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SAMPLED DATA CONTROL OF FLEXIBLE STRUCTURES USING CONSTANT GAIN VELOCITY FEEDBACK[1]

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Abstract

A framework is developed for sampled data control of flexible structures, in terms of discrete time recursive equations in second order form. This framework is used to analyze the sampled data control scheme where the loop is closed using constant gain output velocity feedback. It is well known that the closed loop is stable if colocated velocity feedback with symmetric and positive definite feedback gain is used, so long as the sampling rate is sufficiently high. In this work it is shown that the closed loop can be stablized using sampled data output velocity feedback for arbitrary sampling rate. Our approach leads to explicit stability conditions in terms of the feedback gain matrix, the sampling time, and the matrices describing the flexible structure.

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1. Models for Sampled Data Controlled Flexible Structures

A sampled data controlled flexible structure can be defined as a distributed parameter system, where the structure input is the output of an ideal zero order hold and the structure output is sampled. Although distributed parameter models typically involve infinite dimensional variables, our analysis is based on the finite dimensional model

$$M\ddot{x} + Kx = Bu \quad . \tag{1}$$

For simplicity in the subsequent development no structural damping is included. The structural displacement vector $x = (x_1,\ldots,x_n)$ and the force input vector $u = (u_1,\ldots,u_m)$. The mass matrix M and the structural stiffness matrix K are assumed symmetric and positive definite. Throughout, we consider velocity output of the form

$$y = C\dot{x} \tag{2}$$

where the velocity output vector $y = (y_1,\ldots,y_m)$. The input influence matrix B and output influence matrix C are assumed to be dimensionless.

The structure input u is defined in terms of the input sequence u_k by the ideal zero order hold relation

$$u(t) = u_k,\ kT \le t < kT + T \quad . \tag{3}$$

The output sequence y_k is defined in terms of the structure output y by the ideal sampling relation.

$$y_k = y(kT) \tag{4}$$

The fixed value $T > 0$ is the constant sampling time. This open loop sampled data controlled structure can be viewed as a discrete time system with input sequence u_k and output sequence y_k, where $k = 0,1,\ldots$

Let Φ be n × n nonsingular modal matrix and let Ω^2 be n × n diagonal modal frequency matrix [1] satisfying

$$\Phi^T M\Phi = I, \quad \Phi^T K\Phi = \Omega^2 = diag\,(\omega_1^2,\ldots,\omega_n^2) \quad . \tag{5}$$

Introduce the coordinate change

$$x = \Phi\eta \tag{6}$$

so that (1), (2) can be written in modal coordinates as

$$\ddot{\eta} + \Omega^2\eta = \Phi^T Bu \quad , \tag{7}$$

$$y = C\Phi\,\dot{\eta} \quad . \tag{8}$$

It is an easy task to solve the vector equation (7), using the constraint (3), to obtain

$$\begin{aligned}\eta_{k+1} = {} & \cos\Omega T\;\eta_k + \Omega^{-1}\sin\Omega T\;\dot{\eta}_k \\ & + \Omega^{-2}(I - \cos\Omega T)\;\Phi^T Bu_k \quad ,\end{aligned} \tag{9}$$

$$\begin{aligned}\dot{\eta}_{k+1} = {} & -\Omega\sin\Omega T\;\eta_k + \cos\Omega T\;\dot{\eta}_k \\ & + \Omega^{-1}\sin\Omega T\;\Phi^T Bu_k \quad ,\end{aligned} \tag{10}$$

where $\eta_k = \eta(kT)$, $\dot{\eta} = \dot{\eta}(KT)$, and $\sin\Omega T =$ diag $(\sin\omega_1 T,\ldots,\sin\omega_n T)$, $\cos\Omega T =$ diag $(\cos\omega_1 T,\ldots,\cos\omega_n T)$ [2].

Although the first order recursive equations (9), (10) could be used it is more convenient for our purposes to make use of a second order recursive equations in $\dot{\eta}_k$ alone

$$\dot{\eta}_{k+1} - 2\cos\Omega T\ \dot{\eta}_k + \dot{\eta}_{k-1} = \Omega^{-1}\sin\Omega T\ \Phi^T B(u_k - u_{k-1}) \tag{11}$$

$$y_k = C\Phi\dot{\eta}_k \tag{12}$$

The modal equations (11), (12) form the basis for our subsequent analysis. It is natural to make use of the recursive equation for $\dot{\eta}_k$ alone in considering velocity feedback systems.

It should be noted that relations (11), (12) involve no numerical approximation; they are valid for any sampling time $T > 0$.

