ENGINEERING RESEARCH INSTITUTE

UNIVERSITY OF MICHIGAN, ANN ARBOR

ON THE USE OF THE KIRCHHOFF APPROXIMATION FOR THE SOLUTION OF REFLECTION PROBLEMS

William C. Meecham

Office of Naval Research, U.S. Navy Department
Contract N6onr-23221, ONR Project NR 385-203

November 1955

ON THE USE OF THE KIRCHHOFF APPROXIMATION FOR THE SOLUTION OF REFLECTION PROBLEMS

William C. Meecham
Physics Department
University of Michigan, Ann Arbor, Michigan

I. INTRODUCTION

In the past there has been a considerable amount of work done on the problem of the reflection of radiation from non-plane (frequently termed irregular) surfaces. Of the various approximations used in approaching this problem, perhaps the most popular is the one due to Kirchhoff [1]. (For work using this approximation see references [2], [3], and [6].) It is assumed in this approximation that the field near every region of the surface is essentially what it would have been if the surface had been flat with a slope equal to that of the irregular surface at the point in question. The approximation has been assumed to be useful when the surface variations are in some sense large compared with the wave length of the incident radiation. Using this assumption concerning the value of the field at the bounding surface in conjunction with the Helmholtz formula (see below) it is possible to obtain an estimate of the field in regions removed from the reflecting surface. Despite the prevalence of the Kirchhoff approximation in the treatment of the problem of the reflection of radiation from an irregular surface there does not exist, to the author's knowledge, a systematic derivation of the approximation which demonstrates the way in which it may be corrected, nor for that matter, which shows conclusively the size of the errors incurred through its use. For this reason it is hoped that the derivation in the present paper may prove of some interest.

For simplicity consideration will be restricted to two dimensional problems in which the field function is assumed to vanish at the irregular surface (in acoustical work, a pressure release surface). It is a simple matter to extend the arguments to cover the case where the normal derivative of the field vanishes at the surface as well as to extend them to cover analogous three dimensional (scalar) problems. One wishes to find then a function $\Phi(x,z,t)$ which satisfies the wave equation in two dimensions,

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} \quad , \qquad (1.1)$$

at all points except the source point (x_0, z_0) of the field; c is the phase velocity, and is assumed to be constant. In acoustics, the function Φ represents the velocity potential,

$$\vec{v} = -\nabla\Phi \qquad (1.2)$$

where $\vec{v}(x,z,t)$ is the velocity at (x,z,t).

It will be supposed that the source radiates a single (angular) frequency ω, so that one can write

$$\Phi(x,z,t) = e^{-i\omega t}\phi(x,z) \quad . \qquad (1.3)$$

Substituting Eq. (1.3) in Eq. (1.1) one obtains,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2\right]\phi(x,z) = 0 \quad , \qquad (1.4)$$

with $k = \frac{\omega}{c}$; the function ϕ is to satisfy Eq. (1.4) at all points except at (x_0, z_0). Near this point the field is to behave like

$$\phi(x,z) \approx H_0^{(1)}(kr) \qquad (1.5)$$

where $r = \left[(x - x_0)^2 + (z - z_0)^2\right]^{1/2}$.

For three dimensional problems the Hankel function is replaced by $\frac{e^{ikr}}{r}$. To continue, it is supposed that at the surface $\zeta(x)$, the function ϕ vanishes,

$$\phi(x, \zeta(x)) = 0 . \tag{1.6}$$

The function $\zeta(x)$ is assumed to be continuous, single-valued, and bounded with continuous first and second derivatives; the function is defined for all values of x. Finally let ϕ_i be the value which the field would assume if ζ were not present; in our case $\phi_i = H_0^{(1)}(kr)$. The Kirchhoff approximation consists of supposing that $\frac{\partial \phi}{\partial \nu} \approx 2\frac{\partial \phi_i}{\partial \nu}$ on the boundary, where $\frac{\partial}{\partial \nu}$ is the outward-directed derivative normal to $\zeta(x)$. It is supposed in this work that k is complex, that is that the medium is slightly absorptive. Comment will be made upon this assumption at the end of Section 2 (following Eq. (2.35)). We now proceed with the derivation of the approximation.

