T H E U N I V E R S I T Y O F M I C H I G A N

COLLEGE OF ENGINEERING
Department of Meteorology and Oceanography

Technical Report

ON THE NUMERICAL EVALUATION OF THE STRUCTURE OF VARIOUS VERTICAL MODES IN TRANSIENT PLANETARY WAVES

N. A. McFarlane

Aksel C. Wiin-Nielsen
Project Director

ORA Project 002630

supported by:

NATIONAL SCIENCE FOUNDATION
GRANT NO. GA-16166
WASHINGTON, D.C.

administered through:

OFFICE OF RESEARCH ADMINISTRATION   ANN ARBOR

April 1971

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

ABSTRACT

A numerical method for obtaining approximate solutions to the equation governing the vertical structures of transient planetary waves has been outlined. Approximate solutions are compared with analytical ones for the cases discussed by Jacobs and Wiin-Nielsen (1966).

In the last part of the study approximate solutions to the vertical structure equation are obtained for several cases in which the basic temperature profile has a constant lapse rate in the troposphere and is isothermal from the tropopause to the top of the model. It is found that the vertical modes, in these cases, differ little in their structure from those which arise in an atmosphere with a constant temperature lapse rate everywhere. However, the equivalent depths associated with these modes are larger than those of the constant lapse rate atmosphere but are much smaller than those which occur in an isothermal atmosphere.

1. INTRODUCTION

In a recent investigation of the behavior of transient small amplitude disturbances imposed on a resting atmosphere Wiin-Nielsen (1971a,b) has shown that the vertical structures of these may be obtained as solutions to the following equation:

$$\frac{d}{dp}\left(\frac{1}{\sigma}\frac{dF}{dp}\right) + \frac{1}{gh}F = 0 \qquad (1.1)$$

where the notation used here is identical with that in the above cited papers.

The known, or prescribed, quantities in this equation are the acceleration due to gravity, g, and the static stability, σ. This latter quantity is taken to be a function of pressure alone and may be defined in terms of a horizontal mean temperature, T, in the usual manner:

$$\sigma = -\frac{R}{p}\left(\frac{\partial T}{\partial p} - \frac{\kappa T}{p}\right) \qquad (1.2)$$

The quantity, h, called the equivalent depth in direct analogy with the usage common in tidal theory (c.f. Lindzen (1967)), may be determined as an eigenvalue when appropriate boundary conditions are imposed on the solutions of Equation 1.1. Wiin-Nielsen has considered two different sets of boundary conditions, hereafter labelled BC1 and BC2 as follows:

$$\text{BC1:} \quad \frac{1}{\sigma}\frac{dF}{dp} = 0; \qquad p = 0,\ p_o \qquad (1.3)$$

$$\left.\begin{aligned} \text{BC2:}\quad \frac{1}{\sigma}\frac{dF}{dp} &= 0; \qquad p = 0 \\ \frac{dF}{dp} + \frac{\sigma p_o}{RT_o} F &= 0; \qquad p = p_o \end{aligned}\right\} \tag{1.4}$$

where p_o, σ_o, T_o are the pressure, static stability, and temperature at the lower boundary of the model atmosphere. The BC1 boundary conditions result from requiring that $dp/dt = 0$ at the top ($p = 0$) and the bottom of the model, while the modified lower boundary condition in Equations 1.4 results from requiring the vertical velocity, ω to vanish at the bottom of the model.

Jacobs and Wiin-Nielsen (1966) have examined the solutions to Equation 1.1 for two particular specifications of the static stability: that corresponding to an isothermal atmosphere and that for an atmosphere with a constant temperature lapse rate. In that investigation the lower boundary condition in Equation 1.3 was used in conjunction with a somewhat different upper boundary condition corresponding to the requirement that the vertical flux of energy remain finite at the top of the atmosphere. Wiin-Nielsen (1971a) adapted the solutions for the constant lapse rate case to fit the boundary conditions outlined in Equations 1.3 and 1.4 above.

Investigation of cases where the static stability specification corresponds to more realistic or complicated temperature profiles than either of the constant lapse rate or isothermal cases is frequently hampered by the fact that closed form solutions to Equation 1.1 can only be obtained for certain special functional representations of the static stability. In order to obtain some generality we are forced to use approximation techniques and the purpose of this report, therefore, is to outline a numerical method for obtaining approximate solutions to Equation 1.1 and either of the two sets of boundary conditions, BC1 or BC2. We will also present some results from applications of this method to cases where the temperature profiles are idealized representations of typical profiles in the troposphere and lower stratosphere.

2. ANALYTICAL SOLUTIONS

The numerical method to be used involves obtaining finite difference approximations to Equation 1.1 and the boundary conditions. The accuracy of the numerical procedure is best examined by comparing the approximate solutions with known analytical ones. The isothermal and constant lapse rate cases discussed by Jacobs and Wiin-Nielsen (1966) and Wiin-Nielsen (1971a) are appropriate for this purpose especially because, for both of these cases, Equation 1.1 becomes singular near p = 0 due to the fact that the static stability becomes infinite there. With this in mind, we shall review these analytical results in this section, prior to discussing the numerical method.

2.1 THE ISOTHERMAL CASE

In the absence of temperature variation with height the static stability has the following functional form:

$$\sigma = \sigma_o \left(\frac{p_o}{p}\right)^2 \tag{2.1.1}$$

where $\sigma_o = R^2 T_o / c_p p_o^2$. After making the transformation $p_* = p/p_o$ and substituting from Equation 2.1.1, Equation 1.1 becomes:

$$\frac{d}{dp_*}\left(p_*^2 \frac{dF}{dp_*}\right) + \frac{\sigma_o p_o^2}{gh} F = 0 \tag{2.1.2}$$

Solutions to this equation are of the form

$$F \sim p_*^m \tag{2.1.3}$$

where m may be determined from the indicial equation

$$m^2 + m + \frac{\sigma_o p_o^2}{gh} = 0 \qquad (2.1.4)$$

Solving for m gives the complete solution in the form:

$$F = p_*^{-1/2} (A p_*^{\Gamma/2} + B p_*^{-\Gamma/2}) \qquad (2.1.5)$$

where $\Gamma = (1 - 4\sigma_o p_o^2/gh)^{1/2}$ and A and B are arbitrary constants. The boundar conditions BC1 and BC2 may be written as follows:

$$\text{BC1:} \quad \lim_{p_* \to 0} p_*^2 \frac{dF}{dp_*} = 0$$

$$\frac{dF}{dp_*} = 0; \quad p_* = 1 \qquad (2.1.6)$$

$$\text{BC2:} \quad \lim_{p_* \to 0} p_*^2 \frac{dF}{dp_*} = 0$$

$$\frac{dF}{dp_*} + \frac{\sigma_o p_o^2}{RT_o} F = 0; \quad p_* = 1 \qquad (2.1.7)$$

On applying the lower boundary condition in each of these we have one of the following two possibilities:

$$\text{BC1:} \quad F = \frac{p_*^{-1/2} A}{(\Gamma+1)} [(\Gamma+1) p_*^{\Gamma/2} + (\Gamma-1) p_*^{-\Gamma/2}] \qquad (2.1.8)$$

$$\text{BC2:} \quad F = \frac{p_*^{-1/2} A}{(\Gamma+1-\frac{2\sigma_o p_o^2}{RT_o})} [(\Gamma + 1 - \frac{2\sigma_o p_o^2}{RT_o}) p_*^{\Gamma/2}$$

$$+ (\Gamma - 1 + \frac{2\sigma_o p_o^2}{RT_o}) p_*^{-\Gamma/2}] \qquad (2.1.9)$$

In order that Equation 2.1.8) should satisfy the upper ($p_* = 0$) boundary condition the following must be true:

$$\lim_{p_* \to 0} \frac{A(\Gamma-1)}{2} p_* [p_*^{(\Gamma-1)/2} - p_*^{-(\Gamma+1)/2}] = 0 \qquad (2.1.10)$$

Consequently, for this case, either Γ is pure imaginary or $0 \leq \Gamma \leq 1$. Refering back to the definition of Γ we see that one of these two conditions will be satisfied for all positive values of h. When Γ is pure imaginary the solutions will be oscillatory and this will be so when

$$h < 4\sigma_o p_o^2/g \qquad (2.1.11)$$

A similar analysis applied to Equation 2.1.9 shows that $\Gamma = 1$, corresponding to an infinite equivalent depth, is not permissible. The effect of the lower boundary condition in Equations 2.1.7 is therefore to ensure that h shall remain finite. However this restriction is fairly weak since solutions satisfying the BC2 boundary conditions do exist for all positive finite values of h. When the second term in square brackets in Equation 2.1.9 is positive F will have no zeros. The condition for this to be true is that

$$\Gamma \geq 1 - \frac{2\sigma_o p_o^2}{RT_o} \qquad (2.1.12)$$

from which we have that

$$h \geq \frac{4\sigma_o p_o^2}{g(1-\hat{\Gamma}^2)} \qquad (2.1.13)$$

where $\hat{\Gamma} = 1 - 2\sigma_o p_o^2/RT_o$.

