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ANALYSIS AND SIMPLIFICATIONS OF DISCRETE EVENT SYSTEMS AND JACKSON QUEUING NETWORKS

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ABSTRACT

ANALYSIS AND SIMPLIFICATIONS OF DISCRETE EVENT SYSTEMS AND JACKSON QUEUING NETWORKS

by

Benjamin Melamed

Co-chairmen: Frederick J. Beutler, Bernard P. Zeigler

This dissertation contains studies in two related areas: Discrete Event System Theory and Queuing Network Theory.

In the first line of study, deterministic discrete event systems are modeled by formal automata-like structures, and a hierarchy of morphic relations among them is developed. A canonical representation of stochastic discrete event systems in coordinate probability space is proposed, and a hierarchy of morphic relations among them is constructed by means of measure preserving transformations.

A general conceptual framework for simplifications is proposed, and the morphisms above are shown to fall within its scope. Under this framework, these morphisms are viewed as a mathematical vehicle for simplification.

In the second line of study, several operating characteristics of the class of Jackson queuing networks are investigated. Included are: line sizes (network state), total service awarded to customers, and traffic processes on the arcs.

Special emphasis is placed on rigorous derivations of results from solid mathematical and statistical foundations. In the process,

a number of theoretical gaps in the extant theory of state equilibrium are closed, and Burke's Theorem is extended from M/M/1 queues to Jackson networks with single server nodes. Applications to equilibrium decompositions of Jackson networks are also pointed out.

These results are applied to exemplify a number of structural simplifications that take Jackson networks into Jackson networks, while preserving a variety of operating characteristics. A new methodology, combining statistical tools with system-theoretic ones, is used in some of the aforesaid simplifications.

Finally, simulation complexities of Jackson networks are discussed, and their behavior under various simplifications is investigated.

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by
Benjamin Melamed

A dissertation submitted in partial fulfillment
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To My Family: Wife, Parents and Sister.

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CHAPTER 0

INTRODUCTION

0.0 General

This dissertation has two main themes: a system-theoretic theme and a statistical-theoretic one.

The system-theoretic theme concerns a class of systems - the so-called *discrete event systems*. Such systems can be loosely described as evolving in continuous time in discrete transitions, driven by inputs occurring discretely in time.

The statistical-theoretic theme involves an important class of stochastic discrete event systems - the class of *queuing systems* and especially *Jackson queuing networks*. Loosely speaking, a queuing system models a servicing facility consisting of two types of entities: immobile entities called servers, and mobile ones called customers. In a queuing network the servers are arranged in a graph configuration; the customers move about - among servers, into the network, and out of it - according to certain prescribed rules.

The subjects of study in this dissertation are mathematical abstractions that formalize loose heuristic notions relating to discrete event systems and queuing networks. Based on these formalisms, we then proceed to study *morphic relations* among discrete event systems, and to derive computational results for some operating characteristics of Jackson queuing networks. Finally, we investigate *simplification relations* among Jackson networks; in the process we make a combined use of system-theoretic and statistical-theoretic methods.

0.1 Brief Historical Review

In a series of papers ([Z2], [Z3], [Z4], [Z5], [Z6]) culminating in a book ([Z1]), B. P. Zeigler develops a conceptual framework for modeling and simulation of real-life systems.

His paradigm consists of four basic components:

1. a *real system* which serves as a source of observable data.
2. a *base model* which is a system description based on the totality of currently acquired data.
3. a *lumped model* which is a simplified version of the base model.
4. a *computer* on which the lumped model may be simulated.

The interrelations among the four components may be summarized as follows.

I. The real system and the base model are behaviorally related. Going from the former to the latter is the enterprise of *modeling* (also called *realization* or *identification*).

II. The base model and the lumped model are structurally and behaviorally related. Going from the former to the latter is the enterprise of *simplification*.

III. Finally, the lumped model and a certain computer (program) are structurally related. Running a computer program of the lumped model is the enterprise of *computer simulation*.

The essence of the paradigm, proposed by Zeigler, is that in order to obtain valid results from a modeling and simulation of some real life system, certain preservation relations pertaining to structure and behavior must hold among the four components. These relations are collectively called *morphisms*.

Recently, considerable work has been done within the scope of Zeigler's paradigm. Foo [Fo1] investigates homomorphic simplifications and topological realizations of dynamical systems. He introduces topological structure on this class of systems, in which light the goodness of certain simplifications is examined.

Corynen [Co1] develops a comprehensive framework for modeling and simulation of deterministic and stochastic systems. His treatment is general enough to provide an abstract common foundation, applicable to a large variety of disciplines.

Barto [Ba1) considers discrete modeling of natural systems by means of cellular automata and homomorphic simplifications thereof.

Aggarwal [A1] introduces deterministic models for probabilistic systems. He applies them to simulations of neural networks, in order to construct lumped models that approximately preserve the base model's behavior.

Considerable attention is devoted in this body of work to discrete systems. It is motivated by the fact that a large variety of real-life systems admit of a discrete system realization. To mention a few: biological systems (living cells, photosynthetic units, neural networks), ecosystems, information processing systems (computer hardware and software, pattern recognition), physical phenomena (particle movement), and service systems (queues and queuing networks, production lines, traffic systems). (See e.g. [WZL1], [Z3], [Z7], [Z8], [ZB1], [GZ1]).

