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Final Report

CALCULATION OF THE REFLECTION OF RADIATION FROM SAW-TOOTH SURFACES AND THE REFLECTION AND TRANSMISSION OF RADIATION BY PARALLEL CYLINDER SYSTEMS

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I. INTRODUCTION

The purpose of this report is to present in final form the results of theoretical research conducted in the period of from 1 November 1956 to 30 November 1957 on Contract No. AF49(638)-26.

The report is divided into two main parts, comprising Sections II and III. The first part is concerned with the calculation of the reflection of radiation from saw-tooth surfaces. The calculations performed for this problem are based on a variational method which has been previously published. A review of this method is presented in Section II. The results of the calculations are given in Figs. 2-13. The computations are restricted to the class of reflection problems where the propagation vector of the incident plane wave is normal to the corrugations of the reflectiing surface and where the incident electromagnetic wave is polarized with the electric vector parallel to the corrugations. For an acoustic wave this is equivalent to a pressure release surface.

In Section III the problem considered is that of the reflection and transmission of radiation by an infinite system of parallel, circular, equally-spaced cylinders whose axes lie in a plane. Computations are restricted to the case where the propagation vector of the incident plane wave is perpendicular to the axes of the cylinders, and further, to the case where the wave is normally incident on the plane formed by those axes. The theory of W. von Ignatowsky is used in making calculations. The results reported here are all for the case where the radiation wavelength, λ, is greater than the spacing of the cylinders, D. Hence there are no propagating diffracted orders; in this case the sum of the directly transmitted and reflected intensities must be equal to the incident intensity.

The calculations were performed on MIDAC, The University of Michigan digital computer, and on the IBM-650 operated by the Statistical Research Laboratory of the University. The MIDAC was closed down for lack of funds early in 1957. Consequently some of the results obtained during the course of this work are necessarily incomplete. Nevertheless, all the results which were complete enough to be interesting have been reported here.

II. CALCULATION OF THE REFLECTION OF RADIATION FROM PERIODIC SAW-TOOTH SURFACES USING A VARIATIONAL METHOD

The problem of the reflection of radiation from nonplane surfaces has been treated by a large number of workers in the past.[1-5] A variational method has recently been proposed for the treatment of such problems.[6] The experiments which have been performed to check the results of the earlier calculations have shown encouraging agreement.[7,8] This method has been applied in this work to calculate the reflection properties of saw-tooth surfaces for a wider range of parameters. For convenience the theory will be reviewed here.

A. REVIEW OF THE THEORY

The method may be described as follows. Following Trefftz,[9] a linear combination of known solution to the wave equation is chosen to represent the reflected field. The coefficients will be chosen here so that they minimize the square of the error in the boundary condition. (Trefftz chose them so as to minimize the Rayleigh quotient.) This process of minimization is equivalent to orthogonalizing the set of functions formed by evaluating the trial functions on the boundary. Once this set is orthogonalized, one can easily construct the estimates of the reflection coefficients for the surface involved.

The class of problems to be considered will now be described. It is desired to find a solution ϕ of the two-dimensional, time-independent wave equation,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial x^2} + k^2\right]\phi(x,z) = 0 \qquad (1)$$

in a half-space bounded by a periodic surface $\zeta(x)$ (see Fig.1). Here ζ is assumed to depend only on x. In Eq. (1) $k = \omega/c$ when ω is the angular frequency of the radiation source and c is the phase velocity in the homogeneous medium bounded by $\zeta(x)$. The solution of the time-dependent wave equation is then given by $\phi e^{-i\omega t}$.

Using the method described herein, one may treat either the first or the second boundary value problem. The present calculations are restricted to the case,

$$\phi[x,\zeta(x)] = 0 \qquad (2)$$

It is supposed that the incident radiation consists of a plane wave making an angle Θ with the + z-direction; then one can write the total field as the sum of two components,

$$\phi = \phi_i + \phi_r \qquad (3)$$

where

$$\phi_i = \exp\,(ik[x \sin \Theta_i + z \cos \Theta_i]) \qquad (4)$$

with $\exp\,(x) = e^x$.

The boundary condition given by Eq. (2) is frequently encountered in the treatment of problems involving acoustic and electromagnetic radiation. For acoustic problems, the function ϕ may be taken to represent the (time-independent) velocity potential, with ϕ defined by

$$v = -\nabla\phi\ , \qquad (5)$$

where v is the particle velocity at an arbitrary field point (x,z). Then the first boundary value problem, represented by Eq. (2), corresponds to a physical problem in which $\zeta(x)$ is a pressure release surface. For problems involving electromagnetic radiation, on the other hand, $\zeta(x)$ is assumed to be a perfectly conducting surface. Then for an incident plane wave which has its propagation vector lying in the x-z-plane and which is polarized so that the electric vector is perpendicular to the x-z-plane, one chooses the boundary condition given by Eq. (2) where it is supposed that the electric field, which has but a single Cartesian component, is given by the function ϕ.

To make progress toward a solution of the foregoing class of problems, Rayleigh and others have chosen to represent the reflected field by an infinite set of plane-wave solutions of the wave equation. In addition to homogeneous, one must choose inhomogeneous waves. The waves must be chosen in such a way that they are either outgoing or exponentially damped as $z \to -\infty$. Furthermore, the fact that the boundary is periodic implies that one needs only a discrete set of such waves. Thus, one is led to expect that the reflected field ϕ can be represented by the following type of sum:

$$\sum_{\nu=-\infty}^{\infty} A_\nu \exp[-\,ik \sin \Theta_\nu\, x - ik \cos \Theta_\nu\, z]\ , \qquad (6)$$

where

$$\sin \Theta_\nu = \nu K/k - \sin \Theta_i;$$

$$\cos \Theta_\nu = [1 - \sin^2 \Theta_\nu]^{1/2}$$

where $K = 2\pi/D$ (see Fig. 1), and where the coefficients, A_ν, are to be determined through the use of the boundary condition. The angles Θ_ν of the various reflected orders are just those obtained from the ordinary grating equation.

Lippmann[13] has pointed out that this representation cannot be expected to be valid in the region $\zeta(x) < z < 0$, i. e., in the grooves. So long as the angle ψ of the surface is not too large, there is no difficulty on this point. The present calculations involve surfaces with ψ no greater than 15°. Some other results obtained indicate that for larger angles a refinement of the method would be necessary to obtain accurate results.

To proceed, upon using Eqs. (2), (3), (4), (6), and (7), one finds the following relation for the determination of the constants A:

$$\exp\,[ik \cos \Theta_i\, \zeta\,(x)] - \sum_{\nu=-\infty}^{\infty} A_\nu \exp\,[-\,i\nu Kx - ik \cos \Theta_\nu\, \zeta(x)] = 0\ . \tag{8}$$

To render the treatment of Eq. (8) more systematic, let

$$\begin{aligned} F_1(x) &= \exp\,(0) \\ F_2(x) &= \exp\,(1) \\ F_3(x) &= \exp\,(-1) \\ F_4(x) &= \exp\,(2) \\ &\ \ \vdots \end{aligned}$$

and

$$\begin{aligned} \bar{A}_1 &= A_0 \\ \bar{A}_2 &= A_1 \\ \bar{A}_3 &= A_{-1} \\ &\ \ \vdots\ , \end{aligned} \tag{9}$$

where, in Eq. (9),

$$\exp(\nu) = \exp\,[-\,i\nu Kx - ik \cos \Theta_\nu\, \zeta(x)]\ ;$$

further let

$$\bar{\phi}_i = \exp\,[ik \cos \Theta_i\, \zeta\,(x)]. \tag{10}$$

Then Eq. (8) becomes,

$$\bar{\phi}_i(x) - \sum_{k=1}^{\infty} \bar{A}_k F_k(x) = 0. \tag{11}$$

If the series in Eq. (11) is broken off after the Nth term, as must be done in many problems, the left side of the equation is not in general equal to zero. It is proposed that in such a case the constants A_k be chosen in a way so that the integral over the surface ζ of the absolute square of the left side of Eq. (11) is minimized. Since all quantities in that equation are periodic with period D, it is sufficient to carry the integral from x = 0 to x = D. It is easily seen that this minimization is equivalent to carrying out the corresponding minimization of the error in the boundary condition. It is not difficult to show that if one chooses the coefficients $\bar{A}_k$ so that they satisfy the set of equations (with $\ell = 1,2,\ldots,N$),

$$\sum_{k=1}^{N} \bar{A}_k (F_\ell, F_k) = (F_\ell, \bar{\phi}_i) , \tag{12}$$

when the inner product of two functions, (g,h), is defined by

$$(g,h) = \frac{1}{D} \int_0^D g^* h dx, \tag{13}$$

then the indicated minimization is accomplished.

The problem which remains, once the integrals of the type (g,h) have been calculated, is the inversion of the linear system given in Eq. (12). This is actually accomplished on the computer by the use of an orthogonalizing procedure. A new orthonormal set of functions was constructed from the nonorthogonal set F_k.

The coefficients A_ν obtained in this are not plotted directly. By considering the energy balance within the region of the x-z-plane bounded by $\zeta(x)$, x = 0, x = D, and z = - C, where C is large and positive, one obtains the following relation for the exact solution:

$$\cos \theta_i = \sum_{\nu} \cos \theta_\nu |A_\nu|^2 , \tag{14}$$

where the summation is carried over those values of ν for which $\cos \theta_\nu$ is real. Guided by this relation, the ratio of the reflected energy in a given order, $\cos \theta_\nu |A_\nu|^2$, to the incident energy, $\cos \theta_i$, is the quantity presented in the graphical results shown in Figs. 2-13.

B. DISCUSSION OF THE COMPUTATIONAL RESULTS

Calculations were made for surfaces with $\psi = 5°$, $10°$, and $15°$, and for radiation wavelengths ranging variously from $\frac{D}{\lambda} = 1.25$ to $\frac{D}{\lambda} = 11.00$. Most of the computations are for the surface $\psi = 10°$.

Referring to Figs. 4-11, the effect of increasing the frequency of the incident radiation is seen. Beginning with low frequencies, Fig. 4, one sees that most of the reflected energy is in the zeroth-order component. As the frequency is increased, the energy in the zeroth order is reduced in favor of the plus first order. As the frequency is still further increased, $D = 5.00\lambda$, the plus second order begins to become important. Such strong positive orders are to be expected from the nature of the reflecting surface. Referring to Fig. 1, it is seen that for small values of ψ, much of the surface looks like a mirror slanted to throw the reflected energy to the left; that is, in the direction of the positive orders. It should be emphasized that even the highest frequencies used here are not sufficiently high to allow the use of geometrical, or even physical, optics. Hence the exact strength of the individual orders is still very much a wave property of the reflection process. This is true despite the fact, as pointed out above, that the qualitative behavior is as expected from geometrical optics.

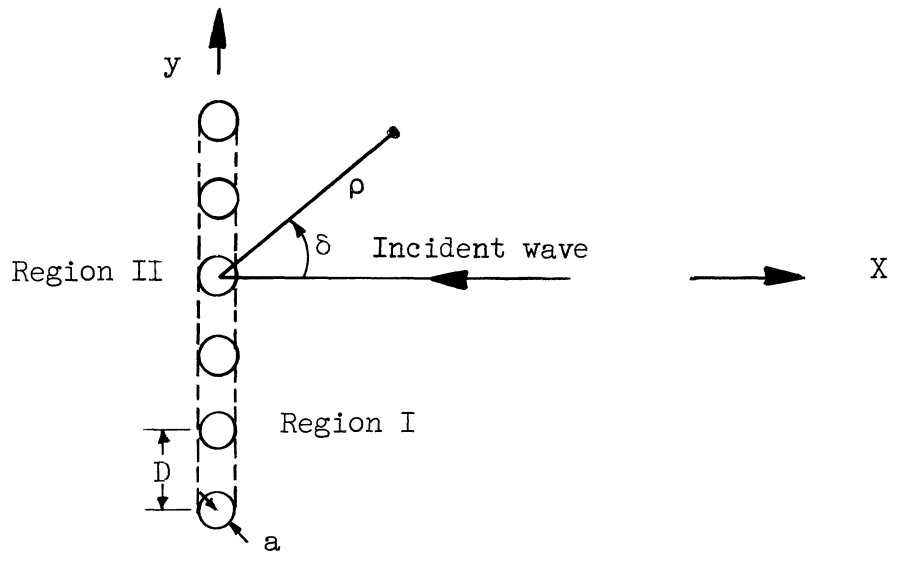
The cusps which occur in many of the curves are known experimentally as Wood anomalies.[14] As the incident angle is increased, the positive orders will successively disappear, whereas the negative orders will appear. For example, this occurs in Fig. 5 for the plus and minus first orders at $-15°$ and $+15°$, respectively. Since this change occurs rapidly, actually in principle with infinite slope, then if energy is to be conserved, the other orders must show discontinuous rates of change. These sudden changes in intensity constitute the Wood anomalies.

Although the results are sketchy for surfaces other than $\psi = 10°$, there is at least one interesting point which can be made concerning the relative reflection properties of surfaces with different angles. Upon comparing Fig. 7, $\psi = 10°$ and $D = 2.00\lambda$, with Fig. 13, $\psi = 15°$ and $D = 2.00\lambda$, it is seen that increasing the surface angle has the effect of causing the shift of energy from the zeroth to other orders to occur for angles nearer to grazing. This is the sort of result which is predicted by Rayleigh's perturbation treatment.[2]

III. REFLECTION AND TRANSMISSION OF RADIATION BY PARALLEL CYLINDER SYSTEMS

In this section we shall review the theory of Ignatowsky[15-20] and discuss the results of calculations based upon that theory.

