac{c_2}{8}\right)}{\left(q_1 + \frac{c_2}{24}\right)^2} J^2}\,. \tag{3.16}$$

Expanding to first order in c_2 gives

$$S = 2\pi\sqrt{Q^3 - J^2}\left(1 + \frac{3}{2}\frac{Q_1 Q_5}{Q^3 - J^2} + \ldots\right)\,, \tag{3.17}$$

where we identified the IIB charges as

$$Q_1 Q_5 = \frac{1}{2}c^{ij} q_i q_j \,, \quad n = q_1 \,, \quad Q^3 - J^2 = Q_1 Q_5 n - J^2 \,. \tag{3.18}$$

If all the charges q_i, J scale equally, the expression to sub-leading order is:

$$S = 2\pi\sqrt{Q_1 Q_5 n}\left(1 + \frac{3}{2n} - \frac{J^2}{2Q_1 Q_5 n} + \ldots\right)\,, \tag{3.19}$$

where the sub-leading dependence on angular momentum is due to leading supergravity result. The higher derivatives terms give rise to corrections proportional to J as displayed in (3.17), but are not important in this regime.

Summarizing, we have an expression for the sub-leading corrections to the entropy (3.19) for rotating five dimensional black holes. These corrections come from the supersymmetric completion of $c_{2I} A^I \wedge \mathrm{Tr}(R^2)$. The macroscopic entropy (3.19) is what we would like to compare with the microscopic counting formula.

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4 The microscopic degeneracy formula

The 5d counting problem of the D1-D5 system on $K3$ is captured by a $(4,4)$ two-dimensional superconformal field theory along the worldvolume $\mathbb{R}\times S^1$ with target space $\mathrm{Sym}^{Q_1Q_5+1}(K3)$ [37]. We denote this sigma model SCFT by

$$X^{5d}=\sigma(\mathrm{Sym}^{Q_1Q_5+1}(K3))\ . \tag{4.1}$$

Two of the charges Q_1, Q_5 that the black hole carries appear in the definition of the sigma model. The third charge momentum n and the angular momentum l appear as the eigenvalues of the hamiltonian L_0 and R-charge $J_0/2$ of the sigma model. The charges are related to the number of D-branes in the following fashion

$$Q_5=N_5\,,\quad Q_1=N_1-N_5\,, \tag{4.2}$$

because there is an effective negative unit one-brane charge generated by the five-brane wrapped on the $K3$. The relevant object which captures the BPS states is the elliptic genus

$$\chi(X^{5d};q,y)\equiv \mathrm{Tr}^{X^{5d}}_{RR}(-1)^{J_0-\widetilde{J}_0}q^{L_0}\widetilde{q}^{\overline{L}_0}y^{J_0}\equiv\sum_{n,l}c^{5d}(Q_1Q_5,n,l)q^ny^l\ . \tag{4.3}$$

To estimate the growth of the coefficients of this SCFT, we can use Cardy's formula and spectral flow in the SCFT

$$\Omega\sim\exp\left(\sqrt{\frac{c}{6}L_0-J^2}\right)+\dots \tag{4.4}$$

Plugging in

$$c=6Q_1Q_5\,,\quad L_0=n\,,\quad J^2=\frac{l^2}{4}\ . \tag{4.5}$$

we get

$$\Omega(Q_1,Q_5,n,l)\sim\exp(2\pi\sqrt{Q_1Q_5n-l^2/4})+\dots\,, \tag{4.6}$$

The approximation (4.6) is valid at high values of L_0, i.e. $n\gg Q_1Q_5$. One can actually systematically compute corrections to this result using an exact formula which determines the fourier coefficients of the elliptic genus of a symmetric product SCFT in terms of the fourier coefficients of the original SCFT (in this case $K3$) [38]. The formula relies on the modular transformation properties of the elliptic genus under $\mathrm{SL}(2,\mathbb{Z})$ and uses the Jacobi-Rademacher expansion [29, 30]. By its nature, it is expressed as a series of corrections to the Cardy formula and can be used as above when $L_0\gg c$, i.e. $n\gg Q_1Q_5$.

On the other hand, the black hole entropy function is valid for large values of charges when *all* the charges scale equally, i.e. $Q_1Q_5\gg n\gg 1$. In order to meaningfully compare the two expressions, we would need to re-sum the Farey tail expansion in Q_1Q_5/n and reexpress it as an expansion in n/Q_1Q_5, which *a priori* seems to be a difficult problem.

However, we can make progress using the relation of the elliptic genus of the symmetric product to the Siegel modular form Φ_{10}. This is known as the Igusa cusp form and is the unique weight 10 modular form of $\mathrm{Sp}(2,\mathbb{Z})$. Using the more powerful Siegel modular transformation properties and a saddle point approximation, we can compute the expansion of the above elliptic genus for any regime of charges, in particular $n/Q_1Q_5 \ll 1$. Physically, this is related to the 4d-5d lift which we shall discuss in a following section. In this section, we shall simply use this relation to our calculational advantage.

The generating function of the elliptic genus of the symmetric product is given by [38]

$$Z(\rho,\sigma,v) \equiv \sum_{k=0}^{\infty} p^k \chi(Sym^k(X);q,y) = \prod_{n>0,m\geq 0,l} \frac{1}{(1-p^n q^m y^l)^{c(nm,l)}} , \tag{4.7}$$

where we have set

$$q = e^{2\pi i \rho}, \quad p = e^{2\pi i \sigma}, \quad y = e^{2\pi i v}, \tag{4.8}$$

and the coefficients $c(n,l)$ are defined through

$$\chi(X;q,y) = \sum_{n,l} c(n,l) q^n y^l . \tag{4.9}$$

For $X = K3$, this generating function is related to the Igusa cusp form Φ_{10} as [22],

$$Z(\rho,\sigma,v) = \frac{f^{\rm KK}(\rho,\sigma,v)}{\Phi_{10}(\rho,\sigma,v)}, \tag{4.10}$$

where

$$\begin{aligned} f^{\rm KK}(\rho,\sigma,v) &= p\,q\,y\,(1-y^{-1})^2 \prod_{m=1}^{\infty} (1-q^m)^{20}(1-q^m y)^2(1-q^m y^{-1})^2 \\ &= p\,\eta^{18}(\rho)\,\vartheta_1^2(v,\rho) . \end{aligned} \tag{4.11}$$

