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TECHNICAL REPORT NO. 3

NEW OPERATIONAL MATHEMATICS
THE OPERATIONAL CALCULUS OF LAGUERRE TRANSFORMS

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ABSTRACT

Let $e^{-x} L_n(x)$ serve as the kernel function for a linear integral transformation, where $L_n(x)$ is the Laguerre polynomial of n^{th} degree. Operational properties, including a convolution property, are derived here. Transforms of particular functions as well as a few examples and applications are given.

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CHAPTER I

INTRODUCTION

1. Integral Transforms

When the function K(a,x) is a known function of the two variables a and x and the integral

$$(1) \qquad I(a) = \int_b^c F(x)\, K(a,x)\, dx$$

is convergent, then the equation (1) defines a function of the variable a. This function is called the integral transform of the function F(x) by the kernel K(a,x). One of the better known examples of such a kernel is

$$(2) \qquad K(a,x) = e^{-ax}$$

which leads to the Laplace transform. Examples of other transforms can be found in Sneddon[14] and Tranter [17].

It follows immediately from the definition (1) that, if F(x) and G(x) are two functions which possess integral transforms by the kernel K(a,x) then the integral transform of their sum is

$$(3) \quad \int_b^c [F(x) + G(x)]\, K(a,x)\, dx = \int_b^c F(x)\, K(a,x)\, dx + \int_b^c G(x)\, K(a,x)\, dx.$$

If d is a scalar,

$$(4) \qquad \int_b^c d\, F(x)\, K(a,x)\, dx \;=\; d\int_b^c F(x)\, K(a,x)\, dx.$$

Equations (3) and (4) express the fact that the integral transform is a linear operator.

We will direct our attention in what follows to a particular choice of the constants b, c, and the kernel function K(a,x) in the definition (1). We will choose $b = 0$, $c = \infty$, and the kernel will involve a Laguerre polynomial.

2. Properties of Laguerre Polynomials

We list here from the literature various properties of the Laguerre polynomials which will be of use to us later.

Following Szegö [16] we define the Laguerre polynomials $L_n(x)$ by the following conditions of orthogonality:

$$(5) \qquad \int_0^\infty e^{-x} L_n(x)\, L_m(x)\, dx \;=\; \begin{cases} 0 & \text{if } m \neq n \\ 1 & \text{if } m = n\,. \end{cases}$$

We note here that Courant and Hilbert [5] denote by $L_n(x)$ a function which is the same as $n!\, L_n(x)$ in our notation. Laguerre uses the notation $F_n(x) = n!\, L_n(-x)$.

We have the differential equations

$$xy'' + (1-x)\, y' + ny = 0, \qquad y = L_n(x),$$

(6)
$$xz'' + (1+x)\, z' + (n+1)\, z = 0, \qquad z = e^{-x} L_n(x),$$

and

$$xu'' + u' + \left(n+\frac{1}{2}-\frac{x}{4}\right) u = 0, \qquad u = e^{-\frac{x}{2}} L_n(x).$$

The Rodriques' formula for Laguerre polynomials is

$$(7) \qquad L_n(x) = \frac{e^x}{n!}\, \frac{d^n}{dx^n}\, (x^n e^{-x}).$$

Other properties which will be of interest are as follows:

(8) $$xL_n'(x) = nL_n(x) - nL_{n-1}(x),$$

(9) $$L_n'(x) = L_{n-1}'(x) - L_{n-1}(x),$$

and

(10) $$L_n'(x) = -\sum_{k=0}^{n-1} L_k(x).$$

It follows immediately from the relation

(11) $$L_n(x) = \sum_{k=0}^{n} \frac{n!\,(-x)^k}{(n-k)!\,(k!)^2}$$

that the first few polynomials are

(12)
$$L_0(x) = 1, \qquad L_3(x) = 1-3x + \frac{3}{2}x^2 - \frac{x^3}{6},$$
$$L_1(x) = 1-x,$$
$$L_2(x) = 1-2x + \frac{x^2}{2}, \quad L_4(x) = 1-4x + 3x^2 - \frac{2}{3}x^3 + \frac{x^4}{24}.$$

The Laguerre polynomials possess the following generating functions:

(13) $$e^t J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!},$$

where $J_0(2\sqrt{xt})$ is the zero order Bessel function;

(14) $$\frac{1}{1-t} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)\, t^n, \qquad |t| < 1.$$

Of particular interest in connection with the convolution property of the Laguerre transforms will be the addition property

(15) $$\pi L_n(x)\, L_n(y) = \int_0^{\pi} e^{\sqrt{xy}\cos\theta} \cos(\sqrt{xy}\sin\theta)\, L_n(x+y-2\sqrt{xy}\cos\theta)\, d\theta.$$

Relation (13) can be found in Rainville [13], the addition property (15) follows from a property on Sonine polynomials in Bateman [1], and the remaining properties can be found in Szegö [16] or Erdelyi [6].

3. Laguerre Kernels

A kernel function K(a,x) will be called a Laguerre kernel if it involves a Laguerre polynomial. We will consider the three possible choices $e^{-x} L_n(x)$, $e^{-\frac{x}{2}} L_n(x)$, and $L_n(x)$ as Laguerre kernel functions.

In this section we will attempt to show how the need for knowledge of properties of an integral transformation based on such a kernel function as one of the above might arise.

Suppose one desired to obtain an advantageous resolution of the differential form

$$L[F(x)] = xF''(x) + (1-x) F'(x) \tag{16}$$

into a simpler form. The variable x will be allowed to range over the semi-infinite interval from zero to infinity.

We will now follow a procedure outlined by R. V. Churchill in 1950 in some unpublished notes.

We will assume that the function F(x) has a continuous derivative of the second order with respect to x, $x \geq 0$, and F(x) is $\mathcal{O}(e^{rx})$, $r < 1$, as x tends to infinity.

We shall determine a kernel K(a,x) such that the linear integral transformation

$$T[F(x)] = \int_0^\infty F(x)\, K(a,x)\, dx \tag{17}$$

resolves the differential form L[F(x)] in terms of the transform T[F(x)].

Let us also assume that K(a,x) has a continuous derivative of the second order with respect to x, on the range $x \geq 0$.

We assume the following form of the resolution:

$$T\{L[F(x)]\} = \lambda(a)\, T[F(x)]. \tag{18}$$

We will consider the self adjoint form of the differential form $L[F(x)]$. To do this we write

(19) $$r(x) = \exp\left[\int \frac{1-x}{x}\,dx\right], \qquad p(x) = e^{-x};$$

then

$$L[F] = e^{x}[xe^{-x}F'' + (1-x)\,e^{-x}F'] = e^{x}(xe^{-x}F')'.$$

We now write the kernel in terms of a new function M,

$$K(a,x) = e^{-x} M(a,x).$$

The equation (17) now becomes

(20) $$T[F(x)] = \int_0^\infty e^{-x} F(x)\, M(a,x)\, dx.$$

We see that for F(x) of the order $\mathcal{O}(e^{rx})$, $r < 1$ that the integral (20) will exist as long as M(a,x) does not become infinite of an order higher than a positive power of x.

By successive integration by parts we can write

$$T\{L[F]\} = \int_0^\infty (xe^{-x}F')'\; M(a,x)\,dx$$

$$= \int_0^\infty (xe^{-x}M')'F\,dx + M(a,x)\,xe^{-x}F'(x)\Big|_0^\infty - M'xe^{-x}F\Big|_0^\infty .$$

In view of form (18) for $T\{L[F]\}$ it follows that

(21) $$\int_0^\infty [(xe^{-x}M')' - \lambda(a)\,e^{-x}M]\,F\,dx = M'xe^{-x}F\Big|_0^\infty - Me^{-x}\,x\,F'\Big|_0^\infty .$$

If we assume that

$$|M'F| < M_1\, e^{sx}, \qquad M_1 \text{ constant}, \quad s < 1, \text{ as } x \to \infty,$$

$$|MF'| < M_2\, e^{tx}, \qquad M_2 \text{ constant}, \quad t < 1, \text{ as } x \to \infty,$$

then the right hand side of equation (20) will be equal to zero. Since the functions M(a,x), xe^{-x}, and λ are independent of F it follows that

(22) $[xe^{-x} M'(a,x)]' - \lambda(a)\ e^{-x} M(a,x) = 0 \qquad (0 \leq x < \infty$

The equation (22) along with the conditions that M(ax) is bounded at the origin and does not become infinite of an order higher than a positive power of x make up a Sturm - Liouville system. The values of λ for which this system has solutions that are not identically zero are the characteristic numbers [see the first one of equations (6)]

$$\lambda = \lambda_n \qquad (n = 0, 1, 2, \ldots)$$

of this system. Courant and Hilbert [5] show that the characteristic numbers here are the negative integers $\lambda = -n$. The characteristic functions corresponding to these values of λ are the so-called Laguerre polynomials. This family of characteristic functions is our kernel.

The integral transformation (20) becomes

(23) $$T[F(x)] = \int_0^\infty e^{-x} L_n(x) F(x)\ dx\ F(n) \qquad (n=0, 1, 2, \cdots);$$

we shall call it the Laguerre transformation and f(n) represents the Laguerre transform of F(x). In view of equation (18) this transformation resolves the form (16) as follows:

(24) $$T\{L[F(x)]\} = -n\ f(n)\ , \qquad (n=0, 1, 2, \cdots).$$

If we would have considered above the differential form [see eqs.(6)]

(25) $$\bar{L}[F] = (xF')' - \frac{x}{4} F$$

instead of the form (16) we would have arrived at the integral transformation

(26) $$\bar{T}[F] = \int_0^\infty e^{-\frac{x}{2}} L_n(x)\ F(x)\ dx = f(n) \qquad (n=0, 1, 2, \cdots)\ .$$

This transformation is seen to resolve the form [25] as follows:

(27) $$\bar{T}\{\bar{L}[F]\} = -(n + \frac{1}{2})\ \bar{T}\ [F]\ , \qquad (n=0, 1, 2, \cdots).$$

Application of the above process to the form [see equations (6)]

(28) $$L^*[F] = xF'' + (1+x)\ F' + F$$

leads to the integral transformation

(29) $$T^*[F] = \int_0^\infty L_n(x)\ F(x)\ dx.$$

The transformation (29) resolves the form (28) as follows:

(30) $$T^* \mathcal{L}^*\{F\} = -nf(n) , \quad (n=0, 1, 2,\cdots).$$

We notice in the three integral transformations (23), (26), and (29) that the function F(x) will have to satisfy different order properties in order for the transformation integral to exist.

We will now abandon this approach and proceed by centering our attention on the kernel function $e^{-x} L_n(x)$ and we will derive various properties for a linear integral transformation built on this kernel function.

CHAPTER II

LAGUERRE TRANSFORMS

1. The Transform

The sequence of numbers f(n) defined by the equation

$$f(n) = \int_0^\infty F(x)\, e^{-x}\, L_n(x)\, dx \qquad (n = 0, 1, 2, \ldots), \tag{31}$$

where $L_n(x)$ denotes the Laguerre polynomial of degree n, is the Laguerre transform of the transform F(x). The integral transformation here will be denoted by $T\{F(x)\}$.

