Analogue Computers for Servo Problems

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INTRODUCTION

In the last few years, a great deal of interest has been centered on the use of high-gain, operational amplifiers in analogue computers for the study of dynamic problems. (Ref. 1, 2, 3 and 4) One of the major applications for such analogue computers has been in the field of servomechanisms. The transient equation of motion of even the simpler servomechanisms is difficult and tedious to solve, and therefore, analogues of servomechanism systems are being used extensively to facilitate the study and design of such systems.

Analogue computers for general dynamic studies normally use operational amplifiers having very high gain, some as high as 1,000,000. When analogue computers are applied to servomechanism problems, certain important characteristics of the servo problem tend to reduce the required gain and the resulting complexity of the operational amplifiers and power supplies used in the computer. Gains as low as several hundred yield very satisfactory performance for servo problems. In the analysis and synthesis of servomechanisms, the dynamic response and stability and the steady-state response are the major factors to be determined. The differential equations of motion of the servomechanism, which are needed to set up the servo analogue on the computer,yield the steady-state respone quite easily without the aid of the computer. Hence, only the dynamic response of the servomechanism need be determined with the computer. As will be shown, the dynamic response accuracy of analogue computers can be made much higher than their steady-state response accuracy for comparatively low values of the operational amplifier gain.

BASIC OPERATIONAL CIRCUITS

Figure 1A is a schematic diagram of the basic operational amplifier circuit. In order to more easily analyze the operation of this circuit, it has been transformed by the use of Thevenin's theorem as shown in Fig. 1B.* Equation (5) of Fig. 1B, which is obtained by the simultaneous solution of equations (1) - (4), states the response of the operational circuit. This basic circuit, when employed in analogue computers, is usually required to perform the three operations of differentiation, integration, and addition.

Differentiation is achieved by setting Z_i and Z_f equal to $\frac{1}{j\,w\,C_i}$ and R_f respectively; equation (5) becoming

$$(6) \quad \frac{e_o}{e_i} = -\,j\,w\,R_f\,C_i \left[\frac{r\,K}{r\,K+1}\right]\left[\frac{1-\frac{R_s}{K\,R_f}}{1+j\,w\,\frac{(R_f+r\,R_s)\,C_i}{r\,K+1}}\right]$$

In equation (6) and in all later equations Z_s has been set equal to R_s.

If the input and feedback branches are a resistor and a condenser respectively, then

$$Z_i = R_i \qquad \text{and} \qquad Z_f = \frac{1}{j\,w\,C_f}$$

and the operational circuit becomes an integrator. For an integrator equation (5) becomes:

$$(7) \quad \frac{e_o}{e_i} = -\frac{1}{j\,w\,R_i C_f}\left[\frac{r\,K}{r\,K+1}\right]\left[\frac{j\,w\,(r\,K+1)\,R_i\,C_f(1-jw\,\frac{R_s\,C_f}{K})}{1+j\,w\left[(r\,K+1)\,R_i+r\,R_s\right]C_f}\right]$$

* Equation (4) of Figure 1B infers that an odd number of phase inversions take place in the amplifier. The use of the same circulating current i in all branches of the circuit implies the input impedance of the amplifier is infinite.

The operation of addition or summation implies the existence of two or more input signals; hence the circuit of Fig. 1B is shown modified in Fig. 2 with three separate inputs. In Fig. 2, the input and feedback branch impedances have been made resistive and equation (15) demonstrates the summation properties of the circuit.

It may be noted, if only one input is used and $\frac{R_f}{R_1} = 1$, the circuit of Fig. 2 degenerates to the special case of a sign changer. Of course, the differentiating, integrating, and adding circuits in their most general form also perform a sign change.

DISTORTIONS OF THE EXACT OPERATIONS

The three basic equations (6), (7) and (15) for the operations of differentiation, integration, and addition all have the same general form and all are composed of three similar terms. For each of these equations the first term represents the exact operation while the remaining two terms in brackets distort the exact operation. The first of these distortion terms is common to all of the basic operation equations and will be denoted by A. A may be rewritten as follows:

$$(16) \qquad A = \frac{r\,K}{r\,K + 1} = 1 - \frac{1}{r\,K + 1}$$

The distortion produced by A is simply that of multiplying the basic operation by a scale factor dependent only upon the effective gain r K.