At the theoretical level, discrete systems may be used to describe cellular automata, and mathematical queues and queuing networks.

Discrete systems come in two flavors. Those operating over a discrete time base comprise the class of *automata* and their variants. (See e.g. [AU1]). Those operating over a continuous time base are called *discrete event systems*. A formal tool for describing such systems has been proposed in [Z1]. The formalism is called a *DEVS (discrete event system specification)*; it has been used elsewhere to describe such systems (see e.g. [Z8], [ZB1], [GZ1]).

The class of queuing systems provides an important instance of discrete event systems. The widespread interest in queuing systems becomes evident as one leafs through the various technical journals in such areas as Computer Science, Applied Probability and Operations Research. This interest stems from the fact that queuing systems provide a mathematical model for a host of real-life systems which have emerged from the technological advancements of this century.

The initiation of mathematical studies of queuing systems is credited to the telephone engineer A. K. Erlang, early in the century. In his pioneering work, Erlang was interested in trunking problems arising in telephone service, where customers model incoming calls and servers model telephone lines. (See any standard book on Queuing Theory such as [Sa1]).

The advent of fast computers and time sharing systems, complex communication systems and intricate manufacturing processes, during the recent thirty years or so, considerably enlarged the applicational scope of queuing systems and especially queuing networks.

The study of queuing networks seems to have originated with R. R. P. Jackson during the 50's. In two papers ([JR1] and [JR2])

Jackson studies a tandem sequence of queues with exponential services and Poisson arrival processes. This work was later subsumed in [JJ1] and [JJ2] by J. R. Jackson who studied open networks of arbitrarily connected queues with Poisson arrivals, exponential servers and Bernoulli switches. The class of networks of such queues with arbitrary topology will be called here *Jackson queuing networks*, after these two workers. These include the closed networks studied by Gordon and Newell in [GN1], in the early 60's. These early studies and others on Jackson queuing networks were motivated by machine repair shops, where a customer (a machine) has to visit more than one server (a repair stage).

Recently, performance studies of computer systems and communication networks have sparked renewed interest in queuing networks resulting in a surge of research effort. (See e.g. [Bu1], [BCMP1], [GM1], [GP1], [K1], and [Ru1]). However, parts of this work have been heuristic in nature, the emphasis being on computational results. Little effort has been devoted to theoretical foundations, and but scant references have been made to the underlying mathematical theory that justifies usage of the computational tools.

The study reported in this dissertation addresses itself to theoretical issues as well as computational ones. It is based on an earlier preliminary study in [BMZ1], [BM1] and [MZB1].

0.2 Organization and Research Scope

This dissertation is organized in three parts. The first part consists of Chapters 1-3 and is concerned with the theory of deterministic and stochastic discrete event systems and morphic relations among them. As such it falls largely within the scope of the base model and lumped model components in Zeigler's paradigm, and the simplification relations connecting them.

Chapter 1 develops a hierarchy of morphic relations among DEVSs and related system-theoretic objects. The morphisms are related to a general simplification notion outlined in Appendix B. Interpretations for their intuitive meaning are also furnished.

Chapter 2 proposes a canonical representation of stochastic discrete event systems in coordinate probability spaces. A connection between Probability Theory and System Theory is shown to reside in the coordinate sample space, where sample histories are described by means of deterministic DEVS-related objects.

Chapter 3 creates a hierarchy of stochastic morphisms among probability spaces, formalized by measure preserving transformations. These morphisms are shown to fit into the simplification framework of Appendix B, by pointing out the lumping effect exerted by them on the base model's sample space and σ-algebra.

The second part consists of Chapters 4-5 and is devoted to the theory of Jackson queuing networks and their simplifications. Special attention is paid to the relevant statistical theory that underlies the computations, and care is taken to base the results on solid theoretical foundations and mathematical rigor.

Chapter 4 contains a detailed study of Jackson queuing networks with single server nodes. The operating characteristics studied include equilibrium line sizes, service obtained by customers and equilibrium traffic processes on the arcs.

Chapter 5 uses results in Chapter 4 to exemplify various simplifications that take Jackson networks into Jackson networks. Considered are simplifications that eliminate feedback arcs or remove arcs within a subnetwork, and some simplifications that lump a subnetwork into a single node. The first type of simplifications makes a combined use of DEVS-theoretic results derived in Chapter 1 as well as the statistical-theoretic treatment of Chapters 2 and 3. Finally, simulation complexities of Jackson networks are discussed, and their behavior under various simplifications is investigated.

Chapters 1-5 are followed by a Conclusion that summarizes the results attained in them and suggests a number of research topics to be pursued.

The third part consists of Appendices A, B and C, which provide mainly background material.

Appendix A contains a digest of elementary System Theory compiled from [Z1], and which serves as an introduction to Chapter 1.