The problem will now be defined. Consider an infinite system of parallel equally spaced, perfectly conducting cylinders of radius a separated by the distance D. A plane electromagnetic wave is incident upon this system with propagation vector perpendicular to the plane formed by the axes of the cylinders. The case where the incident wave is polarized with electric vector parallel to the axes will be called Case I; electric vector perpendicular to the axes will be Case II.* The reader is referred to the figure for definitions of the terms.



A. REVIEW OF THE THEORY OF IGNATOWSKY

Ignatowsky has reported his method very completely in a series of papers.[15-20] It may help the reader to know that of these papers probably the most important is Ref. 19. Despite this very complete report of Ignatowsky's work, it will prove convenient to discuss briefly the major ideas of his method. The development is carried out for Case I; only the result will be given for Case II since the development is entirely analogous.

We let Z represent the z-component of the electric field (Case I). This component satisfies the (time-independent) wave equation,

$$(\nabla^2 + g^2)\, Z = 0, \tag{15}$$

where $g = 2\pi/\lambda$ with λ equal to the incident wavelength. Since the cylinders are perfect conductors, Z vanishes on their surface. The incident field is given by

$$\begin{aligned} Z_{inc.} &= Ae^{ig\rho \cos \delta} \\ &= A \left\{ J_o(g\rho) + 2 \sum_{s=1}^{\infty} i^s\, J_s(g\rho) \cos s\,\delta \right\} \end{aligned} \tag{16}$$

*Ignatowsky considered more general problems than the class described here.

The diffracted field can be written (the general solution of the two-dimensional wave equation):

$$Z_{diff.} = A \sum_{s=0}^{\infty} \left[D_s Q_s (g\rho) \cos s\delta + G_s J_s(g\rho) \cos s \delta \right] \quad (17)$$

with D_s, G_s unknown constants, and

$$Q_s(x) = \frac{i\pi}{2} H_s^{(2)} (x) \quad , \quad (18)$$

where $H_s^{(2)}$ is the Hankel function of the second kind of order s. The J_s in Eq. (17) are Bessel functions. From the boundary condition it follows that the sum of the incident and diffracted fields must vanish when $\rho = a$; then from Eqs. (16) and (17) one finds a relation between G_s and D_s,

$$G_0 = \left[\frac{D_0}{L_0} - 1 \right] \; ; \; \underset{s>0}{G_s} = 2i^s \left[\frac{D_s}{L_s} - 1 \right] , \quad (19)$$

where

$$L_0 = - \frac{J_0(ga)}{Q_0(ga)} \quad ; \quad \underset{s>0}{L_s} = - 2(i)^s \left(\frac{J_s(ga)}{Q_s(ga)} \right) \quad (20)$$

Thus only the D_s remain unknown. These quantities can be determined through the use of a general relation which exists between the D_s and G_s of Eq. (17). To see this relation, we begin by supposing that the cylinders are numbered running from $-\infty$ to $+\infty$. Then let Z_s represent the field diffracted from the s^{th} cylinder. This field can be defined precisely as follows. The diffracted field is defined in terms of an integral over the reflecting surface of the boundary value of the field times a Green's function, as can always be done.[19] Then that part of the diffracted field which results from carrying the integral over the s^{th} cylinder is defined as Z_s. Now for points outside the s^{th} cylinder we can write

$$Z_s = \sum_{\tau=0}^{\infty} E_\tau Q_\tau(g\rho_s) \cos \tau \delta_s \quad , \quad (21)$$

where ρ_s and δ_s are the polar coordinates about the axis of the s^{th} cylinder. For normal incidence, the E_τ are the same for every cylinder. (It is a simple matter to modify the phase for non-normal incidence.) Further, the total diffracted field is given by

$$Z_{diff.} = \sum_{s=-\infty}^{\infty} Z_s \; . \quad (22)$$

We wish now to write each Z_s in terms of polar coordinates measured from the 0^{th} cylinder, these coordinates to be designated by ρ and δ. This can be done through the use of the addition formula for Hankel functions.[20] The result of this transformation for Z_1 is

$$Z_1 = H_0 J_0(g\rho) + \sum_{s=1}^{\infty} \left\{ H_s\left(\cos\frac{s\pi}{2}\cos s\delta - \sin\frac{s\pi}{2}\sin s\delta\right) + K_s\left(\sin\frac{s\pi}{2}\cos s\delta + \cos\frac{s\pi}{2}\sin s\delta\right)\right\} J_s(g\rho), \quad (23)$$

where

$$H_0 = \sum_{\tau=0}^{\infty} E_\tau \cos\frac{\tau\pi}{2} Q_\tau(gD)$$

$$\begin{pmatrix} H_s \\ K_s \end{pmatrix} = (-1)^s \sum_{\tau=0}^{\infty} E_\tau (-1)^\tau \begin{pmatrix} \cos\frac{\tau\pi}{2} \\ \sin\frac{\tau\pi}{2} \end{pmatrix} \times \left\{ Q_{s-\tau}(gD) + (\pm)(-1)^\tau Q_{s+\tau}(gD) \right\} \quad (24)$$

where the upper quantities in the brackets( ), belong together, as do the lower quantities. Proceeding in this way, one finds for Eq. (22),

$$Z_{diff.} = \sum_{s=0}^{\infty} [E_s Q_s(g\rho) + M_s J_s(g\rho)] \cos s\delta \quad , \quad (25)$$

where

$$M_0 = 2 \sum_{\tau=0}^{\infty} (-1)^\tau E_{2\tau} S_{2\tau}$$

$$M_{2s} = 2(-1)^s \sum_{\tau=0}^{\infty} (-1)^\tau E_{2\tau}[S_{2(s-\tau)} + S_{2(s+\tau)}] \quad (26)$$

$$M_{2s-1} = 2(-1)^s \sum_{\tau=1}^{\infty} (-1)^\tau E_{2\tau-1}[S_{2(s-\tau)} + S_{2(s+\tau-1)}] \quad ,$$

with

$$S_{2s}\left(\frac{\lambda}{D}\right) = \sum_{\tau=1}^{\infty} Q_{2s}(gD\tau). \quad (27)$$

This shows the predicted relation between the coefficients in Eq. (17). Upon identifying Eqs. (17) and (25) and using the boundary-condition relations (19) we find for D_n

$$\left[\frac{D_0}{L_0} - 1\right] = 2 \sum_{\tau=0}^{\infty} (-1)^\tau D_{2\tau} S_{2\tau}$$

$$(i)^{2s}\left[\frac{D_{2s}}{L_{2s}} - 1\right] = (-1)^s \sum_{\tau=0}^{\infty} (-1)^\tau D_{2\tau}[S_{2(s-\tau)} + S_{2(s+\tau)}]$$

$$(i)^{2s-1}\left[\frac{D_{2s-1}}{L_{2s-1}} - 1\right] = (-1)^s \sum_{\tau=1}^{\infty} (-1)^{\tau} D_{2\tau-1}\left[S_{2(s-\tau)} + S_{2(s+\tau)}\right] \tag{28}$$

with L_s and S_s defined in Eqs. (20) and (27).

It is now convenient to consider the plane-wave representation of the field. In the region I, one can represent the reflected field for normal incidence by the expansion (aside from the time factor)

$$Z_{diff.} = 2\sum_{n=0}^{\infty} N_n e^{-ig[1-n^2(\lambda^2/D^2)]^{1/2}} x \cos\frac{2\pi ny}{D} \tag{29}$$

and in region II by

$$Z_{diff.} = 2\sum_{n=0}^{\infty} \overline{N}_n e^{ig[1-n^2(\lambda^2/D^2)]^{1/2}} x \cos\frac{2\pi ny}{D} \tag{30}$$

where N_n and $\overline{N}_n$ are constants to be determined. It should be emphasized that this representation is not correct for the region between the cylinders. In that region more complicated functions are required to represent the field.[6] The representation given by Eqs. (29) and (30) is now compared with the representation given by Eq. (17). The plane-wave representation can be considered the Fourier transform, in y, of the cylindrical-wave representation of Eq. (17). In this way one can obtain the relationship between the coefficients of the plane wave and the coefficients D_n now determined by Eq. (28). One finds for region I (the reflected wave),

$$N_o = \frac{\pi A}{2gD}\sum_{s=0}^{\infty} i^{s+1} D_s \tag{31}$$

$$n > 0 \qquad N_n = \frac{\pi A}{gD\cos\theta_n}\sum_{s=0}^{\infty} i^{s+1} D_s \cos s\,\theta_n, \tag{32}$$

and for the transmitted wave in region II,

$$\overline{N}_o = \frac{\pi A}{2gD}\sum_{s=0}^{\infty} i^{s+1} (-1)^s D_s \tag{33}$$

$$n > 0 \qquad \overline{N}_n = \frac{\pi A}{gD\cos\theta_n}\sum_{s=0}^{\infty} i^{s+1} (-1)^s D_s \cos s\,\theta_n, \tag{34}$$

where θ_n is the angle made by the diffracted wave with the normal to the plane of the grating

$$\cos\theta_n = \left(1 - n^2\frac{\lambda^2}{D^2}\right)^{1/2} \tag{35}$$

When $n^2 \frac{\lambda^2}{D^2} > 1$, the diffracted waves are inhomogeneous and do not carry energy away from the grating.

Finally, then, the expressions (28)-(34) determine the diffracted field. This field plus the incident field,

$$Z_{inc.} = Ae^{+igx} \tag{36}$$

gives the total field,

$$Z_{total} = Z_{diff.} + Z_{inc.} \tag{37}$$

To obtain the solution to Case II, where the electric field is perpendicular to the axes of the cylinder, the above results may be used unchanged with one exception. The equations corresponding to Eqs. (28) for Case II are obtained by replacing L_n of those equations by L'_n where

$$L'_o = -\frac{J'_o(ga)}{Q'_o(ga)} \,, \quad \underset{s>0}{L'_s} = -2i^s \frac{J'_s(ga)}{Q'_s(ga)} \,. \tag{38}$$

That is, to determine the D'_s for perpendicular polarization, just replace the Bessel functions by their derivatives.

The functions S_{2s} (p) may be obtained from the functions T_{2s} (p) of Table I. The functions S are related to the functions T by

$$S_{2s}(p) = (2s)! \; T_{2s}(p) + i \begin{cases} \frac{p-\pi}{4} & \text{if } s = 0 \\ \frac{p}{4} & \text{if } s \neq 0 \end{cases} \tag{39}$$

The table gives T_{2s} (p) to eight places for values of 2s, 0(2)30, and values of p, 1.25 (0.25) 6.75.*

Using the values of the functions S_{2s} tabulated in Table I, one can break off the system represented by Eqs. (28) and obtain the quantities D to an accuracy dependent on where the system is terminated. This is done by solving the linear system so formed. The values of D obtained in this way are then substituted in Eqs. (31) and (32) to obtain the coefficients of the diffracted plane waves. For instance, the coefficient of the directly transmitted plane wave is given by

$$Z_{To} = A \left[1 + \frac{\lambda}{2D} \sum_{s=0}^{\infty} i^{s+1} (-1) D_s \right], \tag{40}$$

*A table given in Ref. 2 can be used to find S_o.

and the coefficient of the reflected wave by

$$Z_{RO} = A \frac{\lambda}{2D} \sum_{s=0}^{\infty} i^{s+1} D_s . \qquad (41)$$

For $\lambda > D$, as is the case for the calculations reported here, the directly transmitted and reflected waves are the only propagating waves which occur. As pointed out above, to obtain the corresponding result for perpendicular polarization (Case II), replace L_n of Eq. (28) by L_n' defined in Eqs. (38).

B. DISCUSSION OF THE RESULTS OF THE CALCULATIONS

The calculations were made using coefficients D_n through order n = 5. The results obtained were checked by using the relation

$$|Z_{To}|^2 + |Z_{Ro}|^2 = |A|^2 \quad , \qquad (42)$$

which is the requirement for conservation of energy when $\lambda > D$. In most cases the check was to four or five places, the limit of accuracy of the calculation, indicating that the results can be considered exact. For D/a = 2.5 when $\lambda/D = 1.25$, the energy checked to only two percent, indicating that in the region $\lambda \approx D$ terms of order higher than D_5 are needed when cylinders of larger radius are considered.

There is a further computational check which can be applied. Suppose the following diffraction problem is considered: an electromagnetic (or acoustic) wave is incident on a grating composed of equally spaced, parallel, conducting half-cylinders which are placed on a conducting plane. The amplitudes of the waves diffracted from this system can be calculated from the solution of the full-cylinder problem by taking the difference of the solutions to the full cylinder problem for an incident wave coming from the right and from the left. Then using the requirement of conservation of energy, one finds

$$|Z_{Ro} - Z_{To}|^2 = |A|^2 \quad . \qquad (43)$$

This check when applied gave agreement to within one or two percent.

In Figs. 14-19 the computations are compared with the experimental results of Pursley.[22] Pursley used a klystron source mounted in a paraboloidal reflector for the incident beam. The gratings were constructed of aluminum and brass rods. A crystal detector mounted at the focus of a paraboloidal reflector was used as a receiver. For D/a = 4.0, Pursley checked his results using a small-scale grating and infrared radiation as a source. The results here checked very well with the microwave results.

The agreement between experiment and theory is in general quite good. It is seen in Figs. 14, 15, and 17 that when the incident wave is perpendicularly polarized, Pursley found that there is a region of wavelengths about $\lambda = 1.21$ which gives rise to good transmission.* In our calculations we find that this peak should appear at slightly larger wavelengths than found by Pursley; Figs. 14 and 17 show this, although, of course, more computation is needed in this region to make the case clearer.