The second distortion term is different for each operation and therefore shall be denoted by B_d, B_i, and B_a for differentiation, integration, and addition, respectively. For the case of addition, B_a is also only a scale factor but it is dependent upon all the parameters of the operational circuit as follows:

$$(17) \qquad B_a = \frac{1 - \dfrac{R_s}{K\,R_f}}{1 + \dfrac{R_f + r\,R_s}{(r\,K + 1)\,R_p}}$$

For differentiation, B_d has both magnitude and phase:

(18a) $$\left|B_d\right| = \left(1 - \frac{R_s}{K R_f}\right)\left[1 + \frac{(w\, C_i)^2}{(rK+1)^2}\,(R_f + r\,R_s)^2\right]^{-\frac{1}{2}}$$

(18b) $$\angle B_d = -\tan^{-1}\left[\frac{w\,(R_f + r\,R_s)\,C_i}{(r\,K+1)}\right]$$

Similarly for integration, B_i is

(19a) $$\left|B_i\right| = w\,(r\,K+1)\,R_i\,C_f\left[1 + \left(w\,\frac{R_s\,C_f}{K}\right)^2\right]^{\frac{1}{2}}$$
$$\left\{1 + (w\,C_f)^2\left[(r\,K+1)\,R_i + r\,R_s\right]^2\right\}^{-\frac{1}{2}}$$

(19b) $$\angle B_i = \tan^{-1}\left\{\frac{1}{w\,C_f\left[(r\,K+1)\,R_i + r\,R_s\right]}\right\} - \tan^{-1}\left[\frac{w\,R_s\,C_f}{K}\right]$$

For satisfactorily designed operational amplifiers, some of the quantities in the distortion terms may be small compared to other quantities in the distortion terms. If the amplifier gain is above several hundred and the input, feedback, and load impedances are large compared to the source impedance, the following approximations in the distortion terms may be made:

(20) $$A \cong 1 - \frac{1}{K}$$

(21) $$B_a \cong 1 - \frac{1}{K}\,\frac{R_f}{R_p}$$

(22a) $$\angle B_d \cong -\tan^{-1}\left[\frac{w\,C_i\,R_f}{K}\right] \cong -\frac{w\,C_i\,R_f}{K}$$

(22b) $$\left|B_d\right| \cong \left[1 + \left(\frac{w\,C_i R_f}{K}\right)^2\right]^{-\frac{1}{2}} \cong 1 - \frac{1}{2}\left(\frac{w\,C_i R_f}{K}\right)^2 = 1 - \frac{1}{2}\left(\angle B_d\right)^2$$

(23a) $$\left|B_i\right| \cong K\,w\,R_i\,C_f\left[1 + \left(w\,C_f\,\frac{R_s}{K}\right)^2\right]^{\frac{1}{2}}\left[1 + (K\,w\,R_i C_f)^2\right]^{-\frac{1}{2}}$$

$$(23b)\quad \angle B_i \cong \tan^{-1}\left[\frac{1}{w\,K\,R_i\,C_f}\right] - \tan^{-1}\left[\frac{w\,R_s\,C_f}{K}\right] \cong \frac{1}{K\cdot w\,R_i\,C_f} - w\,C_f\,\frac{R_s}{K}$$

OPERATIONAL AMPLIFIER

Figure 3 is a schematic diagram of an operational amplifier designed for servo application.* It uses only two tubes, two regulated B supplies, and an a.c. filament supply regulated by a conventional constant voltage transformer. The first stage of the amplifier is a twin stage with the amplification of one half of a 6SL7. This twin stage has been used primarily to minimize the drift produced by variations in cathode emission. The second stage operates at higher voltage levels and thus is a simple amplification stage. The third and last stage is a cathode follower stage used to minimize R_s. This circuit shows an equivalent input drift of the order of 1 millivolt over four to six hours.

For the circuit of Fig. 3

$$K \cong 700$$

$$R_s \cong 700$$

and the output stage has been designed to deliver ± 40 volts to a load having a resistance not less than 25,000 ohms. Hence the minimum value of r is

$$r = \frac{25000}{750 + 25000} = 0.97$$

or

$$1 > r > 0.97$$

and for all practical purposes r = 1.

The circuit of Fig. 3 is only one example of a number of elementary basic circuits which may be used to obtain medium gain amplification. Figures 4 and 5 are photographs of one of these amplifiers and ten of these amplifiers plugged into a rack-mounted chassis, respectively.

* Mr. K. C. Mathews, a member of the Controls Group of the Aeronautical Research Center of the University of Michigan, was responsible for the design and development of these amplifiers.

If the power output of this circuit is not sufficient, a 7F8 may be substituted for the second 6SL7.* This substitution will result in approximately a 30 percent reduction in K and an increase in power output by a factor of 2.