Constant gain output velocity feedback has been studied extensively for analog controlled structures. Our interest is in use of constant gain output velocity feedback for sampled data controlled structures.

Consider the closed loop sampled data controlled structure defined by (11), (12), using the control input sequence

$$u_k = -Gy_k \tag{13}$$

where G is a constant $m \times m$ feedback gain matrix. Substituting (13) in (11), (12), a closed loop recursive equation is obtained

$$\begin{aligned} \dot{\eta}_{k+1} &- \left[2\cos\Omega T - \Omega^{-1}\sin\Omega T\ \Phi^T BGC\Phi\right]\dot{\eta}_k \\ &+ \left[I - \Omega^{-1}\sin\Omega T\ \Phi^T BGC\Phi\right]\dot{\eta}_{k-1} = 0\ . \end{aligned} \tag{14}$$

The closed loop characteristic equation can be written as

$$\begin{aligned} d(T,z) = \det&\left[z^2 I - (2\cos\Omega T - \Omega^{-1}\sin\Omega T\ \Phi^T BGC\Phi)z \right.\\ &\left. + (I - \Omega^{-1}\sin\Omega T\ \Phi^T BGC\Phi\right] = 0\ . \end{aligned} \tag{15}$$

The objective of constant gain velocity feedback control is to make the closed loop as described by (14) geometrically stable, i.e. to make the closed loop characteristic zeros lie inside the unit disk.

We use equation (14) as basis for our subsequent analysis of the closed loop. If $\sin\Omega T$ is nonsingular the following implications hold: if $\dot{\eta}_k \to 0$ as $k \to \infty$ then necessarily $u_k \to 0$ as $k \to \infty$ and $\eta_k \to 0$ as $k \to \infty$; consequently $\dot{x}_k \to 0$ and $x_k \to 0$ as $k \to \infty$.

2. Conditions for Closed Loop Stabilization

Recall the following results for constant gain output velocity analog feedback control where $u = -Gy$. If colocated force actuators and velocity sensors are selected so that $C = B^T$ then the closed loop analog controlled structure is asymptotically stable if G is any symmetric, positive definite matrix, and if a certain controllability assumption is satisfied [3,4,5]. Moreover, this result does not depend on the particular values of the modal frequencies and modal functions.

We first mention a rather obvious result that if the sampled data feedback control is chosen according to the analog feedback theory the closed loop is stable for sufficiently small sampling time. The brief proof is included for completeness; it also serves as an introduction to our subsequent development.

Theorem 1.

Assume that

(a) $C = B^T$;

(b) the matrix pair Ω^2, $\Phi^T B$ is complete controllable;

(c) G is symmetric and positive definite.

The closed loop equation (14) is geometrically stable for sufficiently small sampling time $T > 0$.

Proof: Consider the associated polynomial

$$p(T,w) = \det\Big[2(I+\cos\Omega T - \Omega^{-1}\sin\Omega T \; \Phi^T BGC\Phi)\, \frac{T^2 w^2}{4} + 2\Omega^{-1}\sin\Omega T \; \Phi^T BGC\Phi \frac{Tw}{2} + 2(I - \cos\Omega T)\Big] . \tag{16}$$

Using the results in [3] the polynomial defined by

$$\lim_{T\to 0} p(T,w)\frac{1}{T^2}$$

has all zeros in left-half-plane; hence there is $\overline{T} > 0$ such that $p(T,w)$ has all zeros in left-half-plane for $0 < T < \overline{T}$. Using the bilinear transformation

$$z = \frac{1 + \frac{Tw}{2}}{1-\frac{Tw}{2}} \tag{17}$$

it follows that the zeros of $d(T,z)$ are necessarily in the unit desk; hence (14) is stable.

This result has limited application since there is no indication of the range of values of the sampling times, relative to the feedback gain matrix, required for closed loop stability. In [6] conditions are developed which, in principle, characterize a range of values of the sampling time for which the closed loop is stable. Unfortunately, the conditions depend on an a priori computable bound on an exponential matrix; computation of such a bound, in analytical terms, is not considered in [6]. Of course one could perform a numerical search, based on the characteristic polynomial $d(T,z)$ (or equivalently $p(T,w)$) for a specific case, to determine a range of values of the sampling time for which the closed loop is stable. However, for the case of colocated velocity feedback there are no known explicit conditions, in terms of the sampling time and feedback gain matrix, which guarantee stability of the closed loop sampled data system.

We now present the main result of the paper: a set of explicit conditions on the input and output influence matrices, the sampling time and the feedback gain matrix for which the closed loop sampled data controlled structure is stable. The key idea is to suitably modify the assumptions so that the approach used in the proof of Theorem 1 can be followed.