II. DERIVATION OF THE KIRCHHOFF APPROXIMATION

In the treatment to be presented here we shall attempt to develop the Kirchhoff approximation in a systematic way. The Helmholtz formula is considered in connection with the determination of the normal derivative of the field function on the boundary; an integral equation is obtained. The solution by iteration of this integral equation will then be considered. It will appear that using the first term is equivalent to adopting the Kirchhoff approximation. We then investigate the next term of the iteration solution, in particular observing that it has the order property,

$$|I| \leq M' \left|\frac{d\zeta}{dx}^M\right| + M'' \frac{1}{|k|R_m} . \tag{2.1}$$

Here I is the first term neglected in the solution; R_m is the minimum radius of curvature and $\frac{d\zeta}{dx}^M$ is the maximum slope of $\zeta(x)$; M' and M" are constants independent of surface properties. On the basis of this property, it seems reasonable to suppose that the Kirchhoff approximation is valid if --

i) $\left|\frac{d\zeta^M}{dx}\right| \ll 1$,

and ii) $|k| R_m \gg 1$.

We proceed with the outlined program. First the Weber two-dimensional analogue of the Helmholtz formula is needed. By using Green's formula in connection with Eq. (1.4), one is able to derive the following result (see Fig. 1) [1] :

$$\phi(P) = \phi_i(P) + \frac{1}{4i} \int_{\zeta} \left\{ \phi(1) \frac{\partial}{\partial \gamma_1} H_0^{(1)}(kr_{1P}) - H_0^{(1)}(kr_{1P}) \frac{\partial}{\partial \gamma_1} \phi(1) \right\} ds_1 , \tag{2.2}$$

subject to the restrictions that --

a. The function $\phi(x,z)$ possesses a singularity of the type

$$\phi(x,z) \rightarrow H_0^{(1)}(kr_{PO}) , \qquad \text{as } r_{PO} \rightarrow 0;$$

(x_0,z_0) is the source point and $r_{PO} = \left[(x-x_0)^2 + (z-z_0)^2\right]^{1/2}$.

b. The function $\phi(x,z)$ is continuous with continuous first and second order partial derivatives for all points (x,z) satisfying the condition $z \geq \zeta(x)$, with the exception of the source point.

c. The function $\phi(x,z)$ shall be of order $e^{-\alpha r}$ as $r \rightarrow \infty$, where $r = (x^2 + z^2)^{1/2}$. The constant α is the imaginary part of k defined by

$$k = k_0 + i\alpha , \qquad \alpha > 0 . \tag{2.3}$$

The integral in Eq. (2.2) is to be carried over the entire surface $\zeta(x)$. The symbol $\frac{\partial}{\partial \gamma_1}$ indicates the derivative with respect to the outward normal direction at the surface point (1). We have let $\phi(P)$ stand for the value of the function ϕ at some space point P and similarly for other functions appearing in Eq. (2.2) and later. We now use the boundary condition given by Eq. (1.6); Eq. (2.2) becomes

$$\phi(P) = \phi_i(P) - \frac{1}{4i} \int_{\zeta} H_0^{(1)}(kr_{1P}) \frac{\partial}{\partial \gamma_1} \phi(1)\, ds_1 \quad . \tag{2.4}$$

If we allow P to approach the surface at the point (2), we see

$$\phi_i(2) = \frac{1}{4i} \int_{\zeta} H_0^{(1)}(kr_{12}) \frac{\partial}{\partial \gamma_1} \phi(1)\, ds_1 \quad . \tag{2.5}$$