The amplifier of Fig. 3 has been designed for values of R_f, R_i, C_f, and C_i as follows:

$$100{,}000 \text{ ohms} < R_f < 1{,}000{,}000 \text{ ohms}$$

$$100{,}000 \text{ ohms} < R_i < 1{,}000{,}000 \text{ ohms}$$

$$0.1\,\mu f < C_f < 1\,\mu f$$

$$0.01\,\mu f < C_i < 1\,\mu f$$

Therefore $R_f \gg R_s$ and $R_i \gg R_s$

EVALUATION OF THE DISTORTION TERMS FOR THE AMPLIFIER OF FIGURE 3

Using the values of K, R_s, and r given for the amplifier of Fig. 3, the approximate distortion terms can be evaluated as follows.

The distortion term A, which is common to all of the operations, affects only the magnitude of the exact operation reducing it by 1/7 of 1%.

For the case of addition, if the several inputs are summed and amplified, the ratio $\frac{R_f}{R_p}$ will have a maximum value equal to the number of input signals summed. Attenuation of the sum will reduce this ratio. Amplification of the sum will increase $\frac{R_f}{R_p}$ but it is not necessary that amplification be done in the summing amplifier. Consequently, B_a, which also affects the magnitude of the exact summation, reduces it by less than $\frac{1}{7}$ x number of input signals x 1%.

* Usually the loads requiring appreciable power are the multiplier servo potentiometers in the analogue computer. Precise potentiometers for such purposes are commercially available with at least 25,000 ohms resistance.

In differentiation, B_d produces a negative phase shift as well as a reduction in the magnitude of the exact operation. This phase shift is more important than the magnitude change and for satisfactory servo applications should not exceed a maximum value $\angle B_{d_m}$ for all frequencies below a certain frequency w_c. Therefore, from equation (22a)

(24) $$R_f\, C_i < \frac{700}{w_c} \angle B_{d_m}$$

For example, if $\angle B_{d_m}$ is to be kept less than 3° or $\frac{1}{20}$ radian up to 50 cps., then

$$R_f\, C_i < 0.1$$

will satisfy this requirement. This restriction on $R_f\, C_i$ is very easy to achieve because R_f is usually chosen around 100,000 to 500,000 ohms and hence, C_i must be less than about 1 to 0.2 μf. The choice of 3° for $\angle B_{d_m}$ also limits the magnitude reduction produced by $\left| B_d \right|$ to less than $\frac{1}{8}$ of 1%

For the case of integration, there are two phase shifts in B_i, one a positive phase shift occurring at low frequencies and the second a negative phase shift occurring at relatively high frequencies. These two phase shifts occur at such widely separated frequencies that they may be considered separately. The maximum permissible value of $\angle B_i$ at low frequencies and at high frequencies will be denoted by $\angle B_{i_l}$ and $\angle B_{i_h}$, respectively. The frequencies corresponding to these two values of $\angle B_i$ are w_l and w_h, respectively. From equation (23b)

(25a) $$\angle B_{i_l} \geq \frac{1}{700\, w_l R_i\, C_f}$$

(25b) $$\angle B_{i_h} \geq w_h\, C_f$$

If the same high frequency specifications are imposed upon the integrator phase shift as upon the differentiator and $C_f < 1\ \mu f$ then

$w_h \geqq 50{,}000$ rad/sec

or $f_h \geqq 8{,}000$ cps.

Reasonable maximum values for R_i, C_f, and $R_i\ C_f$ are 1,000,000 ohms, 1 μf, and 1, respectively.* Consequently, if $\angle B_{i\ 1}$ is also chosen to be less than 3^o,

$$w_1 \geqq \frac{1}{70} \text{ rad/sec}$$

or

$$f_1 \geqq \frac{1}{400} \text{ cps.}$$

Thus from 1/400 cps. to 8,000 cps., the phase shift of the distortion term B_i will be less than 3^o.

Substitution of equations (25) into equation (23a) yields the following form for $|B_i|$:

$$(26) \qquad |B_i| \cong \frac{w}{w_1}\ \frac{1}{\angle B_{i\ 1}} \left[1 + \left(\frac{w}{w_h} \angle B_{i\ h}\right)^2\right]^{\frac{1}{2}} \left[1 + \left(\frac{w}{w_1}\ \frac{1}{\angle B_{i\ 1}}\right)^2\right]^{-\frac{1}{2}}$$

From equation (26) it can be shown that $|B_i|$ reduces the magnitude of of the exact operation by less than 1/8 of 1% over a frequency range from 1/400 cps to 8,000 cps.