Appendix B proposes a conceptual framework for Simplifications which is in line with Zeigler's paradigm, and into which large tracts of this dissertation are fitted. It provides a common foundation for a variety of simplification problems arising in applied areas such as Modeling and Simulation as well as in theoretical contexts. The central view, expounded by it, is that morphic relations among systems constitute a major mathematical vehicle for formalizing the intuitive simplification notion.

Appendix C is a collection of definitions and facts from the domain of Stochastic Processes. It provides some mathematical foundations for the methods employed in Chapter 4.

0.3 Some Notational Conventions

Each chapter or appendix in this dissertation is divided into sections. Section m of chapter or appendix n is numbered according to the scheme n.m. Theorems, lemmas, corollaries etc. within each section n.m are numbered according to the scheme n.m.ℓ and delimited by the symbol $\square$. Lines are usually tagged by numbers although upper case and lower case letters as well as Latin numerals are occasionally used. References to a line tag made within the scope of a theorem, lemma, corollary etc. are always local, unless otherwise specified. References to a line tag, made outside the above, are always local to the section of occurrence, unless otherwise specified.

The symbol $\triangleq$ means equality by definition. The symbol Pr is an abbreviation for probability, and E - for expectation.

In referencing bibliographic material we occasionally abbreviate the word Chapter as Ch. and the word Section as Sec.

CHAPTER 1

DETERMINISTIC DISCRETE EVENT SYSTEMS

1.0 Introduction

Discrete event systems are characterized by the fact that they evolve in continuous time but change state due to events occurring discretely in time. Such systems respond to discrete stimuli by undergoing state "jumps"; they remain quiescent during the time intervals separating them. Loosely speaking, their state trajectories trace out step functions.

The importance of discrete event systems stems from the fact that they model a variety of real life systems such as software systems, information processing systems, production processes, traffic systems, service facilities - in particular queuing systems - and certain aspects of biological and physical phenomena (see e.g. [Z8] and [GZ1]).

Our interest in discrete event systems is motivated by the fact that queuing systems can be modeled as stochastic discrete event systems, while particular queuing histories are modellable as deterministic discrete event systems. The term stochastic systems (versus deterministic systems) alludes to the fact that the operation of the later is governed by ordinary functions, and that of the former by random variables.

This chapter studies the logic of deterministic discrete event systems and certain preservation relations among them, which are collectively called morphisms.

The applications to queuing systems are twofold. First, to describe accurately their operation (see Chapter 2), and second, to perform simplifications on them (see Chapters 3 and 5).

The formalization of discrete event systems by the DEVS (discrete event system specification) concept is due to B. P. Zeigler (see [Z1] Ch. IX Sec. 9.11). This definition is used here with minor changes as the starting point, and the treatment of morphic relations follows in spirit that of [Z1] and especially Chapters IX and X.

The organization of this chapter is as follows. Sections 1.1 - 1.2 present a hierarchy of deterministic discrete event systems and related structures, which is based on [Z1] Ch. IX, mainly Sec. 9.11. Sections 1.3 - 1.5 present a hierarchy of morphisms and investigate some of their properties. Finally, Sections 1.6 - 1.7 describe operations on discrete event systems and investigate morphic relations among triples of discrete event systems.

The reader is referred to Appendix A and to [Z1] for additional background.

1.1 The DEVS and DEVN Concepts

A DEVS (discrete event system) specification is a special case of an iterative specification of a system, which is itself a special case of the class of time invariant mathematical systems (see Appendix A). By "special case" we mean here that the specialized case induces an instance of the generalized case in a one-one manner.

The salient feature of DEVSs is that they operate in continuous time, but significant state changes occur discretely in time. These changes (or jumps) are caused by discrete occurrence of "events".

Consequently, the evolution of a DEVS can be described by a step function.

The definition of a DEVS follows that of [Z1], (see Ch. IX Sec. 9.11) with rather minor deviations.

Definition 1.1.1

A DEVS (*discrete event system*) specification is a structure $M = \langle X, S, Y, t, \delta, \lambda \rangle$ where

X is the *external event set*

S is the *sequential state set*

Y is the *output value set*

t is the *time advance function*

δ is the *sequential state transition function*

λ is the *output function*

subject to the following restrictions:

a) t is a function $t: S \to [0, \infty]$.

$t(s)$ is the maximal time the system is allowed to stay in sequential state s. This maximum is attained whenever no external events occur while the system is in sequential state s.

b) δ is a function $\delta: Q \times (X \cup \{\phi\}) \to S$ where

$\phi \notin X$ is the external nonevent symbol and

$Q \overset{\Delta}{=} \{(s,e): s \in S \text{ and } 0 \leq e < t(s)\}$ is the full state set of M.

A full state q is a pair (s,e) interpreted as a sequential state s, and the time elapsed e in that state. The e component will be referred to as the *clock*.

The definition of δ has two parts.

b.1) $\forall q \in Q, \forall x \in X, \quad \delta(q,x) \triangleq \delta_M((s,e),x)$

where $\delta_M((s,e),x)$ gives the sequential state to which the system transits from the full state q, under the external event x.

b.2) $\forall q \in Q, \quad \delta(q,\phi) \triangleq \delta_\phi(s)$

where δ_ϕ is the *autonomous transition function* of the system. Such transitions occur whenever the clock exceeds $t(s)$.

c) λ is a function $\lambda: Q \to Y$.