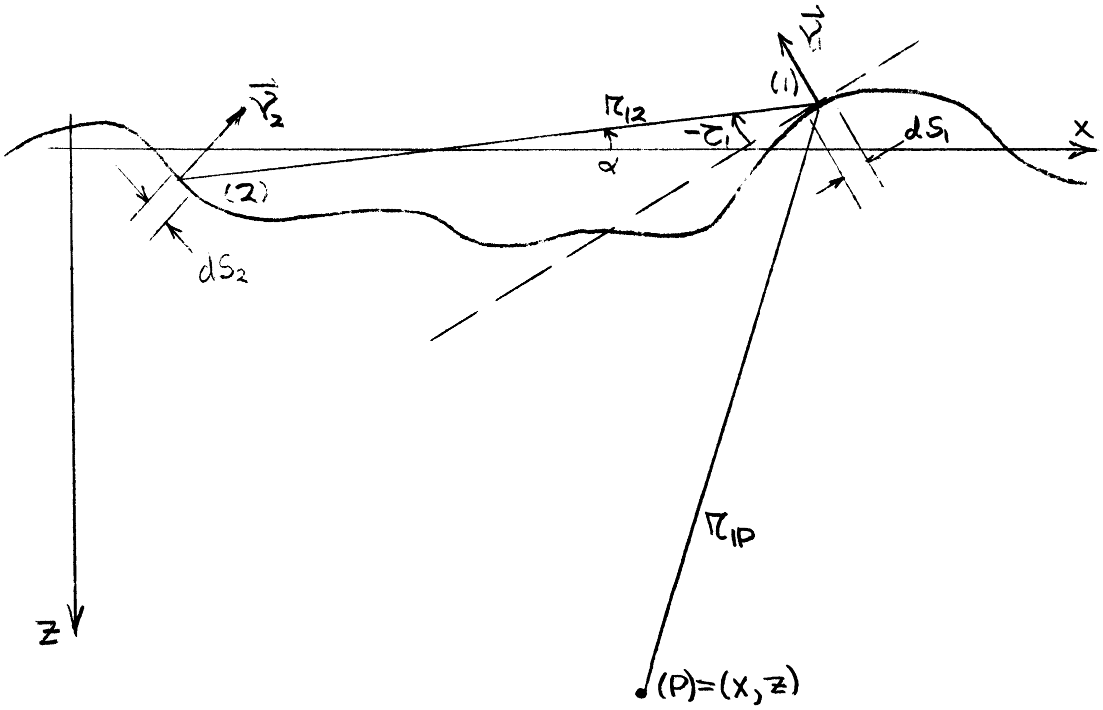

Fig. 1. Diagram used in the discussion of the Helmholtz formula.

Once the function $\frac{\partial \phi(1)}{\partial \gamma_1}$ is known, the solution to the problem can be obtained from Eq. (2.4). In order to obtain an estimate for this function we set up an integral equation of the second kind and propose a solution by iteration.[1]

We take the derivative in some direction, $\frac{\partial}{\partial \mu_P}$ of Eq. (2.4); the point (P) is then allowed to approach the surface along the normal at the point (2); at the same time the derivative $\frac{\partial}{\partial \mu_P}$ approaches $\frac{\partial}{\partial \gamma_2}$. One obtains

---

1. The integral equation to be developed here (Eq. (2.16)) has been obtained previously by Maue [8] for both boundary conditions for the three-dimensional problem. It is repeated here in order to show the detailed development for the two-dimensional case. For the use of integral equations in the treatment of diffraction problems one can also refer to references [4] and [7].

$$\frac{\partial \phi(2)}{\partial \nu_2} = \frac{\partial \phi_i(2)}{\partial \nu_2} - \frac{1}{4i} \lim_{(P)\to(2)} \int_{\zeta} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \frac{\partial \phi(1)}{\partial \nu_1} ds_1 . \tag{2.6}$$

That it is legitimate to differentiate under the integral sign for $r_{P2} \geq \delta > 0$ may be seen in the following way. From the asymptotic formula for the Hankel function one has,

$$H_0^{(1)}(y) \approx \left(\frac{2}{\pi y}\right)^{1/2} e^{i(y)} \qquad |y| \gg 1 . \tag{2.7}$$

Using this representation together with the radiation condition, hypothesis c) above, it is evident that the integrand of Eq. (2.6) is less in absolute value than $Me^{-2\alpha|x_1-x_2|}$ for some M so long as $r_{P2} \geq \delta > 0$. It follows that the integral is uniformly convergent; furthermore $\frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P}$ is continuous for $r_{P2} \geq \delta$, as is $\frac{\partial \phi}{\partial \nu}$ by hypothesis for $z \geq \zeta(x)$. Hence the interchange of differentiation and integration is justified.

The quantity $\frac{\partial H_0^{(1)}}{\partial \mu_P}$ in the integrand represents a distribution of dipoles. It is a well known result of potential theory that limiting processes such as the one in Eq. (2.6) introduce integrable singularities in the integrand. In order to evaluate the limit, the integral of Eq. (2.6) is split up as follows:

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{\zeta} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \frac{\partial \phi(1)}{\partial \nu_1} ds_1 =$$

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12} \geq \xi_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \frac{\partial \phi(1)}{\partial \nu_1} ds_1$$

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12} < \xi_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \frac{\partial \phi(1)}{\partial \nu_1} ds_1 . \tag{2.8}$$

The limit in the first term on the right side of Eq. (2.8) can be moved inside the integral sign. Then we are left with the task of evaluating the second term on the right side of Eq. (2.8). For convenience we write

