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PRELIMINARY REPORT

HERMITE AND LAGUERRE INTEGRAL TRANSFORMS

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INTRODUCTION

When a function $K(a,x)$ is a known function of two variables a and x, and the integral

$$I_f(a) = \int_b^c f(x)\, K(a,x)\, dx$$

is convergent, then $I_f(a)$ defines a function of the variable a. This function is called the integral transform of the function $f(x)$ with kernel $K(a,x)$. When the limits b, c are both finite we speak of $I_f(a)$ as being a finite transform of $f(x)$.

We are going to consider two possible choices of the kernel $K(a,x)$. They are as follows:

$$K_1(n,x) = e^{-x^2} H_n(x), \qquad n = 0, 1, 2, \ldots ,$$

and $$K_2(n,x) = e^{-x} L_n(x), \qquad n = 0, 1, 2, \ldots ,$$

where $H_n(x)$ is the Hermite polynomial and $L_n(x)$ is the simple Laguerre polynomial.

The literature contains more than one definition for the Hermite polynomials. We will give two of the most common here. In [1] the Hermite polynomial of degree n is defined by

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} . \tag{1}$$

In [2] the Hermite polynomial of degree n is defined by

$$H_n(x) = e^{x^2/2} (-1)^n \frac{d^n}{dx^n} e^{-x^2/2} . \tag{2}$$

In this paper we will consider (1) as defining the Hermite polynomial. Formulas such as (1) and (2) are often referred to as Rodriguez formulas.

Following the definition (1) the first few polynomials are:

$$H_0(x) = 1,$$
$$H_1(x) = 2x,$$
(3) $$H_2(x) = 4x^2 - 2,$$
$$H_3(x) = 8x^3 - 12x,$$
$$H_4(x) = 16x^4 - 48x^2 + 12.$$

In what follows we shall be interested in some particular properties of the Hermite polynomials. Among these the Hermite differential equation and two differential recurrence relations, namely:

(4) $$H_n''(x) - 2xH_n'(x) + 2n\,H_n(x) = 0,$$

(5) $$H_n'(x) = 2n\,H_{n-1}(x),$$

(6) $$2xH_n(x) - H_n'(x) = H_{n+1}(x), \qquad n = 0, 1, 2, \ldots .$$

The differential recurrence relations (5) and (6) can be used to arrive at the differential equation (4). To see how one might arrive at relations (5) and (6) let us consider another definition of the Hermite polynomial of degree n.

Consider the expansion in powers of t of the function $\exp(2xt - t^2)$. Since $\exp(2xt - t^2) = \exp(2xt)\exp(-t^2)$, the coefficients of the powers of t in the expansion will be polynomials in x. We will define $H_n(x)$ by

(7) $$\exp(2xt - t^2) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

The polynomials $H_n(x)$ will be Hermite polynomials. From (7) we can readily obtain the result

(8) $$H(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k\, n!\, (2x)^{n-2k}}{k!\,(n-2k)!},$$

where [n/2] is the greatest integer in (n/2). The relation (7) is valid for all x and t in the finite t plane.

By differentiating both sides of (7) with respect to x one can arrive at (5) and the relations

(9) $$H_0'(x) = 0,$$

(5) $$H_n'(x) = 2n\,H_{n-1}(x), \qquad n = 1,2,3,\ldots .$$

If, on the other hand, one differentiates both sides of (7) with respect to t, uses known expansions, and compares coefficients, one arrives at relation (6).

The Hermite polynomials are orthogonal over the interval $(-\infty,\infty)$. The weight function is e^{-x^2} and

(10) $$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\,dx = \begin{cases} 0, & m \neq n, \\ \sqrt{\pi}\,2^n\,n!, & m = n. \end{cases}$$

We define $L_n(x)$ as

(11) $$L_n(x) = \sum_{k=0}^{n} \frac{n!\,(-x)^k}{(n-k)!\,(k!)^2}.$$

We shall be interested in the Laguerre differential equation

(12) $$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0.$$

Other properties which will be of interest are as follows:

(13) $$xL_n'(x) = nL_n(x) - nL_{n-1}(x),$$

(14) $$L_n'(x) = L_{n-1}'(x) - L_{n-1}(x),$$

(15) $$L_n'(x) = -\sum_{k=0}^{n-1} L_k(x),$$

(16) $$L_n(x) = \frac{e^x}{n!}\frac{d^n}{dx^n}(x^n e^{-x}),$$

(17) $$\int_0^{\infty} e^{-x} L_n(x) L_m(x)\,dx = \begin{cases} 0 \text{ if } m \neq n. \\ 1 \text{ if } m = n. \end{cases}$$

From (17) we see that the Laguerre polynomials are orthogonal over the interval $(0,\infty)$. The weight function is e^{-x}.