$\lambda(q)$ is the instantaneous output of the system from full state $q = (s,e)$. □

Definition 1.1.2

A DEVS $\hat{M} = \langle \hat{X}, \hat{S}, Y, \hat{t}, \hat{\delta}, \hat{\lambda} \rangle$ is a *sub-DEVS* of a DEVS $M = \langle X, S, Y, t, \delta, \lambda \rangle$ if

a) $\hat{X} \subset X$

b) $\hat{S} \subset S$

c) $\hat{t} = t|\hat{S}$

d) $\hat{\delta} = \delta|\hat{Q} \times (\hat{X} \cup \{\phi\})$ where $\hat{Q} = \{(s,e) \in Q: s \in \hat{S}\}$

e) $\hat{\lambda} = \lambda|\hat{Q}$

where a vertical bar designates restriction of a function domain. □

Some heuristic remarks concerning the intuitive operating conventions of DEVSs are warranted at this point.

The transition function δ describes a discrete transition structure which is essentially that of a sequential machine, while the time advance function t describes the continuous time component superimposed on it. Consequently the state of a DEVS has a discrete as well as a continuous flavor; its sequential state component changes discretely in time, while the clock component changes continuously in time. A change of the sequential state will be referred to as a *jump* of the system. A DEVS remains in a fixed sequential state s between jumps, whereas the clock e increases from 0 to ŧ(s), thus timing the elapsed time since the last jump to s. Sequential state transitions (jumps) take place from a full state $q = (s,e)$ as a result of either of the following events.

I) An "internal" event occurred due to the fact that the clock value e has reached the value ŧ(s).

If no external event has occurred at that very instant, the system will undergo an instantaneous transition to full state $(\delta_\phi(s),0)$. That is, a jump will take place according to δ_ϕ and the clock is reset to zero.

II) An external event $x \in X$ has occurred but no internal event is scheduled to take place at the same instant.

The system will undergo an instantaneous transition to full state $(\delta_M((s,e),x),0)$. That is, a jump will take place according to δ_M and the clock is reset to zero.

III) An internal event and an external event are scheduled to occur at the same instant.†

In this case the user should devise a tie-breaking rule that specifies the jump to be taken by the system, due to the two

†Unlike [Z1] Ch. IX Sec. 9.13, we do not assume that internal events have priority over external ones.

imminent events above. For example, one can have a δ_ϕ jump preempt a δ_M jump or vice versa. One can also have any combination of δ_ϕ and δ_M jumps ranging from simple composition of δ_ϕ and δ_M to any arbitrary function of δ_ϕ and δ_M.

For our purposes, it is convenient to choose a composition rule for a tie-breaking rule. This is the most natural rule for a wide variety of applications, and queuing theoretic ones in particular. It also enjoys the advantage of being robust with respect to the morphism concept to be defined later. This fact will allow us to disregard the special case of double scheduling in the impending study of morphic relations among deterministic DEVSs.

Simultaneous events and tie-breaking rules are vital in simulation of stochastic systems, especially when the time base has a minimal resolution. In theoretical applications simultaneous events typically occur with zero probability.

The mode of operation of DEVSs requires that all jumps are instantaneous and always reset the clock to zero, whereby the timing process starts all over again till the next jump. The mathematical operating conventions are embedded in the discrete event structures induced by a DEVS, to be discussed in the next section. Typically, the duration $t(s)$ that the system is allowed to stay in sequential state $s \in S$, will appear as a component of s.

Definition 1.1.3

A DEVS $M = \langle X,S,Y,t,\delta,\lambda \rangle$ is said to be in *explicit form* if S is a structured set and every $s \in S$ has the form $s = (\xi,r)$ such that $t(s) = r$.
□

Explicit form DEVSs are handy to work with, as the residual time to the next autonomous transition from full state $q = ((\xi,r),e)$ is $r - e$.

We now introduce the concept of state-DEVS and its behavioral frames in the spirit of Appendix A.

Definition 1.1.4

A *state-DEVS* $M = \langle X,S,\cdot,t,\delta,\cdot\rangle$ is a DEVS with unspecified output value set Y and output function λ. □

Definition 1.1.5

A *behavioral frame* of a state-DEVS $M = \langle X,S,\cdot,t,\delta,\cdot\rangle$ is a structure $\psi = \langle Y,\lambda\rangle$ where the symbols in the angular brackets have the same meaning and constraints as in Definition 1.1.1. □

We will regard a state-DEVS as a representative of the class of all DEVSs with the same underlying state structure. As a matter of fact we refer to it interchangeably as DEVS or state-DEVS whenever the context is clear.

Sequential states are classified as follows.

Definition 1.1.6

Let $M = \langle X,S,Y,t,\delta,\lambda\rangle$ be a DEVS. Let $s \in S$ be any sequential state. Then

a) s is called *transitory* if $t(s) = 0$

b) s is called *passive* if $t(s) = \infty$

c) s is called *regular* if $0 < t(s) \leq \infty$ □

A transitory state is an intermediate state which the system enters, and from which it departs instantly. Such states are extremely important in describing DEVS transitions under composition type tie-breaking rules that are incurred by simultaneous events. A passive state, on the other hand, can change only due to an external event.

