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A STATISTICAL MODEL FOR THE PROPAGATION OF RADIATION IN REFRACTION DUCTS BOUNDED BY ROUGH SURFACES

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PREFACE

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My wife, Barbara Brown Meecham, has assisted in countless ways, among them being the performance of certain of the calculations appearing in this work.

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Abstract

In certain physical problems involving the propagation of acoustic (or electromagnetic) radiation in inhomogeneous media near a bounding surface, conditions are such as to set up a "surface-bounded refraction duct". Such a duct occurs, for example, when the phase velocity of the radiation depends only on the distance from the bounding surface and when it depends upon this distance in such a way that the velocity increases to some finite distance and decreases thereafter. In such a case some of any radiation originating near the bounding surface will be refracted by the medium and subsequently be reflected from the surface; in this way some radiation is trapped by the duct and may propagate to large distances by the combined refraction and reflection process. Exact treatment of this problem is possible (using the "theory of normal modes") when the bounding surface is sufficiently smooth so that little error is made in assuming it plane.

When the bounding surface is so rough that most radiation impinging upon it is diffusely scattered, it is shown that it is possible to treat the problem through the use of a statistical model. The model takes the form of an integro-difference equation which is derived by considering in detail the average amount of energy reflected by an element of surface per second per element of angle. The basic equation

is derived from the wave equation after applying a number of restricting assumptions. The three most important of these are: a) it is assumed that the frequency is high enough or the surface rough enough so that radiation is reflected from the surface primarily diffusely; b) it is supposed that the properties of the material medium vary slowly enough so that one may use geometrical optics to treat the propagation of radiation within the volume; and c) it is required that the surface be such that one can treat an individual reflection event through the use of physical (or geometrical) optics. Criteria for the validity of these assumptions are given.

Two methods of solution of the basic integro-difference equation are presented. The first of these is a solution by iteration, while the second depends upon taking the Laplace transform of the equation. Existence, uniqueness, and continuity properties of the solution are shown. Finally a numerical example is treated, and plots of the field strength versus range are presented for various positions of the source and receiver. It is found that the attenuation of the field depends strongly upon the fraction of energy trapped within the duct after a surface reflection.

A review of the more important methods useful in the treatment of the reflection of radiation from non-plane surfaces is given, the results being used extensively in the derivation of the model equation from the wave equation. First the method of geometrical optics is reviewed; then a perturbation method, due to Rayleigh, is presented. Next the method of physical optics (depending upon the Kirchhoff assumption) is presented by treating an integral equation derived from the Helmholtz formula; it is shown that the usual result of physical optics is obtained from the first term in the solution by iteration of the integral equation while the next term may be used to obtain the criteria of applicability for the method. Following this review a new method depending upon the use of a Fourier transform in connection with a certain approximation to the Helmholtz formula is presented. This last method is applicable when the slope of the reflecting surface is small and when the wavelength of the radiation is greater than, or of the same order of magnitude as, the displacement of the surface from its average value.

CHAPTER I

INTRODUCTION

This thesis is concerned with certain aspects of the propagation of radiation (either acoustic or electromagnetic) in a half-space of inhomogeneous material bounded by an irregular surface. It is assumed that the properties of the material medium at a given space point depend only upon the distance from that point to the bounding surface. When the velocity of propagation increases with distance from the surface over some finite region, some fraction of any radiation originating at or near the surface is refracted back to the surface. In such a case a surface-bounded duct is said to have been set up. The radiation refracted back to the surface is said to be trapped. If no energy is lost from the duct, the intensity of the field within the duct will decrease like $\frac{1}{r}$ where r is the distance between the source and the field point; this is to be compared with the $\frac{1}{r^2}$ dependence for the corresponding propagation within a homogeneous medium with no bounding surface. There are however certain ways in which energy is lost from the duct. Among these may be listed losses due to diffraction out of the duct and losses due to the scattering of energy out of the duct by the surface. This work is concerned with the latter loss mechanism.

The problem of duct propagation has evoked a considerable amount of interest on the part of physicists, largely as a result of the many practical applications which have arisen. The problem of over-the-horizon propagation of radio waves, of such interest at the beginning of this century, probably constitutes the first important example of duct propagation. Since that time, still other important examples have arisen. When radar was developed early in the Second World War it became evident that under certain conditions atmospheric ducts contributed appreciably to the observed propagation characteristics of such centimeter wavelength radiation. At about the same time the effects of ducts on the propagation of acoustic energy underwater were being observed.

Theoretical treatments of the duct propagation problem have almost invariably assumed that the bounding surface is quite regular (usually plane). Probably the most successful treatment is the "theory of normal modes" developed during and after the Second World War [1].* In this theory one first constructs solutions of the wave equation for the given inhomogeneous medium which solutions satisfy some boundary condition on the (assumed plane) bounding surface. These solutions are then added together in such a way as to reproduce the source. This method correctly accounts for losses from the duct due to diffraction. It is evident however that such a treatment cannot account for losses due to surface scattering, since the bounding surface has been assumed to be plane. Under certain conditions, the losses due to surface scattering can be shown to be negligible. In particular is this so if $kh\sin\mu$ is small enough; k is the propagation constant ($k = \frac{2\pi}{\lambda}$ where λ is the radiation wavelength), h is the rms deviation of the surface from its average value, and μ is the largest angle made with the bounding surface by radiation trapped within the duct. That the scattering losses are then negligible can be seen by treating the reflection problem using a perturbation method

* The symbol [1] refers to reference one in the bibliography at the end of this thesis.

(cf. Section 3.3); by so doing, one finds that the reflection of radiation in such a case is essentially specular (that is, as if the bounding surface were plane), with little scattered radiation.

It is possible to assess the losses due to surface scattering when $kh\sin\mu \ll 1$ by using a perturbation treatment based upon the normal mode solution [2]. However when $kh\sin\mu$ is of order unity or greater, the perturbation treatment can no longer be expected to apply and one must resort to other methods. Indeed in such a case, the scattered radiation becomes predominant in a reflection event, the specular component becoming negligible (cf. Section 4.4).

In Chapter II a heuristic presentation will be made of a model which is useful in the treatment of duct propagation problems in which the reflection of trapped radiation from the surface is primarily diffuse.* In such a case the loss of energy from the duct due to surface scattering may become important. In order to set up the model, a two-dimensional problem is considered. It is assumed that all quantities depend upon the two Cartesian coordinates x and z alone; it is supposed that z lies normal to the bounding surface. The bounding surface is supposed to be cylindrical (generated by a line moving parallel to itself and following a curve in a plane, the x-z plane, perpendicular to the line). A representative profile taken from the bounding surface in the physical problem is chosen as the generating curve for the two-dimensional surface.

In order to complete the model, we must define certain quantities connected with the reflected radiation. Let $\mathcal{Q}(\theta,x)d\theta dx$ represent the average energy per second per length,**reflected from the element of

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* Occasionally the word "diffuse", in connection with reflected radiation, is reserved for cases where the reflected radiation obeys Lambert's cosine law. In the present connection diffuse refers to all reflected radiation which is not specular.

** This extra length unit is associated with the direction parallel to the generating element for the bounding surface; in the interests of conciseness, this unit will in general be suppressed in what follows.

surface lying between x and $x+dx$, the radiation making an angle with the positive x direction which lies between θ and $\theta+d\theta$. Similarly $\mathcal{S}(\theta,x)d\theta dx$ is defined as the average energy per second reflected once from the surface with x and θ lying within the above increments. Further suppose that an average of one unit of energy per second is incident upon a section of surface from the direction θ' (the angle being measured from the positive x direction). Then $A(\theta,\theta')d\theta$ is defined as the average energy per second reflected from the section of surface which energy has an angle lying between θ and $\theta+d\theta$. Finally $L(\theta)$ is defined as the distance travelled between surface reflections by a trapped ray. (This function is determined entirely by the dependence of the velocity of propagation upon the spatial position z.) Then it will be shown in Chapter II that the following relationship results,*

$$\mathcal{Q}(\theta,x) = \mathcal{S}(\theta,x) + \int_{\Theta_T} \mathcal{Q}(\theta',x-L(\theta'))A(\theta,\pi-\theta')d\theta' , \tag{1.1}$$

where Θ_T represents the set of all angles made by rays trapped within the duct. Once the function $\mathcal{Q}(\theta,x)$ is known, one obtains the energy received at a field point by adding up the contributions of all surface points sending energy to the point in question. The required formulae are given in Section 2.1.

A model somewhat similar to the one proposed here has been used by Bateman and Pekeris [3] in the treatment of the propagation of radiation in a homogeneous slab of material bounded on either side by a surface which reflects incident energy according to Lambert's cosine law. The authors comment that their model could be extended to cover situations where the bounding surfaces have more general reflection characteristics.

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* Equations will be numbered in the remainder of the work with the first number indicating the chapter, the second indicating the section and the third indicating the equation number within the section. In the present instance, there is no section number.

Chapters III and IV are devoted to the justification of the model and to the derivation of the conditions of applicability for Eq. (1.1). It is evident that the problem of the reflection of radiation from an irregular surface is of central importance in assessing the effect of surface scattering upon duct propagation. Accordingly Chapter III is devoted to this problem. A method of treatment utilizing geometrical optics is presented in Section 3.2;* Section 3.3 is devoted to a perturbation method; in Section 3.4 the method of physical optics, utilizing the Kirchhoff approximation is set forth; in Section 3.5 a new method based upon a Fourier transform is considered.

In Chapter IV the Helmholtz formula for inhomogeneous media is developed (Section 4.1). In Section 4.2 a review of geometrical optics as applied to the propagation of radiation through an inhomogeneous medium is presented; in Section 4.3 the formulae of geometrical optics are specialized to stratified media. Finally in Sections 4.4 and 4.5, using certain restrictions, Eq. (1.1) is derived from the Helmholtz formula. There are three restrictions of primary importance: the first is the requirement that radiation trapped within the duct shall be reflected from the surface primarily diffusely (which has already been mentioned). Secondly, it is required that the properties of the material medium vary slowly enough so that one may use geometrical optics to treat the propagation of radiation within the volume. Finally it is required that the amount of radiation received by a region of the surface from nearby surface regions be small compared with the amount received from more distant regions. This last requirement will be seen to be essentially equivalent to requiring that the conditions for the applicability of the Kirchhoff approximation be fulfilled (see Section 3.4).

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* The term geometrical optics will be used in referring to acoustic as well as electromagnetic problems.

In Chapter V, two methods of solution of Eq. (1.1) are presented. The first of these is a solution by iteration. When the bounding surface is such that it scatters much energy out of the duct, the iteration solution is rapidly convergent in regions near the source.

A second method of solution of Eq. (1.1) is presented in Section 5.2. The Laplace transform of Eq. (1.1) is first taken; one obtains a linear integral equation of the second kind to be solved. It is then assumed that the normalized scattering function $A(\theta,\theta')$ can be represented by a finite sum of products of functions of θ and θ' alone. In such a case the integral equation is said to be degenerate, and is readily solved. Finally the inversion integral is used to represent the inverse Laplace transform. This integral is evaluated by summing up the residues of the poles of the Laplace transform. The Laplace transform method provides a representation of the solution of Eq. (1.1) which converges rapidly when the distance between the source and the field point is large.

A numerical example is presented in Section 5.3. The problem considered is that of the propagation of high frequency acoustic radiation under water in an isothermal duct. It is supposed that the surface reflects radiation according to a modified Lambert's cosine law. The results are compared with the analogous problem in an isovelocity medium.

CHAPTER II

A HEURISTIC PRESENTATION OF THE STATISTICAL MODEL FOR DUCT PROPAGATION

In this chapter we shall first present the model to be used in the treatment of the propagation of radiation in surface-bounded ducts. In the presentation the various necessary approximations will be pointed out. However the justification for these approximations and the establishment of criteria of applicability will be reserved until Chapter IV. Once this presentation is completed, it will be shown in Section 2.2 that the suggested model conserves energy.

2.1 Statement of the Problem and Development of the Model

We shall first state the problem to be solved. A half-space of inhomogeneous material bounded by an irregular surface is given. It is assumed that the properties of the medium depend only on the distance from the bounding surface (the z coordinate). The problem is restricted to be two-dimensional (coordinates x and z). It is assumed that a source of either acoustic or electromagnetic radiation is introduced in the medium. If an acoustic field is being considered, it is supposed that the bounding surface is a "pressure-release surface", (i.e. a surface on which the excess pressure of the acoustic field is zero). On the other hand if the field is electromagnetic, it is supposed that the radiation is polarized with the electric vector parallel to the generating element of the bounding surface and that the bounding surface is perfectly conducting. Hence the electric field must vanish on the surface.

It will be supposed that the field function, Φ , satisfies the wave equation,

$$\nabla^2 \Phi = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} , \tag{2.1.1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$ and c is the phase velocity. Furthermore in this work we restrict the field to a single angular frequency, ω , so that we can write,*

$$\Phi(x,z;t) = e^{-i\omega t} \phi(x,z) \tag{2.1.2}$$

and Eq. (2.1.1) becomes:

$$(\nabla^2 + k^2(z))\phi = 0 \tag{2.1.3}$$

where $k = \frac{\omega}{c}$.

It is assumed that the dependence of k upon z is such that some of any radiation originating at or near the bounding surface will be refracted back to it. For this purpose it is sufficient to have $k(z)$ decrease from $z=0$ to some finite value of z, and increase thereafter.

If an acoustic problem is being considered, Φ represents the velocity potential defined by,

$$\vec{v} = -\nabla\Phi . \tag{2.1.4}$$

It can be shown that Eq. (2.1.1) follows from the Navier-Stokes equation upon linearizing the latter [4].

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* Physical quantities are represented by the real parts of complex quantities

On the other hand for electromagnetic problems Φ is to be taken as the component of the electric field in the direction of the generating element of the bounding surface. It can be shown that Eq. (2.1.1) follows from Maxwell's equations if it is assumed that the permeability of the propagating medium is constant in space and time and that the dielectric constant does not change with time.

It is convenient to write down an expression for the average intensity of the field, or in other words for the average energy flow per second per unit of area. The intensity, $\vec{J}$, is given by

$$\vec{J} = \gamma \operatorname{Re}\left\{\frac{1}{i}\,\phi^{*}\nabla\phi\right\}, \tag{2.1.5}$$

where Re stands for the real part of the expression following the symbol, the star indicates the complex conjugate, ∇ indicates the gradient and for the acoustic field,

$$\gamma = \frac{\rho_0 \omega}{2} \tag{2.1.6}$$

while for the electromagnetic field,

$$\gamma = \frac{1}{2\mu\omega} \tag{2.1.7}$$

where ρ_0 represents the average density of the medium at the point under consideration and μ represents the magnetic permeability of the medium. The Eq. (2.1.5) follows for the acoustic field from the known relationship that,

$$\vec{J} = p\vec{v} \tag{2.1.8}$$

where p is the pressure excess over the average pressure and $\vec{v}$ is the particle velocity. In the case of the electromagnetic field, Eq. (2.1.5) follows from a consideration of the Poynting vector. In dealing with the field intensity, since we are most often interested only in relative intensities, it is frequently convenient to set $\gamma = 1$. (It is supposed that the variation in ρ_0 is negligible in the region of physical interest.) If absolute intensities are desired, the quantity γ can be reintroduced.

To continue with a presentation of the problem, the bounding surface will be described by the function $\zeta(x)$. It is assumed that this function is single-valued, bounded, and continuous; other restrictions on $\zeta(x)$ will be applied as they are needed in later developments. In certain physical problems, the surface is also a function of time. In such a case it will be assumed that $\frac{\left|\frac{\partial\zeta}{\partial t}^M\right|}{c} \ll 1$ where $\frac{\partial\zeta}{\partial t}^M$ is the maximum value of the time derivative of the surface. This of course offers no restriction in the case where electromagnetic radiation is being treated; even for acoustic problems, it is seldom of importance. If the surface velocity fulfills this restriction one can treat each individual reflection event as if the surface were stationary, the time dependence entering only insofar as the surface, from which reflection occurs, changes slowly.

In view of the description of the bounding surface given above we know that,

$$\phi(x,\zeta(x)) = 0 \ . \qquad (2.1.9)$$

This follows for the acoustic field since $p = \rho_0 \frac{\partial \Phi}{\partial t}$.

We now desire a solution of Eq. (2.1.3) satisfying the boundary condition given in Eq. (2.1.9). Furthermore the function ϕ is to reduce to out-going waves at infinity and is to have a singularity of the type,

$$\phi(x,z) \rightarrow H_0^{(1)}(kr), \quad \text{as} \quad r \rightarrow 0 \qquad (2.1.10)$$

at the source point (x_0,z_0); $r = \left[(x-x_0)^2 + (z-z_0)^2\right]^{\frac{1}{2}}$.

As was pointed out in Chapter I, much progress has been made in recent years on the above problem for the special case $\zeta = 0$ (more accurately the problem ordinarily considered in normal mode theory is the three-dimensional analogue of the above problem). However the added complication caused by the introduction of an irregular bounding surface has been sufficient to prevent an exact treatment to the present date. In view of this fact we shall apply a number of simplifying assumptions designed to render the problem more manageable and still yield results of physical interest.

In the treatment of this section the assumptions are for the most part implicit in the definitions of the quantities to be used. The desired quantities will be defined again (they have already been defined in Chapter I) and the underlying assumptions emphasized. To avoid unnecessary repetition it is remarked at the outset that the approximations involved in this section are discussed in Sections 4.4 and 4.5 unless otherwise stated.

The quantity $\mathcal{Q}(\theta,x)d\theta dx$ has been defined as the average energy per second reflected from the surface element dx at x in the angular element $d\theta$ at θ (measured from the horizontal). Similarly $\mathcal{S}(\theta,x)d\theta dx$ was defined as the average once-reflected energy per second reflected from dx in the angular element $d\theta$. The normalized scattering function has been defined by: $A(\theta,\theta')d\theta$ represents the average energy per second scattered in the angular element $d\theta$ from a section of surface on which unit energy per second falls from the direction θ'. It is also convenient to define the scattering function $\tilde{A}$: $\tilde{A}(\theta,\theta')d\theta dx$ represents the average energy per second scattered in the angular element $d\theta$ from the surface element dx due to a beam of unit intensity incident upon the surface from the direction θ'. (It is supposed that the medium in which the reflection occurs is homogeneous.) The two functions A and $\tilde{A}$ are related by,

$$A(\theta,\theta') = \frac{\tilde{A}(\theta,\theta')}{\sin\theta'}, \qquad (2.1.11)$$

as can be seen from their definitions. Finally the function $L(\theta)$ is defined as the horizontal distance travelled between surface reflections by a ray trapped within the duct, where the ray makes an angle θ with the positive x axis upon reflection from the surface. The sign is fixed by:

$$L(\theta) \geq 0, 0 \leq \theta \leq \frac{\pi}{2} \; ; \; L(\theta) \leq 0, \frac{\pi}{2} \leq \theta \leq \pi \quad .$$

To begin the discussion of these quantities, the average referred to above is to be an average either over space or time, depending upon whether or not the bounding surface is fixed in time. It is to be noted that the definitions of the quantities $\mathcal{J}$, $\mathcal{S}$, A , and $\tilde{A}$ depend upon the concept of "the energy contained within an angular element". Thus, if these definitions are to have meaning the energy contained in an angular element must remain within that element. Of course this is not true in general; in particular, it does not hold in the region of space near the element of surface from which the reflection occurs. However it is one of the characteristics of the "far field" that energy per angle is a meaningful quantity; it will be necessary to restrict the problem so that most of the radiation involved in the duct propagation can be characterized, upon reflection, as far-field radiation.

In this connection there is still another implicit assumption. It has been assumed in the above definitions that the angular element is measured from an origin placed on the surface; a consequence of this is that the radiation undergoes geometric spreading from the surface. Again it is not in general true that this is so. Indeed if the reflection is completely specular, that is if the surface is flat, the reflected radiation will appear to emerge from the image (in the surface) of the source point. It will be seen that the conditions sufficient to guarantee diffuse reflection will also mean that reflected radiation does appear to spread from the surface in many cases of interest.

If the definition of the normalized scattering function is to be useful it should be possible to treat a single reflection event as if it occurred in a homogeneous medium, the refraction properties of the medium entering only after reflection. If the function k(z) varies sufficiently rapidly, this is not so. It will be seen that this particular property of the reflection process follows quite naturally from the restrictions fixed to guarantee the far field property mentioned above.

Proceeding with the development, reference is made to Fig. 1.

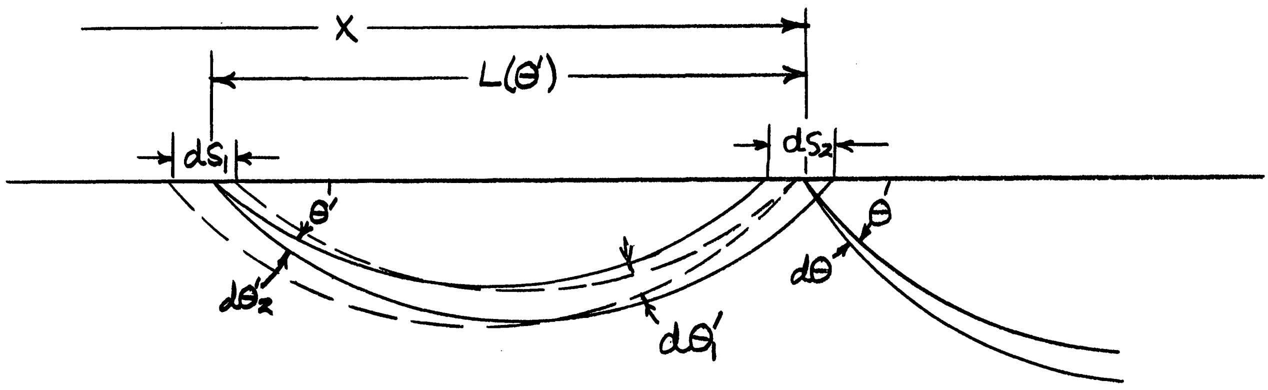

Fig. 1. Diagram used in the derivation of the statistical model equation.

It is desired to find the total amount of energy per second leaving the surface element ds_2 between the angles Θ and $\Theta + d\Theta$. First let us find the total energy falling upon ds_2 from the direction $\pi - \Theta'$.

Now $\mathcal{J}(\Theta', x-L(\Theta'))$ is the energy per angle per length per second leaving the surface at the position $x - L(\Theta')$. Thus the total energy per second falling upon ds_2 from the direction $\pi - \Theta'$ is

$$dE_2 = ds_1 d\Theta_2' \mathcal{J}(\Theta', x-L(\Theta')) , \qquad (2.1.12)$$

where $d\Theta_2'$ is the angle subtended by ds_2 at ds_1 taking into account the refracting properties of the medium. It is to be noted that,

$$ds_1 = \frac{\partial L(\Theta')}{\partial \Theta'} d\Theta_1' , \qquad (2.1.13)$$

and

$$ds_2 = \frac{\partial L(\Theta')}{\partial \Theta'} d\Theta_2' . \qquad (2.1.14)$$

From the definitions of $\mathcal{J}(\Theta,x)$ and of $A(\Theta,\pi-\Theta')$, it is seen that the energy per second reflected from the surface at an angle between Θ and $\Theta+d\Theta$ from the element ds_2 due to radiation incident from the direction $\pi-\Theta'$ is given by

$$d\left[\mathcal{J}(\Theta,x)\right] d\Theta ds_2 = \mathcal{J}(\Theta; x-L(\Theta'))A(\Theta,\pi-\Theta')d\Theta ds_1 d\Theta_2' , \qquad (2.1.15)$$

or

$$d\left[\mathcal{J}(\Theta,x)\right] = \mathcal{J}(\Theta; x-L(\Theta'))A(\Theta,\pi-\Theta')d\Theta_1' , \qquad (2.1.16)$$

using Eqs. (2.1.13) and (2.1.14).

It is to be noted that this treatment has the implicit assumption that radiation enclosed by a pair of rays (defining an angular element $d\Theta$) remains between the pair of rays during refraction. If the frequency is sufficiently high to allow the use of geometrical optics in the treatment of the progress of the radiation through the volume, this will be so (cf. Section 4.2).

In order to proceed, it is desirable to add up the various contributions to the radiation moving in the direction θ as a result of energy incident on ds_2 from different directions θ'. No mention has yet been made of the phase of the reflected radiation; in fact the expressions which have been so far defined treat only the intensity of the radiation and related quantities. In order to obtain the energy leaving the element ds_2, the intensities of the various contributions will be added together. This is justified if the phases of the different contributions are random; this addition property will be seen to hold if the reflected radiation is diffuse. To continue, it is found upon adding up the various quantities obtained from Eq. (2.1.16) for different values of θ',

$$\mathcal{Q}_R(\theta,x) = \int_{\Theta_T} \mathcal{Q}(\theta',x-L(\theta'))A(\theta,\pi-\theta')d\theta' \quad , \qquad (2.1.17)$$

where $\mathcal{Q}_R$ represents the multiply-reflected radiation leaving the element ds_2. When the singly-reflected radiation is added to Eq. (2.1.17) the following result is obtained:

$$\mathcal{Q}(\theta,x) = \mathcal{S}(\theta,x) + \int_{\Theta_T} \mathcal{Q}(\theta',x-L(\theta'))A(\theta,\pi-\theta')d\theta' \quad . \qquad (2.1.18)$$

The symbol Θ_T represents the set of all angles corresponding to rays trapped within the duct; these angles will be termed "trapped" in what follows. Similarly, Θ_{NT} is to represent the set of all angles between 0 and π which are not trapped within the duct.

The function $\mathcal{Q}$ can now be found by solving Eq. (2.1.18), using one of the methods to be given in Chapter V; by so doing one obtains the energy per angle per length per second leaving the surface. This is not however the quantity ordinarily measured in experiments. The remainder of this section will be devoted to determining the experimentally measured quantity once the function $\mathcal{Q}$ has been found.

To begin, it is supposed that the sensing element consists of a circular cylinder of radius a with its axis perpendicular to the x-z plane. The cylinder is assumed to absorb all incident radiation (a "black" cylinder); let the coordinates of the center of the cylinder be (x_R, z_R). It is assumed that the radiation from a given element of surface is approximately constant over the surface of the receiving cylinder (equivalently, $z_R \gg a$) and that the receiving element is small enough so that it does not appreciably affect the field. To find the total energy per second received by the cylinder, one adds up the contributions from all elements of the surface. Reference will be made to Fig. 2.

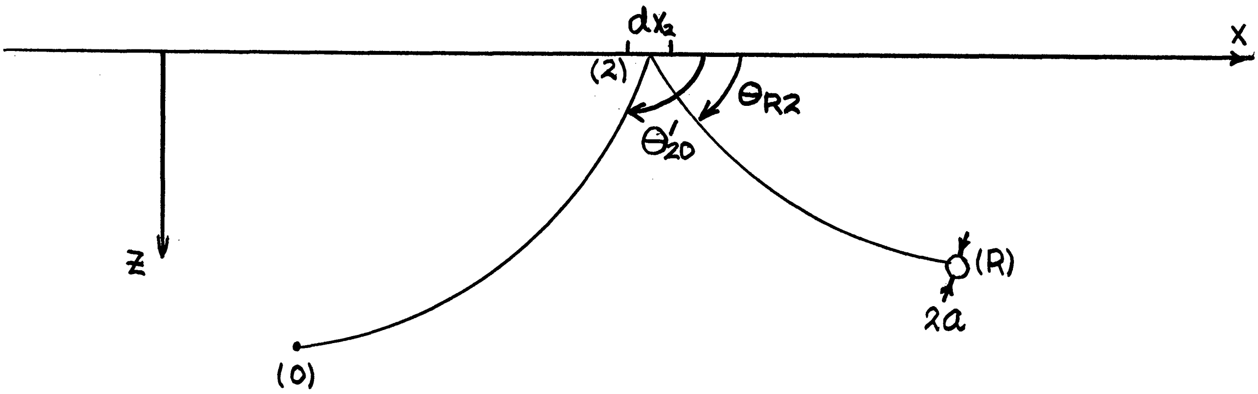

Fig 2. Diagram used in determining the amount of energy received per second.

It is convenient to let J(2,0) stand for the magnitude of the intensity at the point (2) due to a source located at the point (0), and similarly for the function J(R,2). It will be shown (cf. Eq. (4.3.17)) that a singularity of the type given in the relation (2.1.10) radiates $\frac{2}{\pi}$ units of energy per second per angle.* Then using the assumption that

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* The quantity γ of Eq. (2.1.5) has been set equal to unity.

the intensity of the radiation does not vary appreciably over the "black" receiving cylinder and the definition of the scattering function $\tilde{A}$, it is seen that the singly-reflected energy per second received by the cylinder is given by,

$$F_{R1} = 2a \frac{\pi}{2} \int_{\zeta} J(2,0)J(R,2)\tilde{A}(\theta_{R2},\theta'_{20})dx_2 \quad . \tag{2.1.19}$$

There is an alternate expression for the quantity F_{R1}. It is obtained from the definition of the function $\mathcal{S}$,

$$F_{R1} = 2a \frac{\pi}{2} \int_{\zeta} J(R,2) \mathcal{S} (\theta_{R2},x_2)dx_2 \quad . \tag{2.1.20}$$

Similarly the total reflected energy per second received by the cylinder is given by,

$$F_R = 2a \frac{\pi}{2} \int_{\zeta} J(R,2) \mathcal{Q} (\theta_{R2},x_2)dx_2 \quad . \tag{2.1.21}$$

The energy per second (received by the cylinder) which has travelled from the source to the receiver without reflection is seen to be given by

$$F_D = 2a \; J(R,0) \quad , \tag{2.1.22}$$

and finally the total energy per second received by the cylinder is,

$$F = F_D + F_R \quad . \tag{2.1.23}$$

The integrals in Eqs. (2.1.19), (2.1.20), and (2.1.21) are to be carried over all sections of the surface so located that rays exist connecting the point (R). The integrals then ordinarily have finite limits due to the supposed refraction within the medium. Using geometrical optics, an expression will be given for J in Section 4.3 for the case of the stratified medium (k a function of z). Expressions for the angles θ_{R2} and θ'_{20} will also be given in terms of the function k .

To sum up, in this section the basic equation of the model was derived (Eq. (2.1.18)); a function representing the total energy per second received by a black cylinder was also developed (Eq. (2.1.23)).

2.2. Conservation of Energy

It will be shown in this section that Eq. (2.1.18) conserves energy. Of the energy radiated per second by the source, a certain fraction is reflected from the surface; from the definition of $\mathcal{S}$ this quantity is seen to be given by,

$$\int_0^{\pi}\int_{-\infty}^{\infty} \mathcal{S}(\theta,x)dxd\theta, \tag{2.2.1}$$

where it is assumed that the integral converges (as is the case in any problem of physical interest). Some of the radiation once reflected by the surface, is refracted by the duct to be re-reflected one or more times from the surface. However if the bounding surface is such that some energy is reflected out of the duct regardless of the incident angle, then all of the energy reflected from the surface is ultimately reflected out of the duct. The total amount of energy per second reflected out of the duct is seen to be given by,

$$\int_{\Theta_{NT}}\int_{-\infty}^{\infty} \mathcal{Q}(\theta,x)dxd\theta \quad , \tag{2.2.2}$$

it being supposed that the integral converges. It will be seen that in many cases the function $\mathcal{Q}$ has the property that $|\mathcal{Q}| \leq Me^{-\beta|x|}$ where M and β are positive and can be chosen independent of θ ; in such a case convergence is guaranteed. It is the purpose of this section to show that the expressions (2.2.1) and (2.2.2) are equal.

In order to show this equality let Eq. (2.1.18) be integrated over x from $-\infty$ to $+\infty$,

$$\tilde{\mathcal{I}}(\theta) = \tilde{\mathcal{S}}(\theta) + \int_{\Theta_T} A(\theta,\pi-\theta')d\theta' \int_{-\infty}^{\infty} \mathcal{I}(\theta', x-L(\theta'))dx, \qquad (2.2.3)$$

where

$$\tilde{\mathcal{I}}(\theta) = \int_{-\infty}^{\infty} \mathcal{I}(\theta,x)dx \quad , \qquad (2.2.4)$$

and

$$\tilde{\mathcal{S}}(\theta) = \int_{-\infty}^{\infty} \mathcal{S}(\theta,x)dx \quad . \qquad (2.2.5)$$

In Eq. (2.2.3) the order of the integrations over θ' and x has been changed; this is surely possible if the function A is bounded and if the function $\mathcal{I}$ satisfies the exponential condition cited above. Now the variable of integration in the inner integral of Eq. (2.2.3) is changed by a simple translation to give,

$$\tilde{\mathcal{I}}(\theta) = \tilde{\mathcal{S}}(\theta) + \int_{\Theta_T} A(\theta,\pi-\theta')\tilde{\mathcal{I}}(\theta')d\theta' \quad . \qquad (2.2.6)$$

Using Eqs. (2.2.4) and (2.2.5), the expressions (2.2.1) and (2.2.2) become

$$\int_0^{\pi} \tilde{\mathcal{S}}(\theta)d\theta \quad , \qquad (2.2.7)$$

and

$$\int_{\Theta_{NT}} \tilde{\mathcal{I}}(\theta)d\theta \quad . \qquad (2.2.8)$$

To continue let Eq. (2.2.6) be integrated over the set of angles Θ_{NT},

$$\int_{\Theta_{NT}} \tilde{\mathcal{I}}(\theta)d\theta = \int_{\Theta_{NT}} \tilde{\mathcal{S}}(\theta)d\theta + \int_{\Theta_{NT}} d\theta \int_{\Theta_T} A(\theta,\pi-\theta')\tilde{\mathcal{I}}(\theta')d\theta' \quad . \qquad (2.2.9)$$

From the definitions of Θ_T and Θ_{NT} it is known that

$$\Theta_T + \Theta_{NT} = (0,\pi) \ , \qquad (2.2.10)$$

where $(0,\pi)$ represents the set of all angles between 0 and π . Using this fact it is seen that Eq. (2.2.9) can be written

$$\int_{\Theta_{NT}} \tilde{\mathcal{J}}(\theta)d\theta = \int_0^{\pi} \tilde{\mathcal{S}}(\theta)d\theta - \int_{\Theta_T} \tilde{\mathcal{S}}(\theta)d\theta + \int_0^{\pi} d\theta \int_{\Theta_T} A(\theta,\pi-\theta')\tilde{\mathcal{J}}(\theta')d\theta'$$

$$- \int_{\Theta_T} d\theta \int_{\Theta_T} A(\theta,\pi-\theta')\tilde{\mathcal{J}}(\theta')d\theta' \ . \qquad (2.2.11)$$

Now from the definition of the normalized scattering function, A , and from the conservation of energy at a surface reflection (which follows from the boundary condition given in Eq. (2.1.9)) it is seen that,

$$\int_0^{\pi} A(\theta,\theta')d\theta = 1 \ . \qquad (2.2.12)$$

Assuming that the order of integration in the third term on the right side of Eq. (2.2.11) may be changed, and using Eq. (2.2.12), Eq. (2.2.11) becomes,

$$\int_{\Theta_{NT}} \tilde{\mathcal{J}}(\theta)d\theta = \int_0^{\pi} \tilde{\mathcal{S}}(\theta)d\theta + \int_{\Theta_T} \Big\{ \tilde{\mathcal{J}}(\theta) - \tilde{\mathcal{S}}(\theta)$$

$$- \int_{\Theta_T} A(\theta,\pi-\theta')\tilde{\mathcal{J}}(\theta')d\theta' \Big\} d\theta \ . \qquad (2.2.13)$$

When Eq. (2.2.13) is compared with Eq. (2.2.6) it is seen that

$$\int_{\Theta_{NT}} \tilde{\mathcal{J}}(\theta)d\theta = \int_0^{\pi} \tilde{\mathcal{S}}(\theta)d\theta \ , \qquad (2.2.14)$$

as desired.

CHAPTER III

THE REFLECTION OF RADIATION FROM A NON-PLANE SURFACE

In Chapter II an integro-difference equation (Eq. (2.1.18)) was obtained heuristically. Under certain conditions this equation governs the propagation of radiation in a surface-bounded duct. In Chapter IV restrictions will be established under which Eq. (2.1.18) will be derived from the wave equation. In that development an understanding of the basic characteristics of the reflection of radiation from a non-planar surface bounding a homogeneous medium will be essential. It is for this reason that we digress in this chapter and consider such reflection problems.

3.1. The Statement of the Problem

A great amount of work has been done on the problem of the reflection of radiation from non-plane surfaces. Our main interest here is not in the reflection problem itself but rather in gaining the information necessary for the further treatment of the problem of the propagation of radiation in ducts. Accordingly we will content ourselves with a description of some of the more important techniques available. We proceed with a definition of the problem. We assume that the problem is two-dimensional so that all quantities depend on the coordinates x and z alone. A function $\zeta(x)$ is given which bounds a half space of homogeneous, isotropic material, the half space lying in the positive z direction. We assume that we can define a mean, $\langle \zeta \rangle$, of $\zeta(x)$,

$$\langle \zeta \rangle = \lim_{L\to\infty} \frac{1}{2L} \int_{-L}^{L} \zeta(x)dx \quad . \tag{3.1.1}$$

Without loss of generality we may set $\langle \zeta \rangle = 0$. Now we suppose that we are given incident radiation, ϕ_i, falling upon the surface $\zeta(x)$ from the positive z direction. The ϕ_i is here the field function (say the velocity potential) set up by the sources, which would have existed had the surface not been present. We wish to determine the reflected radiation, ϕ_r, subject to the condition that it shall consist only of outgoing radiation at large distances from the surface.

We wish then to find a function $\Phi(x,z,t)$ which satisfies the wave equation in two dimensions,

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 \Phi}{\partial t^2} \quad , \tag{3.1.2}$$

at all points except the source points of the field. The quantity c is the velocity of propagation. It will be supposed that the source radiates a single (angular) frequency ω, so that we can write

$$\Phi(x,z,t) = e^{-i\omega t}\phi(x,z) \quad . \tag{3.1.3}$$

Substituting Eq. (3.1.3) in Eq. (3.1.2) we obtain,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} + k^2 \right)\phi(x,z) = 0 \quad , \tag{3.1.4}$$

with $k = \frac{\omega}{c}$. We define

$$\phi = \phi_i + \phi_r \quad . \tag{3.1.5}$$

We shall also let $\lambda = \frac{2\pi}{k}$. From its definition it follows that ϕ_i satisfies Eq. (3.1.4) everyshere except at the source points of the field; from this we see that the reflected radiation, ϕ_r , must also satisfy Eq. (3.1.4). It will be supposed that the function ϕ vanishes at the surface, so that we have

$$(\phi_i + \phi_r) \Big|_{z = \zeta(x)} = 0 \quad . \qquad (3.1.6)$$

Ideas similar to those set forth below will obtain for other boundary conditions. The reader is referred to the references given.

Two parameters connected with the surface, $\zeta(x)$, will be defined. It is first assumed that $\zeta(x)$ is bounded, so that,

$$|\zeta(x)| \leq \zeta^M \quad . \qquad (3.1.7)$$

It is also assumed that ζ and $\frac{d\zeta}{dx}$ are continuous functions of x; $\overline{\Lambda}$ is defined as the average separation of neighboring maxima of the function ζ . Finally we suppose that ζ is a single-valued function of x.

Before proceeding, it is convenient to discuss the reciprocity theorem. The theorem may be stated as follows: * if a point source acts at (0) and if the value of the field is observed at (P), the field function ϕ has the same value as would be observed if the points (0) and (P) were interchanged. The reciprocity theorem applies to a line source (in two dimensions) as well as to point sources. In some problems it is convenient to choose a plane wave for the source of the field. It is not difficult to extend the theorem to cover such a source. To do this we suppose that the incident field near the reflecting surface is set up by a source sufficiently distant so that its field in the region of physical interest is

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* See Rayleigh, [5], p. 145.

essentially a plane wave; the strength of the source is assumed to be increased so that the amplitude of the field is finite in the physical region.

We now take up a method of solution of the reflection problem based upon geometrical optics.

3.2. The Use of Geometrical Optics in the Solution of the Reflection Problem

Geometrical optics may be used to approximate the field reflected from a non-plane surface if $kR_m \sin\Theta' \gg 1$ where R_m is the minimum radius of curvature of the surface and Θ' is the angle made by the incident radiation with the horizontal [6],[7],[8]. The justification for the given restriction can be found in Section 4.4 (cf. the relation (4.5.27)) . The method of this section has been used in the past in attempts to find the physical foundation for Lambert's cosine law of reflection [9].

In Fig. 3 we have singled out a small section of surface, which has been approximated by a circular arc with radius R . An increment of the incident energy, ΔE , falls on this section. The reflected radiation appears to diverge from the point f with opening angle $\Delta\bar{\Theta}$, lying in a direction determined by the relation that the angle of incidence should equal the angle of reflection. We have

$$\Theta = \pi - \Theta' + 2\chi \quad , \tag{3.2.1}$$

with χ defined as the angle made by the tangent to the surface with the x direction.