We are interested in the microscopic degeneracy of the system with charges (Q_1,Q_5,n,l), which is given by the coefficient $c(n,l)$ of the sigma model (4.1). This can be expressed as an inverse Fourier transform of the generating function $Z(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$

$$\Omega^{5d}(Q_1,Q_5,n,l) = \oint_{\mathcal{C}} d\widetilde{\rho} d\widetilde{\sigma} d\widetilde{v}\, e^{-2i\pi(\widetilde{\rho}n+\widetilde{\sigma}(Q_1Q_5+1)+l\widetilde{v})}\, Z(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) . \tag{4.12}$$

The contour $\mathcal{C}$ in the above integral is presented in appendix §A. In the 4d theory, the choice of contour was important for the analysis of BPS decays and the associated walls of marginal stability. These decays happened precisely when the contour crossed a pole related to the decay. These effects did not affect the power series expansion for the entropy, but were exponentially small corrections in the degeneracy formula.

In five dimensions, it is expected from a supergravity analysis that there are no such decays corresponding to real codimension one walls [39]. Note in this context that the purely v dependent factors in the function $f^{\rm KK}$ which have a zero at $v=0$. These poles therefore do not exist in the 5d partition function. It would be interesting to analyze in more detail all the poles of the partition function in the 5d theory. However, for the purpose of computing power law corrections to the entropy our analysis is sufficient.

4.1 Saddle point approximation

We can solve the integral (4.12) in two steps as in [24, 27, 28]. First, we notice that the dominant pole of the expression $1/\Phi_{10}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$ is *not* factored out by the function $f^{\rm KK}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})$. We can therefore do a contour integral around this pole and the residue is an integral over two remaining coordinates. This can be approximated by the saddle point method to give an asymptotic expansion. We follow the method of [24, 27] of which we present some relevant details in appendix §A. The actual evaluation only relies on the fact that the charges n, Q_1Q_5, l are large and not on the relative magnitude of the two charges.[3]

We are interested in the answer to first order beyond the large charge limit, and to this order it is given by

$$S^{5d}_{\rm stat} = S_0 + S_1\,, \tag{4.13}$$

which is to be evaluated at its extremum. The classical (S_0) and first correction to the large charge limit (S_1) are

$$\begin{aligned} S_0 &= -2\pi i\widetilde{\rho}n - 2\pi i\widetilde{\sigma}(Q_1Q_5+1) + 2\pi i\left(\frac{1}{2}-\tilde{v}\right)l\,, &(4.14)\\ S_1 &= 12\ln\widetilde{\sigma} - \ln\eta^{24}(\rho) - \ln\eta^{24}(\sigma) + \ln f^{\rm KK}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v})\,, &(4.15)\end{aligned}$$

with

$$\widetilde{\rho} = \frac{\rho\sigma}{\rho+\sigma}, \quad \widetilde{\sigma} = -\frac{1}{\rho+\sigma}, \quad \widetilde{v} = \frac{1}{2} - \sqrt{\frac{1}{4}+\widetilde{\rho}\widetilde{\sigma}}\ . \tag{4.16}$$

Since we are interested in the answer to only the first order beyond the large charge limit, we can extremize only the classical part S_0 and evaluate the full expression (4.13) at those values. By extremizing the classical functional S_0 we obtain

$$\begin{aligned} \widetilde{\rho} &= \frac{i}{2}\frac{Q_1Q_5+1}{\sqrt{Q^3-J^2}}\,,\\ \widetilde{\sigma} &= \frac{i}{2}\frac{n}{\sqrt{Q^3-J^2}}\,, \end{aligned} \tag{4.17}$$

where[4]

$$Q^3 - J^2 \equiv (Q_1Q_5+1)n - l^2/4\ . \tag{4.18}$$

Plugging (4.17) in (4.14)–(4.15) gives

$$S_0(Q_1,Q_5,n) = 2\pi\sqrt{Q^3-J^2} \tag{4.19}$$

---

[3] This fact was also used for computing the four dimensional black hole entropy in a region where Q^2, P^2 are large, and one was much larger than the other [24].

[4] The shift of one in Q_1Q_5 is not important to sub-leading order in the black hole regime $Q_1Q_5 \gg n$, note the difference with (3.18). This shift will be important in the Cardy regime which we discuss below.

and

$$
\begin{aligned}
S_1(Q_1,Q_5,n) = & -\pi\frac{n}{\sqrt{Q^3-J^2}} - 24\ln\eta\left(\frac{l+i2\sqrt{Q^3-J^2}}{2n}\right) \\
& -24\ln\eta\left(\frac{-l+i2\sqrt{Q^3-J^2}}{2n}\right) + 18\ln\eta\left(\frac{iQ_1Q_5}{2\sqrt{Q^3-J^2}}\right) \\
& +2\ln\vartheta_1\left(\frac{1}{2}-\frac{il}{4\sqrt{Q^3-J^2}}, \frac{iQ_1Q_5}{2\sqrt{Q^3-J^2}}\right)+\dots\,. \qquad (4.20)
\end{aligned}
$$

4.2 Supergravity limit

In the limit where all the charges (n, Q_1, Q_5, l) are large and scale uniformly, we can use the expansion of the functions $\eta(\tau)$, $\vartheta_1(v,\tau)$ (appendix §B) and after dropping higher terms we get

$$
\begin{aligned}
S_1(Q_1,Q_5,n) &= 4\pi\frac{\sqrt{Q^3-J^2}}{n} - \pi\frac{Q_1Q_5}{\sqrt{Q^3-J^2}}+\dots \\
&= 3\pi\sqrt{\frac{Q_1Q_5}{n}}+\dots \qquad (4.21)
\end{aligned}
$$

Combining (4.19) and (4.21), the full entropy formula reads

$$
S^{5d}(Q_1,Q_5,n) = 2\pi\sqrt{Q_1Q_5n}\left(1+\frac{3}{2n}-\frac{l^2}{8Q_1Q_5n}\right)+\dots \qquad (4.22)
$$

We see that this agrees with the macroscopic result (3.19) in the same regime of large charges.

