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A FOURIER TRANSFORM METHOD FOR THE TREATMENT OF THE PROBLEM OF THE REFLECTION OF RADIATION FROM IRREGULAR SURFACES

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ABSTRACT

A method is presented which can be used for the calculation of the distribution of energy reflected from irregular surfaces. The formulation is useful for the first boundary value problem and can be used in either two-or three-dimensional problems with any given incident field. The solution is reduced to quadrature with negligible error when the average square of the slope of the reflecting surface is small and when the wave length of the incident radiation is not small compared with the displacement of the surface from its average value. A numerical example is worked, the sinusoidal surface, and is compared with experiment and with a method due to Rayleigh. It is found that the Fourier transform method is preferable to previous methods, notably those which can be classified as physical optics (such as Rayleigh's), since the error in the transform method is of second order in the surface slope whereas the error in previous methods is of first order in the same quantity.

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I. INTRODUCTION

In the past there has been a considerable amount of work done on the problem of the reflection of radiation from non-plane, or irregular, surfaces. The attention given this problem has increased in recent years partly as a result of the expanded interest in centimeter wave length electromagnetic and acoustic radiation. The approximations which have been used in the past have centered for the most part around perturbation treatments[1,2,3] and around the Kirchhoff approximation[4] and its adaptations[5,6] which may be classed under the broad title of physical optics.

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1. Lord Rayleigh, Roy. Soc. Proc. 79A, 399 (1907).
2. J.W. Miles, J. Acoust. Soc. Am. 26, 191 (1954).
3. S.O. Rice, Comm. on Pure and Appl. Math. 4, 351 (1951).
4. B.B. Baker and E.T. Copson, The Mathematical Theory of Huygens' Principle (2nd ed.; London:Oxford University Press, 1950).
5. L.M. Brekhovskikh, Zh. exsper. teor. Fiz. (USSR) 23, 275 (1952). Translated by G.N. Goss, U.S.N. Electronics Lab., San Diego, Cal.
6. C. Eckart, J. Acoust. Soc. Am. 25, 566 (1953).

It is the purpose of this paper to present a new method for the treatment of reflection problems. The method is obtained as follows. From the Helmholtz formula (see below) one can obtain, through the use of the boundary condition applicable to the problem, an integral equation of the first kind whose solution can be used to calculate the field reflected from a given surface. It will be shown that under certain conditions (when the square of the slope of the reflecting surface is small and when the wave length of the incident radiation is not small compared with the displacement of the surface from its average value) one can approximate the kernel of the integral equation in such a way that the modified equation can be solved through the use of the Fourier integral transform. It will be seen that the method is primarily suited to the treatment of the type of boundary value problem in which the field function is assumed to vanish at the reflecting surface (often called the first boundary value problem). Peculiarly enough it does not seem that there is a parallel formulation for the second boundary value problem, where the normal derivative of the field function is assumed to vanish at the reflecting surface. This is true because of the special form which the kernel of the integral equation must assume in order that the central approximation of the method be useful. The method can be applied to either two- or three-dimensional (scalar) problems; it will be outlined in detail only for the two-dimensional problem, it being an easy matter to extend the formulation to the analogous three-dimensional problems.

The problem to be considered here will now be described in detail. One is given a half-space of homogeneous material bounded by the two-dimensional surface $\zeta(x)$; the function $\zeta(x)$ is assumed to be continuous, single-valued, bounded, and to possess a piece-wise continuous and bounded first derivative. It is supposed that radiation is incident upon the surface $\zeta(x)$ from the half-space $z < \zeta(x)$ (see Fig. 1).

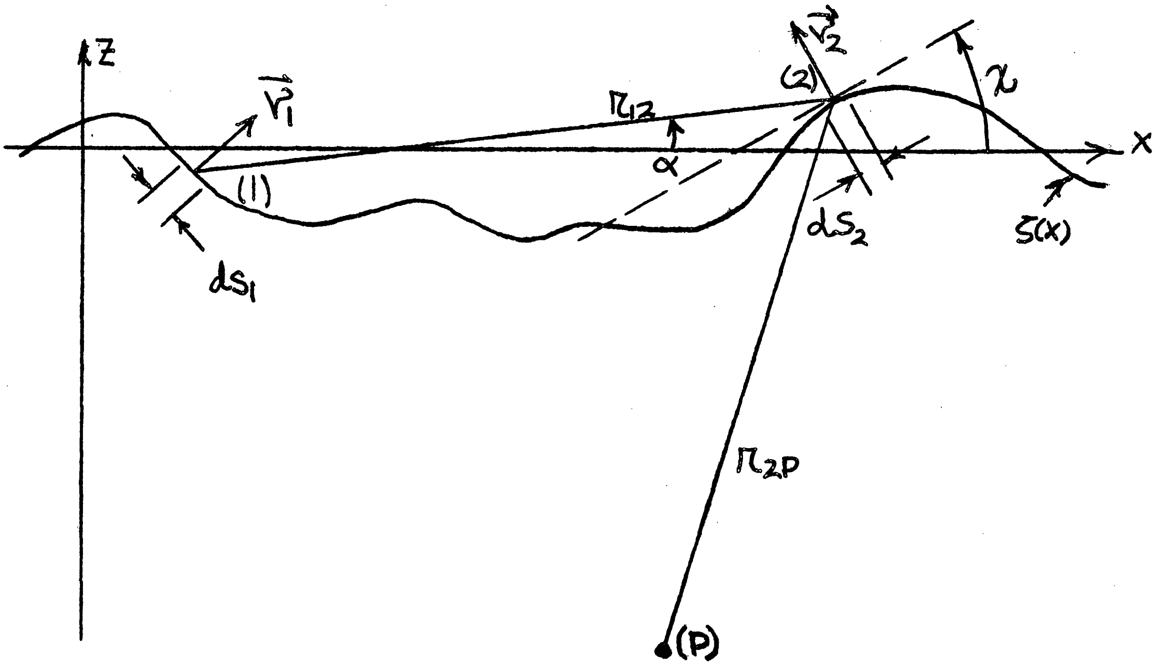

Fig. 1. Diagram used in the description of the Helmholtz formula.