We now consider the combined effect of many such increments of energy falling upon an irregular surface. Consider first the image points f . (Although the image point shown in Fig. 3 is in non-physical space, $z < \zeta(x)$,

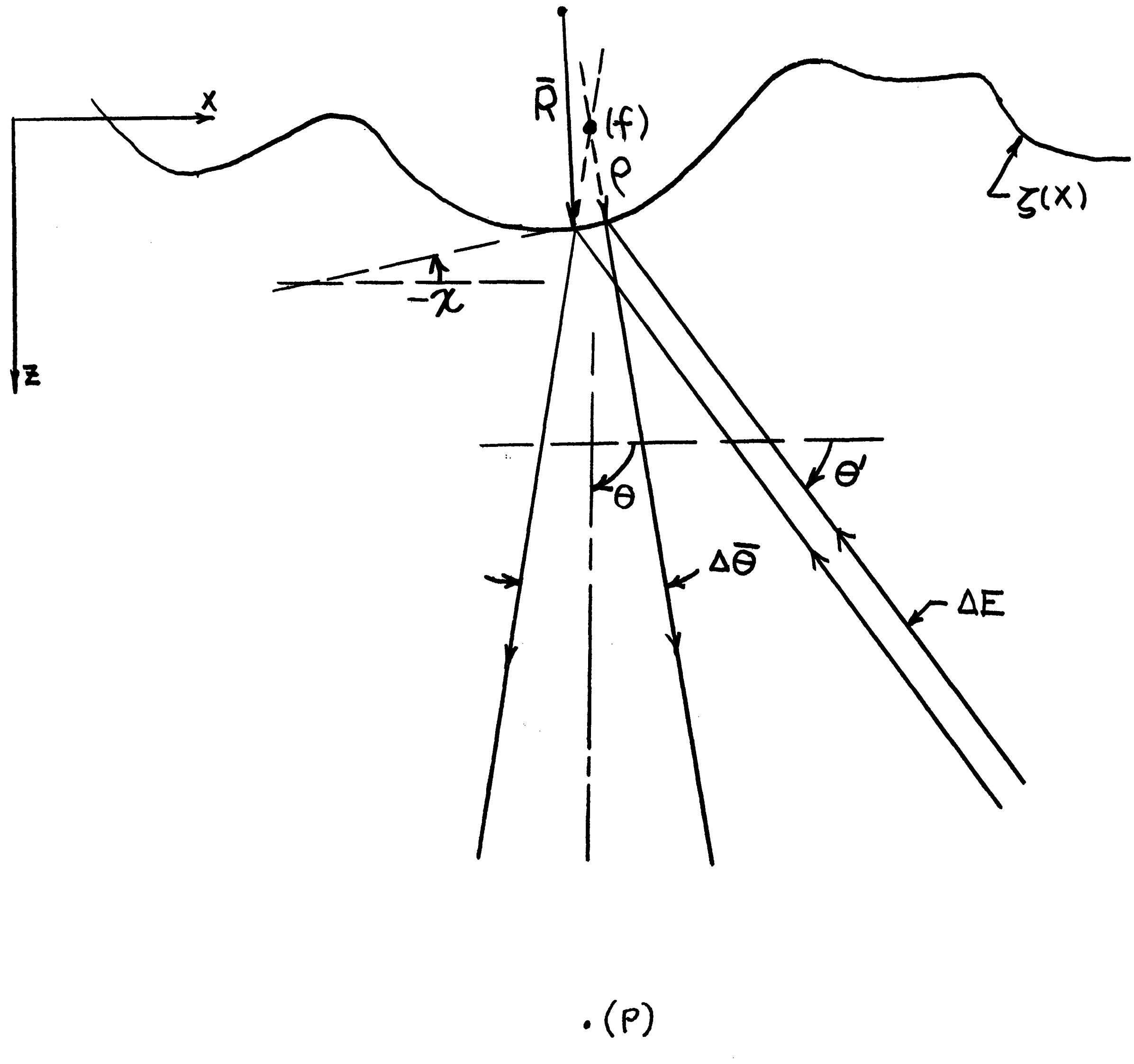

Fig. 3. Diagram used in the derivation of the reflecting properties of a surface through the use of geometrical optics.

it is of course quite possible for image points to fall in the region $z < \zeta(x)$). By the use of simple geometry in connection with Fig. 3 it can be shown that $\rho \leq R/2$. (It is well known that for a parallel beam normally incident on a cylinder, $\rho = R/2$.) Now suppose that the incident radiation consists of a beam of unit intensity and of width ℓ ; the projection of the width on the x axis is then $\ell/\sin\Theta'$. It is supposed that the length ℓ is large enough so that the statistical properties of the surface within this segment are essentially the same as those of the surface as a whole. The statistics of the surfaces considered in this work are discussed in detail in Section 4.4.

The energy reflected from the surface due to the incident beam will be distributed in angle in a manner dependent upon the distribution of slopes of the radiated section of surface. From Fig. 3 it is seen that the radiation received at (P) as a result of reflection occuring at a region of the surface with focal point (f) behaves like $\frac{1}{R(P,f)}$ where $R(P,f)$ is the distance from (f) to (P) . The various values of R obtained from different reflection regions may be replaced by r , where r is the distance from the center of the region of surface $\ell/\sin\Theta'$ to (P) , if $r \gg R^M$ and $r \gg \ell/\sin\Theta'$. Here R^M indicates some maximum radius of curvature of the surface; the error is of order $\frac{R^M}{r}$ or $\frac{\ell/\sin\Theta'}{r}$, whichever is larger. Now suppose that we examine radiation in the angular element $\Delta\Theta$ at a distance r , where we choose the increments of incident energy small enough so that

$$\Delta\Theta \gg \Delta\bar{\Theta} \text{ , for all } \Delta\bar{\Theta} \text{ .}$$

Then the energy contained in the angular element $\Delta\Theta$ divided by the total incident energy will be equal to the probability that the reflected angle

Θ lies between Θ and $\Theta + \Delta\Theta$ (if there is no multiple scattering). We now define $P_{\pi/2}(\chi)d\chi$ as the probability that the surface angle lies between χ and $\chi + d\chi$ when the surface is viewed from the direction $\Theta' = \pi/2$. We wish to find the corresponding probability when the surface is viewed from the direction Θ'. From the definition of $P_{\pi/2}$ it is seen that the projection on the x-axis of those segments of the surface which make an angle lying between χ and $\chi + d\chi$ is given by $\ell/\sin\Theta' P_{\pi/2}(\chi)d\chi$ (it is assumed that the end points of the section of radiated surface lie on the axis). By considering the corresponding projection on an axis normal to the direction of the incident beam one finds

$$P_{\Theta'}(\chi)d\chi = P_{\pi/2}(\chi)\frac{\sin(\Theta'-\chi)}{\sin\Theta'\cos\chi}\,d\chi \quad , \qquad (3.2.2)$$

where the left side is defined as the probability that the surface angle lies between χ and $\chi + d\chi$ when the surface is viewed from the direction Θ'. It is assumed implicitly in Eq. (3.2.2) that there is no shadowing (equivalently that $P_{\pi/2} = 0$, $\chi > \Theta'$ or $\chi < \Theta'-\pi$).

We can now form two expressions for the energy contained in the angular element $\Delta\Theta$, the energy so contained arising from the total incident energy, ℓ. On the one hand we use the definition of the normalized scattering function, $A(\Theta,\Theta')$, and on the other we use the distribution function, given by Eq. (3.2.2), in the manner outlined above. Thus we obtain the relation

$$A(\Theta,\Theta') = \frac{1}{2\sin\Theta'}\,P_{\pi/2}\left(\frac{\Theta+\Theta'-\pi}{2}\right)\,\frac{\cos\left(\frac{\Theta-\Theta'}{2}\right)}{\sin\left(\frac{\Theta+\Theta'}{2}\right)}\,, \qquad (3.2.3)$$

where we have used Eq. (3.2.1). It is supposed that $P_{\pi/2}(\chi)$ vanishes unless χ is such that $0 \leq \Theta \leq \pi$ (cf. Eq. (3.2.1)); this is a necessary condition if there is to be no multiple scattering. It is noted that the quantity $\sin\theta' A(\theta,\theta')$ (which has been defined as $\tilde{A}$) is symmetric in the directions θ and θ'. It can also be seen that the expression,

$$\int_0^{\pi} A(\theta,\theta')d\theta = 1 \quad , \tag{3.2.4}$$

is satisfied, as is necessary from the definition of $A(\theta,\theta')$ (cf. Eq. (2.2.12)). Finally, if $\chi \ll \theta'$ and $\chi \ll 1$, Eq. (3.2.3) reduces to, (see Eq. (3.2.2)),

$$A(\theta,\theta') = \frac{1}{2} P_{\pi/2}\left(\frac{\theta+\theta'-\pi}{2}\right) \quad , \tag{3.2.5}$$

so that, as one might expect, the transformation of the distribution function for non-normal incidence is not necessary in this case.

The normalized scattering function is given correctly by Eq. (3.2.3)--

a. If the condition for the validity of the use of geometrical optics in the reflection problem is fulfilled, or $kR_m \sin\theta' \gg 1$.

b. If there is no shadowing so that Eq. (3.2.2) is correct. With shadowing present, Eq. (3.2.2) should be replaced by the corresponding relation for the section of the surface which is illuminated. The restriction for no shadowing may be written

$$P(\chi) = 0, \quad \chi > \theta' \quad \text{or} \quad \chi < \theta'-\pi \quad .$$

c. If there is no multiple scattering, so that Eq. (3.2.3) is correct. The restriction may be written in the form $P(\chi) = 0$ unless $\frac{\theta'}{2} - \frac{\pi}{2} \leq \chi \leq \frac{\theta'}{2}$. It is possible to have multiple scattering even though χ lies within this range; however the restriction just given will serve in the present connection.

In general, in order to take into account shadowing and multiple scattering, it is necessary to have a more detailed description of the surface than that used here.

3.3. A Perturbation Method for the Reflection Problem

We now consider problems in which one must take into account the wave properties of the radiation. We proceed with a perturbation method. Probably Rayleigh was the first to consider reflection problems for which, in the present notation, $k\zeta^M \ll 1$, [10]. Since then a large amount of work has been done using this approximation [11],[12],[13]. Under this hypothesis the field will be shown to consist mainly of the incident radiation and the specularly reflected radiation. (Specularly reflected radiation is, except for a constant multiplier, that radiation which would have been set up had the bounding surface been flat.) Now although the direct and specularly reflected radiation are the most important part of the field, interest really centers on the diffusely reflected radiation which in general accompanies these two major components. The amplitude of the diffuse radiation is in general of order $k\zeta^M$.

We will now present the perturbation solution. Let us represent the reflected field, ϕ_r, by a sum of plane wave solutions of Eq. (3.1.4) with each elementary plane wave proceeding in the positive z direction.* In order to represent the solution by such a sum, one must include complex directions of propagation (inhomogeneous waves). These must be chosen in a way such that the solution remains bounded for all x and as $z \to +\infty$. The correct plane wave representation is given by

$$\phi_r(x,z) = \int_{-\infty}^{\infty} B(K) e^{iKx + i(k^2-K^2)^{1/2} z} \, dK \quad . \tag{3.3.1}$$

Now if we substitute Eq. (3.3.1) in Eq. (3.1.6) we have

$$\phi_i(x,\zeta(x)) + \int_{-\infty}^{\infty} B(K) e^{iKx + i(k^2-K^2)^{1/2}\zeta(x)} \, dK = 0 \quad . \tag{3.3.2}$$

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* For a discussion of such plane wave representations see Whittaker and Watson; [14], sec. 18.6 .

We wish to find $B(K)$, which in conjunction with Eqs. (3.1.5) and (3.3.1) will define the solution to the problem. Now let us take advantage of the hypothesis, $k\zeta^M \ll 1$. In order to do this we adopt a procedure familiar in the treatment of perturbation problems.*

First replace $\zeta(x)$ by $\varepsilon\zeta(x)$. Then assume that $\phi_i(x,\varepsilon\zeta(x))$ and $B(K)$ are analytic functions of ε in the interval $0 \leq \varepsilon \leq 1$. Expand all functions in powers of ε and equate corresponding coefficients in Eq. (3.3.2). We have first

$$\phi_i(x,\varepsilon\zeta(x)) = \phi_i^{(0)}(x) + \varepsilon\zeta(x)\phi_i^{(1)}(x) + \varepsilon^2\zeta^2(x)\phi_i^{(2)}(x) + \ldots , \quad (3.3.3)$$

and,

$$B(K) = B^{(0)}(K) + \varepsilon B^{(1)}(K) + \varepsilon^2 B^{(2)}(K) + \ldots \quad . \quad (3.3.4)$$

The solution $B(K)$ is obtained by setting $\varepsilon = 1$ in Eq. (3.3.4). We find from Eq. (3.3.2), upon equating equal powers of ε,

$$\left.\begin{aligned}
&\phi_i^{(0)}(x) + \int_{-\infty}^{\infty} B^{(0)}(K)e^{iKx}dK = 0 , \\
&\zeta(x)\phi_i^{(1)}(x) + \int_{-\infty}^{\infty} B^{(0)}(K)e^{iKx} i\zeta(x)(k^2-K^2)^{1/2}dK \\
&\qquad + \int_{-\infty}^{\infty} B^{(1)}(K)e^{iKx}dK = 0 , \\
&\zeta^2(x)\phi_i^{(2)}(x) + \int_{-\infty}^{\infty} B^{(0)}(K)e^{iKx} \frac{\left[i(k^2-K^2)^{1/2}\zeta(x)\right]^2}{2!} dK \\
&\qquad + \int_{-\infty}^{\infty} B^{(1)}(K)e^{iKx}\left[i(k^2-K^2)^{1/2}\zeta(x)\right] dK \\
&\qquad + \int_{-\infty}^{\infty} B^{(2)}(K)e^{iKx}dK = 0 .
\end{aligned}\right\} \quad (3.3.5)$$

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* See Schiff, [15], chap. VII.

We will suppose that ϕ_i and $\zeta(x)$ are such as to cause all of the integrals given in Eqs. (3.3.5) to converge. The question of the convergence of Eq. (3.3.4) (that is, the question of the validity of the assumption of analyticity) will not be considered.

Now consider Eq. (3.3.5). If all of the quantities given there have Fourier transforms we take the Fourier transform of the entire system. The transform is defined by

$$\text{F.T.}\left\{F(x)\right\} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{iKx}F(x)dx \ . \qquad (3.3.6)$$

(If some of the Fourier transforms do not exist, one may use more general transforms; we consider the simpler case here where Eq. (3.3.6) converges for all F(x) considered.) The result of taking the transform of Eqs. (3.3.5) is

$$B^{(0)}(K) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iKx}\,\phi_i^{(0)}(x)dx, \qquad (3.3.7)$$

$$B^{(1)}(K) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iKx}\zeta(x)\phi_i^{(1)}(x)dx \qquad (3.3.8)$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iKx}dx\int_{-\infty}^{\infty} B^{(0)}(K')e^{iK'x}\left[i(k^2-K'^2)^{1/2}\zeta(x)\right]dK' \ ,$$

$$B^{(2)}(K) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iKx}\zeta^2(x)\phi_i^{(2)}(x)dx \qquad (3.3.9)$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iKx}dx\int_{-\infty}^{\infty} B^{(1)}(K')e^{iK'x}\left[i(k^2-K'^2)^{1/2}\zeta(x)\right]dK'$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iKx}dx\int_{-\infty}^{\infty} B^{(0)}(K')e^{iK'x}\frac{\left[i(k^2-K'^2)^{1/2}\zeta(x)\right]^2}{2!}dK' \ .$$

Let us now examine the first two terms in the solution as given by Eqs. (3.3.7) and (3.3.8). First it can be seen that $B^{(0)}(K)$ is the value which $B(K)$ assumes in the problem where $\zeta(x)$ is zero (a plane surface). We have the result

$$\phi_r^{(0)}(x,z) = \int_{-\infty}^{\infty} e^{iKx+i(k^2-K^2)^{1/2}z} B^{(0)}(K)dK$$

$$= - \phi_i(x,-z) \ . \qquad (3.3.10)$$

This is the specularly reflected radiation mentioned above. Now consider the first order term $B^{(1)}(K)$. In Eq. (3.3.8) we see that $B^{(1)}(K)$ is of first order in $\zeta(x)$. This was the result proposed above. We have, to terms of first order in $k\zeta(x)$, the following result:

$$\phi_r(x,z) = - \phi_i(x,-z) + \int_{-\infty}^{\infty} e^{iKx+i(k^2-K^2)^{1/2}z} B^{(1)}(K)dK + \ldots \qquad (3.3.11)$$

where $B^{(1)}(K)$ is given by Eq. (3.3.8). Finally we see that Eq. (3.3.11) together with Eq. (3.1.5) gives the solution to the problem through terms of first order in $k\zeta^M$. We see from Eq. (3.3.9) that the first term neglected is of order $(k\zeta^M)^2$.

Before turning to the next approximation method, let us consider the second term of Eq. (3.3.11). We wish to obtain the far-field approximation (x and z large) for the function $\phi_r(x,z)$ by using the method of stationary phase to evaluate this integral.* We become interested then in those points, K_i, at which the phase of the integrand becomes stationary. They are determined by the relation,

$$\frac{\partial}{\partial K}\left[Kx + (k^2-K^2)^{1/2}z\right] = 0 \qquad (3.3.12)$$

which yields the single stationary point K_0 where,

$$\frac{K_0}{(k^2-K_0^2)^{1/2}} = \frac{x}{z} \ .$$

* See Lamb, [16], sec. 241.

We now expand the phase about the point K_0. We have,

$$Kx + (k^2-K^2)^{1/2}z = K_0x + (k^2-K_0{}^2)^{1/2}z \qquad (3.3.13)$$

$$+ \frac{1}{2!}\frac{\partial^2}{\partial K^2}\left[Kx + (k^2-K^2)^{1/2}z\right]_{K=K_0}(K-K_0)^2$$

$$+ \frac{1}{3!}\frac{\partial^3}{\partial K^3}\left[Kx + (k^2-K^2)^{1/2}z\right]_{K=K_0}(K-K_0)^3 + \ldots$$

with

$$\frac{\partial^2}{\partial K^2}\left[Kx + (k^2-K^2)^{1/2}z\right]_{K=K_0} = -\left(\frac{r}{k\ \sin^2\theta}\right), \qquad (3.3.14)$$

and

$$\frac{\partial^3}{\partial K^3}\left[Kx + (k^2-K^2)^{1/2}z\right]_{K=K_0} = -\left(\frac{3r\cos\theta}{k^2\sin^4\theta}\right), \qquad (3.3.15)$$

where θ is defined by

$$K_0 = k\cos\theta \qquad (3.3.16)$$

and

$$r = (x^2+z^2)^{1/2} \quad . \qquad (3.3.17)$$

If we substitute in Eq. (3.3.11) we have (using the first two terms of Eq. (3.3.13)),

$$\phi_{rs} = B^{(1)}(K_0)e^{i\left[K_0x+(k^2-K_0^2)^{1/2}z\right]}\int_{-\infty}^{\infty} e^{-i\frac{r}{2k\sin^2\theta}(K-K_0)^2}\, dK\ , \qquad (3.3.18)$$

where

$$\phi_r = -\phi_i(x,-z) + \phi_{rs} \quad ,$$

and where it has been assumed that $e^{i[Kx+(k^2-K^2)^{1/2}z]}$ varies much more rapidly than does $B^{(1)}(K)$. The neglect of the third term in Eq. (3.3.13) is justified if,*

$$\frac{\left| \frac{\partial^3}{\partial K^3}\left[Kx+(k^2-K^2)^{1/2}z\right]_{K=K_0}\right|}{\left| \frac{\partial^2}{\partial K^2}\left[Kx+(k^2-K^2)^{1/2}z\right]_{K=K_0}\right|^{3/2}} \ll 1 \quad . \qquad (3.3.19)$$

Substituting Eqs. (3.3.14) and (3.3.15) in the relation (3.3.19) we have the condition,

$$\left|\frac{3 \cot \Theta}{(kr)^{1/2}}\right| \ll 1 \quad . \qquad (3.3.20)$$

Thus the stationary phase approximation fails for angles sufficiently near to grazing ($\sin \Theta \approx 0$). The integral in Eq. (3.3.18) can be carried out to yield

$$\phi_{rs}(x,z) = \sin\Theta \; B^{(1)}(K_0)e^{ikr}\left(\frac{2\pi k}{ir}\right)^{1/2} \quad . \qquad (3.3.21)$$

It is noted that the amplitude of the scattered field falls off like $(r)^{-1/2}$ with range. The restriction given above on the relative rates of variation of the exponential in the integrand of Eq. (3.3.11) and of the function $B^{(1)}(K)$ is sufficient to guarantee this range dependence for the scattered radiation.

The result given by Eq. (3.3.21) is valid--

a. If the perturbation method is applicable, i.e. $k\zeta^M \ll 1$.

b. If x and z are large enough so that $e^{i[Kx+(k^2-K^2)^{1/2}z]}$ is much more rapidly varying than is $B^{(1)}(K)$.

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* Lamb, [16], p. 396.

c. If $\left|\frac{3\cot\theta}{(kr)^{1/2}}\right| \ll 1$ so that only the first term in the stationary phase approximation need be used.

It has already been mentioned that this work is primarily concerned with surfaces for which specularly reflected radiation forms a small part of the total reflected radiation. From the work of this section we see that when $k\zeta^M \ll 1$, the reflected radiation is primarily specular. For this reason we shall not have further occasion to consider problems in which the method of this section is applicable. The method has been presented in order to give a more complete picture of the approximations which have proved to be useful in the solution of reflection problems.

3.4. The Kirchhoff Approximation

We now proceed with a discussion of a method of approximation due to Kirchhoff [17].* We shall begin with the Helmholtz formula; this equation (see below) expresses the value of the field function in terms of an integral over the bounding surface (here $\zeta(x)$) involving the value of the field function and of its normal derivative on the surface. The Kirchhoff approximation in the present connection consists of supposing that the bounding surface is sufficiently smooth to allow one to make the assumption that it is "locally flat". A surface will be termed "locally flat" if one can, to a good approximation, treat radiation reflected from every portion of it as if that portion were flat. This assumption allows one to estimate the normal derivative of the field on the boundary. Furthermore the boundary condition (Eq. (3.1.6)) allows one to fix the value of the field itself on the boundary. The use of the

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* See also Baker and Copson, [18], chap. II.

Helmholtz formula then yields an approximate solution to the problem. The Kirchhoff approximation as presently stated forms the foundation for almost all calculations which have been made for the distribution of energy reflected from diffraction gratings.

The question of the region of validity of the approximation has received much attention. Recently Brekhovskikh has proposed the criterion [19],

$$\rho k \sin \theta' \gg 1$$

where ρ is a (representative) radius of curvature of the surface, and θ' is the grazing angle of incidence. It will be one of our purposes in this section to discuss the validity of the approximation.

In the treatment to be presented here we shall attempt to develop the Kirchhoff approximation in a systematic way. We first consider the Helmholtz formula in connection with the determination of the normal derivative of the field function on the boundary; an integral equation is obtained. The solution by iteration of this integral equation will then be considered. It will appear that using the first term is equivalent to adopting the Kirchhoff approximation. We then investigate the next term of the iteration solution, in particular observing that it has the order property,

$$|I| \leq M \left|\frac{d\zeta}{dx}^M\right| + M' \frac{1}{kR_m} . \qquad (3.4.1)$$

Here I is the first term neglected in the solution; R_m is the minimum radius of curvature and $\frac{d\zeta}{dx}^M$ is the maximum slope of $\zeta(x)$; M and M' are some constants independent of surface properties. On the basis of this order property, we shall assume that the Kirchhoff approximation is valid if--

i) $\left|\frac{d\zeta^M}{dx}\right| \ll 1$,

and ii) $kR_m \gg 1$.

We proceed with the outlined program. We first need the Weber two-dimensional analogue of the Helmholtz formula. By using Green's formula in connection with Eq. (3.1.4), one is able to derive the following result (see Fig. 4) [20]:*

$$\phi(P) = \phi_i(P) + \frac{1}{4i} \int_{\zeta} \left\{ \phi(1) \frac{\partial}{\partial \nu_1} H_0^{(1)}(kr_{1P}) - H_0^{(1)}(kr_{1P}) \frac{\partial}{\partial \nu_1} \phi(1) \right\} ds_1 , \qquad (3.4.2)$$

subject to the restrictions that--

a. The function $\phi(x,z)$ possesses singularities of the type,

$$\phi(x,z) \longrightarrow H_0^{(1)}(kr_{P0}), \qquad \text{as} \quad r_{P0} \longrightarrow 0 ,$$

for a finite set of source points; (x_0,z_0) is a representative source point and $r_{P0} = \left[(x-x_0)^2 + (z-z_0)^2\right]^{1/2}$.

b. The function $\phi(x,z)$ is continuous with continuous first and second order partial derivatives for all points (x,z) satisfying the condition $z \gtrless \zeta(x)$, with the exception of the source points.

c. The function $\phi(x,z)$ shall have the asymptotic form $\frac{e^{ikr}}{(kr)^{1/2}}$ as $r \longrightarrow \infty$, where $r = (x^2 + z^2)^{1/2}$. This is equivalent to requiring that at great distances from the surface the field shall consist of outgoing waves only.

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* For a discussion of the Helmholtz formula as well as of Weber's two-dimensional analogue, see Baker and Copson, [18], chaps. I and II.

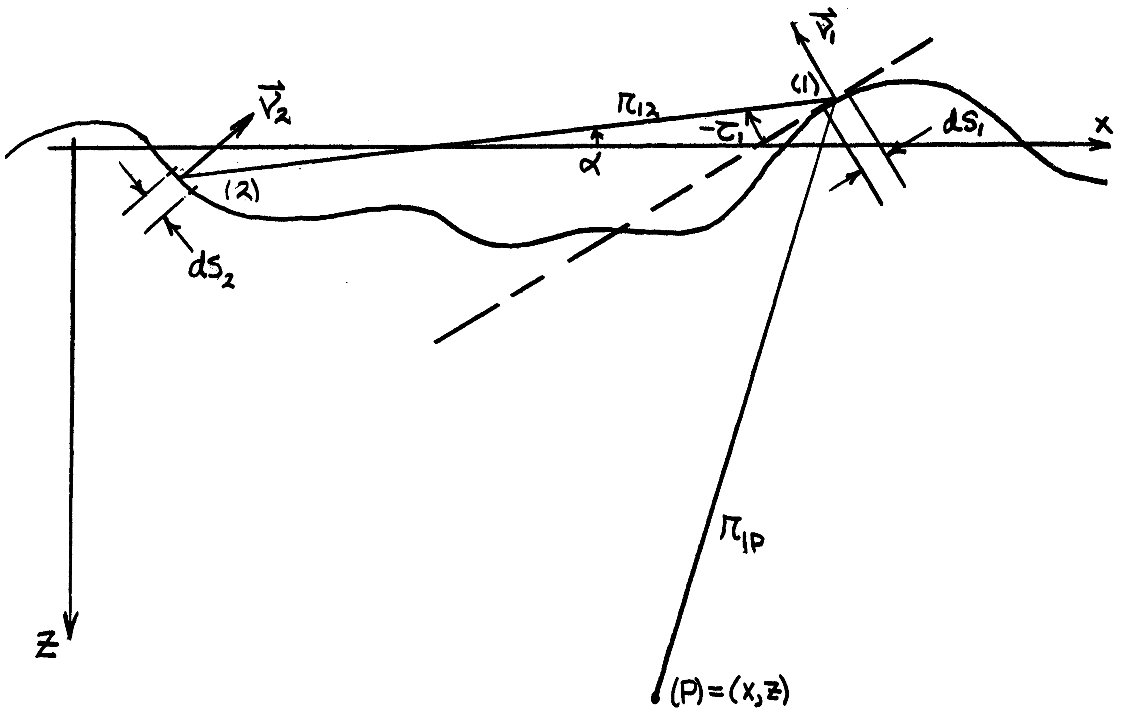

Fig. 4. Diagram used in the discussion of the Helmholtz formula.

The integral in Eq. (3.4.2) is to be carried over the entire surface $\zeta(x)$. The symbol $\frac{\partial}{\partial \nu_1}$ indicates the derivative with respect to the outward normal direction at the surface point (1). We have let $\phi(1)$ stand for the value of the function ϕ at the point (1), $\phi(P)$ represents the value of the function ϕ at some space point P and similarly for other functions appearing in Eq. (3.4.2) and later. The $H_0^{(1)}(y)$ is the Hankel function of the first kind.* We now use the boundary condition given by Eq. (3.1.6); Eq. (3.4.2) becomes

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* Watson, [21], p 73.

$$\phi(P) = \phi_i(P) - \frac{1}{4i}\int_{\zeta} H_0^{(1)}(kr_{1P})\,\frac{\partial}{\partial \nu_1}\phi(1)\,ds_1 \;. \qquad (3.4.3)$$

If we allow P to approach the surface at the point (2), we see

$$\phi_i(2) = \frac{1}{4i}\int_{\zeta} H_0^{(1)}(kr_{12})\,\frac{\partial}{\partial \nu_1}\phi(1)\,ds_1 \quad . \qquad (3.4.4)$$

Once the function $\frac{\partial \phi(1)}{\partial \nu_1}$ is known, the solution to the problem can be obtained from Eq. (3.4.3). In order to obtain an estimate for this function we set up an integral equation of the second kind and propose a solution by iteration.*

We take $\frac{\partial}{\partial \mu_P}$ of Eq. (3.4.3); the point (P) is then allowed to approach the surface along the normal at the point (2); at the same time the derivative $\frac{\partial}{\partial \mu_P}$ approaches $\frac{\partial}{\partial \nu_2}$. It is assumed that the surface is sufficiently regular so that so long as (P) does not lie at (2), we may differentiate under the integral sign, obtaining

$$\frac{\partial \phi(2)}{\partial \nu_2} = \frac{\partial \phi_i(2)}{\partial \nu_2} - \frac{1}{4i}\lim_{(P)\to(2)}\int_{\zeta} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P}\,\frac{\partial \phi(1)}{\partial \nu_1}\,ds_1 \quad . \qquad (3.4.5)$$

This operation is valid if the integral in Eq. (3.4.5) is uniformly convergent for $kr_{2P} \geq \delta > 0$; in this connection it is to be remarked that under hypothesis b) the derivative of the integral in Eq. (3.4.5) is a continuous function of r_{2P} for all (P).

The quantity $\frac{\partial H_0^{(1)}}{\partial \mu_P}$ in the integrand represents a distribution of dipoles. It is a well known result of potential theory that limiting processes such as the one in Eq. (3.4.5) introduce integrable singularities

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* For the use of integral equations in the treatment of more general reflection problems, see a memorandum by C.L. Dolph, [22].

in the integrand [23]. In order to evaluate the limit, we first suppose that the function $\zeta(x)$ is continuous with a continuous first derivative for all x. The integral of Eq. (3.4.5) is split up as follows:

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{\zeta} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \frac{\partial \phi(1)}{\partial \nu_1} ds_1 =$$

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12} \geq \zeta_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \frac{\partial \phi(1)}{\partial \nu_1} ds_1$$

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12} < \zeta_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \frac{\partial \phi(1)}{\partial \nu_1} ds_1 \; . \qquad (3.4.6)$$

It is assumed that the limit in the first term on the right side of Eq. (3.4.6) can be moved inside the integral sign, which is valid if the integral is uniformly convergent in r_{P2}. Then the limit in the first term is trivial and we are left with the task of evaluating the second term on the right side of Eq. (3.4.6). For convenience we write

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12} < \zeta_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \frac{\partial \phi(1)}{\partial \nu_1} ds_1 \qquad (3.4.7)$$

$$= - \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12} < \zeta_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \frac{\partial \phi(2)}{\partial \nu_2} ds_1$$

$$- \frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12} < \zeta_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \left(\frac{\partial \phi(1)}{\partial \nu_1} - \frac{\partial \phi(2)}{\partial \nu_2} \right) ds_1 \; .$$

The Hankel function is defined by,*

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* Watson, [21], pp. 60 and 64.

$$H_0^{(1)}(y) = J_0(y) + \frac{2i}{\pi}\left[\gamma + \ln\left(\frac{1}{2}y\right)\right] J_0(y)$$

$$- \frac{2i}{\pi} \sum_{m=1}^{\infty} \frac{(-1)^m\left(\frac{1}{2}y\right)^{2m}}{(m!)^2} \left\{\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{m}\right\}, \quad (3.4.8)$$

or,

$$H_0^{(1)}(y) = \frac{2i}{\pi} \ln(y)\, J_0(y) + G(y) \quad (3.4.9)$$

where $G(y)$ is analytic at $y = 0$, and γ is Euler's constant, 0.5772..... Using the assumption that $\zeta(x)$ has a continuous derivative, together with the expression for the Hankel function for small values of its argument, one obtains the following result:

$$\lim_{\zeta_0 \to 0}\left\{-\frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12}<\zeta_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \frac{\partial \phi(2)}{\partial \nu_2}\, ds_1\right\} = \frac{1}{2}\frac{\partial \phi(2)}{\partial \nu_2}. \quad (3.4.10)$$

To see this, ζ_0 is chosen small enough so that the region of surface ($r_{12} < \zeta_0$) is essentially flat. Then using Eq. (3.4.9), Eq. (3.4.10) follows directly. Also since $\frac{\partial \phi}{\partial \nu}$ is continuous, we see that

$$\lim_{\zeta_0 \to 0}\left\{-\frac{1}{4i} \lim_{(P)\to(2)} \int_{r_{12}<\zeta_0} \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \mu_P} \left(\frac{\partial \phi(1)}{\partial \nu_1} - \frac{\partial \phi(2)}{\partial \nu_2}\right) ds_1\right\} = 0. \quad (3.4.11)$$

Now allowing $\zeta_0 \to 0$ in Eq. (3.4.6) and substituting in Eq. (3.4.5) we find,

$$\frac{\partial \phi(2)}{\partial \nu_2} = \frac{\partial \phi_i(2)}{\partial \nu_2} + \frac{1}{2}\frac{\partial \phi(2)}{\partial \nu_2} - \frac{1}{4i}\int_{\zeta} \frac{\partial H_0^{(1)}(kr_{12})}{\partial \nu_2} \frac{\partial \phi(1)}{\partial \nu_1}\, ds_1 \quad . \quad (3.4.12)$$

Equivalently Eq. (3.4.12) may be written,

$$\frac{\partial \phi(2)}{\partial \nu_2} = 2\frac{\partial \phi_i(2)}{\partial \nu_2} - \frac{1}{2i}\int_{\zeta} \frac{\partial H_0^{(1)}(kr_{12})}{\partial \nu_2} \frac{\partial \phi(1)}{\partial \nu_1} ds_1 \; . \qquad (3.4.13)$$

Now we consider the solution of Eq. (3.4.13) by iteration. The procedure is as follows:* one substitutes the right side of Eq. (3.4.13) for the quantity $\frac{\partial \phi}{\partial \nu}$ appearing in the integrand. This again gives the quantity $\frac{\partial \phi}{\partial \nu}$ appearing on the right side, now under a double integral. By repeating the substitution process indefinitely, one generates an infinite series of terms in which each term involves, in general, a multiple integral with the known function $\frac{\partial \phi_i}{\partial \nu}$ appearing in the integrand. This process then yields for the first two terms of the series,

$$\frac{\partial \phi(2)}{\partial \nu_2} = 2\frac{\partial \phi_i(2)}{\partial \nu_2} - \frac{1}{2i}\int_{\zeta} \frac{\partial H_0^{(1)}(kr_{12})}{\partial \nu_2} \left[2\frac{\partial \phi_i(1)}{\partial \nu_1} \right] ds_1 + \dots \; . \qquad (3.4.14)$$

It will be assumed that the infinite series, the first two terms of which are given in Eq. (3.4.14), converges (to $\frac{\partial \phi}{\partial \nu}$) for every point on the bounding surface. We now retain only the first term of Eq. (3.4.14), assuming for the moment that the next term is negligible in comparison to it; this gives,

$$\frac{\partial \phi(2)}{\partial \nu_2} \approx 2\frac{\partial \phi_i(2)}{\partial \nu_2} \; . \qquad (3.4.15)$$

As a check on the result, it is not difficult to see that for $\zeta(x) = 0$ (a flat surface), Eq. (3.4.15) is exact. It is the approximation which would be obtained from an application of the assumption of local flatness.

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* See Lovitt, [24], chap. II.

To complete the solution, Eq. (3.4.15) is substituted in Eq. (3.4.3) to obtain,

$$\phi^{(1)}(P) = \phi_i(P) - \frac{1}{4i}\int_{\zeta} H_0^{(1)}(kr_{1P})2\,\frac{\partial\phi_i(1)}{\partial\nu_1}\,ds_1 \quad , \qquad (3.4.16)$$

when $\phi^{(1)}(P)$ represents the first approximation to the field ϕ.

We shall now consider the first term neglected in the approximation represented by Eq. (3.4.15). We make use of the relation,*

$$\frac{d}{dy}H_0^{(1)}(y) = -H_1^{(1)}(y) \quad , \qquad (3.4.17)$$

and also

$$\frac{\partial r_{12}}{\partial\nu_1} = \sin\tau_1 \quad , \qquad (3.4.18)$$

with τ_1 defined as the angle between the tangent to the surface at the point (1) and the radius vector connecting (1) and (2) (see Fig. 4). Then we have for the second term of Eq. (3.4.14)

$$-\frac{1}{2i}\int_{\zeta}\frac{\partial H_0^{(1)}(kr_{12})}{\partial\nu_2}\left[2\frac{\partial\phi_i(1)}{\partial\nu_1}\right]ds_1$$

$$= \frac{1}{i}\int_{\zeta} H_1^{(1)}(kr_{12})k\sin\tau_2\,\frac{\partial\phi_i(1)}{\partial\nu_1}\,ds_1 . \qquad (3.4.19)$$

We shall label this quantity I; it must be small if Eq. (3.4.15) is to be valid.

It will be convenient to define an order property, O(f(t)). We shall write

$$F(t) = O(f(t)) \qquad (3.4.20)$$

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* Jahnke and Emde, [25], p. 145.

if there exist a pair of positive constants M and T such that

$$\frac{|F(t)|}{f(t)} \leq M \quad , \qquad t \geq T \, , \tag{3.4.21}$$

where $f(t)$ is a positive function for $t \geq T$. In what follows, the constants M and T will be chosen so as to be independent of the properties of the surface, $\zeta(x)$. Furthermore we shall use the symbol M (and M') to indicate an unspecified constant, not always the same.

With these definitions in mind, we proceed with the consideration of the function I. We examine separately the regions of the integrand of Eq. (3.4.19) where $kr_{12} \leq \varepsilon$ and where $kr_{12} > \varepsilon$. The constant ε is chosen small enough so that a circle of radius r_{12} centered at $(x_1, \zeta(x_1))$ intersects $\zeta(x)$ at only two points for $kr_{12} \leq \varepsilon$; it is assumed that the surface is sufficiently regular so that there exists such an ε which can be chosen independent of x. We have,

$$I = I_1 + I_2 \, , \tag{3.4.22}$$

where

$$I_1 = \frac{1}{i} \int_{kr_{12} \leq \varepsilon} H_1^{(1)}(kr_{12}) \sin \tau_2 \frac{\partial \phi_i(1)}{\partial \nu_1} \, ds_1 \, , \tag{3.4.23}$$

and

$$I_2 = \frac{1}{i} \int_{kr_{12} > \varepsilon} H_1^{(1)}(kr_{12}) \sin \tau_2 \frac{\partial \phi_i(1)}{\partial \nu_1} \, kds_1 \, . \tag{3.4.24}$$

Two conditions on the function $\frac{\partial \phi_i}{\partial \nu_1}$ are now established. It is first supposed that there exists an M such that for all x,

$$\left| \frac{\partial \phi_i}{\partial \nu} \right| \leq M \quad . \tag{3.4.25}$$

This condition is fulfilled for any problem with a finite number of sources located in the region $z > \zeta$. Secondly, for simplicity, it is supposed that

$$\frac{\partial \phi_i(1)}{\partial \nu_1} = O\left((kx)^{-\frac{1}{2}-\varepsilon'}\right) , \tag{3.4.26}$$

$\varepsilon' > 0$. This second condition can be relaxed in some cases. It is used here to render the integral in Eq. (3.4.24) absolutely convergent.

We now consider the quantity I_2. To begin we note that,

$$\left|\frac{\zeta_2 - \zeta_1}{x_2 - x_1}\right| \leq \left|\frac{d\zeta^M}{dx}\right| , \tag{3.4.27}$$

since the quantity on the left is just the average slope of the segment of surface between the points (1) and (2). By examining Fig. 4, and using the result given by the relation (3.4.27), it is easily seen that

$$\left|\sin \tau_2\right| \leq 2\left|\frac{d\zeta^M}{dx}\right| . \tag{3.4.28}$$

We also need the asymptotic expansion of $H_1^{(1)}(y)$:*

$$H_1^{(1)}(y) = \left(\frac{2}{\pi y}\right)^{1/2} e^{i(y - \frac{3}{4}\pi)} \left[1 + O(\tfrac{1}{y})\right] , \tag{3.4.29}$$

The order property holding here for $y > \varepsilon$ (y real). Then using Eqs. (3.4.29), (3.4.28), and (3.4.26) in Eq. (3.4.24), we have

$$\left|I_2\right| \leq M\left|\frac{d\zeta^M}{dx}\right| . \tag{3.4.30}$$

In obtaining Eq. (3.4.30) we used

$$ds_1 \leq dx_1 M \tag{3.4.31}$$

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* Watson, [21], p. 197.

which follows if we restrict consideration to surfaces with bounded slope; we have also used the obvious relation that $r_{12} \geq |x_1 - x_2|$.

We now proceed by considering Eq. (3.4.23). The following relation is easily established (cf. Fig. 4):

$$|\sin \tau_2| \leq \frac{1}{2} \frac{r_{12}}{R_m} , \qquad (3.4.32)$$

when R_m is the minimum radius of curvature.

We need the expression for the function $H_1^{(1)}$ corresponding to Eq. (3.4.8):*

$$H_1^{(1)}(y) = J_1(y) + \frac{i}{\pi}\left\{2\left[\gamma + \ln\tfrac{1}{2}y\right] J_1(y) - \frac{2}{y}\right. \qquad (3.4.33)$$

$$\left. - \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{1}{2}y)^{1+2m}}{m!\,(m+1)!}\left[\frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m} + \frac{1}{1} + \frac{1}{2} + \ldots + \frac{1}{1+m}\right]\right\} .$$

From Eq. (3.4.33) it follows that for $kr_{12} \leq \varepsilon$,

$$\left|H_1^{(1)}(y)\right| \leq \frac{M}{y} . \qquad (3.4.34)$$

Then using Eqs. (3.4.34), (3.4.32), and (3.4.25) in Eq. (3.4.23) we have

$$|I_1| \leq \frac{M}{kR_m} , \qquad (3.4.35)$$

where it is supposed that the same ε may be used for all surfaces considered. Then taking the absolute value of Eq. (3.4.22) and using Eqs. (3.4.35) and (3.4.30) we have the required relation,

$$|I| \leq M \left|\frac{d\zeta^M}{dx}\right| + M' \frac{1}{kR_m} . \qquad (3.4.36)$$

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* Watson, [21], p. 62 and p. 64.

Hence the method of this section is applicable to surfaces with small slope when the frequency is high enough so that the radiation wavelength is very much less than a representative radius of curvature of the surface.