The Laguerre transform of a function F(x) exists if F(x) is sectionally continuous in every finite interval in the range $x \geqq 0$ and if the function is $\mathcal{O}(e^{ax})$, $a < 1$ as x tends to infinity. Under the conditions stated, the integrand of the Laguerre integral is integrable over the finite interval $0 \leqq x \leqq x_0$ for every positive number x_0, and since $L_n(x)$ does not become infinite of an order higher than a finite power of x

$$|e^{-x}\, F(x)\, L_n(x)| < M\, e^{-bx}\ , \quad b > 0\ ,$$

where M is some constant. The integral of the function on the right exists. Hence the Laguerre integral converges absolutely when $a < 1$.

The inverse of this transformation is represented here by the expansion of F(x) in a series of the Laguerre polynomials. The inversion process here can be thought of as an expansion in an infinite series in terms of the eigenfunctions $L_n(x)$. This differs from the case of continuous spectra where a Fourier integral theorem would replace the eigenfunction expansion.

The inverse of the Laguerre transformation is then

$$F(x) = \sum_{n=0}^{\infty} f(n)\, L_n(x) = T^{-1}\{f(n)\} \qquad (0 < x < \infty). \tag{32}$$

Uspensky [18] gives the following conditions which will guarantee the convergence of the Laguerre series: let

(1) $\int_a^\infty e^{-x} [F(x)]^2 \, dx$ exist for a certain constant a,

(2) $\int_0^b x^{-\frac{1}{4}} |F(x)| \, dx$ exist for a certain value of b,

(3) F(x) be of bounded variation in a certain interval x-d, x+d, and absolutely integrable in any finite interval; then

$$\frac{1}{2}[F(x+0) - F(x-0)] = \sum_{n=0}^{\infty} a_n L_n(x),$$

where

$$a_n = \int_0^\infty e^{-x} L_n(x) F(x) \, dx.$$

It is the necessity of taking into account the infinite values of the variable that constitutes the essential difficulty of the problem of the development of arbitrary functions in series of Laguerre polynomials. The first two conditions above take care of the difficulties brought into the problem in such a way. The summability of the series has been discussed by E. Hille [7] and G. Szegö [16]. The Parseval theorem for the series has been investigated by S. Wigert [19].

Wigert [19] states the following theorem: "If the function F(x) is continuous for $x \geqq 0$, and the integral

$$\int_0^\infty e^{-ax} |F(x)| \, dx$$

converges for $a > \frac{1}{2}$, one has for $x \geqq 0$

$$\lim_{r \to 1} \sum_{n=0}^{\infty} f(n) L_n(x) r^n = F(x)."$$

Wigert shows that the hypotheses given on F(x) imply $\lim \sqrt[n]{|f(n)|} \leq 1$ which condition must be satisfied if

$$\sum_{n=0}^{\infty} f(n) L_n(x)$$

is to be convergent. Wigert demonstrates that the integral condition is necessary by considering the function $F(x) = e^{bx}$, $1 > b > \frac{1}{2}$. He shows the

Laguerre series does not converge. This illustrates a situation which occurs in other linear integral transformations. The function $F(x) = e^{bx}$, $1 > b > \frac{1}{2}$, has a transform, namely:

$$T\left\{e^{bx}\right\} = \frac{(-1)^n}{1-b} \left(\frac{b}{1-b}\right)^n \quad (1 > b > \frac{1}{2}),$$

but its Laguerre series does not converge. In other words conditions on a function insuring the inverse process are more severe than conditions necessary for the existence of the transform.

It follows from the inequality

$$e^{-\frac{x}{2}} |L_n(x)| \leq 1,$$

Szegö [16], that if $\sum_{n=0}^{\infty} |f(n)|$ converges, then

$$\sum_{n=0}^{\infty} f(n)\, L_n(x)$$

will converge and will represent the inverse transformation. That this condition is sufficient and not necessary can be seen from the expansion

$$\frac{1}{2} e^{\frac{x}{2}} = \sum_{n=0}^{\infty} (-1)^n L_n(x).$$

The above expansion is given in Wigert [19].

2. The Basic Operational Property

Let $L[F]$ denote the differential form

$$e^x \left[x e^{-x} F'\right]'. \tag{33}$$

When the integral $T\{L[F]\}$ is integrated successively by parts and $-nL_n(x)$ is substituted for $L[L_n(x)]$ in accordance with Laguerre's differential equation, the following result is easily obtained.

Theorem 1: Let $F(x)$ denote a function that satisfies these conditions: $F'(x)$ is continuous and $F''(x)$ is sectionally continuous over each finite interval contained in the range $x \geqq 0$; $F(x)$ and $F'(x)$ are $\mathcal{O}(e^{ax})$, $a < 1$, as x tends to infinity. Then $T\{L[F(x)]\}$ exists and

$$T\{L[F(x)]\} = -nf(n) \qquad (n=0, 1, 2, \cdots). \tag{34}$$

Formula (34) represents the first basic operational property of the Laguerre transformation T under which the differential operation L[F] defined by equation (33) is replaced by the algebraic operation -nf(n).

We note here that in deriving the first basic operational property we have in a sense reversed the procedure used in section 3 when we were establishing the form of the kernel which would annihilate form (33).

Relations (6) exhibited three forms of Laguerre's differential equation. We have seen in Theorem 1 the result of applying the Laguerre transform to the first of the three forms. In section 3 we obtained the kernels of integral transformations which would annihilate parts of the remaining forms in equation (6). We will now investigate the result of applying the Laguerre transform to the second and third equations in expression (6).

Let R[F] denote the differential form

$$e^{-x}\,[xe^{x}\,F']'. \tag{35}$$

Then by integration by parts we can write

$$T\{R[F]\} = \int_0^\infty e^{-2x}\,L_n(x)\,[xe^{x}\,F']'\,dx = -\int_0^\infty xF'\,e^{-x}\,L'_{n+1}(x)\,dx + \int_0^\infty e^{-x}\,xL_n(x)\,F'\,dx;$$

here we have used property (9) of Laguerre polynomials to write the first integral on the right. Integrating by parts again gives

$$T\{R[F]\} = \int_0^\infty [xe^{-x}\,L'_{n+1}(x)]'\,F(x)\,dx + \int_0^\infty e^{-x}\,xL_n(x)\,F'(x)\,dx.$$

We can replace the expression $[xe^{-x}\,L'_{n+1}(x)]'$ by $-(n+1)\,L_{n+1}(x)$ in the first integral on the right. In order to complete the derivation we must now find $T\{xF'\}$. By integration by parts we can write

$$T\{xF'\} = \int_0^\infty e^{-x}\,xL_n(x)\,F'(x)\,dx = -\int_0^\infty e^{-x}[xL'_n(x) + (1-x)\,L_n(x)]\,F(x)\,dx.$$

Use of properties (8) and (9) leads to the following:

$$xL_n'(x) - xL_n(x) = (n+1)\, L_{n+1}(x) - (n+1)\, L_n(x).$$

Hence

$$T\{xF'\} = -\int_0^\infty e^{-x}\, [(n+1)\, L_{n+1}(x) - nL_n(x)]\, F(x)\, dx$$

$$= -(n+1)\, f(n+1) + nf(n).$$

We can now write $T\{R[F]\}$ as follows:

$$T\{R[F]\} = -2(n+1)\, f(n+1) + nf(n).$$

We summarize the above in the following theorems.

Theorem 2: Let $F(x)$ denote a function that satisfies these conditions: $F'(x)$ is sectionally continuous over each finite interval in the range $x \geqq 0$, $F(x)$ is $\mathcal{O}(e^{ax})$, $a < 1$, as x tends to infinity. Then $T\{xF'\}$ exists and

(36) $$T\{xF'\} = -(n+1)\, f(n+1) + nf(n) \quad (n=0,\ 1,\ 2,\cdots).$$

Theorem 3: Let $F(x)$ denote a function that satisfies the conditions of Theorem 1. Then $T\{R[F]\}$ exists and

(37) $$T\{R[F]\} = -2(n+1)\, f(n+1) + f(n) \quad (n=0,\ 1,\ 2,\cdots).$$

We note in equations (36) and (37) that we are led to difference expressions in the transform.

Formula (37) will be called the second basic operational property of the Laguerre transform.

Let $S[F]$ denote the differential form

(38) $$(xF')'.$$

When the integral $T\{S[F]\}$ is integrated successively by parts and $L_{n+1}'(x)$ is substituted for $L_n'(x) - L_n(x)$ in accordance with property (9) the following result is readily obtained.

Theorem 4: Let $F(x)$ denote a function that satisfies the conditions of Theorem 1. Then $T\{S[F]\}$ exists and

(39) $$T\{S[F]\} = -(n+1)\, f(n+1) \quad (n=0,\ 1,\ 2,\cdots).$$

Formula (39) will be called the third basic operational property of the Laguerre transformation T under which the differential operation $S[F]$ has been replaced by the algebraic operation $-(n+1)\, f(n+1)$.

CHAPTER III

OPERATIONAL PROPERTIES

1. The Iterated Operator

We note here that the differential form of the fourth order $L^2[F(x)]$ is also resolved by the Laguerre transform. If each of the functions $L[F(x)]$ and $F(x)$ satisfy the sufficient conditions for the validity of formula (34) then the transform of the iterated differential form $L[L[F]]$ can be written as

$$(40) \qquad T\left\{L^2[F(x)]\right\} = n^2 f(n) \qquad (n=0, 1, 2, \cdots).$$

The process can be carried on in a similar fashion for iterations of higher order.

2. Differentiation and Indefinite Integration

The operational properties which arise from considering the effect of the Laguerre transform on differentiation and indefinite integration will be given here.

Let $F(x)$ be a continuous function whose first order derivative is bounded and integrable on each finite interval in the range $x \geqq 0$. Let $F(x)$ be $\mathcal{O}(e^{ax})$, $a < 1$, as x tends to infinity. One integration by parts of the integral

$$\int_0^\infty e^{-x} L_n(x) F'(x)\, dx,$$

and use of property (10) leads at once to the formula

$$(41) \qquad T\left\{F'(x)\right\} = \sum_{k=0}^{n} f(k) - F(0).$$

Formula (41) exhibits the image under the Laguerre transform T of the operation of differentiation.