This incident radiation may consist either of a plane wave (called the first incident field) or it may be set up by a line source assumed to be located at (x_0, z_0) (called the second incident field). One must distinguish between these two cases in the general formulation, although the distinction disappears in the special method of solution to be presented below.

If the incident function consists of a plane wave, it is supposed that the propagation vector lies in the x-z plane. One wishes then to find a function $\Phi(x, z, t)$ which satisfies the wave equation in two dimensions,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} \qquad (1)$$

where c is the phase velocity, assumed to be constant. In acoustical problems the function Φ represents the velocity potential,

$$\vec{v} = -\nabla \Phi \qquad (2)$$

where $\vec{v}(x,z,t)$ is the velocity at (x,z,t); the surface is assumed to be pressure release. For electromagnetic problems, it is supposed that the surface $\zeta(x)$ is perfectly conducting, and that the incident radiation is polarized with the electric vector perpendicular to the x-z plane; then the function Φ is taken to represent the electric field, which, under the given assumptions, lies entirely in the direction perpendicular to the x-z plane. It will be supposed that the source radiates a single (angular) frequency ω, so that one can write

$$\Phi(x,z,t) = e^{-i\omega t}\phi(x,z) \ . \qquad (3)$$

Substituting Eq. (3) in Eq. (1) one obtains,

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} + k^2\phi = 0 \ , \qquad (4)$$

with $k = \frac{\omega}{c}$; the function ϕ is to satisfy Eq. (4) throughout the half-space if the incident function consists of a plane wave, and at all points of the half-space except the point (x_0, z_0) if the incident field is set up by a line source. In the latter case the field near the source is to behave like

$$\phi(x,z) \approx H_0^{(1)}(kr_0), \qquad r_0 \to 0 \tag{5}$$

where

$$r_0 = \left[(x-x_0)^2 + (z-z_0)^2 \right]^{1/2} ,$$

and $H_0^{(1)}$ is the zeroth order Hankel function of the first kind. For three-dimensional problems the Hankel function is replaced by $\frac{e^{ikr_0}}{r_0}$. To continue, it is supposed that at the surface $\zeta(x)$ the function ϕ vanishes,

$$\phi(x, \zeta(x)) = 0 \quad ; \tag{6}$$

as stated above this is equivalent to supposing that the surface is pressure release or perfectly conducting, for acoustics and electromagnetics respectively. Finally let $\phi_i(P)$ represent the value which the field would assume at the point (P) if the surface $\zeta(x)$ were not present, that is if the source system were located in an infinite homogeneous medium with phase velocity c. Then one can write,

$$\phi = \phi_i + \phi_r \tag{7}$$

where ϕ_r represents the reflected field (by definition). The function ϕ_i is, of course, given while the function ϕ_r represents the unknown.

II. DERIVATION OF THE METHOD

It will be shown in this section that when

i) $\left(\frac{d\zeta^M}{dx}\right)^2 << 1$ and

ii) $k \zeta^M \lesssim 1$,

it is possible to reduce the problem to quadrature with negligible error. The symbol $\sim$ indicates ''is of the order of magnitude'', ζ^M represents the bound on $\zeta(x)$, and $\frac{d\zeta^M}{dx}$ represents the bound on $\frac{d\zeta(x)}{dx}$. The first of the restrictions i) and ii) is the more important.

To proceed with the derivation, the Weber two-dimensional analogue of the Helmholtz formula is needed. By using Green's formula in connection with Eq. (4), one is able to derive the following result (see Fig. 1):[7]

$$\phi(P) = \phi_i(P) + \frac{1}{4i} \int_{\zeta} \left\{ \phi(2) \frac{\partial}{\partial \nu_1} H_0^{(1)}(kr_{2P}) - H_0^{(1)}(kr_{2P}) \frac{\partial}{\partial \nu_2} \phi(2) \right\} ds_2 \qquad (8)$$

subject to the restrictions that --

a. The function $\phi(x,z)$ is continuous with continuous first and second order partial derivatives for all points (x,z) satisfying the condition $z \leq \zeta(x)$, with the exception of the source point for problems involving the second type of incident field. In this case

$$\phi(x,z) \longrightarrow H_0^{(1)}(kr_0) \quad \text{as} \quad r_0 \longrightarrow 0.$$

b. The function ϕ shall represent outgoing waves at great distances from the surface $\zeta(x)$.

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7. Baker and Copson, loc. cit., chap. II.

The integral in Eq. (8) is to be carried over the entire surface ζ (x). The symbol $\frac{\partial}{\partial \nu_2}$ Indicates the derivative with respect to the outward normal direction at the surface point (2). We have let $\phi(2)$ represent the value of the function ϕ at the point (2), and $\phi(P)$ represent the value of the function at a space point P; similar notation will be used for other functions appearing later.