We now show that the approximate result, expressed by Eq. (3.4.16), satisfies the reciprocity theorem if there is but one source, i.e.,

$$\phi_i(P) = H_0^{(1)}(kr_{PO}) \quad . \tag{3.4.37}$$

A general transformation of Eq. (3.4.16) is first shown. We consider a system having sources giving a field ϕ_i without the bounding surface $\zeta(x)$, so that the sources radiate in an infinite homogeneous medium. Then applying Eq. (3.4.2),

$$\phi_i(P) = \phi_i(P) + \frac{1}{4i} \int_\zeta \left\{ \phi_i(1) \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \nu_1} - H_0^{(1)}(kr_{1P}) \frac{\partial \phi_i(1)}{\partial \nu_1} \right\} ds_1 \quad , \tag{3.4.38}$$

or

$$-\frac{1}{4i} \int_\zeta H_0^{(1)}(kr_{1P}) \frac{\partial \phi_i(1)}{\partial \nu_1} \, ds_1 = -\frac{1}{4i} \int_\zeta \phi_i(1) \frac{\partial H_0^{(1)}(kr_{1P})}{\partial \nu_1} \, ds_1 \tag{3.4.39}$$

Now Eq. (3.4.39) is used in Eq. (3.4.16) to obtain,

$$\phi^{(1)}(P) = \phi_i(P) - \frac{1}{4i} \int_\zeta \frac{\partial}{\partial \nu_1}(H_0^{(1)}(kr_{1P})\phi_i(1))ds_1 \quad . \tag{3.4.40}$$

It is evident that if we specialize the problem by applying Eq. (3.4.37), Eq. (3.4.40) is unchanged by the interchange of (O) and (P), for we have,

$$\phi^{(1)}(P) = H_0^{(1)}(kr_{PO}) - \frac{1}{4i} \int_\zeta \frac{\partial}{\partial \nu_1}(H_0^{(1)}(kr_{1P})\, H_0^{(1)}(kr_{1O}))ds_1 . \tag{3.4.41}$$

Thus when a single source excites the field, Eq. (3.4.16) satisfies the reciprocity theorem.

We now examine the value of the approximate result, given by Eq. (3.4.16), on the boundary. From the boundary condition, Eq. (3.1.6), the function $\phi^{(1)}(2)$ should vanish. Subtracting Eq. (3.4.16) from Eq. (3.4.3) we see,

$$\phi(P) - \phi^{(1)}(P) = -\frac{1}{4i}\int_{\zeta} H_0^{(1)}(kr_{1P})\left[\frac{\partial\phi(1)}{\partial\nu_1} - 2\frac{\partial\phi_i(1)}{\partial\nu_1}\right] ds_1 \quad . \quad (3.4.42)$$

When (P) lies on the boundary at the point (2), Eq. (3.4.42) becomes,

$$-\phi^{(1)}(2) = -\frac{1}{4i}\int_{\zeta} H_0^{(1)}(kr_{12})\left[\frac{\partial\phi(1)}{\partial\nu_1} - 2\frac{\partial\phi_i(1)}{\partial\nu_1}\right] ds_1 \ . \quad (3.4.43)$$

Hence if the error in the determination of the value of $\frac{\partial\phi}{\partial\nu}$ on the boundary is made small by hypotheses i) and ii) , one may expect the value of the approximation given by Eq. (3.4.16) to be small there, and indeed to be of the same order.

There is one further point of interest in connection with the approximation of this section. It is easily seen from Eqs. (3.4.19) and (3.4.28) that the first term neglected in the approximation contains a term of first order in the surface slope. It seems reasonable to suppose that the effect of this error would be greatest in regions where the reflected radiation is far removed from the specular direction. Brekhovskikh [19] finds such deviations in comparing calculations using the present approximation with an exact formulation due to Rayleigh.

This completes our review of the major methods at present available for the solution of the reflection problem. Before beginning our next topic, it is in order to mention briefly two other approximations which have been used.

Rayleigh proposed a unique formulation for reflection problems in which $\zeta(x)$ is a periodic function of x;* such surfaces are of evident interest in the treatment of diffraction gratings. He was led to an infinite system of linear equations to be inverted. By applying the additional restriction that the repeat distance of the surface should be very much greater than the radiation wavelength, Rayleigh was able to solve this system. He states that his result is equivalent to that obtained through the use of the Kirchhoff approximation.

Twersky [26] has recently considered surface reflection problems of a restricted class by making what may be called a single scattering hypothesis. In other words he proposes that the radiation reflected from the surface under certain conditions can be considered as having undergone a single reflection. In later work he has set up a formulation taking into account multiple scattering.

3.5. A Fourier Transform Method

Before proceeding with the duct propagation problem, we should like to take this opportunity to present a method of solution for the reflection problem which depends upon taking the Fourier transform of an approximation to Eq. (3.4.4). It will be seen that the method is useful for

i) $\left|\frac{d\zeta^M}{dx}\right| \ll 1$ and

ii) $k\zeta^M \lesssim 1$;

the symbol $\sim$ indicates "is of the order of magnitude of". Of these two restrictions, the first is the more important.

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* Rayleigh, [5], p. 89.

Let us consider Eq. (3.4.4),

$$\phi_i(1) = \frac{1}{4i}\int_{\zeta} H_0^{(1)}(kr_{12}) \frac{\partial\phi(2)}{\partial\nu_2}\, ds_2 \quad , \tag{3.5.1}$$

where,

$$r_{12} = \left[(x_1-x_2)^2 + (\zeta(x_1) - \zeta(x_2))^2\right]^{1/2} \quad . \tag{3.5.2}$$

If Eq. (3.5.1) could be solved for the function $\frac{\partial\phi}{\partial\nu}$, its value could be substituted in Eq. (3.4.3) to obtain the solution to the problem. One is tempted to let

$$r_{12} \approx |x_1-x_2| \quad , \tag{3.5.3}$$

since in such a case Eq. (3.5.1) could be solved. Let us examine the error incurred by making this assumption. First we let

$$r_{12} = \frac{|x_1-x_2|}{\cos\alpha} \tag{3.5.4}$$

where α is the acute angle between r_{12} and the x direction (see Fig. 4). Then we define the difference between the exact and the approximate kernel for Eq. (3.5.1) as

$$K(x_1,x_2) = H_0^{(1)}(k \frac{|x_1-x_2|}{\cos\alpha}) - H_0^{(1)}(k|x_1-x_2|) . \tag{3.5.5}$$

In considering the error made in adopting the assumption represented by Eq. (3.5.3), it is convenient to treat the following regions separately: I, $k|x_1-x_2| \ll 1$; II, $k|x_1-x_2| \gg 1$; III, all other values of the argument, $k|x_1-x_2|$. We shall also suppose in this connection that $a \ll 1$ indicates $a \leq 0.1$, and similarly $a \gg 1$ indicates $a \geq 10.0$. Because of the (integrable) singularity of $H_0^{(1)}(y)$ at $y=0$, it is to be expected that I is the most important region of the integrand in Eq. (3.5.1). We begin by considering it.

Upon referring to Eqs. (3.4.8) and (3.5.4) we see that in I

$$|K| \sim \frac{2}{\pi} \ln \frac{1}{\cos \alpha} \quad , \tag{3.5.6}$$

or from the definition of α, assuming $\frac{d\zeta^M}{dx}$ small we have,

$$|K| \sim \frac{1}{\pi} \left| \frac{d\zeta^M}{dx} \right|^2 \quad . \tag{3.5.7}$$

For region II we use the asymptotic form for the Hankel function,*

$$H_0^{(1)}(y) = \frac{e^{iy}}{\left(\frac{\pi}{2}iy\right)^{1/2}} \left[1 + O(\tfrac{1}{y})\right] \quad , \tag{3.5.8}$$

in Eq. (3.5.5) to find

$$K \approx \frac{e^{ik|x_1-x_2|}}{\left(\frac{\pi}{2}ik|x_1-x_2|\right)^{1/2}} \left\{(\cos \alpha)^{1/2} \left[\exp\ ik|x_1-x_2|\left(\frac{1}{\cos\alpha} - 1\right)\right] - 1\right\}. \tag{3.5.9}$$

omitting $O(\frac{1}{y})$ as a term of higher order. It is easily seen that

$$|\alpha| \leq \frac{2\zeta^M}{|x_1-x_2|} \quad , \tag{3.5.10}$$

so that Eq. (3.5.9) becomes,

$$K \sim \frac{e^{ik|x_1-x_2|}}{\left(\frac{\pi}{2}ik|x_1-x_2|\right)^{1/2}} \left[- \frac{(k\zeta^M)^2}{(k|x_1-x_2|)^2} + 2i \frac{(k\zeta^M)^2}{k|x_1-x_2|}\right] , \tag{3.5.11}$$

or ignoring the term $-\frac{(k\zeta^M)^2}{(k|x_1-x_2|)^2}$, which is of higher order,

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* Watson, [21], p. 197.

$$K \sim \frac{e^{ik|x_1-x_2|}}{\left(\frac{\pi}{2} ik|x_1-x_2|\right)^{1/2}} \; 2i \, \frac{(k\zeta^M)^2}{k|x_1-x_2|} \quad . \tag{3.5.12}$$

Finally region III is considered. We note first that for α small,

$$K \approx \frac{d}{d(k|x_1-x_2|)} H_0^{(1)}(k|x_1-x_2|)k|x_1-x_2|\left(\frac{1}{\cos\alpha} - 1\right) \quad ,$$

or using Eq. (3.4.17),

$$K \approx - H_1^{(1)}(k|x_1-x_2|)k|x_1-x_2|\left(\frac{1}{\cos\alpha} - 1\right) \quad . \tag{3.5.13}$$

From the pertinent tables in Jahnke and Emde,* we see that the largest value of $\left|H_1^{(1)}(k|x_1-x_2|)k|x_1-x_2|\right|$ for $0.1 \leq k|x_1-x_2| \leq 10$ occurs at $k|x_1-x_2| = 10$ and,

$$\left|H_1^{(1)}(10) \cdot 10\right| = 2.53 \; . \tag{3.5.14}$$

Again from the definition of α and using Eq. (3.5.14) (replacing 2.53 by 3.0), Eq. (3.5.13) becomes

$$|K| \sim \frac{3}{2} \left|\frac{d\zeta^M}{dx}\right|^2 \quad . \tag{3.5.15}$$

To summarize, it is seen that K is negligible in region I if hypothesis i) is fulfilled (cf. Eq. (3.5.7)); the same hypothesis makes K small in region III (cf. Eq. (3.5.15)). Finally from Eq. (3.5.12) we see that K is negligible in region II if hypothesis ii) is fulfilled. It is noted however that in region II the kernel of Eq. (3.5.1) is small so that an error in this region is not so important as errors in the other two. Thus in some problems the approximation may lead to a useful result

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* Jahnke and Emde, [25], pp. 157 and 191.

even though the hypothesis ii) is not fulfilled.

We now proceed with a method of solution based upon the assumption that K is small. Let us rewrite Eq. (3.5.1), using Eq. (3.5.5),

$$\phi_i(1) = \frac{1}{4i}\int_{\zeta}\left[H_0^{(1)}(k|x_1-x_2|) + K(x_1,x_2)\right]\frac{\partial\phi(2)}{\partial\nu_2}\,ds_2. \qquad (3.5.16)$$

Before continuing we make some changes in notation. Let

$$\Psi(x_2) = \frac{1}{\cos\chi(x_2)}\,\frac{\partial\phi(2)}{\partial\nu_2}, \qquad (3.5.17)$$

$$F(x_1) = 4i\,\phi_i(1)\ , \qquad (3.5.18)$$

and note

$$ds_2 = \frac{dx_2}{\cos\chi(x_2)}\ . \qquad (3.5.19)$$

Here $\chi(x)$ is the acute angle made by the tangent to the surface at x_2 with the x-axis. Then substituting in Eq. (3.5.16) we have

$$F(x_1) = \int_{-\infty}^{\infty}\left[H_0^{(1)}(k|x_1-x_2|) + K(x_1,x_2)\right]\Psi(x_2)dx_2. \qquad (3.5.20)$$

As the next step let us consider $K(x_1,x_2)$ as a perturbation. Let

$$K(x_1,x_2) = \varepsilon\,K(x_1,x_2), \qquad (3.5.21)$$

$$\Psi(x) = \Psi^{(0)}(x) + \varepsilon\,\Psi^{(1)}(x) + \varepsilon^2\,\Psi^{(2)}(x) + \dots \qquad (3.5.22)$$

where we assume that $\Psi(x)$ is an analytic function of ε for $0 \leq \varepsilon \leq 1$. The solution is obtained by allowing $\varepsilon \to 1$. By substituting Eqs. (3.5.21) and (3.5.22) in Eq. (3.5.20), and equating equal powers of ε we see:

$$\int_{-\infty}^{\infty} H_0^{(1)}(k|x_1-x_2|)\Psi^{(0)}(x_2)dx_2 = F(x_1) \ , \qquad (3.5.23)$$

$$\left.\begin{aligned} \int_{-\infty}^{\infty} H_0^{(1)}(k|x_1-x_2|)\Psi^{(1)}(x_2)dx_2 &= -\int_{-\infty}^{\infty} K(x_1,x_2)\Psi^{(0)}(x_2)dx_2, \\ \int_{-\infty}^{\infty} H_0^{(1)}(k|x_1-x_2|)\Psi^{(2)}(x_2)dx_2 &= -\int_{-\infty}^{\infty} K(x_1,x_2)\Psi^{(1)}(x_2)dx_2 \ , \\ &\vdots \end{aligned}\right\} (3.4.24)$$

There is a well known method due to Levi-Civita [27] which may be used for the solution of integral equations of the above type in which the kernel is a function of the difference of its two variables.* The method consists of taking the Fourier transform of the integral equation. Accordingly let us take the Fourier transform of Eq. (3.5.23) and Eqs.(3.5.24). We have,**

$$(2\pi)^{1/2}h(t)\Psi^{(0)}(t) = f(t) \ , \qquad (3.5.25)$$

$$(2\pi)^{1/2}h(t)\Psi^{(1)}(t) = -\ \text{F.T.}\left\{\int_{-\infty}^{\infty} K(x_1,x_2)\Psi^{(0)}(x_2)dx_2\right\}, \qquad (3.5.26)$$

$$(2\pi)^{1/2}h(t)\Psi^{(2)}(t) = -\ \text{F.T.}\left\{\int_{-\infty}^{\infty} K(x_1,x_2)\Psi^{(1)}(x_2)dx_2\right\}, \qquad (3.5.27)$$

$$\vdots$$

where

$$\begin{aligned} f(t) &= \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} e^{itx}F(x)dx \ , \qquad (3.5.28) \\ &= \text{F.T.}\left\{F(x)\right\} \ . \end{aligned}$$

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* For a modern presentation see Titchmarsh, [28], chap. XI.

** Titchmarsh, [28], p.59.

Similarly,

$$\left.\begin{aligned} h(t) &= \text{F.T.}\left\{H_0^{(1)}(k|x|)\right\} \;, \\ \psi^{(0)}(t) &= \text{F.T.}\left\{\Psi^{(0)}(x)\right\} \;, \\ \psi^{(1)}(t) &= \text{F.T.}\left\{\Psi^{(1)}(x)\right\} \;, \\ &\vdots \end{aligned}\right\} \qquad (3.5.29)$$

In order to guarantee that the transforms of Eq. (3.5.23) and Eqs. (3.5.24) will exist and yield Eq. (3.5.25) and Eqs. (3.5.26) and (3.5.27), it is sufficient to require that $\Psi^{(0)}, \Psi^{(1)}, \ldots$ and $H_0^{(1)}(k|x|)$ be absolutely integrable and that $F(x_1)$ and $\int_{-\infty}^{\infty} K(x_1,x_2)\Psi^{(n)}(x_2)dx_2$ have Fourier transforms.* If these conditions are not satisfied by $\Psi^{(n)}$, the solution is subject to verification. For the function $H_0^{(1)}(k|x|)$ we will let $k = k' + i\delta$, $\delta > 0$, in order to guarantee absolute integrability, then allow $\delta \to 0$ in the solution.

Now if we add the restriction that $\Psi^{(n)}(x)$ be sectionally continuous we can, after solving Eqs. (3.5.25), (3.5.26), and (3.5.27) for $\psi^{(n)}(t)$, take the inverse Fourier transform.** It is known that,***

$$h(t) = \left(\frac{2}{\pi}\right)^{1/2} (k^2-t^2)^{-1/2} \quad . \qquad (3.5.30)$$

We have then from Eqs. (3.5.25),(3.5.26), and (3.5.27):

$$\Psi^{(0)}(x) = \frac{1}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} (k^2-t^2)^{1/2} e^{-ixt} f(t)dt \;, \qquad (3.5.31)$$

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* Titchmarsh, [28], p.59.

** Churchill, [29], secs. 40 and 41.

*** Campbell and Foster, [30], No. 918.

$$\Psi^{(1)}(x) = -\frac{1}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} (k^2-t^2)^{1/2} e^{-ixt} dt$$

$$\cdot \left\{ \text{F.T.} \left[\int_{-\infty}^{\infty} K(x_1,x_2)\Psi^{(0)}(x_2)dx_2 \right] \right\}, \qquad (3.5.32)$$

$$\Psi^{(2)}(x) = -\frac{1}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} (k^2-t^2)^{1/2} e^{-ixt} dt$$

$$\cdot \left\{ \text{F.T.} \left[\int_{-\infty}^{\infty} K(x_1,x_2)\Psi^{(1)}(x_2)dx_2 \right] \right\}, \qquad (3.5.33)$$

⋮

The solution, $\Psi(x)$, is obtained from

$$\Psi(x) = \Psi^{(0)}(x) + \Psi^{(1)}(x) + \ldots \quad . \qquad (3.5.34)$$

The symbols in Eqs. (3.5.31), (3.5.32), and (3.5.33) are defined by:

$$\Psi(x) = \frac{1}{\cos \chi(x)} \frac{\partial \phi}{\partial \nu} , \qquad (3.5.17)$$

$$f(t) = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{itx} F(x)dx, \qquad (3.5.28)$$

$$F(x) = 4i\ \phi_i(x) , \qquad (3.5.18)$$

$$K(x_1,x_2) = H_0^{(1)}(kr_{12}) - H_0^{(1)}(k|x_1-x_2|) . \qquad (3.5.5)$$

One obtains the solution, $\phi(P)$, from

$$\phi(P) = \phi_i(P) - \frac{1}{4i}\int_{\zeta} H_0^{(1)}(kr_{1P})\frac{\partial \phi(1)}{\partial \nu_1}\, ds_1 \quad . \qquad (3.4.3)$$

We see from Eqs. (3.5.32) and (3.5.33) that the corrections to the first approximation, $\Psi^{(0)}$, are small if K is small enough.

As a check on the above result we can replace hypothesis ii) of this section by the stronger, $k\zeta^M \ll 1$. In such a case the present method and the perturbation method should give the same result. We shall show that this is so by showing that through terms of first order in $k\zeta$ and $\frac{d\zeta}{dx}$, the value of $\left.\frac{\partial\phi}{\partial\nu}\right|_{z=\zeta(x)}$ obtained by the two methods is the same.

We begin with the perturbation method. We assume that the incident radiation consists of a plane wave:

$$\phi_i = e^{iK_i x - i(k^2-K_i^2)^{1/2} z} \tag{3.5.35}$$

We shall not restrict K_i, except to require that it be real, so the incident wave may be inhomogeneous. If we prove that $\left.\frac{\partial\phi}{\partial\nu}\right|_{z=\zeta}$ is the same by the two methods for the single incident plane wave, then it follows for any sum of incident plane waves. Let us now for convenience group together the incident wave and the specularly reflected wave

$$\overline{\phi_i} = \phi_i(x,z) - \phi_i(x,-z)$$

$$= -2i\, e^{iK_i x} \sin\,(k^2-K_i^2)^{1/2} z \quad . \tag{3.5.36}$$

Furthermore,

$$\left.\overline{\phi}_i\right|_{z=\zeta(x)} = -2i\, e^{iK_i x} \sin\,(k^2-K_i^2)^{1/2}\zeta(x) \; . \tag{3.5.37}$$

Referring now to Eq. (3.5.37) and Eqs. (3.3.3) and (3.3.7) we see

$$\overline{\phi}_i^{(0)} = 0 \; , \tag{3.5.38}$$

so that $B^{(0)}(K) = 0$; also,

$$\zeta(x)\, \overline{\phi}_i^{(1)} = -\,2i\, e^{iK_i x} (k^2-K_i^2)^{1/2}\zeta(x) \quad . \tag{3.5.39}$$

Then using Eq. (3.3.8),

$$B^{(1)}(K) = \frac{i}{\pi} (k^2-K_i^2)^{1/2} \int_{-\infty}^{\infty} e^{i(K_i-K)x} \zeta(x) dx \quad . \qquad (3.5.40)$$

When Eq. (3.5.36) is substituted in Eqs. (3.3.11) and (3.1.5) we have for the total field, through terms of first order in $k\zeta$,

$$\begin{aligned} \phi(x,z) &= -2ie^{iK_i x} \sin(k^2-K_i^2)^{1/2} z \\ &\quad + \int_{-\infty}^{\infty} e^{iKx + i(k^2-K^2)^{1/2} z} B^{(1)}(K) dK. \end{aligned} \qquad (3.5.41)$$

Now we wish to calculate $\left.\frac{\partial \phi}{\partial \nu}\right|_{z=\zeta}$. Through terms of first order in $k\zeta$ and $\frac{d\zeta}{dx}$ we have,

$$\begin{aligned} \left.\frac{\partial \phi}{\partial \nu}\right|_{z=\zeta(x)} &= 2i(k^2-K_i^2)^{1/2} e^{iK_i x} \\ &\quad + \frac{(k^2-K_i^2)^{1/2}}{\pi} \int_{-\infty}^{\infty} (k^2-K^2)^{1/2} e^{iKx} dK \int_{-\infty}^{\infty} e^{i(K_i-K)x'} \zeta(x') dx' \quad . \end{aligned} \qquad (3.5.42)$$

We have assumed that we may differentiate under the integral sign in Eq. (3.5.41); this step is valid if $\int_{-\infty}^{\infty} \frac{\partial}{\partial z} e^{iKx+i(k^2-K^2)^{1/2} z} B^{(1)}(K) dK$ is uniformly convergent in z and if $B^{(1)}(K)$ is continuous. It should be noted that in obtaining Eq. (3.5.42) we have dropped terms of order $\zeta \frac{d\zeta}{dx}$.

Let us calculate the same quantity, $\frac{\partial \phi}{\partial \nu}$, using the method of this section. We have,

$$\left.\phi_i\right|_{z=\zeta(x)} = e^{iK_i x - i(k^2-K_i^2)^{1/2} \zeta(x)} \qquad (3.5.43)$$

When Eq. (3.5.43) is substituted in Eq. (3.5.31) we have, through terms of first order in $\frac{d\zeta}{dx}$,

$$\left.\frac{\partial\phi(x)}{\partial\nu}\right|_{z=\zeta} = \frac{1}{2(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{-ixt}(k^2-t^2)^{1/2} \tag{3.5.44}$$

$$\cdot\left\{\frac{4i}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} e^{itx'} e^{iK_i x-i(k^2-K_i^2)^{1/2}\zeta(x')} dx'\right\}dt.$$

We can now apply the restriction $|k\zeta| \ll 1$. This reduces Eq. (3.5.44), after a change of integration variable, to:

$$\left.\frac{\partial\phi(x)}{\partial\nu}\right|_{z=\zeta} = 2i(k^2-K_i^2)^{1/2}e^{iK_i x} \tag{3.5.45}$$

$$+ \frac{(k^2-K_i^2)^{1/2}}{\pi}\int_{-\infty}^{\infty} e^{itx}(k^2-t^2)^{1/2}dt \int_{-\infty}^{\infty} e^{i(K_i-t)x'}\ \zeta(x')dx' ,$$

where we have retained terms through the first order in $k\zeta$ and in $\frac{d\zeta}{dx}$. We have made use of the Fourier representation of the Dirac delta function:*

$$\delta(x-x_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x-x_0)\chi}\, d\chi \quad . \tag{3.5.46}$$

It is seen that Eqs. (3.5.42) and (3.5.45) are identical, establishing the proposed equivalence.

We shall now work a problem using the method of the present section, and check it by the perturbation method.

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* Schiff, [15], p. 51.

We choose as the reflecting surface

$$\zeta(x) = \frac{\beta}{k}\frac{\sin kx}{kx} \quad . \tag{3.5.47}$$

The surface consists of a damped train of sine waves with a wavelength equal to that of the incident radiation; β is a free parameter. We suppose that the incident radiation consists of a plane wave falling normally upon the x-axis; for convenience we will group the incident and the specularly reflected plane wave, calling their sum ϕ_i. The amplitude of the incident wave will be taken as unity. We have

$$\phi_i = -2i\sin kz \ . \tag{3.5.48}$$

Furthermore using Eq. (3.5.47) we find for the value of this function on the surface

$$\phi_i \Big|_{z=\zeta} = -2i\beta \frac{\sin kx}{kx} \ , \qquad \text{for} \quad \beta \ll 1. \tag{3.5.49}$$

Now we substitute Eq. (3.5.49) in Eq. (3.5.31) to obtain,*

$$\Psi^{(0)}(x) = 2\pi\beta \frac{J_1(kx)}{x} \ , \tag{3.5.50}$$

$$\approx \frac{\partial \phi}{\partial \nu} \ , \tag{3.5.51}$$

for the estimate of the normal derivative, $\frac{\partial \phi}{\partial \nu}$, on the surface($\frac{1}{\cos\chi} = 1$ to terms of order $\left(\frac{d\zeta}{dx}\right)^2$). From Eq. (3.5.50) we see that $\Psi^{(0)}(x)$ is bounded and also

$$\Psi^{(0)}(x) = O(|x|^{-3/2}) \tag{3.5.52}$$

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* The needed Fourier transforms may be found in Campbell and Foster, [30], Nos. 622 and 914.7.

so that the function is absolutely integrable. This property, together with the fact that F(x) possesses a Fourier transform, shows that Eq. (3.5.50) is the solution to Eq. (3.5.23). In order to obtain the solution to the reflection problem, we substitute Eqs. (3.5.48) and (3.5.50) in Eq. (3.4.3) to obtain

$$\phi(P) = -2i \sin kz - \frac{\pi\beta}{2i} \int_{-\infty}^{\infty} H_0^{(1)}(kr_{1P}) \frac{J_1(kx_1)}{x_1} \, dx_1 \quad . \qquad (3.5.53)$$

The exact evaluation of the integral in Eq. (3.5.53) is difficult because of the complex dependence of r_{1P} on x_1 . For this reason we shall restrict our attention to the field a great distance from the surface (the far field).

We shall need an estimate of r_{1P}. From Fig. 4 we see

$$r_{1P} = r - \cos\theta x_1 - \sin\theta\, \zeta(x_1) + O\left(\left[\frac{x_1}{r}\right]^2\right) + O\left(\left[\frac{\zeta^M}{r}\right]^2\right). \qquad (3.5.54)$$

The angle θ is defined by $\sin\theta = \frac{z}{r}$ where $r = (x^2+z^2)^{1/2}$. We now suppose that $\left|\frac{\zeta^M}{r}\right| \ll 1$ so that the last term in Eq. (3.5.54) may be neglected. Furthermore it is not difficult to see that there exists an X_1 such that for $|x_1| > X_1$, the integrand of Eq. (3.5.53) may be neglected. Then we choose $\left|\frac{X_1}{r}\right| \ll 1$ so that the term of order $\left[\frac{x_1}{r}\right]^2$ in Eq. (3.5.54) may be neglected. We use the asymptotic expansion of $H_0^{(1)}$ (cf. Eq. (3.5.8)),

$$H_0^{(1)}(kr_{1P}) \approx \left(\frac{2}{\pi i k r_{1P}}\right)^{1/2} e^{ikr_{1P}} \qquad (3.5.55)$$

$$\approx \left(\frac{2}{\pi i k r_{1P}}\right)^{1/2} e^{ik(r-\cos\theta x_1 - \sin\theta\,\zeta(x_1))} , \qquad (3.5.56)$$

where we have substituted from Eq. (3.5.54). The error term in Eq. (3.5.55) is $O((kr)^{-3/2})$; we let $kr_{1P} \gg 1$ in order to make this error negligible. Substituting Eq. (3.5.56) in Eq. (3.5.53) and using $\beta \ll 1$,

$$\phi(P) = -2i\sin kz + \left(\frac{i\pi}{2}\right)^{1/2} \beta \frac{e^{ikr}}{(kr)^{1/2}} \int_{-\infty}^{\infty} e^{-ik\cos\theta x_1}\left(\frac{J_1(kx_1)}{x_1}\right) dx_1 . \quad (3.5.57)$$

The integral in Eq. (3.5.57) is a known Fourier transform,* so that we have finally,

$$\phi(P) = -2i\sin kz + (2\pi i)^{1/2} \beta \frac{e^{ikr}}{(kr)^{1/2}} \sin\theta \quad . \quad (3.5.58)$$

We will now show that Eq. (3.5.58) is also the result which one obtains from the perturbation method. Comparing Eqs. (3.5.47) and (3.5.48) with Eq. (3.3.3) we see, through terms of first order in ε,

$$\phi_i^{(0)} = 0 \quad (3.5.59)$$

$$\zeta(x)\phi_i^{(1)} = -2i\beta \frac{\sin kx}{kx} \quad , \quad (3.5.60)$$

Now Eq. (3.5.59) substituted in Eq. (3.3.7) shows that $B^{(0)}(k)=0$; this fact together with Eq. (3.5.60) and Eq. (3.3.8) gives:

$$B^{(1)}(K) = \frac{i\beta}{\pi} \int_{-\infty}^{\infty} e^{-iKx}\left(\frac{\sin kx}{kx}\right) dx \quad ,$$

or,**

$$\left. \begin{aligned} B^{(1)}(K) &= \frac{i\beta}{k} \quad , \qquad &|K| < k \\ &= 0 \quad , \qquad &|K| > k . \end{aligned} \right\} \quad (3.5.61)$$

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* Campbell and Foster, [30], No. 914.7.

** Campbell and Foster, [30], No. 622.

It is interesting to notice that the coefficients of the inhomogeneous waves, for which $K > k$, vanish for this surface to this order. If the surface consisted of a pure sine wave rather than a damped sine wave, there would be a single homogeneous plane wave reflected, and this wave would have a propagation vector parallel to the z-axis. Thus we see from Eq. (3.5.61) that the effect of damping the surface sine wave (cf. Eq. (3.5.47)) has been to make the coefficients of all reflected homogeneous waves equal.

Substituting Eq. (3.5.61) in Eq. (3.3.21) we have for the first order far field,

$$\phi_{rs}(x,z) \approx (2\pi i)^{1/2} \beta \frac{e^{ikr}}{(kr)^{1/2}} \sin\theta . \qquad (3.5.62)$$

Finally substituting Eqs. (3.5.62) and (3.5.48) in Eq. (3.1.5) we have

$$\phi(P) = -2i \sin kz + (2\pi i)^{1/2} \beta \frac{e^{ikr}}{(kr)^{1/2}} \sin\theta , \qquad (3.5.63)$$

which is identical with Eq. (3.5.58).

To summarize, the method of this section is applicable to surfaces of small slope where the displacement of the surface is of the order of, or smaller than, the radiation wavelength. The result of the approximation may be expected to be valid through terms of first order in the slope of the surface; in this respect the present approximation has an advantage over the Kirchhoff approximation for problems in which they are both applicable. Again, as for the perturbation method, in the remainder of this work we shall not have occasion to consider problems for which the method of this section is applicable. This is so since in the duct propagation problem we are interested in surfaces which reflect little energy specularly. Hypothesis ii) of this section will be seen to be incompatible with the conditions necessary to establish such diffuse reflection.

CHAPTER IV

DERIVATION OF THE MODEL FOR THE PROPAGATION OF RADIATION IN DUCTS

In Chapter II we presented a model for the propagation of radiation in surface bounded ducts; the presentation was largely heuristic. In Chapter III, some of the methods for the solution of the reflection problem in a homogeneous medium were considered. In particular we considered a method based upon geometrical optics, a perturbation method, a method based upon the use of the Kirchhoff assumption, and a method of solution depending upon an approximation of an integral equation. This work on the reflection problem was presented largely as preparation for the present chapter. In this chapter we propose to rigorously derive the equations presented in Chapter II.

First we shall develop a formula for inhomogeneous media, analogous to the Helmholtz formula for the homogeneous medium.* In this development, we shall need a Green's function for the inhomogeneous medium; the properties of this function in the case where the inhomogeneities are such as to allow the use of geometrical optics will then be considered. Following this we shall develop Eqs. (2.1.18) and (2.1.23) on the basis of the Kirchhoff assumption. Some further restrictions on the reflecting surface will be needed. Furthermore we shall institute an averaging, either over space or

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* The formula to be developed is a special case of a much more general result of Hadamard's [31].

over time, whichever is pertinent to the problem at hand. The equations presented in Section 2.1 will then be seen to govern the average intensities. Following this presentation for the Kirchhoff reflection method, we shall show that with minor modifications in the argument the same result is obtained for situations in which the methods of geometrical optics may be applied to the reflection problem (see Section 3.2).

The perturbation method of Section 3.3 and the approximation method of Section 3.5 have been seen to be best suited to problems in which the radiation wavelength is large compared with, or at least comparable to, the surface displacement. It will be seen in Section 4.4 that in order to guarantee that the reflected radiation is largely diffuse, one must establish the condition that the surface displacement is large compared with the radiation wavelength. Hence we shall not consider the reflection methods of Sections 3.3 and 3.5 in connection with the statistical model.

4.1. The Helmholtz Formula for Inhomogeneous Media

We first state the problem to be solved. It is desired to find a solution Φ of the wave equation in two dimensions,*

$$\nabla^2\Phi(x,z;t) = \frac{1}{c^2(x,z)}\,\frac{\partial^2\Phi(x,z,t)}{\partial t^2}\ , \tag{4.1.1}$$

and satisfying the boundary condition,

$$\Phi(x,\zeta(x);t) = 0 \quad . \tag{4.1.2}$$

For the present purpose, the problem has been generalized to include non-stratified media. Further Φ is to have a logarithmic singularity at the source point (0) and is to reduce to outgoing waves as $(x^2+z^2)^{1/2}\to\infty$.

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* A development similar to that presented here can be made for a three dimensional wave equation by suitably modifying the condition (4.1.3).

The symbol ∇^2 is defined by $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$. We now suppose that the source generates a disturbance of a single angular frequency $\omega = 2\pi f$ where f is the frequency in cycles per second; then we let $\Phi = \phi(x,z)e^{-i\omega t}$. The singularity at (0) becomes,

$$\phi \rightarrow H_0^{(1)}\left[k_0((x-x_0)^2 + (z-z_0)^2)^{1/2}\right], \qquad (4.1.3)$$

as $x \rightarrow x_0$ and $z \rightarrow z_0$; $k_0 = k(x_0, z_0)$; and Eq. (4.1.1) can be written,

$$(\nabla^2 + k^2(x,z))\phi(x,z) = 0 \qquad (4.1.4)$$

where $k = \frac{\omega}{c}$, and is defined for all x and for all $z \geq \zeta(x)$. This region will be referred to as physical space; it will be represented by the relation $z \geq \zeta(x)$ in what follows.

In the remainder of the development we follow closely the corresponding development for the homogeneous medium, where k is a constant. Those details which are the same as for the corresponding homogeneous case will be omitted. For such details the reader is referred to the text by Baker and Copson ([18], sec. 6.2).

We consider two solutions of Eq. (4.1.4), $\bar{\phi}$ and Ψ; suppose that the function $k(x,z)$ is sufficiently regular so that these functions possess continuous first and second order derivatives on a contour Γ and in a region A bounded by it. For our purposes the contour Γ will be formed by the function $\zeta(x)$ and a large semicircle of radius R. Then using Green's transformation, which is valid because of the assumed regularity of $\bar{\phi}$ and Ψ,

$$\iint_A (\Psi \nabla^2 \bar{\phi} - \bar{\phi}\nabla^2\Psi)\,dxdy = \int_\Gamma (\Psi \frac{\partial \bar{\phi}}{\partial \nu} - \bar{\phi}\frac{\partial \Psi}{\partial \nu})\,ds, \qquad (4.1.5)$$

where $\frac{\partial}{\partial \nu}$ means differentiation along the outward normal to Γ , and the integrals are taken over the region A and the contour Γ respectively. Since the functions $\bar{\phi}$ and Ψ are assumed to be solutions of Eq. (4.1.4), Eq. (4.1.5) becomes

$$\int_{\Gamma}(\Psi \frac{\partial \bar{\phi}}{\partial \nu} - \bar{\phi} \frac{\partial \Psi}{\partial \nu})ds = 0 \quad . \tag{4.1.6}$$

Let Ψ represent the solution of Eq. (4.1.4) valid over all space ($z < \zeta(x)$ as well as $z \geq \zeta(z)$), which has the singularity represented by the relation (4.1.3) and which represents outgoing waves at infinity. Further we assume that Ψ possesses continuous first and second order derivatives everywhere except at the point (x_0, z_0). Now in order to define such a solution we must extend the definition of k , since values of k outside physical space (in regions where $z < \zeta$) become important. We do this by allowing k to increase smoothly for $z < \zeta$ to some asymptotic positive value as $z \to -\infty$. It is evident that this specification is not in general sufficient to determine the function uniquely. We shall discuss this apparent ambiguity after the present derivation is completed.

If the singularity of the function Ψ lies outside A , we have just seen that Eq. (4.1.6) holds. However if the singularity lies within A we must alter the result somewhat, since in such a case the continuity conditions on the function Ψ no longer hold within the region A . To make this alteration, we enclose the singularity within a small circle. Within the annular region enclosed by the contour and this circle, both $\bar{\phi}$ and Ψ by hypothesis possess continuous first and second order partial derivatives and thus we have;

$$\int_{\sigma}(\Psi \frac{\partial \bar{\phi}}{\partial \nu} - \bar{\phi}\frac{\partial \Psi}{\partial \nu}) ds + \int_{\Gamma}(\Psi \frac{\partial \bar{\phi}}{\partial \nu} - \bar{\phi} \frac{\partial \Psi}{\partial \nu}) ds = 0 \quad , \tag{4.1.7}$$

where σ represents the small circle about the singularity of Ψ . By allowing the radius of the circle σ to shrink to zero, and using the relation (4.1.3) we have finally,

$$\bar{\phi} = \frac{1}{4i}\int_{\Gamma}\left(\bar{\phi}(1)\,\frac{\partial\Psi(P,1)}{\partial\nu_1} - \Psi(P,1)\,\frac{\partial\bar{\phi}(1)}{\partial\nu_1}\right)ds, \qquad (4.1.8)$$

if (P) lies within Γ . Here $\Psi(P,1)$ indicates the value of the Green's function at the point (P) when its singularity lies at the point (1).

Let us suppose that we identify $\bar{\phi}$ with ϕ. In such a case, we must also consider the singularity of ϕ given by relation (4.1.3). If the point (0) lies outside Γ, Eq. (4.1.8) holds without alteration. If the point (0) lies within Γ , we enclose the point with a small circle and allow the radius of the circle to shrink to zero, as above. By this process we obtain

$$\phi(P) = \Psi(P,0) + \frac{1}{4i}\int_{\Gamma}\left(\phi(1)\,\frac{\partial\Psi(P,1)}{\partial\nu_1} - \Psi(P,1)\,\frac{\partial\phi(1)}{\partial\nu_1}\right)ds, \quad (4.1.9)$$

for the desired equation, if both (P) and (0) lie within Γ . We now consider the integral over the part of Γ formed by the semicircle of radius R. We wish to establish conditions under which this part of the integral vanishes for R sufficiently large. In order to avoid difficulty on this point, we shall assume that at distances very much greater than all distances of physical interest, the function $k^2(x,z)$ approaches some constant value. Then the restriction that ϕ should represent outgoing waves at large distances reduces to the restriction that in this asymptotic region it should go like $\frac{e^{i\bar{k}r}}{(\bar{k}r)^{1/2}}$ where $\bar{k}$, independent of x and z , is the asymptotic value of k . Since Ψ at large distance represents outgoing waves only, it has the same form in the far region. Then by examining the part of the integral in Eq. (4.1.9) over the large semicircle, and allowing R to go to infinity, it is seen that this part of the integral vanishes. Thus Eq. (4.1.9) becomes, after using the boundary condition given by Eq. (4.1.2),

$$\phi(P) = \Psi(P,0) - \frac{1}{4i} \int_{\zeta} \Psi(P,1) \frac{\partial \phi(1)}{\partial \nu_1} ds_1 , \qquad (4.1.10)$$

where the integral is to be taken over the contour (surface) ζ .

If one considers the problem where k is constant, it is seen from Eqs. (4.1.3) and (4.1.4) that $\Psi(P,1)$ goes over into $H_0^{(1)}(kr_{1P})$. Then Eq. (4.1.10) becomes Eq. (3.4.3) with $\phi_i(P)$ replaced by the source used here, $\Psi(P,0)$ (which becomes $H_0^{(1)}(kr_{P0})$).

We see that Eq. (4.1.10) is the desired formula, analogous to the Helmholtz formula, which governs propagation in an inhomogeneous medium.

Let us consider the effect produced by different specifications of $k(x,z)$ in non-physical space, that is in the region $z < \zeta$. Different specifications will in general produce Green's functions which differ even in physical space. However, if the conditions outlined at the beginning of this section are sufficient to determine the solution $\phi(P)$ uniquely, the function $\phi(P)$ must be independent of the different choices since we see that Eq. (4.1.10) fulfills all of the conditions on the solution. We will not discuss the question of uniqueness here except to say that it is ordinarily assumed in similar problems that the conditions given at the beginning of this section are sufficient to determine the solution (see for instance Haskell [32]). There is a similar arbitrariness in the choice of Ψ even in a homogeneous medium. For such problems one can use any function which is a solution of Eq. (4.1.4), which reduces to outgoing waves at infinity, which has the singularity (4.1.3), and which is regular in and on Γ except at the point (0).

4.2. A Review of the Development of Geometrical Optics and a Discussion of the Green's Function

In the last section we obtained a representation for the solution of the wave equation in an inhomogeneous medium, where the solution is subject to a boundary condition on some non-plane bounding surface.

