ON A FORMULA FOR EQUIVALENT NUMBER OF INDEPENDENT
NOISE SAMPLES PER UNIT TIME

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Department of Electrical Engineering
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ABSTRACT

An extension is made of a published derivation of a formula for equivalent number of independent noise samples per unit time. The extension reduces the formula from a form containing an infinite integral to a particularly interpretable form. It also discloses the inherent limitations of its application to low-pass noise spectra having relatively large power density at zero frequency. It is shown that under these conditions the equivalent number is just twice the white-noise frequency-bandwidth of the low-pass noise spectral power density produced by rectification, that is, at the output of a detector. This is the interpretable form. The entire equivalent number concept is limited to cases for which noise bandwidth is clearly defined and meaningful.
ON A FORMULA FOR EQUIVALENT NUMBER OF INDEPENDENT
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I. INTRODUCTION

In a report on detection of pulsed signals, C. W. Sherwin\(^1\) has presented a formula which is made the subject of this paper. Relative to false alarm rate, Sherwin considers the problem of integration of the output of a detector whose input is a band of noise and calculates the "improvement in signal-to-noise" effected by a linear integrator operating for a time T. He also calculates the rate \( n_i \) at which discrete independent samples would have to be presented to the same integrator in order to effect the same "improvement in signal to noise." Equating input and output S/N ratios, he obtains for large integration time T the result (his formula (3-9))

\[
n_i = \frac{1}{2 \int_{0}^{\infty} \rho(\tau) d\tau}
\]

(1)
as a general relationship which gives the equivalent rate \( n_i \) of independent samples per unit time in a band of rectified noise whose normalized auto-correlation function is \( \rho(\tau) \), excluding the d.c. component.\(^*\)

\(^1\) Sherwin, C. W., "Detection of Pulsed Signals with a Narrow-band Filter, Detector and Integrator," Report R-42, 15 June 1953, Control Systems Laboratory, University of Illinois; Formula (3-9) p. 19.

\(^*\) This restriction is necessary; otherwise, the infinite integral diverges, giving \( n_i = 0 \).
The "signal-to-noise" of which Sherwin writes is: (before improvement)
(1) the ratio of mean value of rectified noise to rms value of rectified noise;
(2) (after improvement) the ratio of mean value of integrator output at time T
to rms value of integrator output at time T, assuming zero integrator output at
time zero. The latter is also interpreted as the output mean to rms ratio of a
low pass filter following the detector. The filter output feeds a threshold
alarm device.

The derivation of formula (1) will in this paper be extended to arrive
at an alternative simpler interpretation. It will be shown that \( n_1 \) is just twice
the white-noise frequency-bandwidth of the rectified noise output of the detector.
In this interpretation frequency-bandwidth in cycles per unit time must be
understood, not radian-bandwidth in radians per unit time.

II. EXTENSION OF DERIVATION

It is necessary to define normalized auto-correlation function and to
relate it to the auto-correlation function and power density spectrum of a
stationary random process representing the noise. In terms of the auto-
correlation function \( \psi''_n(\tau) \), the normalized auto-correlation function
\( \rho(\tau) \) is defined by

\[
\rho(\tau) = \frac{\psi''_n(\tau)}{\psi''_n(0)}, \quad -\infty < \tau < \infty
\]  

(2)

so that \( \rho(0) = 1 \).  

(3)
The auto-correlation function $\psi_\omega(\tau)$ of a stationary random process is defined by

$$\psi_\omega(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f_i(t)f_i(t+\tau)dt,$$

where $f_i(t)$ represents a member function of the stationary random process.

Wiener's theorem states that the auto-correlation function $\psi_\omega(\tau)$ and power density spectrum $\Phi_\omega(\omega)$ are related as a Fourier transform pair:

$$\Phi_\omega(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_\omega(\tau)e^{-j\omega \tau}d\tau,$$

and

$$\psi_\omega(\tau) = \int_{-\infty}^{\infty} \Phi_\omega(\omega)e^{j\omega \tau}d\omega.$$

It is also true that both functions are even functions, and that $\Phi_\omega(\omega)$ is nonnegative.

---

With these definitions, the extension of Eq (1) can be made as follows.

First, since $\psi_n(\tau)$ is an even function, Eq (2) requires that $\rho(\tau)$ be also an even function. Then Eq (1) may be written in the form

$$n_i = \frac{1}{\int_{-\infty}^{\infty} \rho(\tau) d\tau} = \frac{1}{\int_{-\infty}^{\infty} \rho(\tau) d\tau}.$$  \hspace{1cm} (7)

Second, using Eq (2), Eq (5) may be written in the form

$$\Phi_n(\omega) = \frac{\psi_n(0)}{2\pi} \int_{-\infty}^{\infty} \rho(\tau) e^{-j\omega \tau} d\tau$$  \hspace{1cm} (8)

or

$$\frac{2\pi \Phi_n(\omega)}{\psi_n(0)} = \int_{-\infty}^{\infty} \rho(\tau) e^{-j\omega \tau} d\tau, \quad -\infty < \omega < \infty.$$  \hspace{1cm} (9)