Now allowing (P) to approach the surface point (1) and utilizing the boundary condition given by Eq. (6), Eq. (8) becomes,

$$\phi_1(1) = \frac{1}{4i} \int_\zeta H_o^{(1)}(kr_{12}) \frac{\partial \phi(2)}{\partial \nu_2} ds_2 \quad , \tag{9}$$

where,

$$r_{12} = \left[(x_1-x_2)^2 + (\zeta(x_1) - \zeta(x_2))^2 \right]^{1/2} \quad . \tag{10}$$

If Eq. (9) could be solved for the function $\frac{\partial \phi}{\partial \nu}$, its value could be substituted in Eq. (8) to obtain the solution to the problem. One is tempted to let

$$r_{12} \approx |x_1-x_2| \quad , \tag{11}$$

since in such a case Eq. (9) could be solved. Let us examine the error incurred by making this assumption. First let

$$r_{12} = \frac{|x_1-x_2|}{\cos\alpha} \tag{12}$$

where α is the acute angle between r_{12} and the x direction (see Fig. 1). Then define the difference between the exact and the approximate kernel for Eq. (9) as

$$K(x_1,x_2) = H_o^{(1)}\left(k\,\frac{|x_1-x_2|}{\cos\alpha}\right) - H_o^{(1)}(k|x_1-x_2|). \qquad (13)$$

In considering the error made in adopting the assumption represented by Eq. (11), it is convenient to treat the following regions separately: I, $k|x_1-x_2| \ll 1$; II, $k|x_1-x_2| \gg 1$; III, all other values of the argument, $k|x_1-x_2|$. Also suppose in this connection that $a \ll 1$ indicates $a \leq 0.1$, and similarly $a \gg 1$ indicates $a \geq 10.0$. Because of the (integrable) singularity of $H_o^{(1)}(y)$ at $y = 0$, it is to be expected that I is the most important region of the integrand in Eq. (9). Hence begin by considering it.

The Hankel function is defined by,

$$H_o^{(1)}(y) = J_o(y) + \frac{2i}{\pi}\left[\gamma + \ln(\tfrac{1}{2}y)\right] J_o(y) - \frac{2i}{\pi}\sum_{n_1=1}^{\infty}\frac{(-1)^m(\frac{1}{2}y)^{2m}}{(m!)^2}\left\{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}\right\}, \qquad (14)$$

where $\gamma = 0.5772\ldots$. Upon referring to Eqs. (14) and (12) it is seen that in I

$$|K| \sim \frac{2}{\pi}\ln\frac{1}{\cos\alpha}\quad, \qquad (15)$$

or from the definition of α, assuming $\frac{d\zeta^M}{dx}$ small one has,

$$|K| \sim \frac{1}{\pi}\left|\frac{d\zeta^M}{dx}\right|^2 \qquad (16)$$

For region II use the asymptotic form for the Hankel function,[8]

$$H_o^{(1)}(y) = \frac{e^{iy}}{(\frac{\pi}{2}iy)^{1/2}}\left[1 + O(\tfrac{1}{y})\right]\quad, \qquad (17)$$

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8. E. Jahnke and F. Emde, Tables of Functions (4th ed.; New York: Dover Publications, 1945).

in Eq. (13) to find

$$K \approx \frac{e^{ik|x_1-x_2|}}{\left(\frac{\pi}{2}ik|x_1-x_2|\right)^{1/2}} \left\{ (\cos\alpha)^{1/2} \left[\exp ik|x_1-x_2|\left(\frac{1}{\cos\alpha} - 1\right)\right] - 1 \right\}, \tag{18}$$

omitting $O(\frac{1}{y})$ as a term of higher order. It is easily seen that

$$|\alpha| \leq \frac{2\zeta^M}{|x_1-x_2|}, \tag{19}$$

so that Eq. (18) becomes,

$$K \sim \frac{e^{ik|x_1-x_2|}}{\left(\frac{\pi}{2}ik|x_1-x_2|\right)^{1/2}} \left[- \frac{(k\zeta^M)^2}{(k|x_1-x_2|)^2} + 2i\frac{(k\zeta^M)^2}{k|x_1-x_2|}\right], \tag{20}$$

or ignoring the term $-\frac{(k\zeta^M)^2}{(k|x_1-x_2|)^2}$, which is of higher order,

$$K \sim \frac{e^{ik|x_1-x_2|}}{\left(\frac{\pi}{2}ik|x_1-x_2|\right)^{1/2}} 2i\frac{(k\zeta^M)^2}{k|x_1-x_2|}. \tag{21}$$

Finally region III is considered. Note first that for α small,

$$K \approx \left[\frac{d}{d(k|x_1-x_2|)} H_0^{(1)}(k|x_1-x_2|)\right] k|x_1-x_2|\left(\frac{1}{\cos\alpha} - 1\right),$$

or from the known relation $\frac{d}{dy}H_0^{(1)}(y) = -H_1^{(1)}(y)$,

$$K \approx -H_1^{(1)}(k|x_1-x_2|)k|x_1-x_2|\left(\frac{1}{\cos\alpha} - 1\right). \tag{22}$$

From the pertinent tables in Jahnke and Emde,[9] it is seen that

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9. Jahnke and Emde, loc. cit., pp. 157 and 191.

the largest value of $\left| H_1^{(1)}(k|x_1-x_2|)k|x_1-x_2| \right|$ for $0.1 \leq k|x_1-x_2| \leq 10$ occurs at $k|x_1-x_2| = 10$ and,

$$\left| H_1^{(1)}(10) \cdot 10 \right| = 2.53 \; . \tag{23}$$

Again from the definition of α and using Eq. (23) (replacing 2.53 by 3.0), Eq. (22) becomes

$$|K| \sim \frac{3}{2} \left| \frac{d\,\zeta^M}{dx} \right|^2 \; . \tag{24}$$

To summarize, it is seen that K is negligible in region I if hypothesis i) is fulfilled (cf. Eq. (16)); the same hypothesis makes K small in region III (cf. Eq. (24)). Finally from Eq. (21) we see that K is negligible in region II if hypothesis ii) is fulfilled. It is noted however that in region II the kernel of Eq. (9) is small so that an error in this region is not so important as errors in the other two. Thus in some problems the approximation may lead to a useful result even though the hypothesis ii) is not fulfilled.