In conclusion, it is seen that Ignatowsky's theory gives an exact solution to the parallel cylinder diffraction problem. The computations which must be made to obtain transmission and reflection coefficients are somewhat involved. To obtain efficiently results from the theory, a digital computer is needed. Further computations using Ignatowsky's theory would be very valuable, insofar as the theory constitutes one more complete, exact solution to a diffraction problem.

*Though a detailed explanation of this phenomenon is not available, it is evidently a resonance effect.

TABLE I

TABLE OF THE FUNCTION T_{2s} (p) DEFINED IN EQ. (39)

The order 2s is shown explicitly only for p=1.25

2s	p=1.25	p=2.00	p=2.75
2s=0	.19935161	.31410624	.52021634
2	.17361713	-.01663377	-.17698117
4	-.01116211	-.04474956	-.12644957
6	-.00119387	-.00987102	-.05122495
8	-.00010700	-.00246138	-.02625242
10	-.00001048	-.00072466	-.01531310
12	-.00000118	-.00023179	-.00950548
14	-.00000014	-.00007767	-.00612661
16	-.00000001	-.00002683	-.00405216
18	.00000000	-.00000947	-.00273155
20	.00000000	-.00000340	-.00186845
22	.00000000	-.00000123	-.00129300
24	.00000000	-.00000045	-.00090327
26	.00000000	-.00000016	-.00063597
28	.00000000	-.00000005	-.00045075
30	.00000000	-.00000002	-.00032128

p=1.50	p=2.25	p=3.00
.06593563	.39501555	.57124035
.08685571	-.06674401	-.23849820
-.01795879	-.06540139	-.16942916
-.00259979	-.01781867	-.08227536
-.00033591	-.00582195	-.05102250
-.00005123	-.00221625	-.03569928
-.00000879	-.00090790	-.02649147
-.00000160	-.00038804	-.02038171
-.00000030	-.00017063	-.01607807
-.00000005	-.00007660	-.01291980
-.00000000	-.00003493	-.01053111
.00000000	-.00001612	-.00868212
.00000000	-.00000751	-.00722438
.00000000	-.00000353	-.00605786
.00000000	-.00000167	-.00511285
.00000000	-.00000079	-.00433933

p=1.75	p=2.50	p=3.25
.21122143	.46230017	.61695189
.03255322	-.11988564	-.30469767
-.02920771	-.09222733	-.22266051
-.00521002	-.03080385	-.12810617
-.00095429	-.01277118	-.09446313
-.00020829	-.00609147	-.07803668
-.00005010	-.00310629	-.06819923
-.00001270	-.00164803	-.06172396
-.00000332	-.00089824	-.05724109
-.00000089	-.00049931	-.05405229
-.00000024	-.00028178	-.05176038
-.00000006	-.00016092	-.05012212
-.00000001	-.00009280	-.04898039
-.00000000	-.00005394	-.04822941
.00000000	-.00003157	-.04779574
.00000000	-.00001858	-.04762713

TABLE I
(Continued)

p=3.50	.65842527	p=4.25	.76416129	p=5.00	.85034002
	- .37573051		.61867815		- .90724024
	- .28776991		- .57258867		-1.03538708
	- .19400449		- .58619318		-1.50211034
	- .16764114		- .76249900		-2.73649106
	- .16137192		-1.09232674		-5.45468659
	- .16400900		-1.64591972		- .74277466
	- .17247016		-2.56156362		- .77131012
	- .18574636		-4.07871355		-1.67683876
	- .20362895		-6.60656030		-3.76335601
	- .22632967		-4.44478528		-2.15839721
	- .25434608		-1.25026027		-4.65417290
	- .28841646		-1.83235956		-5.44936961
	- .32951447		-3.08464181		-2.66382473
	- .37886757		- .66203853		-1.31728504
	- .43799115		- .97122370		-2.94698594
p=3.75	.69642815	p=4.50	.79466813	p=5.25	.87591054
	- .45170078		- .70976083		-1.01365933
	- .36651432		- .70408477		-1.24014109
	- .28658718		- .81505996		-1.99654119
	- .28666437		-1.19411911		-4.02133574
	- .31796061		-1.92179976		-8.84862566
	- .37178682		-3.25048435		-9.75656952
	- .44948195		-5.67618338		-4.11357444
	- .55630670		-2.13869405		- .75737015
	- .70067603		-3.41136450		-1.36630201
	- .89459629		-6.26853610		-3.41413659
	-1.15468771		-5.84532020		-2.83383483
	-1.50374317		-5.38421096		-1.65638088
	-1.97293185		-5.03800757		-4.04565436
	-2.60486705		- .25496031		-1.33328985
	-3.45784449		- .16849612		-3.15592771
p=4.00	.73152844	p=4.75	.82331993	p=5.50	.90018455
	- .53266630		- .80594236		-1.12521552
	- .46078133		- .85754787		-1.47447911
	- .41401018		-1.11479411		-2.62073936
	- .47438413		-1.82685401		-5.80728913
	- .60048728		-3.28158497		-1.22710376
	- .80031270		-6.19068761		-1.92096410
	-1.10220515		-2.91020793		-4.98075761
	-1.55347872		-5.17920549		-5.31130791
	-2.22770566		-3.36006013		-6.84767360
	-3.23783850		-6.51763372		-5.66257463
	-4.75703216		-1.87057130		-3.49447476
	-1.71775523		-3.40470693		-3.95053801
	-2.35093417		-2.24089684		-1.09407864
	-3.52868142		-4.56776632		-2.74113680
	-1.06399381		-1.19877207		-3.56206846

TABLE I
(Concluded)

p=5.75	.92329175
	-1.24191401
	-1.74120058
	-3.40079542
	-8.25374174
	-9.01776053
	-2.47260764
	-5.36280754
	-7.61820958
	-.82301047
	-1.33545785
	-4.03521284
	-1.70888806
	-4.85070423
	-1.35359220
	-3.75475517
p=6.00	.94534167
	1.36376284
	-2.04323514
	-4.36657112
	-11.56036743
	-7.69518410
	-10.19266248
	-3.39271577
	-.92312498
	-2.46114678
	-1.65315877
	-5.16786845
	-1.25330570
	-3.57359888
	-2.93973803
	-1.15150129
p=6.25	.96643033
	-1.49076249
	-2.38364263
	-5.55204815
	-15.97490449
	-11.56024942
	-2.32506082
	-5.23689285
	-2.00849925
	-6.10239876
	-2.47953358
	-2.17296951
	-2.26900264
	-3.14008053
	-2.20205566
	-3.54334595
p=6.50	.98664038
	-1.62292237
	-2.76561285
	-6.99569242
	-5.73681505
	-4.76252056
	-4.96110739
	-8.12181216
	-5.70518152
	-6.00709408
	-3.12162774
	-5.37966176
	-4.81657255
	-3.60086150
	-4.60076190
	-.97910196
p=6.75	1.00604371
	-1.76023659
	-3.19246630
	-8.74083380
	-13.34812292
	-8.49306994
	-7.52074353
	-9.58555321
	-5.34650586
	-5.74830897
	-3.85036430
	-3.56878197
	-3.89472303
	-1.37691674
	-.77090093
	-3.15250448

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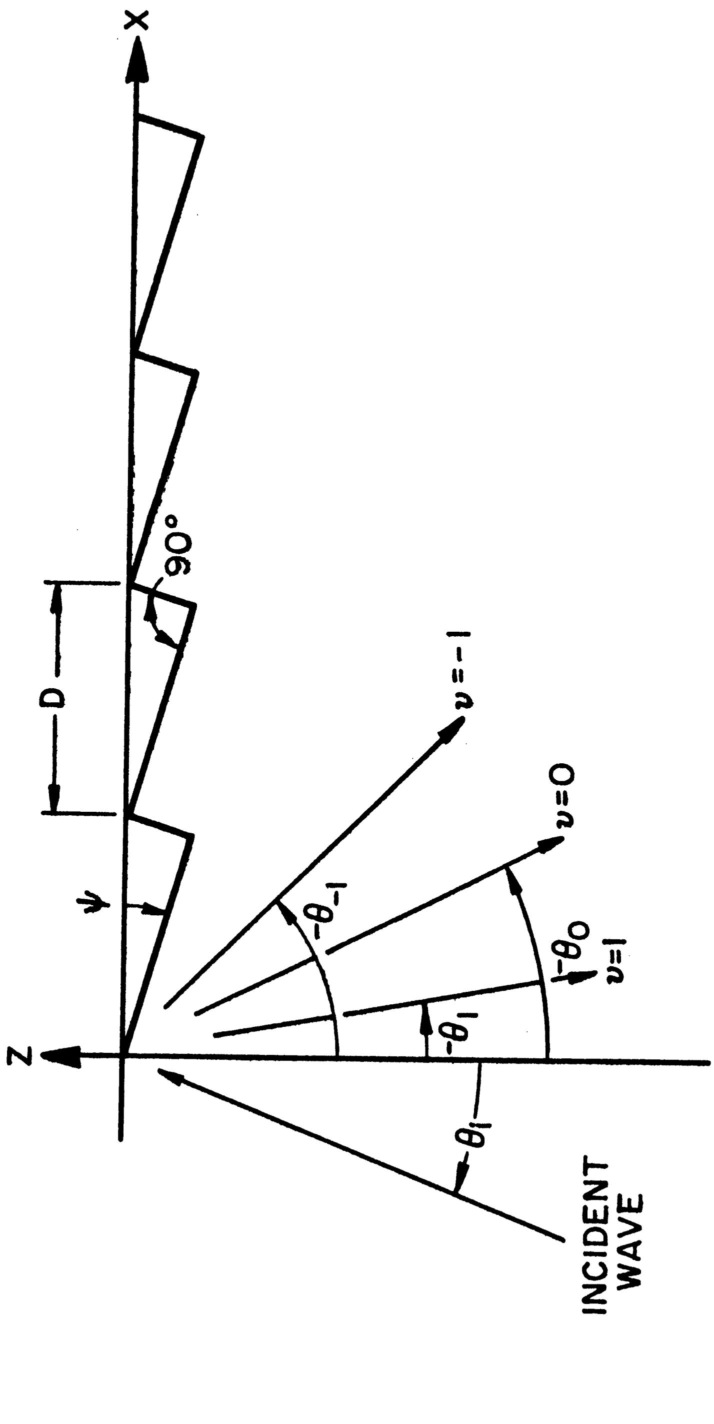

Fig. 1. The definition of notation used in the calculations whose results are given in Figs. 2-13. The θ_γ represents the angle made by the diffracted order with the normal, where γ represents the order.

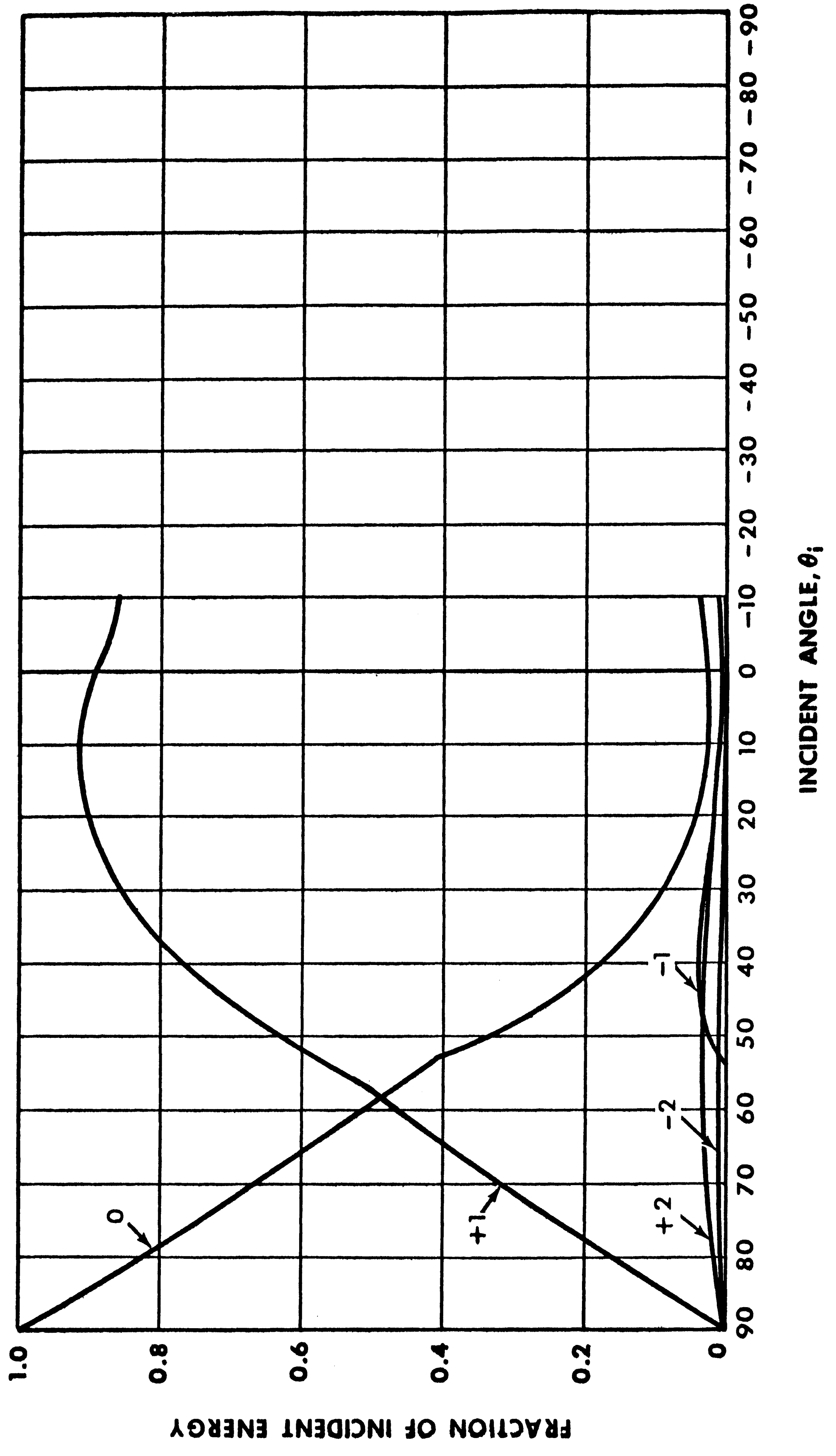

Fig. 2. Reflection with ψ = 5°, D = 5.00λ.