The fractional reduction in magnitude and the phase shift caused by the several distortion terms may be summarized as follows:

Summation:

$$1 - A\ B_a \geqq \frac{1}{700}\ (1 + \text{number of inputs})$$

Differentiation:

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*These values of R_f and C_i can be increased several fold but have been chosen for convenience.

$$1 - A\left|B_d\right| \leq \frac{1}{800} \qquad 0 < f < 50 \text{ cps.}$$

$$\angle B_d \leq 3^\circ$$

Integration:

$$1 - A\left|B_i\right| \leq \frac{1}{800} \qquad \frac{1}{400} < f < 8000 \text{ cps.}$$

$$\angle B_i \leq 3^\circ$$

The reduction in the magnitude of the exact operations produced by the distortion terms of the operational amplifier of Fig. 3 is comparable to the error produced in the exact operation by the use of 0.1% resistors and condensers in the input and feedback branches. For integration, differentiation, and addition, 0.1% resistors and condensers will produce an error of as much as 1/500 in the quantities $R_i C_f$, $R_f C_i$, and $\frac{R_f}{R_i}$ in the exact operation terms.

Thus low gain operational amplifiers which are less sensitive to drift, are simpler in construction, and require fewer and less regulated power supplies than do high gain operational amplifiers, have more than satisfactory dynamic accuracy for servo problems. The magnitude errors in the dynamic response of the operations are less than those obtained (1) in measuring or estimating the coefficients of the differential equations of motion of the servo and (2) in recording the solution of the problem. Similarly, the small distortion phase shifts produced by the integrator and differentiator occur at frequencies which have negligible effect upon the dynamic response, i.e., the effective natural frequency, damping ratio, etc.

On the other hand, the steady-state or zero frequency accuracy of integrators is lower for the low gain operational amplifiers.

STEADY-STATE ACCURACY OF INTEGRATORS

The zero frequency responses of an integrator for different values of operational amplifier gain may be compared by studying its response for a step input

(27) $e_i = 1.$

Equation (7) may be rewritten as follows:

(28) $$\frac{e_o}{e_i} \cong - \frac{K}{1 + p\,K\,R_i\,C_f}, \text{ where } \quad p = \frac{d}{dt}.$$

The approximations used in (28) are the same as used in the previous section with the additional approximation that

$$1 - jw\,\frac{R_s\,C_f}{K} \cong 1$$

at the frequencies under consideration.

Substituting (27) into (28) and integrating yields

(29) $$e_o = - K\,(1 - e^{-t/T}),$$

where

(30) $$T = K\,R_i\,C_f.$$

Substituting a power series in $\frac{t}{T}$ for $e^{-t/T}$ into equation (29) and rearranging terms gives

(31a) $$e_o = - \frac{t}{R_i\,C_f}\left[1 - \frac{1}{2}\left(\frac{t}{T}\right) + \frac{1}{6}\left(\frac{t}{T}\right)^2 - \ldots\right]$$

or

(31b) $$e_o \cong - \frac{t}{R_i\,C_f}\left[1 - \frac{1}{2}\left(\frac{t}{T}\right)\right] \quad \text{for } \frac{t}{T} \ll 1.$$

An ideal integrator would have an output

$$\frac{-t}{R_i\,C_f}$$

Hence the term $\frac{1}{2}\left(\frac{t}{T}\right) \times 100$ is the percentage by which the output of a real integrator deviates from the output of an ideal integrator for short periods after being subjected to a step input of voltage. In other words, $\frac{50}{T}$ may be referred to as the initial rate of percent deviation of an integrator. This rate of deviation is dependent upon T. Inasmuch as the maximum value of R_i C_f is essentially independent of amplifier gain K, the minimum initial rate of percent deviation of integrators is dependent upon only K. Therefore,higher gain operational amplifiers will have a smaller rate of deviation than have low gain operation amplifiers. This smaller rate of deviation is very important in many studies and requires gains as great as 1,000,000. Such high values of gain increase the number of tubes, power supplies, components and the care that must be exercised in the design of them to minimize drift effects.