Example 1.1.1

To illustrate how the DEVS concept may be used to describe real life descrete event systems, we now model a particular queuing history of a FIFO (first in first out) queue, with one server, where the behavioral frame is the stream of departing customers.

Let $C = \{c_i\}_{i=1}^{\infty}$ be a set of customer tags where c_i tags the i-th customer served. Let $\{s_n\}_{n=1}^{\infty}$ be the sequence of service times obtained by the customers, from the server. The modeling DEVS $M = \langle X,S,Y,t,\delta,\lambda \rangle$ is defined in explicit form as follows.

a) $X = \{1_{c_i} : c_i \in C\}$ where 1_{c_i} codes the arrival of customer c_i.

b) $S = \{(\Lambda,n,\infty) : n \in N\} \cup \{(\gamma,n,r) : \gamma \in C^+,\ n \in N,\ 0 \le r < s_n\}$

where Λ is the empty string, C^+ is the set of all finite nonempty strings over C, and N is the set of natural numbers.

c) $Y = \{0,1_{c_i} : c_i \in C\}$ where 1_{c_i} codes the departure of customer c_i and 0 codes a nondeparture.

d) $t : S \to (0,\infty]$ is defined by $t(\gamma,n,r) = r$

e) $\delta : Q \times (X \cup \{\phi\}) \to S$ is defined by e.1) $\delta_M(((\gamma,n,r),e),1_{c_i}) = \begin{cases} (c_i,n,s_n), & \text{if } \gamma = \Lambda \\ (c_i\gamma,n,r-e), & \text{if } \gamma \in C^+ \end{cases}$

e.2) $\delta_\phi(\gamma,n,r) = \begin{cases} (\Lambda,n+1,\infty), & \text{if len}(\gamma) = 1 \\ (\rho,n+1,s_{n+1}), & \text{if } \gamma = \rho c_j \text{ and len}(\gamma) > 1 \end{cases}$

where len(γ) is the length of the string γ.

f) $\lambda: Q \rightarrow Y$ is defined by

$$\lambda((\gamma,n,r),e) = \begin{cases} 1_{c_{n-1}}, & \text{if } e = 0,\ n > 1,\ r = \infty \text{ or } r = s_n \\ 0, & \text{otherwise} \end{cases}$$

To describe a queuing history, one chooses an "initial state" $s_0 = (\gamma,n,r)$ where

$$s_0 = \begin{cases} (\Lambda,1,\infty), & \text{if } \gamma = \Lambda \\ (c_\ell c_{\ell-1}\ldots c_1,1,s_1), & \text{if } \gamma \neq \Lambda \end{cases}$$

Note that external events model arrivals and internal events model service completions.

In any sequential state $s = (\gamma,n,r)$, γ is the line configuration, n is the index of the customer in service or to be served, and r is the residual service time. In particular $s = (\Lambda,n,\infty)$ is a passive state since an empty queue can have a jump only due to an arrival of a customer.

For double scheduling (simultaneous arrival and service completion) the tie-breaking rule is $\delta_M \circ \delta_\phi$. □

Other examples may be found in [Z1] Ch. IX Sec. 9.12.

Next, we introduce a formalism for describing discrete event networks composed of DEVS components.

Definition 1.1.7

A *DEVN* (*discrete event network*) specification is a structure

$$N = \left\langle D, \{M_\alpha\}_{\alpha \in D}, \{I_\alpha\}_{\alpha \in D}, \{Z_{\alpha,\beta}\}_{\substack{\alpha \in D \\ \beta \in I_\alpha}}, \{J_\alpha\}_{\alpha \in D} \right\rangle \text{ where}$$

D is a set of component indices called the *index set*.

$\{M_\alpha\}_{\alpha \in D}$ is a set of state-DEVSs called the *component set*.

$\{I_\alpha\}_{\alpha \in D}$ is a family of subsets of D that specify the components influenced by each component of the network. I_α is called the *influence set* of α.

$\{Z_{\alpha,\beta}\}_{\substack{\alpha \in D \\ \beta \in I_\alpha}}$ is a family of maps that determine the effect of a component on those components it influences in the network.

$Z_{\alpha,\beta}$ is called the *effect function* of α on β.

$\{J_\alpha\}_{\alpha \in D}$ is a family of functions that specify the jump taken by a component due to scheduling of an event or simultaneous events.

J_α is called the *jump function* of α.

The above are subject to the following restrictions:

a) each state-DEVS $M_\alpha = \langle X_\alpha, S_\alpha, \cdot, t_\alpha, \delta_\alpha, \cdot \rangle$, $\alpha \in D$, is in explicit form.

b) for any $\alpha \in D$ and $\beta \in I_\alpha$, $Z_{\alpha,\beta}$ is a partial† map $Z_{\alpha,\beta}: S_\alpha \rightarrow X_\beta$.