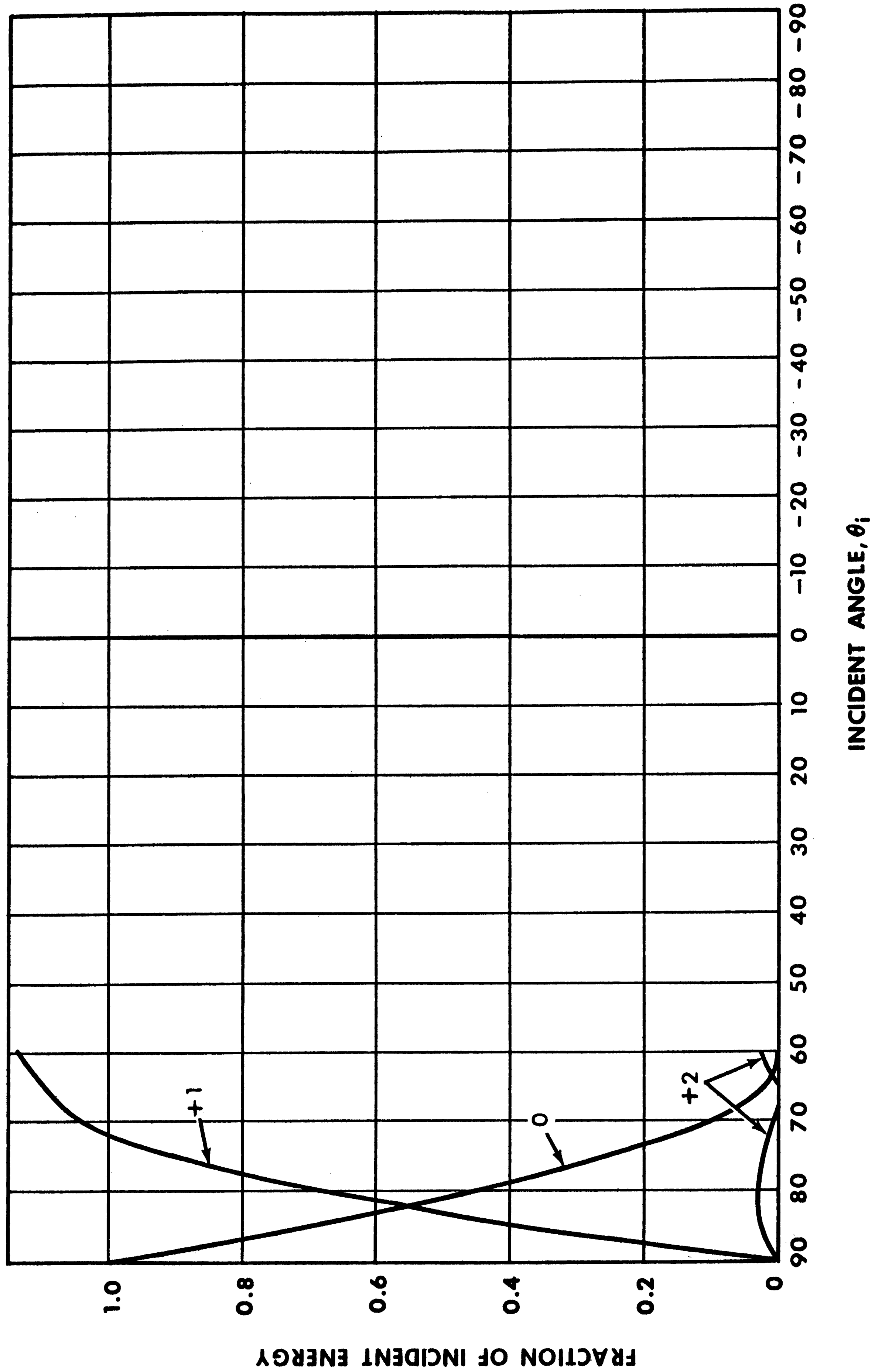

Fig. 3. Reflection with $\psi = 5^\circ$, D = 11.00λ.

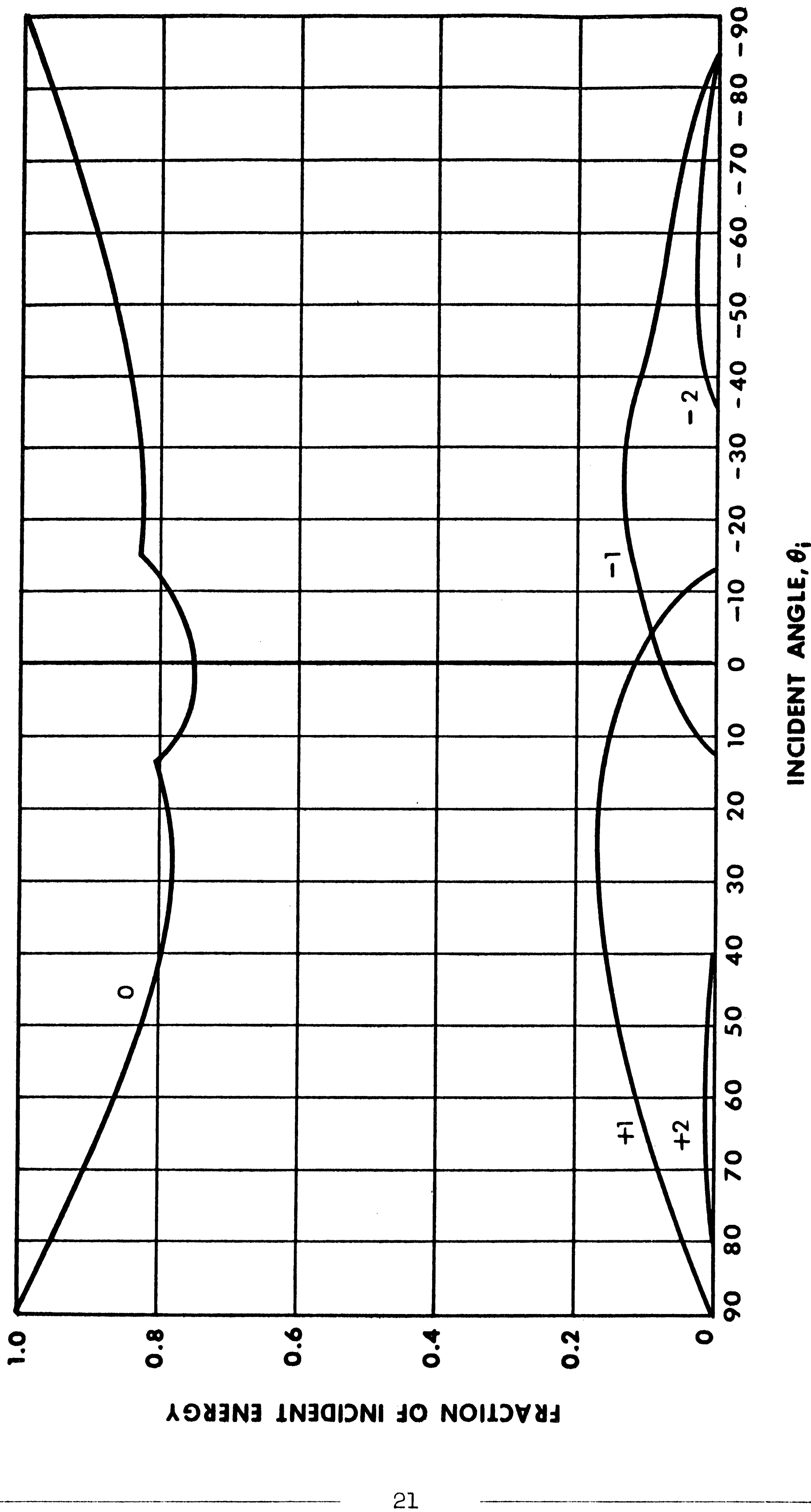

Fig. 4. Reflection with ψ = 10°, D = 1.25λ.

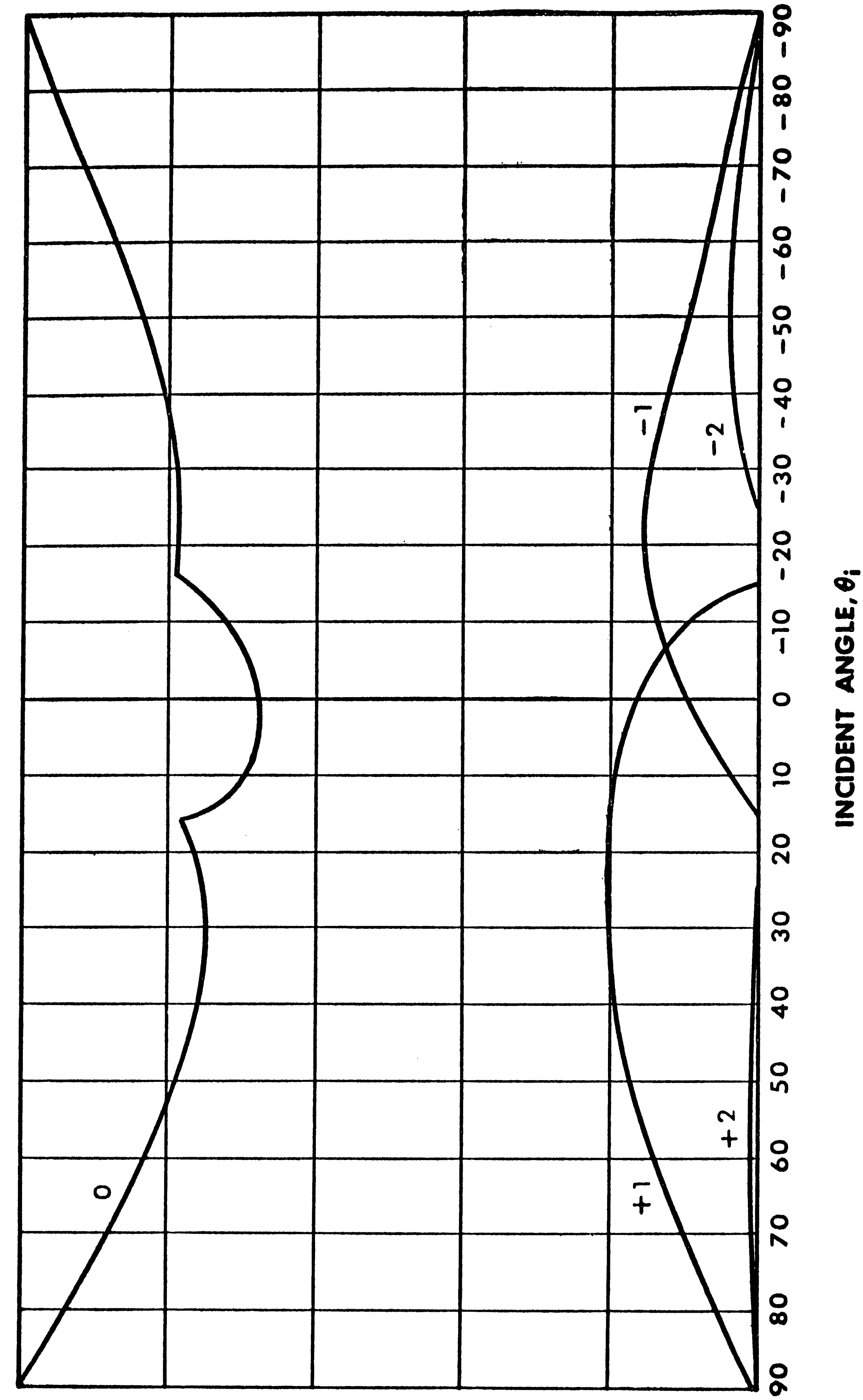

Fig. 5. Reflection with $\psi = 10°$, $D = 1.35\lambda$.