4.3 Cardy limit

In the opposite Cardy limit, when $n \gg Q_1Q_5$, and $Q^3-J^2 \gg 1$ we can also expand the result (4.20) to sub-leading order. In order to do that, we first need to use the modular transformation properties of the various functions (appendix §B)

$$
\begin{aligned}
S_1(Q_1,Q_5,n) = & -\pi\frac{n}{\sqrt{Q^3-J^2}} - 24\ln\eta\left(\frac{l+i2\sqrt{Q^3-J^2}}{2Q_1Q_5}\right) \\
& -24\ln\eta\left(\frac{-l+i2\sqrt{Q^3-J^2}}{2Q_1Q_5}\right) + 18\ln\eta\left(i\frac{2\sqrt{Q^3-J^2}}{Q_1Q_5}\right) \\
& +2\ln\vartheta_1\left(-i\frac{2\sqrt{Q^3-J^2}}{Q_1Q_5}\left[\frac{1}{2}-\frac{il}{4\sqrt{Q^3-J^2}}\right], i\frac{2\sqrt{Q^3-J^2}}{Q_1Q_5}\right) \\
& +2\pi\frac{2n}{\sqrt{Q^3-J^2}}\left[\frac{1}{2}-\frac{l}{i4\sqrt{Q^3-J^2}}\right]^2+\dots \qquad (4.23)
\end{aligned}
$$

Dropping terms of higher order in Q_1Q_5/n in (4.23) we get

$$\begin{aligned} S_1(Q_1,Q_5,n) &= 4\pi\frac{n}{\sqrt{Q^3-J^2}} - 3\pi\frac{\sqrt{Q^3-J^2}}{Q_1Q_5} \\ &\quad +\pi\frac{n}{\sqrt{Q^3-J^2}} - \pi\frac{n}{\sqrt{Q^3-J^2}} - \pi\frac{n}{\sqrt{Q^3-J^2}} + \ldots \\ &= 0 + \mathcal{O}\left(\frac{1}{\sqrt{Q_1Q_5n - l^2/4}}\right) + \ldots, \end{aligned} \tag{4.24}$$

which finally allows us to write the entropy as

$$S^{5d}(Q_1,Q_5,n) = 2\pi\sqrt{(Q_1Q_5+1)n - l^2/4}\left(1+\mathcal{O}\left(\frac{1}{Q_1Q_5n - l^2/4}\right)\right) + \ldots \tag{4.25}$$

Note that unlike in the other limit, all the terms suppressed by $1/Q_1Q_5$ have dropped away, and the first sub-leading term is suppressed by $1/(Q^3-J^2)$. This is exactly in agreement with the more familiar Jacobi-Rademacher expansion to the same order which we have sketched in appendix §B.1.

5 Clarifying the 4d-5d lift

The 4d-5d lift [26, 40–43] is a relation between a black hole in five dimensions carrying three gauge charges plus angular momentum, and a black hole in four dimensions carrying the above charges and in addition, a unit[5] Taub-NUT charge. The angular momentum in the five dimensions becomes a mometum along the Taub-NUT circle at infinity in four dimensions. On application to a rotating BMPV black hole preserving 1/4 supersymmetry, the 5d black hole can be related to a four dimensional 1/4 dyonic black hole. This relation can be used to derive an exact counting formula for 1/4 BPS dyons in $\mathcal{N}=4$ string theory [23, 24].

As a consequence of the attractor mechanism, the entropy of extremal black holes is independent of asymptotic value of moduli. By tuning one of these moduli, one can make the curvature of the Taub-NUT space large or small. Therefore it seems reasonable to relate the entropy of 4d dyonic black holes with 5d black holes and the leading order prescription [23] was

$$S^{4d}(Q_1Q_5+1,n,l) = S^{5d}(Q_1Q_5,n,l)\,, \tag{5.1}$$

This equation however, will receive corrections[6] at sub-leading order

$$S^{4d}(Q_1,Q_5,n,l) = S^{5d}(Q_1,Q_5,n,l)\left(1+\frac{c_1}{Q^2}+\ldots\right)\,. \tag{5.2}$$

The computation of these corrections boils down to computing the sub-leading corrections to the 5d black hole entropy and comparing with the known sub-leading corrections to the 4d black hole entropy. The results of the previous sections fill in this gap, and we can now

[5]This has been extended recently to the case when there are multiple KK monopoles [25].
[6]These corrections are not related to the shift in the charges in (5.1).

explain the origin of the small difference in the 4d and the 5d black hole entropy both from the microscopic and macroscopic viewpoints.

In the regime of charges that all the charges are large and scaled equally, the 5d entropy is (3.17), (4.22)

$$S^{5d}(Q_1, Q_5, n, l) = 2\pi\sqrt{Q_1 Q_5 n}\left(1 + \frac{3}{2n} - \frac{l^2}{8Q_1Q_5n}\right) + \ldots \tag{5.3}$$

In the same limit, the corresponding 4d black hole with one additional Taub-NUT charge is [see the review [28] and references therein]

$$S^{4d}(Q_1, Q_5, n, l) = 2\pi\sqrt{Q_1 Q_5 n}\left(1 + \frac{2}{n} - \frac{l^2}{8Q_1Q_5n}\right) + \ldots \tag{5.4}$$

The discrepancy between the two expressions is essentially accounted for by the Taub-NUT space whose small effects remain at all values of the moduli. The interesting fact is that the actual micro and macro mechanisms are different. As we explain below, in the microscopic theory, the Taub-NUT space gives rise to additional bound states, which changes the degeneracy function, whereas in the macroscopic formalism, the Taub-NUT space changes the final value of entropy because of a Chern-Simon coupling in the effective action. It is a non-trivial reflection of the consistency of string theory that the two mechanisms in different regimes of parameter space account quantitatively for the same effect.