Now proceed with a method of solution based upon the assumption that K is small. Rewrite Eq. (9), using Eq. (13),

$$\phi_i(1) = \frac{1}{4i} \int_{\zeta} \left[H_0^{(1)}(k|x_1-x_2|) + K(x_1,x_2) \right] \frac{\partial \phi(2)}{\partial \nu_2} \, ds_2 \; . \tag{25}$$

Before continuing it is desireable to make some changes in notation. Let

$$\Psi(x_2) = \frac{1}{\cos \chi(x_2)} \frac{\partial \phi(2)}{\partial \nu_2} \; , \tag{26}$$

$$F(x_1) = 4i \; \phi_i(1) \; , \tag{27}$$

and note

$$ds_2 = \frac{dx_2}{\cos \chi(x_2)} \quad . \tag{28}$$

Here $\chi(x)$ is the acute angle made by the tangent to the surface at x_2 with the x-axis. Then substituting in Eq. (25) we have

$$F(x_1) = \int_{-\infty}^{\infty} \left[H_o^{(1)}(k|x_1-x_2|) + K(x_1,x_2) \right] \Psi(x_2)dx_2 . \tag{29}$$

As the next step consider $K(x_1,x_2)$ as a perturbation. Let

$$K(x_1,x_2) = \varepsilon K(x_1,x_2) , \tag{30}$$

$$\Psi(x) = \Psi^{(0)}(x) + \varepsilon \Psi^{(1)}(x) + \varepsilon^2 \Psi^{(2)}(x) + \ldots \tag{31}$$

where it is assumed that $\Psi(x)$ is an analytic function of ε for $\varepsilon \leq 1$. The solution is obtained by allowing $\varepsilon \rightarrow 1$. By substituting Eqs. (30) and (31) in Eq. (29), and equating equal powers of ε it is evident that:

$$\int_{-\infty}^{\infty} H_o^{(1)}(k|x_1-x_2|) \Psi^{(0)}(x_2)dx_2 = F(x_1), \tag{32}$$

$$\left.\begin{aligned} \int_{-\infty}^{\infty} H_o^{(1)}(k|x_1-x_2|) \Psi^{(1)}(x_2)dx_2 &= -\int_{-\infty}^{\infty} K(x_1,x_2) \Psi^{(0)}(x_2)dx_2, \\ \int_{-\infty}^{\infty} H_o^{(1)}(k|x_1-x_2|) \Psi^{(2)}(x_2)dx_2 &= -\int_{-\infty}^{\infty} K(x_1,x_2) \Psi^{(1)}(x_2)dx_2, \end{aligned}\right\} \tag{33}$$

There is a well known method due to Levi-Civita[10] which may be used for the solution of integral equations of the above

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10. T. Levi-Civita, R. Accademia della Scienze di Torino Atti <u>31</u>, 25 (1895). See also E.C. Titchmarsh, <u>Introduction to the Theory of Fourier Integrals</u> (Oxford: The Clarendon Press, 1937), chap. XI.

type in which the kernel is a function of the difference of its two variables. The method begins by taking the Fourier transform of the integral equation. Accordingly take the Fourier transform of Eq. (32) and Eq. (33). We have,

$$(2\pi)^{1/2}h(t)\,\psi^{(0)}(t) = f(t)\,, \tag{34}$$

$$(2\pi)^{1/2}h(t)\,\psi^{(1)}(t) = -\,\text{F.T.}\left\{\int_{-\infty}^{\infty} K(x_1,x_2)\,\Phi^{(0)}(x_2)dx_2,\right\} \tag{35}$$

$$(2\pi)^{1/2}h(t)\,\psi^{(2)}(t) = -\,\text{F.T.}\left\{\int_{-\infty}^{\infty} K(x_1,x_2)\,\Phi^{(1)}(x_2)dx_2,\right\} \tag{36}$$

$$\vdots$$

where

$$f(t) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{itx}\,F(x)dx\,, \tag{37}$$

$$= \text{F.T.}\left\{F(x)\right\}.$$

Similarly,

$$\left.\begin{aligned} h(t) &= \text{F.T.}\left\{H_o^{(1)}(k|x|)\right\}, \\ \psi^{(0)}(t) &= \text{F.T.}\left\{\Phi^{(0)}(x)\right\}, \\ \psi^{(1)}(t) &= \text{F.T.}\left\{\Phi^{(1)}(x)\right\}, \\ &\vdots \end{aligned}\right\} \tag{38}$$

In order to guarantee that the transforms of Eq. (32) and Eq. (33) will exist and yield Eq. (34) and Eqs. (35) and (36), it is sufficient to require that $\Phi^{(0)}$, $\Phi^{(1)}$, ... and $H_o^{(1)}(k|x|)$ be absolutely integrable and that $F(x_1)$ and $\int_{-\infty}^{\infty} K(x_1,x_2)\,\Phi^{(n)}(x_2)dx_2$

have Fourier transforms. If these conditions are not satisfied by $\Psi^{(n)}$, the solution is subject to verification. For the function $H_0^{(1)}(k|x|)$ let $k = k' + i\delta$, $\delta > 0$, in order to guarantee absolute integrability, then allow $\delta \rightarrow 0$ in the solution.

Now if we add the restriction that $\Phi^{(n)}(x)$ be sectionally continuous we can, after solving Eqs. (34), (35), and (36) for $\psi^{(n)}(t)$, take the inverse Fourier transform. It is known that,[11]

$$h(t) = \left(\frac{2}{\pi}\right)^{1/2}(k^2-t^2)^{-1/2} \quad . \tag{39}$$

We have then from Eqs. (34), (35), and (36):

$$\Phi^{(0)}(x) = \frac{1}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} (k^2-t^2)^{1/2} e^{-ixt} f(t)dt, \tag{40}$$

$$\Phi^{(1)}(x) = -\frac{1}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} (k^2-t^2)^{1/2} e^{-ixt} dt \cdot \left\{ \text{F.T.} \left[\int_{-\infty}^{\infty} K(x_1,x_2)\, \Phi^{(0)}(x_2)dx_2 \right] \right\}, \tag{41}$$

$$\Phi^{(2)}(x) = -\frac{1}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} (k^2-t^2)^{1/2} e^{-ixt} dt \cdot \left\{ \text{F.T.} \left[\int_{-\infty}^{\infty} K(x_1,x_2)\, \Phi^{(1)}(x_2)dx_2 \right] \right\}, \tag{42}$$

⋮

The solution, $\Phi(x)$, is obtained from

$$\Phi(x) = \Phi^{(0)}(x) + \Phi^{(1)}(x) + \ldots \quad . \tag{43}$$

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11. G.A. Campbell and R.M. Foster, Fourier Integrals for Practical Applications. Bell Telephone System Tech. Pub., 1931; Monograph B-584, No. 918.