In looking for a property in connection with indefinite integration we would desire a relation which would give the transform of the integral

$$\int_0^x F(t)\, dt$$

in terms of the transform of $F(x)$. Let $F(x)$ be a sectionally continuous function over every finite interval in range $x \geqq 0$, and let $G(x)$ denote the continuous function

$$G(x) = \int_0^x F(t)\, dt.$$

Then

$$f(n) = \int_0^\infty e^{-x}\, G'(x)\, L_n(x)\, dx = \int_0^\infty e^{-x}\, L'_{n+1}(x)\, G(x)\, dx$$

and it follows from relation (9) for $L_n(x)$ that

$$(42)\quad f(n-1) - f(n) = -\int_0^\infty e^{-x}\, G(x)[L'_{n+1}(x) - L'_n(x)]\, dx = -g(n) \quad (n=1,2,3,\cdots);$$

also,

$$\begin{aligned} f(0) &= \int_0^\infty e^{-x}\, G'(x)\, dx \\ &= e^{-x}\, G(x)\Big|_0^\infty + \int_0^\infty e^{-x}\, G(x)\, dx \\ &= g(0), \end{aligned}$$

and

$$\begin{aligned} f(1) &= \int_0^\infty e^{-x}\, (1-x)\, G'(x)\, dx \\ &= e^{-x}\, (1-x)\, G(x)\Big|_0^\infty + \int_0^\infty e^{-x} G(x) dx + \int_0^\infty e^{-x}(1-x) \\ &\qquad G(x)\, dx \\ &= g(0) + g(1). \end{aligned}$$

We have used in the above that $L_0(x) = 1$ and that $L_1(x) = 1-x$.

From the difference equation (42) for $g(n)$ and $f(n)$ we have the following conclusion:

Theorem 5: If $F(x)$ is sectionally continuous on each finite interval over the range $0 \leq x < \infty$, and $O(e^{ax})$, $a < 1$, as x tends to infinity, then

(43) $$T\left\{\int_0^x F(t)\,dt\right\} = f(n) - f(n-1) \qquad (n=1, 2, 3, \cdots),$$

and for n = 0, $g(0) = f(0)$.

In the above derivation we have used the fact that $G(x)$ is $O(e^{ax})$, $a < 1$, as x tends to infinity. Since $F(x)$ has the desired order property it follows by the use of simple properties of the Riemann integral that $G(x)$ does have the aforementioned order property.

Solution of the difference equation (42) for $f(n)$ leads to the conclusion:

Theorem 6: If $G(x)$ is continuous and $G'(x)$ sectionally continuous, and if $G(0) = 0$ and $G(x)$ and $G'(x)$ are $\mathcal{O}(e^{ax})$, $a < 1$, as x tends to infinity, then

(44) $$T\{G'(x)\} = g(0) + g(1) + g(2) + \cdots + g(n) \qquad (n=1, 2, \cdots),$$

$$= g(0) \qquad (n = 0).$$

It is interesting now to compare formula (44) with formula (41). We see that we have arrived at the same expression for the transform of a derivative as we had previously with the exception that the term $G(0)$ does not appear here.

3. The Inverse Operator

We will consider now the transform of the function $L^{-1}[F]$, where L^{-1} is the inverse of the differential operator L. Let $Y(x)$ denote the function $L^{-1}[F(x)]$; then $Y(x)$ is a solution of the differential equation

(45 $$L[Y(x)] = F(x).$$

Suppose that $F(x)$ is a function which is sectionally continuous in every finite interval in the range $x \geqq 0$, and that

(46) $$\int_0^\infty e^{-x} F(x)\,dx = 0;$$

that is, that the zero transform of $F(x)$ is zero. It follows from equation (45) that

(47) $$xe^{-x}\,Y'(x) = \int_0^x F(t)\,e^{-t}\,dt$$

is then a continuous function of x and has the limit zero as x tends to infinity. Hence $Y'(x)$ is continuous and $\mathcal{O}(e^{ax})$, $a < 1$, as x tends to infinity. The second integral can be written

$$(48) \qquad Y(x) = \int_0^x \frac{e^y}{y} \int_0^y F(t)\, e^{-t}\, dt\, dy + C = L^{-1}[F],$$

where C is an arbitrary constant. The function (48) is continuous and can be shown to have the necessary order property as x tends to infinity and hence $T\{Y\}$ exists.

According to Theorem 1 and equation (45) then

$$T\{L[Y]\} = -n\, T\{Y\} = f(n);$$

thus

$$(49) \qquad T\{L^{-1}[F]\} = -\frac{f(n)}{n} \qquad (n=1, 2, 3, \cdots).$$

The value of the transform of $L^{-1}[F]$ at $n = 0$ is given by

$$(50) \qquad T\{L^{-1}[F]\} = \int_0^\infty e^{-x} \int_0^x \frac{e^y}{y} \int_0^y F(t)\, e^{-t}\, dt\, dy\, dx + C .$$

The operational property concerning L^{-1} can be stated as follows:

Theorem 7: Let F(x) denote a function which is sectionally continuous in every finite interval in the range $x \geqq 0$, and let $f(0) = 0$; also let F(x) satisfy a certain order condition. Then f(n) exists and for each constant C,

$$(51) \quad T^{-1}\left\{\frac{-f(n)}{n}\right\} = L^{-1}[F(x)] = \int_0^x \frac{e^y}{y} \int_0^y F(t)\, e^{-t}\, dt\, dy + C \quad (n=1,2,\cdots).$$

4. Miscellaneous Remarks

Theorem 2 gave us the transform of xF'. We will now, for the sake of completeness, derive the transform of xF. Suppose that F(x) is a function which satisfies sufficient conditions for its Laguerre transform to exist. The integral

$$\int_0^\infty e^{-x} L_n(x)[xF(x)]\, dx$$

can be written as

$$\int_0^\infty e^{-x} [-(n+1) L_{n+1}(x) + (2n+1) L_n(x) - nL_{n-1}(x)]\, F(x)\, dx$$

by means of properties (8) and (9) of $L_n(x)$. Hence

$$(52)\qquad T\{xF\} = -(n+1)\, f(n+1) + (2n+1)\, f(n) - nf(n-1) \qquad (n=1, 2, 3, \cdots),$$

$$= \int_0^\infty xe^{-x} F(x)\, dx \qquad (n = 0).$$

Subtraction of equation (36) from (52) leads one immediately to the following operational property:

$$(53)\qquad T\{x(F-F')\} = (n+1)\, f(n) - nf(n-1) \qquad (n=1, 2, 3, \cdots).$$

Equations (52) and (53) are noted to be difference relations.

The following theorem follows from the linearity of T:

Theorem 8: If $T\{F(x)\}$ and $T\{G(x)\}$ exist, then

$$(54)\qquad T\{C_1 F(x) + C_2 G(x)\} = C_1 F(n) + C_2 g(n),$$

where C_1 and C_2 are constants.

When $G(x) = 1$, then $g(n) = 0$ for $n \neq 0$ and $g(0) = 1$; according to equation (54) then, if C is a constant,

$$(55)\qquad T\{F(x) + C\} = f(n) \qquad \text{when } n = 1, 2, \cdots,$$

$$= f(0) + C \qquad \text{when } n = 0\,.$$

The convolution property will be discussed in the next chapter. In a later chapter on Sonine transforms a property will be given which relates Laguerre transforms to Sonine transforms.

CHAPTER IV

THE LAGUERRE CONVOLUTION

1. Introduction

The convolution property of the transformation is one that expresses the inverse transform of the product of two transforms in terms of the two object functions without use of the inversion formula. We quote now from Churchill and Dolph [4].

"As in the operational calculus based upon Fourier and Laplace transforms, the convolution property makes possible a substantial extension of the tables of transforms and it leads to alternate forms, even closed forms of solutions of many boundary value problems."

Let F(x) and G(x) be two functions which are sectionally continuous over each finite interval in the range $x \geqq 0$, and $\mathcal{O}(e^{ax})$, $a < 1$, as x tends to infinity, and let

$$f(n) = L\{F(x)\}, \quad g(n) = L\{G(x)\} .$$

Then,

$$f(n)\, g(n) = \int_0^\infty e^{-x} L_n(x)\, F(x)\, dx \int_0^\infty e^{-y} L_n(y)\, G(y)\, dy \tag{56}$$

$$= \int_0^\infty e^{-x} F(x) \int_0^\infty e^{-y} G(y)\, L_n(x)\, L_n(y)\, dy\, dx.$$

It will be our aim to write equation (56) in the form

$$f(n)\, g(n) = \int_0^\infty e^{-t} L_n(t)\, [H(t)]\, dt. \tag{57}$$

The function H(t) will be the so-called convolution of the functions F(x) and G(x).

2. The Addition Property

We see in equation (56) that if we could express the product $L_n(x)\cdot L_n(y)$ in terms of a single Laguerre polynomial, we would then have a means of obtaining the form (57).

With the aid of the following addition formula from Bateman [1]:

$$(58)\qquad 2\pi\,\Gamma(m+n+1)(-1)^n\,[2(-xy)^{\frac{1}{2}}K]^m\,T_m^n(2i\,x\,k)\,T_m^n(2i\,y\,k) = \int_0^{2\pi} \exp\,[2(-xy)^{\frac{1}{2}}ke^{i\gamma} - im\gamma]\,T_o^m[2ik(x+y) - 4(-xy)^{\frac{1}{2}}k\cos\gamma]\,d\gamma$$

we will establish the convolution property.

Let us now simplify equation (58). Since

$$(59)\qquad T_m^n(x) = (-1)^n \frac{1}{\Gamma(m+n+1)} L_n^m(x)$$

we can write relation (58) in terms of Laguerre polynomials. We will also make the following substitution:

$$2ixk = x', \qquad 2iyk = y'.$$

Expression (58) now becomes

$$(60)\qquad 2\pi\,L_n(x)\,L_n(y) = \int_0^{2\pi} \exp\,(\sqrt{xy}\,e^{i\gamma})\,L_n(x+y - 2\sqrt{xy}\cos\gamma)\,d\gamma,$$

where we have dropped the primes.

We now assert that the imaginary part of the integral in expression (60) is zero. Since

$$(61)\qquad e^{i\gamma} = \cos\gamma + i\sin\gamma$$

the imaginary part of equation (60) can be written as

$$(62)\qquad \frac{i}{2\pi}\left[\int_0^{\pi} e^{\sqrt{xy}\cos\gamma}\sin(\sqrt{xy}\sin\gamma)\,L_n(x+y - 2\sqrt{xy}\cos\gamma)\,d\gamma + \int_{\pi}^{2\pi} e^{\sqrt{xy}\cos\gamma}\sin(\sqrt{xy}\sin\gamma)\,L_n(x+y - 2\sqrt{xy}\cos\gamma)\,d\gamma\right].$$

In the second integral in the brackets make the following substitution: let $2\pi - \phi = \gamma$. Then the second integral becomes

$$-\int_0^{\pi} e^{\sqrt{xy}\cos\phi} \sin(\sqrt{xy}\sin\phi)\, L_n(x+y-2\sqrt{xy}\cos\phi)\, d\phi,$$

and the imaginary part of equation (60) is zero. We now have

$$(63) \quad L_n(x)\, L_n(y) = \frac{1}{\pi}\int_0^{\pi} e^{\sqrt{xy}\cos\gamma}\cos(\sqrt{xy}\sin\gamma)\, L_n(x+y-2\sqrt{xy}\cos\gamma)\, d\gamma.$$

The form (63) follows by writing the real part of equation (60) in two parts and letting $\phi = 2\pi - \gamma$ in the integral with limits from π to 2π.