It will be our purpose in this section to give a representation for the two-dimensional Green's function in the special case where geometrical optics may be applied. It will turn out that the amplitude of Ψ (2,1) is, except for a constant, equal to the square root of the intensity obtained by the methods of ray tracing. The phase of the function $\Psi(2,1)$ will be seen to be given under suitable restrictions by $\int_{(1)}^{(2)} k(x,z)ds - \frac{\pi}{4}$ where the integral is to be taken along the ray path connecting (1) and (2).

It will prove advantageous to first present a brief review of the development of the method of geometrical optics.

Let us consider a function χ which satisfies the equation,

$$\left[\nabla^2 + k^2(x,y,z)\right]\chi = 0 , \qquad (4.2.1)$$

where the symbol ∇^2 is defined by $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$; we have generalized the problem to three dimensions since this entails no extra difficulty. We also suppose that the function χ has a singularity of the type given by the relation (4.1.3) if we consider the problem specialized to two dimensions. Similarly we suppose that for a three dimensional problem the function χ has a singularity of the type

$$\chi(x_b, y_b, z_b) \rightarrow \frac{\exp(ik_b r_{b0})}{r_{b0}} \qquad \text{as} \quad r_{b0} \rightarrow 0 \ , \qquad (4.2.2)$$

where r_{b0} is the distance between (x_b, y_b, z_b) and (x_0, y_0, z_0). Properly speaking the function χ satisfies Eq. (4.2.1) at all points except (0). It is further required that χ should represent outgoing waves at infinity. We also restrict k^2 to be sufficiently regular to allow a solution χ which possesses continuous first and second order derivatives at all points of space except (0).

Two real functions, G and V, are now defined through the relation,*

$$\chi = G\exp(i \frac{\omega}{c_a} V) \ , \qquad (4.2.3)$$

Also let,

$$g = \frac{\omega}{c_a} V \ . \qquad (4.2.4)$$

Here c_a is the velocity of propagation at some reference point, (a), in the medium. Now if we substitute Eq. (4.2.3) in Eq. (4.2.1) we have, upon equating real and imaginary parts to zero,

$$(\nabla V)^2 = n^2 + \frac{\lambda_a^2}{4\pi} \cdot \frac{\nabla^2 G}{G} \qquad (4.2.5)$$

and

$$\nabla^2 V + \nabla V \cdot \nabla(\ln G^2) = 0 \ , \qquad (4.2.6)$$

where the symbol ∇ represents the gradient, c is the velocity of propagation at an arbitrary point, λ_a is the wavelength at (a), and $n = c_a/c$.

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* This method of attack follows that used in "Physics of Sound in the Sea" [33], sec. 3.6.1.

So far we have imposed no restricting assumptions upon the function k. Indeed Eqs. (4.2.5) and (4.2.6) could be used instead of Eq. (4.2.1).

We now add the restriction,

$$\left| \frac{\lambda_a^2}{4\pi} \frac{\nabla^2 G}{G} \right| \ll n^2 \quad . \tag{4.2.7}$$

Using this restriction, Eq. (4.2.5) becomes,

$$(\nabla V)^2 = n^2 \tag{4.2.8}$$

which is often termed the eikonal equation; it is the basic equation of geometrical optics. Through the use of Eqs. (4.2.6) and (4.2.8) one can show that the restriction (4.2.7) leads to the following necessary conditions for the applicability of geometrical optics:* a) the radius of curvature of all rays (a ray to be defined below)should be very much greater than λ; b) the per unit change of the velocity of propagation should be slight over a distance λ ; c) the per unit change of the function G should be negligible over a distance λ . It is noted that Eqs. (4.2.6) and (4.2.8) are independent of ω .

The lines which are orthogonal to the surfaces of constant phase, that is to the surfaces V = constant, are called rays. Using this definition of a ray, it is now a simple matter to obtain the function V . First we note from Eq. (4.2.8) that,

$$\nabla V(x,y,z) = n(x,y,z) \; \vec{a}_n \tag{4.2.9}$$

where $\vec{a}_n$ is a unit vector parallel to the ray passing through the point (x,y,z), positive in the direction of increasing V. Now we take the line

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* See "Physics of Sound in the Sea", [33], sec. 3.6.2.

integral

$$\int_{(b)}^{(c)} \nabla V \cdot \vec{ds} \quad , \tag{4.2.10}$$

where the points (b) and (c) lie on the same ray, and $\vec{ds}$ is the differential of path length with its direction parallel to $\vec{a}_n$. Then using Eq. (4.2.9) in the expression (4.2.10) we have the result:

$$V(c) - V(b) = \int_{(b)}^{(c)} n \, ds \quad , \tag{4.2.11}$$

the integral being taken along the ray connecting (b) and (c) with ds taken as positive in the direction from the source to the receiver. We see the phase, cf. Eq. (4.2.4), is given by:

$$g(c) - g(b) = \frac{\omega}{c_a} (V(c) - V(b)) = \int_{(b)}^{(c)} k ds \quad . \tag{4.2.12}$$

One can show that Eq. (4.2.6) expresses the conservation of energy within a tube of rays.* This may be done by integrating Eq. (4.2.6) throughout a volume formed by a tube of rays bounded on the ends by surfaces of constant phase. Upon using Green's theorem and Eq. (4.2.8) it is found that the "flow"

$$\vec{J_1} = kG^2 \vec{a}_n \quad , \tag{4.2.13}$$

is conserved in any tube of rays. We now show that under certain conditions $\vec{J_1}$ is identical with the quantity $\vec{J}$, the average intensity. It is remembered that $\vec{J}$ was defined by

$$\vec{J} = \mathrm{Re}\left[\frac{1}{i} \phi^* \nabla \phi\right] \quad . \tag{4.2.14}$$

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* For the proof of this statement for electromagnetic radiation, see Friedlander, [34] .

Substituting Eq. (4.2.3) for ϕ in Eq. (4.2.14) and using Eq. (4.2.9) we see:

$$\mathrm{Re}\left[\frac{1}{i}\chi^* \nabla \chi\right] = kG^2 \vec{a}_n + G\nabla G \ . \qquad (4.2.15)$$

Now if

$$\left|\frac{\nabla G}{kG}\right| \ll 1 \ , \qquad (4.2.16)$$

it is seen that $\vec{J_1} = \vec{J}$, that is, the quantity conserved in ray theory is the energy. The condition (4.2.16) is identical with the restriction c) already established above for the validity of geometrical optics.

To sum up, the amplitude, G, of the field function χ is obtained through the use of the conservation property of the quantity $\vec{J_1}$ (cf. Eq. (4.2.13)). The phase, g, of the function χ is obtained from Eq. (4.2.12).

Before taking up our next topic it is worth commenting that geometrical optics cannot be applied in a shadow zone, that is in a region which cannot be reached by a ray from the source. Under the approximations leading to geometrical optics, the field vanishes within a shadow zone by virtue of the energy conservation within the radiated region.

We now proceed with the specification of the Green's function, using the results of this section. Once again the problem is specialized to two dimensions, which will as usual be designated by (x,z). The problem is the following: we desire a solution Ψ of the equation,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)\Psi + k^2(x,z)\,\Psi = 0 \qquad (4.2.17)$$

where we suppose that the function $k^2(x,z)$ has been extended in some (at the moment) arbitrary way into the non-physical region $z < \zeta(x)$. It is assumed that Ψ has the further property that,

$$\Psi(b,0) \rightarrow H_0^{(1)}(k_0 r_{b0}) \qquad \text{as} \quad r_{b0} \rightarrow 0 \tag{4.2.18}$$

where r_{b0} is the distance from the point (x_0,z_0) to the point (x_b,z_b) and $k_0 = k(x_0,z_0)$. Also Ψ consists solely of outgoing waves at infinity. It is supposed that k^2 is sufficiently regular so that a function exists satisfying these conditions, possessing continuous first and second order derivatives at all points of space except the source point, (x_0,z_0).

It is now assumed that the variation of k is sufficiently small so that the limiting relation (4.2.18) may be extended to the following relation:

$$\Psi(b,0) \approx H_0^{(1)}(k_0 r_{b0}), \qquad r_{b0} \leq \rho \tag{4.2.19}$$

where $\rho \gg 1/k_0$. Then for $1/k_0 \ll r_{b0} \leq \rho$ we have from Eq. (4.2.3) and from the asymptotic form for the Hankel function, (see Eq. (3.5.8)),

$$\Psi(b,0) = G(b,0)\exp\left(\frac{i\omega}{c_a}V(b,0)\right) \approx \frac{\exp\left[ik_0 r_{b0} - i\frac{\pi}{4}\right]}{\left(\frac{\pi}{2}\, k_0 r_{b0}\right)^{1/2}} \tag{4.2.20}$$

The phase g of the Green's function, Ψ , can then be written

$$g = \int_{(0)}^{(b)} k ds \; - \; \frac{\pi}{4} \; . \tag{4.2.21}$$

For a more general point (c) not necessarily fulfilling the restriction $r_{c0} \leq \rho$ we see from Eq. (4.2.12) that we may write

$$g(c,0) = \int_{(0)}^{(c)} k ds - \frac{\pi}{4} \qquad , \tag{4.2.22}$$

which is valid for $r_{c0} \gg 1/k$ (cf. Eq. (4.2.20)). The constant $\pi/4$ is the contribution to the phase from that region near (0) in which geometrical optics does not hold. The function G may be found through the use of the conservation relation satisfied by $\vec{J}_1$ (cf. Eq. (4.2.13)), where the value of G at the end of a given tube of rays near the source is given (see Eq. (4.2.20) as,

$$G(b,0) = \left(\frac{\pi}{2} k_0 r_{b0}\right)^{-1/2} \tag{4.2.23}$$

for $1/k_0 \ll r_{b0} \leq \rho$.

Thus the approximate Green's function is determined through the use of Eqs. (4.2.22) and (4.2.23). The approximation is valid if --

i) It is possible to define ρ fulfilling condition (4.2.19).

ii) If the problem is such that geometrical optics may be applied; see restrictions a), b), and c) of this section.

In most applications, i) is fulfilled if ii) is.

4.3. Geometrical Optics in a Stratified Medium

In this section we shall consider the consequences of a further restriction upon the inhomogeneity of the propagating medium. We suppose that the function k^2 depends upon the coordinate z alone; the medium is said to be stratified in such a case. The problem is restricted to two dimensions. We wish to apply the results of Section 4.2 to this problem, supposing that the restrictions of that section are satisfied so that we may apply geometrical optics. We shall develop certain expressions which will be useful later. Finally we shall consider the effect of the above restriction of k^2 on the Green's function.

The assumption of stratification simplifies considerably problems of propagation where geometrical optics may be applied; this is true since such an assumption leads to a "constant of the motion". This constant corresponds to the conservation of the horizontal component of linear momentum in the analogous problems of mechanics. To derive the constant consider again Eq. (4.2.8) with the restriction on k^2 ,

$$(\nabla V)^2 = n^2(z) \quad . \tag{4.2.8}$$

Now we assume a solution of the form

$$V = X(x) + Z(z) \quad , \tag{4.3.1}$$

and substituting in Eq. (4.2.8) obtain

$$\left(\frac{dX}{dx}\right)^2 + \left(\frac{dZ}{dz}\right)^2 = n^2\,(z) \quad . \tag{4.3.2}$$

By the familiar argument of the separation of variables we have,

$$\left(\frac{dX}{dx}\right) = n \cos \Theta_z = d \tag{4.3.3}$$

and

$$\left(\frac{dZ}{dz}\right) = n \sin \Theta_z \tag{4.3.4}$$

where we have utilized Eq. (4.2.9); Θ_z is the angle which the ray makes with the x-direction at the point (x,z) and d is the above mentioned constant of motion.

Let us consider now those rays which originate at $z = 0$. Consider the particular ray making an angle θ with the x-axis as it leaves. Then we can evaluate the constant in Eq. (4.3.3) obtaining,

$$n(z)\cos\theta_z = d = \cos\theta \quad . \qquad (4.3.5)$$

We have chosen the reference point of Eq. (4.2.3) on the x-axis so that $n = 1$ at the point where the ray originates.

Since in this work we are primarily interested in those functions $n(z)$ which give rise to a surface-bounded duct, we shall restrict the functions considered accordingly. We suppose that n decreases continuously and monotonically as z goes from $z = 0$ to $z = D$; and increases continuously and monotonically for $z > D$, approaching some finite (positive) value asymptotically. This is equivalent to saying that the velocity of propagation increases monotonically to $z = D$ and decreases monotonically thereafter. It is to be noted that the assumption that the velocity decreases monotonically for $z > D$ rules out the possibility of reflections from surfaces other than $\zeta(x)$; in particular, in the case of the propagation of acoustic waves under water, possible reflections from the ocean floor are not taken into account. Although by considering a more complicated model with two bounding surfaces, one could treat problems where such reflections are of importance, this will not be done here.

It is true that surface-bounded ducts are found physically with velocity profiles (functions $c(z)$) more complicated than the one suggested here. Nevertheless the above outlined profile covers many of the cases of duct propagation which are of physical interest. For simplicity then we restrict attention to this profile.

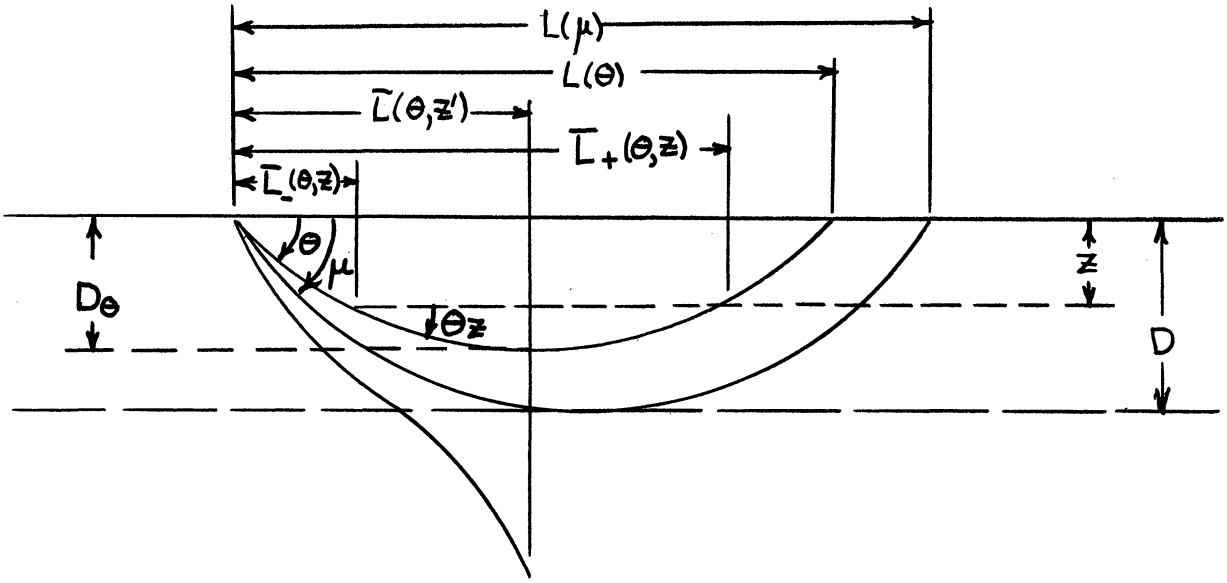

Fig. 5. Diagram showing the distances travelled by trapped and untrapped rays.

We are now prepared to derive expressions for certain functions which will prove useful;* reference is made to Fig. 5. To begin we define μ as the largest angle made by a ray trapped within the duct. It is determined implicitly by the relation

$$\cos(\mu) = \frac{c(0)}{c(D)} \quad , \qquad (4.3.6)$$

it being remembered that $c(z) < c(D)$. In Fig. 5 it is seen that $\overline{L}(\theta,z)$ is defined as the horizontal distance travelled by an untrapped ray in going from the surface to the depth z . Then if $\theta > \mu$ we have from the definition of θ_z and $\overline{L}$,

$$\overline{L}(\theta,z) = \int_0^z \cot\theta'_z \, dz' \quad . \qquad (4.3.7)$$

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* The treatment follows that to be found in "Physics of Sound in the Sea", [33], sec. 3.4.1.

We can use Eq. (4.3.5) and the definition of $n(z)$ to transform Eq. (4.3.7) into

$$\overline{L}(\Theta,z) = \cos\Theta \int_0^z \frac{c(z')dz'}{(c^2(0)-c^2(z')\cos^2\Theta)^{1/2}} \quad . \qquad (4.3.8)$$

If $\Theta < \mu$, $\overline{L}$ becomes a double-valued function of z ; we designate the smaller of its two values by $\overline{L}_-$ and the larger by $\overline{L}_+$. Then it is evident that,

$$\overline{L}_-(\Theta,z) = \cos\Theta \int_0^z \frac{c(z')dz'}{(c^2(0)-c^2(z')\cos^2\Theta)^{1/2}} \quad , \qquad (4.3.9)$$

and

$$\overline{L}_+(\Theta,z) = \cos\Theta \left\{ \int_0^z \frac{c(z')dz'}{(c^2(0)-c^2(z')\cos^2\Theta)^{1/2}} + \int_z^{D_\Theta} \frac{c(z')dz'}{(c^2(0)-c^2(z')\cos^2\Theta)^{1/2}} \right\} \quad , \qquad (4.3.10)$$

when D_Θ is the maximum depth reached by a trapped ray. The function $L(\Theta)$, defined as the distance travelled by a trapped ray between surface reflections, is given by

$$L(\Theta) = 2\cos\Theta \int_0^{D_\Theta} \frac{c(z')dz'}{(c^2(0)-c^2(z')\cos^2\Theta)^{1/2}} \quad , \qquad (4.3.11)$$

which is seen from Eq. (4.3.10) to be the same as $\overline{L}_+(\Theta,0)$, as it should be. Finally, we have D_Θ given by

$$c(D_\Theta) = \frac{c(0)}{\cos\Theta} \quad . \qquad (4.3.12)$$

We now concentrate attention upon the form of the Green's function for the special type of velocity profile assumed in this section. First, expressions for the amplitude and the phase of the Green's function will be derived; then the results of this section will be applied to the special case of the linear velocity profile. Finally the effect of small variations of the argument of the Green's function will be examined.

Before proceeding with this program, we again consider the extension of the function k^2, now a function of z alone, into the region $z < \zeta(x)$. It has already been proposed that we extend k^2 by allowing it to increase smoothly to some finite positive value. It is assumed that the extension varies slowly enough so that the conditions of Sec. 4.2 for the application of geometrical optics are met. It can be seen from Eq. (4.3.5) that such a specification of k^2 has as a consequence the following fact: if a ray originates in physical space and is so directed that it penetrates non-physical space, it will be refracted away from ζ and move toward $z = -\infty$. Thus in the case where one may apply geometrical optics it is possible to extend k^2 in such a way that its value in non-physical space has little effect upon the value of the Green's function in physical space.

We now consider the Green's function. From Eq. (4.2.22) we have:

$$\Psi(b,0) = G(b,0)\ \exp\left\{i\left(\int_{(0)}^{(b)} k ds - \frac{\pi}{4}\right)\right\} . \qquad (4.3.13)$$

It is seen that,

$$dx = \cot \Theta_z dz \qquad , \qquad (4.3.14)$$

from which it follows that,

$$\int_{(0)}^{(b)} k(z')ds = \left| \int_{(0)}^{(b)} \frac{k(z')c_0 dz'}{(c_0^2 - c^2(z')\cos^2\theta_0)^{1/2}} \right| . \qquad (4.3.15)$$

Thus the phase of the Green's function is given by,

$$g(b,0) = \left| \int_{(0)}^{(b)} \frac{kc_0 dz'}{(c_0^2 - c^2 \cos^2\theta_0)^{1/2}} \right| - \frac{\pi}{4} . \qquad (4.3.16)$$

The angle θ_0 is the angle between the ray and the horizontal at the point (0); also c_0 is the velocity of propagation at (0). If the denominator of the integrand in Eq. (4.3.16) vanishes within the interval of integration, one must exercise some care. In such a case the integral is broken into two parts exactly as was done for the function $\overline{L}_+$ (see Eq. (4.3.10)).

Now we consider the amplitude function, G . We shall refer to Fig. 6; it is assumed in that figure that the source is located at z = 0.

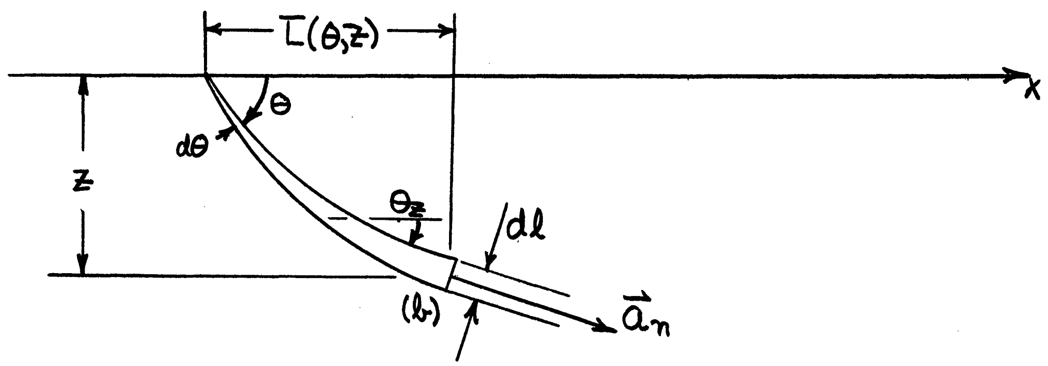

Fig. 6. Diagram for the determination of the amplitude of the Green's function.

In order to find the function G, we make use of the conservation relation for $\vec{J}$. We first see from Eqs. (4.2.13) and (4.2.23) that the energy per second per angle radiated by the source is given by:

$$e_{\Theta} = \frac{2}{\pi} \quad . \tag{4.3.17}$$

Then using the definition of $\vec{J}$ given in Eq. (4.2.13) (dropping the subscript) and referring to Fig. 6 it is seen that,

$$k(b)G^2(b,0) = \frac{2}{\pi}\left|\frac{d\Theta}{d\ell}\right| \quad . \tag{4.3.18}$$

If the source is located at the surface, as shown in Fig. 6, this can be further simplified by the use of the relation,

$$\left|\frac{d\ell}{d\Theta}\right| = \left|\sin\Theta_z \left.\frac{\partial\overline{L}(\Theta,z)}{\partial\Theta}\right]_z\right| , \tag{4.3.19}$$

where the symbol $\left.\frac{\partial\overline{L}}{\partial\Theta}\right]_z$ indicates the partial derivative of $\overline{L}$ with respect to Θ with z held constant. If (b) lies on a trapped ray, one of the functions $\overline{L}_{\pm}$ is used in place of $\overline{L}$; the choice as before is determined by where the point (b) is located on the ray. Substituting Eq. (4.3.19) in Eq. (4.3.18), we see,

$$G(b,0) = \left|\frac{\pi}{2}\; k(b)\, \sin\Theta_z \left.\frac{\partial\overline{L}(\Theta,z)}{\partial\Theta}\right]_z\right|^{-1/2} \quad . \tag{4.3.20}$$

If the source does not lie at $z = 0$, one must suitably modify the function $\overline{L}$ used.

A case of great practical interest is that for which the velocity is linear function of z within the duct. We now apply the results of this section to this special problem.* Let the source lie on the surface at the point (0) with coordinates $(x_0, 0)$; let the field point be (R) with coordinates (x_R, z). The angles θ and θ_z are shown in Fig. 6. The results are applicable for $z \leq D$. It is further supposed that (R) does not lie within a shadow.

We now let

$$c(z) = c(0)(1 + \alpha z), \qquad \alpha > 0,\ z \leq D\ . \tag{4.3.21}$$

Using the constant of the motion (cf. Eq. (4.3.5)) we have

$$\frac{\cos\theta}{\cos\theta_z} = \frac{c(0)}{c(z)} \quad . \tag{4.3.22}$$

From Eq. (4.3.11) it follows that,

$$L(\theta) = \frac{2}{\alpha} \tan\theta \quad . \tag{4.3.23}$$

It is easily seen that,

$$\overline{L}(\theta, z) = \overline{L}_{\pm}(\theta, z) = \frac{1}{2}\left[L(\theta) - \underline{L(\theta_z)}\right] \quad . \tag{4.3.24}$$
$$\qquad\qquad\qquad\qquad\qquad\qquad n$$

Further let R_2 represent the radius of curvature of a ray. Then by observing that

$$R_2 = \frac{ds}{d\theta_z} \quad , \tag{4.3.25}$$

where ds is defined as a differential element of length along the ray, and by using the relation,

$$ds = \frac{dz}{\sin\theta_z} \quad , \tag{4.3.26}$$

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* The special case of the linear velocity profile is considered in detail in "Physics of Sound in the Sea", [33], secs. 3.3.1 and 3.4 2.

together with Eq (4.3.22), one finds

$$R_2 = \frac{1}{\alpha \cos \Theta} \quad . \qquad (4.3.27)$$

By using Eq. (4.3.20) it can be shown that,*

$$G = \left[\frac{2}{\pi}\,\frac{1}{k(R)}\,\frac{\cos\Theta}{x_R - x_0}\right]^{1/2} \quad . \qquad (4.3.28)$$

Then using Eqs. (4.3.28) and (4.2.13) the intensity is seen to be given by

$$J(R,0) = \frac{2}{\pi}\,\frac{\cos\Theta}{x_R - x_0} \quad . \qquad (4.3.29)$$

Because of the extreme smallness of α in most physical applications, it frequently happens that the angles Θ and Θ_z are small for trapped rays. For instance in the numerical example to be considered in Chapter V we shall find $\alpha = 3.73 \times 10^{-6}$ per foot, and consequently $\mu = 3.12°$ for D = 400 feet. As a result the approximations for small angles to the equations just obtained will be of interest. These approximations are:

$$\Theta_z = \left(\Theta^2 - 2\alpha z\right)^{1/2}, \qquad (4.3.30)$$

$$L(\Theta) = \frac{2}{\alpha}\,\Theta \quad , \qquad (4.3.31)$$

$$\overline{L}(\Theta,z) = \overline{L}_-(\Theta,z) = \frac{1}{\alpha}\left(\Theta - \sqrt{\Theta^2 - 2\alpha z}\right) \quad , \qquad (4.3.32)$$

$$\overline{L}_+(\Theta,z) = \frac{1}{\alpha}\left(\Theta + \sqrt{\Theta^2 - 2\alpha z}\right) \quad , \qquad (4.3.33)$$

$$R_2 = \frac{1}{\alpha} \quad , \qquad (4.3.34)$$

---

* This problem is considered in "Physics of Sound in the Sea", [33], secs. 3.4.1 and 3.4.2.

$$G(R,0) = \left[\frac{2}{\pi}\frac{1}{k(R)}\frac{1}{|x_R-x_0|}\right]^{1/2} , \qquad (4.3.35)$$

$$J(R,0) = \frac{2}{\pi}\frac{1}{|x_R-x_0|} . \qquad (4.3.36)$$

Let R_1 be defined as the radius of curvature of the wave front. Then by utilizing Eq. (4.3.34) and considering the angular change of the normal to a ray upon allowing an increment $d\Theta$ in the initial angle of the ray, one finds,

$$R_1 = |x_R-x_0| . \qquad (4.3.37)$$

We consider now expansions of the functions $g(1,0)$ and $G(1,0)$ about the point (1). Let $\vec{\Delta r}$ represent a small variation in the position of the field point (1). Then since, by hypothesis, both g and G possess continuous first and second order derivatives, we know that for Δr small enough,

$$g(\vec{r_1}+\vec{\Delta r},\vec{r}_0) \approx g(\vec{r_1},\vec{r_0}) + \vec{\Delta r}\cdot\nabla_1 g(\vec{r_1},\vec{r_0}) + \frac{1}{2}(\Delta r\cdot\nabla_1)^2 g(\vec{r_1},\vec{r_0}) , \qquad (4.3.38)$$

and

$$G(\vec{r_1}+\vec{\Delta r},\vec{r_0}) \approx G(\vec{r_1},\vec{r_0}) + \vec{\Delta r}\cdot\nabla_1 G(\vec{r_1},\vec{r_0}) ; \qquad (4.3.39)$$

$\vec{r_0}$ and $\vec{r_1}$ are the position vectors of the source and of the field point, respectively, of the Green's function; ∇_1 represents the gradient with respect to the coordinates of the field point, and the symbol $\approx$ means "equals approximately". We have let

$$g(1,0) = g(\vec{r_1},\vec{r_0}) , \qquad (4.3.40)$$

and similarly for G .

It will be convenient to write Eq. (4.3.38) in a different form. Referring to Eqs. (4.2.4) and (4.2.9) we see that

$$\nabla_1 g(1,0) = k\,\vec{a}_n \quad , \tag{4.3.41}$$

where $\vec{a}_n$ is a unit vector with the direction of the ray from the point (0) to the point (1) at (1) , (cf. Fig. 7). Now we must calculate second derivatives of g . To do this we observe that the operator $(\vec{\Delta r}\cdot\nabla_1)^2$ is invariant with respect to rotation.

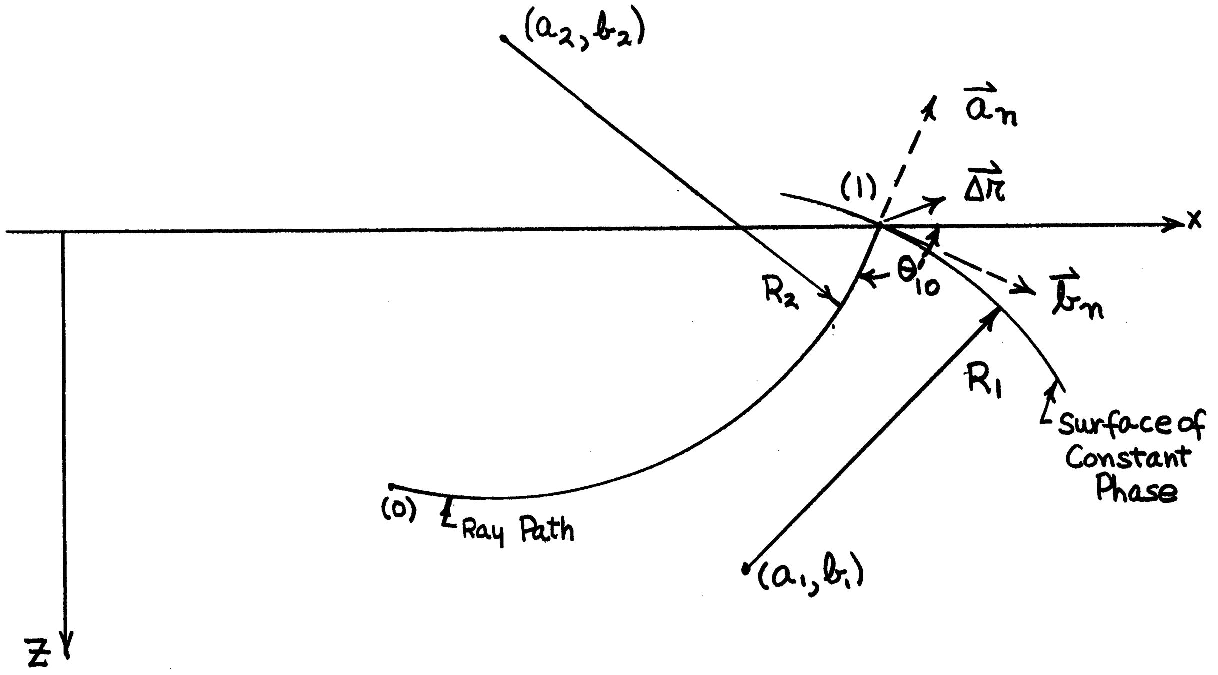

Fig. 7. Diagram used in estimating the effect of $\vec{\Delta r}$ on the functions G and g .

For convenience we choose a special coordinate system, (see Fig. 7). Let $\vec{b}_n$ be a unit vector perpendicular to $\vec{a}_n$ (therefore tangent to the surface of constant phase) and positive in the direction of +z . Using

Eq. (4.3.41), the fact that $\vec{\Delta r}$ is constant, and the known vector identity,

$$\nabla(\vec{a}\cdot\vec{b}) = (\vec{a}\cdot\nabla)\vec{b} + (\vec{b}\cdot\nabla)\vec{a} + \vec{a}\times(\nabla\times\vec{b}) + \vec{b}\times(\nabla\times\vec{a}) \ , \qquad (4.3.42)$$

we obtain the following relation:

$$(\vec{\Delta r}\cdot\nabla_1)^2 g(\vec{r}_1,\vec{r}_0) = \vec{\Delta r}\cdot(\vec{\Delta r}\cdot\nabla_1)k\vec{a}_n \quad . \qquad (4.3.43)$$

We define, as before, R_1 and R_2 as the radius of curvature of the surface of constant phase and the radius of curvature of the ray at the point (1) . It is supposed that if (a_1,b_1) are the coordinates of the center of curvature of the surface of constant phase, and if (a_2,b_2) are the coordinates of the center of curvature of the ray in the $(\vec{a}_n,\vec{b}_n)$ coordinate system, that $a_1 < 0$ and $b_2 < 0$. If $a_1 > 0$, then the sign of R_1 in Eq. (4.3.44) below must be reversed, and similarly for b_2 and R_2 . Using these definitions, Eq. (4.3.43) becomes,

$$(\vec{\Delta r}\cdot\nabla_1)^2 g = (\Delta r_a)^2 \frac{\partial k}{\partial a} + \Delta r_a \Delta r_b \left[\frac{\partial k}{\partial b} - \frac{k}{R_2}\right] + (\Delta r_b)^2 \frac{k}{R_1} \qquad (4.3.44)$$

where Δr_a and Δr_b represent the components of $\vec{\Delta r}$ on the a_n and b_n axes respectively. It will be convenient to transform the expressions back to the (x,z) coordinate system. We first define θ_{10}' as the angle which the ray makes with the positive x-axis upon arrival (see Fig. 7). Also let Δr_x and Δr_z be the components of the displacement in the (x,z) coordinate system. Then using Eq. (4.3.44) and remembering that k is a function of z alone, we have for Eq. (4.3.43),

$$(\vec{\Delta r}\cdot\nabla_1)^2 g = -(\Delta r_x \cos\Theta_{10}' + \Delta r_z \sin\Theta_{10}')^2 \frac{\partial k}{\partial z} \sin\Theta_{10}'$$

$$+ (\Delta r_x \sin\Theta_{10}' - \Delta r_z \cos\Theta_{10}')^2 \frac{k}{R_1}$$

$$+ (\Delta r_x \cos\Theta_{10}' + \Delta r_z \sin\Theta_{10}')(\Delta r_x \sin\Theta_{10}' - \Delta r_z \cos\Theta_{10}')$$

$$\cdot\left[\frac{\partial k}{\partial z}\cos\Theta_{10}' + \frac{k}{R_2}\right] \quad . \qquad (4.3.45)$$

Using Eqs. (4.3.41) and (4.3.45) we have for Eq. (4.3.38)

$$g(\vec{r_1}+\vec{\Delta r},\vec{r_0}) \approx g(\vec{r_1},\vec{r_0}) \qquad (4.3.46)$$
$$- (\Delta r_x \cos\Theta_{10}' + \Delta r_z \sin\Theta_{10}')k(0) + \frac{1}{2}(\vec{\Delta r}\cdot\nabla_1)^2 g \ .$$

We now consider the effect on $g(\vec{r_1},\vec{r_0})$ of varying the argument $\vec{r_0}$. Probably the easiest approach here is to make use of the known reciprocity theorem (see Section 3.1) which in this connection leads to the relation,

$$\Psi(\vec{r_1},\vec{r_0}) = \Psi(\vec{r_0},\vec{r_1}) \quad . \qquad (4.3.47)$$

(It should be remarked that the reciprocity theorem can be applied to in-homogeneous media.) From Eq. (4.3.47) it follows that,

$$g(\vec{r_1},\vec{r_0}+\vec{\Delta r}) = g(\vec{r_0}+\vec{\Delta r},\vec{r_1}) \quad , \qquad (4.3.48)$$

so that we can apply the result given in Eq. (4.3.46). We can use the reciprocity theorem in the same way in treating the quantity $G(\vec{r},\vec{r_0}+\vec{\Delta r})$ (cf. Eq. (4.3.39)).

In the next section an expansion of $\vec{a}_n$ about the point (1) will also be useful. In order to bring out the dependence of $\vec{a}_n$ upon its arguments we write it as $\vec{a}_n(\vec{r}_1,\vec{r}_0)$. Then proceeding as before we have

$$\vec{a}_n(\vec{r}_1 + \vec{\Delta r},\vec{r}_0) \approx \vec{a}_n(\vec{r}_1,\vec{r}_0) + (\vec{\Delta r}\cdot\nabla_1)\vec{a}_n(\vec{r}_1,\vec{r}_0) \tag{4.3.49}$$

or referring to Fig. 7,

$$\vec{a}_n(\vec{r}_1 + \vec{\Delta r},\vec{r}_0) \approx \vec{a}_n(\vec{r}_1,\vec{r}_0) + \vec{b}_n \left\{\frac{\Delta r_b}{R_1} - \frac{\Delta r_a}{R_2}\right\}. \tag{4.3.50}$$

Finally transforming to (x,z) coordinates as in the treatment of $g(\vec{r}_1 + \vec{\Delta r},\vec{r}_0)$ we see

$$\vec{a}_n(\vec{r}_1+\vec{\Delta r},\vec{r}_0) \approx \vec{a}_n(\vec{r}_1,\vec{r}_0) - \vec{b}_n \left\{+\frac{(\Delta r_x\cos\Theta_{i0} + \Delta r_z\sin\Theta_{i0})}{R_1} \quad \frac{(\Delta r_x\sin\Theta_{i0} - \Delta r_z\cos\Theta_{i0})}{R_2}\right\}. \tag{4.3.51}$$

In the remainder of this chapter the following approximations will be used for the amplitude and phase of the Green's function and for the unit vector in the direction of a ray:

$$G(\vec{r}_1 + \vec{\Delta r},\vec{r}_0) \approx G(\vec{r}_1,\vec{r}_0) , \tag{4.3.52}$$

$$g(\vec{r}_1 + \vec{\Delta r},\vec{r}_0) \approx g(\vec{r}_1,\vec{r}_0) - (\Delta r_x\cos\Theta_{i0} + \Delta r_z\sin\Theta_{i0})k(0), \tag{4.3.53}$$

$$\vec{a}_n(\vec{r}_1 + \vec{\Delta r},\vec{r}_0) \approx \vec{a}_n(\vec{r}_1,\vec{r}_0) . \tag{4.3.54}$$

If these approximations are to be valid we must have (cf. Eqs. (4.3.39), (4.3.46), and (4.3.51)),

$$\left| \frac{\vec{\Delta r} \cdot \nabla_1 G(\vec{r}_1, \vec{r}_0)}{G(\vec{r}_1, \vec{r}_0)} \right| \ll 1 \,, \tag{4.3.55}$$

$$\left| (\vec{\Delta r} \cdot \nabla_1)^2 g(\vec{r}_1, \vec{r}_0) \right| \ll 1 \,, \tag{4.3.56}$$

$$\left| -b_n \left\{ \frac{(\Delta r_x \cos\theta'_{10} + \Delta r_z \sin\theta'_{10})}{R_1} + \frac{(\Delta r_x \sin\theta'_{10} - \Delta r_z \cos\theta'_{10})}{R_2} \right\} \right| \ll 1 \,. \tag{4.3.57}$$

It is seen that so long as $\vec{\Delta r}$ is small enough so that the restrictions (4.3.55), (4.3.56), and (4.3.57) are satisfied the approximations given by Eqs. (4.3.52), (4.3.53), and (4.3.54) amount to supposing that changes in the Green's function may be represented as though it were a plane wave for small changes in distance. It is desirable to put the restrictions (4.3.55), (4.3.56), and (4.3.57) in a more manageable form. To do this it is supposed that the phase velocity near the surface may be represented by a linear function (cf. Eq. (4.3.21)) and that the angle θ'_{10} is small. Then letting $\Delta r_x = \bar{\ell}$ and $\Delta r_z = h$, and using the results for the small angle, linear velocity approximations given in Eqs. (4.3.30) to (4.3.37), it is found that the restrictions (4.3.55) through (4.3.57) may be written respectively,

$$\frac{\bar{\ell} + h}{|x_1 - x_0|} \ll 1 \,, \tag{4.3.58}$$

$$(\bar{\ell} + h\theta'_{10})^2 \alpha k \theta'_{10} + \frac{(\bar{\ell}\,\theta'_{10} + h)^2 k}{|x_1 - x_0|} + (\bar{\ell} + h\theta'_{10})(\bar{\ell}\theta'_{10} + h) 2\alpha k \ll 1 \,, \tag{4.3.59}$$

$$\frac{\bar{\ell} + h\theta'_{10}}{|x_1 - x_0|} + (\bar{\ell}\theta'_{10} + h)\alpha \ll 1 \ . \qquad (4.3.60)$$

For present purposes the relations $\bar{\ell}\theta'_{10} \gtrsim h$ and $\bar{\ell}k > 1$ will hold. For instance $\bar{\ell}$ in the next section will be defined as the distance of separation beyond which two points on the reflecting surface are statistically independent and h will be taken as the rms displacement of the surface. Then using values for the parameters appropriate to the numerical example to be presented in Chapter V: $\bar{\ell} \approx 25$ yards, $k \approx 94$ yards^{-1}, $\theta'_{10} \sim \frac{1}{20}$, and $h \sim 1$ yard it is seen that the relations $\bar{\ell}\theta'_{10} \gtrsim h$ and $\bar{\ell}k > 1$ follow. Using these relations, the restrictions (4.3.58), (4.3.59), and (4.3.60) can be replaced by the equivalent restrictions,

$$\frac{\bar{\ell}}{|x_1 - x_0|} \ll 1 \ , \qquad (4.3.61)$$

$$\bar{\ell}^2 \alpha k \theta'_{10} \ll 1 \ , \qquad (4.3.62)$$

$$\frac{(\bar{\ell}\theta'_{10})^2 k}{|x_1 - x_0|} \ll 1 \ , \qquad (4.3.63)$$

If the point (0) lies on the surface, one can alter the restriction (4.3.63) through the use of Eq. (4.3.31) to obtain,

$$\frac{\bar{\ell}^2 \alpha^2 |x_1 - x_0| k}{4} \ll 1 \ . \qquad (4.3.64)$$

To review, in this section we first restricted the function k, allowing it to be a function of z alone. Then Eqs. (4.3.3) and (4.3.4) were derived which could be used to trace the trajectories of rays in a stratified medium. Following this, the function k was further restricted so as to produce a surface-bounded duct. For this structure (and further, assuming a linear velocity variation) various functions to be used later were derived (see Eqs. (4.3.30) to (4.3.37)). Finally after obtaining expressions for the amplitude and phase of the Green's function, the effect of small variations of an argument of the function was considered.