I. HERMITE TRANSFORMS

1. The Transform

We will define the Hermite transformation of the function F(x) to be

$$(1.1) \qquad T_n\left\{F(x)\right\} = \int_{-\infty}^{\infty} F(x)\left\{e^{-x^2} H_n(x)\right\} dx .$$

We would, of course, expect some restrictions on the function F (x) since, by definition, the integral must necessarily be convergent. More will be said later as to the character of F (x).

For an integral transformation, it would be desirable to have for an inversion process a Tauberian theorem, however, in our case we will settle for an Abelian theorem.

Since we are dealing with a set of orthogonal polynomials we are led to consider the expansion of an "arbitrary" function in an infinite series of these polynomials as a possible means of obtaining an inversion formula.

If we assume

$$(1.2) \qquad F(x) = \sum_{n=0}^{\infty} a_n H_n(x) , \qquad -\infty \leqslant x \leqslant \infty ,$$

we can find the coefficients a_n. If we assume the interchange of integration and summation and take advantage of orthogonality property (10) of Hermite polynomials we will have

$$(1.3) \qquad a_n = \frac{T_n\{F\}}{\sqrt{\pi}\, 2^n\, n!} .$$

We therefore have an inversion formula.

$$(1.4) \qquad F(x) = \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{2^n\, n!} T_n\{F\} H_n(x) .$$

The fundamental problem here is seen to be the determination of the conditions which will assure the convergence of the series

$$\sum_{n=0}^{\infty} a_n H_n(x)$$

to F (x). In answer to this question, much has been written. Probably the most complete work being done in Chapter 9 of Szego [3]. (More work is to be done later in order to obtain a workable set of conditions on F (x).)

2. The Basic Operational Property

The Hermite polynomial is known to satisfy the differential equation (4). This leads us to try our transformation on the expression

$$L[V] = V''(x) - 2xV'(x) . \tag{1.5}$$

We will assume in what follows that V (x), V' (x) are continuous and that V" (x) is bounded and integrable on each finite interval. We also assume V (x) and V' (x) are such that $|V(x)| < Me^{ax^2}$ and $|V'(x)| < Me^{ax^2}$ for large values of x and where $a < 1/2$.

Successive integration by parts of the integral

$$\int_{-\infty}^{\infty} L[V]\, e^{-x^2} H_n(x)\, dx \tag{1.6}$$

along with the information from (4) that

$$[e^{-x^2} H_n'(x)]' = -e^{-x^2} 2nH_n(x) \tag{1.7}$$

will lead to the fundamental property of the Hermite transform

$$T_n\{L[V]\} = \begin{cases} 0 , & n = 0 , \\ -2n\, T_n\{V\} , & n = 1, 2, 3, \ldots . \end{cases} \tag{1.8}$$

For a modification of formula (1.8) suppose V' (x) has an ordinary discontinuity at $x = x_o$. A process similar to the one followed in arriving at formula (1.8) gives

$$T_n\{L[V]\} = \tag{1.9}$$

$$\begin{cases} e^{-x_o^2}[V'(x_o+0) - V'(x_o-0)], & n = 0 , \\ e^{-x_o^2} H_n(x_o)[V'(x_o+0) - V'(x_o-0)] - 2n\, T_n[V], & n = 1, 2, 3 \ldots . \end{cases}$$

3. Effect of Hermite Transform on Derivatives

We assume in the following that $F(x)$ has continuous derivatives $F^{(m-1)}(x)$ and a sectionally continuous derivative $F^{(m)}(x)$. We will assume $F(x)$ and its continuous $F^{(m-1)}(x)$ derivatives all are such that $|F(x)| < Me^{\alpha x^2}$ for large values of x and $\alpha < 1/2$. One integration by parts of the integral

$$(1.10) \qquad \int_{-\infty}^{\infty} e^{-x^2} F'(x) H_n(x)\, dx$$

and the use of property (6) leads to

$$(1.11) \qquad T_n\{F'(x)\} = T_{n+1}\{F(x)\}, \qquad n = 0, 1, 2, \ldots .$$

Now, applying (1.11) to F' we have

$$T_n\{F''(x)\} = T_{n+1}\{F'(x)\} = T_{n+2}\{F(x)\} .$$

Continued inductively, this process gives the property

$$(1.12) \qquad T_n\{F^{(m)}(x)\} = T_{n+m}\{F(x)\}, \qquad n = 0, 1, 2, \ldots ,$$

4. The Transform of x^r

It is known that when r is a fixed integer the following expression is true.

$$(1.13) \qquad x^r = \frac{1}{2^r} \sum_{k=0}^{[r/2]} \frac{r!\, H_{r-2k}(x)}{k!\,(r-2k)!},$$

where $[r/2]$ indicates the greatest integer in $(r/2)$.

Now let us consider $T_n\{x^r\}$ for possible choices of r and n (r fixed while $n = 0, 1, 2, \ldots$).