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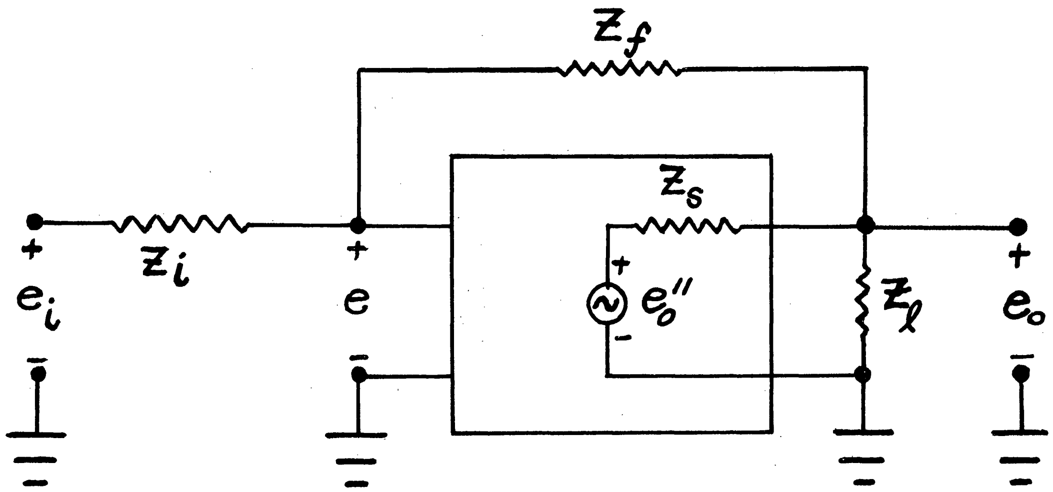


Z_i = input branch impedance

Z_f = feedback branch impedance

Z_s = output stage source impedance

Z_l = output load impedance

K = amplifier internal gain

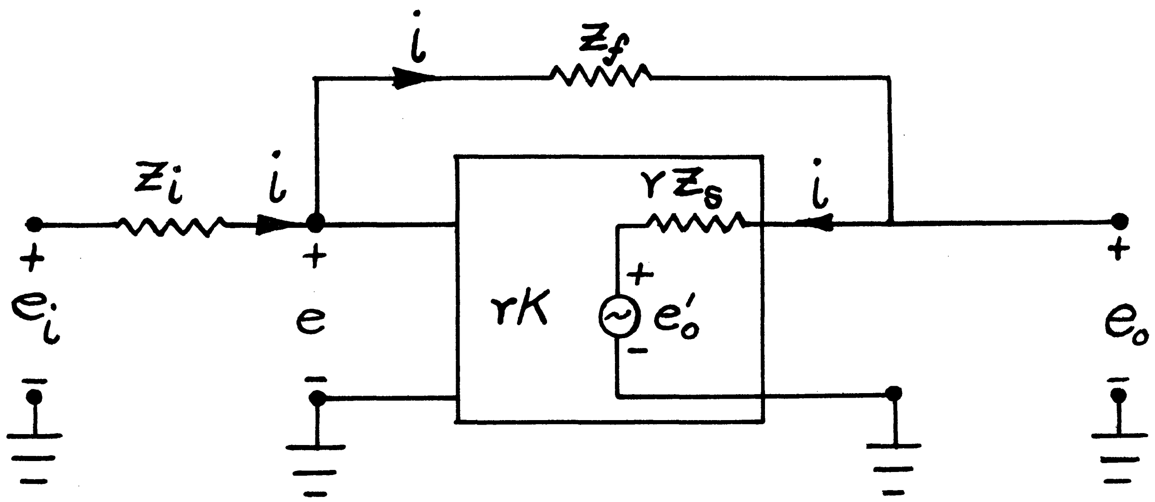
e_i = input signal voltage

e = amplifier input voltage

e_o = output voltage

e_o'' = open circuit or no load output voltage, i.e., if $Z_f = \infty$ and $Z_l = \infty$ then $e_o'' = e_o$