4.4. Justification for the Duct Propagation Model; Treatment of Terms with Incoherent Phase

We are now prepared to derive the basic equations for the statistical model presented in Chapter II. The statistics of the surface $\zeta(x)$ will be discussed first; it will be seen that the equations presented heuristically in Chapter II govern the average intensities. Then the central restriction upon the bounding surface will be introduced; it consists of supposing that the amount of energy reflected specularly from the surface is negligible compared with that which is reflected diffusely. It can be seen from the work on the reflection problem presented in Section 3.3 that for sufficiently small incident angles, the reflected radiation is specular; then the restriction on the surface amounts to supposing that it is rough enough so that for most trapped angles, the reflection is not specular.

The development is begun by considering the Helmholtz formula (Eq. (4.1.10)). Using this equation the intensity of the field at the point (R) is constructed and its average taken. It will be seen that two kinds of terms are obtained in this way. The first of these are those terms which upon averaging will be seen to vanish because of incoherent phase (these will be called the cross-terms). There will also be two terms which have coherent phase and thus will not vanish when the average is taken.

These terms will give rise to the equations obtained heuristically in Chapter II. The cross-terms will be treated in this section, the treatment of the coherent terms being reserved for Section 4.5.

In the development it will be supposed that the index of refraction of the medium varies sufficiently slowly so that geometrical optics may be used to approximate the required Green's function. The function $k = \omega/c(z)$ is assumed to depend upon z in the manner postulated in Section 4.3 so that a surface-bounded duct is set up.

In what follows in this section we shall state the needed restrictions in terms of the small-angle, linear-velocity structure (cf. Eq. (4.3.21)). The restrictions will be illustrated by using the numerical example to be presented in Chapter V. In this example the propagation of high frequency (25kc) sound in an isothermal layer of the ocean will be considered with the ocean surface forming the boundary. In the ocean $c \approx 1670$ yards per second, then $k \approx 94$ yards^{-1}; in an isothermal layer, the gradient parameter α is given by $1/\alpha \approx 89{,}000$ yards. The depth of the isothermal layer is assumed to be 400 feet; then from Eq. (4.3.30) one finds $\mu = 3.12°$. It follows from Eq. (4.3.31) that $L(\mu) = 9760$ yards. Further it will be supposed that $\bar{\ell} = 25$ yards (where $\bar{\ell}$ has already been defined as that separation beyond which the statistical properties of two surface regions become independent.)

We restate the problem. It is desired to find a function $\phi(x,z)$ which satisfies Eq. (4.1.4), which vanishes on the boundary $z = \zeta(x)$, which has a singularity of the following type at the point (0):

$$\phi(R) \rightarrow H_0^{(1)}(kr_{R0}) , \qquad \text{as } r_{R0} \rightarrow 0 , \qquad (4.4.1)$$

and which reduces to out-going waves at infinity. We begin the development with Eq. (4.1.10):

$$\phi(R) = \Psi(R,0) - \frac{1}{4i}\int_{\zeta} \frac{\partial\phi(1)}{\partial\nu_1}\Psi(R,1)ds_1 , \qquad (4.1.10)$$

where $\Psi(R,0)$ is the Green's function for the infinite inhomogeneous medium. As suggested above we suppose that the medium satisfies the conditions of Section 4.2 (for geometrical optics) so that we can use the approximation developed in Section 4.3 to represent the function.

We pause now to discuss the statistics of the surface and to define the averages to be employed. In some problems occuring physically the bounding surface, $\zeta(x)$, remains fixed over times long compared with those in which the experiments are made; an example would be the propagation of radiation over the surface of the earth. In other problems, such as the propagation of radiation over (or under) the sea surface, the surface is continually changing during the time in which the experiments are carried out.

One of the quantities easily obtainable from the measurements in the latter case would be the (time) average intensity observed at some fixed point. (We shall be concerned only with such averages in this work. Fluctuations about the average will not be considered.) It will further be supposed that the time varying surface is statistically homogeneous; that is, we suppose that the statistical properties of the surface are neither dependent upon the spatial position nor upon the time at which they are taken.

In the case where the bounding surface is fixed in time, one could expect the observed intensities to suffer spatial variations which would be connected with individual projections on the surface $\zeta(x)$. Although such effects are of interest, it is clear that in order to take them into account, a detailed knowledge of the reflecting surface is required. In order to avoid such a specialized treatment, we shall institute a space average of the intensity. Again we do not consider fluctuations about such an average. The average is to be found as follows: we suppose that the source and receiver are located at fixed positions and the intensity measured. It is then supposed that the entire experiment is transported parallel to the x-axis and the intensity again measured. This process is repeated again and again over the whole surface and the average intensity recorded. Of course in most physical applications, such an ambitious program need not be carried out. If the surface considered has the not uncommon property that its statistical properties in the region of physical interest can be inferred from a section of the surface of moderate size, one can obtain substantially the correct result by averaging over a region the size of the representative element.

Now consider the distribution function, $Y_n(\zeta_1, \zeta'_1; \zeta_2, \zeta'_2; \ldots; \zeta_n, \zeta'_n; |x_1-x_2|, |x_1-x_3|, \ldots, |x_{n-1}-x_n|)$. The function Y_n is defined by: $Y_n(\zeta_1, \zeta'_1; \zeta_2, \zeta'_2; \ldots; \zeta_n, \zeta'_n; |x_1-x_2|, \ldots, |x_{n-1}-x_n|)$ $d\zeta_1 d\zeta'_1 \ldots d\zeta_n d\zeta'_n$ is the probability that the following conditions occur: the displacement and the slope of the surface at x_1 lie respectively in the ranges $(\zeta_1, \zeta_1 + d\zeta_1)$ and $(\zeta'_1, \zeta'_1 + d\zeta'_1)$; similarly the displacement and the slope at x_2 lie in the ranges $(\zeta_2, \zeta_2 + d\zeta_2)$ and $(\zeta'_2, \zeta'_2 + d\zeta'_2)$; and so on through ζ_n and ζ'_n. It is noted that the distribution function depends at most upon the distances between the various points involved in its definition; this is a direct consequence of the assumption of statistical homogeneity. We can now define the average of a function F of $\zeta_1, \zeta'_1; \zeta_2, \zeta'_2; \ldots; \zeta_n, \zeta'_n$ by

$$\langle F(\zeta_1, \zeta'_1; \ldots; \zeta_n, \zeta'_n) \rangle = \underbrace{\int \cdots \int}_{2n} Y_n F d\zeta_1 d\zeta'_1 \ldots d\zeta_n d\zeta'_n, \quad (4.4.2)$$

where the integrals are carried over all values of ζ_n and ζ_n' . The brackets, $\langle \rangle$, will be used to indicate averages. It will be assumed that this averaging process can be legitimately interchanged with any integration. We also remark that when convenient the function Y_n will be represented by $Y_n(1,2,\ldots,n)$.

In this work we confine our attention to surfaces with a special, though not unusual, statistical property. It is supposed that the surface is such that the statistical properties at two points are independent if the two points are separated by a sufficient distance. To state the property mathematically we consider again the distribution function, Y_n . It is supposed that for,

$$|x_1 - x_i| > \bar{\ell} \qquad , \quad i = 2,3,\ldots,n \ , \tag{4.4.3}$$

we have

$$Y_n(1,2,\ldots,n) \approx Y_1(1) Y_{n-1}(2,3,\ldots,n) \ , \tag{4.4.4}$$

and similarly for (1) replaced by any other point. It is interesting to note that a periodic surface, or set of surfaces, does not have this property, hence the restriction represented by the relations (4.4.3) and (4.4.4) rules periodic surfaces out of consideration in what follows.

It is in order to discuss briefly the value of $\bar{\ell}$ appropriate to the sea surface, since this is the bounding surface in the numerical example to be considered in Chapter V. Unfortunately there has been relatively little work done on the determination of the spatial statistics of the sea surface. In any discussion of the properties of the ocean surface it is necessary to distinguish between what is known as "sea" and "swell". In the former case the surface is quite irregular [35], so that it seems reasonable to choose for $\bar{\ell}$ the width of a representative protuberance;

we choose 25 yards for this case. In the case of swell the surface may take on an almost periodic structure, in which case one must know at what distance the order breaks down. For simplicity we restrict ourselves here to the former case.

We return to a consideration of Eq. (4.1.10), proceeding in exactly the same way as in the development of the Kirchhoff approximation in Section 3.4. We first take $\frac{\partial}{\partial\mu_R}$ of Eq. (4.1.10); then allow $(R)\rightarrow(2)$ (where (2) lies on the surface) and at the same time $\frac{\partial}{\partial\mu_R}\rightarrow\frac{\partial}{\partial\nu_2}$; then, since $\Psi(R,2)$ and $H_0^{(1)}(r_{R2})$ have the same singularity for $r_{R2}\rightarrow 0$ (cf. Eq. (4.2.18)), we find,

$$\frac{\partial\phi(2)}{\partial\nu_2} = 2\,\frac{\partial\Psi(2,0)}{\partial\nu_2} + 2\,\overline{\frac{\partial\phi_r(2)}{\partial\nu_2}} \qquad (4.4.5)$$

where

$$\overline{\frac{\partial\phi_r(2)}{\partial\nu_2}} = -\frac{1}{4i}\int_{\zeta}\frac{\partial\Psi(2,1)}{\partial\nu_2}\;\frac{\partial\phi(1)}{\partial\nu_1}\,ds_1 \quad . \qquad (4.4.6)$$

When Eq. (4.4.5) is substituted in Eq. (4.1.10) the following result is obtained:

$$\phi(R) = \Psi(R,0) - \frac{1}{4i}\int_{\zeta}\Psi(R,1)\;2\left[\frac{\partial\Psi(1,0)}{\partial\nu_1} + \overline{\frac{\partial\phi_r(1)}{\partial\nu_1}}\right]. \qquad (4.4.7)$$

The quantity $\overline{\frac{\partial\phi_r(2)}{\partial\nu_2}}$ is seen from Eqs. (4.4.5) and (4.4.6) to represent the contribution to the normal derivative of the field at the point (2), from radiation previously reflected from the surface.

We now substitute Eq. (4.4.7) in the expression for the intensity of the field (cf. Eq. (4.2.14)) and take the average of the result to obtain for the average intensity:

$$\langle\vec{J}(R)\rangle = \mathrm{Re}\,\frac{1}{i}\left\{\langle\rangle_{00} + \langle\rangle_{10} + \langle\rangle_{01} + \langle\rangle_{11}\right\} , \qquad (4.4.8)$$

where,

$$\langle\,\rangle_{00} = \langle \Psi^*(R,0)\ \nabla_R \Psi(R,0) \rangle \quad , \qquad (4.4.9)$$

$$\langle\,\rangle_{10} = \langle \tfrac{1}{4i} \left\{ \int_\zeta \Psi^*(R,(1))2 \left[\frac{\partial \Psi^*((1),0)}{\partial \nu_{(1)}} + \frac{\overline{\partial \phi^*_r((1))}}{\partial \nu_{(1)}} \right] ds_{(1)} \right\} \nabla_R \Psi(R,0) \rangle \quad , \qquad (4.4.10)$$

$$\langle\,\rangle_{01} = \langle \tfrac{1}{4i} \nabla_R \left\{ \int_\zeta \Psi(R,(1))2 \left[\frac{\partial \Psi((1),0)}{\partial \nu_{(1)}} + \frac{\overline{\partial \phi_r((1))}}{\partial \nu_{(1)}} \right] ds_{(1)} \right\} \Psi^*(R,o) \rangle \quad , \qquad (4.4.11)$$

$$\langle\,\rangle_{11} = \langle \tfrac{1}{16} \int_\zeta \Psi^*(R,(1))2 \left[\frac{\partial \Psi^*((1),0)}{\partial \nu_{(1)}} + \frac{\overline{\partial \phi^*_r((1))}}{\partial \nu_{(1)}} \right] ds_{(1)}$$
$$\cdot \nabla_R \int_\zeta \Psi(R,(2))2 \left[\frac{\partial \Psi((2),0)}{\partial \nu_{(2)}} + \frac{\overline{\partial \phi_r((2))}}{\partial \nu_{(2)}} \right] ds_{(2)} \rangle \quad . \qquad (4.4.12)$$

A slight change in notation has been made. The arguments of those quantities which depend upon the surface displacement (or slope) have been enclosed in an extra set of brackets; those quantities whose arguments are not so enclosed depend upon points not on the surface. This has been done in order to indicate clearly which functions will be involved in the averaging process. As an example of the notation, $\Psi((1),0)$ indicates that the function Ψ depends upon ζ at the point (1) on the surface.

The first term in Eq. (4.4.8) is easily evaluated since from Eq. (4.4.9) we see that the surface function ζ is not involved. The following result is obtained upon using Eqs. (4.2.3) and (4.2.9),

$$\mathrm{Re}\,\frac{1}{i}\langle\,\rangle_{00} = G^2(R,0)k(R)\vec{a}_{nRO} \quad . \tag{4.4.13}$$

In obtaining Eq. (4.4.13) it has been supposed that $kr_{RO} \gg 1$; this restriction is equivalent to restriction c) made in Section 4.2 in connection with the applicability of geometrical optics. The quantity $\vec{a}_{nRO}$ was previously defined as $\vec{a}_n$; it represents the unit vector at (R) in the direction of the ray from (0) to (R). The Eq. (4.4.13) of course represents just that radiation which arrives at (R) directly from the source at (0). It may be compared with Eq. (2.1.22) when the usual "black" receiving cylinder is considered.

Before proceeding we establish some further definitions in connection with Eq. (4.4.12). Let

$$\langle\rangle_{11} = \langle\rangle_{ii} + \langle\rangle_{ri} + \langle\rangle_{ir} + \langle\rangle_{rr} \quad , \tag{4.4.14}$$

where,

$$\langle\rangle_{ii} = \left\langle \frac{1}{4}\int_{\zeta}\int_{\zeta} \Psi^*(R,(1))\nabla_R\Psi(R,(2))\frac{\partial\Psi^*((1),0)}{\partial\nu_{(1)}}\frac{\partial\Psi((2),0)}{\partial\nu_{(2)}}\,ds_{(1)}ds_{(2)}\right\rangle , \tag{4.4.15}$$

$$\langle\rangle_{ri} = \left\langle \frac{1}{4}\int_{\zeta}\int_{\zeta} \Psi^*(R,(1))\nabla_R\Psi(R,(2))\overline{\frac{\partial\phi^*((1))}{\partial\nu_{(1)}}}\frac{\partial\Psi((2),0)}{\partial\nu_{(2)}}\,ds_{(1)}ds_{(2)}\right\rangle , \tag{4.4.16}$$

$$\langle\rangle_{ir} = \left\langle \frac{1}{4} \int_{\zeta}\int_{\zeta} \psi^*(R,(1))\nabla_R\psi(R,(2)) \frac{\partial\psi^*((1),0)}{\partial\nu_{(1)}} \overline{\frac{\partial\phi_r((2))}{\partial\nu_{(2)}}} \; ds_{(1)}ds_{(2)} \right\rangle ,$$

(4.4.17)

$$\langle\rangle_{rr} = \left\langle \frac{1}{4} \int_{\zeta}\int_{\zeta} \psi^*(R,(1))\nabla_R\psi(R,(2)) \overline{\frac{\partial\phi_r^*((1))}{\partial\nu_{(1)}}} \; \overline{\frac{\partial\phi_r((2))}{\partial\nu_{(2)}}} \; ds_{(1)}ds_{(2)} \right\rangle .$$

(4.4.18)

For Eq. (4.4.8), the term involving $\langle\rangle_{00}$ has already been evaluated. This leaves the six terms defined in Eqs. (4.4.10) and (4.4.11) and Eqs. (4.4.15) through (4.4.18). Of these six, four will vanish due to phase incoherence; the two terms $\langle\rangle_{ii}$ and $\langle\rangle_{rr}$ will remain. The former will be seen to represent the contribution of the singly-reflected radiation while the latter represents the contribution of multiply-reflected radiation. It will be shown in this section that the cross-terms vanish; this will be done for a representative term of the group (the one given in Eq. (4.4.10)) the proof for the other cross-terms being analogous. The two non-vanishing terms, $\langle\rangle_{ii}$ and $\langle\rangle_{rr}$, will be treated in Section 4.5.

It is first convenient to write down expressions for ψ and $\frac{\partial\psi}{\partial\nu}$ utilizing the results of Section 4.3. We begin with the function $\psi((2),(1))$,

$$\psi((2),(1)) = G((2),(1))e^{ig((2),(1))} \quad . \tag{4.4.19}$$

We then wish to utilize the plane wave approximations for G and g for small changes in their arguments given respectively in Eqs. (4.3.52) and

(4.3.53). The conditions for the applicability of these approximations have been given in the relations (4.3.61), (4.3.62), and (4.3.63). Using the values of the parameters (given at the beginning of this section) which are appropriate to the numerical example of Chapter V and assuming $\Theta_{21}' \sim \mu$ it is found that (4.3.61) becomes $|x_1-x_2| \gg$ 25 yards; the left side of (4.3.62) becomes 0.036 which satisfies that relation; and (4.3.63) is $|x_1-x_2| \gg$ 175 yards. If, as indicated in Eq. (4.4.19), both (1) and (2) are surface points, then the last relation can be replaced by (4.3.64) which leads to the restriction $|x_1-x_2| \ll$ 540,000 yards. This last condition is seen to be fulfilled for all surface points since $L(\mu)$ = 9760 yards, and no ray connecting two points lying on the surface travels farther than this distance.

Accepting these restrictions on the horizontal separation of the two points (1) and (2) and using Eqs. (4.3.52) and (4.3.53), Eq. (4.4.19) can be written

$$\Psi((2),(1)) = G(2,1)\exp i\left[g(2,1)-k(0)\sin\Theta'_{21} \left(\zeta(2)+\zeta(1)\right)\right] . \quad (4.4.20)$$

In obtaining Eq. (4.4.20) it has been assumed that $\Delta r_x = 0$ and $\Delta r_z = \zeta$. Furthermore, since both of the points (1) and (2) are assumed to lie on the surface, one needs a correction to the function g both at the source and the field point. This correction is found through the use of the reciprocity theorem as indicated in Section 4.3.

Dropping the extra brackets around the symbols (1) and (2) in Eq. (4.4.20) indicates that the functions in which they appear are to be evaluated for $\zeta(1) = \zeta(2) = 0$. The angle Θ'_{21} is defined in Fig. 8.

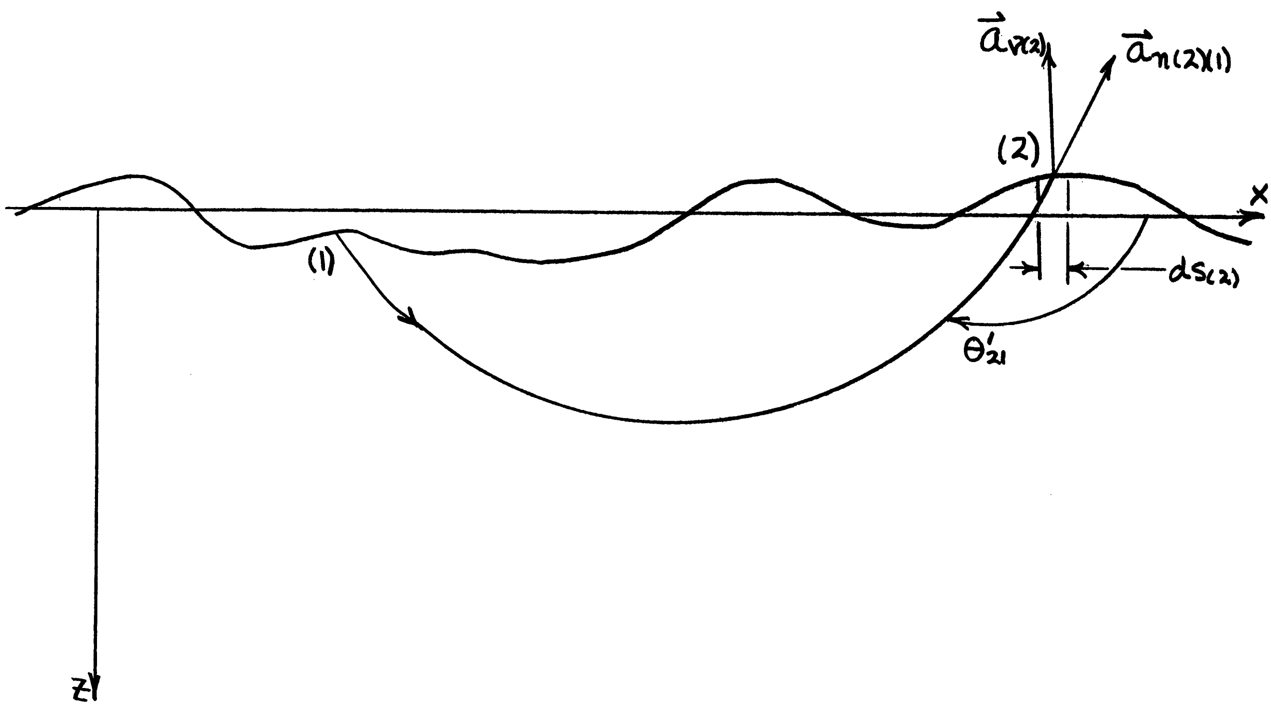

Fig. 8. Diagram used in the development of approximate expressions for Ψ and $\frac{\partial \Psi}{\partial \nu}$ at the surface.

We now consider $\frac{\partial \Psi}{\partial \nu}$; it is useful to treat the quantity $\frac{\partial \Psi}{\partial \nu}$ ds. Since $\frac{\partial}{\partial \nu}$ represents the derivative along the outward-directed normal, we have,

$$\frac{\partial \Psi((2),(1))}{\partial \nu_{(2)}} ds_{(2)} = ds_{(2)} \vec{a}_{\nu(2)} \cdot \Big[\big(\nabla_2 G((2),(1))\big)\, e^{ig((2),(1))} + i\big(G((2),(1))\nabla_2 g((2),(1))\big)\, e^{ig((2),(1))}\Big], \tag{4.4.21}$$

where $\vec{a}_{\nu(2)}$ is a unit vector normal to $\zeta(x)$ (cf. Fig. 8). Now if $\left|\frac{\nabla G}{(\nabla g)G}\right| \ll 1$ we can neglect the first term in the brackets, [ ] , on the right side of Eq. (4.4.21). Since $|\nabla g| \sim k$ (cf. Eqs. (4.2.4) and

(4.2.9)), it is seen that this condition is identical with the requirement that the per unit change of the function G be negligible over a distance of one radiation wavelength (already required for the applicability of geometrical optics). Then using this restriction and Eq. (4.2.9), we have,

$$\frac{\partial \Psi((2),(1))}{\partial \nu_{(2)}} ds_{(2)} = \vec{a}_{\nu(2)} \cdot \vec{a}_{n(2)(1)} ik((2))G((2),(1))e^{ig((2),(1))} ds_{(2)} \tag{4.4.22}$$

where $\vec{a}_{n(2)(1)}$ is again a unit vector in the direction of the ray from (1) to (2) (cf. Fig. 8). Now using the approximations already made in connection with the development for Ψ, Eq. (4.3.54) for the quantity $\vec{a}_{n(2)(1)}$, and the definitions of the quantities shown in Fig. 8, Eq. (4.4.22) can be written,

$$\frac{\partial \Psi((2),(1))}{\partial \nu_{(2)}} ds_{(2)} = ik(0)G(2,1)\exp\left\{i\left[g(2,1)-k(0)\sin\theta'_{21}\,(\zeta(2)+\zeta(1))\right]\right\} \cdot f(\zeta'(2);2,1)dx_2 \,, \tag{4.4.23}$$

where,

$$f(\zeta'(2);2,1) = -\ \zeta'(2)\cos\theta'_{21} + \sin\theta'_{21} \quad . \tag{4.4.24}$$

Once again, dropping the extra set of parentheses around the argument of a function indicates that the function is to be evaluated at the projection on the x-axis of the point involved.

In the treatment of both Ψ and $\frac{\partial \Psi}{\partial \nu}$ it has been assumed that both arguments of each were surface points. If either one of the arguments of either one of the functions does not lie on the surface, the quantity ζ

connected with that argument is to be set equal to zero in Eqs. (4.4.20) and (4.4.23).

We proceed now with the evaluation of Eq. (4.4.10) by defining,

$$\langle\rangle_{10} = \langle\rangle_{10i} + \langle\rangle_{10r} \quad , \tag{4.4.25}$$

where

$$\langle\rangle_{10i} = \left\langle \frac{1}{4i} \int_{\zeta} \psi^*(R,(1)) 2 \frac{\partial \psi^*((1),0)}{\partial \nu_{(1)}} \, ds_{(1)} \right\rangle \nabla_R \Psi(R,0) \tag{4.4.26}$$

and

$$\langle\rangle_{10r} = \left\langle \frac{1}{4i} \int_{\zeta} \psi^*(R,(1)) 2 \frac{\overline{\partial \phi^*((1))}}{\partial \nu_{(1)}} \, ds_{(1)} \right\rangle \nabla_R \Psi(R,0) \quad , \tag{4.4.27}$$

the quantity $\nabla_R \Psi(R,0)$ having been moved outside the brackets, $\langle\,\rangle$, since it does not depend on ζ . Using Eqs. (4.4.20) and (4.4.23), Eq. (4.4.26) can be written

$$\langle\rangle_{10i} = \frac{1}{2}\left[\nabla_R \Psi(R,0)\right] \int_{\zeta} G(R,1)G(1,0)k(0)e^{-i[g(R,1)+g(1,0)]} \langle R10\rangle \, dx_1 \, , \tag{4.4.28}$$

where

$$\langle R10\rangle = \left\langle e^{ik(0)S(R10)\zeta(1)} f(\zeta'(1);1,0)\right\rangle \quad , \tag{4.4.29}$$

and

$$S(R10) = \sin\theta_{R1} + \sin\theta'_{10} \quad . \tag{4.4.30}$$

We shall also let

$$C(R10) = \cos\theta_{R1} + \cos\theta'_{10} \, . \tag{4.4.31}$$

We now establish conditions under which the quantity $\langle R10\rangle$ may be neglected.

First if ζ is continuous and bounded and if, as has been assumed, $\langle\zeta\rangle = 0$, then it is not difficult to show that,*

$$\langle \zeta' G(\zeta) \rangle = 0 \quad . \tag{4.4.32}$$

Then it follows that Eq. (4.4.29) can be written (cf. Eq. (4.4.24)),

$$\langle R10 \rangle = \sin\theta_{10}' \left\langle e^{ik(0)S(R10)\zeta} \right\rangle \quad . \tag{4.4.33}$$

The quantity $\left\langle e^{-ik(0)S(R10)\zeta} \right\rangle$ is termed the characteristic function associated with the (displacement) distribution function,

$$W(\zeta) = \int Y_1(\zeta, \zeta')d\zeta' \quad . \tag{4.4.34}$$

The symbol Q is frequently used to denote the characteristic function; thus,

$$\langle R10 \rangle = \sin\theta_{10}' \, Q(-k(0)S(R10)) \quad . \tag{4.4.35}$$

The quantity Q is closely connected with the coherently reflected radiation [36]. This can be seen by observing that if we adopt the Kirchhoff approximation, the integral in Eq. (4.4.28) represents, except for a constant, the average reflected amplitude observed at (R) as a result of a source placed at (0) (cf. Eq. (4.4.26)).

In order to estimate the size of $\langle R10 \rangle$ we must make some assumption concerning the distribution function W. Following Eckart we choose

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* To see this we consider the time-fixed surface (the proof for the time-varying surface is analogous). It is supposed that a pair of lines separated by a distance $\Delta\zeta$ is drawn parallel to the x-axis. Since it is assumed that $\langle\zeta(x)\rangle = 0$ and that $\zeta(x)$ is continuous and bounded, it is clear that the average slope of the segments of surface contained between the pair of lines vanishes. This is true regardless of the position or the separation of the lines so that Eq. (4.4.32) is seen to follow.

a Gaussian distribution,

$$W(\zeta) = \left[(2\pi)^{1/2}h\right]^{-1} \exp(-\frac{\zeta^2}{2h^2}) , \qquad (4.4.36)$$

where h is the rms value of ζ. Then

$$Q(-k(0)S(R10)) = \exp\left[-\tfrac{1}{2}k^2(0)S^2(R10)h^2\right] . \qquad (4.4.37)$$

For,

$$k(0)S(R10)h \geq 3 \qquad (4.4.38)$$

we have,

$$Q \leq 0.012.$$

The corresponding quantity in the expressions for the non-vanishing terms ($\langle\rangle_{ii}$ and $\langle\rangle_{rr}$) is of order unity, as will be seen in the next section. Hence we shall assume that if the condition (4.4.38) is fulfilled, $\langle R10\rangle$ may be neglected, and therefore also $\langle\rangle_{10i}$.

If the values of the parameters given at the beginning of this section are used, one finds that the left side of the relation (4.4.38) is equal to 10.2 (where it is assumed that $\sin\theta'_{10}$ and $\sin\theta_{R1}$ are of the same order as μ). Thus it is seen that the reflected radiation in the numerical example is primarily of incoherent phase (and hence diffuse).

It is worth remarking in connection with (4.4.38) that even if this relation is not fulfilled in a given problem, the reflected radiation cannot always be considered specular. From the work of Section 3.3 on the perturbation method, it can be seen that the condition

$$k(0)\ \sin\theta'_{10}\ h \ll 1 \qquad (4.4.39)$$

must be fulfilled before the reflected radiation becomes specular.

We now turn to the quantity defined in Eq. (4.4.27). Using Eq. (4.4.6) we can write,

$$\langle\rangle_{10r} = \left\langle \frac{1}{4i}\int_{\zeta} \psi^*(R,(1))\, 2\left\{\frac{1}{4i}\int_{\zeta} \frac{\partial\Psi^*((1),(2))}{\partial\nu_{(1)}}\frac{\partial\phi^*((2))}{\partial\nu_{(2)}}\, ds_{(2)}\right\} ds_{(1)}\right\rangle$$

$$\cdot\nabla_R\,\Psi(R,0) \quad . \tag{4.4.40}$$

The quantity,

$$2\left\{\frac{1}{4i}\int_{\zeta}\frac{\partial\Psi^*((1),(2))}{\partial\nu_{(1)}}\frac{\partial\phi^*((2))}{\partial\nu_{(2)}}\, ds_{(2)}\right\}, \tag{4.4.41}$$

represents the complex conjugate of the normal derivative of that part of the field function at the point (1) which arises from radiation reflected from all surface points (2). Now it is evident upon examining the quantity (4.4.41), that there is a region ((1) near to (2)) of the integrand for which $x_1 - x_2$ is too small to fulfill the condition (4.3.61) required for the validity of the plane wave approximation to the Green's function. (Since both (1) and (2) are assumed to lie on the surface, the other restriction involving the separation of the two points, (4.3.63), can be replaced by the weaker (4.3.64).) However under certain conditions the contribution to the total integral from this region of the integrand may be neglected.

We now put this statement in mathematical terms, afterward deriving the conditions under which it is valid. Under certain conditions, to be given below,

$$\left|\int_{|x_1-x_2|\le l}\frac{\partial\Psi((1),(2))}{\partial\nu_{(1)}}\frac{\partial\phi((2))}{\partial\nu_{(2)}}\, ds_{(2)}\right| \ll \left|\int_{|x_1-x_2|>l}\frac{\partial\Psi((1),(2))}{\partial\nu_{(1)}}\frac{\partial\phi((2))}{\partial\nu_{(2)}}\, ds_{(2)}\right|, \tag{4.4.42}$$

where ℓ is defined by

$$\ell \gg \bar{\ell} \qquad (4.4.43)$$

(this definition of course embraces the weaker, $\ell > \bar{\ell}$). Whenever conditions are such that the relation (4.4.42) is valid, one may use the plane wave approximation for the Green's function in the important region of the integral given in (4.4.41); moreover in such a case for the same region of the integral the two points (1) and (2) are far enough separated so that the surface properties are statistically independent (cf. the relations (4.4.3) and (4.4.4)).

Conditions will now be derived under which the relation (4.4.42) holds. The relation is seen to be equivalent to supposing that the amount of reflected radiation received by a region of the surface from nearby regions is negligible compared with the radiation received from more distant regions.

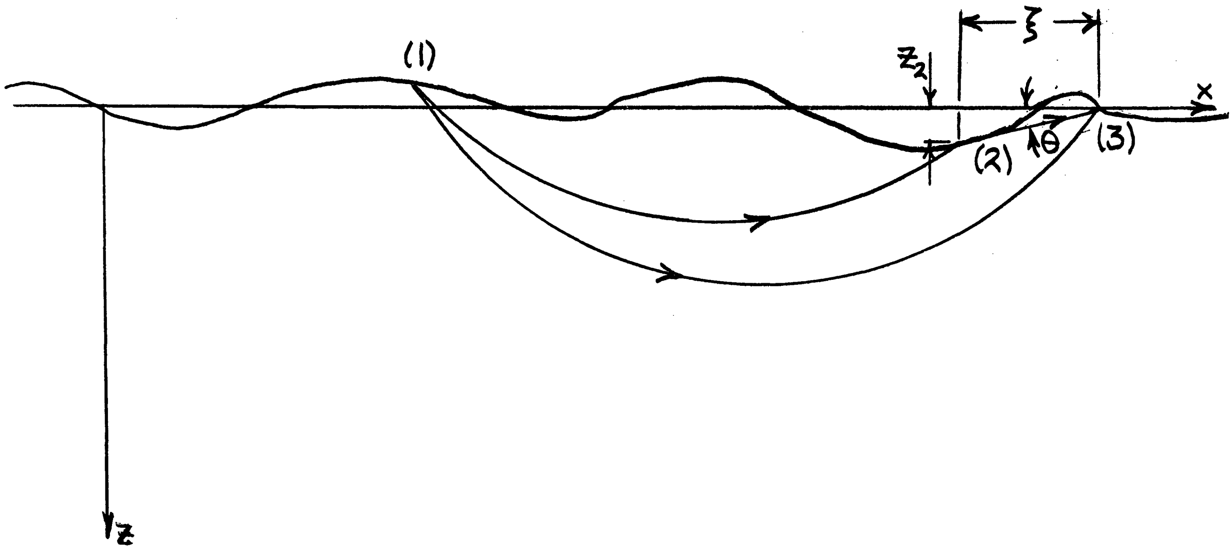

Fig. 9. Diagram showing the reflection of energy from one point to a neighboring point on the surface.

We refer to Fig. 9. It is supposed that radiation is reflected from the surface point (1) to the surface point (2) and then re-reflected to the point (3). It is desired to compare this radiation with the radiation arriving at (3) directly from (1). So long as (2) and (3) lie close enough to one another so that the refracting properties of the medium may be neglected, the results of Section 3.4 (the Kirchhoff approximation) are directly applicable provided that the conditions i) and ii) of that section are fulfilled. It is remembered that in the Kirchhoff approximation the first term neglected was made small (in part) because the assumption of small surface slope implied that angles corresponding to the angle Θ of Fig. 9 were small. Then it seems appropriate to use the angle Θ as the criterion for determining when the refracting properties of the medium are unimportant in the present connection. Assuming a linear-velocity structure and that all angles are small, one finds from Eq. (4.3.32),

$$\Theta = \frac{\alpha}{2}\xi + \frac{z_2}{\xi} \quad . \qquad (4.4.44)$$

Hence for

$$\xi < \left(\frac{2h}{\alpha}\right)^{1/2} \qquad (4.4.45)$$

it is seen that the refracting properties of the medium play a minor role in determining the angle Θ; the specific displacement z_2 has been replaced by the rms displacement h . It is assumed then that when (4.4.45) is fulfilled, the discussion of the error term (the integral in Eq. (3.4.14)) given in Section 3.4 may be applied here.

We now refer to Eq. (3.4.14); in this equation it is supposed that the point (1) is replaced by (2) and the point (2) is replaced by (3) . Then the left side of the equation represents the normal derivative of the total field at (3). The first term on the right side represents the normal derivative of the field coming directly from (1) to (3). The second term can be interpreted then as the normal derivative of that part of the radiation field which has been reflected from the surface in travelling from (1) to (3). It was shown in Section 3.4 that it is reasonable to assume that this second term is of order $\frac{1}{kR_m}$ or $\left|\frac{d\zeta^M}{dx}\right|$ when these two are small. Hence we shall suppose here that relation (4.4.42) is fulfilled for $\ell < \xi$ if $\frac{1}{kR_m} \ll 1$ and $\left|\frac{d\zeta^M}{dx}\right| \ll 1$ where R_m is the minimum radius of curvature of the surface.

To sum up the argument, it is found that the restriction (4.4.42) applies if

$$\left(\frac{2h}{\alpha}\right)^{1/2} \gg \bar{\ell} \quad . \tag{4.4.46}$$

and if $\frac{1}{kR_m}$ and $\left|\frac{d\zeta^M}{dx}\right|$ are small. By examining more closely the effect of refraction in the medium upon the argument presented in Section 3.4 for the neglect of the multiply-reflected radiation terms, one can considerably increase the value of ℓ which can be used in the relation (4.4.42). However the present discussion will serve here. It is of interest to substitute the values of the parameters given at the beginning of this section in the expression (4.4.46). The left side becomes 422 yards and the right side is 25 yards and hence the relation is fulfilled.

We return now to the treatment of Eq. (4.4.40); we had set out to show the restrictions under which the quantity defined there is negligible. Utilizing the relation (4.4.42) with the definition of ℓ given in (4.4.43) and the definition of $\bar{\ell}$ given in Eqs. (4.4.3) and (4.4.4), we find that Eq. (4.4.40) can be written:

$$\langle\rangle_{10r} = \left\{\frac{1}{4i}\int_\zeta \left\langle \frac{\partial\phi^*(2)}{\partial\nu_{(2)}} \, ds_{(2)}\right\rangle \left[\left\langle \frac{2}{4i}\int_\zeta \Psi^*(R,(1)) \frac{\partial\Psi^*((1),(2))}{\partial\nu_{(1)}} \, ds_{(1)}\right\rangle\right]\right\} \cdot \nabla_R \Psi(R,0) \quad , \tag{4.4.47}$$

where we have assumed that we may interchange the integrations over (1) and (2). In connection with Eq. (4.4.26) it has already been shown that under the conditions imposed in this section,

$$\left|\left\langle\frac{2}{4i}\int_{\zeta}\Psi^*(R,(1))\,\frac{\partial\Psi^*((1),0)}{\partial\nu_{(1)}}\,ds_{(1)}\right\rangle\right|\approx 0 \quad . \qquad (4.4.48)$$

By allowing $(0)\rightarrow(2)$ here it is seen that the quantity in the brackets, $[\;]$, in the integrand of Eq. (4.4.47) is the same as the left side of Eq. (4.4.48), and hence the quantity $\langle\,\rangle_{10r}$ may be assumed to be negligible.

This completes our consideration of the quantity $\langle\,\rangle_{10}$, defined in Eq. (4.4.25) as the sum of two terms which have now been shown to be negligible. As stated at the outset, the other cross-terms, $\langle\rangle_{01}$, $\langle\rangle_{ri}$, and $\langle\rangle_{ir}$ of Eqs. (4.4.8) and (4.4.14) can be shown to vanish in a similar way.

4.5. Completion of the Justification of the Statistical Model

In the consideration of Eq. (4.4.8) we are left now with the two non-vanishing terms defined in Eqs. (4.4.15) and (4.4.18). These two terms will lead to the equations presented in Chapter II.

We begin the consideration of the term $\langle\rangle_{ii}$ by noting first that for $|x_1-x_2|>\bar{\ell}$ in the integrand of Eq. (4.4.15) we may write for the right side of that equation,

$$\left\langle\frac{1}{2}\int_{\zeta}\Psi^*(R,(1))\,\frac{\partial\Psi^*((1),0)}{\partial\nu_{(1)}}\,ds_{(1)}\right\rangle$$

$$\cdot\left\langle\frac{1}{2}\int_{\zeta}\nabla_R\Psi(R,(2))\,\frac{\partial\Psi((2),0)}{\partial\nu_{(2)}}\,ds_{(2)}\right\rangle, \qquad (4.5.1)$$

upon using the statistical property represented by Eq. (4.4.4). The first factor on the right side of Eq. (4.5.1) has already been shown to be negligible in connection with Eq. (4.4.26). It is also seen that the second factor is just the gradient of the conjugate of the first, and may also be assumed to be negligible. Thus in Eq. (4.4.15) we need only consider those parts of the integrand for which $|x_1 - x_2| \leq \bar{\ell}$.

The first step in the treatment of Eq. (4.4.15) is to make use of the approximate expressions for Ψ and $\frac{\partial \Psi}{\partial \nu}$ given respectively in Eqs. (4.4.20) and (4.4.23). We have,

$$\langle\rangle_{ii} = \frac{ik^2(0)k(z_R)}{4} \vec{a}_R \int_\zeta\int_\zeta G(R,1)G(R,2)G(1,0)G(2,0)(\vec{a}_R \cdot \vec{a}_{nR2})$$
$$\cdot \exp\left\{ i\left[g(R,2)-g(R,1)+g(2,0)-g(1,0)\right]\right\} \langle\rangle_{iia} dx_1 dx_2 \ , \qquad (4.5.2)$$

where

$$\langle\rangle_{iia} = \left\langle \exp\left\{ ik(0)\left[\zeta(1)S(R10)-\zeta(2)S(R20)\right]\right\} f(\zeta'(1);1,0)f(\zeta'(2);2,0)\right\rangle \ , \qquad (4.5.3)$$

and $\vec{a}_R$ is a unit vector in the direction of ∇_R .

