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ON A FORMULA FOR EQUIVALENT NUMBER OF INDEPENDENT NOISE SAMPLES PER UNIT TIME

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ABSTRACT

An extension is made of a published derivation of a formula for equivalent number of independent noise samples per unit time. The extension reduces the formula from a form containing an infinite integral to a particularly interpretable form. It also discloses the inherent limitations of its application to low-pass noise spectra having relatively large power density at zero frequency. It is shown that under these conditions the equivalent number is just twice the white-noise frequency-bandwidth of the low-pass noise spectral power density produced by rectification, that is, at the output of a detector. This is the interpretable form. The entire equivalent number concept is limited to cases for which noise bandwidth is clearly defined and meaningful.

ON A FORMULA FOR EQUIVALENT NUMBER OF INDEPENDENT NOISE SAMPLES PER UNIT TIME

I. INTRODUCTION

In a report on detection of pulsed signals, C. W. Sherwin[1] has presented a formula which is made the subject of this paper. Relative to false alarm rate, Sherwin considers the problem of integration of the output of a detector whose input is a band of noise and calculates the "improvement in signal-to-noise" effected by a linear integrator operating for a time T. He also calculates the rate n_i at which discrete independent samples would have to be presented to the same integrator in order to effect the same "improvement in signal to noise." Equating input and output S/N ratios, he obtains for large integration time T the result (his formula (3-9))

$$n_i = \frac{1}{2\int_0^\infty \rho(\tau)\,d\tau} \tag{1}$$

as a general relationship which gives the equivalent rate n_i of independent samples per unit time in a band of rectified noise whose normalized auto-correlation function is $\rho(\tau)$, excluding the d.c. component.*

[1] Sherwin, C. W., "Detection of Pulsed Signals with a Narrow-band Filter, Detector and Integrator," Report R-42, 15 June 1953, Control Systems Laboratory, University of Illinois; Formula (3-9) p. 19.

* This restriction is necessary; otherwise, the infinite integral diverges, giving $n_i = 0$.

The "signal-to-noise" of which Sherwin writes is: (before improvement) (1) the ratio of mean value of rectified noise to rms value of rectified noise; (2) (after improvement) the ratio of mean value of integrator output at time T to rms value of integrator output at time T, assuming zero integrator output at time zero. The latter is also interpreted as the output mean to rms ratio of a low pass filter following the detector. The filter output feeds a threshold alarm device.

The derivation of formula (1) will in this paper be extended to arrive at an alternative simpler interpretation. It will be shown that n_i is just twice the white-noise frequency-bandwidth of the rectified noise output of the detector. In this interpretation frequency-bandwidth in cycles per unit time must be understood, not radian-bandwidth in radians per unit time.

II. EXTENSION OF DERIVATION

It is necessary to define normalized auto-correlation function and to relate it to the auto-correlation function and power density spectrum of a stationary random process representing the noise. In terms of the auto-correlation function $\psi_{11}(\tau)$, the normalized auto-correlation function $\rho(\tau)$ is defined by

$$\rho(\tau) = \frac{\psi_{11}(\tau)}{\psi_{11}(0)}, \quad -\infty < \tau < \infty \tag{2}$$

so that $\rho(0) = 1.$ (3)

The auto-correlation function $\psi_{11}(\tau)$ of a stationary random process is defined[2] by

$$\psi_{11}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_1(t) f_1(t+\tau)\, dt, \tag{4}$$

where $f_1(t)$ represents a member function of the stationary random process. Wiener's theorem[2] states that the auto-correlation function $\psi_{11}(\tau)$ and power density spectrum $\Phi_{11}(\omega)$ are related as a Fourier transform pair:

$$\Phi_{11}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_{11}(\tau) e^{-j\omega\tau}\, d\tau, \tag{5}$$

and

$$\psi_{11}(\tau) = \int_{-\infty}^{\infty} \Phi_{11}(\omega) e^{j\omega\tau}\, d\omega. \tag{6}$$

It is also true that both functions are even functions, and that $\Phi_{11}(\omega)$ is nonnegative.

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2 Y. W. Lee, T. P. Cheatham, Jr., and J. B. Wiesner, "Application of Correlation Analysis to the Detection of Periodic Signals in Noise," Proc. IRE vol. 38, pp. 1165-1171, Oct. 1950.