The symbols in Eqs. (40), (41), and (42) are defined by:

$$\Psi(x) = \frac{1}{\cos \chi(x)} \frac{\partial \phi}{\partial \nu} , \qquad (26)$$

$$f(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{itx} F(x)dx, \qquad (37)$$

$$F(x) = 4i\ \phi_i(x) , \qquad (27)$$

$$K(x_1,x_2) = H_0^{(1)}(kr_{12}) - H_0^{(1)}(k|x_1-x_2|). \qquad (13)$$

One obtains the solution, $\phi(P)$, from

$$\phi(P) = \phi_i(P) - \frac{1}{4i} \int_{S} H_0^{(1)}(kr_{1P}) \frac{\partial \phi(1)}{\partial \nu_1} ds_1 \quad ; \qquad (44)$$

this equation is obtained by applying Eq. (6) to Eq. (8).

It is seen from Eqs. (41) and (42) that the corrections to the first approximation, $\Psi^{(0)}$, are small if K is small enough.

III. SPECIALIZATION TO PROBLEMS INVOLVING PLANE WAVES INCIDENT ON PERIODIC SURFACES

In this section attention will be restricted to the class of problems involving plane waves incident upon periodic surfaces. For such problems the result given by Eqs. (40), (26) and (44) may be simplified somewhat. Consider only the zeroth order result, as given in Eq. (40); the development will be formal.

Let it be supposed that a plane wave is incident upon a periodic surface from the negative z direction (see Fig. 2);

the propagation vector for the plane wave makes an angle θ_i with the z axis. Then,

$$\phi_i = e^{ik(x \sin \theta_i + z \cos \theta_i)} \tag{45}$$

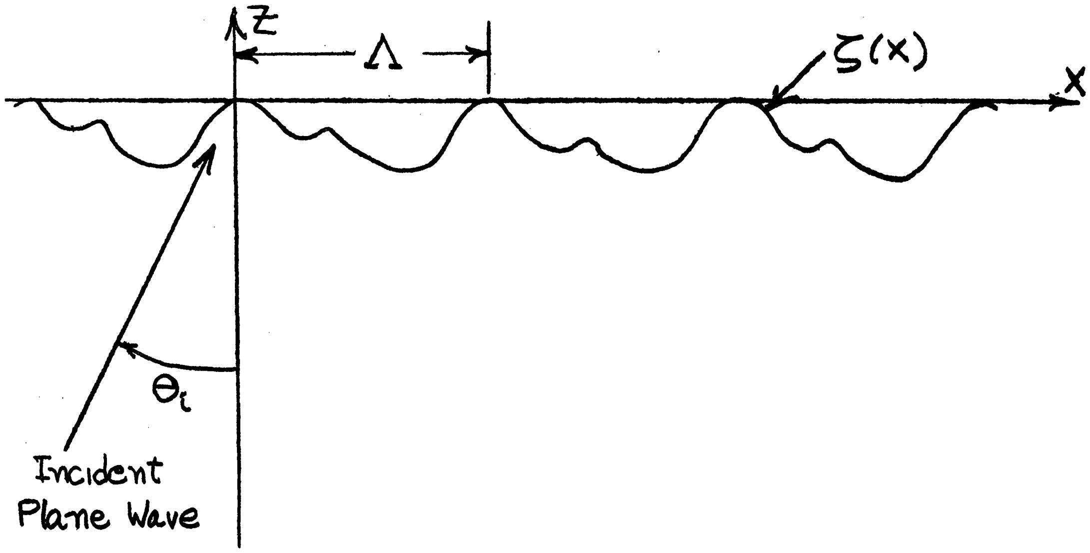

Fig. 2. The figure shows a plane wave incident upon a periodic surface.

Upon substituting Eq. (45) in Eq. (37), one finds,

$$f(t) = \frac{4i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \exp \left\{ i \left[tx' + k \sin\theta_i x' + k \cos\theta_i \zeta(x') \right] \right\} dx', \tag{46}$$

or

$$f(t) = \frac{4i}{(2\pi)^{1/2}} \sum_{m=-\infty}^{\infty} \int_{0}^{\Lambda} \exp \left\{ i \left[tx' + k \sin\theta_i x' + k \cos\theta_i \zeta(x') + (t + k \sin\theta_i) m\Lambda \right] \right\} dx', \tag{47}$$

where Λ is the repeat distance of the periodic surface. Now if one makes use of the identity (which is easily established)

$$\sum_{m=-\infty}^{\infty} e^{-ims\Lambda} = K \sum_{m=-\infty}^{\infty} \delta(s-mK) \qquad (48)$$

where $\delta(x)$ is the Dirac delta function and $K = \frac{2\pi}{\Lambda}$, it is found, upon interchanging the summation and integration, that Eq. (47) can be written

$$f(t) = \frac{4i}{(2\pi)^{1/2}} \sigma(t) K \sum_{m=-\infty}^{\infty} \delta(t - mK + k \sin \theta_i) \qquad (49)$$

when

$$\sigma(t) = \int_0^{\Lambda} \exp\left\{i\left[tx' + k \sin \theta_i \, x' + k \cos \theta_i \, \zeta(x')\right]\right\} dx'. \qquad (50)$$

Now substituting Eq. (49) in Eq. (40) and carrying out the indicated integration, one obtains

$$\Phi^{(0)}(x) = 2i \sum_{n=-\infty}^{\infty} e^{-ix(nK - k\sin\theta_i)} k \cos \theta_n \, \mu_n(\cos \theta_i) \qquad (51)$$

when

$$k \cos \theta_n = \left[k^2 - (nK - k \sin \theta_i)^2\right]^{1/2} \qquad (52)$$

and

$$\mu_n(\cos \theta_i) = \frac{1}{\Lambda} \int_0^{\Lambda} e^{inKx' + ik\cos\theta \, \zeta(x')} dx'. \qquad (53)$$

These latter quantities are, aside from a constant, the complex Fourier coefficients of $e^{ik \cos \theta_i \, \zeta(x)}$.