5.1 Microscopic mechanism

The microscopic setup in type IIB string theory on $K3$ has a D1-D5-p system with the D5 branes wrapping the $K3$, and the effective D1-D5 string with momentum p wrapping a circle S^1. The rest of the five dimensions is a KK monopole (Taub-NUT geometry) which asymptotes to $\mathbb{R}^{3,1} \times \widetilde{S}^1$. The branes sit at the center of the Taub-NUT space where spacetime looks like $\mathbb{R}^{4,1}$. The counting of 1/4 BPS dyons is done by looking at low energy excitations of this system. The counting problem effectively becomes a product of three decoupled systems [24] which we can paraphrase as computing the modified elliptic genus of the following 2d SCFT:

$$\begin{aligned} X^{4d} &= X^{5d} \times \sigma(TN_1) \times \sigma_L(KK - P) & (5.5)\\ X^{5d} &= \sigma(Sym^{Q_1Q_5+1}(K3)) & (5.6) \end{aligned}$$

The first factor which is a symmetric product theory which controls the 5d BPS counting problem of the D1-D5 system. The piece $\sigma(TN_1)$ describes the bound states of the center of mass of the $D1$-$D5$ with the KK monopole. The piece σ_L(KK-P) describes the bound states of the KK monopole and momentum and is a conformal field theory of 24 left-moving bosons of the heterotic string, which can be deduced from the duality between the Type-IIB KK-P system and the heterotic F1-P system. The presence of the second and third factor is crucial for establishing S-duality and the wall-crossing phenomena in 4d.

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The degeneracy of the BPS states of the theory X^{4d} is given in terms of the partition function which is the inverse of the Igusa cusp form, the unique weight 10 modular form of $\mathrm{Sp}(2,\mathbb{Z})$.

$$\Omega^{4d}(Q_1, Q_5, n, l) = \oint_{\mathcal{C}} d\rho d\sigma dv\, e^{-2i\pi(\rho n + \sigma Q_1 Q_5 + lv)} \frac{1}{\Phi_{10}(\rho, \sigma, v)}. \tag{5.7}$$

This partition function is understood by separately counting the three decoupled pieces in the formula (5.5) above. The degeneracies of the theory X^{5d} is given by a similar inverse fourier transform (4.12) with a partition function $Z(\rho, \sigma, v)$ which differs sightly from that of the 4d theory.

The discrepancy between the two partition functions (4.10) is due to the factors $\sigma(TN_1) \times \sigma_L(KK - P)$ which completes the 5d system into the 4d system. The BPS partition function of the extra piece related to the KK monopole is precisely $f^{\mathrm{KK}}(\rho, \sigma, v)$ (4.10), (4.11). Physically, most of the entropy of the dyonic black hole comes from the first factor in (5.5) which governs the 5d black hole, but a small fraction of the entropy of the 4d black hole comes from the bound states of momentum and center of mass with the KK monopole itself. This small fraction precisely accounts for the sub-leading corrections to the 4d-5d lift formula.[7]

5.2 Macroscopic mechanism

As we reviewed in section §3 the macroscopic entropy of the black hole is given by Wald's entropy formula. In principle, one should find the full black hole solution and compute (3.1) to obtain the entropy, but in the presence of higher derivative corrections this can be a very difficult task. For extremal black holes the attractor mechanism greatly simplifies the procedure, since only the value of the moduli at the horizon determines the entropy. Further the entropy function formalism gives a simplified prescription to evaluate (3.1) and obtain the black hole entropy.

In [14, 16] it was first noted that by using this procedure the sub-leading corrections to the entropy for a 4d and 5d black hole differ and the difference is due to a shift of the charges as given by (2.12). The shift is a consequence of the mixed gauge-gravitational Chern-Simons term, where the curvature of spacetime acts as a source for electric charge. In the 4d setup, the Taub-NUT space thus effectively absorbs some of the charge which is placed at the center. If we measure charge using different Gauss spheres, the measurement near the center of the taub-NUT space (5d) is different from that at infinity (4d).

6 Concluding remarks

We would like to finish the discussion by highlighting some of the implications of our results and future directions for both the microscopic and macroscopic approaches.

[7]In the limit of large charges in which we evaluate the integral, the contribution from the KK monopole piece comes purely from the ground state, and one can explicitly see the equivalence to the macroscopic mechanism already at this level of the calculation. We thank the referee for pointing this out.

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The microscopic corrections to the black hole entropy that we found agreed with the macroscopic supergravity theory with higher derivative corrections. The off-shell formalism used to derive such corrections assures that the action is supersymmetric and insensitive to field redefinitions because the off-shell algebra does not mix different orders. The match with the microscopics to this order in α' is therefore a more stringent test of string theory than in the four dimensional case.

The equivalence between Wald's formula (3.1) and the entropy function (3.3) relies on the gauge invariance of the action. In this case, the conserved charge can be identified from the near horizon data and it is defined as

$$Q_I = \frac{\partial f}{\partial e^I} \ . \tag{6.1}$$

In order to define the entropy function in the presence of Chern-Simons terms, one restores gauge invariance by first adding total derivatives to the action and then dimensionally reducing it [36]. For black holes on Taub-NUT this procedure will inevitably define a four dimensional charge and the effects on the charges from the delocalized sources due to the curvature of Taub-NUT will be overlooked. As it stands it seems as if in the presence of the mixed gauge-gravitational Chern-Simons term a five dimensional charge cannot be defined using f, and there is no extremization principle. Nevertheless, because of the attractor mechanism, (3.3) evaluated on the solution will determine the same entropy as defined by Wald's formula. It will be interesting to determine in the semiclassical theory the appropriate generalization of the entropy function that will capture delocalized effects and define an extremization procedure.