With these definitions, the extension of Eq (1) can be made as follows. First, since $\psi_{11}(\tau)$ is an even function, Eq (2) requires that $\rho(\tau)$ be also an even function. Then Eq (1) may be written in the form

$$n_i = \frac{1}{2\int_0^{\infty}\rho(\tau)\,d\tau} = \frac{1}{\int_{-\infty}^{\infty}\rho(\tau)\,d\tau} . \tag{7}$$

Second, using Eq (2), Eq (5) may be written in the form

$$\Phi_{11}(\omega) = \frac{\psi_{11}(0)}{2\pi}\int_{-\infty}^{\infty}\rho(\tau)e^{-j\omega\tau}d\tau \tag{8}$$

or

$$\frac{2\pi\Phi_{11}(\omega)}{\psi_{11}(0)} = \int_{-\infty}^{\infty}\rho(\tau)e^{-j\omega\tau}\,d\tau, \quad -\infty < \omega < \infty . \tag{9}$$

In particular, at $\omega = 0$ we have

$$\frac{2\pi\Phi_{11}(0)}{\psi_{11}(0)} = \int_{-\infty}^{\infty}\rho(\tau)\,d\tau , \tag{10}$$

so that Eq (7) gives

$$n_i = \frac{1}{\int_{-\infty}^{\infty}\rho(\tau)\,d\tau} = \frac{\psi_{11}(0)}{2\pi\Phi_{11}(0)} . \tag{11}$$

Formula (11) may now be interpreted and certain restrictions to its general application made evident. From Eq (6) the numerator factor of the right member is seen to be the total power of the stationary random process, that is, the integral of power density over its entire defined range of the doubly infinite frequency variable ω :

$$\psi_{11}(0) = \int_{-\infty}^{\infty} \Phi_{11}(\omega)d\omega \tag{12}$$

$$= \text{total noise power.}$$

Since the frequency variable ω is a radian frequency variable, the denominator factor $\Phi_{11}(0)$ is the zero-frequency power density in, say, watt-sec. per radian, while the denominator factor $2\pi\Phi_{11}(0)$ is the zero-frequency power density in, say, watt-sec. per cycle:

$$2\pi\Phi_{11}(0) = \text{zero-frequency power density, } \frac{\text{watt}}{\text{c/s}}. \tag{13}$$

Thus Eq (11) defines a frequency-bandwidth n_i centered at zero frequency which, with constant power density $2\pi\Phi_{11}(0)$, has the same total power $\psi_{11}(0)$ as has the given stationary random process. Since $\Phi_{11}(\omega)$ is an even function, the frequency-bandwidth n_i given by Eq (11) is just twice the positive half from zero up to the band edge. The latter band is equal by definition to the white noise bandwidth of the stationary random process when it is derived by rectification of noise giving maximum power density at zero frequency.

It is easy to cook up situations which play hob with this interpretation, just as with the concept of bandwidth of complicated networks. However, for the cases studied by Sherwin[1], the interpretation is correct. Our result, extending the subject formula (1), is

$$n_i = \frac{\psi_{11}(0)}{2\pi\Phi_{11}(0)} = 2\Delta f \,, \qquad (14)$$

where Δf = white noise bandwidth of the noise after rectification. (15)

III. EXAMPLES

The results for the cases calculated in the report of Reference 1 can be obtained directly from the given noise power density spectra at the rectifier output, by the use of Eq. (14). Figure 1 illustrates the cascaded systems of interest, the arrows marking the points for which the calculation of n_i is to be made.