Equation (63) is the final form of the addition property as we shall use it in obtaining the convolution property of the Laguerre transforms.

Since

$$\int_0^{\pi} e^{\sqrt{xy}\cos\gamma}\cos(\sqrt{xy}\sin\gamma)\, d\gamma = \pi,$$

and $L_0(x) = 1$ and $L_n(x) = 1-x$ we see that the above property will check for $n = 0$ and $n = 1$. When $n = 0$ we have

$$1 = \frac{1}{\pi}\int_0^{\pi} e^{\sqrt{xy}\cos\gamma}\cos(\sqrt{xy}\sin\gamma)\, d\gamma.$$

When $n = 1$ we have

$$\begin{aligned}(1-x)(1-y) &= \frac{1}{\pi}\int_0^{\pi} e^{\sqrt{xy}\cos\gamma}\cos(\sqrt{xy}\sin\gamma)[1-(x+y)+2\sqrt{xy}\cos\gamma]\, d\gamma \\ &= \frac{1}{\pi}\int_0^{\pi} e^{\sqrt{xy}\cos\gamma}\cos(\sqrt{xy}\sin\gamma)\, d\gamma \\ &\quad - \frac{(x+y)}{\pi}\int_0^{\pi} e^{\sqrt{xy}\cos\gamma}\cos(\sqrt{xy}\sin\gamma)\, d\gamma \\ &\quad + \frac{2}{\pi}\sqrt{xy}\int_0^{\pi} e^{\sqrt{xy}\cos\gamma}\cos(\sqrt{xy}\sin\gamma)\cos\gamma\, d\gamma.\end{aligned}$$

This expression leads to the identity

$$(1-x)(1-y) = 1-(x+y)+xy.$$

3. The Convolution Property

In light of equation (57) we will now write the product (63) in a form where the Laguerre polynomial will have an argument of a single variable. Equation (63) will be transformed into

$$(64)\quad \pi L_n(x)\, L_n(y) = \int_{x+y-2\sqrt{xy}}^{x+y+2\sqrt{xy}} e^{\frac{1}{2}(x+y-t)}\, \frac{\cos\frac{1}{2}\left\{[4xy-(x+y+t)^2]^{\frac{1}{2}}\right\}}{[4xy-(x+y-t)^2]^{\frac{1}{2}}}\, L_n(t)\, dt$$

by letting

$$t = x+y - 2\sqrt{xy}\cos\gamma.$$

The product (56) now takes the following form:

$$(65)\quad \pi f(n)\, g(n) = \int_0^\infty \int_0^\infty \int_{x+y-2\sqrt{xy}}^{x+y+2\sqrt{xy}} e^{-t}\, L_n(t)\, F(x)\, G(y)\, H(x,y,t)\, dt\, dy\, dx,$$

where

$$H(x,y,t) = \frac{e^{-\frac{1}{2}(x+y-t)} \cos\frac{1}{2}\left\{[4xy-(x+y-t)^2]^{\frac{1}{2}}\right\}}{[4xy - (x+y-t)^2]^{\frac{1}{2}}}$$

We will now proceed to interchange the order of integration in equation (65) since we are aiming for a form similar to equation (57).

We see from Figure 1 that the interchange of the order of integration with respect to y and t affects the inner two integrals as follows:

$$\int_0^\infty \int_{x+y-2\sqrt{xy}}^{x+y+2\sqrt{xy}} e^{-t}\, L_n(t)\, F(x)\, G(y)\, H(x,y,t)\, dt\, dy =$$

$$\int_0^\infty \int_{x+t-2\sqrt{xt}}^{x+t+2\sqrt{xt}} e^{-t}\, L_n(t)\, F(x)\, G(y)\, H(x,y,t)\, dy\, dt.$$

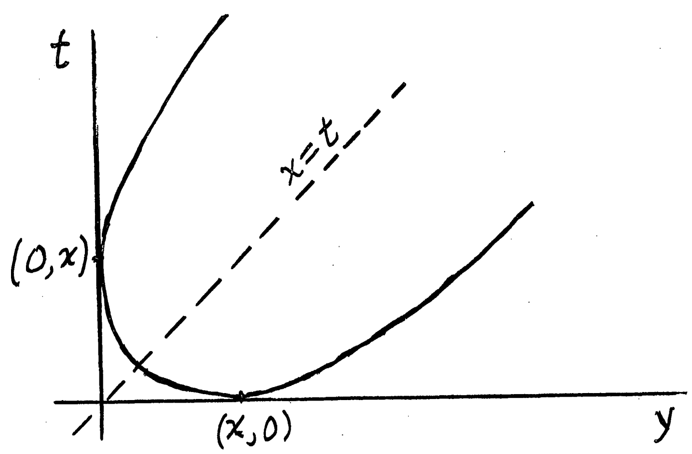

Figure 1.

We can complete the picture now by interchanging the order of integration with respect to x and t. The product (57) can now be written in the following form:

$$(66)\quad \pi f(n)\, g(n) = \int_0^\infty e^{-t} L_n(t) \left[\int_0^\infty \int_{x+t-2\sqrt{xt}}^{x+t+2\sqrt{xt}} F(x)\, G(y)\, H(x,y,t) dy\, dx \right] dt,$$

where

$$H(x,y,t) = \frac{e^{-\frac{1}{2}(x+y-t)} \cos\frac{1}{2} \left\{[4xy-(x+y-t)^2]^{\frac{1}{2}}\right\}}{[4xy-(x+y-t)^2]^{\frac{1}{2}}}$$

The expression in the square brackets in the product (66) is a function N(t) whose Laguerre transform is the product $\pi f(n)\, g(n)$. In this sense N(t) is the convolution F(x) *G(x) of the functions F(x) and G(x).

The product (66) is not in a convenient form for checking our result. With this end in mind we will attempt a simplification in the form of the function N(t).

Consider the region of integration for N(t) as shown in Figure 2.

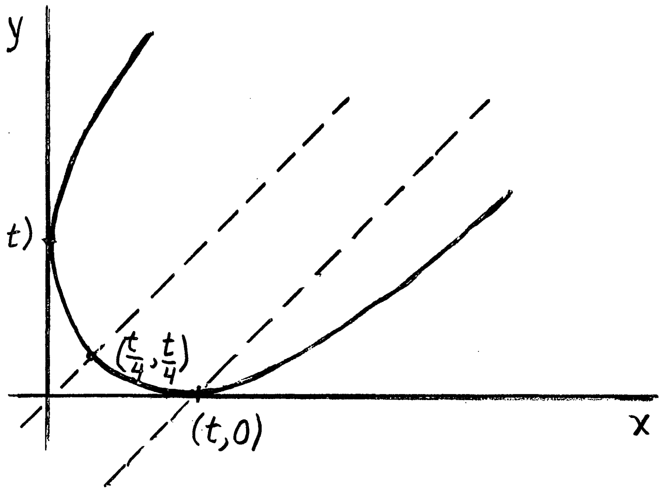

Figure 2.

In the integral

$$(67)\quad N(t) = \int_0^\infty F(x) \int_{x+t-2\sqrt{xt}}^{x+t+2\sqrt{xt}} e^{-\frac{1}{2}(x+y-t)} \frac{\cos\frac{1}{2}\left\{[4xy-(x+y-t)^2]^{\frac{1}{2}}\right\}}{[4xy-(x+y-t)^2]^{\frac{1}{2}}} G(y) dy\, dx.$$

We will make the change of variable

$$(68)\qquad 4xy-(x+y-t)^2 = 4y'^2, \qquad y' > 0.$$

Then,

$$(x-y+t)\,dy = 4y'\,dy' .$$

It follows from equation (68) that

$$(y-x-t)^2 = 4(tx-y'^2),$$

or

$$y-x-t = \pm 2\sqrt{tx-y'^2}.$$

Hence if

$$y > x+t \quad \text{then} \quad y-x-t = 2\sqrt{tx-y'^2},$$

and if

$$y < x+t \quad \text{then} \quad y-x-t = -2\sqrt{tx-y'^2}.$$

It is easily seen that $tx - y'^2 \geqq 0$. We consider

$$\begin{aligned} 4tx - 4y'^2 = 4tx - 4xy + (y+x-t)^2 &= (x+y)^2 - 2t(x+y) + t^2 - 4xy + 4xt \\ &= (y-x)^2 + t^2 - 2ty + 2tx \\ &= (x-y+t)^2 \geq 0 . \end{aligned}$$

In light of the above we write the inner integral in equation (67) as follows:

$$(69) \qquad \int_{x+t-2\sqrt{xt}}^{x+t} e^{-\frac{1}{2}(x+y-t)} \frac{\cos\frac{1}{2}\{[4xy-(x+y-t)^2]^{\frac{1}{2}}\}}{[4xy-(x+y-t)^2]^{\frac{1}{2}}} G(y)\,dy \; +$$

$$\int_{x+t}^{x+t+2\sqrt{xt}} e^{-\frac{1}{2}(x+y-t)} \frac{\cos\frac{1}{2}\{[4xy-(x+y-t)^2]^{\frac{1}{2}}\}}{[4xy-(x+y-t)^2]^{\frac{1}{2}}} G(y)\,dy .$$

In the expression (69) we will now make the change of variable indicated by equation (68). The resulting integral is

$$(70) \qquad e^{-x}\int_0^{\sqrt{xt}} \frac{\cos y'}{\sqrt{tx-y'^2}}\, e^{\sqrt{tx-y'^2}}\, G(x+t-2\sqrt{tx-y'^2})\,dy'$$

$$+e^{-x}\int_{\sqrt{xt}}^{0} e^{-\sqrt{tx-y'^2}} \frac{\cos y'}{-\sqrt{tx-y'^2}}\, G(x+t+2\sqrt{tx-y'^2})\,dy' .$$

Combining the two integrals in expression (70) we obtain

$$(71) \qquad e^{-x} \int_0^{\sqrt{xt}} \frac{\cos y'}{\sqrt{tx-y'^2}} \left[e^{\sqrt{tx-y'^2}}\, G(x+t-2\sqrt{tx-y'^2}) + e^{-\sqrt{tx-y'^2}}\, G(x+t+2\sqrt{tx-y'^2}) \right] dy' \; .$$

We can now write a second form of the Laguerre convolution, namely,

$$(72) \quad \pi f(n)\, g(n) = \int_0^{\infty} e^{-t} L_n(t) \left\{ \int_0^{\infty} e^{-x} F(x) \int_0^{\sqrt{xt}} \frac{\cos y}{\sqrt{tx-y^2}} \left[e^{\sqrt{tx-y^2}}\, G(x+t-2\sqrt{tx-y^2}) + e^{-\sqrt{tx-y^2}}\, G(x+t+2\sqrt{tx-y^2}) \right] dy\, dx \right\} dt \; .$$

In the above expression we have dropped the primes on the variable y.