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12}\leq\xi_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial\mu_P} \frac{\partial\phi(1)}{\partial\nu_1} ds_1 =$$

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12}<\xi_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial\mu_P} \frac{\partial\phi(2)}{\partial\nu_2} ds_1$$

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12}<\xi_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial\mu_P} \left(\frac{\partial\phi(1)}{\partial\nu_1} - \frac{\partial\phi(2)}{\partial\nu_2}\right) ds_1 . \quad (2.9)$$

The Hankel function is defined by,

$$H_0^{(1)}(y) = J_0(y) + \frac{2i}{\pi}\left[\gamma + \ln\left(\frac{1}{2}y\right)\right] J_0(y)$$

$$- \frac{2i}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m\left(\frac{1}{2}y\right)^{2m}}{(m!)^2} \left\{\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m}\right\} , \quad (2.10)$$

or

$$H_0^{(1)}(y) = \frac{2i}{\pi} \ln(y)\, J_0(y) + G(y) \quad (2.11)$$

where $G(y)$ is analytic at $y = 0$, and γ is Euler's constant, 0.5772... , Using the assumption that $\zeta(x)$ has a continuous derivative, together with the expression for the Hankel function for small values of its argument, one obtains the following result:

$$\lim_{\xi_0\to 0}\left\{- \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12}<\xi_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial\mu_P} \frac{\partial\phi(2)}{\partial\nu_2} ds_1\right\} = \frac{1}{2}\frac{\partial\phi(2)}{\partial\nu_2} , \quad (2.12)$$

To see this, ξ_0 is chosen small enough so that the region of surface ($r_{12} \leq \xi_0$) is essentially flat. Then using Eq. (2.11), Eq. (2.12) follows directly. Also since by hypothesis $\frac{\partial\phi}{\partial\nu}$ is continuous, we see that

$$\lim_{\xi_c \to 0} \left\{ -\frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12}<\xi_c} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \left(\frac{\partial \phi(1)}{\partial \nu_1} - \frac{\partial \phi(2)}{\partial \nu_2} \right) ds_1 \right. = 0. \tag{2.13}$$

Now allowing $\xi_0 \to 0$ in Eq. (2.8) and substituting in Eq. (2.6) we find,

$$\frac{\partial \phi(2)}{\partial \nu_2} = \frac{\partial \phi_i(2)}{\partial \nu_2} + \frac{1}{2} \frac{\partial \phi(2)}{\partial \nu_2} - \frac{1}{4i} \int_{\zeta} \frac{\partial H_0^{(1)}(kr_{12})}{\partial \nu_2} \frac{\partial \phi(1)}{\partial \nu_1} ds_1 . \tag{2.14}$$

Equivalently Eq. (2.14) may be written,

$$\frac{\partial \phi(2)}{\partial \nu_2} = 2\frac{\partial \phi_i(2)}{\partial \nu_2} - \frac{1}{2i} \int_{\zeta} \frac{\partial H_0^{(1)}(kr_{12})}{\partial \nu_2} \frac{\partial \phi(1)}{\partial \nu_1} ds_1 . \tag{2.15}$$

Now we consider the solution of Eq. (2.15) by iteration. The procedure is as follows [6]: one substitutes the right side of Eq. (2.15) for the quantity $\frac{\partial \phi}{\partial \nu}$ appearing in the integrand. This again gives the quantity $\frac{\partial \phi}{\partial \nu}$ appearing on the right side, now under a double integral. By repeating the substitution process indefinitely, one generates an infinite series of terms in which each term involves, in general, a multiple integral with the known function appearing in the integrand. This process then yields the series,

$$\frac{\partial \phi(2)}{\partial \nu_2} = 2\frac{\partial \phi_i(2)}{\partial \nu_2} - \frac{1}{2i} \int_{\zeta} \frac{\partial H_0^{(1)}(kr_{12})}{\partial \nu_2}\, 2\frac{\partial \phi_i(1)}{\partial \nu_1} ds_1$$

$$- \frac{1}{2i} \int_{\zeta} \frac{\partial H_0^{(1)}(kr_{12})}{\partial \nu_1} \left\{ -\frac{1}{2i} \int_{\zeta} \frac{\partial H_0^{(1)}(kr_{13})}{\partial \nu_1} \left[2\frac{\partial \phi_i(3)}{\partial \nu_3} \right] ds_3 \right\} ds_1 , \tag{2.16}$$