Case 1. $n \geq r$

$$(1.14) \qquad T_n\{x^r\} = \frac{1}{2^r} \int_{-\infty}^{\infty} e^{-x^2} H_n(x) \sum_{k=0}^{[r/2]} \frac{r!\, H_{r-2k}(x)}{k!\,(r-2k)!}\, dx$$

Since $n > r$ we see that in the finite sum no $H_s(x)$ can occur with an index equal to n. Hence by property (10) we have the following formula:

(1.15) $$T_n\left\{x^r\right\} = 0\,, \qquad n > r\,.$$

Case 2. $n = r$.

From (1.14) we see that

$$T_n\left\{x^n\right\} = \sqrt{\pi}\, n!\,, \qquad n = r\,.$$

Case 3. $n \leq r$.

We see from equation (1.14) that in order to obtain a contribution,

$$n = r - 2k \qquad \text{or} \qquad k = \frac{r-n}{2}\,.$$

When $n = r - 2k$

$$k = 0, 1, 2, \ldots, [r/2]\,,$$

$$n = 0, 1, 2, \ldots,$$

$$n < r\,,$$

equation (1.14) gives

(1.16) $$T_n\left\{x^r\right\} = \frac{2^{n-r}\, r!\, \sqrt{\pi}}{[(r-n)/2]!}\,, \qquad \text{for } n = 0, 1, 2, \ldots, \quad n < r\,, \quad n = r - 2k\,.$$

If, however, $n = r - 2k - 1$ there will be no contribution and we will have

(1.17) $$T_n\left\{x^r\right\} = 0\,, \qquad n = 0, 1, 2, \ldots \quad n < r\,, \quad n = r - 2k - 1\,.$$

where $k = 0, 1, 2, \ldots [r/2]$.

5. Tables

The remaining pages of Chapter I contain tables of some simple transforms and of simple operations which have already been considered or will be considered in Chapter II.

Table of Transforms

$$f(n) = \int_{-\infty}^{\infty} e^{-x^2} H_n(x) F(x) \, dx$$

f (n)	F (x)
(1) $\begin{cases} 0 & n \neq 0 \\ \sqrt{\pi} & n = 0 \end{cases}$	(1) 1
(2) $Kf(n)$	(2) $KF(x)$
(3) $\begin{cases} 0 , & n > r \\ \sqrt{\pi}\, r! , & n = r \\ \dfrac{\sqrt{\pi}\, r!\, (2^{n-r})}{[(r-n)/2]!} , & n < r , \; n = r - 2k \\ 0 , & n < r , \; n = r-2k-1 \\ \text{where } k = 0, 1, 2, \ldots [r/2] \end{cases}$	(3) x^r
(4) $\begin{cases} 0 , & n = 2p , \; p = 0,1,2.. \\ (-1)^p k^n e^{-(k^2/4)} \sqrt{\pi} , & n = 2p + 1 \end{cases}$	(4) $\sin kx$
(5) $\begin{cases} (-1)^p k^{n+1} e^{-(k^2/4)} \sqrt{\pi} , & n = 2p \\ 0 , & n = 2p + 1 \\ \text{where } p = 0, 1, 2, \ldots \end{cases}$	(5) $\cos kx$

Table of Transforms (cont.)

f (n)	F (x)
(6) $q^n e^{q^2/4} \sqrt{\pi}$, $n = 0, 1, 2, ..$	(6) e^{qx}
(7) $-2ne^{1/4} \sqrt{\pi}$, $n = 0, 1, 2, ..$	(7) $e^x (1 - 2x)$
(8) $(-1)^{n+1} 2ne^{1/4} \sqrt{\pi}$, $n = 0, 1, 2, ..$	(8) $e^{-x} (1 + 2x)$
(9) $\begin{cases} (-1)^{p+1} 2nk^{n+1} e^{-(k^2/4)} \sqrt{\pi}, & n = 2p \\ 0, & n = 2p + 1 \\ \text{where } p = 0, 1, 2, \ldots \end{cases}$	(9) $2kx \sin kx - k^2 \cos kx$
(10) $\begin{cases} 0 & n = 2p \\ (-1)^{p+1} 2nk^n e^{-(k^2/4)} \sqrt{\pi}, & n = 2p + 1 \\ \text{where } p = 0, 1, 2, \ldots \end{cases}$	(10) $-(k^2 \sin kx + 2kx \cos kx)$

Table of Operations

F (x)	f (n)
(1) F (x)	(1) $\int_{-\infty}^{\infty} e^{-x^2} H_n(x) F(x)\, dx$
(2) F' (x)	(2) f (n + 1) $n = 0, 1, 2, ..$
(3) $\frac{d^m}{dx^m} F(x)$	(3) f (n + m) , $n = 0, 1, 2, ..$

Table of Operations (cont.)