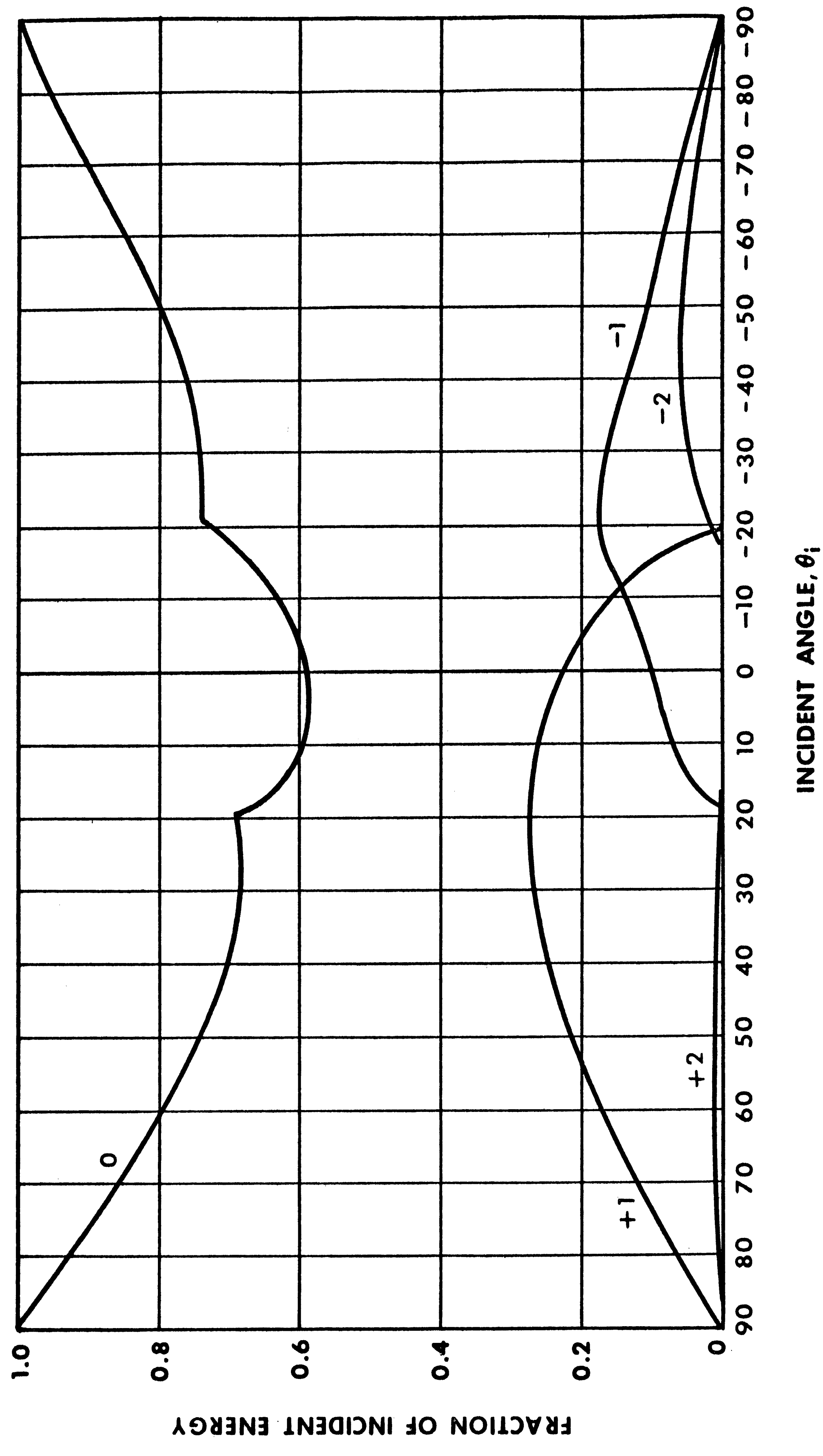

Fig. 6. Reflection with ψ = 10°, D = 1.50λ.

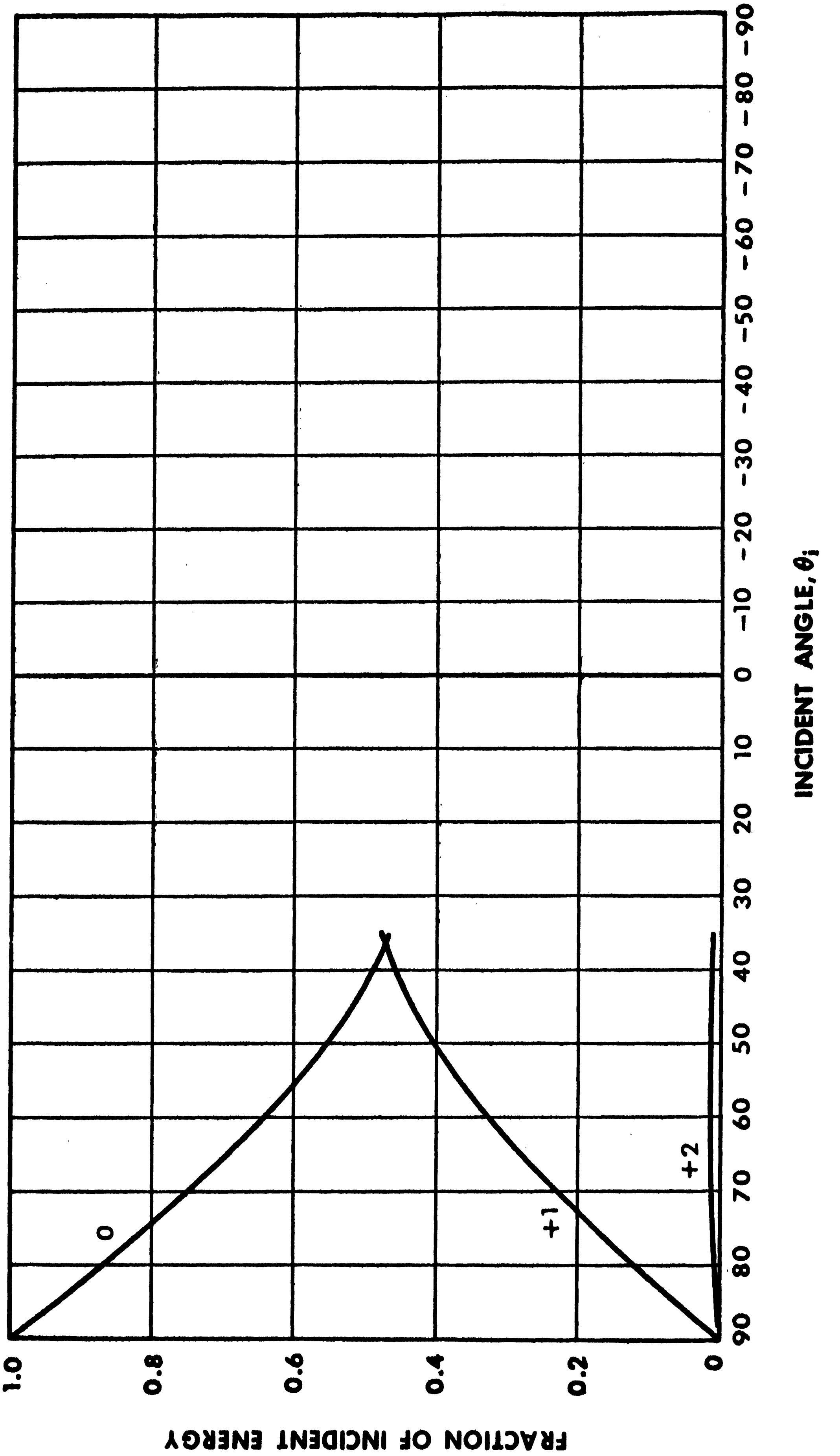

Fig. 7. Reflection with $\psi = 10°$, D = 2.00λ.

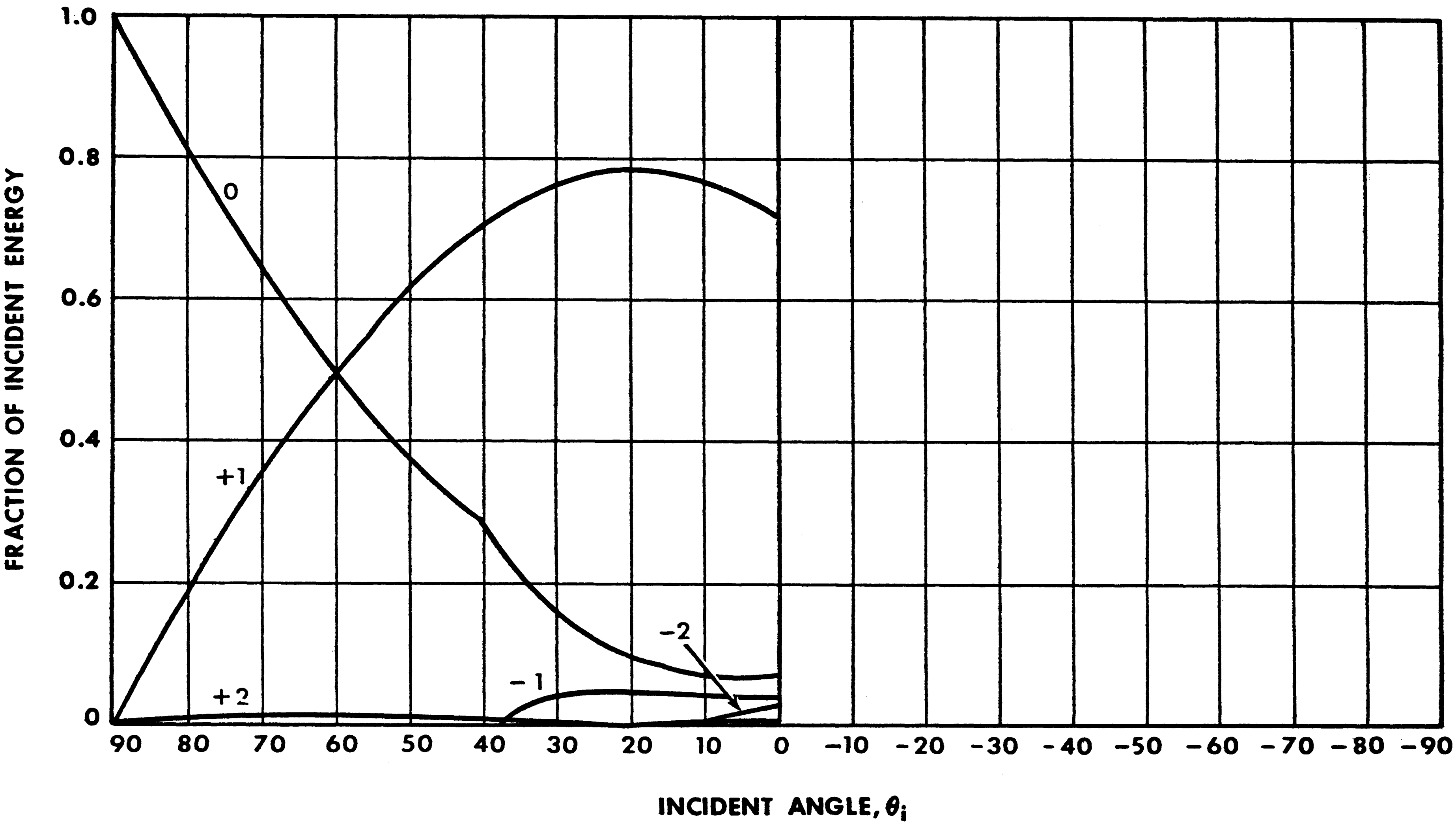

Fig. 8. Reflection with $\psi = 10°$, $D = 2.50\lambda$.

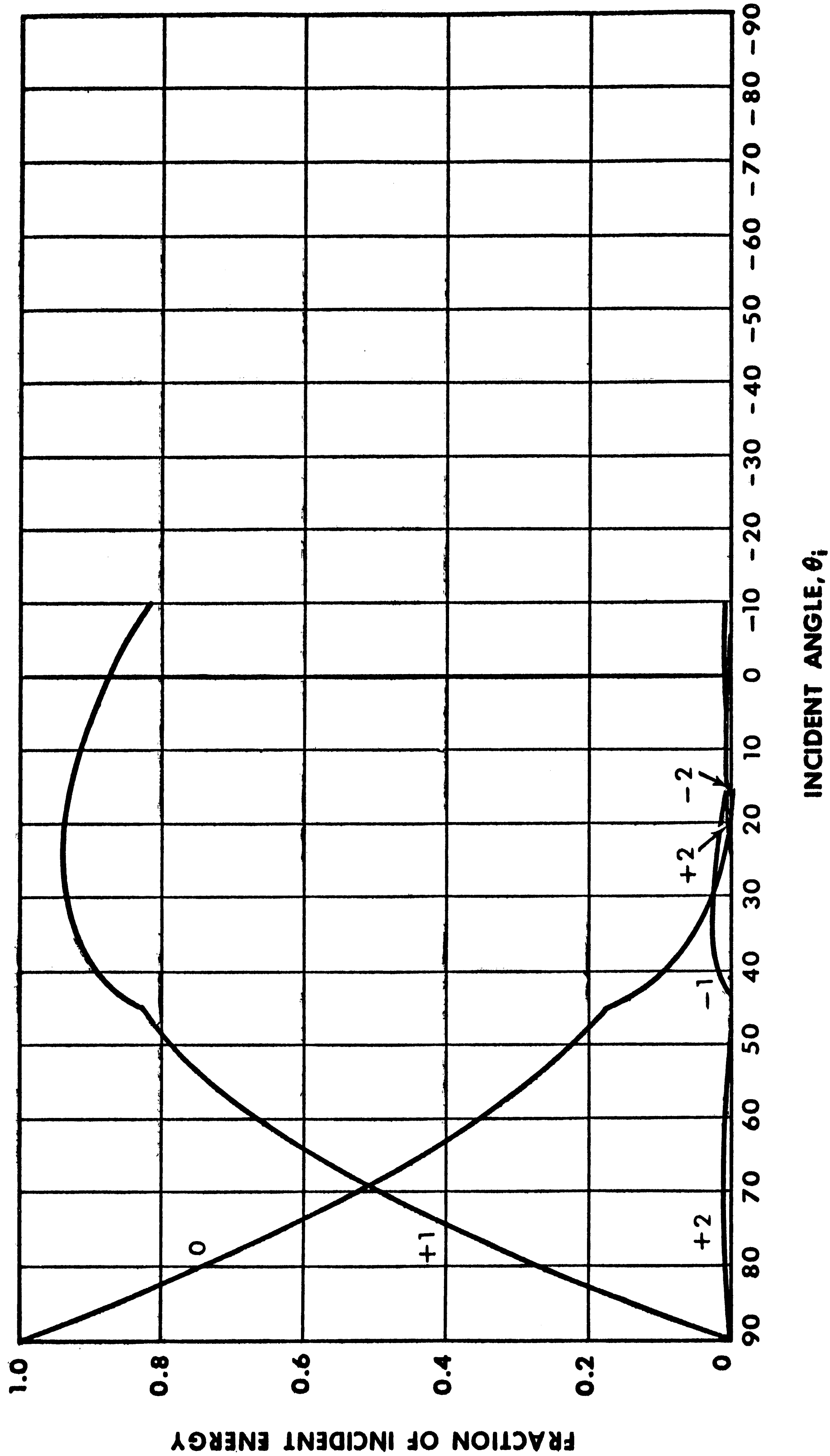

Fig. 9. Reflection with ψ = 10°, D = 3.00λ.