Under the assumptions on $\zeta(x)$ and using Eq. (2.10) it is not difficult to show that $\frac{\partial H_0^{(1)}(kr_{12})}{\partial \nu_2}$ is continuous in x_1 and x_2 with the exception of a removable singularity at $x_1 = x_2$. Then from Eqs. (2.3) and (2.7) it is seen that each integral in each term of the series given in Eq. (2.16) is absolutely and uniformly convergent and moreover that each term is continuous. Finally it will be seen below that for $\left|\frac{d\zeta^M}{dx}\right|$ sufficiently small and for $|k|R^M$ sufficiently large, the series which forms the iteration solution is absolutely and uniformly convergent. Then one sees that Eq. (2.16) is a solution to Eq. (2.15). The series can be integrated term by term since each term is continuous and the series is uniformly convergent. It also follows that under the given conditions the series represents a continuous function of x_2. Furthermore, it is not difficult to show that the iteration solution when it converges is the only continuous solution of Eq. (2.15). We now retain only the first term of Eq. (2.16) assuming for the moment that the remainder of the series is negligible in comparison to it; this gives,

$$\frac{\partial \phi(2)}{\partial \nu_2} \approx 2\,\frac{\partial \phi_i(2)}{\partial \nu_2} \quad . \qquad (2.17)$$

As a check on the result, it is not difficult to see that for $\zeta(x) = 0$ (a flat surface), Eq. (2.17) is exact.

To complete the solution, Eq. (2.17) is substituted in Eq. (2.4) to obtain

$$\phi^{(1)}(P) = \phi_i(P) - \frac{1}{4i}\int_{\zeta} H_0^{(1)}(kr_{1P})\; 2\,\frac{\partial \phi_i(1)}{\partial \nu_1}\, ds_1 \quad , \qquad (2.18)$$

when $\phi^{(1)}(P)$ represents the first approximation to the field ϕ.

We shall now consider the first term neglected in the approximation represented by Eq. (2.17). We make use of the relation

$$\frac{d}{dy} H_0^{(1)}(y) = - H_1^{(1)}(y) \quad , \qquad (2.19)$$

and also

$$\frac{\partial r_{12}}{\partial \nu_1} = \sin \tau_1 \quad , \qquad (2.20)$$

with τ_1 defined as the angle between the tangent to the surface at the point (1) and the radius vector connecting (1) and (2) (see Fig. 1). Then we have for the second term of Eq. (2.16)

$$-\frac{1}{2i}\int_{\varsigma}\frac{\partial H_0^{(1)}(kr_{12})}{\partial \nu_2}\left[2\,\frac{\partial \phi_i(1)}{\partial \nu_1}\right] ds_1 = \int_{\varsigma} H_1^{(1)}(kr_{12})k\,\sin\tau_2\frac{\partial \phi_i(1)}{\partial \nu_1}\,ds_1\,. \tag{2.21}$$

We shall label this quantity I; it must be small if Eq. (2.17) is to be valid.

We need the expression for the function $H_1^{(1)}$ corresponding to Eq. (2.10):

$$H_1^{(1)}(y) = J_1(y) + \frac{i}{\pi}\left\{2\left[\gamma + \ln\frac{1}{2}y\right] J_1(y) - \frac{2}{y} - \sum_{m=0}^{\infty}\frac{(-1)^m\left(\frac{1}{2}y\right)^{1+2m}}{m!\,(m+1)!}\left[\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{m}+\frac{1}{1}+\frac{1}{2}+\ldots+\frac{1}{1+m}\right]\right\}, \tag{2.22}$$

From Eq. (2.22) it follows that for $kr_{12} \leq 1$,

$$\left|H_1^{(1)}(y)\right| \leq \frac{m_H}{|y|}\,. \tag{2.23}$$

Using the asymptotic expansion for the function $H_1^{(1)}$,

$$H_1^{(1)}(y) = \left(\frac{2}{\pi y}\right)^{1/2} e^{i(y-\frac{3}{4}\pi)}\left[1 + 0\left(\frac{1}{y}\right)\right] \tag{2.24}$$

one finds that for $kr_{12} \geq 1$,

$$\left|H_1^{(1)}(kr_{12})\right| \leq M_H e^{-\alpha r_{12}}\,. \tag{2.25}$$

Finally from the definition of ϕ_i; it follows that,

$$\left|\frac{\partial \phi_i}{\partial \nu}\right| \leq |k| M_i\,. \tag{2.26}$$

In these expressions m_H, M_H, and M_i are all of order unity.