Theorem 2.

Assume that $\sin\Omega T$ is nonsingular and

(a) the matrix pair $[I + \cos\Omega T]^{-1}\,[I - \cos\Omega T]$, $\Phi^T B$ is completely controllable,

(b) The matrix $\Omega^{-1}\sin\Omega T\ \Phi^T BGC\Phi$ is symmetric and positive definite;

(c) the matrix $I + \cos\Omega T - \Omega^{-1}\sin\Omega T\ \Phi^T BCG\Phi$ is symmetric and positive definite.

The closed loop equation (14) is geometrically stable.

Proof: The assumptions, as in the proof of Theorem 1, guarantee that the zeros of $p(T,w)$ defined in equation (16) are in the left-half of the complex plane. The bilinear transformation defined in (17) guarantees that the zeros of $d(T,z)$ are necessarily inside the unit disk in the complex plane. Hence equation (14) is stable.

This general result gives sufficient conditions on the influence matrices B and C, the feedback gain matrix G, and the sampling time T. for which the closed loop system is stable.

A few general statements can be made regarding satisfaction of conditions (b) and (c) of Theorem 2. First, note that it is the matrix product BCG which appears in the conditions; in general this matrix product is required to be neither symmetric nor positive definite. Also, for fixed influence matrices B and C conditions (b) and (c) of Theorem 2 can be viewed as characterizing the relation between the feedback gain matrix G and the sampling time T for which the closed loop is stable. Informally, note that condition (c) implies that if the sampling time $T > 0$ is "small" then the feedback gain matrix G may be "large", while if the sampling time is "large" the feedback gain must be "small". In addition as the sampling time satisfies $T \to 0$ condition (c) becomes trivially satisfied and condition (b) implies that BGC tends toward a symmetric matrix, just as required in Theorem 1. Satisfaction of the conditions in Theorem 2 do require explicit knowledge of the modal data. Note also that for fixed influence matrices B and C, e.g. $C = B^T$, and a fixed sampling time T there is no guarantee that there is a feedback gain matrix G which satisfies the above stability conditions.

There are two special cases where the existence of a stabilizing feedback gain matrix can be guaranteed. These two cases are indicated in the following two corollaries.

Corollary 1

Assume that $\sin\Omega T$ is nonsingular and

(a) the matrix pair $[I + \cos\Omega T]^{-1}\,[I - \cos\Omega T],\ \Phi^T B$ is completely controllable;

(b) the influence matrices C and B satisfy

$$C = B^T \Phi \sin\Omega T\ \Omega^{-1} T^{-1} \Phi^{-1} \ .$$

Then there exists a feedback gain matrix G satisfying

(c) the gain matrix G is symmetric and positive definite;

(d) the matrix $I + \cos\Omega T - T\Phi^T C^T G C \Phi$ is symmetric and positive definite

such that the closed loop equation (14) is geometrically stable.

The assumptions that B and C satisfy condition (b) implies that the force actuators and the velocity sensors be selected in a specific way; the actuators and sensors would generally **not** be co-located.

Corollary 2.

Assume that $\sin\Omega T$ is nonsingular and

(a) rank B = rank $C = n$.

Then there exists a feedback gain matrix G satisfying

(b) the matrix $\Omega^{-1}\sin\Omega T\ \Phi^T B G C \Phi$ is symmetric and positive definite;

(c) the matrix $I + \cos\Omega T - \Omega^{-1}\sin\Omega T\ \Phi^T B C G \Phi$ is symmetric and positive definite

such that the closed loop equation (14) is geometrically stable.

In this case, with at least as many actuators and sensors as there are modes to be controlled, there is no need for an explicit condition on the influence matrices. The feedback gain matrix can always be suitably chosen to satisfy the stabilization conditions. But in general the feedback gain matrix, satisfying the conditions of Corollary 2, would be neither symmetric nor positive definite.

These several sufficient conditions for stability of sampled data controlled flexible structures indicate the importance of the sampling constraint. The general results are now illustrated with an example.