2.2 THE CONSTANT LAPSE RATE CASE

If the temperature profile is characterized by a constant lapse rate, $\partial T/\partial z = -\gamma$, application of the hydrostatic equation permits the static stability to be written as follows:

$$\sigma = \sigma_o \, p_*^{-(2 - \frac{R\gamma}{g})} \qquad (2.2.1)$$

where

$$\sigma_o = \frac{RT_o}{p_o^2} \left(\frac{R}{cp} - \frac{R\gamma}{g}\right).$$

Denoting

$$\frac{g}{R\gamma} = n + 1 \qquad (2.2.2)$$

Equation 1.1 becomes:

$$\frac{d}{dp_*}\left(p_*^{\frac{2n+1}{n+1}} \frac{dF}{dp_*}\right) + \frac{\sigma_o p_o^2}{gh} F = 0 \qquad (2.2.3)$$

As outlined by Wiin-Nielsen (1971a) the solutions to this equation are in terms of Bessel functions and it is possible to choose one which satisfies the somewhat stronger upper boundary condition:

$$\frac{dF}{dp_*} = 0; \qquad p_* = 0 \qquad (2.2.4)$$

In this case the solution may be written as follows:

$$F = A \, p_*^{-\frac{n}{2(n+1)}} J_n\left[2(n+1) \, s p_*^{\frac{1}{2(n+1)}}\right] \qquad (2.2.5)$$

where $s^2 = \frac{\sigma_o^2 p_o^2}{gh}$ and A is an arbitrary constant. Application of either of the lower boundary conditions then determines s, and therefore h, for a given value of n. Details for the cases n = 3 (γ = 8.54°K/km) and n = 4 (γ = 6.82°K/km) are presented in Wiin-Nielsen's (1971a) paper.

3. THE NUMERICAL METHOD

To obtain finite difference approximations to Equations 1.1 and the boundary conditions we divide the atmosphere into layers as shown in Figure 3.1. The static stability for each layer is defined on a pressure level in the middle of the layer, while F is defined on those pressure levels which bound the layers.

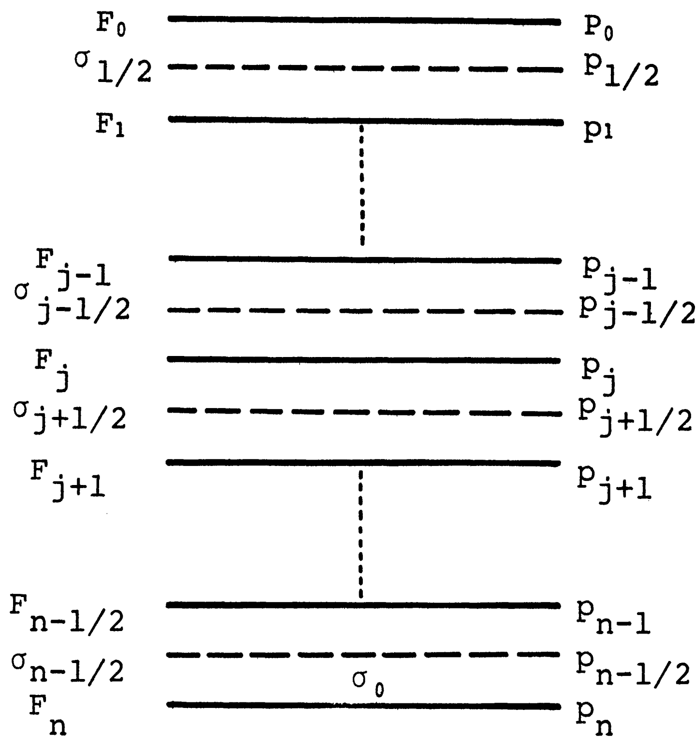

Figure 3.1. The model finite difference levels.

A finite difference approximation to Equation 1.1, for the j'th level, may be written formally as follows:

$$\frac{1}{p_{j+1/2} - p_{j-1/2}} \left[\left(\frac{1}{\sigma} \frac{dF}{dp} \right)_{j+1/2} - \left(\frac{1}{\sigma} \frac{dF}{dp} \right)_{j-1/2} \right] + \frac{1}{gh} F = 0 \qquad (3.1)$$

We now specify the pressure at the middle of the layer (j,j+1) as follows:

$$p_{j+1/2} = \frac{p_{j+1} + p_j}{2} \qquad (0 \leq j \leq n-1) \tag{3.2}$$

Using centered difference approximations for the first derivatives at levels j+1/2 and j-1/2, Equations 3.1 and 3.2 combine to give

$$\frac{2}{p_{j+1} - p_{j-1}} \left[\frac{1}{\sigma_{j+1/2}} \frac{F_{j+1} - F_j}{p_{j+1} - p_j} - \frac{1}{\sigma_{j-1/2}} \frac{F_j - F_{j-1}}{p_j - p_{j-1}}\right] + \frac{1}{gh} F_j = 0 \tag{3.3}$$

In what follows it will be assumed that all layers have equal width and we will write $p_{j+1} - p_j = \Delta p$.

Equation 3.3 may be applied to levels j = 1 to n = 1. The equations for levels j = 0 and j = n must be obtained by application of appropriate difference approximations to the boundary conditions. We consider the lower ($p = p_n$) boundary condition first. This will be of the form:

$$\left(\frac{dF}{dp}\right)_n = \mu F_n \tag{3.4}$$

where $\mu = 0$ if the BC1 condition is used and $\mu = -\sigma_o p_n / RT_o$ if the BC2 condition is used. Let us assume that the function

$$H = \frac{1}{\sigma} \frac{dF}{dp} \tag{3.5}$$

has a convergent Taylor Series expansion near $p = p_n$, so that

$$H(p) = H(p_n) + \left(\frac{dH}{dp}\right)_n s + \left(\frac{d^2H}{dp^2}\right)_n \frac{s^2}{2} + 0\left(\frac{s^3}{6}\right) \qquad 3.6)$$

where $s = p - p_n$. Applying this to the midpoint of the layer (n-1,n) gives

$$H_{n-1/2} = H_n - \left(\frac{dH}{dp}\right)_n \frac{\Delta p}{2} + \left(\frac{d^2H}{dp^2}\right)_n \left(\frac{(\Delta p)^2}{8}\right) + O\left(\frac{(\Delta p)^3}{48}\right) \qquad (3.7)$$

Neglecting terms which are $O(\Delta p^3)$ and solving for $(dH/dp)n$ gives

$$\left(\frac{dH}{dp}\right)_n = \frac{2}{\Delta p}\left[H_n - H_{n-1/2}\right] + \frac{\Delta p}{4}\left(\frac{d^2H}{dp^2}\right)_n$$

$$= \frac{-1}{gh} F_n \qquad (3.8)$$

where the second equality follows from the definition of H. Now we also have the following:

$$\left.\begin{aligned} H_{n-1/2} &= \left(\frac{1}{\sigma}\frac{dF}{dp}\right)_{n-1/2} \\ &= \frac{1}{\sigma_{n-1/2}} \frac{F_n - F_{n-1}}{\Delta p} \end{aligned}\right\} \qquad (3.9)$$

$$\left(\frac{d^2H}{dp^2}\right)_n = \frac{-1}{gh}\left(\frac{dF}{dp}\right)_n = \frac{-\mu}{gh} F_n \qquad (3.10)$$

$$H_n = \frac{\mu}{\sigma_o} F_n \qquad (3.11)$$

where, as in previous sections, σ_o is the static stability at $p = p_n$. Equations 3.8 to 3.11 may be combined to give

$$\frac{2}{\sigma_{n-1/2}\,\Delta p^2\left(1 - \frac{\mu\Delta p}{4}\right)}\left[F_{n-1} - \left(1 - \frac{\mu\Delta p\sigma_{n-1/2}}{\sigma_o}\right)F_n\right] + \frac{1}{gh}F_n = 0 \qquad (3.12)$$

This is the finite difference approximation to Equation 1.1 at $j = n$, the bottom level.