On another front, it would be interesting to see if there is a way to understand — as for the 4d case — the entropy of the 5d black hole in string theory when the various charges in the system are not equally large (but their product is large). This would necessarily involve taking into account the corrections due to worldsheet/membrane instantons which are delocalized in the five dimensions.

Perhaps the above two questions can be attacked using a generalization of the entropy function formalism to an integral over paths instead of minimization of a functional, as suggested in [28]. It is possible that the 5d D1-D5-p black hole could be once again be used as a testing ground for certain fundamental principles in string theory.

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A Some details of the evaluation of the contour and saddle point integral

In this appendix, we shall sketch some relevant details about the evaluation of the integral (4.12) which we recall here. Consider

$$\Omega^{5d}(Q_1, Q_5, n, l) = \oint_{\mathcal{C}} d\widetilde{\rho} d\widetilde{\sigma} d\widetilde{v}\, e^{-2i\pi(\widetilde{\rho}n+\widetilde{\sigma}(Q_1Q_5+1)+l\widetilde{v})}\, Z(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) \ . \tag{A.1}$$

The integral above is over the contour

$$\begin{aligned} &0 < \mathrm{Re}(\widetilde{\rho}) \le 1\,, \qquad && 0 < \mathrm{Re}(\widetilde{\sigma}) \le 1\,, \qquad && 0 < \mathrm{Re}(\widetilde{v}) \le 1\,, \\ &\mathrm{Im}(\widetilde{\rho}) \gg 1\,, && \mathrm{Im}(\widetilde{\sigma}) \gg 1\,, && \mathrm{Im}(\widetilde{v}) \gg 1\,, \end{aligned} \tag{A.2}$$

over the three coordinates, where Re and Im denote the real and imaginary parts. This defines the integration curve $\mathcal{C}$ as a 3-torus in the Siegel upper half-plane. The imaginary parts are taken to be large to guarentee convergence. As we shall see below, the dominant pole in the function is not affected, and we can therefore perform the contour integral around that pole. This gives a prescription for the contour. As mentioned in the text, it is expected that there is no dependence on the moduli in the 5d theory, and therefore there are no other poles where wall-crossing behavior occurs in the 5d integral. A precise analysis of the contour as was done in 4d [44] remains to be done.

We mostly follow [27] in the evaluation of the integral. First we need to do a contour integral in the $\widetilde{v}$ coordinate, which picks up the residue at various poles. These poles occur at zeros of the function Φ_{10} and the poles of the function f^{KK}. For large charges, the dominant contribution when the exponent takes its largest value at its saddle point. This was analyzed in [22]. When f^{KK} is not present, this dominant divisor is

$$\widetilde{\rho}\widetilde{\sigma} - \widetilde{v}^2 + \widetilde{v} = 0 \ . \tag{A.3}$$

We can check that the function (4.11)

$$f^{\mathrm{KK}}(\rho,\sigma,v) = p\,\eta^{18}(\rho)\,\vartheta_1^2(v,\rho)\,, \tag{A.4}$$

does not take away this pole, and does not alter the dominance of this pole. We can now carry out the contour integration in the variable $\widetilde{v}$ around the zero of the above divisor

$$\widetilde{v}_\pm = \frac{1}{2} \pm \Lambda(\widetilde{\rho},\widetilde{\sigma})\,, \qquad \Lambda(\widetilde{\rho},\widetilde{\sigma}) = \sqrt{\frac{1}{4} + \widetilde{\rho}\widetilde{\sigma}} \ . \tag{A.5}$$

In the contour integration, the variables $\widetilde{\rho}$ and $\widetilde{\sigma}$ are held fixed and we choose the negative value of the square root $\widetilde{v}_-$.

The modular properties of the function Φ_{10} under $\mathrm{Sp}(2,\mathbb{Z})$ allow us to factorize it around the value $\widetilde{v} = \widetilde{v}_-$. The integrand in (4.12) behaves like:

$$C \exp\left(-2\pi i(\widetilde{\rho}n + \widetilde{\sigma}(Q_1Q_5+1) + 2\widetilde{v}l)\right) \widetilde{\sigma}^{12}(\widetilde{v}-\widetilde{v}_+)^{-2}(\widetilde{v}-\widetilde{v}_-)^{-2}\eta^{-24}(\rho)\eta^{-24}(\sigma) f^{\mathrm{KK}}(\widetilde{\rho},\widetilde{\sigma},\widetilde{v}) \ . \tag{A.6}$$

Using this factorization, we can evaluate the contour integral, and then perform a saddle point analysis of the remaining integral over $(\widetilde{\rho}, \widetilde{\sigma})$. The contour integral for $\widetilde{v}$ gives

$$\Omega^{5d}(Q_1, Q_5, n, l) = (-1)^{Q.P} K \int d\widetilde{\rho}\, d\widetilde{\sigma}\, e^{X(\tilde{\rho},\widetilde{\sigma}) + \ln \Delta(\tilde{\rho},\widetilde{\sigma})}, \tag{A.7}$$

where K is a numerical constant and

$$\begin{aligned} X(\tilde{\rho}, \widetilde{\sigma}) &= -2\pi i \left(\widetilde{\rho} n + \widetilde{\sigma}(Q_1 Q_5 + 1) - \Lambda(\widetilde{\rho}, \widetilde{\sigma}) l \right) \\ &\qquad + 12 \ln \widetilde{\sigma} - \ln \eta^{24}(\rho) - \ln \eta^{24}(\sigma) + \ln f^{\mathrm{KK}}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}_-)\,, \end{aligned} \tag{A.8}$$