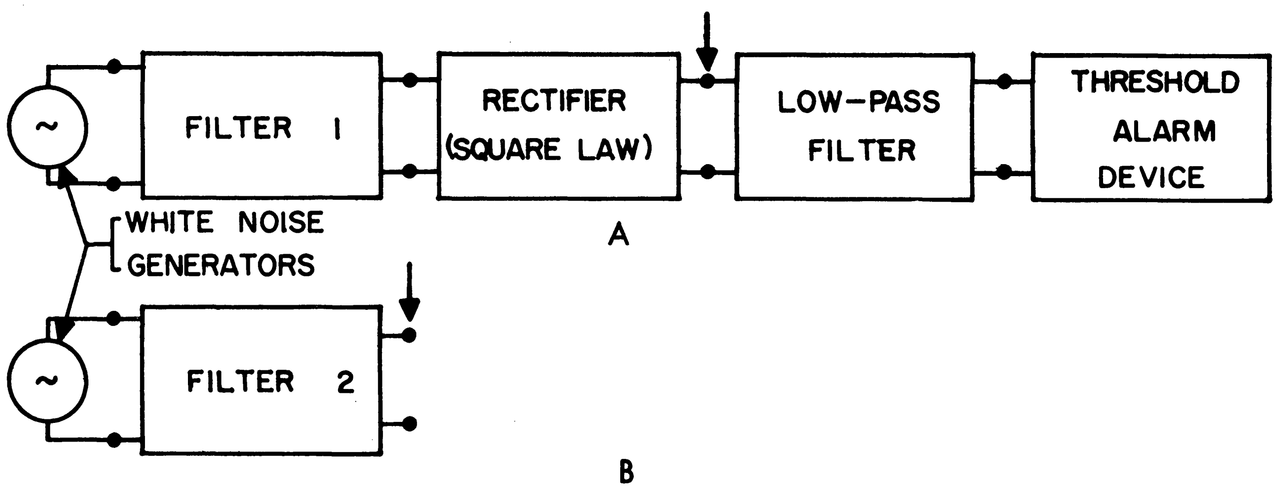


FIG. I. CASCADE SYSTEMS

System (A) corresponds to the first two cases of square rf band Δf_1 and of single-tuned rf band Δf_2 (3-db), while system (B) corresponds to the last two cases of low-pass square band f_1 and single R-C low-pass band $\frac{1}{2\pi RC}$ (3-db). Chart I depicts the power density spectra at the arrow marking points of Fig. 1, and lists the corresponding noise bandwidths Δf, together with the calculation of

$$n_i = 2\Delta f. \qquad (14)$$

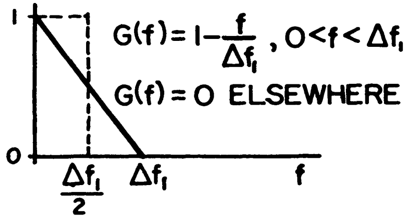
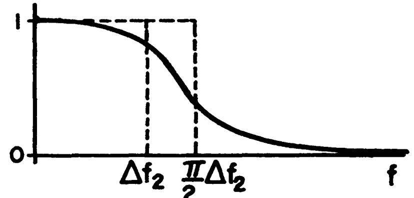
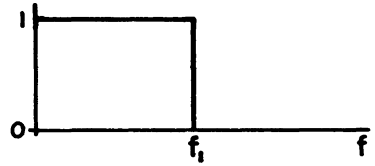
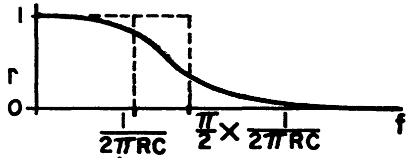
TYPE OF FILTER 1	POWER-DENSITY SPECTRUM (ONE-SIDED)	NOISE BANDWIDTH	EQUIVALENT NUMBER n_i OF INDEPENDENT SAMPLES PER UNIT TIME. $n_i = 2\Delta f$
SQUARE RF BAND Δf_1	 $G(f) = 1 - \frac{f}{\Delta f_1}$, $0 < f < \Delta f_1$; $G(f) = 0$ ELSEWHERE	$\frac{\Delta f_1}{2}$	$n_i = 2 \times \frac{\Delta f_1}{2} = \Delta f_1$
SINGLE - TUNED RF BAND Δf_2 (3db)	 $G(f) = \frac{1}{1+\left(\frac{f}{\Delta f_2}\right)^2}$, $0 < f$; $G(f) = 0, f < 0$	$\frac{\pi}{2}\Delta f_2$	$n_i = 2 \times \frac{\pi}{2}\Delta f_2 = \pi \Delta f_2$
TYPE OF FILTER 2	POWER - DENSITY SPECTRUM (ONE - SIDED)	NOISE BANDWIDTH	EQUIVALENT NUMBER n_i OF INDEPENDENT SAMPLES PER UNIT TIME. $n_i = 2\Delta f$
SQUARE BAND LOW PASS f_1	 $G(f) = 1$, $0 < f < f_1$; $G(f) = 0$, ELSEWHERE	f_1	$n_i = 2 \times f_1 = 2 f_1$
SINGLE R-C LOW-PASS BAND $\frac{1}{2\pi RC}$ (3db)	 $G(f) = \frac{1}{1+(2\pi f RC)^2}$, $0 < f$; $G(f) = 0$, $f < 0$	$\frac{\pi}{2} \times \frac{1}{2\pi RC}$	$n = 2 \times \frac{\pi}{2} \times \frac{1}{2\pi RC} = \frac{1}{2RC}$

CHART I. EXAMPLE CALCULATIONS OF EQUIVALENT NUMBER n_i OF INDEPENDENT SAMPLES PER UNIT TIME AT POINTS MARKED BY ARROW IN FIG. I.

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