With these definitions in mind, we proceed with the consideration of the function I. We examine separately the regions of the integrand of Eq. (2.21) where $|k|r_{12} \leq 1$ and where $|k|r_{12} > 1$. We have

$$I = I_1 + I_2 , \qquad (2.27)$$

where

$$I_1 = \frac{1}{i} \int_{|k|r_{12} \leq 1} H_1^{(1)}(kr_{12}) \sin \tau_2 \frac{\partial \phi_i(1)}{\partial \gamma_1} \, kds_1 , \qquad (2.28)$$

and

$$I_2 = \frac{1}{i} \int_{|k|r_{12} > 1} H_1^{(1)}(kr_{12}) \sin \tau_2 \frac{\partial \phi_i(1)}{\partial \gamma_1} \, kds_1 . \qquad (2.29)$$

Let I_2 be considered first. To begin we note that,

$$\left| \frac{\zeta_2 - \zeta_1}{x_2 - x_1} \right| \leq \left| \frac{d\zeta^M}{dx} \right| , \qquad (2.30)$$

since the quantity on the left is just the average slope of the segment of surface between the points (1) and (2). By examining Fig. 1, and using the result given by the relation (2.30), it is easily seen that

$$\left| \sin \tau_2 \right| \leq 2 \left| \frac{d\zeta^M}{dx} \right| . \qquad (2.31)$$

Upon using Eqs. (2.25), (2.31), and (2.26) one then finds without difficulty that

$$\left| I_2 \right| \leq |k| \, M_i \cdot 4M_H \left[1 + \left(\frac{d\zeta^M}{dx} \right)^2 \right]^{1/2} \frac{|k|}{\alpha} \left| \frac{d\zeta^M}{dx} \right| . \qquad (2.32)$$

We now proceed by considering Eq. (2.28). The following relation is easily established (cf. Fig. 1):

$$\left| \sin \tau_2 \right| \leq \frac{1}{2} \frac{r_{12}}{R_m} , \qquad (2.33)$$

when R_m is the minimum radius of curvature. One then finds upon using Eqs. (2.23), (2.33), and (2.26) that,

$$\left| I_1 \right| \leq |k| \, M_i m_H \left[1 + \left(\frac{d\zeta^M}{dx} \right)^2 \right]^{1/2} \frac{1}{|k| R_m} . \qquad (2.34)$$

Hence from Eq. (2.27) it follows that,

$$|I| \leq |k| M_i \left[1 + \left(\frac{d\xi^M}{dx}\right)^2\right]^{1/2} \left\{4M_H \frac{|k|}{\alpha} \left|\frac{d\xi^M}{dx}\right| + \frac{m_H}{|k| R_m}\right\} . \tag{2.35}$$

It is noted here that as one considers media of smaller attenuation (α), one is also forced to restrict consideration to surfaces with smaller slope in order to guarantee a given bound on the error using the present development. In order to remove this difficulty, one must deal with conditionally convergent integrals and in particular consider the question of whether or not they are uniformly convergent. To continue, one sees upon examining Eq. (2.16) that,

$$\left|\frac{\partial \phi(2)}{\partial \gamma_2}\right| \leq 2 |k| M_i \left\{1 + s + s^2 + \ldots\right\} \tag{2.36}$$

where

$$s = \frac{1}{2} \left(1 + \left(\frac{d\xi^M}{dx}\right)^2\right)^{1/2} \left\{4M_H \frac{|k|}{\alpha} \left|\frac{d\xi^M}{dx}\right| + \frac{m_H}{|k| R_m}\right\} \quad ; \tag{2.37}$$

the series in Eq. (2.36) converges for

$$s < 1 .$$

This proves the assertion made above that for $\left|\frac{d\xi^M}{dx}\right|$ sufficiently small and for $|k| R_m$ sufficiently large, the series in Eq. (2.16) converges absolutely and uniformly.

It is worth commenting that for $|k| R_m \ll 1$ the terms neglected in the approximation represented by Eq. (2.17) are of the same order as the terms retained for regions of the surface where the radius of curvature is small.