3. Example

Consider the two mass and three spring connection indicated in Figure 1, with notation also given in Figure 1. This is the same example studied in [7].,

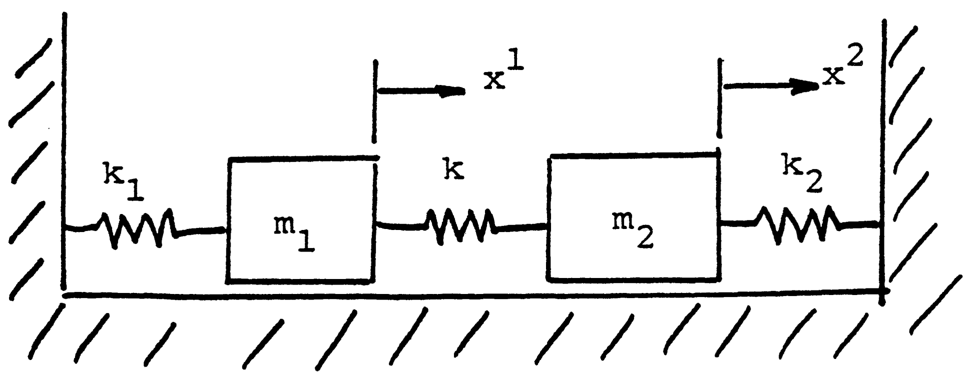

Figure 1.

where analog feedback was used to stabilize the closed loop. Our objective is to consider the use of sampled data feedback to achieve stabilization of the spring - mass system.

For simplicity,the numerical values for the masses and spring stiffnesses are chosen as $m_1 = m_2 = 1,\ k_1 = 1,\ k_2 = 4,\ k = 2$, so that

$$M = \begin{bmatrix} 1. & 0. \\ 0. & 1. \end{bmatrix} \qquad K = \begin{bmatrix} 3. & -2. \\ -2. & 6. \end{bmatrix}$$

Suppose that the control forces u_1 on mass m_1 and u_2 on mass m_2 are given by the **analog feedback** relations

$$u^1 = -g_1\dot{x}^1 - g(\dot{x}^1 - \dot{x}^2)$$

$$u^2 = -g(\dot{x}^2 - \dot{x}^1) - g_2\dot{x}^2$$

so that g_1, g_2, g can be viewed as damping parameters for three **analog dampers** (or dashpots) as shown in Figure 2. From the results in [7] it follows that the system is stable if

$$g_1 \geq 0,\ \ g_2 \geq 0,\ \ g \geq 0$$

with at least one strict inequality. Further, this conclusion does not depend on the particular numerical values of the masses and spring stiffnesses considered.

Now suppose that the control forces are given according to the **sampled data feedback** relations

$$\dot{u}_k^1 = -g_1\dot{x}_k^1 - g(\dot{x}_k^1 - \dot{x}_k^2)$$

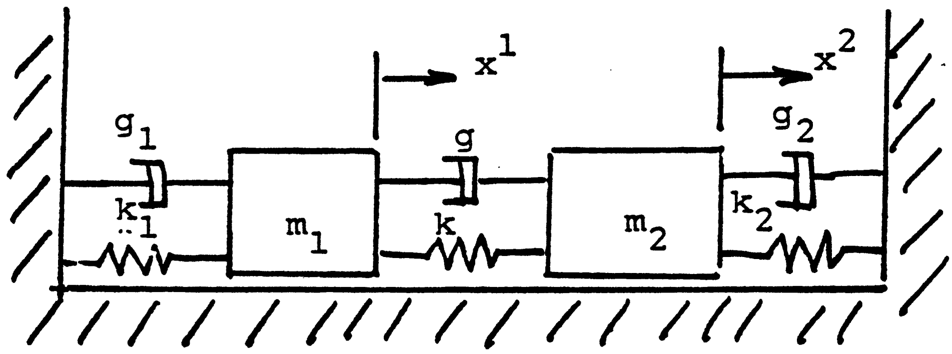

Figure 2.

$$u_k^2 = -g(\dot{x}_k^2 - \dot{x}_k^1) - g_2\dot{x}_k^2$$

corresponding to equation (13) with sampling time $T > 0$. Hence the parameters g_1, g_2, g can be viewed as damping parameters for three **digital dampers**, located as shown in Figure 2. Corollary 2 can be used to show that the spring - mass system, controlled by the indicated digital dampers, is stable if, in addition to the previously stated requirements for the analog feedback case, the feedback gains also satisfy the equality

$$2g_2 - 2g_1 - 3g = 0$$

and the inequalities

$$1. + \cos\sqrt{2}T - \frac{\sin\sqrt{2}T}{5\sqrt{2}}(4g_1 + g_2 + g) > 0,$$

$$1. + \cos\sqrt{7}T - \frac{\sin\sqrt{7}T}{5\sqrt{7}}\,(g_1 + 4g_2 + 9_g) > 0.$$

These conditions define an explicit region in the four dimensional parameter space for g_1, g_2, g, T for which the closed loop is stable.