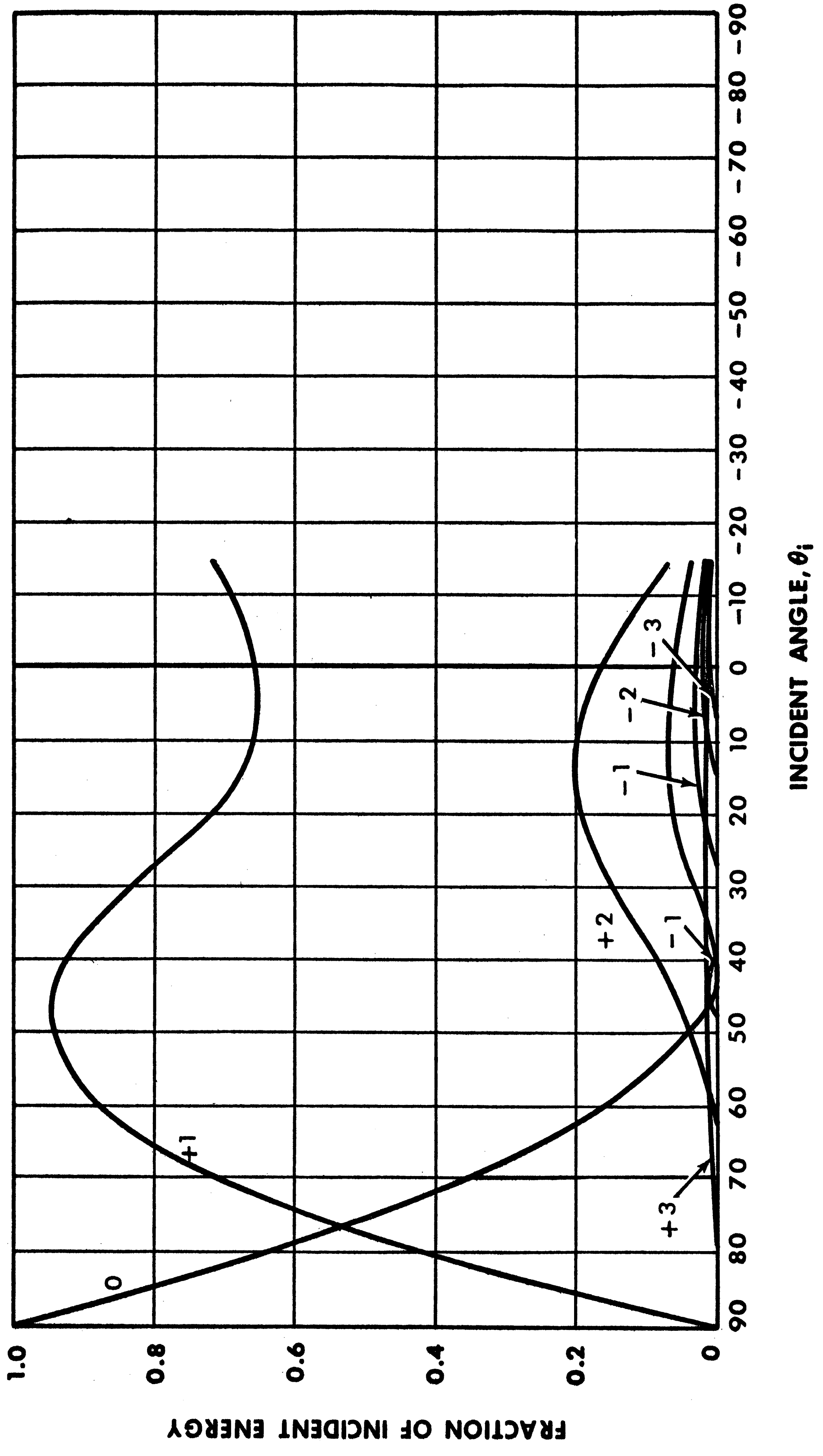

Fig. 10. Reflection with ψ = 10°, D = 4.00 λ.

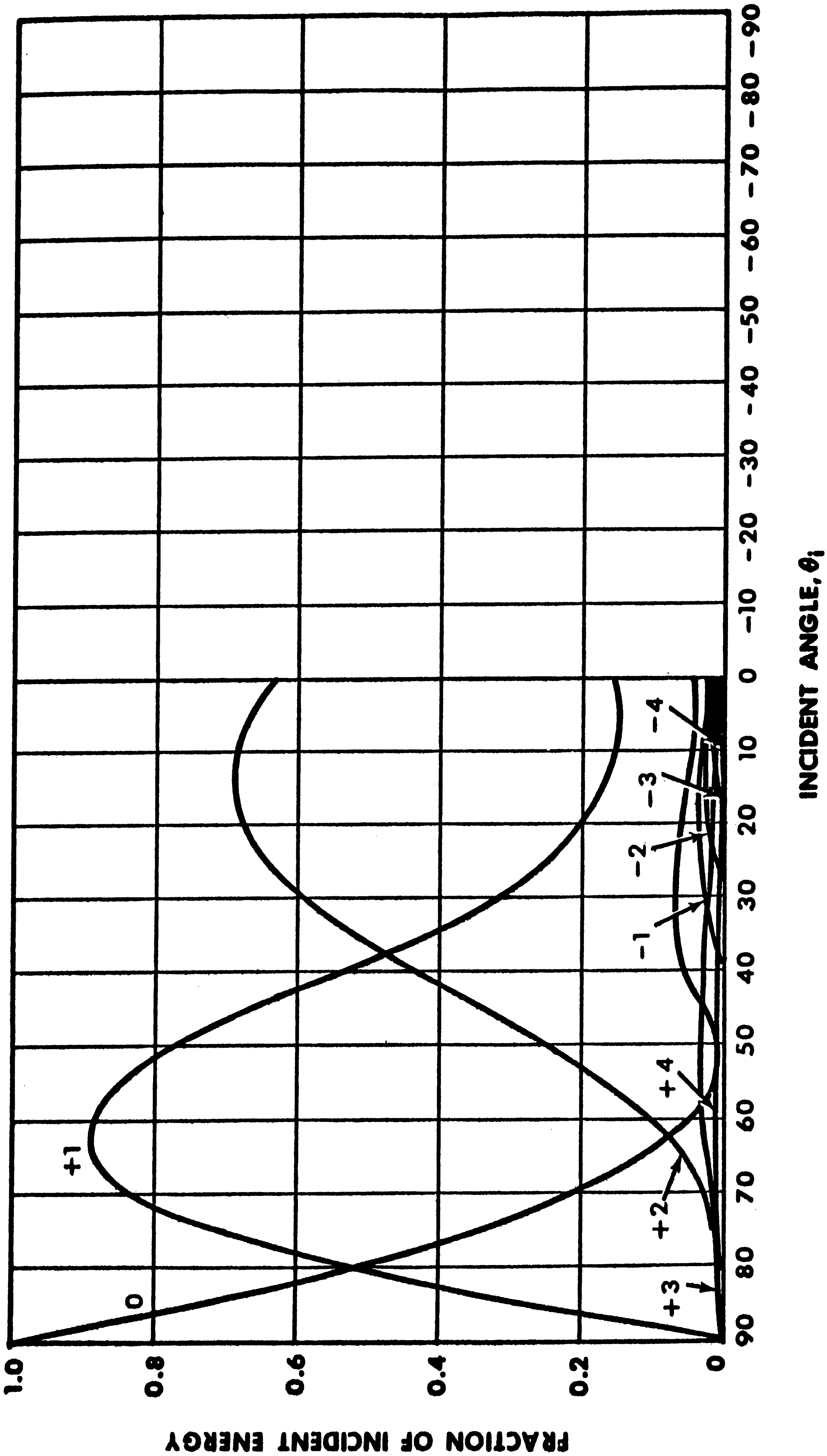

Fig. 11. Reflection with ψ = 10°, D = 5.00λ.

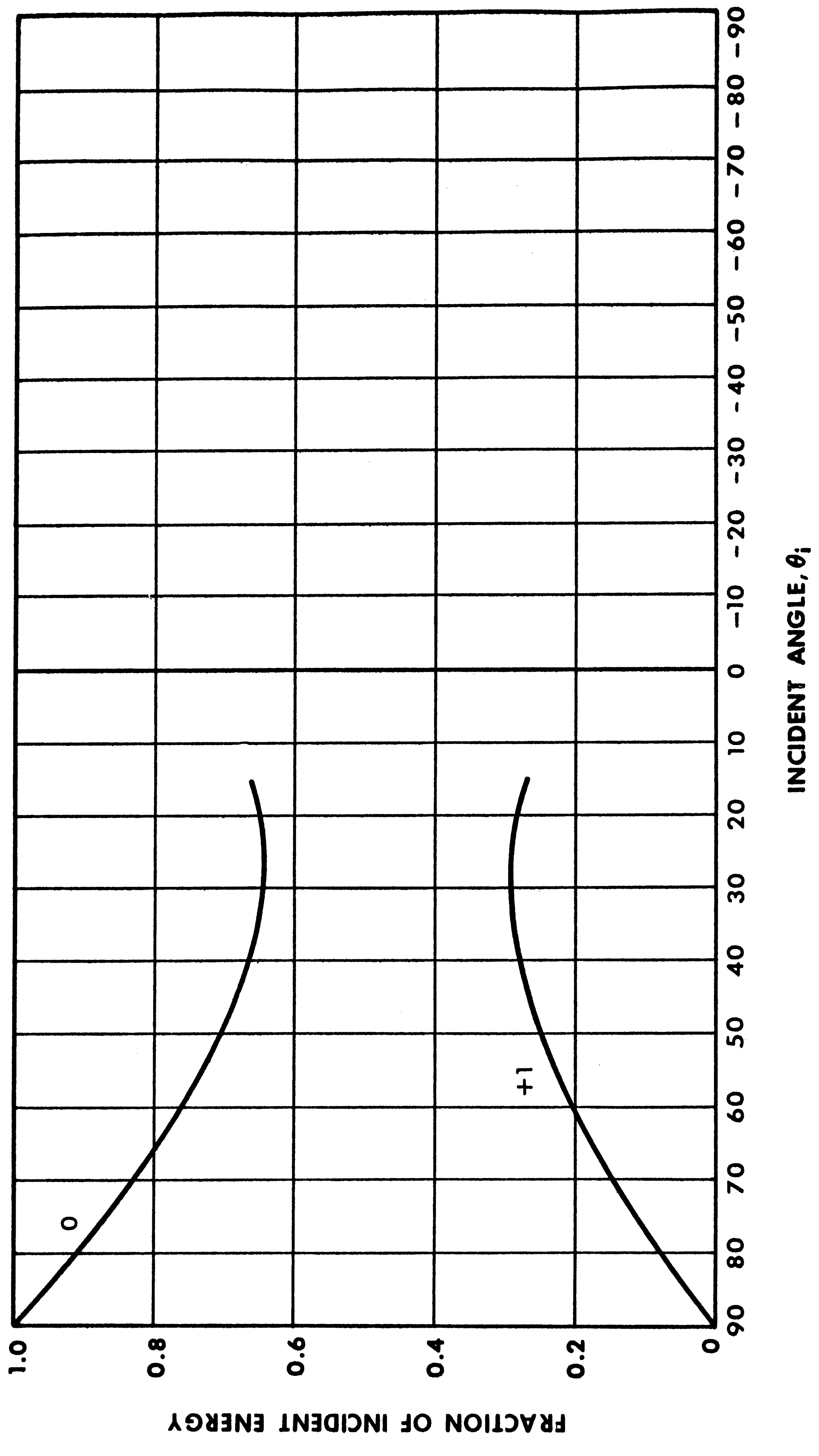

Fig. 12. Reflection with $\psi = 15°$, D = 1.55λ.