The upper boundary condition states that the quantity H defined above must vanish at $p = 0$. Reference to the analytical results for the isothermal case shows that F may not satisfy a Lipschitz condition near p = 0, and in addition, for n > 2,

$$\frac{1}{n!}\frac{d^nH}{dp^n} \approx \frac{1}{4}\frac{(\Gamma^2-1)}{2^{n-1}n!} p^{-n} \left(\frac{p}{p_o}\right)^{1/2} [(\Gamma-1)\ldots(\Gamma-2n+3)\ A\left(\frac{p}{p_o}\right)^{\Gamma/2} + (-1)^{n-1}(\Gamma+1)\ldots(\Gamma+2n-3)\ B\left(\frac{p}{p_o}\right)^{-\Gamma/2}] \qquad (3.13)$$

In this case, therefore, the Taylor Series for H does not converge rapidly near p = 0 for all permissible values of Γ. Despite this, we choose to proceed as above and truncate the series at the third term, noting that the resulting misrepresentation of F should be confined mainly to a relatively small neighborhood of p = 0. Moreover, we shall justify this a Posteriori through comparison of the numerical and analytical solutions for the isothermal case. Therefore, using the condition that H(0) = 0 we may write:

$$\left(\frac{dH}{dp}\right)_o = \frac{2}{\Delta p} H_{1/2} - \frac{\Delta p}{4}\left(\frac{d^2H}{dp^2}\right)_o = \frac{-1}{gh} F_o \qquad (3.14)$$

Noting that

$$\left.\begin{aligned} H_{1/2} &= \frac{F_1 - F_o}{\sigma_{1/2}\,\Delta p} \\ \left(\frac{d^2H}{dp^2}\right)_o &= \frac{-1}{gh}\left(\frac{dF}{dp}\right)_o \end{aligned}\right\} \qquad (3.15)$$

we have

$$\frac{2}{\sigma_{1/2}\,\Delta p^2}[F_1 - F_o] + \frac{1}{gh}[F_o + \left(\frac{dF}{dp}\right)_o \frac{\Delta p}{4}] = 0 \qquad (3.16)$$

If $\frac{dF}{dp} = 0$ at $p = 0$, as in the constant lapse rate case, then

$$\frac{2}{\sigma_{1/2}\,\Delta p^2}(F_1 - F_o) + \frac{1}{gh}F_o = 0 \qquad (3.17)$$

Alternatively we let

$$\left(\frac{dF}{dp}\right)_o = \frac{F_1 - F_o}{\Delta p}$$

which gives:

$$\frac{2}{\sigma_{1/2}\,\Delta p^2}(F_1 - F_o) + \frac{1}{gh}\left[F_o + \frac{1}{4}(F_1 - F_o)\right] = 0 \qquad (3.18)$$

From Equation 3.1 we also have:

$$\frac{1}{\sigma_{3/2}\,\Delta p^2}(F_2 - F_1) - \frac{1}{\sigma_{1/2}\,\Delta p^2}(F_1 - F_o) + \frac{1}{gh}F_1 = 0 \qquad (3.19)$$

Eliminating F_1/gh between 3.18 and 3.19 gives the desired finite difference equation:

$$\frac{1}{3}\left[\frac{9(F_1 - F_o)}{\sigma_{1/2}\,\Delta p^2} - \frac{(F_1 - F_2)}{\sigma_{3/2}\,\Delta p^2}\right] + \frac{1}{gh}F_o = 0 \qquad (3.20)$$

Equation 3.17 is the boundary equation appropriate to the condition $dF/dp = 0$ at $p = 0$, while Equation 3.20 is appropriate for the less stringent condition that $\frac{1}{\sigma}\,dF/dp = 0$ at $p = 0$ when $\lim_{p \to 0} \sigma \to \infty$.

Equations 3.1, 3.12, and either 3.17 or 3.20 define a closed system of n+1 equations in the n+1 unknowns $F_o,\ldots,F_n$. The system may be written in matrix form:

$$(\underline{\underline{A}} - \frac{1}{gh}\underline{\underline{I}})\ \underline{F} = 0 \qquad (3.21)$$

where $\underline{\underline{A}}$ is a square real matrix and $\underline{\underline{I}}$ is the identity matrix. Equation 3.21 is in the classical form of an algebraic eigen-system whose eigenvalues, λ_j, define equivalent depths:

$$h_j = \frac{1}{g\lambda_j} \qquad (3.22)$$

The corresponding eigenvectors are numerical approximations to the eigenfunctions defined by the solutions to Equation 1.1 and one of the sets of boundary conditions.

The numerical procedure is, then, to compute the eigenvalues and eigenvectors associated with $\underline{\underline{A}}$. In all the numerical calculations the model contained 40 layers of equal pressure width with p_n = 100 cb. The computations were carried out utilizing IBM SSP programmes for computing the eigenvalues and eigenvectors of a general real matrix. The details of these programmes are contained in the IBM SSP manuals, and the mathematical background for the algorithms used, and for several alternative ones, is presented in Wilkinson's (1965) book.

4. COMPARISON OF NUMERICAL AND ANALYTICAL RESULTS

4.1 THE ISOTHERMAL ATMOSPHERE

As shown in section 2 above, for the isothermal atmosphere, solutions satisfying Equation 1.1 and the boundary conditions exist for all finite equivalent depths. The numerical model can resolve only as many of these as there are levels where F is defined. Consequently in comparing the approximate solutions with analytical ones we first compute those values of h which correspond to eigenvalues of the numerical model. Using one of these values we may then compute the vertical structure of F from either of Equations 2.18 or 2.19. The result can be compared with the numerical approximation to F for this particular value of h. In the computational results to be presented p_n and T have been chosen to be 100 cb and 285°K, respectively.

The four largest equivalent depths for each of the two lower boundary conditions are shown in the first section of Table 4.1. The BC2 boundary condition gives rise to the external mode referred to by Wiin-Nielsen (1971a). There is also a mode of very large equivalent depth for the BC1 boundary condition. This mode is the model approximation to the solution $F \equiv$ constant corresponding to an infinite equivalent depth. This solution is not permissible when the BC2 lower boundary condition is used. In this case the largest equivalent depth obtained turns out to be very close to that for which the second term in 2.1.9 vanishes, i.e., from Equation 2.1.12,

$$h \simeq \frac{4\sigma_n p_n^2}{g(1-\hat{\Gamma}^2)} \qquad (4.1.1)$$

Lapse Rate (°K.km)	Mode	Eq. Depth (m)		LND (cb)	
		BC1	BC2	BC_1	BC_2
ISOTHERMAL	1 2 3 4	1297×10^3 5014 2124 944	11360 4497 1984 905	- 21.25 1.25, 31.25 1.25, 6.25, 43.75	- 11.25 1.25, 23.75 1.25, 6.25, 38.75
6.82°K/km	EXT 1 2 3	- 968 430 188	8996 947 423 186	- 23.25 1.25, 36.25 1.25, 6.25, 50	- 23.25 1.25, 36.25 1.25, 7.0, 50
8.54°K/km	EXT 1 2 3	- 349 157 71	8881 346 156 70	- 23.5 1.25, 36.25 1.25, 6.25, 50	- 20 1.25, 33.75 1.25, 6.25, 50
6.82°K/km ($p_0 \geq p \geq 25$cb) ISOTHERMAL (25cb $> p \geq 0$)	EXT 1 2 3	- 1168 500 226	9016 1143 492 224	- 21.25 1.25, 33.75 1.25, 6.25, 46.25	- 18.75 1.25, 31.25 1.25, 6.25, 43.75
8.54°K/km ($p_0 \geq p \geq 25$cb) ISOTHERMAL (25cb $> p \geq 0$)	EXT 1 2 3	- 460 197 90	8551 456 196 90	- 21.25 1.25, 33.75 1.25, 6.25, 46.25	- 18.75 1.25, 31.25 1.25, 6.25, 41.25

Table 4.1. BC1: $dF/dp = 0$ at $p = p$; BC2: $dF/dp + \sigma_o p_o/RT_o\, F = 0$ at $p = p_o$. LND: nondivergence levels (zeros) in the vertical structure of F. EXT: external modes for the BC2 condition.

Structures of the numerical and analytical solutions for the first three modes obtained with the BC1 boundary condition are shown in Figures 4.1 and 4.2, respectively. The results for the BC2 boundary condition are shown in Figures 4.3 and 4.4. In both cases there is good agreement between numerical and analytical solutions. However, we do note from Figures 4.1 and 4.2, that there is noticeable disagreement between numerical and analytical results near p = 0 for the mode having the largest equivalent depth in the BC1 case. For this case the analytical solution becomes infinite at p = 0. The point plotted near p = 0 in Figure 4.2 corresponds to the value obtained for $p = 10^{-3}$ cb. It is not surprising, of course, that the numerical model should misrepresent the behavior of F in this case. However, the misrepresentation is confined to the top layer of the model and is not serious.

4.2 THE CONSTANT LAPSE RATE ATMOSPHERE

For the isothermal cases just discussed the appropriate upper boundary condition was that in Equations 1.3 and 1.4. Consequently Equation 3.20 was used for the uppermost level in the model. For a constant lapse rate Equation 3.17, corresponding to dF/dp = 0 at p = 0, is appropriate.