An illustration is now given for the case where a single digital damper is located between the two masses so that equal and opposite control forces are applied to the two masses. Consider the **sampled data feedback** relation

$$u_k^1 = -u_k^2 = -g\left[c_1\dot{x}_k^1 + c_2\dot{x}_k^2\right]$$

where c_1, c_2 are the output influence coefficients which must be explicitly chosen to satisfy condition (b) of Corollary 1 with $n = 2$, $m = 1$, and

$$B = \begin{bmatrix} 1. \\ -1. \end{bmatrix} \qquad C = [c_1 \;\; c_2], \qquad G = [g].$$

From condition (b) of Corollary 1 it follows that

$$c_1 = \frac{1}{5}\left[\frac{2\sin\sqrt{2}T}{\sqrt{2}T} + \frac{3\sin\sqrt{7}T}{\sqrt{7}T}\right]$$

$$c_2 = \frac{1}{5}\left[\frac{\sin\sqrt{2}T}{\sqrt{2}T} - \frac{6\sin\sqrt{7}T}{\sqrt{7}T}\right]$$

The additional conditions of Corollary 1 require that the feedback gain satisfy $g > 0$ and that the matrix

$$\begin{bmatrix} 1 + \cos\sqrt{2}T & 0 \\ 0 & 1 + \cos\sqrt{7}T \end{bmatrix} - \frac{g}{5T}\begin{bmatrix} \frac{1}{2}\sin^2\sqrt{2}T & \frac{3}{\sqrt{14}}\sin\sqrt{2}T\sin\sqrt{7}T \\ \frac{3}{\sqrt{14}}\sin\sqrt{2}T\sin\sqrt{7}T & \frac{9}{7}\sin^2\sqrt{7}T \end{bmatrix}$$

be positive definite. Thus, on the basis of Corollary 1 the spring - mass system, controlled by the single digital damper as indicated, is stable if the above conditions on c_1, c_2 and g are satisfied. Notice the important feature that the control force from the digital damper does not depend on the relative velocity $\dot{x}_k^1 - \dot{x}_k^2$. But rather, to compensate for the sampling effects, the control force depends on the determined linear combination of the mass velocities. These conditions define an explicit region in the two dimensional parameter space for g, T for which the closed loop is stable.

It should be mentioned that the closed loop characteristic polynomial, of fourth degree with coefficients depending on the feedback gains and the sampling time, could in principle be used as a basis for stability analysis. However, the resulting necessary and sufficient conditions for stability have an exceedingly complicated dependence on the feedback gains and the sampling time. Although our conditions above are only sufficient conditions for stability, they expose rather clearly the dependence on the feedback gains and the sampling time.

4. Conclusions

We have presented two results which can serve as guidelines for choice of the feedback gains and the sampling time to guarantee that the sampled data controlled structure is stable. In Corollary 1 the stability conditions are that the feedback gain matrix be symmetric and positive definite, plus satisfy additional constraints, while the input and output influence matrices satisfy a certain matrix equation. In Corollary 2 the stability conditions are that the input and the output influence matrices have rank n, while the feedback gain matrix satisfy conditions which do not require it to be symmetric or positive definite. In each of the theorems the dependence on the sampling time is made explicit. The complexity of these results, in comparison with the simple results for stabilization using analog velocity feedback, is due to the complicated dependence on the sampling time.

5. Appendix

The main result of the paper, namely Theorem 2, was stated in terms of the modal matrices Φ and Ω, as characterizations of the flexible structure. By appropriately defining certain matrix functions as power series that result can be expressed in terms of the mass matrix M and the stiffness matrix K of the flexible structure.

Define the $n \times n$ matrix functions

$$\Psi_1 = \Phi \cos\Omega T \Phi^{-1}$$
$$\Psi_1 = \sum_{i=0}^{\infty} \frac{(-1)^i (M^{-1}K)^i T^{2i}}{(2i)!}$$

$$\Psi_2 = \Phi \sin\Omega T \Omega^{-1} T^{-1} \Phi^{-1}$$
$$\Psi_2 = \sum_{i=0}^{\infty} \frac{(M^{-1}K)^i T^{2i}}{(2i+1)!}$$

Then it can be shown that Theorem 2 can be restated as

Theorem 3.

Assume that Ψ_2 is nonsingular and

(a) the matrix pair $M[I+\Psi_1]^{-1}[I-\Psi_1]$, B is completely controllable;

(b) the matrix $M\Psi_2 M^{-1} BGC$ is symmetric and positive definite,

(c) the matrix $M + M\Psi_1 - TM\Psi_2 M^{-1} BGC$ is symmetric and positive definite.

The closed loop equation (14) is geometrically stable.

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