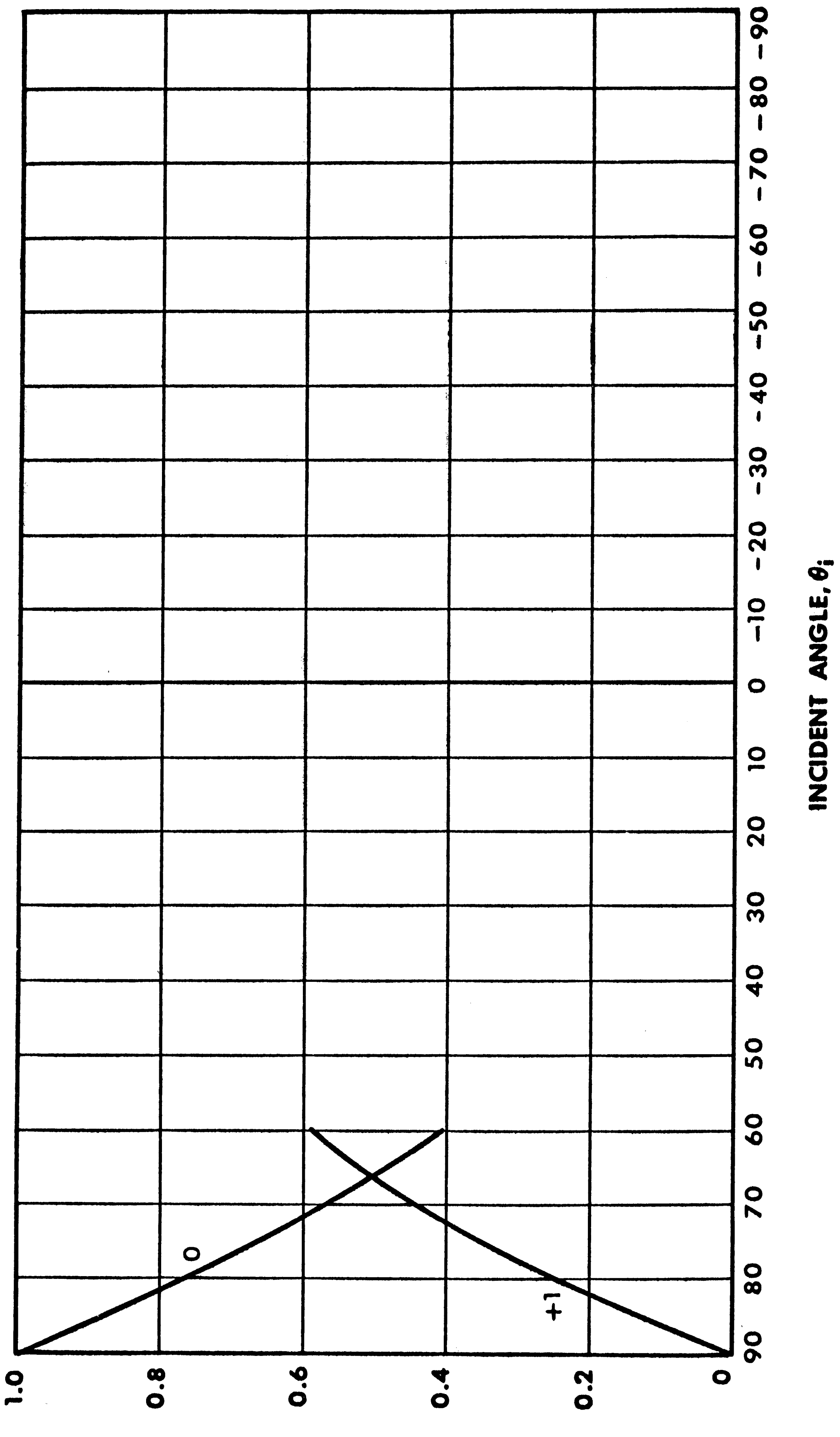

Fig. 13. Reflection with $\psi = 15°$, $D = 2.00\lambda$.

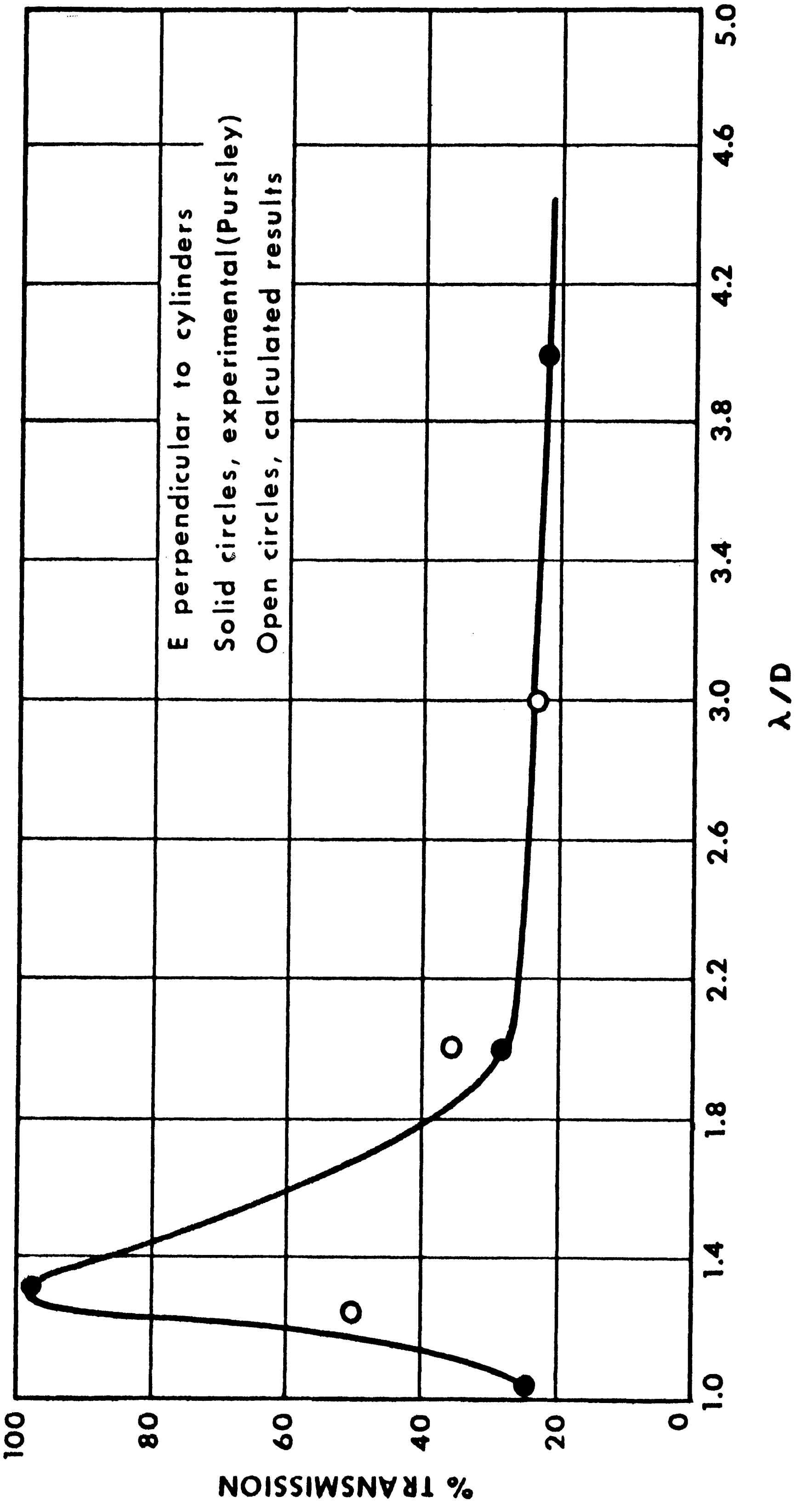

Fig. 14. Observed and calculated transmission of grating with D/a = 2.5, E perpendicular.

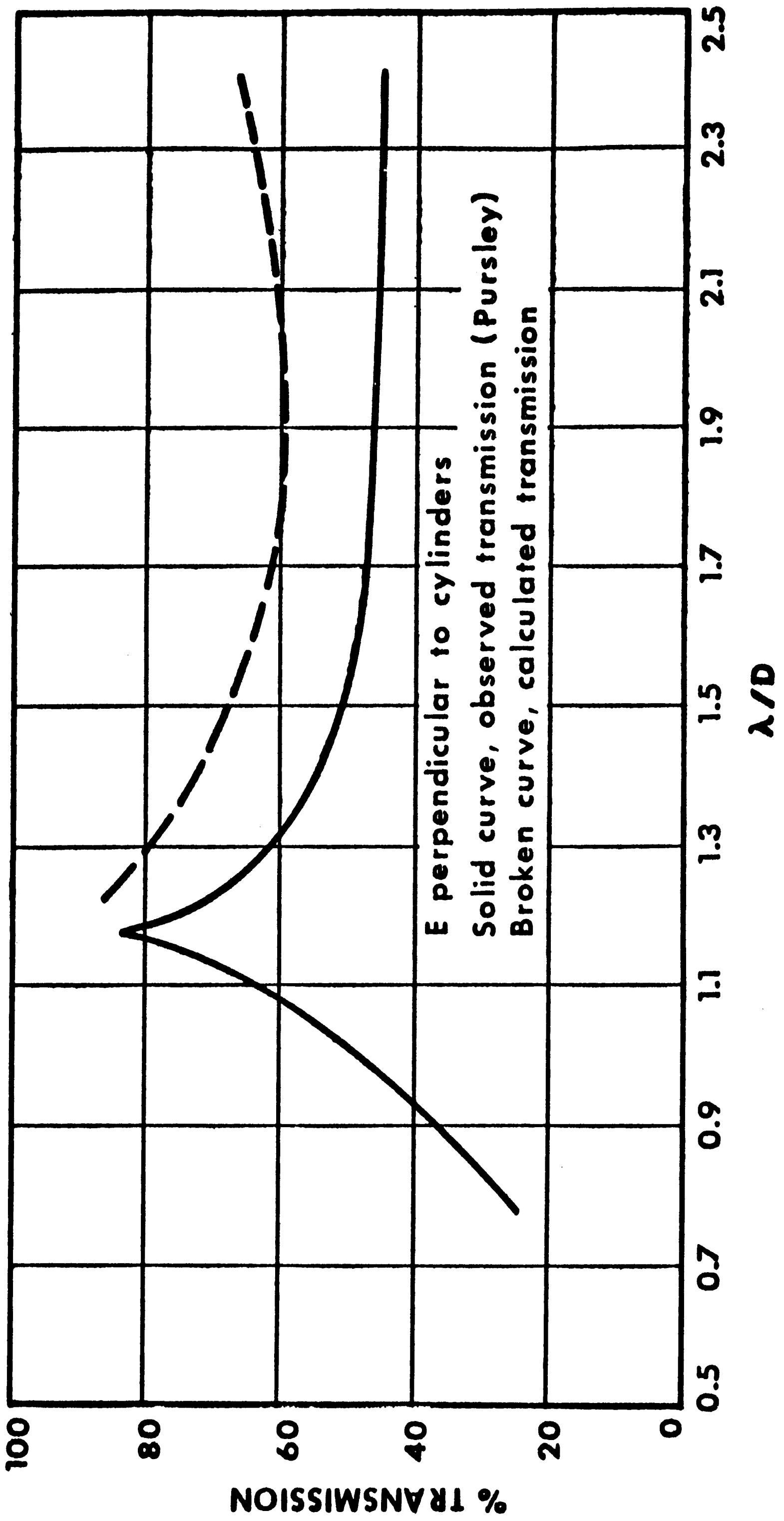

Fig. 15. Observed and calculated transmission of grating with D/a = 3.3, E perpendicular.

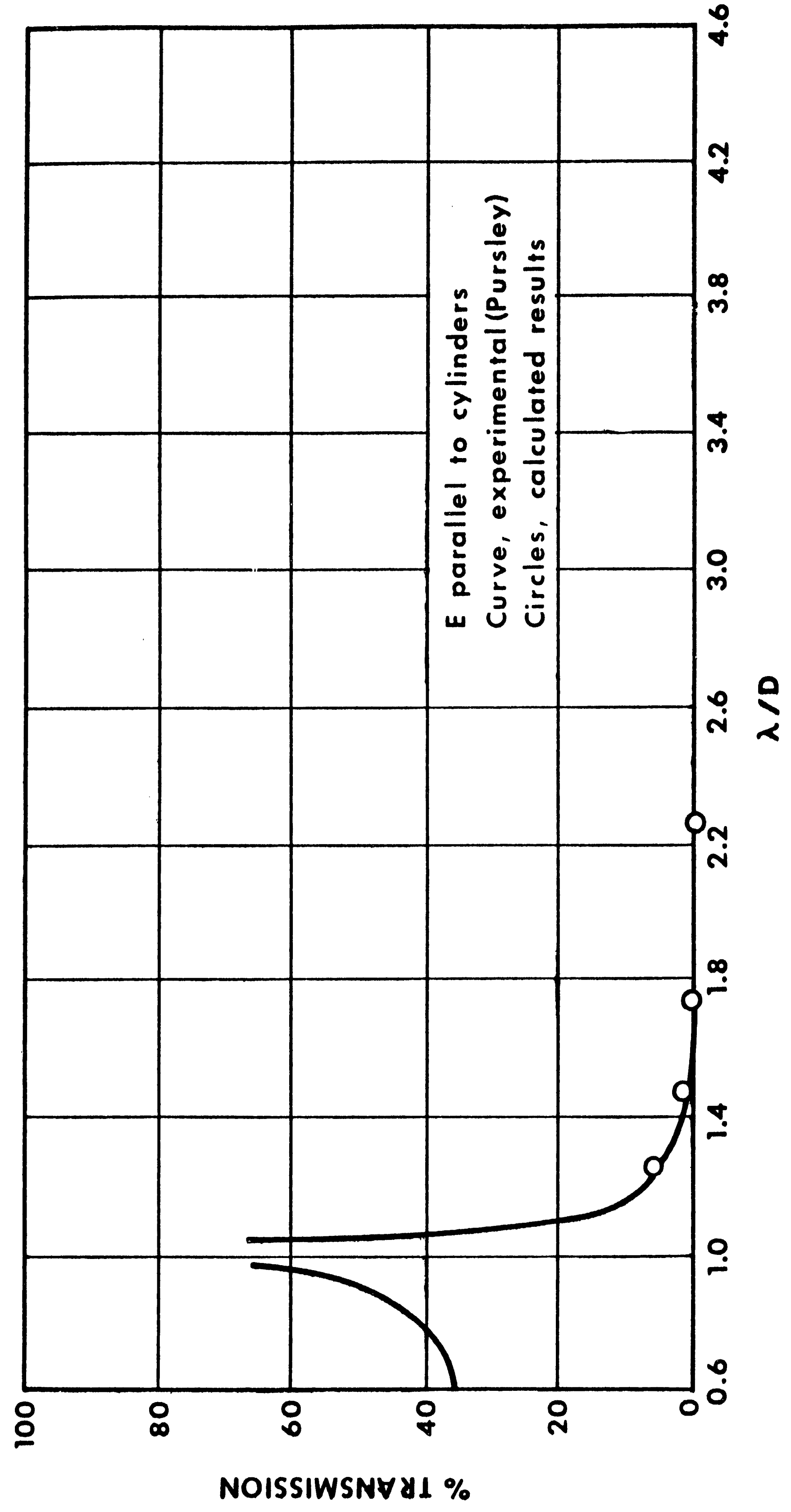

Fig. 16. Observed and calculated transmission of grating with D/a = 4.0, E parallel.