f (x)	f (n)
(4) $\int_0^x F(t)\,dt$	(4) $\begin{cases} f(n-1), & n = 1,2,3,\ldots, \\ \int_{-\infty}^{\infty} e^{-x^2}\left\{\int_0^x F(t)\,dt\right\} dx, & n = 0 \end{cases}$
(5) $F(x) + K$, K a constant	(5) $\begin{cases} f(n), & n = 1,2,3,\ldots \\ f(0) + \sqrt{\pi}, & n = 0 \end{cases}$
(6) $F(x) + kx$, K a constant	(6) $\begin{cases} f(n), & n = 0,2,3,\ldots, \\ f(n) + K\sqrt{\pi}, & n = 1 \end{cases}$
(7) $F(x) + H_m(x)$	(7) $\begin{cases} f(n), & n \neq m \\ f(n) + 2^n n!\sqrt{\pi}, & n = m \end{cases}$
(8) $F''(x) - 2x\,F'(x)$	(8)* $\begin{cases} -2n\,f(n), & n = 1,2,3,\ldots \\ 0, & n = 0 \end{cases}$

*This is (1.8). See conditions on F, $d/dx(F)$, $d^2/dx^2(F)$ there.

II. FURTHER PROPERTIES AND EXAMPLES OF HERMITE TRANSFORMS

1. Example: Find $T_{n+1}\left\{\int_0^x V(t)\,dt\right\}$.

We will assume V (t) continuous and $|V(t)| < Me^{at^2}$ for large t and $a < 1/2$.

$$F(x) = \int_0^x V(t)\,dt \,. \tag{2.1}$$

Then F (x) is continuous and since V (t) has the desired order property it is easily seen, by the use of simple properties of the Riemann integral, that $|F(x)| < Me^{ax^2}$ for large x and $a < 1/2$. Also $F'(x) = V(x)$. Then $T\{F'(x)\} = T_n\{V(x)\}$, but since $F'(x)$ is continuous and $F'(x)$ possesses the correct order property we can use property (1.12) on transforms of derivatives and write

$$\begin{aligned} T_n\left\{\int_0^x F(t)\,dt\right\} &= T_{n-1}\{F(x)\}\,, \qquad n = 1, 2, 3, \ldots, \\ T_0\left\{\int_0^x F(t)\,dt\right\} &= \int_{-\infty}^{\infty} e^{-x^2}\int_0^x V(t)\,dt\,dx\,, \qquad n = 0\,. \end{aligned} \tag{2.2}$$

2. Inverse Operator

We consider now the operator L [V] given by (1.5) where V (x) and V' (x) are continuous and $|V(x)| < Me^{ax^2}$, $|V'(x)| < Ne^{bx^2}$ for large values of x and a, $b < 1/2$. Also $V''(x)$ is bounded and integrable on each finite interval.

We are now interested in considering $T_n\{L^{-1}[V]\}$. Proceeding formally we have

$$L^{-1}[V] = F(x) \qquad \text{or} \qquad V = L[F]\,.$$

Hence

$$F''(x) - 2x\,F'(x) = V(x)\,. \tag{2.3}$$

We now apply the Hermite Transform to equation (2.3). If F, F' and F" possess the necessary properties we can apply (1.8) and write

$$-2n\, f\,(n) = V\,(n)$$

or

(2.4) $$f\,(n) = -\frac{V\,(n)}{2n}\,, \qquad n = 1, 2, 3, \ldots .$$

For the case n = 0 we have 0 = v (0).

In other words, the zero transform of V (x) must be zero. We see then that if we are to be able to use (1.8), V (x) must be such that

(2.5) $$\int_{-\infty}^{\infty} e^{-x^2}\, V\,(x)\, dx = 0\,.$$

We will encounter equation (2.5) again when we investigate the order property of F' (x). It will be seen that unless equation (2.5) is true no function F(x) will exist for equation (2.3) such that F' (x) will have the necessary order property as $x \to -\infty$.

3. Order and Continuity Properties of F, F', and F"

We will obtain F (x) from the differential equation

(2.6) $$[e^{-x^2}\, F'\,(x)]' = V\,(x)\, e^{-x^2}\,.$$

It now follows that

(2.7) $$e^{-x^2}\, F'\,(x) = \int_{0}^{x} V\,(t)\, e^{-t^2}\, dt + C_1\,,$$

where C_1 is a constant of integration.

We see that if F' (x) is to have the correct order property as $x \to +\infty$,

(2.8) $$F'\,(x) = -e^{x^2} \int_{x}^{\infty} V\,(t)\, e^{-t^2}\, dt\,.$$

From equation (2.8) we arrive at

(2.9) $$F(x) = -\int_0^x e^{y^2} \int_y^\infty V(t)\, e^{-t^2}\, dt\, dx + C_2 ,$$

where C_2 is a constant of integration.

It is easily seen from equation (2.8) that $F'(x)$ has the correct order property as $x \to +\infty$. When $x \to -\infty$ we have

(2.10) $$e^{-x^2} F'(x) \to K \neq 0$$

and hence $F'(x)$ does not possess the desired order property when $x \to -\infty$ unless (2.5) is true.

That $F(x)$ has the correct order property follows immediately from the order property of $F'(x)$.

Continuity properties of F, and F' follow from properties of $V(t)$ and of the Riemann integral. Hence we have the following result.