By using Sommerfeld's contour integral representation for the Hankel function one may construct the plane wave representation for that function,

$$H_0^{(1)}(kr_{1P}) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{e^{is(x_p - x_1) - i(k^2 - s^2)^{1/2}(z_p - z_1)}}{(k^2 - s^2)^{1/2}} ds, \qquad (54)$$

valid when $z_P - z_1 < 0$; for $z_P - z_1 > 0$ one changes the sign of the radical in the exponential of the integrand. Upon substituting the representation given by Eq. (54) for the Hankel function in the integrand of Eq. (44) and remembering the definition of ϕ_r (see Eq. (7)), one obtains

$$\phi_r^{(0)}(P) = -\frac{1}{4\pi i}\int_{-\infty}^{\infty}\Psi^{(0)}(x_1) \cdot \left\{\int_{-\infty}^{\infty}\frac{\exp\left[is(x_P-x_1)-i(k^2-s^2)^{1/2}(z_P-\zeta(x_1))\right]}{(k^2-s^2)^{1/2}}\,ds\right\}dx_1 \tag{55}$$

where $\Psi^{(0)}$ is defined in Eq. (51). It is supposed that $z_P < -\zeta^M$ so that Eq. (54) is valid for all surface points x_1. Now noting from Eq. (51) that,

$$\Psi^{(0)}(x_1+\Lambda) = \Psi^{(0)}(x_1)e^{i\Lambda k\sin\theta_i} \tag{56}$$

and breaking the integration into parts as was done in Eq. (47), one finds

$$\phi_r(P) = -\frac{1}{4\pi i}\sum_{m=-\infty}^{\infty}\int_0^{\Lambda}\Psi^{(0)}(x_1)e^{im\Lambda k\sin\theta i}\,dx_1 \cdot \left\{\int_{-\infty}^{\infty}ds\,\frac{\exp\left[is(x_P-x_1-m\Lambda)-i(k^2-s^2)^{1/2}(z_P-\zeta(x_1))\right]}{(k^2-s^2)^{1/2}}\right\}; \tag{57}$$

the superscript on ϕ_r has been dropped for convenience. Utilizing the identity given by Eq. (48), assuming the validity of the required interchanges of summations and integrations, and substituting the value of $\Psi^{(0)}$ given by Eq. (51), Eq. (57) can be written,

$$\phi_r(P) = \sum_{m=-\infty}^{\infty} R^{(-,m)}(x_P,z_P)\,A_m \tag{58}$$

when

$$A_m = -\sum_{n=-\infty}^{\infty} \frac{\cos\,\theta_n}{\cos\,\theta_m}\,\mu_n(\cos\,\theta_i)\,\mu_{m-n}(\cos\,\theta_m) \qquad (59)$$

and

$$P\ell^{(-,m)} = \exp\left[-i(mK-k\,\sin\,\theta_i)x_P-ik\,\cos\,\theta_m z_P\right]. \qquad (60)$$

The quantities $P\ell^{(-,m)}$ represent plane waves moving in the negative z direction; the waves are homogeneous when

$$(mK - k\,\sin\,\theta_i)^2 \leq k^2 \qquad (61)$$

and inhomogeneous for other values of m. The quantities A_m are then recognized as the reflection coefficients for the plane waves (m^{th} order diffracted waves). The quantities $\cos\,\theta_n$ are defined in Eq. (52); μ_n in Eq. (53).

Before proceeding, it is of interest to show that for a given m the series solution for the reflection coefficients given by Eq. (59) always converges, under the restrictions on $\zeta(x)$ which have already been given. We begin, as usual, by observing from Eq. (59) that

$$|A_m| \leq \frac{1}{|\cos\theta_m|} \sum_{n=-\infty}^{\infty} |\cos\theta_n|\cdot|\mu_{m-n}(\cos\theta_m)|\cdot|\mu_n(\cos\theta_i)|. \qquad (62)$$

Now from Eq. (52) it is easily seen that

$$|\cos\,\theta_n| \leq K'|n| \qquad (63)$$

for $n\neq 0$; K' is some positive constant independent of n. Also upon integrating by parts one sees from Eq. (53) that

$$|\mu_{m-n}(\cos\,\theta_m)| \leq \frac{K''_m}{|n|}, \qquad m\neq n \qquad (64)$$

where the piecewise continuity and the boundedness of $\frac{d\zeta}{dx}$ have been utilized (of course the boundedness follows from the assumption of piecewise continuity if attention is restricted to periodic surfaces as above); K_m'' is some positive constant dependent on m. Finally since $\zeta(x)$ is continuous with a piecewise continuous first derivative and since $\mu_n(\cos\theta_i)$ are essentially the Fourier coefficients of $e^{ik\cos\theta_i\zeta(x)}$ it follows that,[12]

$$\sum_{n=-\infty}^{\infty} \left|\mu_n(\cos\theta_i)\right| \leq K''' \qquad (65)$$

Upon using Eqs. (63), (64), and (65) it is seen that

$$\left|A_m\right| \leq \frac{K'K_m''K'''}{\left|\cos\theta_m\right|} + c_m \qquad (66)$$

which is the desired relation. The positive constant c_m arises from the terms n=0 and n=m.