In view of the fact that we need only consider those values of the integrand of Eq. (4.5.2) for which $|x_1 - x_2| \leq \bar{\ell}$, we can make use of the plane wave approximation for the Green's function, where it is assumed that $\Delta r_x = x_1 - x_2$ and $\Delta r_z = 0$. Then from Eqs. (4.3.52), (4.3.53), and (4.3.54) (the conditions of applicability have already been discussed) it is seen that Eqs. (4.5.2) and (4.5.3) can be written

$$\langle\rangle_{ii} = \frac{ik^2(0)k(z_R)}{4} \vec{a}_R \int_\zeta dx_2 G^2(R,2)G^2(2,0)(\vec{a}_R \cdot \vec{a}_{nR2})$$
$$\cdot \left\{ \int_{x_1-x_2=-\bar{\ell}}^{x_1-x_2=\bar{\ell}} \langle |x_1-x_2| \rangle_{iia} \exp\left[ik(0)(x_1-x_2)C(R20)\right] dx_1 \right\} \ , \qquad (4.5.4)$$

and

$$\langle |x_1-x_2| \rangle_{iia} = \langle f(\zeta'(1);2,0) f(\zeta'(2);2,0)$$

$$\exp i\, k(0)(\zeta(1)-\zeta(2)) S(R20) \rangle \; . \qquad (4.5.5)$$

It is recognized in Eq. (4.5.5) that the average can depend on x_1 only through $|x_1-x_2|$ (this follows from the property of statistical homogeneity). The limits of the integration over x_1 in Eq. (4.5.4) have been fixed by the result that the integrand vanishes for $|x_1-x_2| > \bar{\ell}$. We note that $\langle\rangle_{iia}$ is real from Eq. (4.5.5); this is seen by observing that the distribution function is symmetric in the interchange of (1) and (2) as a result of the assumption of statistical homogeneity. Then if we introduce the quantity being averaged in Eq. (4.5.5) in the definition of the average (cf. Eq. (4.4.2)) and interchange $\zeta(1)$ and $\zeta(2)$ (and likewise $\zeta'(1)$ and $\zeta'(2)$) it is seen that the imaginary part of Eq. (4.5.5) vanishes. Likewise the quantity in the brackets $\{\}$ in the integrand of Eq. (4.5.4) is real. To see this we let $x_1-x_2 = \xi$ and then allow the change of integration variable, $\xi \rightarrow -\xi$.

Now designating the contribution of $\langle\,\rangle_{ii}$ to the average intensity observed at (R) by $\langle\vec{J}\rangle_{ii}$ (cf. Eq. (4.4.8)) we find from Eq. (4.5.4),

$$\langle\vec{J}\rangle_{ii} = k(0)k(z_R)\vec{a}_R \frac{\pi}{2}\int_{\zeta} G^2(R,2)G^2(2,0)(\vec{a}_R\cdot\vec{a}_{nR2})\tilde{A}(\theta_{R2},\theta'_{20})dx_2 \qquad (4.5.6)$$

where

$$\tilde{A}(\theta_{R2},\theta_{20}) = \frac{k(0)}{2\pi}\int_{-\bar{\ell}}^{\bar{\ell}} \langle|\xi|\rangle_{iia} \exp\left[ik(0)\xi\, C(R20)\right] d\xi \; . \qquad (4.5.7)$$

Using Eq. (4.5.6), the differential contribution to $\langle\vec{J}\rangle_{ii}$ from (2) is considered; we choose $\vec{a}_R = \vec{a}_{nR2}$ and find for the intensity of the radiation coming from (2) to (R):

$$d\langle\vec{J}\rangle_{ii} = k(0)G^2(2,0)k(z_R)\vec{a}_{nR2}\frac{\pi}{2}G^2(R,2)\tilde{A}(\theta_{R2},\theta'_{20})dx_2 \quad . \qquad (4.5.8)$$

Using Eq. (4.2.13) it is seen that $k(0)G^2(2,0)$ represents the magnitude of the intensity of the radiation at (2) coming from the source at (0). Similarly upon using Eq. (4.3.17) it is seen that $k(z_R)\vec{a}_{nR2}\frac{\pi}{2}G^2(R,2)$ represents the intensity at (R) as a result of a source at (2) radiating unit energy per second per angle. It follows then from Eq. (4.5.8) that $\tilde{A}(\theta_{R2},\theta'_{20})dx_2$ represents the energy per second per angle reflected in the direction θ_{R2} as a result of radiation of unit intensity incident from the direction θ'_{20} . Hence, the function $\tilde{A}$ is the same as that defined in Section 2.1. Furthermore we can see that

$$\mathcal{S}(\theta_{R2},x_2)dx_2 = J(2,0)\tilde{A}(\theta_{R2},\theta'_{20})dx_2 \quad , \qquad (4.5.9)$$

where $\mathcal{S}(\theta,x)dx$ is the energy per second per angle reflected (once) from the surface, coming from the element dx_2 ; this is the source function defined in Section 2.1. Finally upon introducing the "black" cylinder of radius a as the receiving element and using Eq. (4.2.13) we have for the once reflected energy received per second (per length of cylinder):

$$F_{R1}(R,0) = 2a\,\frac{\pi}{2}\int_{\zeta} J(R,2)J(2,0)\tilde{A}(\theta,\theta')dx_2 \quad . \qquad (4.5.10)$$

This result can be compared with Eq. (2.1.19). The integral is to be carried over all portions of the surface which contribute singly-reflected radiation to the point (R). The subscripts on the angles θ and θ' have been dropped, it being understood that they represent the angles made with the horizontal by the reflected and the incident rays respectively.

The function $\tilde{A}$ is seen from Eq. (4.5.7) to be independent of the velocity structure, depending only upon $k(0)$. Furthermore, it depends upon the two points (0) and (R) only through the incident and reflected angles θ' and θ; in particular the function does not depend upon the distances r_{R2} and r_{20} . It can be directly verified that $\tilde{A}$ is the scattering function obtained for the surface ζ in a homogeneous medium, upon using the Kirchhoff approximation. Actually this verification is not necessary since the homogeneous medium is contained as a special case of the present treatment. The result can be compared with the corresponding result obtained by Eckart for the three dimensional reflection problem [36].

We consider now the symmetry properties of Eq. (4.5.10). Using the reciprocity theorem on the function G in the definition of the intensity function J (cf. Eq. (4.2.13)) we see from Eq. (4.5.10) that if,

$$\tilde{A}(\theta,\theta') = \tilde{A}(\theta',\theta) \quad , \qquad (4.5.11)$$

then it follows that

$$\frac{F_{R1}(R,0)}{k(z_R)} = \frac{F_{R1}(0,R)}{k(z_0)} \quad . \qquad (4.5.12)$$

The conclusion is then that if Eq. (4.5.11) is fulfilled, the total once-reflected energy received by the postulated black cylinder satisfies a reciprocity relation of the type given by Eq. (4.5.12). By virtue of the reciprocity theorem it seems reasonable to suppose that Eq. (4.5.11) is satisfied, however it has now been possible to prove this.

We now consider the term $\langle\rangle_{rr}$ defined in Eq. (4.4.18). As in the treatment of $\langle\rangle_{ii}$ it can be seen that if $|x_1-x_2|\rangle\bar{\ell}$, the average of the integrand in Eq. (4.4.18) will vanish as a result of the diffuse reflecting property of the surface. We use the definition of $\overline{\frac{\partial\phi_r}{\partial\nu}}$ given in

Eq. (4.4.6) and the approximations for Ψ and $\frac{\partial\Psi}{\partial\nu}$ given by Eqs. (4.4.20) and (4.4.23) to obtain from Eq. (4.4.18),

$$\langle\rangle_{rr} = \frac{ik(z_R)k^2(0)\vec{a}_R}{64} \int dx_1 \int dx_2 \int dx_3 \int dx_4 \; (\vec{a}_R \cdot \vec{a}_{nR2})$$

$$|x_1-x_2| \leq \bar{\ell}; \quad |x_2-x_4| > \ell; \quad |x_1-x_3| > \ell \qquad (4.5.13)$$

$$\cdot \langle\rangle_{rra} G(R,1)G(R,2)G(1,3)G(2,4)\exp\left\{ i\left[g(R,2)-g(R,1)+g(2,4)-g(1,3)\right]\right\} ,$$

where

$$\langle\rangle_{rra} = \Big\langle f(\zeta'(1));1,3)f(\zeta'(2);2,4)(1+\zeta'^2(3))^{1/2}(1+\zeta'^2(4))^{1/2}$$

$$\frac{\partial\phi^*((3))}{\partial\nu_{(3)}} \frac{\partial\phi((4))}{\partial\nu_{(4)}} \exp\Big\{ik(0)\Big[\zeta(1)S(R13)$$

$$- \zeta(2)S(R24) + \zeta(3)\sin\theta_{13} - \zeta(4)\sin\theta_{24}\Big]\Big\}\Big\rangle . \qquad (4.5.14)$$

The factors $(1+\zeta'^2(3))^{1/2}$ and $(1+\zeta'^2(4))^{1/2}$ in Eq. (4.5.14) appear when ds is replaced by dx; the restrictions $|x_2-x_4| > \ell$ and $|x_1-x_3| > \ell$ on the range of integration in Eq. (4.5.13) appear as a result of the assumption that the surface reflects little energy from a point to nearby surface points (cf. the relation (4.4.42)).

From the restrictions on the integration given in Eq. (4.5.13) we deduce that $|x_1-x_4| > \bar{\ell}$ and $|x_2-x_3| > \bar{\ell}$. Then from the diffuse reflecting property of the surface it follows that we need only consider regions where $|x_1-x_2| \leq \bar{\ell}$ and $|x_3-x_4| \leq \bar{\ell}$ in the integrals of Eq. (4.5.13). Because of these properties and the statistical property given in Eqs. (4.4.3) and (4.4.4), we can then write for Eq. (4.5.14),

$$\langle\rangle_{rra} = \langle f(\zeta'(1);2,4)f(\zeta'(2);2,4)\exp\left[ik(0)S(R24) \cdot (\zeta(1)-\zeta(2))\right\rangle \cdot \Big\langle (1+\zeta'^2(3))^{1/2}(1+\zeta'^2(4))^{1/2} \cdot \frac{\partial\phi^*((3))}{\partial\nu_{(3)}} \frac{\partial\phi((4))}{\partial\nu_{(4)}} \exp\left[ik(0)\sin\theta_{24}(\zeta(3)-\zeta(4))\right]\Big\rangle \quad , \quad (4.5.15)$$

where we have made use of the approximation given in Eq. (4.3.54). Now using the definition of the scattering function given in Eq. (4.5.7) we find for Eq. (4.5.13),

$$\langle\rangle_{rr} = ik(z_R)k(0)\,\frac{2\pi}{64}\,\vec{a}_R \iint_{\zeta\,\zeta} dx_2dx_4\tilde{A}(\theta_{R2},\theta'_{24})(\vec{a}_R\cdot\vec{a}_{nR2})G^2(R,2)G^2(2,4)$$
$$\cdot\int_{-\bar{l}}^{\bar{l}} d\eta\, e^{-ik(0)\eta\cos\theta'_{24}} \Big\langle(1+\zeta'^2(3))^{1/2}(1+\zeta'^2(4))^{1/2}$$
$$\cdot\left\{\frac{\partial\phi^*((3))}{\partial\nu_{(3)}}\Bigg|_{x_3=x_4+\eta}\right\}\frac{\partial\phi((4))}{\partial\nu_{(4)}} \exp\left[ik(0)\sin\theta_{24}(\zeta(3)-\zeta(4))\right]\Big\rangle, \qquad (4.5.16)$$

where we set $\xi = x_1-x_2$ (in the definition of $\tilde{A}$) and $\eta = x_3-x_4$. We now define the quantity,

$$\mathcal{Q}(\theta_{24},x_4) = \frac{1}{8\pi}\int_{-\bar{l}}^{\bar{l}} d\eta\, e^{-ik(0)\eta\cos\theta'_{24}} \Big\langle(1+\zeta'^2(3))^{1/2}(1+\zeta'^2(4))^{1/2}$$
$$\cdot\left\{\frac{\partial\phi^*((3))}{\partial\nu_{(3)}}\Bigg|_{x_3=x_4+\eta}\right\}\frac{\partial\phi((4))}{\partial\nu_{(4)}}\, e^{ik(0)\sin\theta_{24}(\zeta(3)-\zeta(4))}\Big\rangle, \qquad (4.5.17)$$

which when substituted in Eq. (4.5.16) yields,

$$\langle\rangle_{rr} = i\left(\frac{\pi}{2}\right)^2 \vec{a}_R \int_\zeta dx_2 J(R,2) \int_\zeta \tilde{A}(\theta_{R2},\theta_{24}')J(2,4)\mathcal{L}(\theta_{24},x_4)dx_4 \ , \qquad (4.5.18)$$

where we have let $\vec{a}_R = \vec{a}_{nR2}$. It can be shown that $\mathcal{L}(\theta,x)$ is real by substituting the definition of $\frac{\partial\phi}{\partial\nu}$ (cf. Eq. (4.4.5)) in Eq. (4.5.17) and repeating the procedure carried through in this section. This process would give a real term and another term similar to Eq. (4.5.17). The whole procedure would then be repeated again; finally a series of real terms would be obtained for $\mathcal{L}$. These terms would represent the contribution of various multiple reflections to the quantity $\mathcal{L}$.

From Eq. (4.5.18), using the definition of the intensity given in Eq. (4.4.8) and remembering that the energy per angle per second radiated by a unit source is $2/\pi$, it is seen that the quantity $\mathcal{L}(\theta_{24},x_4)$ defined above is the same as that defined in Section 2.1. It is just the (average) energy per angle per second per length reflected from the position x_4 in the direction θ_{24}. Furthermore, we see that $\mathcal{L}_R(\theta,x)$, the corresponding quantity for energy which has been multiply-reflected, is given by

$$\mathcal{L}_R(\theta_{R2},x_2) = \frac{\pi}{2}\int_\zeta \tilde{A}(\theta_{R2},\theta_{24}')J(2,4)\mathcal{L}(\theta_{24},x_4)dx_4 \ . \qquad (4.5.19)$$

Now adding the singly-reflected radiation given by Eq. (4.5.9) and the multiply-reflected radiation given in Eq. (4.5.19) we find:

$$\mathcal{L}(\theta_{R2},x_2) = \mathcal{S}(\theta_{R2},x_2) + \frac{\pi}{2}\int_\zeta \tilde{A}(\theta_{R2},\theta_{24}')J(2,4)\mathcal{L}(\theta_{24},x_4)dx_4 \ . \quad (4.5.20)$$

It is convenient to make a change of the variable of integration through the relation,

$$L(\theta_{24}) = x_2 - x_4 \tag{4.5.21}$$

or

$$dx_4 = -\frac{\partial L(\theta_{24})}{\partial \theta_{24}} \, d\theta_{24} \tag{4.5.22}$$

Then using the definition of $J(2,4)$ given in Eq. (4.2.13) and the ray approximation to G given in Eq. (4.3.20), it is seen that Eq. (4.5.20) may be written,

$$\mathcal{Q}(\theta,x) = \mathcal{S}(\theta,x) + \int_{\Theta_T} \tilde{A}(\theta,\pi-\theta')\mathcal{Q}(\theta',x-L(\theta'))d\theta' \quad , \tag{4.5.23}$$

and this is identical with the result given in Eq. (2.1.18). The function $\tilde{A}$ is the normalized scattering function, defined by

$$\tilde{A}(\theta,\theta') = \sin\theta' \, A(\theta,\theta') \; , \tag{4.5.24}$$

and the symbol Θ_T in Eq. (4.5.23) represents, as before, the set of all trapped angles.

Furthermore from Eqs. (4.5.9), (4.5.10), (4.5.18), and (4.5.19) we see that the total reflected energy per second received by the cylinder of radius a is given by,

$$F_R = 2a\,\frac{\pi}{2} \int_{\zeta} J(R,2)\mathcal{Q}(\theta_{R2},x_2)dx_2 \; , \tag{4.5.25}$$

in agreement with Eq. (2.1.21).

Finally it can be shown that the total reflected radiation (cf. Eq. (4.5.25)) satisfies the reciprocity relation given for singly-reflected radiation in Eq. (4.5.12), if the symmetry condition given in Eq. (4.5.11) is fulfilled. This is probably most easily seen by considering the iteration solution of Eq. (4.5.20) (or of Eq. (4.5.23)). This iteration solution is discussed in Section 5.1. From the solution in this form it can be seen that the above statement concerning the reciprocity relation is true. We conclude then that if the restriction given in Eq. (4.5.11) is fulfilled for a given surface, the total reflected energy obeys the relation:

$$\frac{F_R(R,0)}{k(z_R)} = \frac{F_R(0,R)}{k(z_0)} \quad . \tag{4.5.26}$$

Before proceeding with the solution of Eq. (4.5.23) it is in order to discuss the effect of raising the frequency on the approximations involved in the derivation of this section. It is easily shown that upon raising the frequency all approximations which are affected are improved, with the exception of those which were established in order to guarantee that the phase of the radiation from some source point is essentially that of a plane wave over some finite region of the surface (the relations (4.3.62) and (4.3.63)). However, if the frequency is high enough to permit the use of geometrical optics in describing the reflection process, these approximations give no difficulty. This is so since for such frequencies, as is well known [37], the regions active in the reflection process are narrowed to include only the "highlights" of the surface. (A "highlight" is defined as a region of the surface for which the slope is such that the angle of incidence is approximately equal to the angle of reflection for a given source and receiver point.)

It is first important to establish a criterion for the applicability of geometrical optics to the reflection. In order to do this one can begin with the Kirchhoff approximation, simultaneously restricting the problem to be such that the conditions i) and ii) of Section 3.4 are satisfied. Beginning then with Eq. (3.4.16), using the asymptotic approximation to the

Hankel function (cf. Eq. (3.5.8)), and evaluating the resultant integral through the use of the method of stationary phase (cf. Section 3.3), it is not difficult to verify that a sufficient condition for the applicability of geometrical optics is,

$$kR_m \sin \Theta' \gg 1 \qquad (4.5.27)$$

where R_m is the minimum radius of curvature of the surface and Θ' is the angle made by the incident radiation with the horizontal. One at the same time finds that the horizontal dimension of the highlight is given by,

$$\Delta x_h \sim \left(\frac{R_m}{k \sin \Theta'} \right)^{1/2} \qquad (4.5.28)$$

and the vertical dimension is

$$\Delta z_h \sim \frac{1}{2k \sin \Theta'} \quad . \qquad (4.5.29)$$

All other regions of the surface offer no contribution to the reflected radiation by virtue of the rapidly changing phase of the integrand of Eq. (3.4.16). Hence, one can replace the quantity $\bar{\ell}$ in the restrictions (4.3.62) and (4.3.63) using the result, (4.5.28), obtaining,

$$\alpha R_m \ll 1 \, , \qquad (4.5.30)$$

and

$$\frac{R_m \Theta_{10}}{|x_1 - x_0|} \ll 1 \quad . \qquad (4.5.31)$$

These restrictions can be used, respectively, in place of the restrictions (4.3.62) and (4.3.63), whenever the restriction (4.5.27) is fulfilled for most angles Θ' involved in the reflection process.

We summarize what has been accomplished in the last two sections. The purpose of these sections was to derive the equations which had been obtained heuristically in Chapter II, and to simultaneously present the necessary restrictions for the applicability of the model. It is found that there are three central conditions necessary. First, the frequency must be high enough so that geometrical optics can be used to treat the propagation of radiation in the volume of the duct. Second, the surface must be rough enough or the frequency high enough so that there is a negligible specular component in that portion of the radiation which is reflected from the surface and trapped within the duct. Finally the conditions i) and ii) of Section 3.4 must be fulfilled so that one can use the Kirchhoff approximation in treating the reflection process.

CHAPTER V

THE GENERAL SOLUTION OF THE MODEL EQUATION AND A NUMERICAL EXAMPLE

In the last chapter the results of Section 2.1 were derived from the Helmholtz equation under certain conditions which were discussed in Sections 4.4 and 4.5. In this chapter the solution of Eq. (2.1.18) will be presented in Sections 5.1 and 5.2. In Section 5.3 a numerical example will be considered.

We begin by noting that once the solution of Eq. (2.1.18) is obtained, the problem of finding the energy received by a "black" cylinder is reduced to quadrature (see Eqs. (2.1.21) and (2.1.23)). As a result we will concentrate our attention here on the solution of Eq. (2.1.18); the equation is repeated for convenience,

$$\mathcal{J}(\theta,x) = \mathcal{S}(\theta,x) + \int_{\Theta_T} \mathcal{J}(\theta',x-L(\theta'))A(\theta,\pi-\theta')d\theta' \quad . \qquad (2.1.18)$$

This is a linear integro-difference equation in two variables; an integral equation in θ and a difference equation in x. Schürer [38] has considered certain types of integro-difference equations, in particular homogeneous equations with a single independent variable. Because of these restrictions it will prove convenient here to treat the problem ab initio.

Two representations for the solution of Eq. (2.1.18) will be considered. The first of these is obtained by iteration. While producing this solution it will be possible to show sufficient conditions for the uniqueness of the solution and to show some continuity properties of the solution. The second form is obtained by taking the Laplace transform of Eq. (2.1.18).

5.1. The Solution by Successive Substitutions

The procedure to be used follows closely that employed in the solution of a linear integral equation of the second kind by successive substitutions.* (This is frequently called "the solution by iteration".)

It is convenient to consider the multiply-reflected radiation, which has been defined as

$$\mathcal{I}_R(\theta,x) = \mathcal{I}(\theta,x) - \mathcal{S}(\theta,x) \ . \qquad (5.1.1)$$

Now by replacing $\mathcal{I}$ by $\mathcal{I} + \mathcal{S} - \mathcal{S}$ in the integrand of Eq. (2.1.18) we obtain

$$\mathcal{I}_R(\theta,x) = \mathcal{T}(\theta,x) + \int_{\Theta_T} \mathcal{I}_R(\theta',x-L(\theta'))A(\theta,\pi-\theta')d\theta' \ , \qquad (5.1.2)$$

where

$$\mathcal{T}(\theta,x) = \int_{\Theta_T} \mathcal{S}(\theta',x-L(\theta'))A(\theta,\pi-\theta')d\theta' \ . \qquad (5.1.3)$$

Adopting the velocity profile of Section 4.3, where it was assumed that $c(z)$ is a monotonic increasing function to $z = D$ and thereafter a monotonic decreasing function, one can write,

$$\Theta_T = (0,\mu) + (\pi-\mu,\pi) \qquad (5.1.4)$$

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* See Lovitt, [24], pp. 9 - 13.

with the symbol (a,b), as before, defined as the set of all angles from a to b ; μ is again the largest angle made by a ray trapped within the duct.

It is convenient to recall certain properties of the functions A and $\mathcal{S}$. From the definitions of these functions it is seen that both are real and positive. It follows then from Eq. (5.1.3) that the function $\mathcal{T}$ is real and positive. The conservation property of A is also needed (cf. Eq. (2.2.12))

$$\int_0^{\pi} A(\theta,\theta')d\theta = 1 \quad . \tag{5.1.5}$$

It is now assumed that for all incident angles some of the radiation reflected from the surface is scattered out of the duct. Then,

$$\int_{\Theta_r} A(\theta,\theta')d\theta = \beta(\theta') \leq \beta^M < 1, \qquad \text{for all } \theta' \ . \tag{5.1.6}$$

Continuity conditions on the functions A and $\mathcal{T}$ are now established. It is assumed that--

a) $A(\theta,\theta')$ is positive, continuous (and therefore bounded) for θ and θ' in $(0,\pi)$. Let the bound on A be A^M.

b) $\mathcal{T}(\theta,x)$ is a positive, continuous, and bounded function of θ and of x for θ in $(0,\pi)$ and for all x. Let its bound be $\mathcal{T}^M$.

Under these conditions it will be shown that there exists a solution of Eq. (5.1.2) (the solution by iteration) and that this solution is continuous and bounded. Furthermore it will be shown that this is the only continuous, bounded solution.

We begin by iterating Eq. (5.1.2) n times. This process is carried out, as in Section 3.4, by substituting the right side of Eq. (5.1.2) for the quantity $\mathcal{J}_R$ appearing under the integral sign of that equation. The quantity $\mathcal{J}_R$ now appears under a double integral on the right side of the new equation. The process of substitution is then repeated; after n such substitutions, one obtains the following equation:

$$\begin{aligned}\mathcal{J}_R(\theta,x) = \mathcal{T}(\theta,x) &+ \int_{\Theta_T} A(\theta,\pi-\theta_1)\,\mathcal{T}(\theta_1,x-L(\theta_1))d\theta_1 \\ &+ \int_{\Theta_T} A(\theta,\pi-\theta_1)\int_{\Theta_T} A(\theta_1,\pi-\theta_2)\,\mathcal{T}(\theta_2,x-L(\theta_1)-L(\theta_2))d\theta_2 d\theta_1 \\ &+ \ldots \\ &+ \int_{\Theta_T} A(\theta,\pi-\theta_1)\int_{\Theta_T} A(\theta_1,\pi-\theta_2)\ldots \\ &\qquad \cdot \int_{\Theta_T} A(\theta_{n-1},\pi-\theta_n)\,\mathcal{T}(\theta_n,x-L(\theta_1)-\ldots-L(\theta_n))d\theta_n\ldots d\theta_1 \\ &+ R_{n+1} \quad ,\end{aligned} \tag{5.1.7}$$

where

$$\begin{aligned}R_{n+1} = \int_{\Theta_T} A(\theta,\pi-\theta_1)\int_{\Theta_T} A(\theta_1,\pi-\theta_2)\ldots \\ \cdot \int_{\Theta_T} A(\theta_n,\pi-\theta_{n+1})\mathcal{J}_R(\theta_{n+1},x-L(\theta_1)-\ldots-L(\theta_{n+1}))d\theta_{n+1}\ldots d\theta_1\end{aligned} \tag{5.1.8}$$

This leads one to consider the infinite series

$$\begin{aligned}K(\theta,x) = \mathcal{T}(\theta,x) &+ \int_{\Theta_T} A(\theta,\pi-\theta_1)\mathcal{T}(\theta_1,x-L(\theta_1))d\theta_1 \\ &+ \int_{\Theta_T} A(\theta,\pi-\theta_1)\int_{\Theta_T} A(\theta_1,\pi-\theta_2)\mathcal{T}(\theta_2,x-L(\theta_1)-L(\theta_2))d\theta_2 d\theta_1 \\ &+ \ldots\end{aligned} \tag{5.1.9}$$

Using Eq. (5.1.6) and the hypotheses a) and b) it is seen that

$$|K(\theta,x)| \leq \mathcal{T}^M + A^M[\Theta_T]\mathcal{T}^M + A^M[\Theta_T]\mathcal{T}^M\beta^M$$

$$+ \ldots + A^M[\Theta_T]\mathcal{T}^M(\beta^M)^m + \ldots \quad , \tag{5.1.10}$$

or

$$|K(\theta,x)| \leq \mathcal{T}^M + A^M[\Theta_T]\mathcal{T}^M \frac{1}{1-(\beta^M)} , \tag{5.1.11}$$

since by hypothesis $\beta^M < 1$. The symbol $[\Theta_T]$ represents $\int_{\Theta_T} d\theta$; for the velocity profile proposed in Section 4.3 we find from Eq. (5.1.4):

$$[\Theta_T] = 2\mu . \tag{5.1.12}$$

Thus the series given in Eq. (5.1.9) converges absolutely and uniformly (in both θ and x). From hypotheses a) and b) it follows that every term in the series on the right side of Eq. (5.1.9) is continuous in θ and x.* Then since the series converges uniformly, it follows that the function $K(\theta,x)$ is continuous in θ and x.** The function is also seen to be bounded from Eq. (5.1.11) and seen to be positive from hypotheses a) and b). It can be verified by direct substitution that the function $K(\theta,x)$ is a solution of Eq. (5.1.2). Since every term in the series is continuous and since the series converges uniformly, the series may be integrated term by term.*** Then by substituting Eq. (5.1.9) for the

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* Carslaw, [39], p. 188.

** Sokolnikoff, [40], pp. 256 - 258.

*** Sokolnikoff, [40], pp. 258 - 261.

function $\mathcal{L}_R$ in Eq. (5.1.2) one obtains an identity. Hence,

$$\mathcal{L}_R(\theta,x) = K(\theta,x) \tag{5.1.13}$$

is a solution. Thus the existence of a continuous, bounded solution has been shown, and in fact is the solution constructed. We shall henceforward treat only this solution.

We now show that this is the only continuous, bounded solution. Let it be assumed first that there exist two such solutions designated by $\mathcal{L}_{R1}$ and $\mathcal{L}_{R2}$. They must both satisfy Eq. (5.1.7) for any finite value of n. Then it is supposed that we subtract Eq. (5.1.7) with $\mathcal{L}_{R2}$ substituted for $\mathcal{L}_R$ from Eq. (5.1.7) with $\mathcal{L}_{R1}$ substituted for $\mathcal{L}_R$. There is left the difference of the remainder terms,

$$\mathcal{L}_{R1}(\theta,x) - \mathcal{L}_{R2}(\theta,x) = \int_{\Theta_T} A(\theta,\pi-\theta_1) \int_{\Theta_T} A(\theta_1,\pi-\theta_2)\ldots$$

$$\cdot \int_{\Theta_T} A(\theta_n,\pi-\theta_{n+1})(\mathcal{L}_{R1}-\mathcal{L}_{R2})d\theta_{n+1}\ldots d\theta_1 \, . \tag{5.1.14}$$

Taking the absolute value of Eq. (5.1.14) and proceeding as in the treatment of Eq. (5.1.9) we find

$$|\mathcal{L}_{R1} - \mathcal{L}_{R2}| \leq A^M[\Theta_T] \, (\beta^M)^n(\mathcal{L}^M_{R1} + \mathcal{L}^M_{R2}) \, , \tag{5.1.15}$$

where $\mathcal{L}^M_{R1}$ and $\mathcal{L}^M_{R2}$ are the bounds on $\mathcal{L}_{R1}$ and $\mathcal{L}_{R2}$. This relation must hold for all n. Thus using Eq. (5.1.6), we deduce that

$$\lim_{n\to\infty} |\mathcal{L}_{R1} - \mathcal{L}_{R2}| = 0 \, , \tag{5.1.16}$$

so that $\mathcal{L}_{R1}$ and $\mathcal{L}_{R2}$ must be the same solution. Thus the solution is unique, which is what we set out to prove.

It can be noted from Eqs. (5.1.9) and (5.1.1) that since $\mathcal{T}$ and A are real positive functions by hypothesis, so also is the function $\mathcal{L}_R$ (and hence $\mathcal{L}$) as is required by its definition.

5.2. The Solution Obtained by the Use of the Laplace Transform

In this section we shall find the solution of Eq. (2.1.18),

$$\mathcal{L}(\theta,x) = \mathcal{S}(\theta,x) + \int_{\Theta_T} \mathcal{L}(\theta',x-L(\theta'))A(\theta,\pi-\theta')d\theta', \qquad (2.1.18)$$

by taking its Laplace transform with respect to x. In order to do this it is necessary that the quantities to be transformed shall vanish for $x < 0$. If the surface scatters trapped radiation primarily in the forward direction, it is possible to separate the problem so that the terms of Eq. (2.1.18) vanish when $x < 0$. Because of the small angles made by rays trapped in ducts occurring physically, this will be a reasonable assumption.*

The assumption can be stated as follows: it is supposed that--

a) $A(\theta,\pi-\theta') = 0$, $\qquad \frac{\pi}{2} \leq \theta \leq \pi$.

b) $\mathcal{S}(\theta,x) = 0$, $\qquad \frac{\pi}{2} \leq \theta \leq \pi \qquad$ or $\quad x < 0$.

The restriction a) is only needed for θ' in Θ_T . If in the physical problem the source radiates energy in both the -x and the +x directions, the region $x < 0$ is treated in a way analogous to that to be used here for $x > 0$. The assumption that there is no back scattering guarantees,

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* It should be remarked that if one took the Fourier rather than the Laplace transform of Eq. (2.1.18), this assumption would not be necessary. However in such a case the analytical difficulties increase considerably and inasmuch as the assumption of only forward scattering is ordinarily a weak restriction physically, the Laplace transform will be used.

as we shall see, that these two parts of the problem do not interact.

We now consider

$$\mathcal{T}(\Theta,x) = \int_{\Theta_T} \mathcal{S}(\Theta',x-L(\Theta'))A(\Theta,\pi-\Theta')d\Theta' . \qquad (5.1.3)$$

From the hypotheses a) and b) it follows that

$$\mathcal{T}(\Theta,x) = 0, \qquad \frac{\pi}{2} \leq \Theta \leq \pi \qquad \text{or } x < 0, \qquad (5.2.1)$$

since $L(\Theta) \geq 0$ when $0 \leq \Theta \leq \frac{\pi}{2}$. By repeating the argument it is seen that every term of the series given in Eq. (5.1.9) vanishes for $\frac{\pi}{2} \leq \Theta \leq \pi$ or $x < 0$. If, as is supposed, the hypotheses a) and b) and Eq. (5.1.6) of Section 5.1 are fulfilled, the series in Eq. (5.1.9) represents the function $\mathcal{L}_R(\Theta,x)$. Hence

$$\mathcal{L}_R(\Theta,x) = 0 \quad , \qquad x < 0 \quad \text{or } \frac{\pi}{2} \leq \Theta \leq \pi .$$

From hypothesis b) above and Eq. (5.1.1) it follows that

$$\mathcal{L}(\Theta,x) = 0 , \qquad x < 0 \quad \text{or } \frac{\pi}{2} \leq \Theta \leq \pi . \qquad (5.2.2)$$

Thus every term in Eq. (2.1.18) vanishes for $x < 0$ or $\frac{\pi}{2} \leq \Theta \leq \pi$.

We now adopt the velocity profile proposed in Section 4.3; that is, it is supposed that the velocity of propagation is a continuous function of z and increases monotonically for $0 \leq z \leq D$, decreases monotonically for $z > D$, and approaches some positive value asymptotically as $z \rightarrow +\infty$. In this case, referring to Eqs. (5.1.4) and (5.2.2), it is seen that Eq. (2.1.18) can be written

$$\mathcal{L}(\Theta,x) = \mathcal{S}(\Theta ,x) + \int_0^{\mu} \mathcal{L}(\Theta',x-L(\Theta'))A(\Theta,\pi-\Theta')d\Theta' . \qquad (5.2.3)$$

We now impose a further restriction on the source function. It is supposed that --

c) $\mathcal{S}(\theta,x)$ is a positive, continuous function of θ for $0 \leq \theta \leq \mu$ and is a piece-wise continuous function of x in every finite interval of x.

It should be remarked that in the hypotheses a) and b) of Section 5.1 we are now only concerned with angles in $(0,\mu)$.

We now impose restrictions on the function $L(\theta)$ which together with hypothesis c) of this section and hypothesis a) of Section 5.1 will be seen to be equivalent to hypothesis b) of Section 5.1. It is assumed that --

d) $L(\theta)$ is a continuous, monotonic increasing function of θ for $0 \leq \theta \leq \mu$.

From a) of Section 5.1 and c) and d) of this section it can be seen that $\mathcal{T}(\theta,x)$ defined in Eq. (5.1.3) is a continuous function of θ.* In considering $\mathcal{T}(\theta,x)$ as a function of x, it is convenient to consider each continuous section of the function $\mathcal{S}(\theta,x)$ separately. Then from Eq. (5.1.3), using a simple extension of an argument of Carslaw's,** it can be seen that $\mathcal{T}(\theta,x)$ is a continuous function of x. Hence c) and d) of this section together with a) of Section 5.1 are seen to be equivalent to b) of Section 5.1.

In Section 5.1 it was seen that $\mathcal{I}_R(\theta,x)$ is a continuous, bounded function of x; under hypothesis c) $\mathcal{S}(\theta,x)$ is a piece-wise continuous and bounded function of x. Hence from Eq. (5.1.1) it is seen that $\mathcal{I}(\theta,x)$ is a piece-wise continuous and bounded function of x. It follows then

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* Carslaw, [39], p. 189.

** Carslaw, [39], p. 188.

that one can take the Laplace transform of each term in Eq. (5.2.3).* The following notation is adopted for the Laplace transform:

$$I(\Theta,y) = \int_0^\infty \mathcal{I}(\Theta,x)\, e^{-xy} dx \;, \tag{5.2.4}$$

$$S(\Theta,y) = \int_0^\infty \mathcal{S}(\Theta,x)\, e^{-xy} dx \;. \tag{5.2.5}$$

From the bounded property of $\mathcal{I}$ and $\mathcal{S}$, it follows that I and S exist and are analytic for $\mathrm{Re}(y) > 0$.**

Upon taking the Laplace transform of Eq. (5.2.3) we find,

$$I(\Theta,y) = S(\Theta,y) + \int_0^\infty e^{-xy} \left\{ \int_0^\mu \mathcal{I}(\Theta',x-L(\Theta'))A(\Theta,\pi-\Theta')d\Theta' \right\} dx. \tag{5.2.6}$$

Since the integral over Θ' is uniformly convergent in x and since $\mathcal{I}$ is a bounded function of x, it follows that we can interchange the order of integration in the double integral appearing in Eq. (5.2.6). Then we use the property of the Laplace transform,***

$$\int_0^\infty \mathcal{I}(\Theta',x-L(\Theta'))e^{-xy}dx = I(\Theta',y)e^{-L(\Theta')y} \;, \tag{5.2.7}$$

to find for Eq. (5.2.6),

$$I(\Theta,y) = S(\Theta,y) + \int_0^\mu I(\Theta',y)A(\Theta,\pi-\Theta')e^{-L(\Theta')y}d\Theta' \quad . \tag{5.2.8}$$

This is a linear integral equation of the second kind to be solved for I with y as a parameter. That it has a solution follows from the fact that it was derived from Eq. (5.2.3).

Now we could proceed with the solution of Eq. (5.2.8) through the use of one of the standard methods of solution for such equations. For example

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* Churchill, [41], p. 5.

** Churchill, [41], p. 151.

*** Churchill, [41], p. 21.

Eq. (5.2.8) could be solved by iteration (in which case we would obtain the Laplace transform of the solution by iteration of Eq. (5.2.3)). However we desire another, and in some cases more useful, representation for the solution. To obtain this we shall suppose that the function $A(\theta,\pi-\theta')$ is representable by a (finite) sum of products of functions of θ and θ'. In such a case the integral equation is said to be degenerate, and admits a particularly simple solution.* To this end we suppose that --

e) $$A(\theta,\pi-\theta') = \sum_{n=1}^{N} a_n(\theta)a_n'(\theta') \qquad (5.2.9)$$

where a_n and a_n' are continuous functions of θ and θ' for all n and for θ and θ' in $(0,\mu)$. The $a_n(\theta) = 0, \pi-\mu \leq \theta \leq \pi$, in accord with a) of this section.

Further, the functions a_n and a_n' are such that A is positive. By choosing a sufficient number of terms in the series of Eq. (5.2.9) we can expect to be able to approximate as closely as we wish most of the scattering functions of the type met in physical applications.

Substituting Eq. (5.2.9) in Eq. (5.2.3) we obtain,

$$\mathcal{L}(\theta,x) = \mathcal{S}(\theta,x) + \int_0^{\mu} \mathcal{L}(\theta',x-L(\theta')) \sum_{n=1}^{N} a_n(\theta)a_n'(\theta')d\theta' . \qquad (5.2.10)$$

Since the functions under the integral sign in Eq. (5.2.10) are bounded, we can interchange the summation and the integration to obtain,

$$\mathcal{L}(\theta,x) = \mathcal{S}(\theta,x) - \sum_{n=1}^{N} a_n(\theta)U_n(x) \qquad (5.2.11)$$

where

$$-U_n(x) = \int_0^{\mu} \mathcal{L}(\theta',x-L(\theta'))a_n'(\theta')d\theta' . \qquad (5.2.12)$$

Since there is a unique (and bounded) function $\mathcal{L}(\theta,x)$ which satisfies Eq. (5.2.3), it follows from Eq. (5.2.12) that the functions U_n are unique.

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* Lovitt, [24], p. 22.

The function $\mathcal{U}$ has been shown to be bounded and to be a continuous function of Θ and a piece-wise continuous function of x which vanishes when $x < 0$; then by repeating the argument just carried out in connection with $\mathcal{X}(\Theta,x)$ we can show that $U_n(x)$ is a bounded, continuous function of x (which vanishes for $x < 0$). Thus U_n possesses a Laplace transform,

$$u_n(y) = \int_0^\infty e^{-xy} U_n(x)dx \quad , \tag{5.2.13}$$

which exists, and is in fact analytic, for $Re(y) > 0$. Then we take the Laplace transform of Eq. (5.2.11) to obtain

$$I(\Theta,y) = S(\Theta,y) - \sum_{n=1}^{N} a_n(\Theta)u_n(y) \tag{5.2.14}$$

where the symbols have already been defined. From Eqs. (5.2.12) and (5.2.14) we find the following system of equations for the determination of the function $u_n(y)$:

$$\sum_{m=1}^{N} u_m(y)M_{mn}(y) - u_n(y) = s_n(y) \quad , \tag{5.2.15}$$

where

$$s_n(y) = \int_0^\mu e^{-L(\Theta')y} a_n'(\Theta')S(\Theta',y)d\Theta' \quad , \tag{5.2.16}$$

and

$$M_{mn}(y) = \int_0^\mu e^{-L(\Theta')y} a_m(\Theta')a_n'(\Theta')d\Theta' \quad . \tag{5.2.17}$$

It is of interest to investigate the properties of the functions s_n and M_{mn}. Before doing this it is supposed that --

f) $\mathcal{S}(\Theta,x)$ vanishes outside some finite region of the x-axis. This condition is imposed in addition to the hypotheses b) and c) already fulfilled by the function $\mathcal{S}(\Theta,x)$. From hypotheses b), c), and f) it follows that $S(\Theta,y)$ (cf. Eq. (5.2.5)) is an analytic function of y for every finite y.* From hypotheses c) and f) it is easy to show that $S(\Theta,y)$ is a continuous function of Θ for $0 \leq \Theta \leq \mu$. From hypothesis d), using an argument essentially the same as the one needed to show that $S(\Theta,y)$ is an analytic function for all y , one can show that $s_n(y)$ and $M_{mn}(y)$ are analytic functions of y for every finite y.