$$\begin{aligned} \Delta(\tilde{\rho}, \widetilde{\sigma}) &= \frac{1}{4\Lambda(\widetilde{\rho}, \widetilde{\sigma})^2} \Bigg[-2\pi i l + 2\frac{\widetilde{v}_-}{\widetilde{\sigma}} \frac{\partial}{\partial \rho} \ln \eta^{24}(\rho) - 2\frac{\widetilde{v}_+}{\widetilde{\sigma}} \frac{\partial}{\partial \sigma} \ln \eta^{24}(\sigma) \\ &\qquad + \frac{1}{\Lambda(\widetilde{\rho}, \widetilde{\sigma})} + \left(\frac{\partial}{\partial \widetilde{v}} \ln f^{\mathrm{KK}}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) \right)_{\widetilde{v} = \widetilde{v}_-} \Bigg]. \end{aligned} \tag{A.9}$$

The above expression has to be evaluated at the saddle point. In the large charge limit

$$Q_1 Q_5 \gg 0\,, \quad n \gg 0\,, \quad \sqrt{Q^3 - J^2} \gg 0\,, \tag{A.10}$$

the saddle point of (A.7) is well approximated by the first line of (A.8), and in this limit it is located at

$$\widetilde{\rho} = \frac{i}{2} \frac{Q_1 Q_5 + 1}{\sqrt{Q^3 - J^2}}\,, \quad \widetilde{\sigma} = \frac{i}{2} \frac{n}{\sqrt{Q^3 - J^2}}\,, \tag{A.11}$$

which is the extremum given by (4.17). We can now estimate the above expressions (A.8), (A.9) for the two relevant limits used in §4.1: the Supergravity regime, i.e. $Q_1 Q_5 \sim N^2$, $n \sim N$ and $l \sim N$; and the Cardy regime, i.e. $Q_1 Q_5 \sim N$, $n \sim N^2$ and $l \sim N$ with $N \gg 1$. For both regimes, (A.8) evaluated at (A.11) behaves as

$$X(\widetilde{\rho}, \widetilde{\sigma}) = (\mathrm{const}) N^{3/2} + (\mathrm{const}) N^{1/2} + \mathcal{O}(1)\,, \tag{A.12}$$

where the precise values of the constants are computed in section §4.1 for each regime. Next, the subleading behavior of (A.8), (A.9) relevant for the saddle point approximation are

$$\begin{aligned} \ln \Delta &= -\ln |l| + \ln \left(\frac{1}{4} + \widetilde{\rho} \widetilde{\sigma} \right) + \mathcal{O}(1)\,, \\ \ln \left(\det |\partial^2 X| \right) &= \ln |l| - \ln \left(\frac{1}{4} + \widetilde{\rho} \widetilde{\sigma} \right) + \mathcal{O}(1)\,, \end{aligned} \tag{A.13}$$

where $\partial^2 X$ is the matrix of second derivatives of X with respect to $\widetilde{\rho}$ and $\widetilde{\sigma}$. Here, $\mathcal{O}(1)$ refers to the above large charge expansion, and refers to the scaling as a function of N. Finally, integrating (A.7) using the saddle point approximation, the statistical entropy is given by

$$\begin{aligned} S^{5d}_{\mathrm{stat}} &= \ln \left(\Omega^{5d}(Q_1, Q_5, n, l) \right) \\ &= -2\pi i \widetilde{\rho} n - 2\pi i \widetilde{\sigma}(Q_1 Q_5 + 1) + 2\pi i \left(\frac{1}{2} - \tilde{v} \right) l \\ &\qquad + 12 \ln \widetilde{\sigma} - \ln \eta^{24}(\rho) - \ln \eta^{24}(\sigma) + \ln f^{\mathrm{KK}}(\widetilde{\rho}, \widetilde{\sigma}, \widetilde{v}) + \mathcal{O}(1)\,, \end{aligned} \tag{A.14}$$

evaluated at (A.11). From the above analysis, $S^{5d}_{\rm stat}$ allows a systematic expansion for both the Supergravity and Cardy regime.

Note that the function $f^{\rm KK}$ does not have any poles in the interior of the region we are considering, but has many zeroes. These zeroes do not include the divisor (A.3). Therefore the dominant pole of Φ_{10}^{-1} remains the dominant pole of the 5d integrand Z. Note however that $f^{\rm KK}$ does have a zero at $\widetilde{v}=0$ which takes away the pole at the same value of the function Φ_{10}^{-1}. This means that there is no wall crossing behaviour in the five dimensional theory due to this pole. For the evaluation of the integral, these observations mean that the presence of the function $f^{\rm KK}$ changes the analysis only through its appearance in the entropy function (4.13) to be extremized.

B The Jacobi η and ϑ functions and their properties

We define

$$q = e^{2\pi i\tau}\,, \quad y = e^{2\pi i v}\,. \tag{B.1}$$

The Jacobi eta function is defined as

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)\,. \tag{B.2}$$

The odd Jacobi theta function is

$$\vartheta_1(v,\tau) = -2q^{\frac{1}{8}}\,\sin(\pi v)\prod_{m=1}^{\infty}(1-q^m)(1-q^m y)(1-q^m y^{-1})\,. \tag{B.3}$$

For large imaginary values of $\tau = it$, $t\to\infty$, we have $q\to 0$ most of the terms in the product become unity and these functions admit an expansion of the form

$$\eta(\tau) = -\frac{\pi}{12}t + \ldots \tag{B.4}$$

These functions satisfy the modular properties:

$$\begin{aligned} \eta\Big(-\frac{1}{\tau}\Big) &= \sqrt{-i\tau}\,\eta(\tau) \\ \vartheta_1\Big(\frac{v}{\tau},-\frac{1}{\tau}\Big) &= i\sqrt{-i\tau}\,e^{i\pi v^2/\tau}\vartheta_1(v,\tau)\,. \end{aligned} \tag{B.5}$$

For the ϑ function, the expansion depends on the value of v compared to τ, but similar expansions are possible.