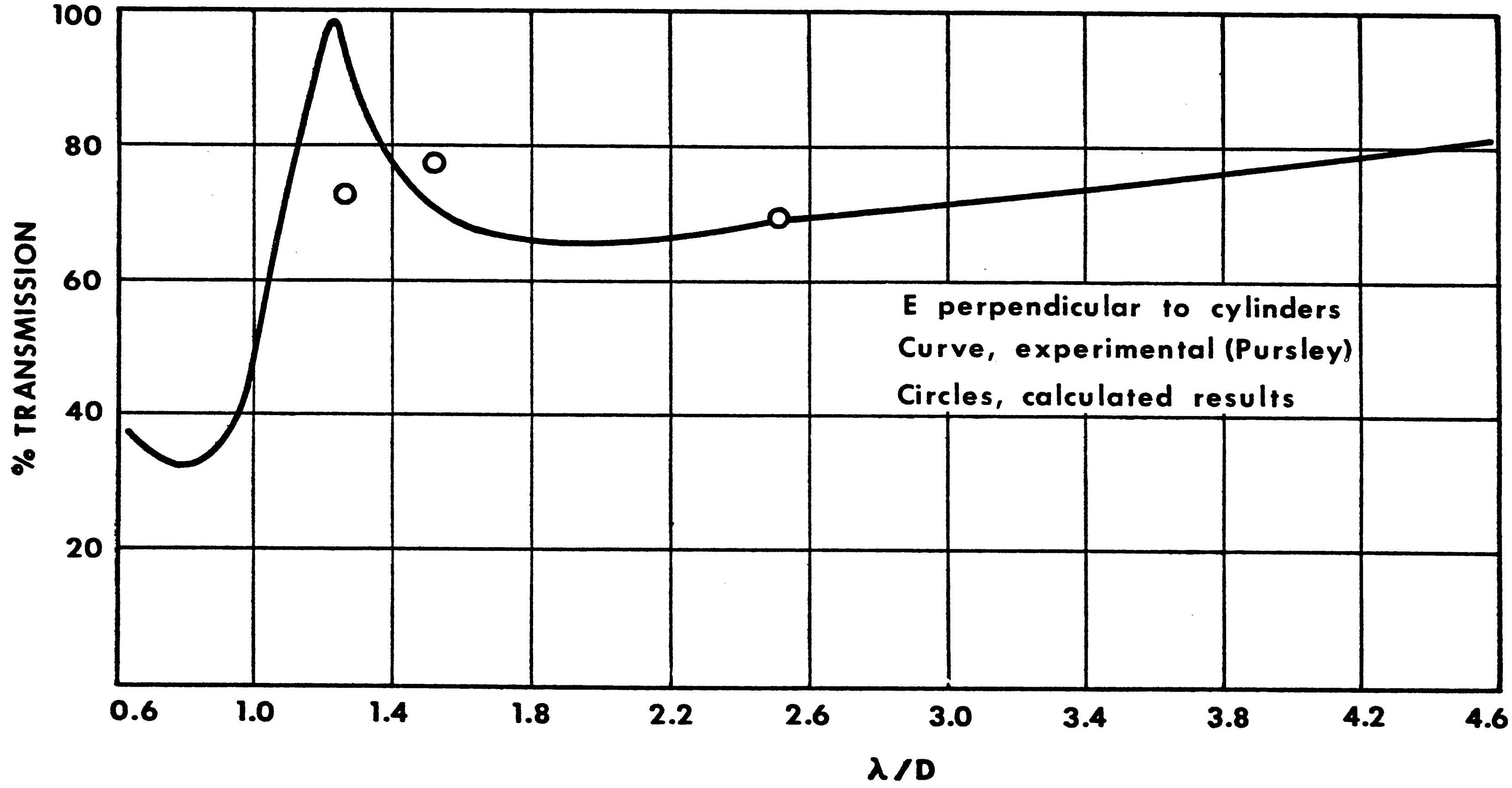

Fig. 17. Observed and calculated transmission of grating with D/a = 4.0, E perpendicular.

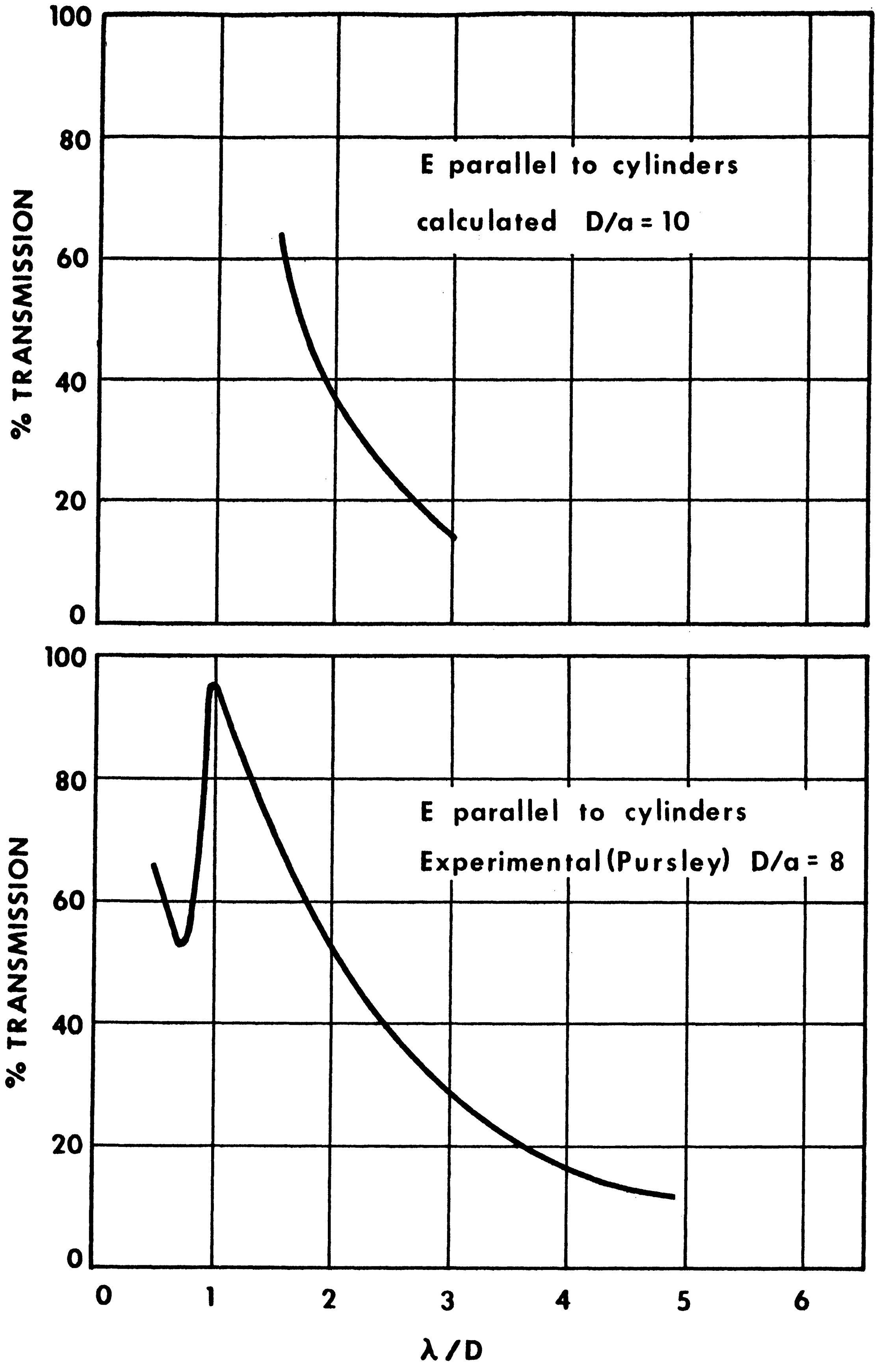

Fig. 18. Comparison of the calculated transmission for D/a = 10 to the measured transmission for D/a = 8, E parallel in both cases.

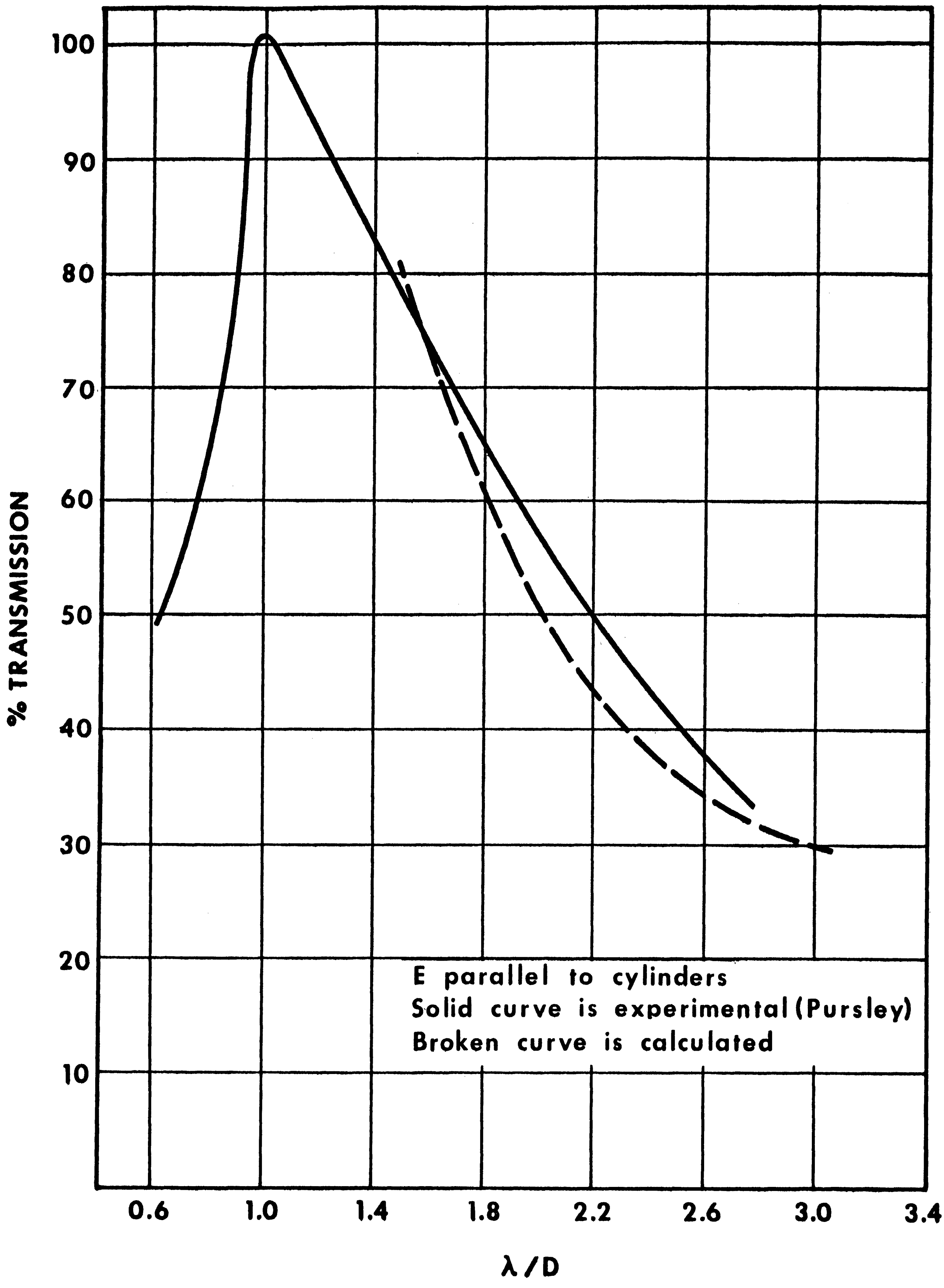

Fig. 19. Observed and calculated transmission of grating with D/a = 16.0, E parallel.