Equivalent depths for the external mode and the first three internal modes for two particular constant lapse rate cases are shown in the second and third sections of Table 4.1. These lapse rates were chosen to facilitate comparison with Wiin-Nielsen's (1971a) results for n = 3 (γ = 8.54°K/km) and n = 4 (γ = 6.82°K/km), respectively. The mode structures for these cases, using the BC2 boundary condition, are shown in Figures 4.5 and 4.6.

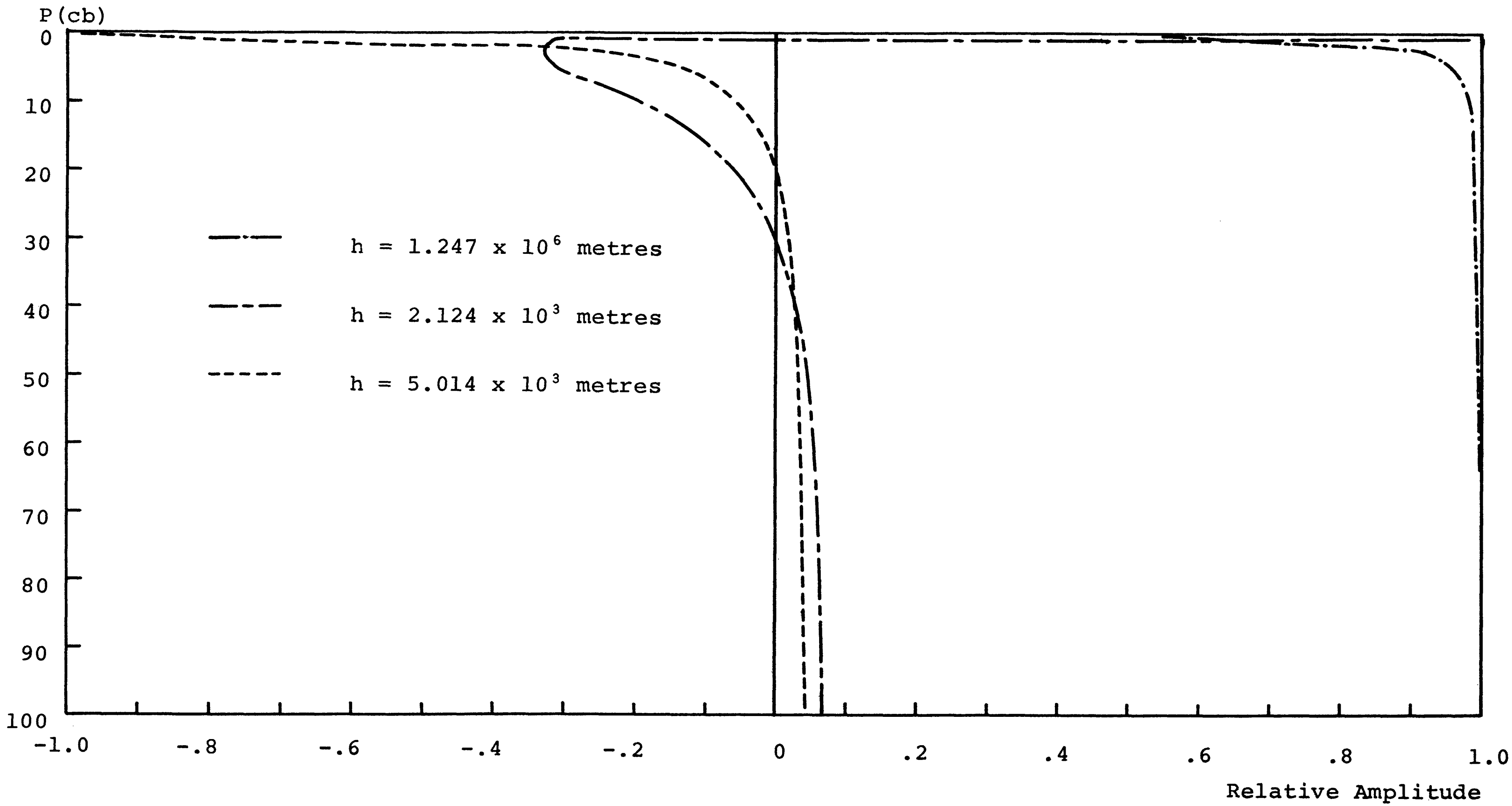

Figure 4.1. Numerical solutions for the first three modes in an isothermal atmosphere. BC1 boundary condition.

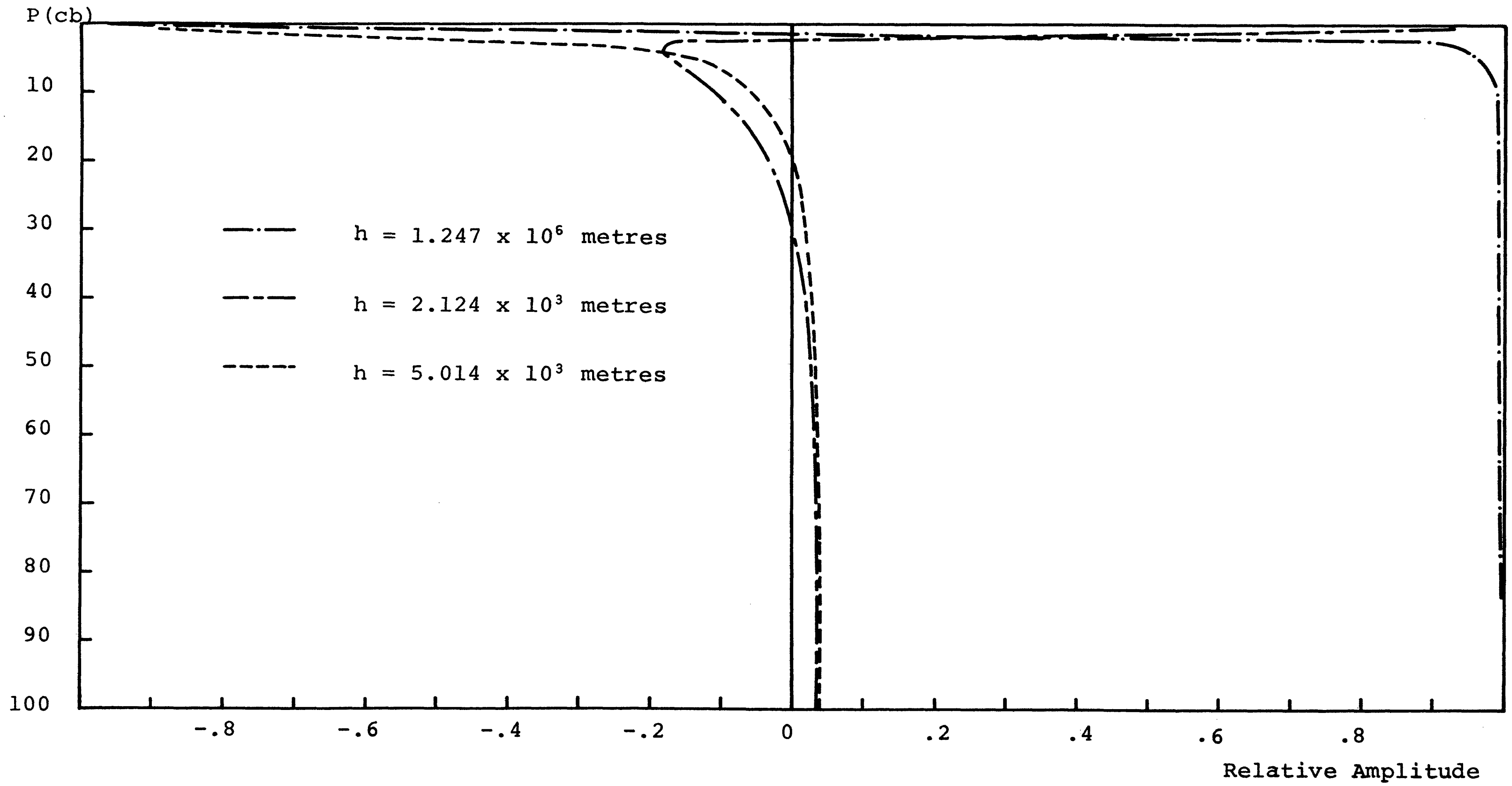

Figure 4.2. Analytical solutions for the first three modes in an isothermal atmosphere. BC1 boundary condition.