---

†a partial map is allowed to be undefined on a subset of its domain.

c) for any $\alpha \in D$, J_α is a function $J_\alpha : Q_\alpha \times 2^{\tilde{X}_\alpha} \rightarrow S_\alpha$ where $\tilde{X}_\alpha = X_\alpha \cup \{\phi_\alpha\}$ and $\phi_\alpha \notin X_\alpha$ codes an internal event in M_α.

Furthermore,[†] J_α is constrained by

$$J_\alpha(((\xi_\alpha, r_\alpha), e), E_\alpha) = \begin{cases} \delta_{\alpha,\phi}(\xi_\alpha, r_\alpha), & \text{if } E_\alpha = \{\phi_\alpha\} \\ \delta_{\alpha,M}(((\xi_\alpha, r_\alpha), e), x_\alpha), & \text{if } E_\alpha = \{x_\alpha\},\ x_\alpha \in X_\alpha \\ (\xi_\alpha, r_\alpha - e), & \text{if } E_\alpha = \Phi \\ (\xi'_\alpha, r'_\alpha), & \text{otherwise} \end{cases}$$

□

To describe the operation of a DEVN N, we associate with it a state-DEVS $M_N = \langle X_N, S_N, \cdot, t_N, \delta_N, \cdot \rangle$ defined by

$X_N = \underset{\alpha \in D}{\times} \tilde{X}_\alpha - \{\phi_\alpha\}_{\alpha \in D}$

$S_N = \underset{\alpha \in D}{\times} S_\alpha$

$t_N : S_N \rightarrow (0, \infty]$ is defined by $t_N(\{(\xi_\alpha, r_\alpha)\}_{\alpha \in D}) = \inf_{\alpha \in D}\{r_\alpha\}$

$\delta_N : Q_N \times X_N \rightarrow S_N$ is determined by the following procedure.

Take any $((s,e),\tilde{x}) = ((\{(\xi_\alpha, r_\alpha)\}_{\alpha \in D}, e), \{x_\alpha\}_{\alpha \in D}) \in Q_N \times X_N$

Define a family of event sets $\{E_\alpha\}_{\alpha \in D}$ as follows:

If an external event $\tilde{x} \in X_N$ is scheduled, then for any $\alpha \in D$

1. set $E_\alpha = \begin{cases} \{x_\alpha\}, & \text{if } x_\alpha \neq \phi_\alpha \\ \Phi, & \text{otherwise} \end{cases}$

[†] We point out that unlike [Z1] Ch. IX Sec. 9.17, the tie-breaking rule does not select a component to be activated but is embedded in $\{J_\alpha\}_{\alpha \in D}$.

If an internal event ϕ_α is scheduled in at least one component before the occurrence of an external event, from sequential state $s \in S_N$, then let $IMM(s) \triangleq \{\alpha \in D: r_\alpha = t_N(s)\}$ be the set of imminent components (i.e. those scheduled to undergo an autonomous jump simultaneously).

Next, for any $\alpha \in D$

2. whenever $\alpha \in IMM(s)$, put ϕ_α in E_α

   and

3. whenever $\alpha \in I_\beta$ for some $\beta \in IMM(s)$, put $Z_{\beta,\alpha}(s_\beta) \in X_\alpha$ in E_α.

4. Finally, compute $J_\alpha(((\xi_\alpha, r_\alpha), e), E_\alpha)$ for each $\alpha \in D$.

The transition function δ_N is then defined in terms of the jump functions $\{J_\alpha\}_{\alpha \in D}$ by

$$\delta_N((\{(\xi_\alpha, r_\alpha)\}_{\alpha \in D}, e), \{\tilde{x}_\alpha\}_{\alpha \in D}) = \{J_\alpha(((\xi_\alpha, r_\alpha), e), E_\alpha)\}_{\alpha \in D}$$

Notice that the symbol ϕ_α is interpreted in N as a nonevent, while in E_α it stands for an internal event in component α.

Definition 1.1.8

Let N be a DEVN and let M_N be the state-DEVS associated with it. A pair $\psi = \langle Y, \lambda \rangle$ is called a *behavioral frame* of N if it is a behavioral frame of M_N. □

Intuitively, a DEVN is composed of a set of DEVSs operating concurrently and interactively. The influence functions describe the topology of the network in terms of influence relations. The residual time to next jump is the infimum of the residual times of all components. A jump occurs whenever one or more components are activated by events in E_α. These may be external to components or internal to them. The external ones are due either to environment stimuli or to events generated by influencers of components. The internal events, symbolized in E_α by ϕ_α, trigger autonomous transitions prompted by a clock reading of $e = t_\alpha(\xi_\alpha, r_\alpha)$ in M_α. The jump function J_α takes all these events into account when determining the jump from state $((\xi_\alpha, r_\alpha), e)$, by means of some tie-breaking rule. In most cases, including queuing situations, J_α reduces to a composition rule that applies $\delta_{\alpha,M}$ and $\delta_{\alpha,\phi}$ sequentially in some order, according to the events in E_α. In this case E_α must be finite, in order that J_α be well-defined. This always happens in a DEVN with a finite index set D. In statistical-theoretic contexts multiple scheduling (i.e. $|E_\alpha| > 1)^\dagger$ occurs in most cases with probability zero anyway.