III. THE RECIPROCITY THEOREM

We now show that the approximate result, expressed by Eq. (2.18), satisfies the reciprocity theorem if there is but one source, i.e.,

$$\phi_i(P) = H_0^{(1)}(kr_{PO}) \quad . \tag{3.1}$$

A general transformation of Eq. (2.18) is first shown. We consider a system having sources giving a field ϕ_i without the bounding surface $\zeta(x)$, so that the sources radiate in an infinite homogeneous medium. Then applying Eq. (2.2)

$$\phi_i(P) = \phi_i(P) + \frac{1}{4i} \int_\zeta \left\{ \phi_i(1) \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \nu_1} - H_0^{(1)}(kr_{1P}) \frac{\partial \phi_i(1)}{\partial \nu_1} \right\} ds_1 \quad , \tag{3.2}$$

or

$$-\frac{1}{4i} \int_\zeta H_0^{(1)}(kr_{1P}) \frac{\partial \phi_i(1)}{\partial \nu_1} ds_1 = -\frac{1}{4i} \int_\zeta \phi\,(1) \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \nu_1} ds_1 \ . \tag{3.3}$$

Now Eq. (3.3) is used in Eq. (2.18) to obtain,

$$\phi^{(1)}(P) = \phi_i(P) - \frac{1}{4i} \int_\zeta \frac{\partial}{\partial \nu_1} (H_0^{(1)}(kr_{1P})\, \phi_i(1))\, ds_1 \quad . \tag{3.4}$$

It is evident that if we specialize the problem by applying Eq. (3.1), Eq. (3.4) is unchanged by the interchange of (0) and (P), for we have,

$$\phi^{(1)}(P) = H_0^{(1)}(kr_{PO}) - \frac{1}{4i} \int_\zeta \frac{\partial}{\partial \nu_1} (H_0^{(1)}(kr_{1P}) H_0^{(1)}(kr_{10}))\, ds_1 \quad . \tag{3.5}$$

Thus when a single source excites the field, Eq. (2.18) satisfies the reciprocity theorem.

We now examine the value of the approximate result, given by Eq. (2.18), on the boundary. From the boundary condition, Eq. (1.6), the function $\phi^{(1)}(2)$ should vanish. Subtracting Eq. (2.18) from Eq. (2.4) we see,

$$\phi(P) - \phi^{(1)}(P) = -\frac{1}{4i} \int_\zeta H_0^{(1)}(kr_{1P}) \left[\frac{\partial \phi(1)}{\partial \nu_1} - 2\frac{\partial \phi_i(1)}{\partial \nu_1} \right] ds_1 \ . \tag{3.6}$$

When (P) lies on the boundary at the point (2), Eq. (3.6) becomes,

$$-\phi^{(1)}(2) = -\frac{1}{4i} \int_\zeta H_0^{(1)}(kr_{12}) \left[\frac{\partial \phi(1)}{\partial \nu_1} - 2\frac{\partial \phi_i(1)}{\partial \nu_1} \right] ds_1 \quad . \tag{3.7}$$

Hence if the error in the determination of the value of $\frac{\partial \phi}{\partial \nu}$ on the boundary is made small by hypotheses i) and ii), one may expect the value of the approximation given by Eq. (2.18) to be small there, and indeed to be of the same order.

IV. CONCLUSIONS AND ACKNOWLEDGEMENT

It is shown that the Kirchhoff approximation is really the first term in the solution by iteration of an integral equation governing the value of the normal derivative of the field function at the surface. The neglect of the remaining terms of the solution is seen to be justified for $\left|\frac{d\zeta^M}{dx}\right|$ sufficiently small and for $|k|R_m$ sufficiently large. Correction terms for the approximation are given. Finally it is shown that the approximation to the solution obtained from the Kirchhoff assumption satisfies the reciprocity theorem.

There is one further point of interest in connection with the present approximation. It is easily seen from Eqs. (2.21) and (2.31) that the first term neglected in the approximation contains a term of first order in the surface slope. It seems reasonable to suppose that the effect of this error would be greatest in regions where the reflected radiation is far removed from the specular direction. Brekhovskikh, cited in reference [2], finds such deviations in comparing calculations using the present approximation with an exact formulation due to Rayleigh.

It is a pleasure to acknowledge the helpful criticism of Dr. David Mintzer and of Dr. C.L. Dolph on several points concerning this work.

BIBLIOGRAPHY

1. Baker, B.B., and Copson, E.T., The Mathematical Theory of Huygens' Principle, (2nd ed.; London: Oxford University Press, 1950), Chap. II.

2. Brekhovskikh, L.M., Zh. Exsper, Teor. Fiz. (USSR) 23, 275, and 289 (1952) translated by G.N. Goss, U.S. Navy Electronics Laboratory, San Diego, California.