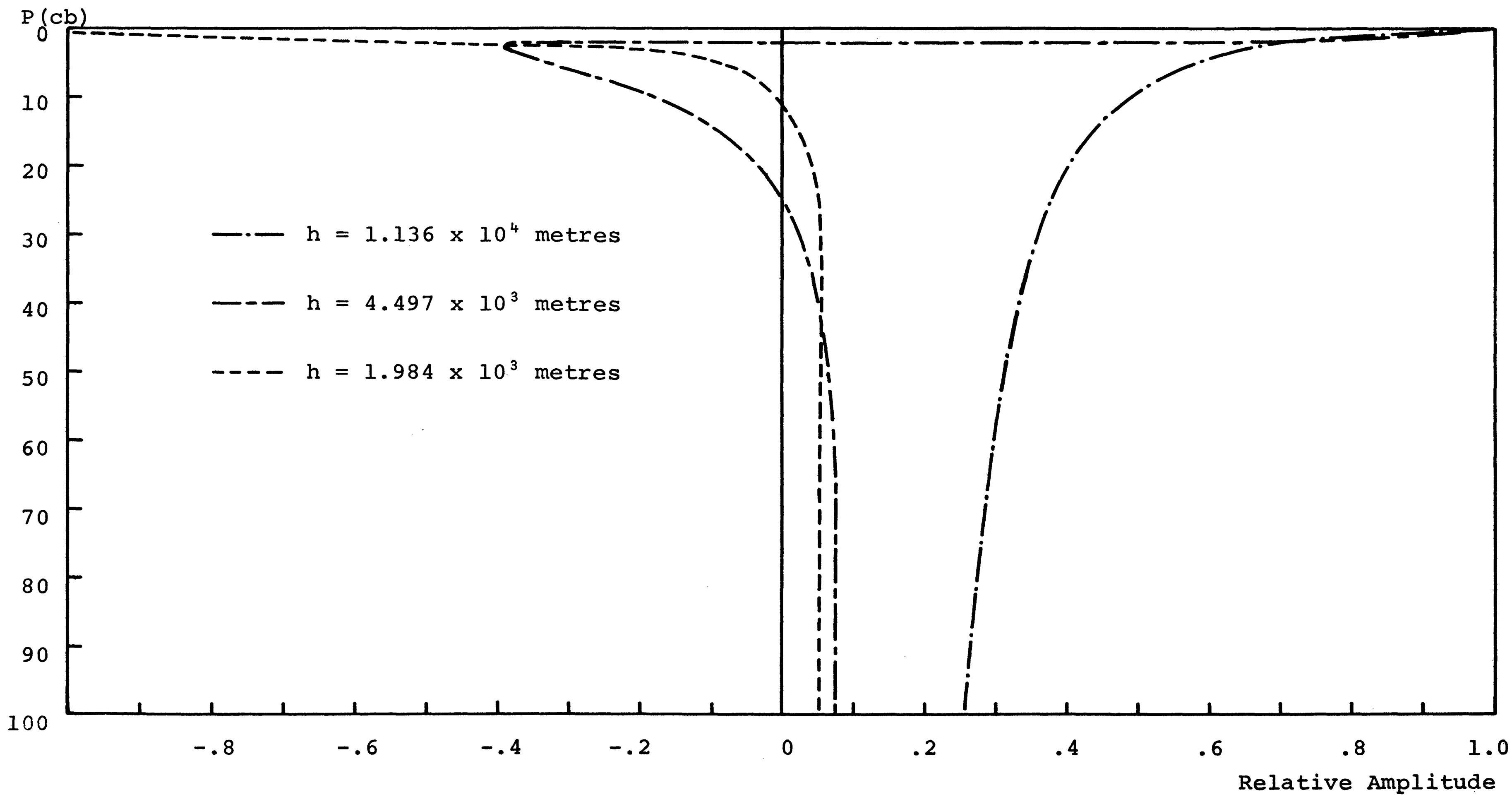

Figure 4.3. Numerical solutions for the first three modes in an isothermal atmosphere. BC2 boundary condition.

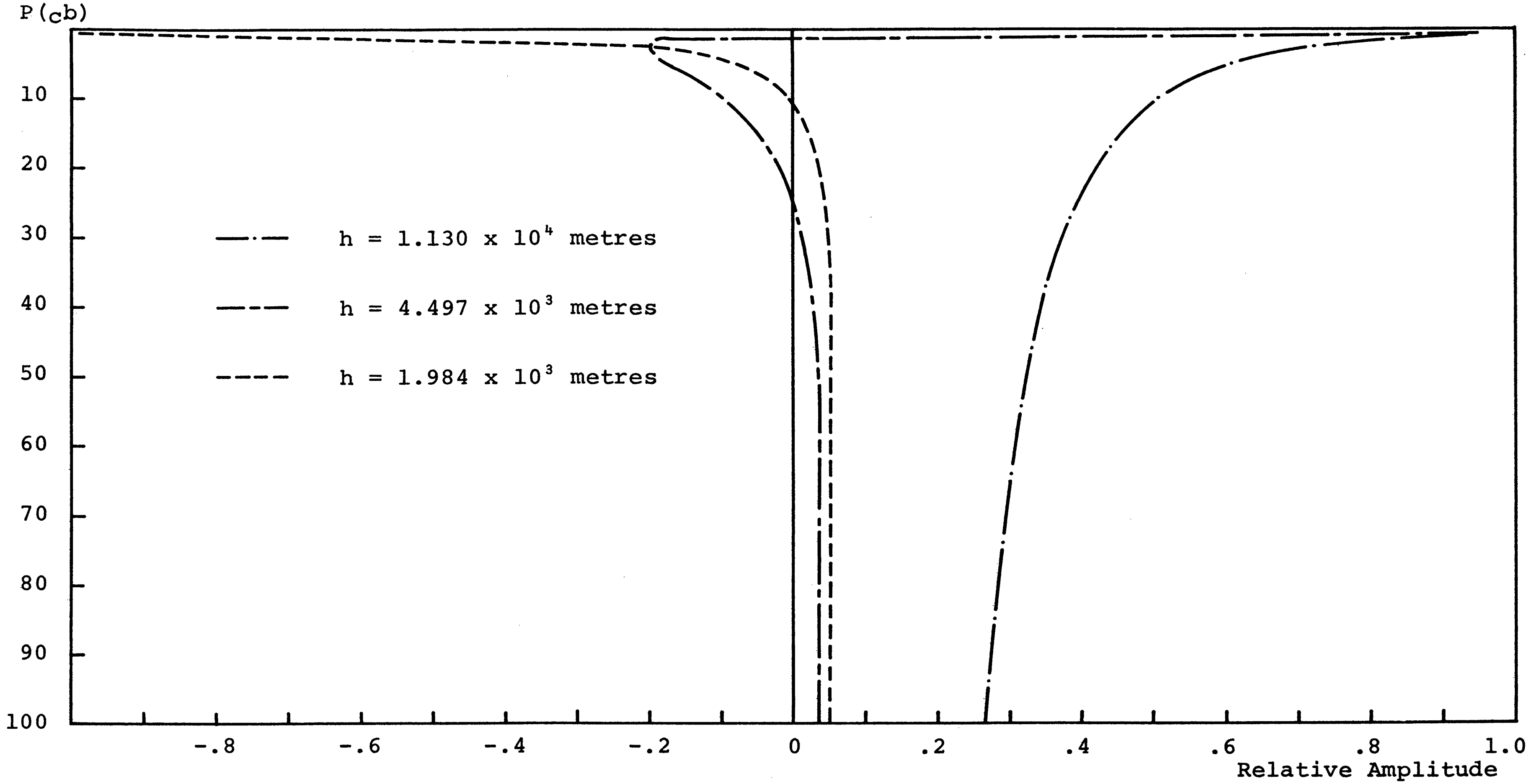

Figure 4.4. Analytical solutions for the first three modes in an isothermal atmosphere. BC2 boundary condition.