Theorem 1. If $V(x)$, $V'(x)$ are continuous and $V''(x)$ is bounded and integrable on each finite interval, and if

$$\int_{-\infty}^{\infty} e^{-x^2} V(x)\, dx = 0$$

and if

$$|V(x)| < M_1\, e^{ax^2} \quad , \qquad M_1,\ a \text{ constants},$$

$$|V'(x)| < M_2\, e^{bx^2} \quad , \qquad M_2,\ b \text{ constants}$$

for $x \to +\infty$ and $a, b < 1/2$ then

(2.11)
$$T_n \left\{L^{-1}[V]\right\} = -\frac{v(n)}{2n} \quad , \qquad n = 1, 2, 3, \ldots .$$

$$T_0 \left\{L^{-1}[V]\right\} = -\int_{-\infty}^{\infty} e^{-x^2} \int_0^x e^{y^2} \int_y^\infty V(t)\, e^{-t^2}\, dt\, dy\, dx ,$$

where $$L^{-1}[V] = -\int_0^x e^{y^2} \int_y^\infty V(t)\, e^{-t^2}\, dt\, dy .$$

4. Example: $V(x) = -x^r$ in Equation (2.3)

We see that only when r is an odd integer will the differential equation have solutions with the desired properties necessary for the use of formula (1.8). To see that this is true it is only necessary to consider the integral

$$(2.12) \qquad \int_{-\infty}^{\infty} e^{-t^2} t^n \, dt = \begin{cases} \dfrac{(n-1)(n-3)(n-5)\ldots 1}{2^{n/2}} \sqrt{\pi} & , \quad n \text{ even} \\ 0 & , \quad n \text{ odd} . \end{cases}$$

Hence, using transform (3) in the Table of Transforms and formula (2.11) from the previous section, we can write some new transforms. For equation (2.3) with $V(x) = -x^r$ we find

$$(2.13) \qquad F(x) = \int_0^x e^{y^2} \int_y^{\infty} e^{-t^2} t^r \, dt \, dy + C_2 ,$$

where C_2 is a constant of integration. From transform (3) of the Table of Transforms we find

$$T_n \left\{ F(x) \right\} = 0 \qquad n > r$$

$$T_n \left\{ F(x) \right\} = \frac{\sqrt{\pi}\, r!}{2r} , \qquad n = r$$

$$(2.14) \qquad T_n \left\{ F(x) \right\} = \frac{2^{n-r-1}\, r!\, \sqrt{\pi}}{n\, [(r-n)/2]\, !} , \qquad \begin{cases} n < r , \\ n = r - 2k , \end{cases}$$

$$T_n \left\{ F(x) \right\} = 0 , \qquad \begin{matrix} n < r , \\ n = r - 2k - 1 , \end{matrix}$$

where $n = 1, 2, 3, \ldots$, r is an odd integer, and

$k = 0, 1, 2, \ldots, [r/2]$.

When $n = 0$,

$$(2.15) \qquad T_0 \left\{ F(x) \right\} = \int_{-\infty}^{\infty} e^{-x^2} \int_0^x e^{y^2} \int_y^{\infty} e^{-t^2} t^r \, dt \, dy \, dx + C_2 \int_{-\infty}^{\infty} e^{-x^2} \, dx ,$$

where
$$C_2 \int_{-\infty}^{\infty} e^{-x^2} \, dx = C_2 \sqrt{\pi} .$$

From (5) of the Table of Operations, page 9 we see that we have here an infinity of functions which will have the same transform at $n = 1, 2, 3, \ldots$

Hence when $n \neq 0$ we can take C_2 as zero.

5. Some Simple Transforms

Let $F(x) = 1$, $-\infty < x < +\infty$. Since $H_0(x) = 1$ we use the orthogonality property of Hermite polynomials to arrive at

$$T_n\{1\} = \begin{cases} 0, & n \neq 0, \\ \sqrt{\pi}, & n = 0. \end{cases} \tag{2.16}$$

Since T_n is a linear transformation we can write the transform of a constant K.

$$T_n\{K \cdot 1\} = KT_n\{1\}$$

and using formula (2.16)

$$T_n\{K\} = \begin{cases} 0, & n \neq 0, \\ K\sqrt{\pi}, & n = 0. \end{cases} \tag{2.17}$$

6. A Possible Use of Some Previous Results

Using formula (1.12), we find

$$T_n\{\cos x\} = T_{n+1}\{\sin x\} \tag{2.18}$$

and

$$-T_n\{\sin x\} = T_{n+1}\{\cos x\}. \tag{2.19}$$

Combining equations (2.18) and (2.19) we obtain

$$T_n\{\cos x\} = -T_{n+2}\{\cos x\}, \qquad n = 0, 1, 2, \ldots, \tag{2.20}$$

and

$$T_n\{\sin x\} = -T_{n+2}\{\sin x\}, \qquad n = 0, 1, 2, \ldots. \tag{2.21}$$

We also obtain, using property (1.12),

(2.22) $$T_n \{e^x\} = T_{n+1} \{e^x\} , \qquad n = 0, 1, 2, \ldots ,$$

which tells us that $T_n \{e^x\}$ is constant. Similarly,

(2.23) $$-T_n \{e^{-x}\} = T_{n+1} \{e^{-x}\}$$

and

(2.24) $$rT_n \{x^{r-1} e^{ax}\} + aT_n \{x^r e^{ax}\} = T_{n+1} \{x^r e^{ax}\} .$$