We now return to Eq. (5.2.15). It has been seen that there exists one and only one set of functions $U_n(x)$, and hence $u_n(y)$, which leads to a bounded, continuous function $\vartheta_R(\Theta,x)$. On the basis of this it may be supposed that $\|M_{mn} - \delta_{mn}\|$ does not vanish identically; $\|N_{mn}\|$ indicates the determinant of the matrix N_{mn} , and

$$\begin{aligned} \delta_{mn} &= 1 , & m &= n , \\ &= 0 & m &\neq n . \end{aligned} \tag{5.2.18}$$

Then the system of equations represented by Eq. (5.2.15) can be solved to obtain,

$$u_n(y) = \frac{\sum_{\ell=1}^{N} \overline{\|M_{n\ell}(y) - \delta_{n\ell}\|} \, s_\ell(y)}{\|M_{\alpha\beta}(y) - \delta_{\alpha\beta}\|} , \tag{5.2.19}$$

where $\overline{\|N_{mn}\|}$ indicates the minor of the element (m,n) in the determinant $\|N_{mn}\|$.

It is possible that the representation given in Eq. (5.2.19) will be recognizable as the transform of some known function in a particular problem. It is more probably though that the function will not appear in such a simple form. In such a case the inversion integral for the inverse

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* Churchill, [41], pp. 148-151. The function $\vartheta(\Theta,x)$ is of order $e^{x_0 x}$, in Churchill's notation, where x_0 is any finite real number. It follows that $S(\Theta,y)$ is analytic for every finite y .

Laplace transform will be useful. It has been shown that the functions $U_n(x)$ are continuous, bounded functions of x which vanish for $x < 0$. Under such conditions [42] the following relation holds,

$$U_n(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{xy} \frac{\sum_{\ell=1}^{N} s_\ell(y) \, \overline{\| M_{n\ell}(y) - \delta_{n\ell} \|}}{\| M_{\alpha\beta}(y) - \delta_{\alpha\beta} \|} \, dy, \qquad (5.2.20)$$

where the integral is to be taken along the line $\mathrm{Re}(y) = \gamma$ with $\gamma > 0$.

In some special cases one might attempt to carry out the integrals of Eq. (5.2.20). However, frequently it is advantageous to apply the Cauchy residue theorem.*

Before doing this we consider some properties of the singularities of the integrand of Eq. (5.2.20). It has been shown that $s_n(y)$ and $M_{mn}(y)$ are analytic functions of y for all finite y. Then the determinant, $\|M_{\alpha\beta}(y) - \delta_{\alpha\beta}\|$, and the minors, $\overline{\|M_{n\ell}(y) - \delta_{n\ell}\|}$, are analytic functions of y for all finite y, since they are composed of finite sums of products of analytic functions. It is seen then that the integrand in Eq. (5.2.20) is an analytic function of y for all finite y except at those points, if any, where

$$\|M_{\alpha\beta}(y) - \delta_{\alpha\beta}\| = 0 . \qquad (5.2.21)$$

Since we know that $u_n(y)$ are analytic functions of y for $\mathrm{Re}(y) > 0$, it follows that all values of y which satisfy Eq. (5.2.21) must also satisfy $\mathrm{Re}(y) \leq 0$.

It is seen then that the singularities of the integrand in Eq. (5.2.20) can only be poles (not necessarily simple). As a further property of the

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* Churchill, [41], p. 142.

singularities (poles) of the integrand of Eq. (5.2.20) we note that for a given set of functions a_n and a_n' , the positions of the singularities are the same for all $u_n(y)$ by Eq. (5.2.19). Hence the positions of the poles need be calculated but once for all of the functions U_n . It is evident however that in general the residues at a given pole position depend upon the particular function U_n being considered. Finally from Eqs. (5.2.5), (5.2.16), and (5.2.17) we see that $s_n^*(y) = s_n(y^*)$ and $M_{mn}^*(y) = M_{mn}(y^*)$. Using this, it is seen upon examining Eq. (5.2.21) that the poles must be located symmetrically about the real axis. Further, from Eq. (5.2.20) one finds that the residues of a pair of poles which are symmetrically placed (about the x-axis) are complex conjugates of one another. This relationship for the positions and residues of the poles is a consequence of the fact that the $U_n(x)$ are real.

The Cauchy residue theorem is now applied. It is supposed that the poles of the integrand in Eq. (5.2.20) are infinite in number and are located at y_0, y_1, y_1^*, y_2, y_2^*, ... where $|y_0| < |y_1| < |y_2|$ ... (see Fig. 10) and $|y_m| \rightarrow \infty$ as $m \rightarrow \infty$. It is a simple matter to alter the argument to cover other cases. Let the residues of the integrand of Eq. (5.2.20) be $\rho_0^{(n)}(x)$, $\rho_1^{(n)}(x)$, $\rho_1^{(n)*}(x)$, $\rho_2^{(n)}(x)$, $\rho_2^{(n)*}(x)$,... respectively; the subscript indicates the pole to which the residue belongs while the superscript indicates to which function, U_n, the residue belongs. The contour C_m is chosen so that together with the line $y = \gamma$ it encloses the poles at y_0, y_1, y_1^*, ..., y_m, y_m^*. The contour begins at the point $\gamma + ic_m$ and ends at the point $\gamma - ic_m$. It is further supposed that $c_m \rightarrow \infty$ as $m \rightarrow \infty$.

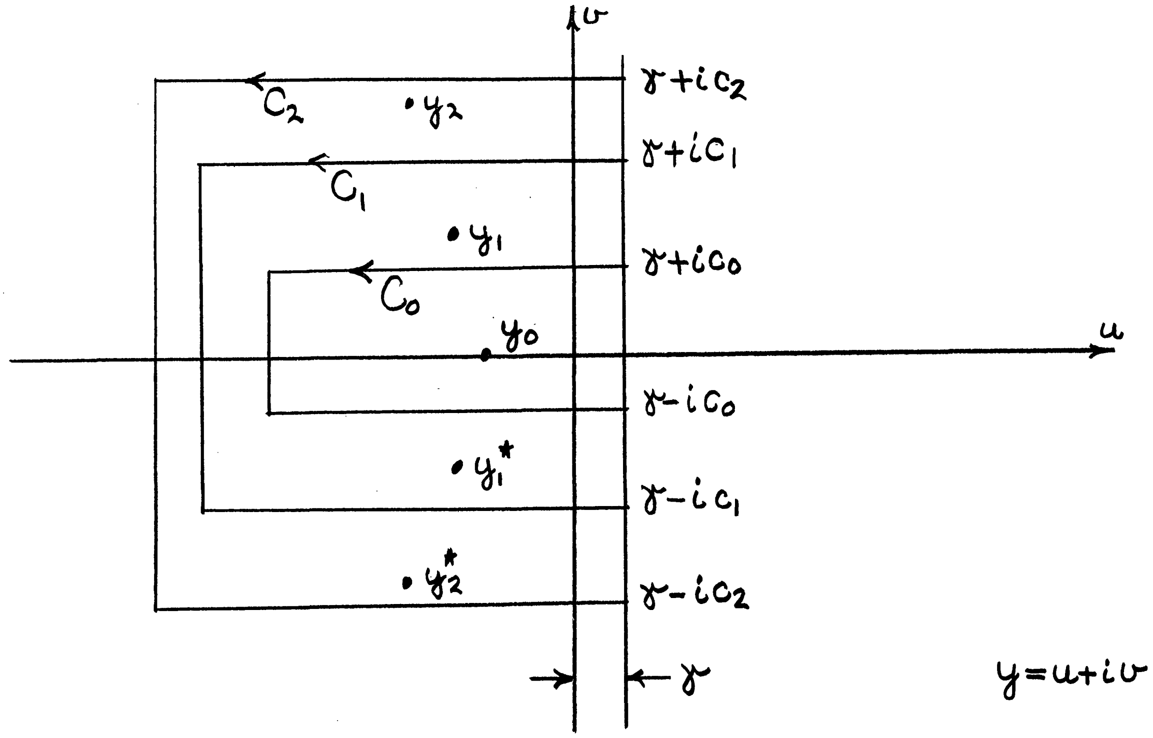

Fig. 10. Diagram in the complex y plane showing the contours chosen for the evaluation of the inversion integral of the Laplace transform.

Employing these definitions it is found, upon using the Cauchy residue theorem, that Eq. (5.2.20) can be written

$$U_n(x) = \rho_0^{(n)}(x) + \sum_{m=1}^{\infty} \left[\rho_m^{(n)}(x) + \rho_m^{(n)*}(x)\right] + \lim_{m\to\infty} I_m^{(n)}(x) , \tag{5.2.22}$$

if the limit of $I_m^{(n)}$ exists, where,

$$I_m^{(n)}(x) = -\frac{1}{2\pi i} \int_{C_m} e^{xy} \frac{\sum_{\ell=1}^{N} s_\ell(y)\ \overline{\|M_{n\ell}(y) - \delta_{n\ell}\|}}{\|M_{\alpha\beta}(y) - \delta_{\alpha\beta}\|}\, dy, \tag{5.2.23}$$

and where the integral is to be taken along the contour C_m from $\gamma + ic_m$ to $\gamma - ic_m$. The first three contours are shown in Fig. 10. It sometimes happens that

$$\lim_{m\to\infty} I_m^{(n)}(x) = 0 \quad , \tag{5.2.24}$$

in which case

$$U_n(x) = \rho_0^{(n)}(x) + \sum_{m=1}^{\infty} \left[\rho_m^{(n)}(x) + \rho_m^{(n)*}(x)\right] \quad . \tag{5.2.25}$$

Finally, if all of the poles y_m are simple, one can write,*

$$\rho_m^{(n)}(x) = \frac{e^{xy_m} \sum_{l=1}^{N} s_l(y_m) \overline{\left\| M_{nl}(y_m) - \delta_{nl} \right\|}}{\left. \frac{d}{dy} \left\| M_{\alpha\beta}(y) - \delta_{\alpha\beta} \right\| \right|_{y=y_m}} \quad . \tag{5.2.26}$$

To sum up, through the use of the Laplace transform we have obtained a solution of Eq. (2.1.18) for surfaces which possess no back scattering. To put the solution in a manageable form, it was assumed that the normalized scattering function could be expressed in a special, although not particularly restrictive, way (cf. Eq. (5.2.9)). Finally the solution was obtained, under certain restrictions, in the form of the sum of the residues of a set of poles.

The solution expressed by Eqs. (5.2.11) and (5.2.25) is valid if hypotheses b) through f) of this section are fulfilled and if Eqs. (5.1.6) and (5.2.24) are valid.

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* Churchill, [41], p. 141.

5.3. A Numerical Example

A numerical problem will be worked in this section to illustrate the foregoing theory. The problem of the propagation of high frequency (25 kc) acoustic radiation in an isothermal, surface-bounded duct in the ocean is considered. In isothermal water there is an increase in the phase velocity of sound with depth as a result of increasing pressure; this change in velocity can be expressed by,*

$$c = c_0(1 + \alpha z) \qquad (5.3.1)$$

where c_0 = 4878 feet per second (for a temperature of 50° F) and $\alpha = 3.73 \times 10^{-6}$ per foot. It will be supposed that the phase velocity is determined by Eq. (5.3.1) for $z \leq 400$ feet. For $z > 400$ feet it is supposed that the temperature decreases in such a way that

$$c = c_0(1 + 2\alpha D - \alpha z), \qquad (5.3.2)$$

D = 400 feet, so that the velocity gradient below the duct is equal in magnitude and opposite in sign to that within the duct. (Of course it must be assumed that for very great z Eq. (5.3.2) changes form so that the phase velocity remains positive.) From Eq. (4.3.30) it is found that $\mu = 3.12°$ where μ is the maximum angle made by a ray trapped within the duct.

It has already been assumed that the rms deviation of the surface from its average plane is one yard. Furthermore it has been supposed that the surface is quite irregular (a "sea") and that the statistical properties of two regions of surface become independent when they are separated by a distance exceeding 25 yards. For the surface scattering function, the

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* "Physics of Sound in the Sea", [33], p. 60.

following (single parameter) form is chosen,

$$\tilde{A}(\theta,\theta') = \gamma \sin\theta \sin\theta' , \qquad (5.3.3)$$

which for $\gamma = \frac{1}{2}$ is seen to be the two-dimensional equivalent of Lambert's cosine law of reflection. There is very little evidence on the reflecting properties of the sea surface; Eq. (5.3.3) seems a reasonable choice for the type of diffuse reflection being considered in this work. From Eq. (5.3.3) it is seen that,

$$A(\theta,\theta') = \gamma \sin\theta \quad . \qquad (5.3.4)$$

In view of the smallness of μ, small angle approximations will be used throughout these calculations. Then Eq. (5.3.4) becomes,

$$A(\theta,\theta') = \gamma\theta \quad . \qquad (5.3.5)$$

The fraction of incident energy which is reflected into the duct is given by,

$$\beta = \int_0^{\mu} \gamma\theta d\theta \qquad (5.3.6)$$

or

$$\beta = \frac{1}{2}\gamma\mu^2 \quad . \qquad (5.3.7)$$

It is proposed to alter Eq. (5.3.4) in two ways. First it is assumed in the numerical example that $\beta = 0.1$, so that rather than letting $\gamma = \frac{1}{2}$, γ will be determined by Eq. (5.3.7) ($\gamma = \frac{0.2}{\mu^2}$). Furthermore, for physical reasons to be given below as well as to fulfill the conservation relation (cf. Eq. (2.2.12)), it is supposed that,

$$A(\theta,\theta') = 0 , \qquad \theta > \theta^M \qquad (5.3.8)$$

where

$$\frac{1}{2}\gamma(\theta^M)^2 = 1 \quad , \tag{5.3.9}$$

or

$$\frac{\theta^M}{\mu} = (10)^{1/2}. \tag{5.3.10}$$

It is frequently convenient to assume that the free parameters are μ and β (and hence the duct depth and velocity gradient within the duct on the one hand and the fraction of incident energy reflected into the duct on the other). This of course implies that if one uses the reflection law given in Eq. (5.3.3), the quantities γ and θ^M are determined through Eqs. (5.3.7) and (5.3.10) once the choice of μ and β has been made.

It is now shown that $\beta \sim \frac{1}{10}$ is a reasonable estimate. Let it be supposed that the reflection may be estimated by geometrical optics. Then from the results of Section 3.2 it is known that the distribution of surface slopes determines the scattering function. The maximum value of the slope of the water surface is of order 1/7; hence the maximum reflection angle is approximately 2/7, or about 15°. The maximum trapped angle has been given as about 3°; it seems reasonable to suppose from this that the fraction of energy trapped within the duct after reflection is of order 1/10.

To continue, using the given value of α in Eq. (4.3.31) it is seen that $L(\mu)$ = 9760 yards. Further, it is supposed for the present calculation that both the source and the receiver have an angular dependence. For simplicity it is assumed that the energy radiated per second per angle by the source is constant for angles (measured from the horizontal) which are less in absolute value than $\frac{\Delta\theta_0}{2}$. For greater angles it is supposed that no energy is radiated. A similar requirement is fixed on the receiver. For the present problem it is assumed that

$$\frac{\Delta\theta_0}{2\mu} = 3 \quad , \tag{5.3.11}$$

or using the value of μ already given, $\frac{\Delta\Theta_0}{2} = 9.36°$.

Let us first consider the Laplace transform method of solution. One must obtain the function M(y) (see Eq. (5.2.17)),

$$M(y) = \int_0^{\mu} e^{-L(\Theta')y} a(\Theta)\ a'(\Theta')d\Theta' \quad , \qquad (5.3.12)$$

it being noted that the scattering function, A , is represented by a single term. It is convenient to reduce the transform variable by $y' = L(\mu)y$; furthermore let $x' = \frac{x}{L(\mu)}$ and $z' = \frac{z}{D}$. Primes will be used in the remainder of this section to indicate such reduced variables; the unit for vertical distance is D and for horizontal distance is $L(\mu)$. Upon using Eq. (5.3.5) in Eq. (5.3.12), it is found that,

$$M(y') = 2\beta\left[\frac{1}{y'^2} - e^{-y'}\left(\frac{1}{y'^2} + \frac{1}{y'}\right)\right] . \qquad (5.3.13)$$

Now the roots of the equation,

$$M(y') - 1 = 0 \quad , \qquad (5.3.14)$$

constitute the positions of the poles of the transform of the function U(x). By putting

$$y' = u + iv \qquad (5.3.15)$$

one can put Eq. (5.3.14) in the form,

$$e^{-2u} = \frac{\left[1+u+\frac{1}{2\beta}(v^2-\rho^2u-u^2)\right]^2 + v^2\left[1+\frac{1}{2\beta}(\rho^2+2u)\right]^2}{\left[(1+u)^2+v^2\right]^2} , \qquad (5.3.16)$$

and

$$\tan v = \frac{v\left[1+\frac{1}{2\beta}(\rho^2+2u)\right]}{1+u+\frac{1}{2\beta}(v^2-\rho^2u-u^2)} , \qquad (5.3.17)$$

where

$$\rho^2 = u^2 + v^2 \quad . \tag{5.3.18}$$

The roots of Eq. (5.3.14) can now be found through the use of an iteration procedure. One first makes an estimate of a pair of values of u and v which will satisfy Eqs. (5.3.16) and (5.3.17), substitutes these values in the right sides of those equations and calculates a new pair of values from the left sides. These now values are now taken as a new estimate and the process repeated. By this procedure the roots of Eq. (5.3.14) can be found; the first fifteen for $\beta = 0.1$ are given in Table I.

TABLE I

$$y_0' = -3.111$$

$$y_1' , y_1'^* = -3.764 \pm 7.264\ i$$

$$y_2' , y_2'^* = -4.298 \pm 13.768\ i$$

$$y_3' , y_3'^* = -4.649 \pm 20.146\ i$$

$$y_4' , y_4'^* = -4.909 \pm 26.484\ i$$

$$y_5' , y_5'^* = -5.116 \pm 32.802\ i$$

$$y_6' , y_6'^* = -5.288 \pm 39.110\ i$$

$$y_7' , y_7'^* = -5.435 \pm 45.412\ i$$

The first fifteen roots of $M(y')-1 = 0$ for $\beta = 0.1$.

By using the small angle approximations (Eqs. (4.3.30) through (4.3.37)), the definition,

$$\mathcal{S}(\theta, x_1) = J(1,0)\widetilde{A}(\theta, \theta_{10}') \ , \tag{4.5.9}$$

and the assumed form of $\tilde{A}$ (Eq. (5.3.3)), and still supposing small angles, one finds when $z_0' \leq 1$,

$$\mathcal{S}(\theta,x) = \frac{\alpha \gamma \theta}{\pi x'}\left(\frac{z_0'}{4x'} + x'\right), \quad \text{for } x_1' \leq x' \leq x_2' \qquad (5.3.19)$$

$$= 0 \qquad \text{other } x' .$$

In Eq. (5.3.19) x_1 and x_2 are defined as the two limiting surface points for the surface region which is directly radiated by the source (see Fig. 11) and $x_{1,2}' = x_{1,2}/L(\mu)$; z_0 is the depth of the source with $z_0' = z_0/D$.

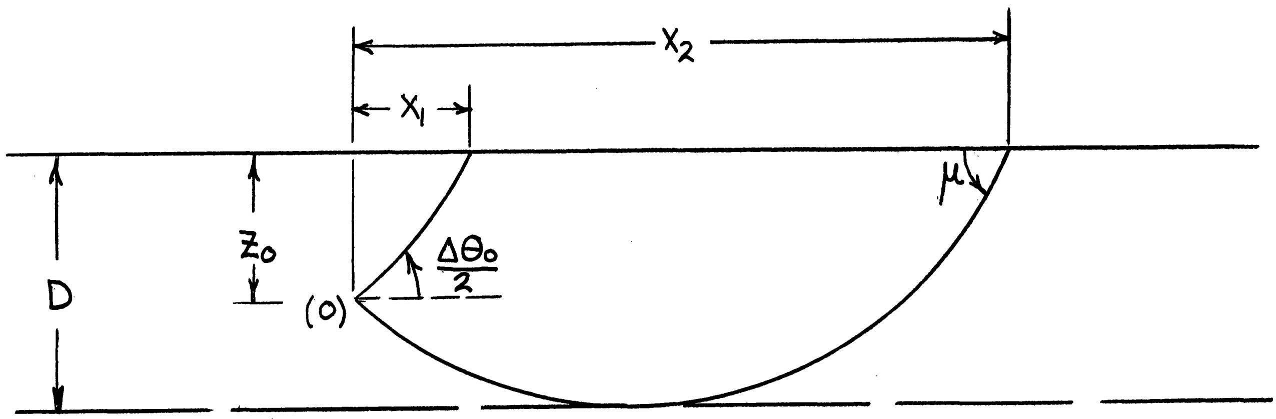

Fig. 11. Diagram showing the region of the surface radiated by energy coming directly from the source.

Now Eq. (5.3.19) can be substituted in Eq. (5.2.5),

$$S(\theta,y) = \int_0^{\infty} \mathcal{S}(\theta,x)e^{-xy}dx , \qquad (5.2.5)$$

to obtain the Laplace transform of the function $\mathcal{S}$; the transform is in turn substituted in,

$$s(y) = \int_0^{\mu} e^{-L(\theta')y}\, a'(\theta')S(\theta',y)d\theta' , \qquad (5.3.20)$$

(see Eq. (5.2.16)) to obtain for the function $s(y')$:

$$s(y') = \mu \frac{2}{\pi} M(y') \left\{ \left(\frac{z_0'}{4x_1'} + \frac{1}{y'}\right) e^{-x_1'y'} - \left(\frac{z_0'}{4x_2'} + \frac{1}{y'}\right) e^{-x_2'y'} - \frac{y'z_0'}{4} \int_{x_1'}^{x_2'} \frac{e^{-\xi y'}}{\xi} \, d\xi \right\} . \tag{5.3.21}$$

When the source lies below the layer the corresponding expression is in general more complicated, although it may be found in a straightforward manner from the results of Section 4.3. For the special case $z_0' = 2$, it is found that $s(y')$ is given by setting $z_0' = 4$ in Eq. (5.3.21) and multiplying the expression on the right side by $1/2$.

It will now be shown that it is possible to choose a set of contours C_m such that the integral in Eq. (5.2.23) vanishes for $m \rightarrow \infty$, when $x' > x_2'$. The contours (cf. Fig. 10) are formed by the lines $y = \pm 2m\pi i$ and $y = -u_L$, where $u_L \rightarrow +\infty$ for each C_m .

From Eq. (5.2.23) it is seen that,

$$I_m(x) = -\frac{1}{2\pi i} \frac{1}{L(\mu)} \int_{C_m} \frac{e^{x'y'} s(y')}{M(y')-1} \, dy' . \tag{5.3.22}$$

Then if $Re(y') \leq 0$ (as is the case on C_m for, in Fig. 10, $\gamma \rightarrow 0$) and if $|y'| \gg 1$ (as will be the case since we consider C_m only when m is large) it is found from Eq. (5.3.13) that

$$M(y') \approx -2\beta \frac{e^{-y'}}{y'} , \tag{5.3.23}$$

and from Eq. (5.3.21)

$$s(y') \approx \frac{2}{\pi} \mu \left(-2\beta \frac{e^{-y'}}{y'}\right) \left\{ \left[\frac{1}{(x_1')^2} \frac{z_0'}{4} + 1\right] \frac{e^{-x_1'y'}}{y'} - \left[\frac{1}{(x_2')^2} \frac{z_0'}{4} + 1\right] \frac{e^{-x_2'y'}}{y'} \right\} . \tag{5.3.24}$$

Now using Eqs. (5.3.23) and (5.3.24) in Eq. (5.3.22), it is seen that the part of the integral along $y = -u_L$ converges and goes to zero as $u_L \to +\infty$ for all m (if $x' > x_2'$). Furthermore the integral taken over the part of the contour, C_m, for which $y = +2m\pi i$ converges and goes to zero as $m \to \infty$, when $x' > x_2'$. (It is easily seen for large m, see Eq. (5.3.23), that when $y = u + 2m\pi i$ the denominator of the integrand of Eq. (5.3.22) does not vanish.) A similar discussion applies when the part of the contour, $y = -2m\pi i$, is considered. Thus $\lim_{m\to\infty} I_m = 0$ as required, and hence Eq. (5.2.25) may be used to represent the function $U(x)$.

It is now shown that the poles lie so that

$$(2n-1)\pi < \mathrm{Im}(y_n) < (2n+1)\pi, \qquad n = 0,+1,+2,\ldots \qquad (5.3.25)$$

Further it is shown that all of the poles are simple. To derive these two properties, one utilizes the following theorem for analytic functions:*

$$N = \frac{1}{2\pi i} \oint \frac{\phi'(z)}{\phi(z)}\, dz \quad , \qquad (5.3.26)$$

where N is the number of zeros of $\phi(z) = 0$ (contained within the closed contour C) less the number of poles so contained, each being counted according to its multiplicity. Then let

$$\phi(y') = M(y') - 1 \quad , \qquad (5.3.27)$$

and since $M(y')$ is analytic in the finite plane, N of Eq. (5.3.26) represents the number of zeros of Eq. (5.3.27) counted according to their multiplicity. For the contour C one chooses the four straight-line segments joining the four points $(-\infty,-(2n+1)\pi i)$, $(-\infty,+(2n+1)\pi i)$,

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* Whittaker and Watson, [14], sec. 6.31.

$(+\infty, +(2n+1)\pi i)$, and $(+\infty, -(2n+1)\pi i)$; $n = 0, 1, 2, \ldots$. One finds that there are $2n+1$ poles within each contour so specified; utilizing the fact that the poles must come in complex conjugate pairs it follows that the poles are simple and that the relation (5.3.25) holds (together with the added fact that one pole lies on the real axis).

Since the poles are simple, Eq. (5.2.26) can be used in calculating the residues of the poles y_n. One finds for the residues,

$$\rho_n(x') = \frac{\left(e^{x'y_n'}\right) s(y_n')}{L(\mu) \frac{d}{dy'} M(y') \Big|_{y' = y_n'}} , \qquad (5.3.28)$$

where the values of the pole positions, y_n' , are to be found in Table I. The residues are now substituted in Eq. (5.2.25),

$$U(x') = \rho_0(x') + \sum_{n=1}^{\infty} \left[\rho_n(x') + \rho_n^*(x')\right] , \qquad (5.3.29)$$

and finally the function $U(x)$ is substituted in Eq. (5.2.11) to obtain the solution, $\mathcal{I}(\theta,x')$,

$$\mathcal{I}(\theta,x') = \mathcal{J}(\theta,x') - \gamma\,\theta U(x') , \qquad (5.3.30)$$

utilizing Eq. (5.3.5).

Now in order to obtain the total reflected energy per second received by the "black" cylindrical receiving element, one substitutes Eq. (5.3.30) in Eq. (2.1.21) to obtain,

$$F_R = 2a\,\frac{\pi}{2} \int_{\zeta} J(R,3)\,\mathcal{I}(\theta_{R3}, x_3)\,dx_3 . \qquad (2.1.21)$$

The functions $J(R,3)$ and Θ_{R3} are obtained in exactly the same way as in the derivation of the function $s(y)$; indeed it is just here that it becomes evident that the function F_R satisfies the reciprocity relation given in Eq. (4.5.26). It turns out that the distinction between the denominators in that equation disappears as a result of the small angle approximations made in the present calculations. It is worth emphasizing here that the distance between the source and the receiver must be great enough so that the entire integral in Eq. (2.1.21) can be carried out subject to the restriction $x_3 > x_2$. For shorter ranges, the iteration solution (see below) is used. To the reflected energy given in Eq. (2.1.21) one must add the contribution of that radiation which proceeds directly from the source to the receiver,

$$F_D = 2aJ(R,0) \ , \tag{2.1.22}$$

obtaining then the total energy per second received by the black cylinder,

$$F = F_D + F_R \quad . \tag{2.1.23}$$

For shorter ranges the iteration solution presented in Section 5.1 is utilized. It is seen from Eqs. (5.1.1) and (5.1.7) that

$$\mathcal{J}(\theta,x) = \mathcal{S}(\theta,x) + \mathcal{T}(\theta,x) + \ldots \quad , \tag{5.3.31}$$

where $\mathcal{S}(\theta,x)$ has already been defined in Eq. (5.3.19) and (see Eq. (5.1.3)),

$$\mathcal{T}(\theta,x) = \int_0^{\mu} \mathcal{S}(\theta',x-\frac{2}{\alpha}\theta')\gamma\,\theta d\theta' \quad . \tag{5.3.32}$$

In the present calculation the iteration solution was carried through the first two terms (shown in Eq. (5.3.31)) since this reduced the error (in the region near the source) to a few per cent which was considered sufficient. In order to obtain the reflected energy per second received by the "black" cylinder, Eq. (5.3.31) is substituted in Eq. (2.1.21),

$$F_R = 2a\,\frac{\pi}{2}\int_{\zeta} J(R,3)\mathcal{S}(\theta_{R3},x_3)dx_3 + 2a\,\frac{\pi}{2}\int_{\zeta} J(R,3)\mathcal{T}(\theta_{R3},x_3)dx_3 + \ldots \tag{5.3.33}$$

Again, the angle θ_{R3} and the function J are obtained just as for Eq. (5.3.19). Upon using the values already found for these functions, one finds for the first term of Eq. (5.3.33),

$$F_R^{(1)} = \frac{4\beta\alpha a}{\pi\mu}\int\left(\frac{z_R'}{4x^2}+1\right)\left(\frac{z_0'}{4(x_R'-x)^2}+1\right)dx \tag{5.3.34}$$

where the integral is to be carried over all portions of the surface for which there are rays connecting both the source and the receiver; (x_R, z_R) are the coordinates of the receiver. The Eq. (5.3.34) is valid for $z_0', z_R' \leq 1$; for regions below the duct, one must alter the form of the functions J and θ as already indicated in connection with Eq. (5.3.21).

Similarly (for $z_0', z_R' \leq 1$) Eq. (5.3.32) becomes,

$$\mathcal{T}(\theta,x') = \frac{2\alpha\beta\gamma\theta}{\pi}\int_{[x'-1,x_1']^G}^{[x',x_2']^L}\left(\frac{z_0'}{4\xi^2}+1\right)(x'-\xi)d\xi\,, \quad [\;]^L \leq [\;]^G$$

$$= 0\,, \qquad [\;]^L \leq [\;]^G\,, \tag{5.3.35}$$

where the symbols $[a,b]^G$ and $[a,b]^L$ mean respectively the greater and the lesser of a and b. Also the second term of Eq. (5.3.33) becomes,

$$F_R^{(2)} = \frac{8a\beta^2\alpha}{\pi\mu}\int_{x'=x_R'-x_{2R}'}^{x'=x_R'-x_{1R}'}\left[\frac{z_R'}{4(x_R'-x')^2}+1\right]\left\{\int_{[x'-1,x_1']^G}^{[x',x_2']^L}\left[\frac{z_0'}{4\xi^2}+1\right](x'-\xi)d\xi\right\}dx'. \tag{5.3.36}$$

The inner integral is defined as zero when its upper limit is less than its lower limit. The integrals in Eq. (5.3.36) can be carried out in terms of elementary functions. However, in view of the complicated behavior of the limits of the inner integral and the rather complicated way in which the elementary functions appear, it seemed simpler to carry out the inner integral analytically and to do the second integral numerically. This was what was done.

The results of the calculations for selected positions of the source and receiver are shown in Figs. 12 through 17. There is a strong check on the calculations in those regions (usually for $1 < x_R' < 2$) where the two methods of calculation overlap. The received energies are plotted in decibels with the energy per second received at one yard as a reference. It should be remarked that the results are reciprocal in that they are unaffected by interchanging the source and receiver positions.

When the receiver is moved out of that region of space in which there is radiation coming directly from the source (without reflection from the bounding surface) there is a sudden loss in the amount of energy received per second. This loss results in the discontinuities shown in the graphs. The slight oscillations shown result from the maxima appearing successively in the once-reflected, twice-reflected, etc., received energies.

For comparison the levels for the corresponding problem with a homogeneous medium are plotted on every figure. It is seen that as a result of the large amount of energy scattered from the duct by the surface, the field is stronger in the homogeneous medium (with the exception in some cases of a short range interval near the source).

In Fig. 18 is shown the effect of varying the value of β (cf. Eq. (5.3.7)). The position of the pole on the real axis, which position controls the attenuation of the field at large ranges, is shown plotted against values of β from 0.05 to 1.0. Also plotted in the same figure is the real value of the first pole off the real axis $(\mathrm{Re}(y_1'))$. The relative sizes of these two quantities for a given value of β may be used to estimate the range at which the effect of the series in Eq. (5.2.25) may be neglected in comparison with the term $\rho_0(x)$ of that equation (in which case the field decays exponentially with x_R). If it is assumed that the residue given in Eq. (5.2.26) is of order e^{xy_m},* then one can estimate that range at which the first term in the series of Eq. (5.2.25) is small compared with the term $\rho_0(x)$. Those ranges at which the first series term is 20 per cent of the term $\rho_0(x)$ are shown plotted in Fig. 19 against different values of β; the ranges are measured in units of maximum skip distances $(L(\mu))$.

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* This has been the case in the numerical examples considered.

CHAPTER VI

DISCUSSION AND CONCLUSIONS

It has been seen that under certain restrictions it is possible to treat the problem of the propagation of radiation in a duct which is bounded on one side by a rough surface by using a relatively simple statistical model (see Eq. (2.1.18)). There are three major restrictions: first it is supposed that the properties of the medium vary but little within a distance of one radiation wavelength so that geometrical optics may be used to trace the progress of radiation within the volume. Secondly it is assumed that the surface is rough enough so that most of the reflected radiation is diffuse (see the restriction (4.4.38)). Finally it is supposed that $\left|\frac{d\zeta^M}{dx}\right| \ll 1$ and $\frac{1}{kR_m} \ll 1$ where $\frac{d\zeta^M}{dx}$ is the maximum value of the slope of the surface and R_m is the minimum radius of curvature of the surface; these restrictions are imposed so that the Kirchhoff approximation may be used in treating the reflection process. In such a case it is guaranteed that most of the radiation reflected from the surface is singly-scattered; this fact ensured the validity of the restriction (4.4.42) which was important in the development.

Once the statistical model was justified, the model was used to treat a numerical example (Section 5.3) through the use of the methods of solution outlined in Sections 5.1 and 5.2.

It is now in order to briefly discuss the solution. Probably the outstanding characteristic to be noted from Figs. 12 through 17 is the fact that the decibel loss is nearly a linear function of range for ranges exceeding one skip distance. This of course indicates that for such ranges the field has already assumed its asymptotic form. Upon examining Fig. 19 it is seen that by the criterion used in constructing that graph the asymptotic field could not be expected to be predominant for ranges of less than about 2.5 skip distances (for $\beta = 0.1$). Hence it would appear that that criterion overestimates the range to the asymptotic field.

The effect on the solution of varying the parameters α, D, and γ is now considered; α measures the velocity gradient within the duct, $c = c_0(1+\alpha z)$; D is the depth of the duct; γ measures the reflection properties of the surface (see Eq. (5.3.3)). It is first supposed that conditions are such that the two-dimensional solution may be converted to the three-dimensional solution in the way indicated in Appendix A. Varying the parameters affects the solution in three separate ways. First, it is emphasized that the distances appearing in the solution are measured in units of D for vertical distances and of $L(\mu)$ (the maximum skip distance) for horizontal distances. Noting that $L(\mu) = \left(\frac{8D}{\alpha}\right)^{1/2}$ it is seen that upon changing D or α, corresponding points appear at different spatial positions. Secondly it is remarked that since $\beta = \frac{1}{2}\gamma\mu^2$ or $\beta = \gamma\alpha D$, changing any of the parameters α, D, or γ will in general alter the value of β involved in the problem and thus alter the entire form of the solution since, for instance, the positions of the poles in the Laplace transform of the solution depend upon the parameter β (see Eq. (5.3.13)). It is remembered that β represents the fraction of energy trapped within the duct after a reflection. Finally it is not difficult to see from

Appendix A and the form of the two-dimensional solution that as a result of using $L(\mu)$ as the unit of horizontal distance, the decibel loss appears in the form $10\mathrm{Log}L^2(\mu)H(z_0';x_R',z_R';\beta)$ where H is a function of reduced distances and β only. Hence, since $L(\mu)$ depends upon D and α, it is seen that changing these parameters has, among other effects, the effect of adding the constant $10\mathrm{Log}\frac{8D}{\alpha}$ to the loss.

If the parameters α, D, and γ are changed in such a way that $\beta = \gamma\alpha D$ remains constant, one can obtain a family of solutions from a given calculated solution (for instance the one of this work). If the subscript 1 indicates the parameters used in a given calculation, and if the subscript 2 indicates a new set of parameters (subject to the restriction just mentioned) then the decibel loss for the new solution is obtained from the old by adding the constant $10\mathrm{Log}\,\frac{D_2}{D_1}\frac{\alpha_1}{\alpha_2}$. In this process it is to be noted that the distances z_0', x_R', and z_R' are reduced and that the units used in the reduction in general change when the parameters are changed.

A brief comparison with experiment will now be given. In the propagation of acoustic energy underwater, large attenuations have regularly been observed which could not be explained by the known attenuation mechanisms (for instance the attenuation due to viscosity). As an example, the average of a large number of measurements at 25kc shows an attenuation of about 4 db per kiloyard.* It seems probable on the basis of the work in this thesis that a considerable part of this attenuation is due to the surface's scattering energy from the duct; for the given attenuation, assuming for simplicity that all of the attenuation is due to surface scattering, the value of β obtained from Fig. 18 is in the range of 0.01 to 0.1 which has been seen to be reasonable for the type of sea surface met in

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* "Physics of Sound in the Sea", [33], p. 105.

practice. A second interesting confirmation occurs when a plot of the attenuation against the depth of the duct is made.* Such a plot reveals that the attenuation increases as the depth of the duct decreases as predicted by the present theory. A good quantitative fit can be made for a value of β in the above range, by utilizing the results shown in Fig. 18.

Before closing this work, it seems appropriate to discuss the ways in which it might be profitably extended. In the first place, as more knowledge is gained concerning the reflection properties of the surfaces met in experiment, it may prove desirable to carry out calculations based on the present theory using surface scattering functions more complicated than the one employed in making the calculations of this work. This is a straightforward procedure and may ultimately yield an interesting connection between surface properties of interest, for instance to the oceanographer, and the observed characteristics of radiation fields in surface-bounded ducts.

Secondly, it is not difficult to set up a statistical model, analogous to that employed in this work, which is applicable to three-dimensional problems. For certain problems this may prove desirable, although of course there is a not inconsiderable increase in complication as a result of this generalization. Another way in which the model might be extended would be to introduce a second bounding surface (for instance the bottom of the ocean); the formalism then becomes quite similar to that of Bateman and Pekeris [3].

Finally it might be of interest to alter the model so as to take into account some specular reflection at the bounding surface.

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* "Physics of Sound in the Sea", [33], p. 129.