In the inner integral in equation (72) we will make the following trigonometric substitution:

$$y = \sqrt{xt} \sin \theta,$$

then,

$$dy = \sqrt{xt} \cos \theta \, d\theta \; .$$

The inner integral now assumes the form

$$(73) \qquad \int_0^{\frac{\pi}{2}} \cos(\sqrt{xt} \sin\theta) \left[e^{\sqrt{xt}\cos\theta}\, G(x+t-2\sqrt{xt}\cos\theta) + e^{-\sqrt{xt}\cos\theta}\, G(x+t+2\sqrt{xt}\cos\theta) \right] d\theta \; .$$

The substitution $\theta = \pi - \phi$ in the integral

$$\int_0^{\frac{\pi}{2}} \cos(\sqrt{xt}\sin\theta)\, e^{-\sqrt{xt}\cos\theta}\, G(x+t+2\sqrt{xt}\cos\theta)\, d\theta$$

leads to the form

$$(74) \qquad \int_0^{\pi} e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta)\, G(x+t-2\sqrt{xt}\cos\theta)\, d\theta$$

for the integral (73).

We can now write a third form for the product (57). Use of the integral (74) leads to the form

$$(75)\quad \pi f(n)\, g(n) = \int_0^\infty e^{-t} L_n(t) \left[\int_0^\infty e^{-x} F(x) \int_0^\pi e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta) \cdot G(x+t-2\sqrt{xt}\cos\theta)\, d\theta\, dx \right] dt.$$

The foregoing can be summarized in the following theorem.

Theorem 9: Let $F(x)$ and $G(x)$ be sectionally continuous functions in every finite interval in the range $x \geqq 0$, and $\mathcal{O}(e^{ax})$, $a < 1$, as x tends to infinity. Then the product $f(n)\, g(n)$ of their Laguerre transforms is the transform of the function $H(t)$; that is,

$$(76)\qquad T^{-1}\{f(n)\, g(n)\} = H(t)$$

where $H(t)$ is given by the following formula:

$$(77)\quad H(t) = \int_0^\infty e^{-x} F(x) \int_0^\pi e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta)\, G(x+t-2\sqrt{xt}\cos\theta)\, d\theta\, dx.$$

4. Remarks

Let us note what has taken place in the previous section. We started by considering the form

$$(78)\quad \pi f(n)\, g(n) = \int_0^\infty e^{-x} F(x) \int_0^\infty e^{-y} G(y) \int_0^\pi e^{\sqrt{xy}\cos\theta} \cos(\sqrt{xy}\sin\theta, \; L_n(x+y-2\sqrt{xy}\cos\theta)\, d\theta\, dy\, dx\, .$$

After a moderate amount of manipulation we arrived at the form

$$(79)\quad \pi f(n)\, g(n) = \int_0^\infty e^{-t} L_n(t) \int_0^\infty e^{-x} F(x) \int_0^\pi e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta) \; G(x+t-2\sqrt{xt}\cos\theta)\, d\theta\, dx\, dt\, .$$

We now notice that we have obtained the form (79) from the product (78) by an interchange of role of one of the functions $F(x)$, $G(x)$ and the Laguerre polynomial involved. One might conjecture at this point that given an addition formula of the type

$$P_n(x)\, P_n(y) \;=\; C\int_0^{\pi} F(x,y,\theta)\, P_n(x,y,\theta)\, d\theta,$$

where C is a constant, for the kernel functions one could immediately write a convolution property for a calculus based upon these particular kernel functions. At the present the only two addition formulas known to the author to have been used to arrive at a convolution property are the Laguerre in the present paper and the Legendre by Churchill and Dolph [4]. The aforementioned interchange has occurred in both places.

Churchill and Dolph [4] consider the following product:

$$f(n)\, g(n) \;=\; \int_0^{\pi} F(\cos\mu)\, P_n(\cos\mu)\, \sin\mu\, d\mu \int_0^{\pi} G(\cos\lambda) P_n(\cos\lambda) \sin\lambda\, d\lambda,$$

where $P_n(x)$ is the Legendre polynomial of degree n. They use the addition formula

$$P_n(\cos\lambda)\, P_n(\cos\mu) \;=\; \frac{1}{\pi}\int_0^{\pi} P_n(\cos\alpha)\, d\nu\,,$$

$$\cos\alpha \;=\; \cos\lambda\, \cos\mu + \sin\lambda\, \sin\mu\, \cos\nu\,,$$

to derive their convolution property.

If we rewrite the form of the product f(n) g(n) and take advantage of the addition property we can by the above conjecture immediately write the convolution for the calculus of the Legendre transform.

We have

$$f(n)\, g(n) \;=\; \frac{1}{\pi}\int_0^{\pi} P_n(\cos\nu)\, \sin\nu \left[\int_0^{\pi} F(\cos\mu)\, \sin\mu \int_0^{\pi} G(\cos\mu\, \cos\nu + \sin\mu\, \sin\nu\, \cos\beta)\, d\beta\, d\mu\right] d\nu.$$

This is the form given by Churchill and Dolph [4] in expression (9) on page 96.

One difficulty at the present time in pursuing this further is the lack of such addition properties in the literature.

To give confidence in the work of section 13 and to get acquainted with the convolution property we will consider here a check of the property in a few simple cases and then suggest a possible check in the general case.

Suppose first that both F(x) and G(x) are constant functions. In this case H(t) should be a constant function, since f(n) g(n) will be different from zero only when n=0. When F(x) and G(x) are both constant, H(t) takes the following form:

(80)
$$H(t) = \int_0^\infty e^{-x} M_1 \int_0^\pi e^{\sqrt{xt}\cos\Theta} \cos(\sqrt{xt}\sin\Theta) M_2 \, d\Theta \, dx$$
$$= M \int_0^\infty e^{-x} \int_0^\pi e^{\sqrt{xt}\cos\Theta} \cos(\sqrt{xt}\sin\Theta) \, d\Theta \, dx,$$

where
$$F(x) = M_1, \quad G(x) = M_2 \text{ and } M_1M_2 = M .$$

Since
$$\int_0^\pi e^{\sqrt{xt}\cos\Theta} \cos(\sqrt{xt}\sin\Theta) \, d\Theta = \pi$$

we have
$$H(t) = M\pi \int_0^\infty e^{-x} dx = M\pi .$$

Hence H(x) is a constant and will have a transform equal to zero for $n \neq 0$ and a transform of $M\pi$ for $n = 0$. This result is seen to check with the product $\pi f(n)\, g(n)$ for F(x) and G(x) constant functions. The product would be zero for $n \neq 0$ and $\pi M_1 M_2 = \pi M$ for $n = 0$.

As a second example let us consider the case $G(x) \equiv 1$ and F(x) arbitrary. H(t) again should turn out to be a constant function. For F(x) and G(x) having the above forms

(81)
$$H(t) = \int_0^\infty e^{-x} F(x) \int_0^\pi e^{\sqrt{xt}\cos\Theta} \cos(\sqrt{xt}\sin\Theta) \, d\Theta \, dx$$
$$= \pi \int_0^\infty e^{-x} F(x) \, dx$$
$$= \pi f(0).$$

Hence H(t) is a constant and also is the constant we would hope for since in this case the product $\pi f(n)\, g(n)$ has exactly the value $\pi f(0)$.

Consider now the special case when F(x) is arbitrary but $G(x) = L_m(x)$. Then by property (15) we can write

$$(82)\quad f(n)\,g(n) = \int_0^\infty e^{-t} L_n(t)\, L_m(t)\, dt \int_0^\infty e^{-y} L_m(x)\, F(x)\, dx$$

$$= 0 \quad \text{if} \quad m \neq n$$

$$= f(m) \quad \text{if} \quad n = m.$$

On the other hand $g(n) = 0$ when $n \neq m$ and $g(m) = 1$. Hence the left-hand side of expression (82) is $f(n)$ when $n = m$, and 0 when $n \neq m$.

Let us assume that $F(x)$ and $G(x)$ are arbitrary functions in the sense that their Laguerre transforms exist and

$$\sum_{n=0}^{\infty} |g(n)| \quad , \qquad \sum_{n=0}^{\infty} |f(n)\, g(n)|$$

converge and $|F(x)| \leq M\, e^{ax}$, $a < \frac{1}{2}$.

From the product (79) we consider the function $G(x+t-2\sqrt{xt}\cos\theta)$. We write this function in terms of its series expansion

$$(83)\quad G(x+t-2\sqrt{xt}\cos\theta) = \sum_{m=0}^{\infty} \left[\int_0^\infty e^{-y}\, G(y)\, L_m(y)\, dy\right] L_m(x+t-2\sqrt{xt}\cos\theta).$$

Substitution of the expression (83) into the product (79) leads to the product

$$(84)\quad \pi f(n)\,g(n) = \int_0^\infty e^{-t} L_n(t) \left[\int_0^\infty e^{-x} F(x) \sum_{m=0}^{\infty} \int_0^\infty e^{-y}\, G(y)\, L_m(y)\, dy \right.$$

$$\left. \int_0^\infty e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta)\, L_m(x+y-2\sqrt{xt}\cos\theta)\, d\theta\, dx\right] dt.$$

Here we have interchanged the order of integration with respect to θ and summation. This is legitimate since

$$\left|e^{-\frac{t}{2}-\frac{x}{2}+\sqrt{xt}\cos\theta}\, L_n(x+t-2\sqrt{xt}\cos\theta)\cos(\sqrt{xt}\sin\theta)\, g(m)\, e^{\frac{t+x}{2}}\right| < e^{\frac{t+x}{2}} |g(m)|,$$

which is independent of θ, and from the assumption on $\sum_{n=0}^{\infty} |g(n)|$ this series of constants converge.

We have used the inequality

$$e^{-\frac{x}{2}} |L_n(x)| \leq 1 \quad , \qquad (n=0,\ 1,\ 2, \cdots ,\ x \geq 0).$$

This inequality can be found in Szegö [16].

By the addition property (15) we can simplify the expression (84) as follows:

$$(85)\quad f(n)\,g(n) = \int_0^\infty e^{-t} L_n(t) \left[\int_0^\infty e^{-x} F(x) \sum_{m=0}^\infty \int_0^\infty e^{-y} G(y) L_n(y)\,dy \; L_m(x)\,dx \right] L_m(t)\,dt.$$

We now wish to interchange the order of summation and integration with respect to x. This can be done since

$$|e^{-x} F(x)\, g(m)\, L_m(x)| = e^{-\frac{x}{2}} |F(x)| e^{-\frac{x}{2}} |L_m(x)|\; |g(m)| < M|g(m)| \;.$$

Hence

$$f(n)\,g(n) = \int_0^\infty \sum_{m=0}^\infty e^{-t} L_n(t)\, f(m)\, g(m)\, L_m(t)\,dt.$$

We can interchange the order of integration and summation here since

$$|e^{-\frac{t}{2}} L_n(t)\, e^{-\frac{t}{2}} L_m(t)\, f(m)\, g(m)| \leq |f(m)\, g(m)| \;.$$

The product (84) can now be written

$$(86)\quad f(n)\,g(n) = \sum_{m=0}^\infty \int_0^\infty e^{-t} L_n(t) L_m(t)\,dt \int_0^\infty e^{-x} F(x) L_m(x)\,dx \int_0^\infty e^{-y} G(y) L_m(y)\,dy.$$

By the orthogonality property (5) of $L_n(x)$ we see that every term in the expansion is zero except when $m = n$. Hence

$$(87)\qquad f(n)\,g(n) = \int_0^\infty e^{-x} F(x) L_n(x)\,dx \int_0^\infty e^{-y} G(y) L_n(y)\,dy \;.$$

Expression (87) is an identity by definition of the Laguerre transform.