FIG. 1A



$$r = \frac{Z_1}{Z_s + Z_1}$$

r K = equivalent amplifier internal gain

The equations for this system are

(1) $e_i = i\, Z_i + e$

(2) $e = i\, Z_f + e_o$

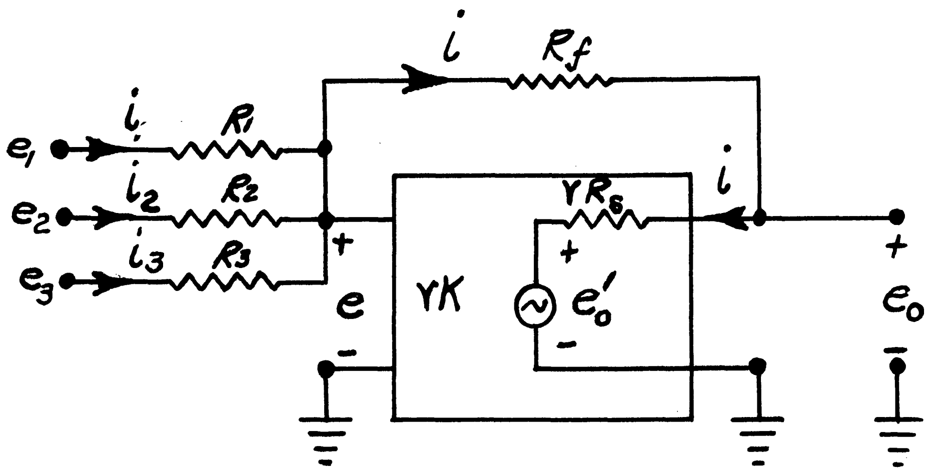
(3) $e_o = i\, r\, Z_s + e'_o$

(4) $e'_o = -\, r\, K\, e$

Solving these equations simultaneously for $\frac{e_o}{e_i}$ yields:

(5) $$\frac{e_o}{e_i} = -\frac{Z_f}{Z_i}\left[\frac{r\,K}{r\,K + 1}\right]\left[\frac{1 - \dfrac{Z_s}{K\,Z_f}}{1 + \dfrac{Z_f + r\,Z_s}{(r\,K + 1)\,Z_i}}\right]$$

Fig. 1B



(8) $e_1 = i_1 R_1 + e$

(9) $e_2 = i_2 R_2 + e$

(10) $e_3 = i_3 R_3 + e$

(11) $i = i_1 + i_2 + i_3$

(12) $e = i Z_f + e_o$

(13) $e_o = i r Z_s + e_o'$

(14) $e_o' = - r K e$

Solving (8) - (14) inclusive simultaneously yields:

(15) $$e_o = -\left(\frac{R_f}{R_1} e_1 + \frac{R_f}{R_2} e_2 + \frac{R_f}{R_3} e_3\right)\left[\frac{r K}{r K + 1}\right]\left[\frac{1 - \frac{R_s}{K R_f}}{1 + \frac{R_f + r R_s}{(r K + 1) R_p}}\right]$$

where $$R_p = \frac{R_1 R_2 R_3}{R_1 R_2 + R_1 R_3 + R_2 R_3}$$

and is the resistance of all three input resistors in parallel.