<u>Example 1.1.2</u>

To illustrate the use of a DEVN model consider a network of finitely many queues in tandem where each single queue is as in Example 1.1.1 (refer to Figure 1.1.1). The DEVN model is

$$N = \left\langle D, \{M_\alpha\}_{\alpha \in D}, \{I_\alpha\}_{\alpha \in D}, \{Z_{\alpha,\beta}\}_{\substack{\alpha \in D \\ \beta \in I_\alpha}}, \{J_\alpha\}_{\alpha \in D} \right\rangle$$

where

a) $D = \{1,2,\ldots m\}$

b) $M_\alpha = \langle X_\alpha, S_\alpha, \cdot, t_\alpha, \delta_\alpha, \cdot \rangle$, is the DEVS modeling the α-th queue.

$^\dagger |\cdot|$ is the cardinality symbol.

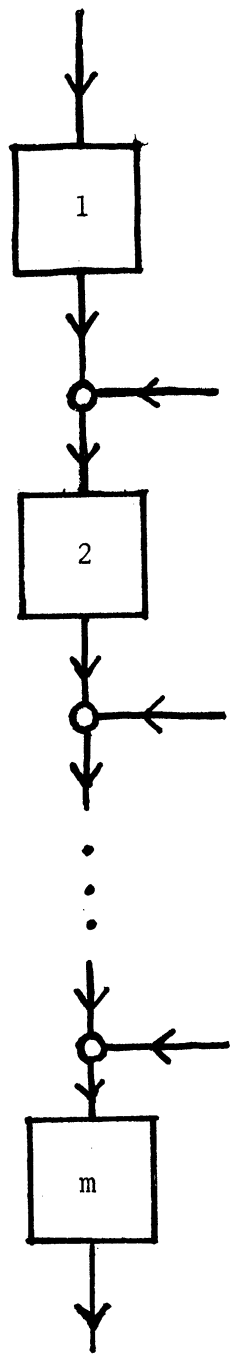

Figure 1.1.1: A Sequence of Queues in Tandem.

It is defined as in Example 1.1.1 except that $X_\alpha = \{1_{c_{\alpha,i}} : c_{\alpha,i} \in C_\alpha\}$, where $C_\alpha = (c_{\alpha,i})_{i=1}^{\infty}$ is the set of customers whose first service occurred in M_α, and $\{s_{\alpha,n}\}_{n=1}^{\infty}$ is the sequence of services awarded at M_α.

c) $$I_\alpha = \begin{cases} \{\alpha + 1\}, & \text{if } 1 \leq \alpha < m \\ \Phi, & \text{if } \alpha = m \end{cases}$$

d) $$Z_{\alpha,\beta}(\xi_\alpha, r_\alpha) = \begin{cases} 1_{c_{\beta,i}}, & \text{if } \xi_\alpha = (\gamma,n,r) \text{ where } \gamma = \rho c_{\beta,i} \\ \text{undefined, otherwise} \end{cases}$$

e) For each $\alpha \in D$, the event sets E_α have the form:

$E_\alpha = \{\phi_\alpha\}$ or $E_\alpha = \{1_{c_{\beta,i}}\}$ or $E_\alpha = \{\phi_\alpha, 1_{c_{\beta,i}}\}$, $\beta \leq \alpha$.

For any $1 \leq \alpha \leq m$

$$J_\alpha(((\xi_\alpha, r_\alpha), e), E_\alpha) = \begin{cases} \delta_{\alpha,\phi}(\xi_\alpha, r_\alpha), & \text{if } E_\alpha = \{\phi_\alpha\} \\ \delta_{\alpha,M}(((\xi_\alpha, r_\alpha), e), 1_{c_{\beta,i}}), & \text{if } E_\alpha = \{1_{c_{\beta,i}}\} \\ (\xi_\alpha, r_\alpha - e), & \text{if } E_\alpha = \Phi \\ \delta_{\alpha,M}((\delta_{\alpha,\phi}(\xi_\alpha, r_\alpha), 0), 1_{c_{\beta,i}}), & \text{if } E_\alpha = \{\phi_\alpha, 1_{c_{\beta,i}}\} \end{cases}$$

Notice that in our DEVN, internal events (denoted ϕ_α) represent service completions and subsequent departures. External events (denoted $1_{c_{\alpha,i}}$) represent customer arrivals. For $\alpha = 1$ these are arrivals from an external source only, while for $1 < \alpha \leq m$ the arrivals originate from an external source or from component $\alpha - 1$. The tie-breaking rule for multiply scheduled events in a component, is a composition whereby departures precede arrivals.

□

1.2 Discrete Event Structures Induced by a DEVS

In order to gain a precise understanding of the operating conventions of DEVSs and their behavior under complex input segments, we need to translate the DEVS structure into the iterative specification and the mathematical system induced by it. We follow the procedure in [Z1] Ch. IX Sec. 9.13, with minor changes.

In the following definitions N denotes the set of natural numbers.