3. Eckart, C., J. Acoust. Soc. Am. 25, 566 (1953).

4. Franz, W., Proc. Phys. Soc. Lond. A, 63, 925 (1950).

5. Lovitt, W.V., Linear Integral Equations (New York: McGraw-Hill Book Company, 1924), Chap. II.

6. Rowland, H.A., Phil. Mag. 35, 397 (1893).

7. Dolph, C.L., Memorandum, University of Michigan, EMF-26, Willow Run Research Center (1951).

8. Maue, A-W., Z. Phys. 126, 601 (1949).

DISTRIBUTION LIST

Chief of Naval Research
Undersea Warfare Branch (Code 466)
Washington 25, D.C. (1)

Director
Naval Research Laboratory
Technical Information Officer
Washington 25, D.C. (6)

Commanding Officer
Office of Naval Research Branch Office
Box 39, Navy 100, c/o Fleet Post Office
New York, New York (2)

Commanding Officer
Office of Naval Research Branch Office
10th Floor
The John Crerar Library Building
86 East Randolph Street
Chicago 1, Illinois (1)

Chief, Bureau of Ships
Code 847, Navy Department
Washington 25, D.C. (1)

Office of Naval Research
Acoustics Branch (Code 411)
Washington 25, D.C. (2)

Director
Woods Hole Oceanographic Institution
Woods Hole, Massachusetts (1)
Attn: Dr. C.B. Officer

Director
Marine Physical Laboratory
University of California
U.S. Navy Electronics Laboratory
San Diego 52, California (1)

Commanding Officer and Director
U S. Navy Underwater Sound Laboratory
Fort Trumbull
New London, Connecticut (1)
Attn: Dr. H.W. Marsh

Director
Naval Ordnance Laboratory
White Oak, Maryland
Attn: Sound Division

Commanding Officer and Director
U S. Navy Electronics Laboratory
San Diego 52, California (1)

Superintendent
U.S. Navy Postgraduate School
Physics Department
Monterey, California (1)
Attn: Prof. L.E. Kinsler

Commander
Naval Air Development Center
Johnsville, Pennsylvania (1)

Hudson Laboratories
Columbia University
145 Palisades Street
Dobbs Ferry, New York (1)
Attn: Dr. I. Tolstoy

Massachusetts Institute of Technology
Acoustics Laboratory
Cambridge 39, Massachusetts (1)

Lamont Geological Observatory
Columbia University
Torre Cliffs
Palisades, New York (1)

NRC Undersea Warfare Committee
Navy Department
Washington 25, D.C. (1)

Catholic University of America
Department of Physics
Washington 17, D.C. (1)

Harvard University
Acoustics Laboratory
Division of Applied Science
Cambridge 38, Massachusetts (1)

Office of Technical Services
Department of Commerce
Washington, D.C. (1)

Yale University
Edwards Street Laboratory
New Haven, Connecticut (1)

DISTRIBUTION LIST (cont.)

Commanding Officer
Office of Naval Research Branch Office
1030 E. Green Street
Pasadena 1, California (1)

Commanding Officer
Office of Naval Research Branch Office
346 Broadway
New York 13, New York (1)

Armed Services Technical Information Agency
Document Service Center
Knott Building
Dayton 2, Ohio (5)

Director, Ordnance Research Laboratory
Post Office Box 30
Pennsylvania State University
State College, Pennsylvania (1)
Attn: Mr. R.J. Urick

University of Texas
Box 1, Defense Research Laboratory
Austin, Texas (1)

David Taylor Model Basin
Washington 25, D.C. (1)

Director
Scripps Oceanographic Institute
La Jolla, California (1)
Attn: Dr. C. Eckart

Applied Physics Laboratory
Johns Hopkins University
Baltimore, Maryland (1)

Director
Underwater Sound Reference Laboratory
Orlando, Florida (1)

Bureau of Standards
Washington 25, D C. (1)
Attn: Mr. Cook

Marine Laboratory
University of Miami
Coral Gables, Florida (1)

Operations Evaluation Group (Op-374)
Navy Department
Washington 25, D.C.

U.S. Naval Academy
Physics Department
Annapolis, Maryland (1)

Dr. R. Mittra
Antenna Laboratory
University of Toronto
Toronto 5, Canada (1)

Dr. C.L. Pekeris
Weizmann Institute
Rehovst, Isreal. (1)

Physics Department
Brown University
Providence 12, Rhode Island (1)