IV. REFLECTION OF A PLANE WAVE FROM A SINUSOIDAL SURFACE

In order to illustrate the foregoing theory, the reflection of radiation from a sinusoidal surface will be considered. This problem was first treated systematically by Rayleigh.[13] For a normally incident plane wave he reduced the problem to the solution of an infinite system of linear equations. Under the restriction that $\frac{K}{k} \ll 1$ he was able to invert this system.

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12. See R.V. Churchill, Fourier Series and Boundary Value Problems (1st ed;New York:McGraw-Hill Book Co., 1941), pp. 83-85.

13. Lord Rayleigh, Theory of Sound (2nd ed; New York: Dover Publications, 1945), vol. II, p. 89.

(Actually there is another, implicit assumption in Rayleigh's development, the effects of which are difficult to evaluate.[14] The implicit assumption concerns the question of the representation of the reflected field near the reflecting surface in terms of plane waves; one of the restrictions necessary for the validity of Rayleigh's treatment is the additional requirement that the slope of the reflecting surface be small.) The reflection coefficients obtained by Rayleigh are given by (for the first boundary value problem):[15]

$$A_m^{(R)} = -(i)^m J_m(2ka \cos \theta_i) \tag{67}$$

The constant a is defined by the assumption that the reflecting surface is

$$\zeta(x) = a \cos Kx \,. \tag{68}$$

The reflected plane wave, whose coefficient is given by Eq. (67), has a propagation vector whose direction cosine in the z direction is given by Eq. (52) with n replaced by m.

Now let the method presented in this paper be considered. Substituting Eq. (68) in Eq. (53), and using a known integral representation for the Bessel function of order n, one finds,

$$\mu_n(\cos \theta_i) = (i)^n J_n(ka \cos \theta_i) \,. \tag{69}$$

Then using Eq. (69) in connection with Eq. (59), one finds for the reflection coefficients,

14. B.A. Lippmann, J. Opt. Soc. Am. 43, 408 (1953).

15. Actually Rayleigh treated only the case of normal incidence, $\theta_i=0$. He states the result given by Eq. (62) for $m=0$; it is not difficult to extend his method to cover other orders, $m \neq 0$.

$$A_m = -\sum_{n=-\infty}^{\infty} \frac{\cos\theta_n}{\cos\theta_m}(i)^m J_n(ka\cos\theta_i) J_{m-n}(ka\cos\theta_m). \qquad (70)$$

The series is rapidly convergent so long as ka is not too large.

Now, as a check, let the assumption $\frac{K}{k} \ll 1$ be adopted. Then neglecting terms of order $\frac{K}{k}$ (see Eq. (52)), Eq. (70) becomes,

$$A_m^{(0)} = -i^m \sum_{n=-\infty}^{\infty} J_n(ka\cos\theta_i) J_{m-n}(ka\cos\theta_i) , \qquad (71)$$

or upon using a known addition formula for Bessel functions

$$A_m^{(0)} = -(i)^m J_m(2ka\cos\theta_i) , \qquad (72)$$

which is in agreement with Rayleigh's result given by Eq. (67).

V. COMPARISON WITH EXPERIMENT AND WITH THE RAYLEIGH THEORY

Experiments have recently been performed by LaCasce and Tamarkin[16] which can be used as a check on the above theory. In these experiments a directional beam of ultrasonic energy originating in water was allowed to impinge upon a sinusoidally-shaped cork surface which floated on the surface of the water. The amplitude of the reflected radiation as a function of angle was then recorded.

In Figs. 3, 4, and 5 appear some of the data obtained by LaCasce and Tamarkin, these data being compared with calculations

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16. E.O. LaCasce, Jr. and P. Tamarkin, Underwater Sound Scattering from a Corrugated Surface, to be published.

made on the basis of the formulation in the present work. The data presented were obtained by LaCasce and Tamarkin from their 'surface number three', for normal incidence. For the sake of comparison, the corresponding calculations based upon the Rayleigh development are also presented.

For the present purpose it has proved convenient to express the results in terms of energy rather than amplitude. For an infinite surface E_m, the fraction of the total incident energy in the m^{th} order, is related to the amplitude of that order by,[13]