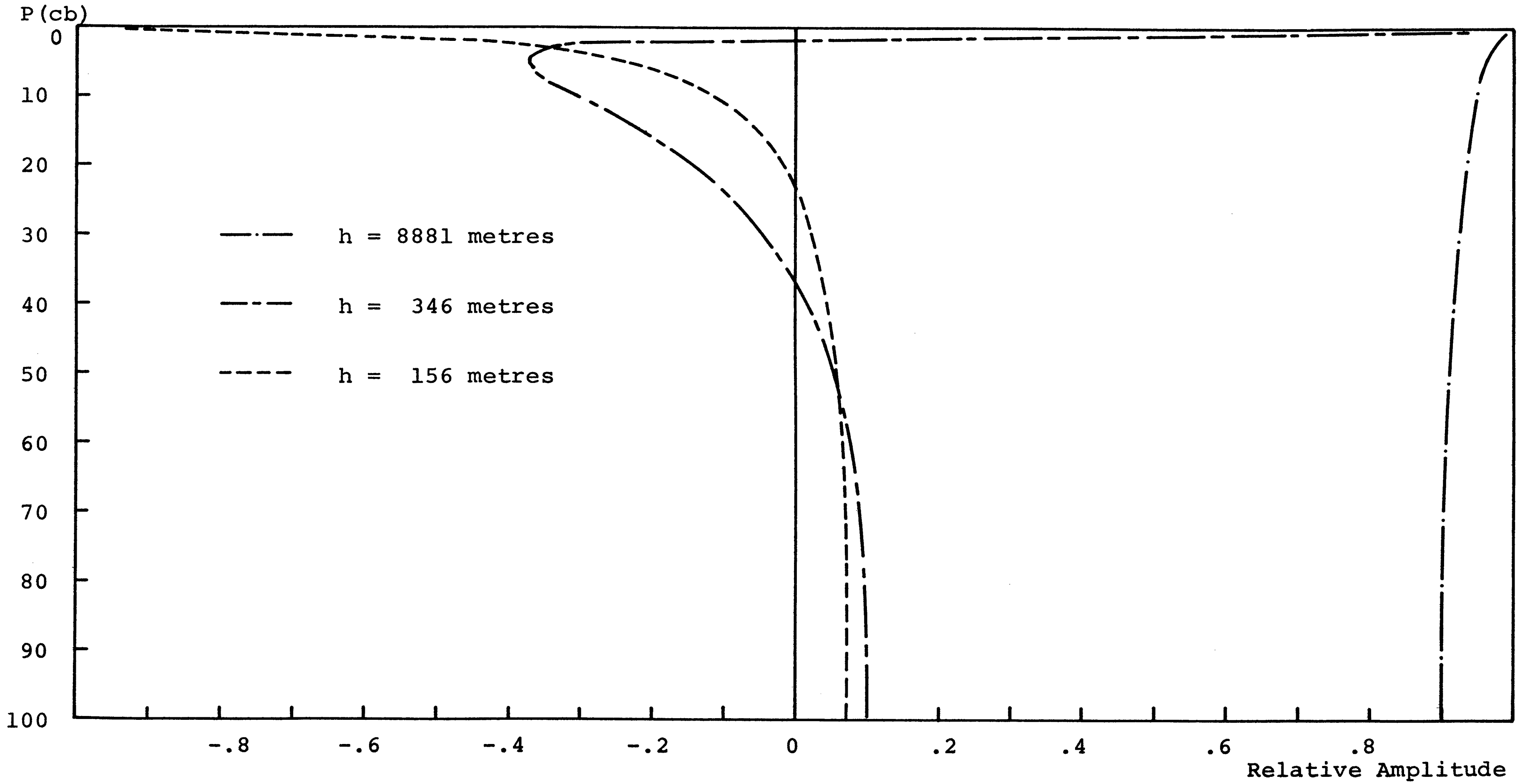

Figure 4.5. Numerical solutions for the first three modes in a constant lapse rate atmosphere, $\gamma = 8.54°K/km$. BC2 boundary condition.

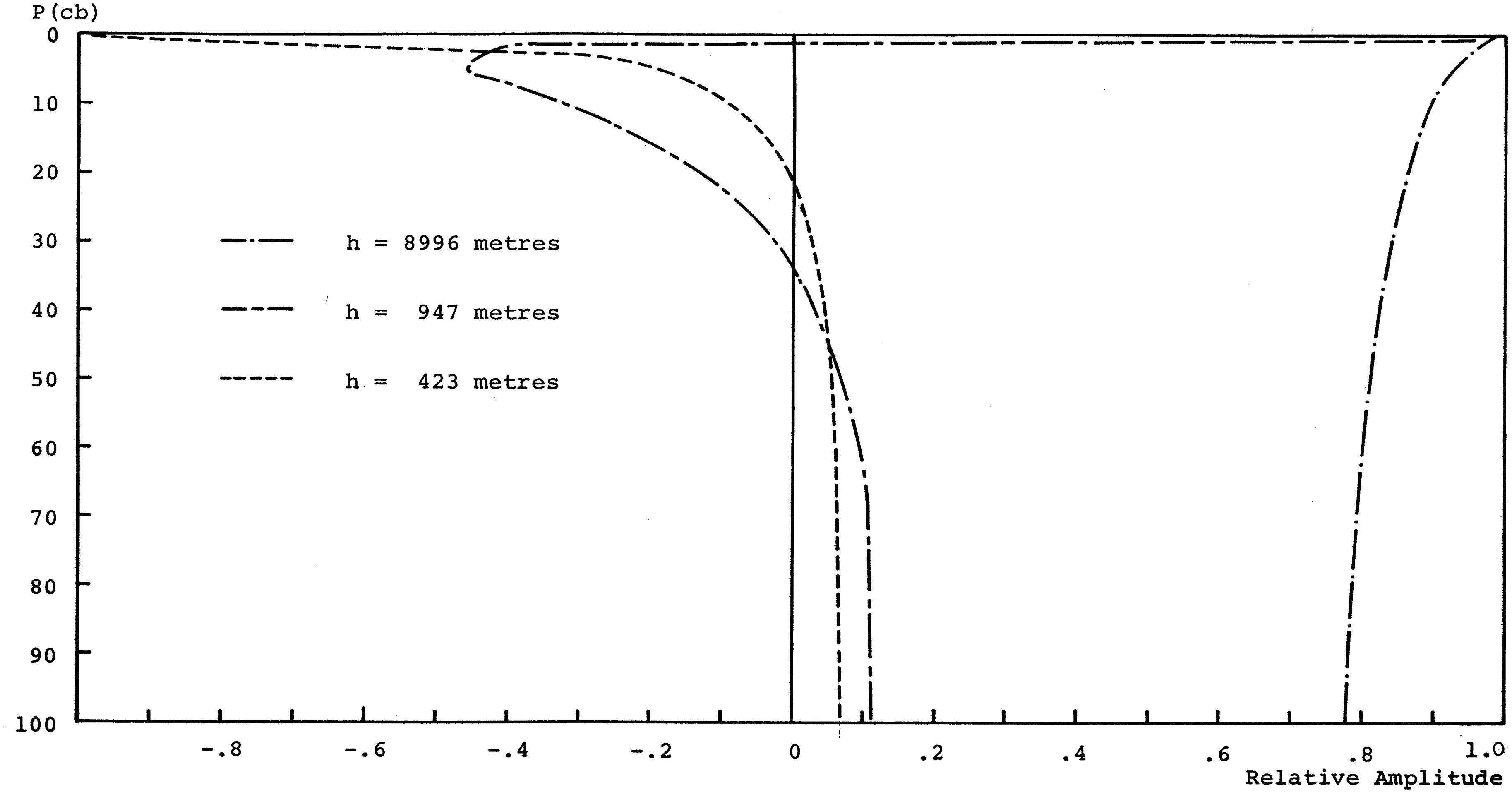

Figure 4.6. Numerical solutions for the first three modes in a constant lapse rate atmosphere, $\gamma = 6.82°K/km$. BC2 boundary condition.

Comparison with Wiin-Nielsen's results again shows that there is good agreement between the numerical and analytical results.

Again it should be noted that in all BCl cases examined there is a mode with a very large equivalent depth (usually $\sim 10^6$ meters) for which $F \simeq$ constant. As mentioned above these are numerical approximations to the infinite equivalent depth case and are of very little practical interest in the context of this investigation. Consequently they are not listed for any of the cases discussed in the following sections.

5. AN ATMOSPHERE BOUNDED ABOVE BY AN ISOTHERMAL LAYER

A more realistic model of typical atmosphere temperature profiles than either of the constant lapse rate or isothermal cases is one in which the temperature lapse rate is constant from the bottom of the model ($p = p_n$) to some pressure level (chosen here to be 25 cb) and isothermal from there to the top of the model ($p = 0$). This isothermal layer then models the effect of a stratosphere. Numerical results for two cases of this type are shown in the bottom sections of Table 4.1.

The results listed in Table 4.1 provide an illustration of the effect of variations in the temperature profile which is used to calculate the static stability. The isothermal case gives rise to relatively large equivalent depths. These are considerably reduced in size for the constant lapse rate case. This is especially true for the internal modes. Not surprisingly, the cases in the last two sections in Table 4.1, corresponding to temperature profiles which are combinations of the constant lapse rate and isothermal cases, have equivalent depths which are bounded (for the internal modes) above by the isothermal ones and below by the constant lapse rate ones.