The conditions here could be weakened by using the fact that if the series is multiplied by e^{-x} it can be integrated term by term from zero to infinity and it is only necessary to check the resulting series for ordinary convergence. In view of this we write expression (85) as

$$f(n)\ g(n) = \int_0^\infty e^{-t}\ L_n(t) \sum_{m=0}^{\infty} g(m) \int_0^\infty e^{-x}\ F(x)\ L_m(x)\ dx\ L_m(t)\ dt$$

$$= \int_0^\infty e^{-t}\ L_n(t) \sum_{m=0}^{\infty} g(m)\ f(m)\ L_m(t)\ dt\ .$$

Hence had we required $\sum_{n=0}^{\infty} g(m)\ f(m)$ to converge instead of $\sum_{n=0}^{\infty} |g(m)\ f(m)|$ we would have had sufficient behavior to integrate the series. The same argument will hold for the remaining interchange of integration with respect to t and summation.

The convolution integral can be given a geometric interpretation. Consider Figure 3 in connection with a possible means of obtaining the interpretation.

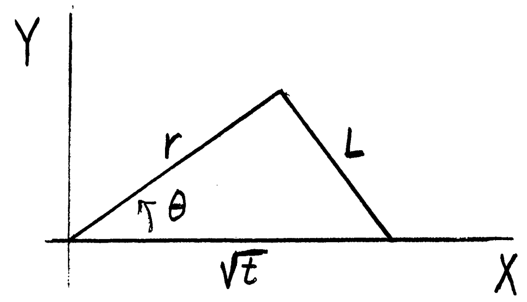

Figure 3.

In the integral

$$(88)\quad H(t) = 2\int_0^\infty e^{-x}\ F(x) \int_0^\pi e^{\sqrt{xt}\ \cos\theta} \cos(\sqrt{xt}\ \sin\theta)\, G(x+t-2\sqrt{xt}\ \cos\theta) \frac{d\theta\ dx}{2}$$

let

$$\sqrt{x} = r\ ,$$

and

$$L^2 = r^2 + t - 2r\sqrt{t}\ \cos\theta;$$

then

$$(89)\quad H(t) = 2 \iint_{(Y>0)} F(r^2)\ G(L^2)\ e^{-r^2}\ e^{\sqrt{t}\ X} \cos(\sqrt{t}\ Y)\ dA.$$

We can write equation (89) as

$$(90) \qquad H(t) = 2e^{t} \iint_{(Y>0)} G(L^2)\ F(r^2)\ e^{-(L^2+r^2)} \cos(\sqrt{t}\ Y)\ d\ .$$

Hence we have the integral over the upper half plane where X, Y are coordinates of the point in question and r and L are distances from the origin and the point $(\sqrt{t},0)$ for fixed t, respectively.

Since the finding of the convolution property has been so closely tied up with the addition property for the Laguerre polynomials it seems natural to consider the question concerning the possibility of obtaining the addition property from the convolution formula.

Consider the product

$$\pi f(n)\ g(n) = \int_0^\infty e^{-t} L_n(t) \int_0^\infty e^{-x} F(x) \int_0^\pi e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta)$$

$$G(x+t-2\sqrt{xt}\cos\theta)\ d\theta\ dx\ dt$$

when $F(x) = L_m(x)$ and $G(y) = L_m(y)$. The product then becomes

$$(91) \qquad \pi f(n)\ g(n) = \int_0^\infty e^{-t} L_n(t) \int_0^\infty e^{-x} L_m(x) \int_0^\pi e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta)$$

$$L_m(x+t-2\sqrt{xt}\cos\theta)\ d\theta\ dx\ dt\ .$$

The left-hand side of the expression (91) is 0 if $n \neq m$ and is π if $m = n$. Hence if we write

$$H(x,t) = \int_0^\pi e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta)\ L_m(x+t-2\sqrt{xt}\cos\theta)\ d\theta\ ,$$

the product (91) can be written

$$(92) \qquad \pi \int_0^\infty e^{-t} L_n(t)\ L_m(t)\ dt = \int_0^\infty e^{-t} L_n(t) \left[\int_0^\infty e^{-x} L_m(x)\ H(x,t)\ dx\right] dt\ .$$

We conclude from the expression (92) that

$$\pi L_m(t) = \int_0^\infty e^{-x} L_m(x)\ H(x,t)\ dt.$$

If we assume the uniqueness of the Laguerre transform we can immediately conclude that

$$\frac{1}{\pi} H(x,t) = L_m(x)\, L_m(t),$$

since this form of $H(x,t)$ will have the Laguerre transform $L_m(t)$.

In light of the above we have obtained the addition property (15) of the Laguerre polynomials. That is

$$\pi L_m(t)\, L_m(x) = \int_0^\infty e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta)\, L_m(x+t-2\sqrt{xt}\cos\theta)\, d\theta.$$

The polynomial $L_m(t)$ has an expansion, a finite one, and if it can be shown that the function

$$\int_0^\infty e^{-x} L_m(x)\, H(x,t)\, dx$$

has a convergent expansion, then we are justified in using the uniqueness property.

CHAPTER V

TRANSFORMS OF PARTICULAR FUNCTIONS

1. Simple Transforms

It follows at once from the orthogonality property of the Laguerre polynomials that when $F(x) = K$, K a constant, then

(93) $$f(n) = 0 \qquad (n \neq 0)$$
$$f(0) = K$$

The orthogonality property also shows that when $F(x) = KL_m(x)$, $(m=0, 1, 2\cdots)$, K a constant, then

(94) $$f(n) = 0 \qquad (n \neq m)$$
$$f(m) = K$$

The following integral

(95) $$\int_0^\infty e^{-(1-t)x} L_n(x)\, dx = \frac{(-1)^n}{1-t}\left(\frac{t}{1-t}\right)^n, \quad (n=0, 1, 2,\cdots),$$

where $0 < t < 1$, leads to the transform

(96) $$T\left\{e^{tx}\right\} = \frac{(-1)^n}{1-t}\left(\frac{t}{1-t}\right)^n, \qquad (n=0, 1, 2\cdots).$$

Since

(97) $$\int_0^\infty x^m e^{-x} L_n(x)\, dx = (-1)^n \binom{m}{n} m! \quad (m \geqq n),$$
$$= 0 \qquad (m < n),$$

m an integer, we can write the transform of x^m.

Since for $x \geqq 0$ and λ any real number > 0, we have the absolutely convergent expansion

$$(98) \qquad \frac{x^\lambda}{\Gamma(\lambda + 1)} = \sum_{n=0}^{\infty} (-1)^n \binom{\lambda}{n} L_n(x)$$

we have

$$(99) \qquad T\{x^\lambda\} = (-1)^n \Gamma(\lambda+1) \binom{\lambda}{n} .$$

Expressions (95), (97), and (98) can be found in Wigert [19].

2. Generating Functions and Laguerre Transforms

From the uniformly convergent power series

$$(100) \qquad \frac{1}{1-t} \exp\left(\frac{-xt}{1-t}\right) = \sum_{n=0}^{\infty} L_n(x)\, t^n , \quad (|t| < 1) ,$$

it follows that

$$(101) \qquad \int_0^\infty e^{-x} L_n(x) \left[\frac{1}{1-t} \exp\left(\frac{-xt}{1-t}\right)\right] dx = t^n \quad (|t| < 1) .$$

From the generating function

$$(102) \qquad e^t J_0(2\sqrt{xt}) = \sum_{n=0}^{\infty} L_n(x) \frac{t^n}{n!}$$

it follows that

$$(103) \qquad \int_0^\infty e^{-x} L_n(x) \left[e^t J_0(2\sqrt{xt})\right] dx = \frac{t^n}{n!} .$$

From Erdelyi [6] we have the generating function

$$(104) \qquad \frac{1}{1-z} \exp\left(-z\,\frac{x+y}{1-z}\right) I_0\left[2\,\frac{(xyz)^{\frac{1}{2}}}{1-z}\right] = \sum_{n=0}^{\infty} L_n(x)\, L_n(y)\, z^n , \quad (|z|<1).$$

It follows from equation (104) that

$$(105) \qquad \int_0^\infty e^{-x} L_n(x) \left\{\frac{1}{1-z} \exp\left(-z\,\frac{x+y}{1-z}\right) I_0\left[2\,\frac{(xyz)^{\frac{1}{2}}}{1-z}\right]\right\} dx = L_n(y)\, z^n \quad (|z|<1)$$

3. Products of Transforms

According to the convolution property (75) we can write some new transforms. From properties (96) and (98) we find

$$(106)\quad T\left\{\int_0^\infty e^{-x}\, x^m \int_0^\pi e^{\sqrt{xt}\cos\theta}\cos(\sqrt{xt}\sin\theta)\, e^{r(x+t-2\sqrt{xt}\cos\theta)}d\theta\, dx\right\}$$

$$= 0 \quad \text{when} \quad (n > m),$$

$$= \pi\,\frac{m!}{1-r}\left(\frac{r}{1-r}\right)^m \left(\frac{r}{1-r}\right)^m (n = m).$$

and

$$= \pi\,\frac{(-1)^{n+m}m!}{1-r}\left(\frac{r}{1-r}\right)^m \binom{n}{m-n} \quad (n < m).$$

From properties (96) and (101) it follows that

$$(107)\left\{T\int_0^\infty e^{-x}\left[\frac{1}{1-r}\exp\left(\frac{-xr}{1-r}\right)\right]\int_0^\pi e^{\sqrt{xt}\cos\theta}\cos(\sqrt{xt}\sin\theta)\right.$$

$$\left. e^{r(x+t-2\sqrt{xt}\cos\theta)}\, d\theta\, dx\right\}$$

$$= \pi\, r^n\,\frac{(-1)^n}{1-r}\left(\frac{r}{1-r}\right)^n$$

$$= (-1)^n\,\pi\,\frac{r^{2n}}{(1-r)^{n+1}}, \quad (|r| < 1).$$

This method can be used to write other transforms; however, the integrals will be in many cases difficult to evaluate.