$$E_m = \frac{\cos \theta_m}{\cos \theta_i} \left| A_m \right|^2 . \qquad (73)$$

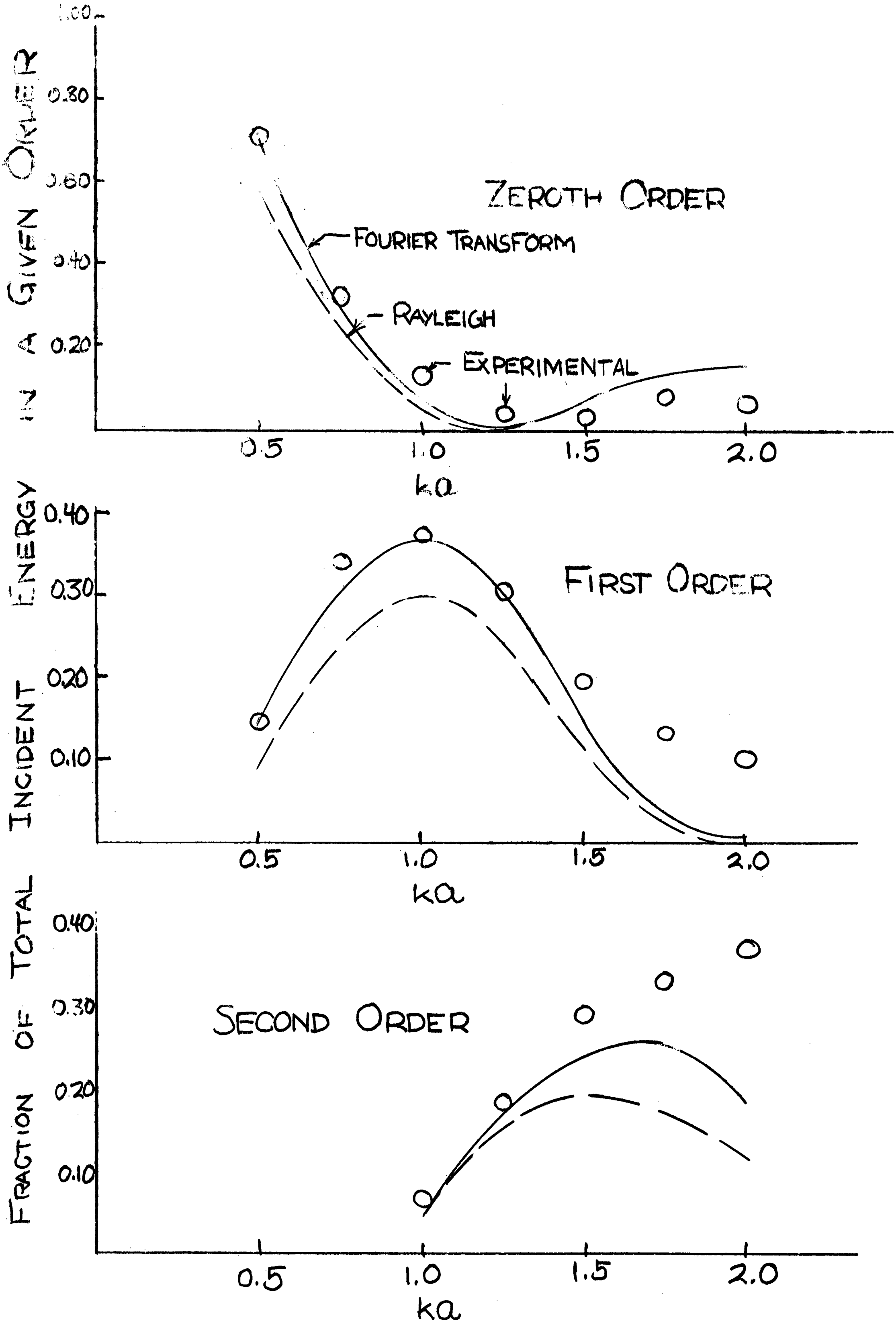
Furthermore from the principle of the conservation of energy one finds the following relation,

$$\cos \theta_i = \sum_m \cos \theta_m \left| A_m \right|^2 , \qquad (74)$$

where the sum runs over real values of $\cos \theta_m$. For a reflecting surface of finite size, or equivalently for a surface radiated by a finite beam, it is possible to obtain $E_m^{(f)}$, the fraction of the incident energy in the m^{th} order for a finite reflecting surface, as follows. Suppose that the reflecting region of surface is of width L; then the m^{th} order will have a diffraction pattern of angular width

$$\Delta \theta_m \sim \frac{\lambda}{L \cos \theta_m} \qquad (75)$$

Suppose that a_m is the maximum amplitude of the m^{th} order diffraction pattern, and that a_{oi} is the maximum amplitude of the diffraction pattern (in the specular direction) when the reflecting surface is replaced by a plane. Then it follows by



Figs. 3, 4, and 5. The figures show the energies calculated and observed in the zeroth, first and second diffracted orders reflected from a sinusoidal surface under the excitation of a normally incident plane wave; aK = 0.47.

integrating the energy contained in the m^{th} order over a large cylinder, whose axis is parallel to the generating element of the reflecting surface, that,

$$E_m^{(f)} = \frac{\cos\theta_i}{\cos\theta_m} \left| \frac{a_m}{a_{io}} \right|^2 \qquad (76)$$

Using again the conservation of energy requirement one sees,

$$\sum_m \frac{\cos\theta_i}{\cos\theta_m} \left| \frac{a_m}{a_{io}} \right|^2 = 1 \, , \qquad (77)$$

the sum being carried over those m for which $\cos\theta_m$ is real.

Returning now to a consideration of the data, the relation (76) was used to determine the relative energy in each order (LaCasce and Tamarkin reported the quantities $\left| \frac{a_m}{a_{io}} \right|$). It is found that their results yield between 1.0 and 2.0 for the left side of Eq. (77); accordingly, the results were divided by the sum on the left hand side of Eq. (77). The adjusted quantities are the plotted experimental values in Figs. 3, 4, and 5.

It is seen from the figures that the agreement between experiment and the Fourier transform theory is good for values of ka less than 1.5. It is to be emphasized that for values of ka above 1.5, orders higher than the second may appear. Since these orders were not reported in the experiments (they are difficult to measure because of the large angles which they make with the normal) it is to be expected that the true normalized experimental values are somewhat lower than those shown in the figures. Furthermore it is seen in the figures that the Rayleigh method gives results which are in error by as much as 20% (referred to the energy in a given order). One finds an energy deficit of twenty-five percent upon summing up the energy carried away by all real orders as calculated using the Rayleigh method. The corresponding deficit for the Fourier transform method is ten percent.

Finally it is remarked that the results of the Rayleigh theory are in many respects almost identical with those obtained from the methods of Brekhovskikh[5] and Eckart[6] (see LaCasce and Tamarkin[16]). For the reasons given above, then, the present method is to be preferred over these other methods as well.

VI. CONCLUSIONS AND ACKNOWLEDGEMENTS

The advantage of the present method over previous methods applicable to the same class of problems lies in the fact that the error incurred through its use is of second order in the slope of the reflecting surface (see assumption (i) of Section II). The error incurred through the use of physical optics is on the other hand of first order in the surface slope.[17] (The error in Rayleigh's method is of the same order as that in physical optics.)

It is a pleasure to acknowledge many helpful discussions with Dr. David Mintzer during the progress of this work. The author wishes also to acknowledge the help of Mr. C.H. Church in the performance of some of the calculations.

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17. W.C. Meecham, On the Use of the Kirchhoff Approximation for the Solution of Reflection Problems, to be published in the Journal of Rational Mechanics.

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