B.1 The Jacobi-Rademacher expansion

The Jacobi-Rademacher expansion [29, 30] is a very powerful (exact) expansion containing both power law and exponential corrections to the Cardy estimate. Here, we are only interested in the first power law correction, which can be estimated by using a Jacobi modular transformation and a saddle point expansion.

The counting of 1/4 BPS states of the D1-D5 system on $K3$ is summarized by the elliptic genus of the 2d SCFT $Sym^k(K3)$ with $k = Q_1Q_5 + 1$. This elliptic genus can be expanded in a theta function decomposition

$$\chi(Sym^k(K3);\tau,z) = -\sum_{l=-k+1}^{k}\sum_{n\in\mathbb{Z}} c(n,\mu)\, q^{n-l^2/4k}\,\theta_{l,k}(z,\tau) \tag{B.6}$$

$$\equiv -\sum_{l=-k+1}^{k} h_l(\tau)\,\theta_{l,k}(z,\tau)\ . \tag{B.7}$$

We write

$$h_l(\tau) = \sum_{m=0}^{\infty} H_l(m)\, q^{m-\frac{l^2}{4k}}\ . \tag{B.8}$$

We can estimate the value of the coefficients $H_l(n)$ when $n \gg k$ using the Cardy's formula after doing a modular transformation on the elliptic genus and performing a saddle point expansion

$$H_l(n) = (const)\, e^{\pi i l}\frac{k}{(4nk-l^2)^{\frac{1}{2}}}\, I_{3/2}(2\pi\sqrt{nk - l^2/4}) + \dots\ , \tag{B.9}$$

where the dots denote terms which are exponentially suppressed. There is actually an exact formula which captures all the exponentially sub-leading terms [29, 30] which we don't need here.

Here $I_{3/2}$ is the modified Bessel function of the first type. The index 3/2 appears because the weight of the vector valued modular form $H_\mu(z)$ is $w = -\frac{1}{2}$. Note that by definition, the elliptic genus has weight zero, but the θ functions have weight $+\frac{1}{2}$, so the functions H_μ have weight $-\frac{1}{2}$. This function in fact has an expression in terms of elementary functions

$$I_{3/2}(z) = \sqrt{\frac{2}{\pi z}}\left(\cosh(z) - \frac{\sinh(z)}{z}\right)\ . \tag{B.10}$$

The entropy is the logarithm of the degeneracy $H_\mu(n)$. With $k = Q_1Q_5 + 1$, we have $z = 2\pi\sqrt{(Q_1Q_5+1)n - \ell^2/4}$. The entropy is equal to

$$S^{5d} = \ln\left(e^z\left[1 - \frac{1}{z}\right]\right) + \dots \tag{B.11}$$

$$= 2\pi\sqrt{(Q_1Q_5+1)n - l^2/4}\left(1 + \frac{1}{4\pi^2(Q_1Q_5n - l^2/4)} + \dots\right)\ , \tag{B.12}$$

which is in agreement with (4.25).

References

[1] K. Behrndt et al., *Higher-order black-hole solutions in $N = 2$ supergravity and Calabi-Yau string backgrounds*, *Phys. Lett.* **B 429** (1998) 289 [hep-th/9801081] [SPIRES].

[2] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Corrections to macroscopic supersymmetric black-hole entropy*, *Phys. Lett.* **B 451** (1999) 309 [hep-th/9812082] [SPIRES].

[3] G. Lopes Cardoso, B. de Wit and T. Mohaupt, *Macroscopic entropy formulae and non-holomorphic corrections for supersymmetric black holes*, Nucl. Phys. **B 567** (2000) 87 [hep-th/9906094] [SPIRES].

[4] G. Lopes Cardoso, B. de Wit, J. Kappeli and T. Mohaupt, *Stationary BPS solutions in $N = 2$ supergravity with R^2 interactions*, JHEP **12** (2000) 019 [hep-th/0009234] [SPIRES].

[5] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, Phys. Rev. **D 70** (2004) 106007 [hep-th/0405146] [SPIRES].

[6] A. Sen, *Black holes, elementary strings and holomorphic anomaly*, JHEP **07** (2005) 063 [hep-th/0502126] [SPIRES].

[7] A. Sen, *Entropy function for heterotic black holes*, JHEP **03** (2006) 008 [hep-th/0508042] [SPIRES].

[8] B. Sahoo and A. Sen, *α'-corrections to extremal dyonic black holes in heterotic string theory*, JHEP **01** (2007) 010 [hep-th/0608182] [SPIRES].

[9] R.M. Wald, *Black hole entropy is the Nöther charge*, Phys. Rev. **D 48** (1993) 3427 [gr-qc/9307038] [SPIRES].

[10] V. Iyer and R.M. Wald, *Some properties of Nöther charge and a proposal for dynamical black hole entropy*, Phys. Rev. **D 50** (1994) 846 [gr-qc/9403028] [SPIRES].

[11] V. Iyer and R.M. Wald, *A comparison of Nöther charge and euclidean methods for computing the entropy of stationary black holes*, Phys. Rev. **D 52** (1995) 4430 [gr-qc/9503052] [SPIRES].

[12] A. Sen, *Black hole entropy function and the attractor mechanism in higher derivative gravity*, JHEP **09** (2005) 038 [hep-th/0506177] [SPIRES].

[13] K. Hanaki, K. Ohashi and Y. Tachikawa, *Supersymmetric completion of an R^2 term in five-dimensional supergravity*, Prog. Theor. Phys. **117** (2007) 533 [hep-th/0611329] [SPIRES].

[14] A. Castro, J.L. Davis, P. Kraus and F. Larsen, *5D black holes and strings with higher derivatives*, JHEP **06** (2007) 007 [hep-th/0703087] [SPIRES].