FIG. 2

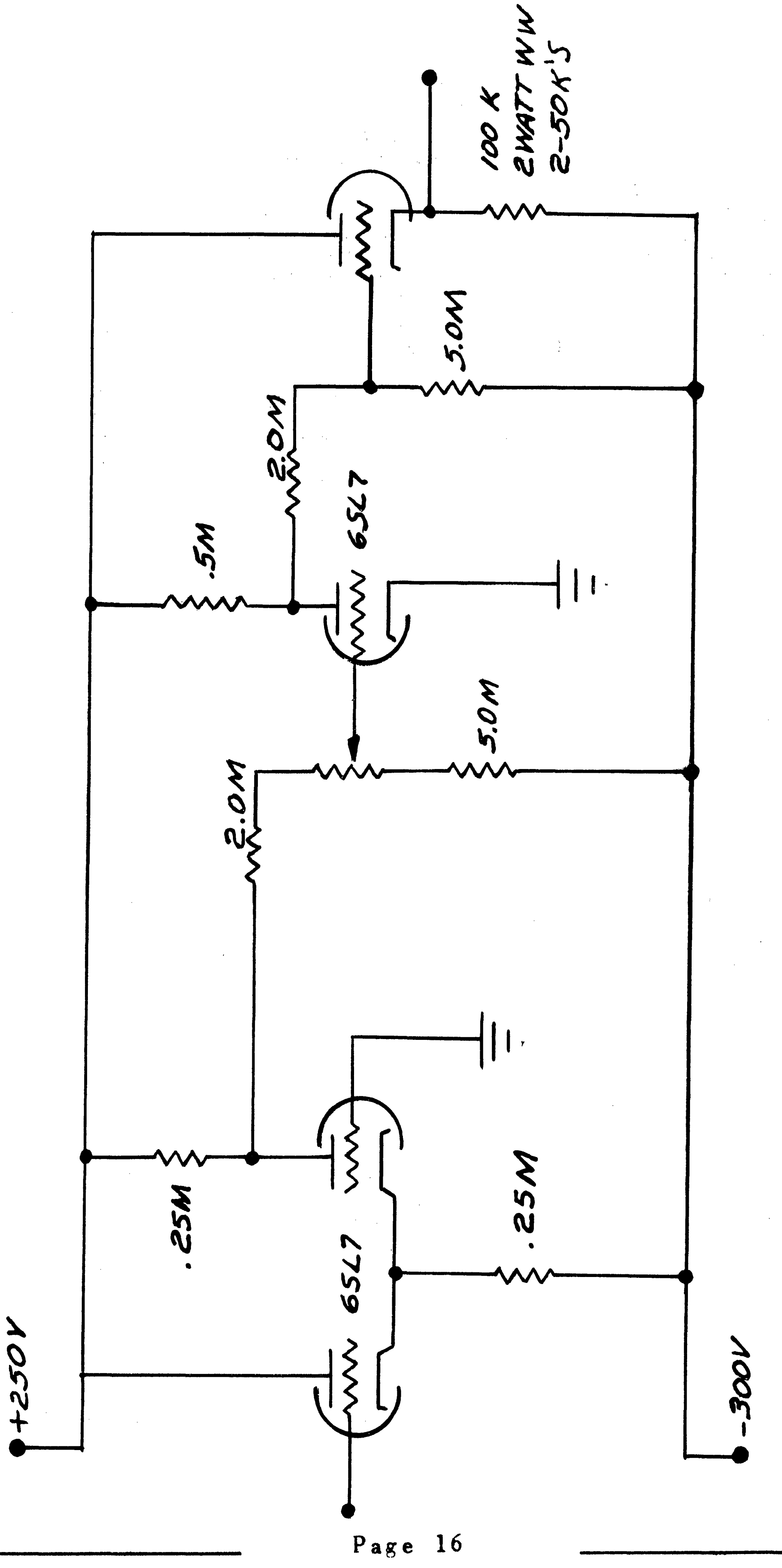


FIG. 3 - CIRCUIT DIAGRAM OF MEDIUM GAIN OPERATIONAL AMPLIFIER

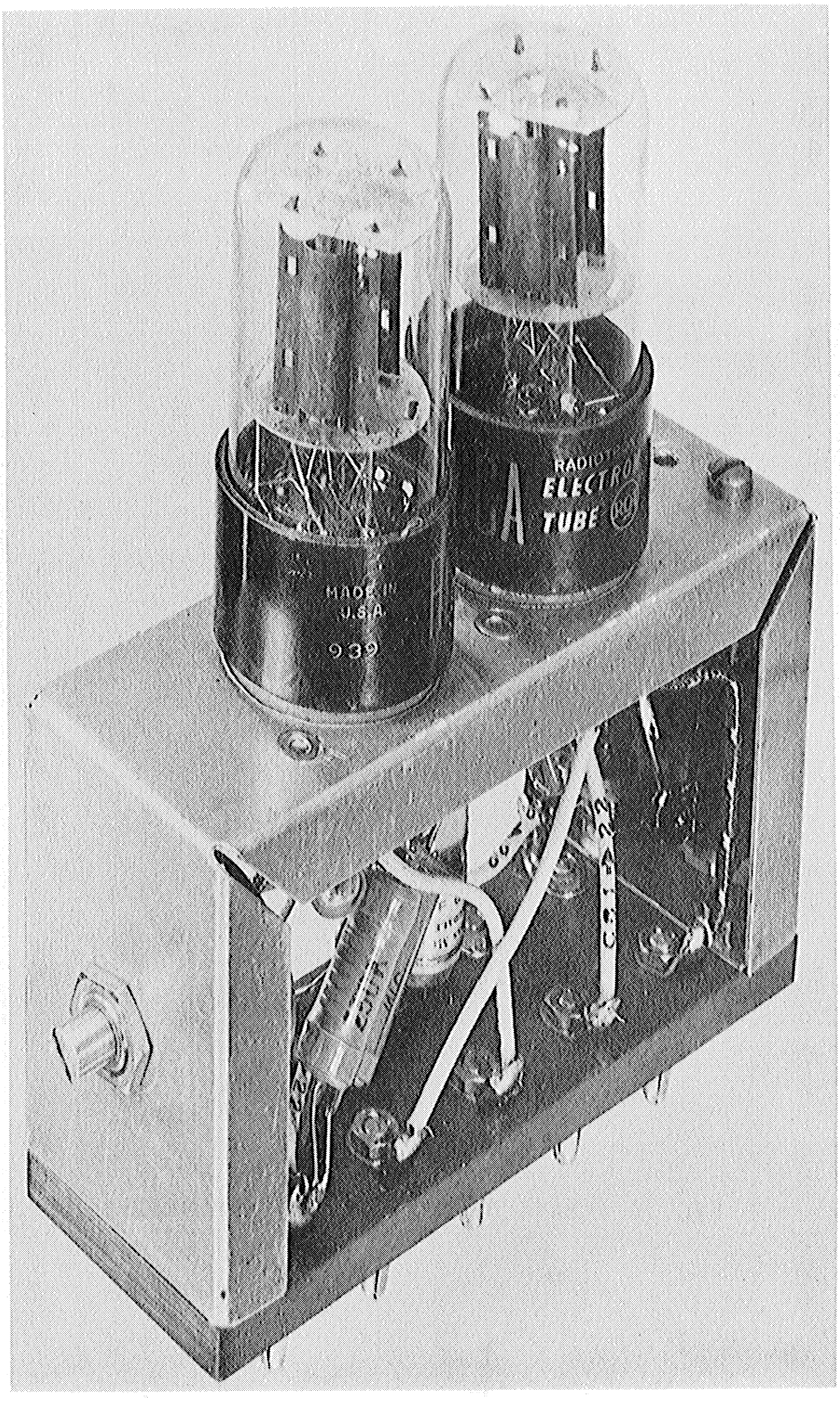