Since

(2.25) $$\int_{-\infty}^{\infty} e^{-x^2} \sin kx \, dx = 0 ,$$

(2.26) $$\int_{-\infty}^{\infty} e^{-x^2} \sin kx \, 2x \, dx = ke^{-(k^2/4)} \sqrt{\pi} , \quad [4]$$

(2.27) $$\int_{-\infty}^{\infty} e^{-x^2} \cos kx \, 2x \, dx = 0$$

we can write the actual transforms for sin kx and cos kx:

(2.28)
$$T_n \{\sin kx\} = 0 \qquad \text{for } n = 2p , \; p = 0, 1, 2, \ldots ,$$
$$T_n \{\sin kx\} = (-1)^p k^{-n} e^{-(k^2/4)} \sqrt{\pi} , \quad n = 2p + 1 .$$

$$T_n \{\cos kx\} = (-1)^p k^{n+1} e^{-(k^2/4)} \sqrt{\pi} , \quad n = 2p ,$$
$$T_n \{\cos kx\} = 0 , \qquad n = 2p + 1 , \; p = 0, 1, 2, \ldots .$$

Since

$$\int_{-\infty}^{\infty} e^{-x^2 + qx} \, dx = e^{(q^2/4)} \sqrt{\pi} , \quad [4]$$

we can obtain the transforms of e^{qx} and e^{-qx}

(2.30) $$T_n \{e^{qx}\} = q^n e^{(q^2/4)} \sqrt{\pi} , \qquad n = 0, 1, 2, \ldots .$$

III. APPLICATIONS

1. Problem

In this section we shall discuss a type of boundary value problem for which it might be advantageous to use the Hermite transform.

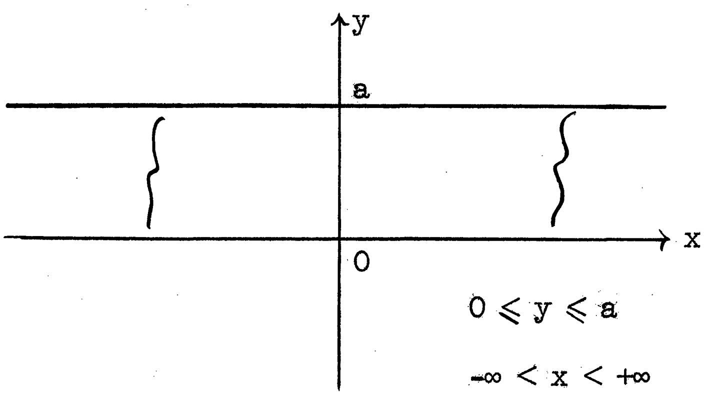

Fig. 1.

Consider the steady-state, two-dimensional case of the heat equation

$$(3.1) \qquad \frac{\partial}{\partial x}\left(K \frac{\partial U\ (xy)}{\partial x}\right) + \frac{\partial}{\partial y}\left(K \frac{\partial U\ (x,y)}{\partial y}\right) = P\ (x)\ ,$$

where the thermal conductivity K is proportional to e^{-x^2} and P (x) is a continuous source of heat within the solid. We shall consider the case in which $P\ (x) = e^{-x^2}\ G\ (x)$, where $|G\ (x)| < Me^{ax^2}$ for large x and $a < 1/2$.

In light of the preceeding equation, (3.1) becomes

$$(3.2) \qquad \frac{\partial^2 U}{\partial x^2} - 2x \frac{\partial U}{\partial x} + \frac{\partial^2 U}{\partial y^2} = G\ (x)\ ,$$

where $|U\ (x,y)| < Me^{bx^2}$ for large x, $b < 1/2$. We will assume boundary conditions

$$(3.3) \qquad U\ (x,0) = 0\ , \qquad U\ (x,a) = 0\ .$$

The transformed problem becomes

(3.4) $$u_{yy}(n,y) - 2n\,u(n,y) = g(n)$$

and

(3.5) $$u(n,0) = 0 \quad , \qquad u(n,a) = 0 .$$

Then

$$u(n,y) = C_1 e^{-y\sqrt{2n}} + C_2 e^{y\sqrt{2n}} - \frac{g(n)}{2n} \quad , \qquad n = 1, 2, 3, \ldots .$$