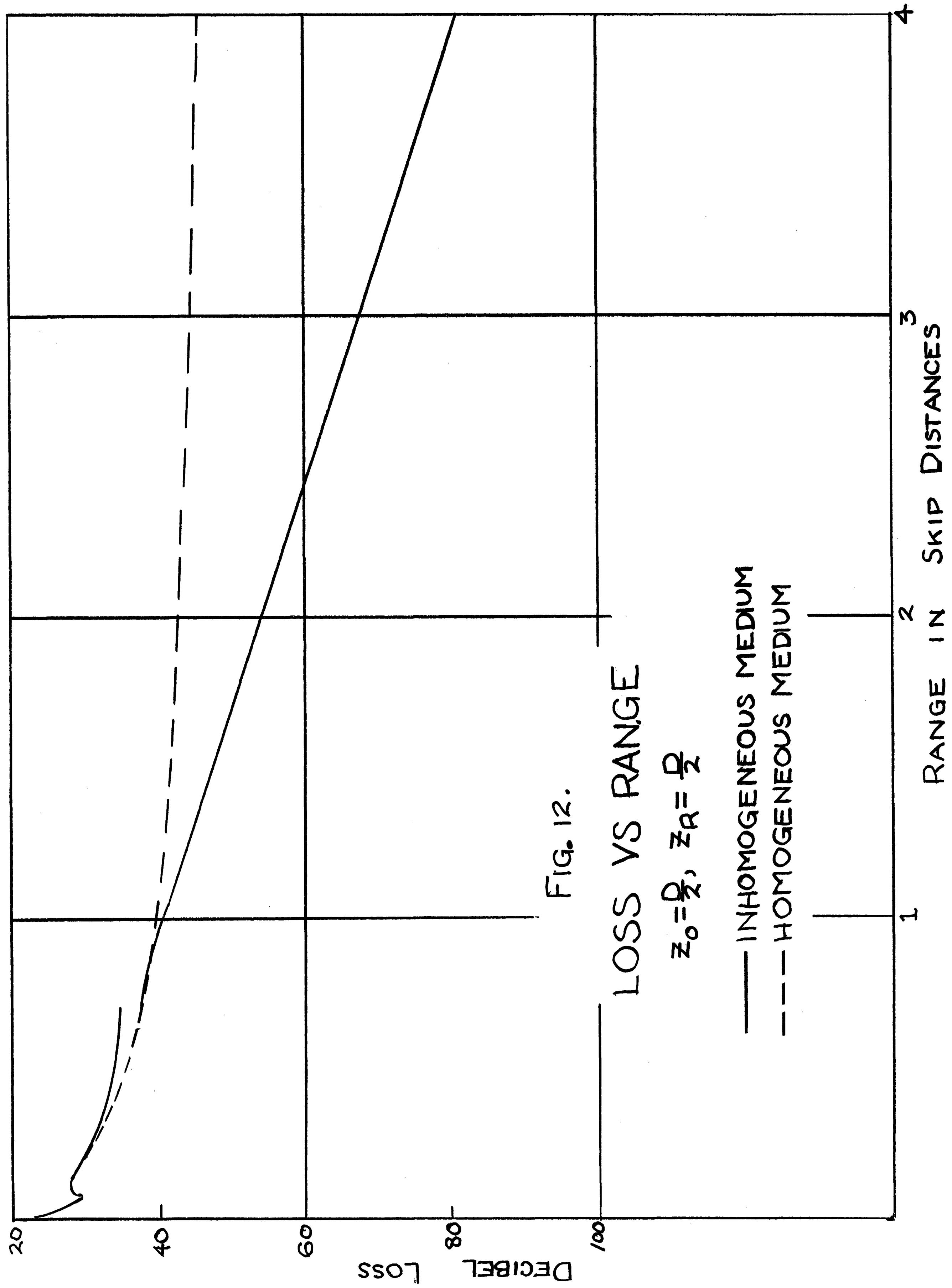

FIG. 12.
LOSS VS RANGE

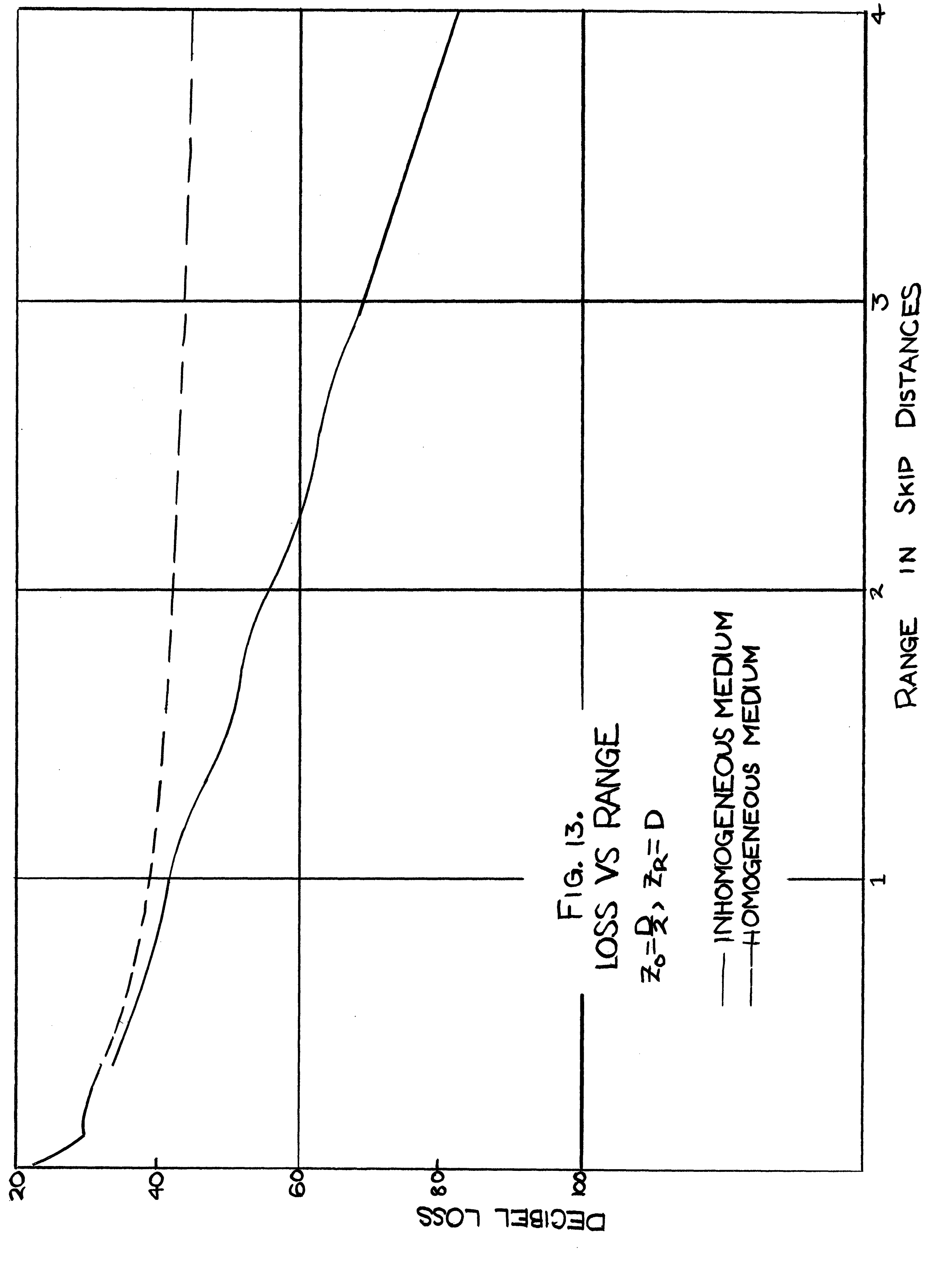

FIG. 13.
LOSS VS RANGE
$z_0 = \frac{D}{2}$, $z_R = D$

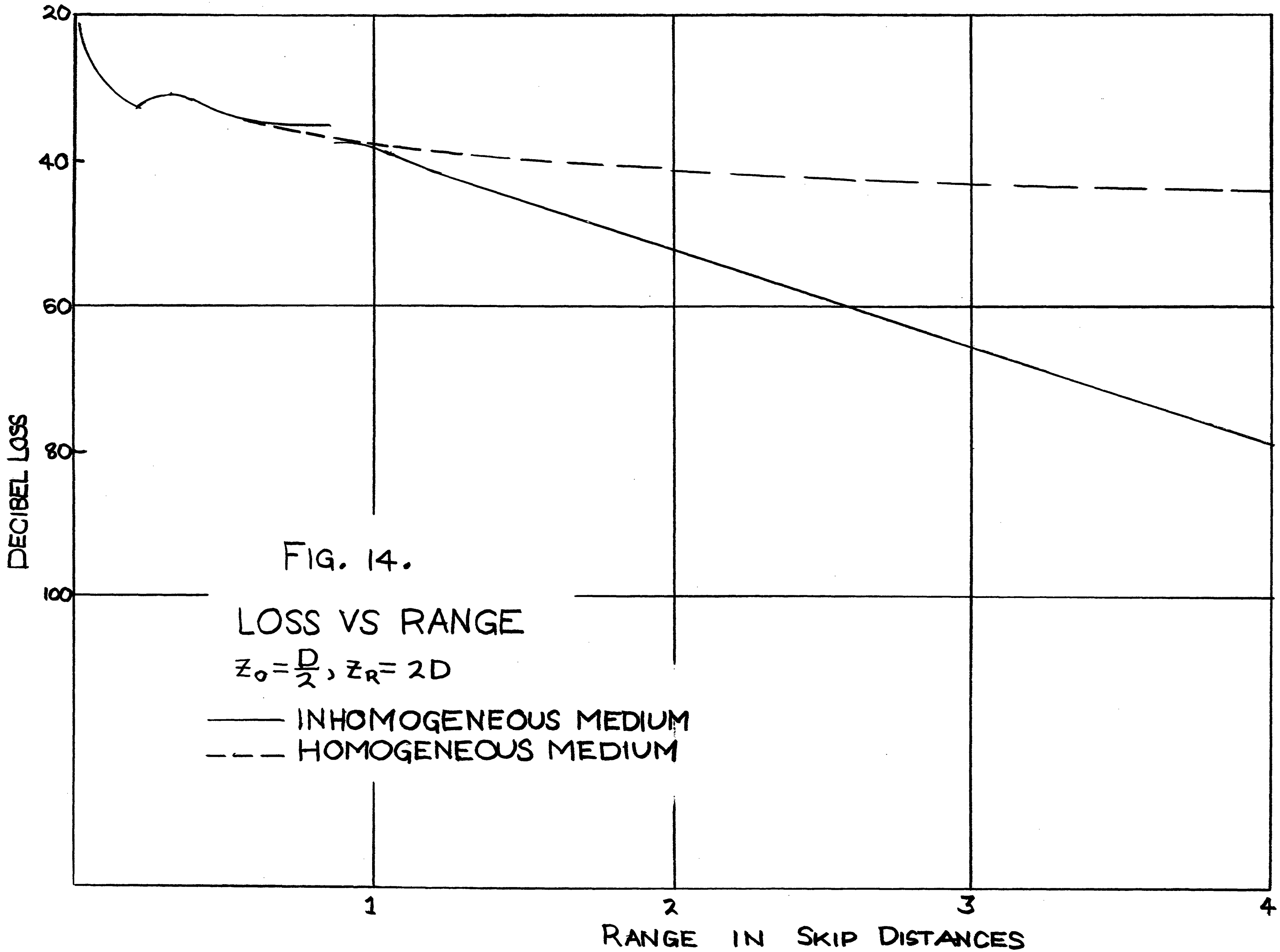

FIG. 14.

LOSS VS RANGE

$z_0 = \frac{D}{2}$, $z_R = 2D$

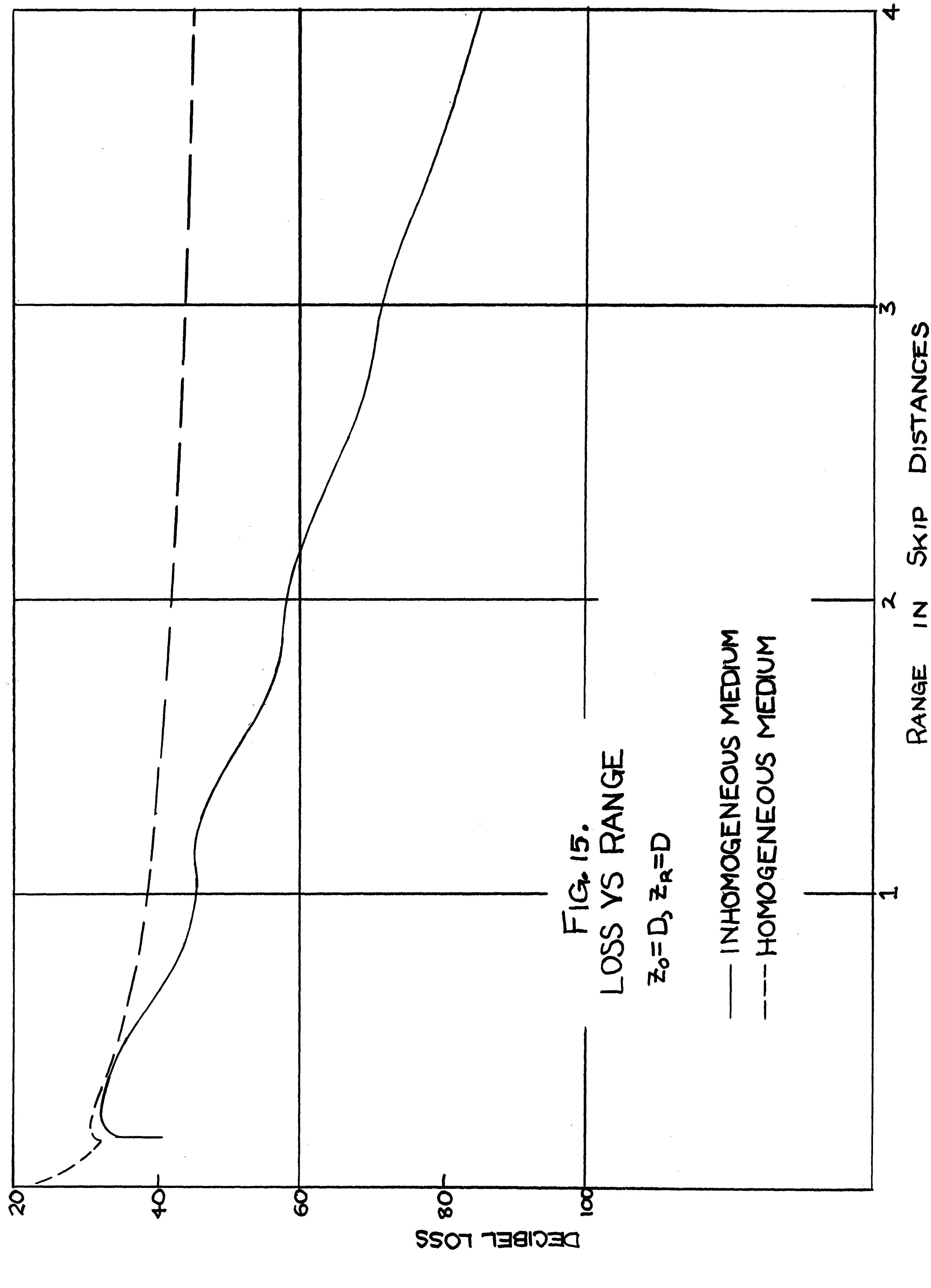

FIG. 15.
LOSS VS RANGE
$z_0 = D$, $z_R = D$

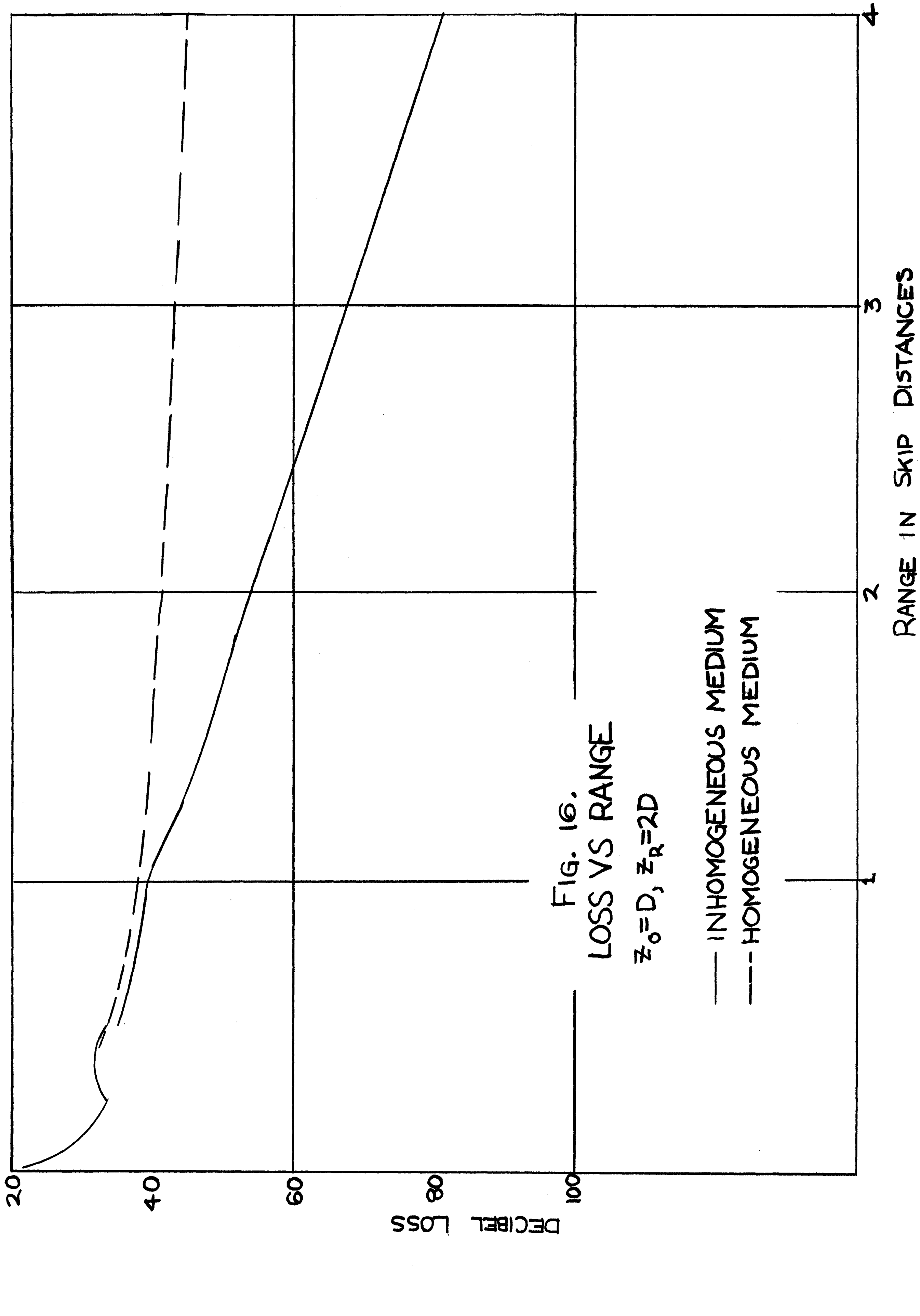

FIG. 16.
LOSS VS RANGE
$z_0 = D,\ z_R = 2D$

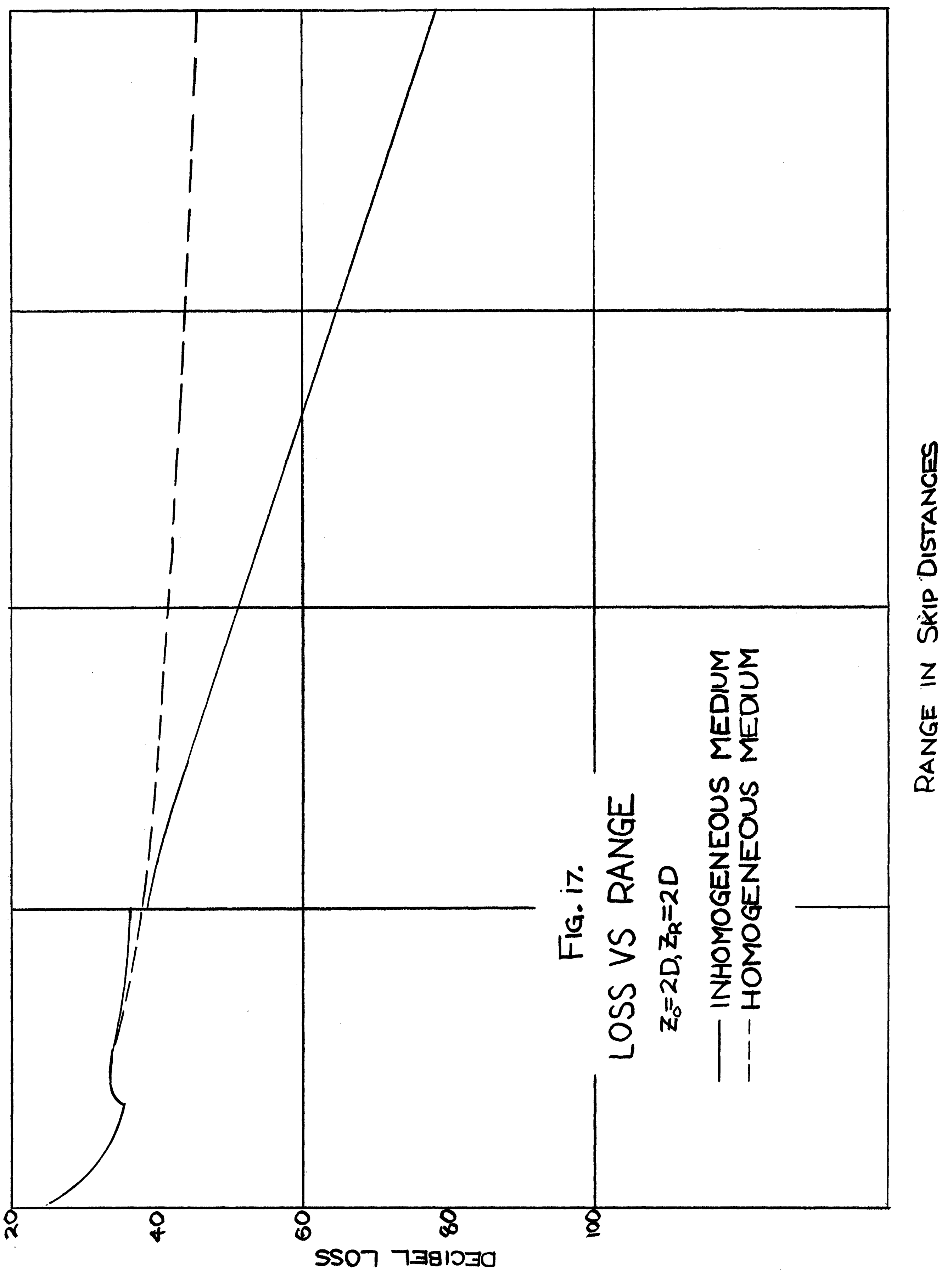

FIG. 17. LOSS VS RANGE

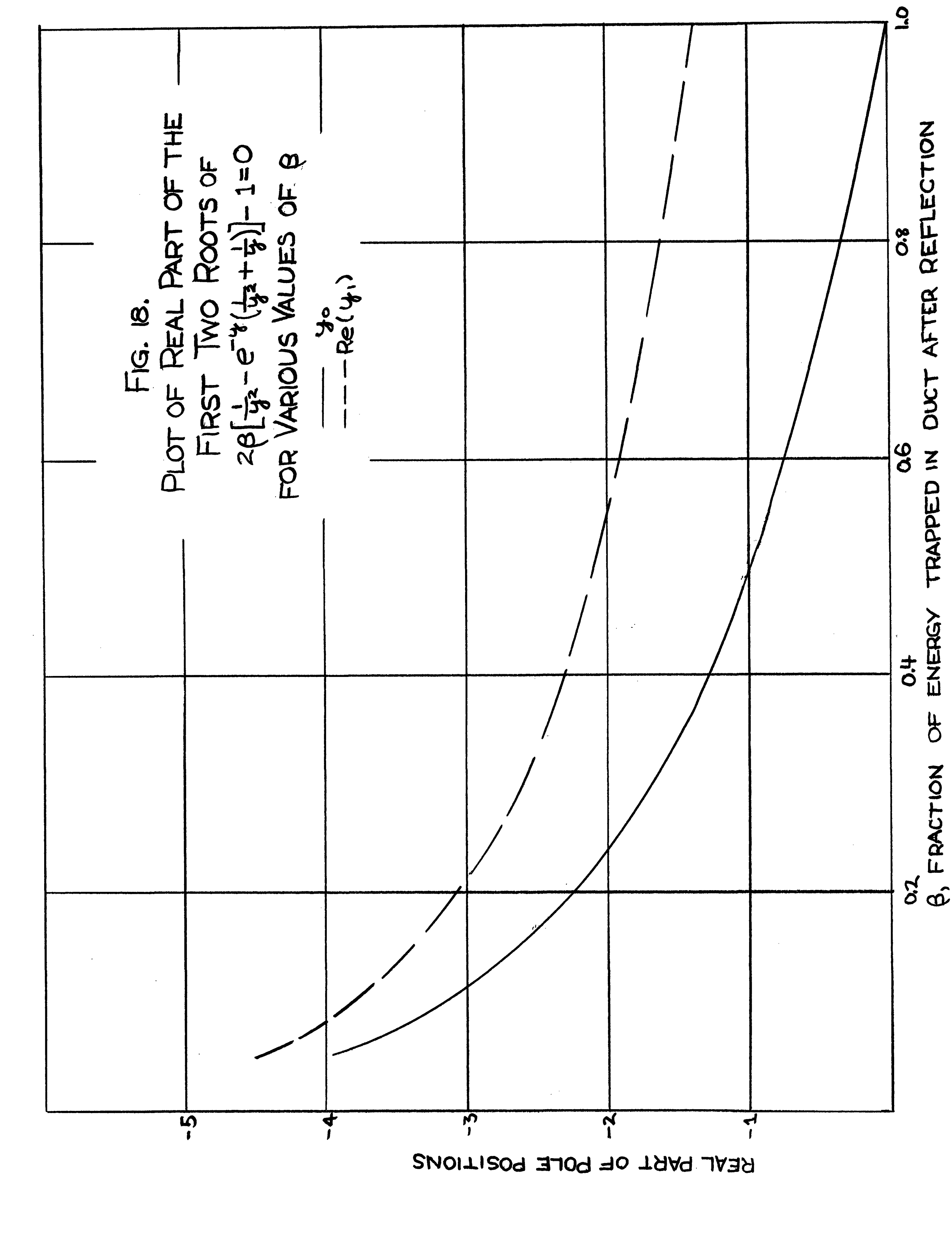

FIG. 18.
PLOT OF REAL PART OF THE FIRST TWO ROOTS OF $2\beta\left[\frac{1}{y^2} - e^{-y}\left(\frac{1}{y^2}+\frac{1}{y}\right)\right] - 1 = 0$ FOR VARIOUS VALUES OF β

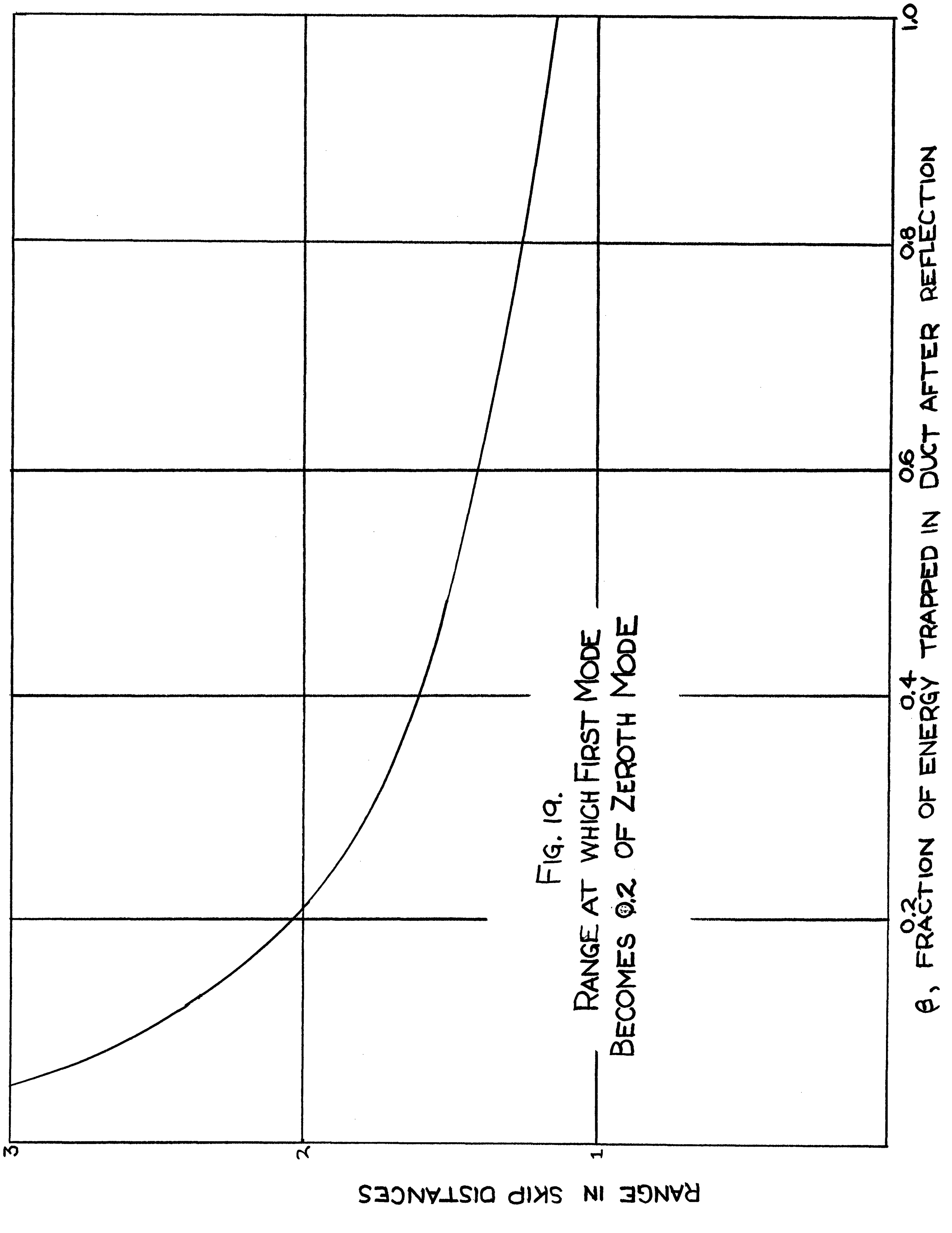

Fig. 19.
Range at which First Mode Becomes 0.2 of Zeroth Mode

APPENDIX A

In this appendix a sufficient condition will be shown under which one may convert quite simply the results of the foregoing theory so that they are applicalbe to certain propagation problems in three dimensions.

The problem to be considered is one in which the bounding surface of a half space is a function of two cartesian coordinates, $\zeta(x,y)$. It is supposed that the phase velocity depends only on the distance, z, from the average surface, and that it depends on this distance in the way postulated in Section 4.3; that is, the velocity increases to some depth D, and decreases thereafter. Thus a surface-bounded duct is set up.

To solve the problem one must find a function Ψ, for instance the velocity potential, which satisfies the equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2(z)\right] \Psi = 0 , \tag{A1}$$

which has a singularity of the type,

$$\lim_{r_0 \to 0} \Psi(x,y,z) = \frac{ce^{ikr_0}}{r_0} , \text{ where } r_0 = \left[(x-x_0)^2+(y-y_0)^2+(z-z_0)^2\right]^{1/2} \tag{A2}$$

and where (x_0, y_0, z_0) represents the source point, and c is a constant determining the strength of the source. The function Ψ is to vanish at the free surface,

$$\Psi(x, y, \zeta(x, y)) = 0 \tag{A3}$$

and to represent outgoing waves at infinity. It has been supposed that the source radiates a single angular frequency, ω .

It is assumed that the restrictions of the body of this work apply in the present problem. In addition to these, two further restrictions on the reflection characteristics of the surface are needed. First it is supposed that the reflection characteristics are independent of the azimuthal angle of the incident radiation. This might be rephrased as an assumption of statistical isotropy in addition to (the already assumed) statistical homogeneity of the surface. The second restriction on the surface consists of supposing that all radiation reflected from the surface lies in the plane of incidence (where the plane of incidence is defined by the direction of the incident radiation and the z axis). The significance of this assumption will be discussed after the present derivation is completed.

From these two restrictions, it follows that if the z axis is assumed to pass through the source, then all radiation reflected from a given region of the surface lies in the plane formed by the radius vector to the reflecting region and the z axis. The derivation of the equations governing the propagation of radiation in the present three-dimensional problem is directly analogous to that presented in Chapter II for the two-dimensional problem. Consequently the argument will only be sketched here and the results given, it being understood that the details of the discussion may be filled in in a way analogous to that used in Section 2.1.

Reference will be made to Fig. 20 (which can be compared with Fig. 1).

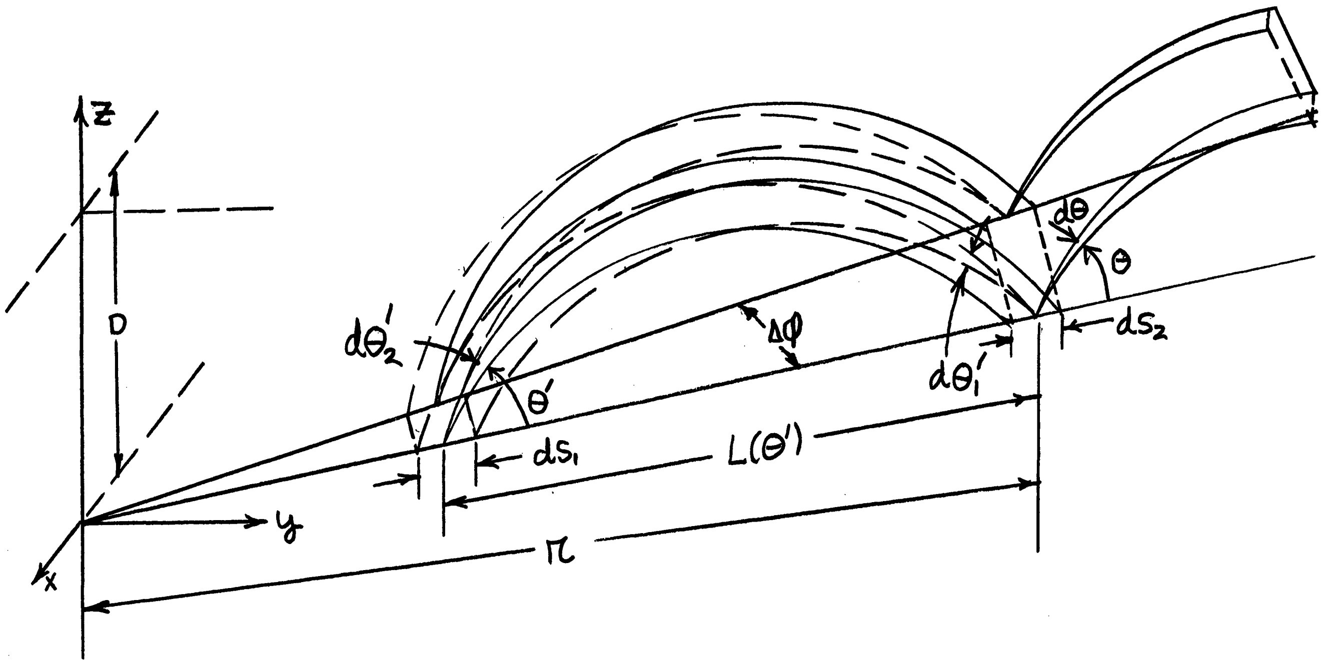

Fig. 20. Three-dimensional diagram showing the progress of radiation in a surface-bounded duct.

Let the following definitions be established: $\ell_{3D}(\Theta,r)d\Theta dA$ is defined as the average amount of energe per second reflected from the element of surface dA (located a distance r from the origin), which energy is contained within the angular element $d\Theta$ at an angle Θ measured from the horizontal. The quantity $\mathcal{S}_{3D}(\Theta,r)d\Theta dA$ is defined similarly with the exception that the radiation considered is assumed to undergo its first reflection at the element dA. The function $A(\Theta,\Theta')d\Theta$ is defined as the average energy per second reflected from some region of surface in the angular element $d\Theta$, making an angle Θ with the horizontal, when unit energy per second falls on the region from the direction Θ'. It is supposed that the incident radiation has a single azimuthal angle (and by hypothesis then the reflected radiation all lies in the plane of incidence).

From the definition of $\mathcal{I}_{3D}(\theta,r)$ just given and from Fig. 20 it is seen that the (average) amount of reflected energy per second incident upon the surface element $ds_2 r\Delta\phi$ is,

$$dE_2 = \mathcal{I}_{3D}(\theta',r-L(\theta'))\left[r-L(\theta')\right] \Delta\phi ds_1 d\theta_2' \quad . \tag{A4}$$

It is assumed that $\Delta\phi \ll 1$; also $L(\theta)$ is defined, as before, as the distance travelled by a trapped ray making an angle θ, between surface reflections. Then using the definitions of $\mathcal{S}_{3D}$ and A one finds, as in Section 2.1, the following equation governing the function $\mathcal{I}_{3D}$:

$$\mathcal{I}_{3D}(\theta,r)r = \mathcal{S}_{3D}(\theta,r)r + \int_{\Theta_T} \mathcal{I}_{3D}(\theta',r-L(\theta'))\left[r-L(\theta')\right] A(\theta,\pi-\theta')d\theta', \tag{A5}$$

where, as before, Θ_T represents the set of all trapped angles.

It is now of interest to show the connection between the function $\mathcal{S}_{3D}$ and the corresponding function for the two-dimensional problem, $\mathcal{S}$.

To do this, Fig. 21 is considered. It is supposed that the constant c of Eq. (A2) is adjusted so that the source radiates $2/\pi$ units of energy per second, per solid angle (then $c = \left(\frac{2}{\pi k}\right)^{1/2}$). Considering first the quantity $\mathcal{S}(\theta,x)$ it is seen from Eq. (4.5.9) that,

$$\mathcal{S}(\theta,x_1) = J(1,0)A(\theta,\theta_{10}')\sin\theta_{10}' \tag{A6}$$

where θ_{10}' is the angle made with the x axis at the surface by the ray connecting the source and the surface point (1). Then one finds upon using Eq. (4.3.18), the definition of $\mathcal{S}_{3D}$, and Fig. 21 that,

$$\mathcal{S}(\theta,r)\cos\theta_{z0} = r\,\mathcal{S}_{3D}(\theta,r) \tag{A7}$$

where θ_{z0} is defined in Fig. 21 as the angle made with the horizontal at the source by the ray connecting the source and the surface point (1).

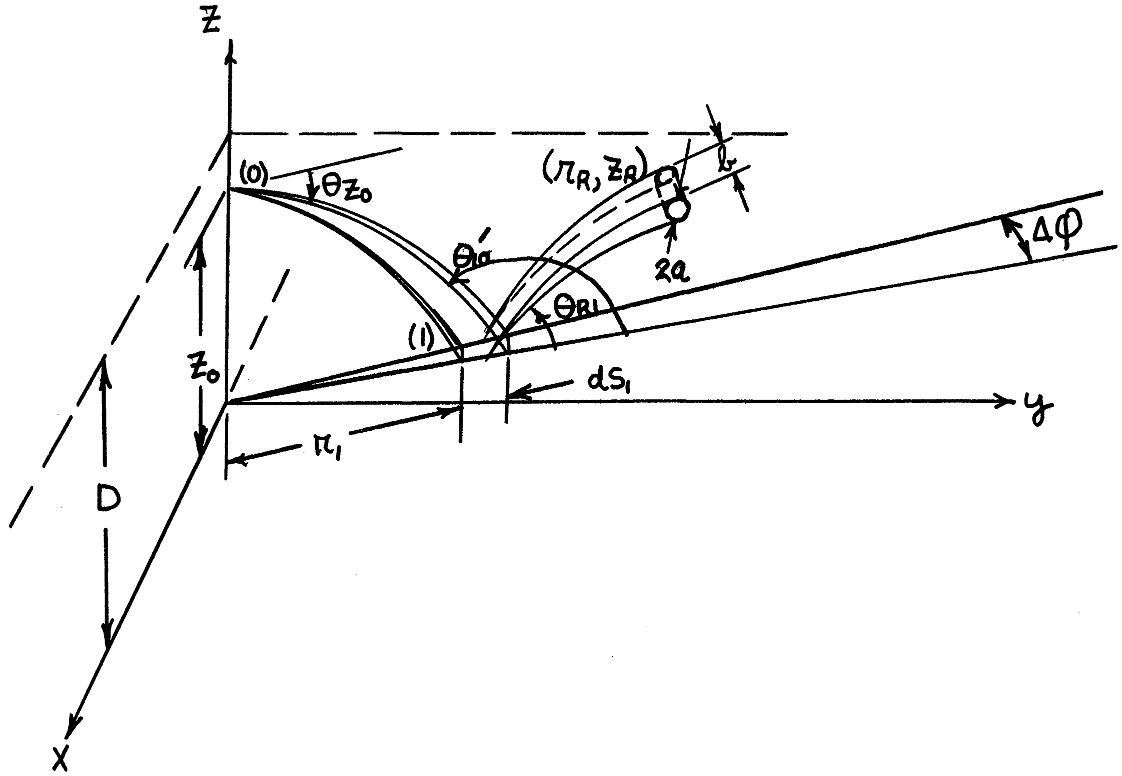

Fig. 21. Diagram showing singly-reflected radiation received by the cylindrical element of radius a and length b .

If it is assumed that the source radiates little energy into directions such that θ_{z0} is not small (as was done in Section 5.3 for the numerical example and as is often the case in experiments to which the present theory is applicable), one sees from Eq. (A7) that

$$r\,\mathcal{S}_{3D}(\theta,r) \approx \mathcal{S}(\theta,r) \quad . \tag{A8}$$

In such a case one finds from Eqs. (A5) and (2.1.18) that $\mathcal{I}_{3D}r$ and $\mathcal{I}$ satisfy the same equation and hence that the solution, $\mathcal{I}_{3D}$, to the present three-dimensional problem can be obtained from that of the corresponding two-dimensional problem by dividing the latter by the range.

If the source radiates an appreciable amount of energy per second in directions for which the angle Θ_{z0} is not small, one can treat the three-dimensional problem by modifying the source function as indicated in Eq. (A7).

The experimentally measured quantity, the amount of energy per second received by the monitoring element, is now considered. A section of circular cylinder of length b and radius a is chosen for the receiving element. It is supposed that a and b are small compared with other distances in the problem, and that the cylindrical section is "black". The receiver is assumed (as shown in Fig. 21) to be oriented with its axis perpendicular to the plane formed by the z-axis and the radius vector to the field point, (r_R, z_R). Furthermore $\Delta\phi$ is chosen so that $\Delta\phi = \frac{b}{r_R}$. Then upon considering Fig. 21 and using the definition of the quantity $\mathcal{S}_{3D}$, the following result for the singly-reflected energy per second, received by the section of cylinder, is found:

$$F_{R1(3d)}(r_R, z_R) = \frac{b}{r_R}\, 2a\, \frac{\pi}{2} \int_{\zeta} \mathcal{S}_{3D}(\Theta_{R1}, r_1(\Theta_{R1}))\, r_1 J(R,1)\, dr_1 \ , \qquad \text{(A9)}$$

where the integral is to be carried over all portions of the surface for which rays connecting both the source and the receiver exist. The function $J(R,1)$ is the magnitude of the intensity defined in Section 2.1. In a similar way the following result is found for the total reflected energy per second received:

$$F_{R(3D)}(r_R, z_R) = 2a\, \frac{\pi}{2}\, \frac{b}{r_R} \int_{\zeta} \mathcal{J}_{3D}(\Theta_{R1}, r_1(\Theta_{R1}))\, r_1 J(R,1)\, dr_1 \ . \qquad \text{(A10)}$$

Finally one finds for the energy per second travelling directly from the source to the receiving element,

$$F_{D(3D)}(r_R, z_R) = 2a \frac{b}{r_R} \cos \Theta_{z0} J(R,0) \ . \tag{A11}$$

Again if all of the energy is radiated at small angles Θ_{z0}, this can be simplified to,

$$F_{D(3D)}(r_R, z_R) = 2a \frac{b}{r_R} J(R,0) \ . \tag{A12}$$

By using the equivalence already shown between $\mathcal{L}_{3D}r$ and $\mathcal{L}$ it is seen upon comparing Eqs. (A10) and (2.1.21) and also Eqs. (A12) and (2.1.22) that the average total energy per second received in the three-dimensional problem is obtained from the corresponding quantity of the corresponding two-dimensional problem by multiplying the latter by $\frac{b}{r_R}$.

It is desirable to again consider the assumption that the surface is such that it reflects all energy in the plane of incidence. Of course it is not expected that such a surface is common in physical problems (if indeed it ever occurs). However, in certain physical problems this situation is approximated. It has been shown by Eckart* that in problems for which one can use geometrical optics to treat the reflection process, the reflected radiation lies predominantly near the plane of incidence when the grazing angle (measured from the horizontal) is small. Since small angles are of major interest in duct propagation problems, this result is useful in the present connection.

On the basis of the results of this appendix, it may be supposed that one can obtain an approximation to the three-dimensional problem in the manner indicated above if the azimuthal spread of the reflected radiation is small (when a uni-directional beam is incident on the surface).

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* Eckart, [8] , p. 11.

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