The structures of first three modes of the cases in the bottom two sections of Table 4.1 are shown in Figures 5.1 and 5.2. As before, these are the results for the BC2 boundary conditions. In these cases, as well as for the constant lapse rate cases, the BC1 boundary condition gives rise to internal modes which are virtually identical with those of the BC2 condition. In addition the BC1 condition gives rise to a numerical approximation to the

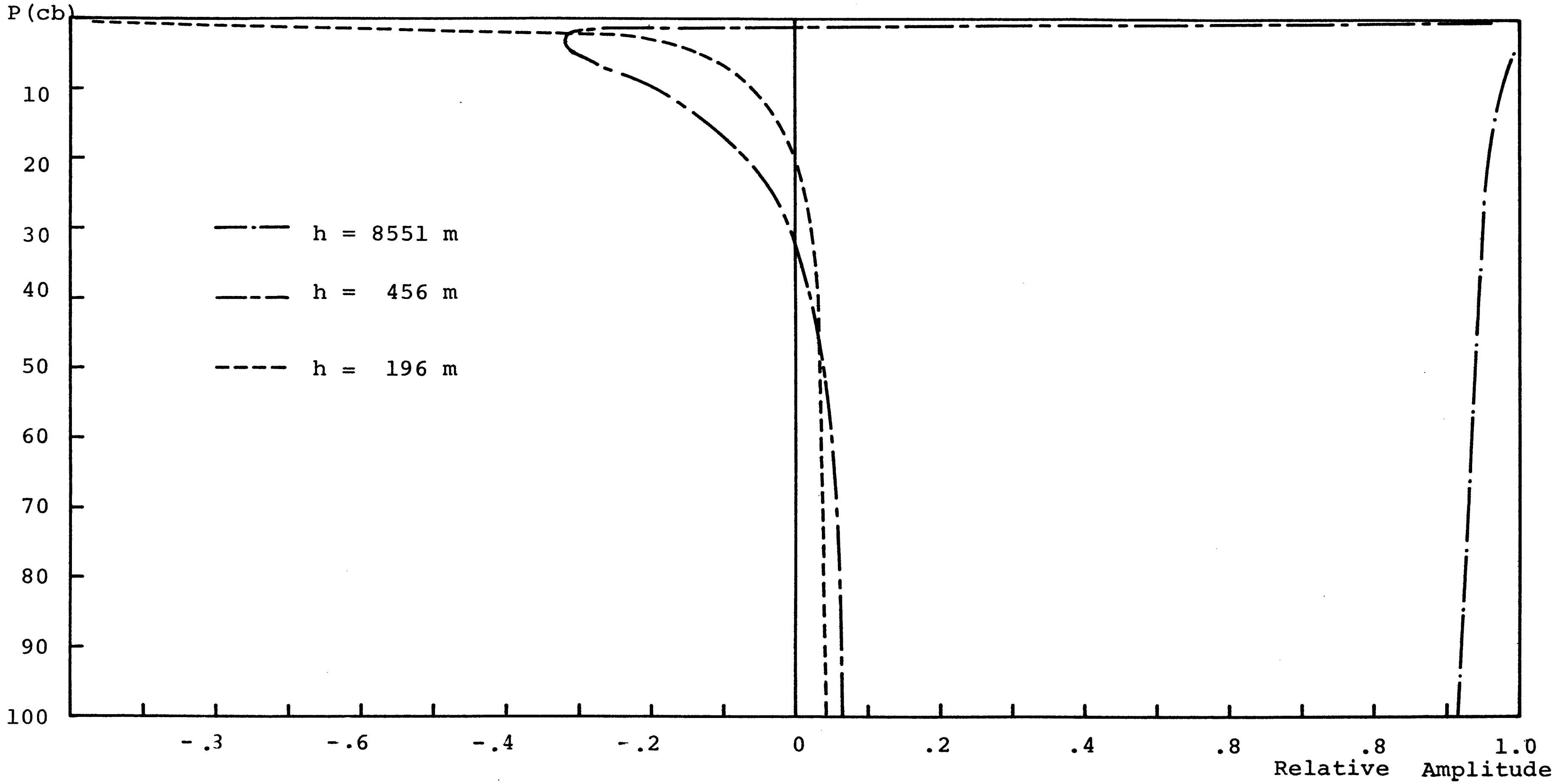

Figure 5.1. Numerical solutions for an atmosphere with a temperature profile defined by: $\gamma = 8.54°K/km$, $100 \text{ cb} \geq P \geq 25 \text{ cb}$; $\gamma = 0$, $25 \text{ cb} \geq P \geq 0$. BC2 boundary condition.

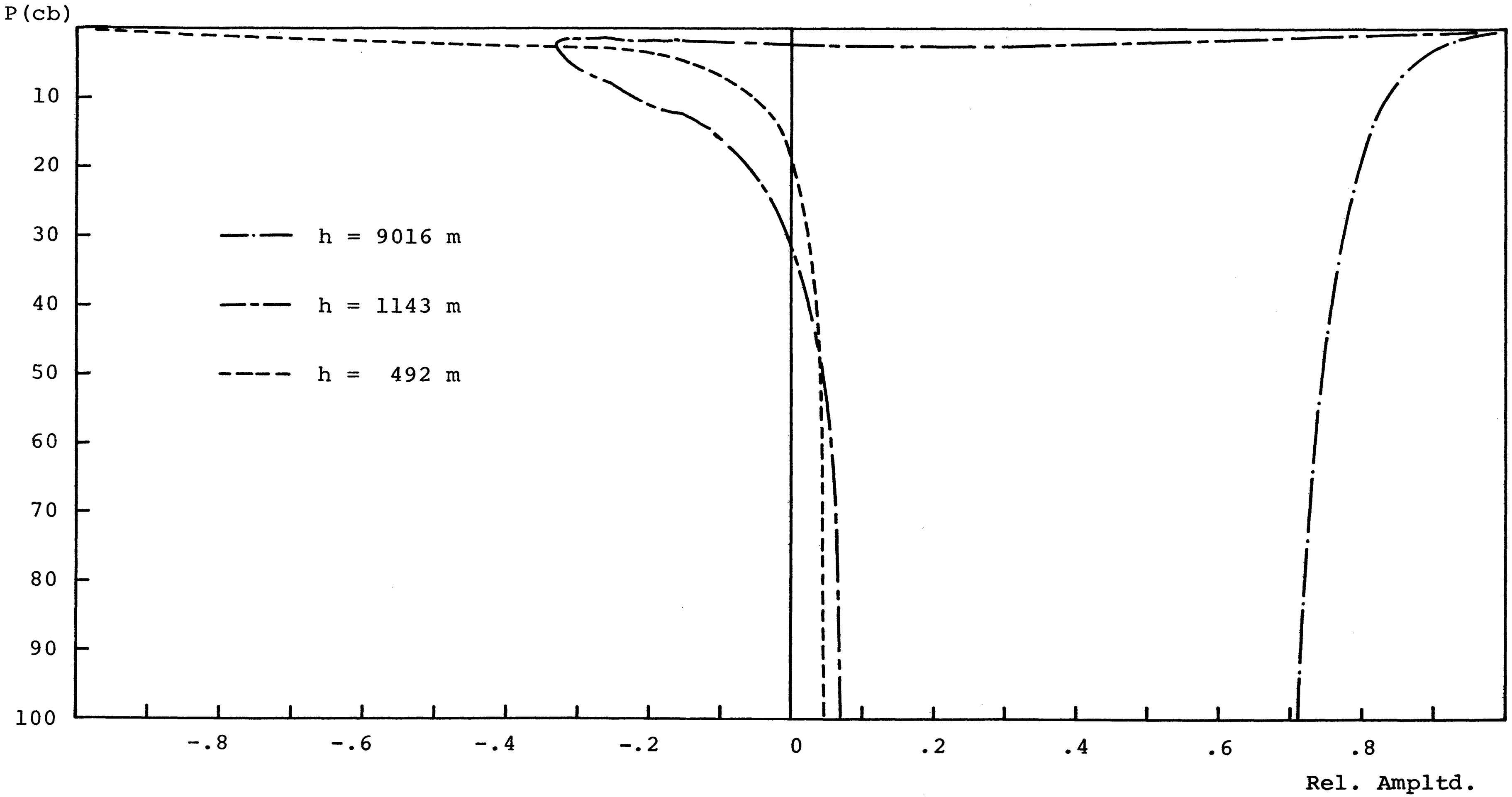

Figure 5.2. Numerical solutions for an atmosphere with a temperature profile defined by: $\gamma = 6.82°K/km$, $100\ cb \geq P \geq 25\ cb$; $\gamma = 0$, $25\ cb \geq P \geq 0$. BC2 boundary condition.

solution F $\equiv$ constant. The BC2 condition, on the other hand, gives rise to the external mode.

In Figures 5.3 and 5.4 we have displayed the behavior of the equivalent depths of the external and first three internal modes as a function of the lapse rate in the lower part of the mode. The behavior of the external mode equivalent depths is less uniform than that of the internal mode depths. However, it should be noted that the total variation in the external mode equivalent depth, over the range of lapse rates shown in Figure 5.3 is less than 20% of the value of the smallest one plotted.