Using conditions (3.5) we find

(3.6)
$$C_1 = \frac{g(n)}{2n}\left[1 - \frac{1 - e^{-a\sqrt{2n}}}{2\sinh(a\sqrt{2n})}\right] ,$$

$$C_2 = \frac{g(n)}{2n}\,\frac{1 - e^{-a\sqrt{2n}}}{2\sinh(a\sqrt{2n})} .$$

Therefore,

(3.7) $$u(n,y) = \frac{g(n)}{2n}\left[e^{-y\sqrt{2n}} + \frac{\sinh y\sqrt{2n}}{\sinh a\sqrt{2n}}\left(1 - e^{-a\sqrt{2n}}\right) - 1\right]$$

for $n = 1, 2, 3, \ldots .$

When $n = 0$,

$$u_{yy}(0,y) = g(0)$$

and

$$u(0,y) = \frac{1}{2} g(0)\, y^2 + C_1 y + C_2 .$$

Again using conditions (3.5) we find

(3.8) $$u(0,y) = \frac{1}{2}\, g(0)\,[y^2 - ay] , \qquad n = 0 .$$

Hence,

(3.9) $$U(x,y) = \frac{1}{\sqrt{\pi}} u(0,y) + \frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{2^n\, n!} u(n,y)\, H_n(x)$$

and

(3.10)
$$U(x,y) = \frac{1}{2\sqrt{\pi}}\,[y(y-a)] \int_{-\infty}^{\infty} e^{-x^2} G(x)\, dx$$

$$+ \sum_{n=1}^{\infty} \frac{g(n)}{\sqrt{\pi}\, 2^n\, n!\, 2n} \left\{N(n,y)\right\} H_n(x) ,$$

where $$N(n,y) = \left[e^{-y\sqrt{2n}} + \frac{\sinh y\sqrt{2n}}{\sinh a\sqrt{2n}} \left(1 - e^{-a\sqrt{2n}}\right) - 1 \right]$$

2. Remarks

The Laplace transformation process is not suited to the above problem because the partial differential equation involves the transform of a second derivative and this transform involves the initial values of both the function and its derivative. For separation of variables we would need $P(x) = 0$.

We note here the expression $[g(n)/2n]$ occurring. If we would demand

$$\int_{-\infty}^{\infty} e^{-x^2} G(x)\, dx = 0$$

we might then be able to use (2.11) in our inversion process.

This problem also points out the need of having a convolution property [5]. More is to be done with this.

As the work in obtaining (5.10) was formal, a verification of the solution would now be necessary. It can easily be seen that the boundary conditions are satisfied. Also if we allow differentiation of infinite series, $U(x,y)$ will satisfy the differential equation.

We would, however, like to have some conditions on $u(n,y)$ which would ensure the existence of $U(x,y)$. At present we can offer a necessary condition on $u(n,y)$:

(3.11) $$\lim_{n \to \infty} \frac{u(n, y)}{2^{(n/2)}\sqrt{n!}} = 0 .$$

This follows from

(3.12) $$\left|H_n(x)\right| < \sqrt[4]{\pi}\ C_2^{(n/2)} \sqrt{n!}\ e^{(x^2/2)}$$

C a constant, for all values of x and n [6]. More is to be done toward finding conditions on $u(n,y)$.

IV. LAGUERRE TRANSFORMS

1. The Transform

We will define the Laguerre Transform of the function F (x) to be

$$(4.1) \qquad T_n \left\{F(x)\right\} = \int_0^\infty F(x)\, e^{-x}\, L_n(x)\, dx .$$

Again as in the case of the Hermite transform, we are faced with the necessity of having some kind of an inversion process. We will proceed as before and consider the possibility of the expansion of an "arbitrary" function in an infinite series of Laguerre polynomials.

If we assume

$$(4.2) \qquad F(x) = \sum_{n=0}^{\infty} a_n L_n(x) , \qquad 0 \leqslant x < \infty ,$$

where

$$(4.3) \qquad a_n = \int_0^\infty F(x)\, e^{-x}\, L_n(x)\, dx ,$$

our inversion formula for the transform is

$$(4.4) \qquad F(x) = \sum_{n=0}^{\infty} T_n \left\{F(x)\right\} L_n(x) .$$

This is arrived at by a process analogous to the one used for Hermite polynomials.

We seem again to have the following fundamental problem: What conditions assure the convergence of the series

$$\sum_{n=0}^{\infty} a_n L_n(x)$$

to the function F (x)? We again refer to [3].

2. The Basic Operational Property

Consider the self-adjoint form

(4.5) $$L[V] = [xe^{-x} V'(x)]' e^{x} .$$

The Laguerre polynomial is known to satisfy the differential equation (12) which can also be written

(4.6) $$[xe^{-x} L_n'(x)]' + ne^{-x} L_n(x) = 0 .$$

Hence it is natural to try our transform on L [V].

We will assume in what follows that V (x), V' (x) are continuous and that V" (x) is bounded and integrable on each finite interval. We assume V (x) and V' (x) are such that $|V(x)| < Me^{ax}$ and $|V'(x)| < Me^{ax}$ for x large and $a < 1/2$.