4. TABLE OF LAGUERRE TRANSFORMS

$f(n) \quad (n=0, 1, 2, \cdots)$	$F(x) \quad (0 < x < \infty)$
1. 1 if $n = m$, 0 if $n \neq m$	$L_m(x) \qquad (m = 0, 1, 2, \cdots)$
2. K if $n = 0$, 0 if $n \neq 0$	$K \qquad$ (constant)
3. $\frac{(-1)^n}{1-t}\left(\frac{t}{1-t}\right)^n$	$e^{tx} \qquad (1 < t < 0)$
4. 0 if $n > m$ $(-1)^m m!$ if $n = m$ $(-1)^n m! \binom{m}{m-n}$ if $n < m$	x^m
5. t^n	$\frac{1}{1-t} \exp\left(\frac{-xt}{1-t}\right) \qquad (\lvert t \rvert < 1)$
6. $\frac{t^n}{n!}$	$e^t J_0 (2\sqrt{xt})$
7. $L_n(y)\, t^n$	$\frac{1}{1-t} \exp\left(-t\,\frac{x+y}{1-t}\right) I_0 \left[\frac{2(xyt)^{\frac{1}{2}}}{1-t}\right] \qquad (\lvert t \rvert < 1)$
8. 0 if $n > m$ $\frac{\pi m!}{1-r}\left(\frac{r}{1-r}\right)^m$ if $n = m$ $\pi \frac{(-1)^{n+m}\, m!\, r^m}{(1-r)^{m+1}} \binom{m}{m-n}$ when $n < m$	$\int_0^\infty e^{-t}\, t^m \int_0^\pi e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta)\, e^{r(x+t-2\sqrt{xt}\cos\theta)}\, d\theta\, dt$
9. $\frac{(-1)^n \pi r^{2n}}{(1-r)^{n+1}}$	$\int_0^\infty e^{-t} \left[\frac{1}{1-r} \exp\left(\frac{-tr}{1-r}\right)\right] \int_0^\pi e^{\sqrt{xt}\cos\theta} \cos(\sqrt{xt}\sin\theta)\, e^{r(x+t-2\sqrt{xt}\cos\theta)}\, d\theta\, dt$ $(\lvert r \rvert < 1)$
10. $(-1)^n \Gamma(\lambda+1)\binom{\lambda}{n}$	x^λ, $(x \geqq 0,\ \lambda > 0$ a real number$)$

5. TABLE OF OPERATIONAL PROPERTIES

	F(x)	f(n)
1.	$F(x)$	$\int_0^\infty e^{-x} L_n(x) F(x)\, dx$
2.	$F(x) + C$, C a constant	$f(n) \quad (n=1, 2, 3, \cdots)$ $f(0) + C \quad (n=0)$
3.	$F(x) + L_m(x)$	$f(n) \quad (n \neq m)$ $f(n) + 1 \quad (n=m)$
4.	$F'(x)$	$\sum_{k=0}^{n} f(k) - F(0)$
5.	$\int_0^x F(t)\, dt$	$f(n) - f(n-1) \quad (n=1, 2, 3, \cdots)$ $f(0) \quad (n=0)$
6.	$xF''(x) + (1-x) F'(x)$	$-nf(n) \quad (n=0, 1, 2, \cdots)$
7.	$xF(x)$	$-(n+1) f(n+1) + (2n+1) f(n) - nf(n-1)$
8.	$xF'(x)$	$-(n+1) f(n+1) + nf(n)$
9.	$x[F(x) - F'(x)]$	$(n+1) f(n) - nf(n-1)$
10.	$[xF'(x)]'$	$-(n+1) f(n+1)$
11.	$e^{-x} [xe^{x} F'(x)]'$	$-2(n+1) f(n+1) + nf(n)$
12.	$\int_0^\infty e^{-x} F(x) \int_0^\pi e^{\sqrt{xt} \cos \theta} \cos(\sqrt{xt} \sin \theta)\, G(x+t-2\sqrt{xt} \cos \theta)\, d\theta\, dx$	$\pi f(n)\, g(n)$

CHAPTER VI

EXAMPLES AND APPLICATIONS

1. Introduction

In this chapter we will indicate possible uses of the Laguerre transform. It was for a while thought that the Laplacian operator in paraloidal coordinates would lead to a natural application of the Laguerre transform. Up to this time, however, nothing promising has resulted.

2. The Transform and Laguerre's Equation

Consider the differential equation with the parameter λ

(108) $$xF''(x) + (1-x) F'(x) + \lambda F(x) = 0.$$

The Laguerre transform applied to equation (108) gives

(109) $$(\lambda-n) f(n) = 0.$$

From equation (109) we see that for $\lambda \neq n$ $f(n) = 0$ for all n. Hence there is no function F(x) satisfying equation (109) and the conditions under which the first basic operational property is valid. For $\lambda = n$ the equation has the polynomial solutions known as Laguerre polynomials.

The transformation, when applied to the following differential equation:

(110) $$xV''(x) + (1-x) V'(x) + \lambda V(x) = F(x)$$

gives rise to

$$v(n) = \frac{f(n)}{\lambda-n}, \qquad (n=0, 1, 2,\cdots).$$

If $\lambda = n$ we see that $f(n) = 0$ for all n. This is connected with a theorem from differential equations to the effect that if $\lambda = n$, n an integer, then the system (5) can have a solution only if F(x) is orthogonal to the solution of the homogeneous case. In this case F(x) would have to be orthogonal to all the $L_n(x)$. The Laguerre polynomials are complete, however, and hence $F(x) = 0$.

For $\lambda \neq n$ we can appeal to the expansion theorem to find V(x). Hence

$$V(x) = \sum_{n=0}^{\infty} \frac{f(n)}{\lambda - n} L_n(x).$$

The third basic operational property offers an example which leads to a known generating function for the Laguerre polynomials. We can consider here that the Laguerre transform can be used to establish that a particular differential equation is a form of Bessel's equation of index zero.

The Laguerre transform applied to the differential equation

(111) $$xV''(x) + V'(x) + \lambda V(x) = 0$$

immediately gives

(112) $$-(n+1)\, v(n+1) = -\lambda v(n), \quad (n=0, 1, 2, \cdots).$$

We note here that if $\lambda = n$ we would obtain

$$\frac{v(n+1)}{v(n)} = \frac{n}{n+1}$$

and hence $v(n) = 0$ for $(n=1, 2, 3, \cdots)$. Hence the only possible choice of V(x) would be a constant function. The only possible choice of $v(0) = C$ which would satisfy the equation would be $C = 0$.

When $\lambda \neq n$ we obtain,

(113) $$\frac{v(n+1)}{v(n)} = \frac{\lambda}{n+1} \qquad (n=0, 1, 2, \cdots).$$

From the difference equation (113) we see that

$$v(n) = \frac{\lambda^n}{n!} v(0).$$

Hence appealing to the inversion formula we write

(114)
$$V(x) = V(0) \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} L_n(x),$$

but
$$T\{e^{\lambda} J_0 (2\sqrt{\lambda x})\} = \frac{\lambda^n}{n!} .$$

Hence
$$\frac{V(x)}{V(0)} = e^{\lambda} J_0 (2\sqrt{\lambda x}) .$$

If the equation (111) is solved by series method one will obtain Rainville's Case II [12]. The nonlogarithmic solution will be

(115)
$$V(x) = a_0 \sum_{n=0}^{\infty} (-1)^n \frac{\lambda^n}{(n!)^2} x^n .$$

but
$$\sum_{n=0}^{\infty} (-1)^n \frac{(\lambda x)^n}{(n!)^2} = J_0 (2\sqrt{\lambda x}).$$

Hence for $a_0 = e^{\lambda}$ and $v(0) = 1$ we have a previous result obtained from the generating function of the Laguerre polynomials.

For $a_0 = e^{\lambda}$ and $v(0) = 1$ we have

$$\int_0^{\infty} e^{-x} e^{\lambda} J_0 (2\sqrt{x\lambda})\, dx = 1$$

or

$$e^{\lambda} \int_0^{\infty} e^{-x} J_0 (2\sqrt{x\lambda})\, dx = 1$$

or

$$\int_0^{\infty} e^{-x} J_0 (2\sqrt{x\lambda})\, dx = e^{-\lambda} .$$

We note here that if one makes the change of independent variable, in equation (111), $2\sqrt{\lambda x} = z$, equation (111) will reduce to Bessel's equation of index zero.

Application of property (8) in the Table of Operational Properties to the simple differential equation

(116) $$xF'(x) - F = 0$$

will serve to illustrate the use of another property which involves difference relations in the transforms.

The transformed problem becomes

(117) $$-(n+1)\, f(n+1) + nf(n) - f(n) = 0 \; ,$$

or

$$f(n+1) = \frac{n-1}{n+1}\, f(n) \; .$$

Hence f(0) is arbitrary and for

$$\begin{aligned} n = 0: &\quad f(1) = -f(0) \\ n = 1: &\quad f(2) = 0 \\ n \geqq 2: &\quad f(n) = 0 \; . \end{aligned}$$

Hence

$$\begin{aligned} F(x) &= \sum_{n=0}^{\infty} f(n)\, L_n(x) \\ &= f(0) + f(1)\,(1-x) \\ &= f(0) - f(0)\,(1-x) \\ &= f(0)\, x \; , \end{aligned}$$

and the general solution of equation(116) is an arbitrary constant times x.

We note here that the equation

$$xF'(x) + F(x) = 0$$

can not be solved by use of the Laguerre transform. Here we would obtain f(n) = f(0) for all n, and f(0) arbitrary. The series $f(0)\sum_{n=0}^{\infty} L_n(x)$ does not converge. The solution of this equation is $F(x) = \frac{c}{x}$ which is not bounded at the origin.

The use of properties which introduce difference relations seem possibly to be of use in considering problems in ordinary differential equations but the use of such properties when considering partial differential equations does not appear promising.

3. Partial Differential Equations and the Laguerre Transform

We will apply the Laguerre transform to the following problem:

$$\frac{\partial^2 u}{\partial y^2} + x\frac{\partial^2 u}{\partial x^2} + (1-x)\frac{\partial u}{\partial x} = 0, \tag{119}$$

$$u(x,0) = F(x) \qquad \lim_{y\to\infty} u(x,y) = 0, \tag{120}$$

$$|u(0,y)| < M \qquad |u(xy)| < M\,e^{ax},\ a<1,\ \text{as } x \to \infty, \tag{121}$$

where F(x) is a function such that its Laguerre series converges and

$$\sum_{n=0}^{\infty} |n^2 f(n)|$$

converges.

Let $\overline{u}(n,y) = T\{u(x,y)\}$. The transformed problem becomes

$$\frac{d^2\overline{u}}{dy^2} - n\,\overline{u}(n,y) = 0 \tag{122}$$

$$\overline{u}(n,0) = f(n) \qquad \lim_{y\to\infty} \overline{u}(n,y) = 0 . \tag{123}$$

Here we use the symbol for ordinary rather than partial differentiation since n is involved in the new problem only as a parameter. Differentiation occurs only with respect to y. We have used the conditions (121) already in writing the transformed problem.