In particular, at $\omega = 0$ we have

$$\frac{2\pi \Phi_n(0)}{\psi_n(0)} = \int_{-\infty}^{\infty} \rho(\tau) d\tau,$$  \hspace{1cm} (10)

so that Eq (7) gives

$$n_i = \frac{1}{\int_{-\infty}^{\infty} \rho(\tau) d\tau} = \frac{\psi_n(0)}{2\pi \Phi_n(0)}.$$  \hspace{1cm} (11)
Formula (11) may now be interpreted and certain restrictions to its general application made evident. From Eq (6) the numerator factor of the right member is seen to be the total power of the stationary random process, that is, the integral of power density over its entire defined range of the doubly infinite frequency variable $\omega$:

$$\psi_n(0) = \int_{-\infty}^{\infty} \Phi_n(\omega) d\omega$$

= total noise power.

Since the frequency variable $\omega$ is a radian frequency variable, the denominator factor $\Phi_{11}(0)$ is the zero-frequency power density in, say, watt-sec. per radian, while the denominator factor $2\pi \Phi_{11}(0)$ is the zero-frequency power density in, say, watt-sec. per cycle:

$$2\pi \Phi_n(0) = \text{zero-frequency power density, watt, (13)}$$

Thus Eq (11) defines a frequency-bandwidth $n_1$ centered at zero frequency which, with constant power density $2\pi \Phi_{11}(0)$, has the same total power $\psi_{11}(0)$ as has the given stationary random process. Since $\Phi_n(\omega)$ is an even function, the frequency-bandwidth $n_1$ given by Eq (11) is just twice the positive half from zero up to the band edge. The latter band is equal by definition to the white noise bandwidth of the stationary random process when it is derived by rectification of noise giving maximum power density at zero frequency.
It is easy to cook up situations which play hob with this interpretation, just as with the concept of bandwidth of complicated networks. However, for the cases studied by Sherwin\textsuperscript{1}, the interpretation is correct. Our result, extending the subject formula (1), is

\[ n_i = \frac{\psi_n(0)}{2\pi \Phi_n(0)} = 2\Delta f, \quad (14) \]

where \( \Delta f \) = white noise bandwidth of the noise after rectification.  \( (15) \)

**III. EXAMPLES**

The results for the cases calculated in the report of Reference 1 can be obtained directly from the given noise power density spectra at the rectifier output, by the use of Eq. (14). Figure 1 illustrates the cascaded systems of interest, the arrows marking the points for which the calculation of \( n_i \) is to be made.

**FIG. I. CASCADE SYSTEMS**
System (A) corresponds to the first two cases of square rf band $\Delta f_1$ and of single-tuned rf band $\Delta f_2$ (3-db), while system (B) corresponds to the last two cases of low-pass square band $f_1$ and single R-C low-pass band $\frac{1}{2\pi RC}$ (3-db).

Chart I depicts the power density spectra at the arrow marking points of Fig. 1, and lists the corresponding noise bandwidths $\Delta f_i$, together with the calculation of

$$n_i = 2\Delta f_i$$

(14)
<table>
<thead>
<tr>
<th>Type of Filter 1</th>
<th>Power-Density Spectrum (One-Sided)</th>
<th>Noise Bandwidth</th>
<th>Equivalent Number $n_i$ of Independent Samples per Unit Time, $n_i = 2\Delta f$</th>
</tr>
</thead>
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<tr>
<td>Square RF Band</td>
<td>$G(f) = 1 - \frac{f}{\Delta f}, 0 &lt; f &lt; \Delta f$, $G(f) = 0$ elsewhere</td>
<td>$\frac{\Delta f_i}{2}$</td>
<td>$n_i = 2 \times \frac{\Delta f_i}{2} = \Delta f_i$</td>
</tr>
<tr>
<td>Single-Tuned RF Band</td>
<td>$G(f) = \frac{1}{1 + \left(f/\Delta f_2\right)^2}, 0 &lt; f_2$, $G(f) = 0$, $f &lt; 0$</td>
<td>$\pi \Delta f_2$</td>
<td>$n_i = 2 \times \frac{\pi \Delta f_2}{2} = \pi \Delta f_2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Type of Filter 2</th>
<th>Power-Density Spectrum (One-Sided)</th>
<th>Noise Bandwidth</th>
<th>Equivalent Number $n_i$ of Independent Samples per Unit Time, $n_i = 2\Delta f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Band</td>
<td>$G(f) = 1$, $0 &lt; f &lt; f_1$, $G(f) = 0$, elsewhere</td>
<td>$f_1$</td>
<td>$n_i = 2 \times f_1 = 2f_1$</td>
</tr>
<tr>
<td>Low-Pass Band</td>
<td>$G(f) = \frac{1}{1 + \left(f/\Delta f_2\right)^2}$, $0 &lt; f$</td>
<td>$\frac{\pi}{2} \times \frac{1}{2\pi RC}$</td>
<td>$n = 2 \times \frac{\pi}{2} \times \frac{1}{2\pi RC} = \frac{1}{2RC}$</td>
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**Chart I. Example Calculations of Equivalent Number $n_i$ of Independent Samples per Unit Time at Points Marked by Arrow in Fig. I.**
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