Successive integration by parts of the integral

(4.7) $$\int_0^\infty (xe^{-x} V')' L_n(x)\, dx$$

as well as the use of equation (4.6) leads to

(4.8) $$T_n\{L[V]\} = \begin{cases} 0 , & n = 0 , \\ -n\, T_n\{V(x)\} , & n = 1, 2, 3, \ldots . \end{cases}$$

A modification of (4.8) is obtained if V' (x) is permitted to have an ordinary discontinuity at $x = x_o$:

(4.9) $$T_n\{L[V]\} = \begin{cases} x_o e^{-x_o} [\frac{d}{dx} V(x_o + 0) - \frac{d}{dx} V(x_o - 0)] , \quad n = 0 , \\ x_o e^{-x_o} L_n(x_o) [\frac{d}{dx} V(x_o + 0) - \frac{d}{dx} V(x_o - 0)] - nT_n\{V(x)\}, \\ n = 1, 2, \ldots . \end{cases}$$

3. Laguerre Transforms and Derivatives

We will assume F (x) continuous and F' (x) bounded and integrable. Also we assume $|F(x)| < Me^{ax}$ for large x and $a < 1/2$. One integration by parts of the integral

(4.10) $$\int_0^\infty F' (x)\, e^{-x} L_n (x)\, dx$$

and the use of property (15) leads to the following formula:

(4.11) $$T \left\{F' (x)\right\} = \sum_{k=0}^{n} T_k \left\{F\right\} - F (0) \, .$$

V. THE INVERSE OPERATOR AND THE LAGUERRE TRANSFORM

1. The Inverse Operator

We consider here the operator

$$L[V] = x V''(x) + (1-x) V'(x) , \tag{5.1}$$

where V, V' are continuous, V" (x) bounded and integrable on each finite interval, and

$$\begin{aligned} |V| &< M_1 e^{ax} , \\ |V'| &< M_2 e^{bx} , \end{aligned} \tag{5.2}$$

for M_1 and M_2 constant, a, b < 1/2, and $x \to +\infty$. In section 2 of Chapter IV it has been shown that

$$\begin{aligned} T_n\left\{L[V]\right\} &= -n T_n\left\{V\right\} , & n &= 1, 2, 3, \ldots , \\ T_0\left\{L[V]\right\} &= 0 , & n &= 0 . \end{aligned} \tag{4.8}$$

We are now interested in considering

$$T_n\left\{L^{-1}[V]\right\} . \tag{5.3}$$

Proceeding formally, we obtain the differential equation

$$x F''(x) + (1-x) F'(x) = V(x) , \tag{5.4}$$

where $F(x) = L^{-1}[V]$.

We apply the Laguerre Transform and obtain

$$T_n\left\{L[F]\right\} = T_n\left\{V\right\} . \tag{5.5}$$

If F (x) and F' (x) are continuous, F" (x) bounded and integrable on each finite interval and F and F' satisfy property (5.2), we can use property (4.8) and write

$$-n f(n) = v(n) , \qquad n = 1, 2, 3, \ldots . \tag{5.6}$$

For the case n = 0 we obtain v (0) = 0, which indicates that the zero transform of V (x) must be zero in order that such a function F (x) exist.

2. Properties of F (x)

We can obtain F (x) from the differential equation

$$(xe^{-x} F')' = Ve^{-x} . \tag{5.7}$$

It follows from equation (5.7) that

$$xe^{-x} F' = -\int_x^{\infty} V (t) e^{-t} dt + C_1 \tag{5.8}$$

where C_1 is a constant of integration. We see that C_1 must be zero if F' is to satisfy a property similar to property (5.2). When $x \to 0$ we see that

$$\int_0^{\infty} V (t) e^{-t} dt = 0 . \tag{5.9}$$

Equation (5.9) is the condition on the zero transform of V (x) which we noted in section 1 of this chapter. That F (x) and F' (x) have the correct order property (5.2) is a consequence of the order property of V (x) and simple properties of the Riemann integral. The continuity properties follow as a consequence of elementary properties of the Riemann integral.

3. Summary

If V (x), V' (x) are continuous, V" (x) bounded and integrable on each finite interval, if

$$\int_0^{\infty} e^{-x} V (x) dx = 0 ,$$

and V (x) and V' (x) satisfy property (5.2), then

$$T_n \left\{L^{-1} [V]\right\} = -\frac{T_n \{V\}}{n} , \qquad n = 1, 2, 3, \ldots ,$$

(5.10)

$$T_o \left\{L^{-1} [V]\right\} = -\int_0^{\infty} e^{-x} \int_0^{x} \frac{e^y}{y} \int_y^{\infty} V (t) e^{-t} dt\, dy\, dx ,$$

where

$$L^{-1} [V] = -\int_0^{x} \frac{e^y}{y} \int_y^{\infty} V (t) e^{-t} dt\, dy .$$

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