The general solution of equation (122) is

$$\overline{u}(n,y) = C_1\,e^{-\sqrt{ny}} + C_2\,e^{\sqrt{ny}}, \tag{124}$$

where C_1 and C_2 may be functions of n.

In obtaining equations (123) we have interchanged the order of taking the limit as $y \to \infty$ and integration with respect to x. If we verify our final result, we need not be concerned with conditions under which these processes may be interchanged. The simplest conditions under which this interchange of order of operations is valid, as well as the one above concerning partial differentiation with respect to y, involve the uniform convergence with respect to y of the Laguerre integrals and the continuity of the integrals with respect to the two variables x and y.

We see from the transformed boundary conditions above that we must take $C_2 = 0$. The first condition above will give C_1. Since $\bar{u}(n,0) = C_1$ we have $C_1 = f(n)$, and thus

(125) $$\bar{u}(n,y) = f(n)\, e^{-\sqrt{ny}} .$$

We can appeal to the inversion formula to write

(126) $$u(xy) = \sum_{n=0}^{\infty} [f(n)\, e^{-\sqrt{ny}}]\, L_n(x) .$$

Our aim now is to show that the series (126) found above represents a function u(x,t) which satisfies all the conditions of the boundary value problem.

The above representation of u(x,) is seen to satisfy the boundary conditions. If y = 0

$$u(x,0) = \sum_{n=0}^{\infty} f(n)\, L_n(x)$$

which converges to F(x), and

$$\lim_{y \to \infty} u(x,y) = \sum_{n=0}^{\infty} [0]\, L_n(x) .$$

We will for the present assume the interchange of two infinite processes, namely differentiation and summation. We will see under this interchange that u(x,y) satisfies the differential equation. Consider the following derivatives:

$$\frac{\partial^2 u}{\partial y^2} = \sum_{n=0}^{\infty} [n\, f(n)\, e^{-\sqrt{ny}}]\, L_n(x) ,$$

$$\frac{\partial u}{\partial x} = \sum_{n=0}^{\infty} [f(n)\ e^{-\sqrt{ny}}]\ L_n'(x)$$

$$\frac{\partial^2 u}{\partial x^2} = \sum_{n=0}^{\infty} [f(n)\ e^{-\sqrt{ny}}]\ L_n''(x)\ .$$

Hence the differential equation becomes

$$\sum_{n=0}^{\infty} [n(f)\ e^{-\sqrt{ny}}]\ L_n(x) + \sum_{n=0}^{\infty} [f(n)\ e^{-\sqrt{ny}}]\ x\ L_n''(x) + \sum_{n=0}^{\infty} [f(n)\ e^{-\sqrt{ny}}](1-x)L_n'(x),$$

or

$$\sum_{n=0}^{\infty} [f(n)\ e^{-\sqrt{ny}}][x\ L_n''(x) + (1-x)\ L_n'(x) + n\ L_n(x)]\ .$$

But since $xL_n''(x) + (1-x)\ L_n'(x) + nL_n(x) = 0$ is Laguerre's differential equation we see that the function

$$u(x,y) = \sum_{n=0}^{\infty} [f(n)\ e^{-\sqrt{ny}}]\ L_n(x)$$

satisfies the differential equation.

In order to justify the above operations we must show uniform convergence of the derived series.

We first consider the series

(127) $$\sum_{n=0}^{\infty} [n\ f(n)\ e^{-\sqrt{ny}}]\ L_n(x)\ .$$

This series must converge uniformly with respect to y. We have by assumption on F(x) that for each fixed $x \geq 0$, the series

(128) $$\sum_{n=0}^{\infty} n\ f(n)\ L_n(x)$$

converges. According to Abel's test, the new series formed by multiplying the terms of a convergent series by the corresponding members of a bounded sequence of functions of y, such as $e^{-\sqrt{ny}}$, whose functions never increase in value with n, converges uniformly with respect to y. Series (127) therefore converges uniformly with respect to y.

The terms of (127) are continuous functions of y, hence the function $\partial^2 u(xy)/\partial y^2$ represented by that series is continuous with respect to y.

We next consider the series

(129) $$\sum_{n=0}^{\infty} [f(n)\, e^{-\sqrt{ny}}]\, L_n'(x) \; ,$$

and

$$\sum_{n=0}^{\infty} [f(n)\, e^{-\sqrt{ny}}]\, L_n''(x) \; .$$

Since $|L_n(x)| \leq e^{\frac{x}{2}}$ for all n and all x we have for $x \leq x_0$, where x_0 is some fixed value of x,

$$|f(n)\, L_n'(x)| < |n\, f(n)|\, M$$

we have

$$|e^{-\sqrt{ny}}\, f(n)\, L_n'(x)| < M_1\, e^{-\sqrt{ny}}$$

and hence the series (129) converges uniformly with respect to x, $0 \leq x \leq x_0$. This same statement holds for series (130). Hence the procedure used to show that u(x,y) satisfied the differential equation was justified.

Let us consider a possible physical interpretation of the preceding problem.

Write equation (119) in the form

(131) $$\frac{\partial}{\partial y}\left(e^{-x}\, \frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial x}\left(xe^{-x}\, \frac{\partial u}{\partial x}\right) = 0.$$

Thus we have a problem in steady state temperatures with conductivity $K_1 = e^{-x}$ in the y direction and $K_2 = xe^{-x}$ in the x direction.

CHAPTER VII

SONINE TRANSFORMS

1. Introduction

In this chapter we will introduce a generalized Laguerre transform which we will call the Sonine Transform. We will derive a few properties and then show how the Laguerre transform and Sonine transform are connected through a property on transforms of derivatives.

2. Sonine Transforms

The sequence of numbers $f_a(n)$ defined by the equation

$$f_a(n) = \int_0^\infty e^{-x} x^a L_n^a(x) F(x) dx \qquad (n=0, 1, 2, \cdots), \tag{132}$$

where $L_n^a(x)$ denotes the generalized Laguerre polynomial of degree n, is the Sonine transform of the function F(x). The integral transformation here will be represented by the symbol $T\{F(x)\}$.

For functions satisfying fairly general conditions, Uspensky [18], on the interval $0 < x < \infty$ the inverse of this transformation is represented by the expansion of F(x) in a series of the generalized Laguerre polynomials

$$F(x) = \sum_{n=0}^{\infty} \frac{n!}{\Gamma(a+n+1)} f(n) L_n^a(x) = T^{-1}\{f_a(n)\} (0 < x < \infty). \tag{133}$$

3. Properties of Sonine Polynomials

The following list of properties of Sonine polynomials will be useful.

$$\int_0^\infty e^{-x} x^a L_n^a(x) L_m^a(x) dx = \frac{\Gamma(n+1+a)}{n!} \quad (n = m), \tag{134}$$
$$= 0 \quad (n \neq m).$$

(135) $$L_0^a(x) = 1 \qquad L_1^a(x) = a+1-x .$$

(136) $$L_n^a(x) = \sum_{m=0}^{n} \binom{n+a}{n-m} \frac{(-x)^m}{m!} .$$

(137) $$(n+1)\, L_{n+1}^a(x) - (2n+a+1-x)\, L_n^a(x) + (n+a)\, L_{n-1}^a(x) = 0.$$

(138) $$xy'' + (a+1-x)y' + ny = 0 \qquad y = L_n^a(x).$$

(139) $$L_n^a(0) = \binom{n+a}{n} .$$

(140) $$\frac{d}{dx} L_n^a(x) = -L_{n-1}^{a+1}(x) .$$

(141) $$xL_n^{a+1}(x) = (n+a+1)\, L_n^a(x) - (n+1)\, L_{n+1}^a(x) .$$

(142) $$L_n^{a-1}(x) = L_n^a(x) - L_{n-1}^a(x) .$$

4. Operational Properties

Let R [F] denote the differential form

(143) $$R\,[F(x)] = \frac{e^x}{x^a}\,[x^{a+1}\, e^{-x}\, F'(x)]' .$$

When the integral $T\{R[F]\}$ is integrated successively by parts and $-n\, L_n^a(x)$ is substituted for $R[L_n^a(x)]$ in accordance with the differential equation (138), the following result is obtained.

Theorem 10: Let F(x) denote a function that satisfies these conditions: $F'(x)$ is continuous and $F''(x)$ is sectionally continuous over each finite interval in the range $x \geq 0$, $F(x)$ and $F'(x)$ are $O(e^{ax})$, $a < 1$, as x tends to infinity.

Then $T\{R\,[F(x)]\}$ exists and

(144) $$T\{R\,[F(x)]\} = -n\, f_a(n) \qquad (n=0, 1, 2, \cdots) .$$

We note here that the basic operational property for the Sonine transform is the same as the first basic operational property for the Laguerre transform.

The differential form of the fourth order $R^2[F(x)]$ obtained by applying the operator

$$R = x\frac{d^2}{dx^2} + (a+1-x)\frac{d}{dx}$$

to $R[F(x)]$ is also resolved by the Sonine transform $T\,F(x)$. The resolution can be written at once as

(145) $$T\{R^2[F]\} = -nT\{R[F]\} = n^2 f_a(n), \qquad (n=0,\ 1,\ 2,\cdots).$$

The addition property (58) is not appropriate for use in finding a Sonine convolution property since the polynomial under the integral sign will turn out to be a Laguerre polynomial and hence if we multiply two Sonine transforms of the same order together we will find the function which has this product as its n^{th} Laguerre transform.

Let us suppose now that $F(x)$ is continuous and $F'(x)$ bounded and integrable. We also assume $F(x)$ is $O(e^{ax})$, $a < 1$, as x tends to infinity.

Integration by parts of the integral

(146) $$\int_0^\infty F'(x)\, e^{-x}\, x^a\, L_n^a(x)\, dx$$

will lead to a property of the Sonine transform which involves derivatives.

Let $F'(x)\,dx = dv$ and $e^{-x}\, x^a\, L_n^a(x) = u$. This leads to

(147) $$T\{F'(x)\} = e^{-x}\, x^a\, L_n^a(x)\, F(x)\Big|_0^\infty - \int_0^\infty (n+a)\, e^{-x}\, x^{a-1}\, L_n^{a-1}(x)\, F(x)\, dx + \int_0^\infty e^{-x}\, x^a\, L_n^a(x)\, F(x)\, dx,$$

or

(148) $$T\{F'(x)\} = f_a(n) - (n+a)\, f_{a-1}(n), \qquad (n=0,\ 1,\ 2,\cdots).$$

In obtaining equation (148) we have used properties (140), (141), and (142).

Since $d/dx\, L_n(x) = -L_{n-1}^1(x)$ we have

(149) $$T_n\{F'\} = -F(0) + f_1(n-1), \qquad (n=1,\ 2,\ 3,\cdots),$$

where $f_1(n-1)$ is the $n-1$ Sonine transform with $a = 1$. When $n = 0$

$$T\{F'(x)\} = F(0) - f(0) .$$

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