FIG. 4 - OPERATIONAL AMPLIFIER

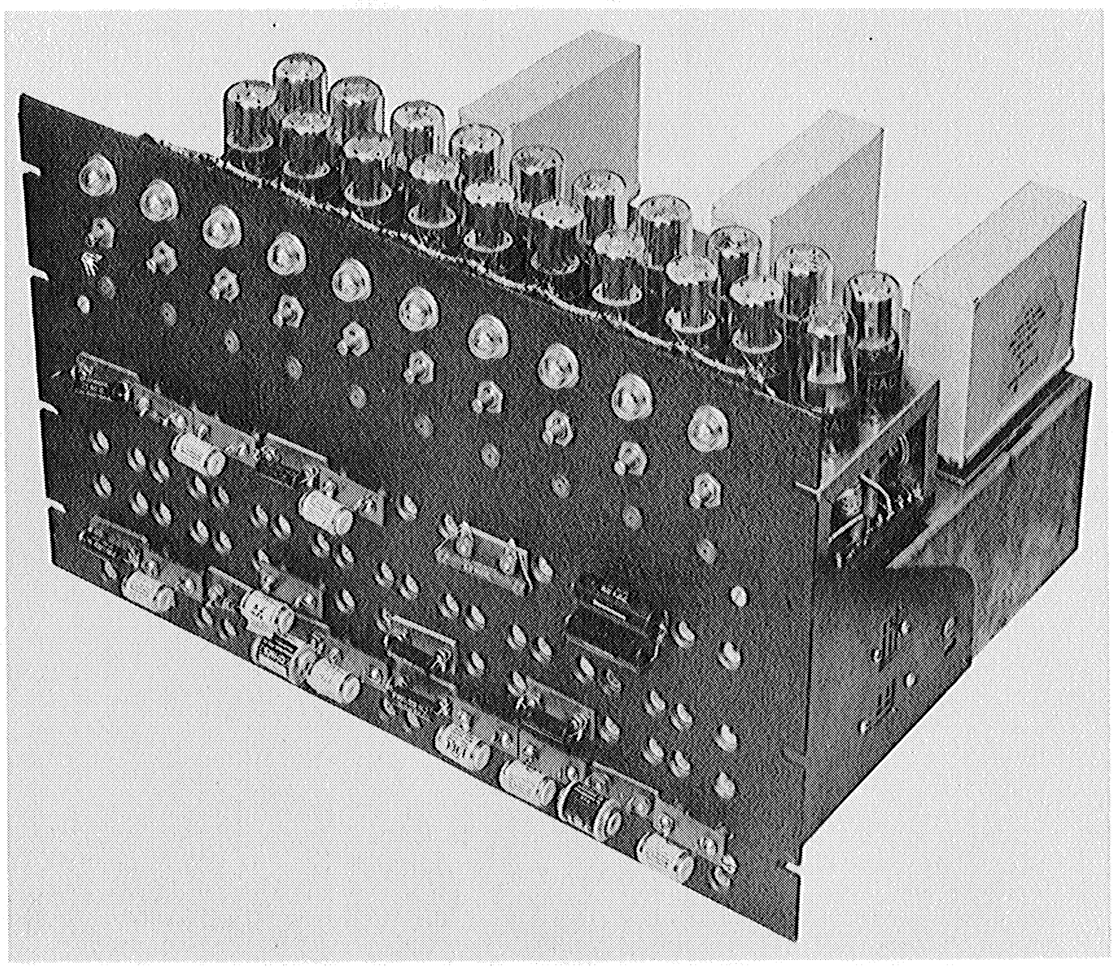


FIG. 5 - BANK OF OPERATIONAL AMPLIFIERS

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