[15] M. Alishahiha, *On R^2 corrections for 5D black holes*, JHEP **08** (2007) 094 [hep-th/0703099] [SPIRES].

[16] A. Castro, J.L. Davis, P. Kraus and F. Larsen, *Precision entropy of spinning black holes*, JHEP **09** (2007) 003 [arXiv:0705.1847] [SPIRES].

[17] A. Castro, J.L. Davis, P. Kraus and F. Larsen, *String theory effects on five-dimensional black hole physics*, Int. J. Mod. Phys. **A 23** (2008) 613 [arXiv:0801.1863] [SPIRES].

[18] M. Guica, L. Huang, W.W. Li and A. Strominger, *R^2 corrections for 5D black holes and rings*, JHEP **10** (2006) 036 [hep-th/0505188] [SPIRES].

[19] M. Cvitan, P.D. Prester and A. Ficnar, *α'^2-corrections to extremal dyonic black holes in heterotic string theory*, JHEP **05** (2008) 063 [arXiv:0710.3886] [SPIRES].

[20] P.D. Prester and T. Terzic, *α'-exact entropies for BPS and non-BPS extremal dyonic black holes in heterotic string theory from ten-dimensional supersymmetry*, JHEP **12** (2008) 088 [arXiv:0809.4954] [SPIRES].

[21] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*, *Phys. Lett.* **B 379** (1996) 99 [hep-th/9601029] [SPIRES].

[22] R. Dijkgraaf, E.P. Verlinde and H.L. Verlinde, *Counting dyons in $N = 4$ string theory*, *Nucl. Phys.* **B 484** (1997) 543 [hep-th/9607026] [SPIRES].

[23] D. Shih, A. Strominger and X. Yin, *Recounting dyons in $N = 4$ string theory*, *JHEP* **10** (2006) 087 [hep-th/0505094] [SPIRES].

[24] J.R. David and A. Sen, *CHL dyons and statistical entropy function from D1-D5 system*, *JHEP* **11** (2006) 072 [hep-th/0605210] [SPIRES].

[25] A. Dabholkar, J. Gomes and S. Murthy, *Counting all dyons in $N = 4$ string theory*, arXiv:0803.2692 [SPIRES].

[26] D. Gaiotto, A. Strominger and X. Yin, *New connections between 4D and 5D black holes*, *JHEP* **02** (2006) 024 [hep-th/0503217] [SPIRES].

[27] D.P. Jatkar and A. Sen, *Dyon spectrum in CHL models*, *JHEP* **04** (2006) 018 [hep-th/0510147] [SPIRES].

[28] A. Sen, *Black hole entropy function, attractors and precision counting of microstates*, *Gen. Rel. Grav.* **40** (2008) 2249 [arXiv:0708.1270] [SPIRES].

[29] R. Dijkgraaf, J.M. Maldacena, G.W. Moore and E.P. Verlinde, *A black hole Farey tail*, hep-th/0005003 [SPIRES].

[30] J. Manschot and G.W. Moore, *A modern Farey tail*, arXiv:0712.0573 [SPIRES].

[31] M.J. Duff, J.T. Liu and R. Minasian, *Eleven-dimensional origin of string/string duality: a one-loop test*, *Nucl. Phys.* **B 452** (1995) 261 [hep-th/9506126] [SPIRES].

[32] S. Ferrara, R. Kallosh and A. Strominger, *$N = 2$ extremal black holes*, *Phys. Rev.* **D 52** (1995) 5412 [hep-th/9508072] [SPIRES].

[33] S. Ferrara and R. Kallosh, *Supersymmetry and attractors*, *Phys. Rev.* **D 54** (1996) 1514 [hep-th/9602136] [SPIRES].

[34] S. Ferrara and R. Kallosh, *Universality of supersymmetric attractors*, *Phys. Rev.* **D 54** (1996) 1525 [hep-th/9603090] [SPIRES].

[35] A.H. Chamseddine, S. Ferrara, G.W. Gibbons and R. Kallosh, *Enhancement of supersymmetry near 5D black hole horizon*, *Phys. Rev.* **D 55** (1997) 3647 [hep-th/9610155] [SPIRES].

[36] B. Sahoo and A. Sen, *BTZ black hole with Chern-Simons and higher derivative terms*, *JHEP* **07** (2006) 008 [hep-th/0601228] [SPIRES].

[37] C. Vafa, *Instantons on D-branes*, *Nucl. Phys.* **B 463** (1996) 435 [hep-th/9512078] [SPIRES].

[38] R. Dijkgraaf, G.W. Moore, E.P. Verlinde and H.L. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, *Commun. Math. Phys.* **185** (1997) 197 [hep-th/9608096]. [SPIRES].

[39] I. Bena and P. Kraus, *Microstates of the D1-D5-KK system*, *Phys. Rev.* **D 72** (2005) 025007 [hep-th/0503053] [SPIRES].

[40] H. Elvang, R. Emparan, D. Mateos and H.S. Reall, *Supersymmetric 4D rotating black holes from 5D black rings*, *JHEP* **08** (2005) 042 [hep-th/0504125] [SPIRES].

[41] D. Gaiotto, A. Strominger and X. Yin, *5D black rings and 4D black holes*, *JHEP* **02** (2006) 023 [hep-th/0504126] [SPIRES].

[42] I. Bena, P. Kraus and N.P. Warner, *Black rings in Taub-NUT*, *Phys. Rev.* **D 72** (2005) 084019 [hep-th/0504142] [SPIRES].

[43] K. Behrndt, G. Lopes Cardoso and S. Mahapatra, *Exploring the relation between 4D and 5D BPS solutions*, *Nucl. Phys.* **B 732** (2006) 200 [hep-th/0506251] [SPIRES].

[44] M.C.N. Cheng and E. Verlinde, *Dying dyons don't count*, *JHEP* **09** (2007) 070 [arXiv:0706.2363] [SPIRES].