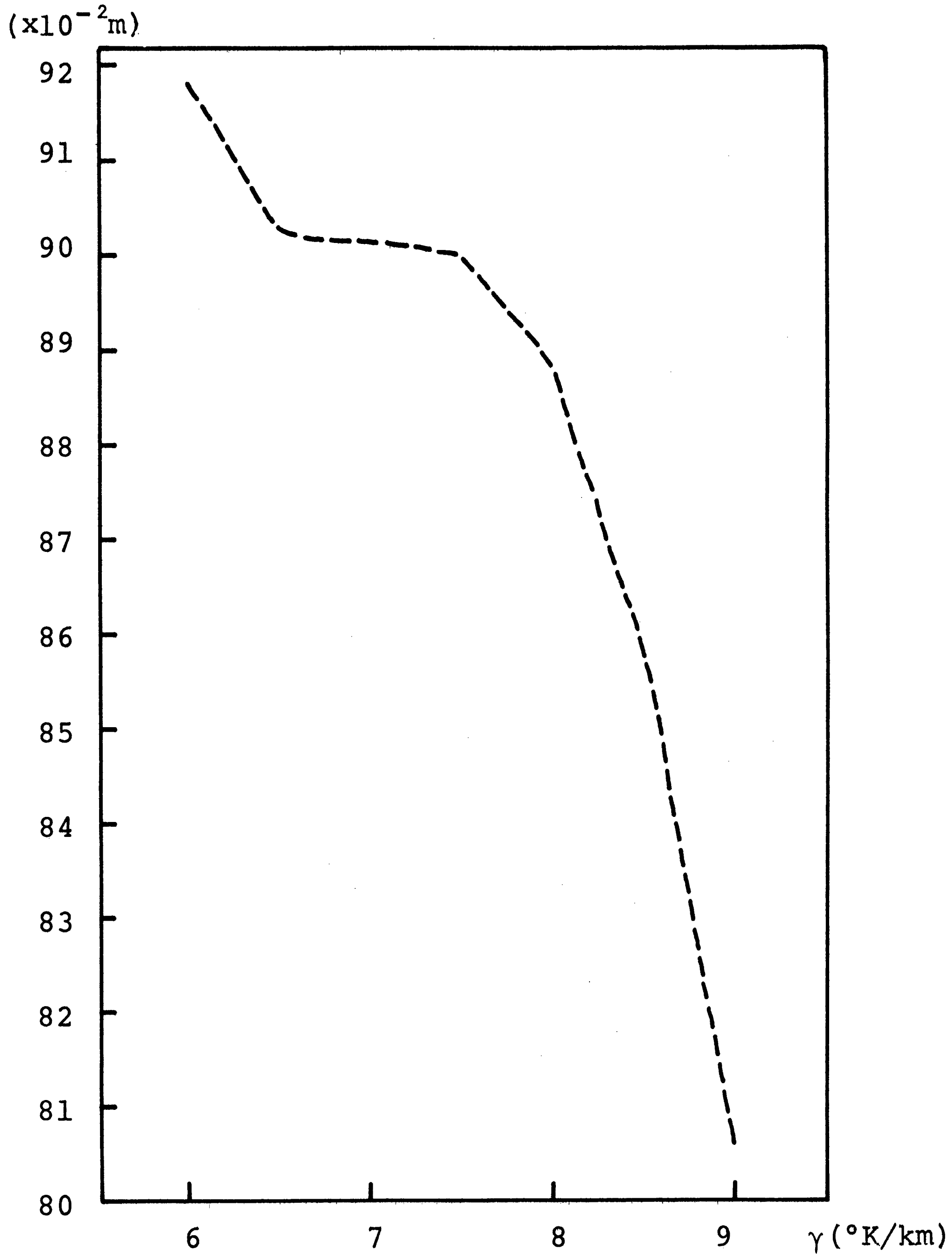

Figure 5.3. External mode equivalent depths as a function of the troposphere lapse rate.

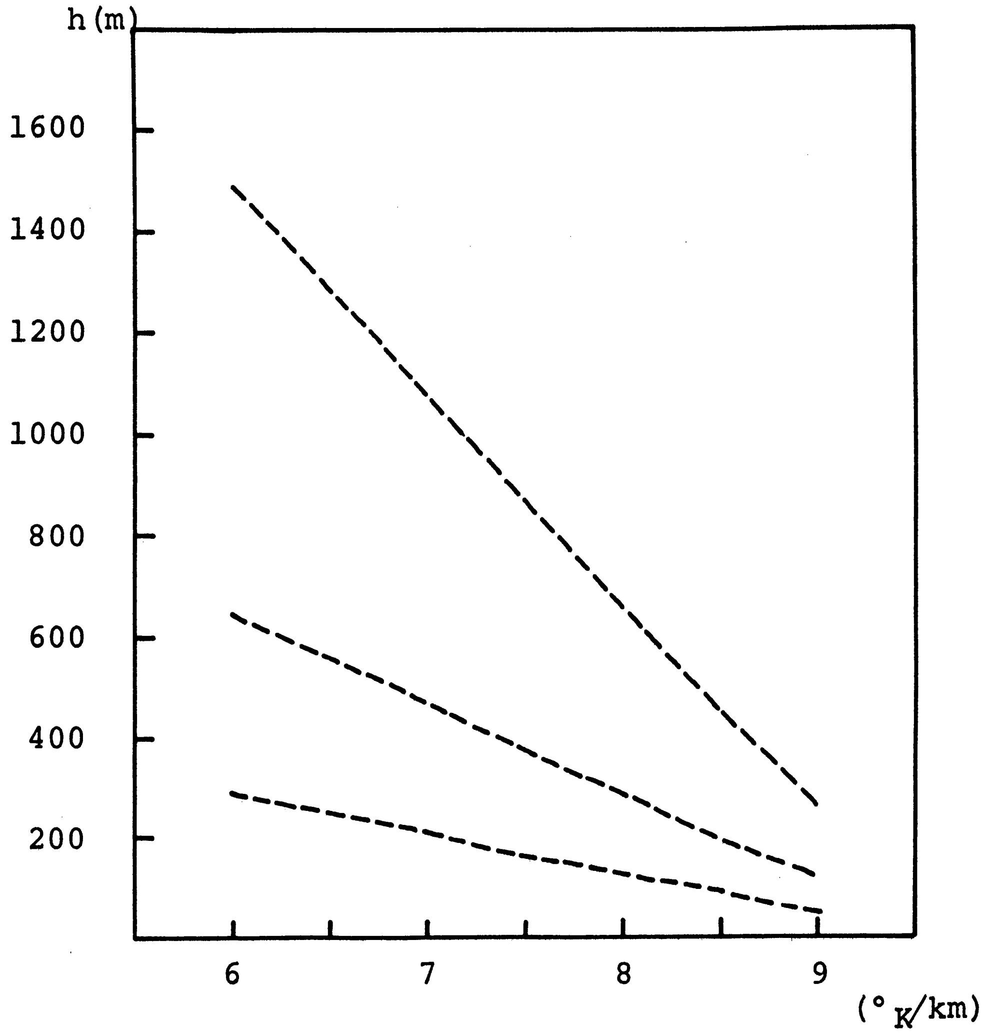

Figure 5.4. Internal mode equivalent depths as a function of lapse rate.

6. CONCLUSIONS

In this study we have demonstrated that accurate approximate solutions to the vertical structure equation for small amplitude transient planetary waves can be obtained numerically. The numerical technique has been used to obtain solutions for cases where the temperature profile has been characterized by a constant lapse rate in the troposphere bounded above by an isothermal layer. It is found that the equivalent depths for internal modes are greater than those of the constant lapse rate atmosphere but considerably less, for lapse rates characteristic of those observed in the troposphere, than the equivalent depths obtained for the isothermal atmosphere. The structures of both the internal and external modes are qualitatively the same as those of the constant lapse rate modes.

ACKNOWLEDGMENTS

The author expresses thanks to Mr. J. Pfaendtner for assistance in programming the calculations in this paper and also for his excellent work in drafting the figures.

REFERENCES

Jacobs, S. J. and A. Wiin-Nielsen, 1966: "On the Stability of Barotropic Basic Flow in a Stratified Atmosphere," Journal of Atmospheric Sciences, Vol. 23, pp. 682-687.

Lindzen, R. D., 1967: "Planetary Waves on Beta Planes," Monthly Weather Review, Vol. 95, pp. 441-451.

Wiin-Nielsen, A., 1971a: "On the Motion of Various Vertical Modes of Transient Very Long Waves," Part I. Tellus, Vol. 23, No. 1, pp. 87-98.

Wiin-Nielsen, A., 1971b: "On the Motion of Various Vertical Modes of Transient Very Long Waves," Part II (Tellus, in press).

Wilkinson, J. H., 1965: The Algebraic Eigenvalue Problem, London, Oxford University Press.