Definition 1.2.1

The *extended autonomous transition function* of a DEVS $M = \langle X,S,Y,t,\delta,\lambda \rangle$ is a function $\overline{\delta}_\phi : S\times(N\cup\{0\}) \to S$ defined recursively by

$$\overline{\delta}_\phi(s,0) \stackrel{\Delta}{=} s$$

$$\overline{\delta}_\phi(s,n+1) \stackrel{\Delta}{=} \delta_\phi(\overline{\delta}_\phi(s,n)) \qquad \square$$

$\overline{\delta}_\phi(s,n)$ gives the sequential state reached autonomously from s after n jumps under a sufficiently long nonevent segment.

Definition 1.2.2

The *total time advance function* of a DEVS $M = \langle X,S,Y,t,\delta,\lambda \rangle$ is the function $\sigma : S\times(N\cup\{0\}) \to [0,\infty]$ defined by

$$\sigma(s,n) \stackrel{\Delta}{=} \begin{cases} 0 , & \text{if } n = 0 \\ \sum_{i=0}^{n-1} t(\overline{\delta}_\phi(s,i)) , & \text{if } n > 0 \end{cases}$$

$\square$

$\sigma(s,n)$ gives the total time it takes the system to evolve autonomously from state $(s,0)$ to state $(\overline{\delta}_\phi(s,n),0)$ i.e. the total time spanning n jumps from state $(s,0)$.

<u>Definition 1.2.3</u>

The *jump counter function* of a DEVS $M = \langle X,S,Y,t,\delta,\lambda\rangle$ is a function $m: Q\times[0,\infty] \to N \cup \{0\} \cup \{\infty\}$ defined by

$$m((s,e),\tau) \triangleq \sup\{n:\ \sigma(s,n) \leq e + \tau\}$$ □

$m((s,e),\tau)$ gives the number of jumps taken by the system when evolving autonomously from state (s,e) for τ time units.

We are now ready to define the iterative specification induced by a DEVS. The input segment generators will be functions ω of the form $\omega: (0,\tau] \to X \cup \{\phi\}$ such that either

a) $\omega = x_\tau$ where $x_\tau(t) \triangleq \begin{cases} \phi, & \text{if } 0 < t < \tau \\ x, & \text{if } t = \tau \end{cases}$

or

b) $\omega = \phi_\tau$ where $\phi_\tau(t) \overset{\Delta}{\equiv} \phi$, $t \in (0,\tau]$.

<u>Definition 1.2.4</u>

The *iterative specification induced by a DEVS* $M = \langle X,S,Y,t,\delta,\lambda\rangle$ is $G(M) = \langle T,X_G,\Omega_G,Q,Y,\delta_G,\lambda\rangle$ where

a) $T \triangleq [0,\infty)$

b) $X_G \triangleq X \cup \{\phi\}$

c) $\Omega_G \triangleq \Omega_X \cup \Omega_\phi$ where $\Omega_X \triangleq \{x_\tau:\ \tau > 0\}$ and $\Omega_\phi \triangleq \{\phi_\tau:\ \tau > 0\}$

d) $Q \triangleq \{(s,e):\ s \in S,\ 0 \leq e < t(s)\}$

e) $\delta_G: Q \times \Omega_G \rightarrow Q$ is defined recursively by

$\forall (s,e) \in Q$, $\forall x \in X$ and $\forall \tau > 0$

$$\text{e.1)} \quad \delta_G((s,e),\phi_\tau) \triangleq \begin{cases} (s, e+\tau), & \text{if } e+\tau < t(s) \\ (\overline{\delta}_\phi(s, m((s,e),\tau)), & \text{if } e+\tau = t(s) \\ \delta_G((\delta_\phi(s),0),\phi_{e+\tau-t(s)}), & \text{if } e+\tau > t(s) \end{cases}$$

$$\text{e.2)} \quad \delta_G((s,e),x_\tau) \triangleq (\delta_M(\delta_G((s,e),\phi_\tau),x),0)$$

An iterative specification G(M) thus derived is called a *discrete event iterative specification* (abbreviated *DEIS*). □

Comment 1.2.1

The symbol ϕ_0 denotes the empty function and is not a generator. However, for notational convenience we shall occasionally use in this chapter the notation $\delta_G((s,e),\phi_0) \triangleq (s,e)$. □

Since δ_G in Definition 1.2.4 has a recursive definition, we need to determine the conditions that render it a well-defined function. Clearly, this happens iff the DEIS G(M) has a finite number of jumps when started from any state (s,e) under any input segment.

Now, recall that a jump occurs either under an external event $x \in X$ according to δ_M, or autonomously according to δ_ϕ. Since input segments are composed of generators, they give rise to at most one jump according to δ_M. It remains to ensure that jumps according to δ_ϕ are also finitely many.

Formally we have

Definition 1.2.5

A DEVS $M = \langle X,S,Y,t,\delta,\lambda\rangle$ is called *legitimate* if

$\forall q \in Q,\ \forall\tau \geq 0,\ m(q,\tau) < \infty$. □

Theorem 1.2.1

A DEIS G(M) is well defined iff the inducing DEVS M is legitimate.

Proof

See [Z1] Ch. IX Sec. 9.11. □

For a legitimate DEVS M, the autonomous part of the transition function of G(M) may be specified explicitly as follows.

Lemma 1.2.1

If $G(M) = \langle T,X_G,\Omega_G,Q,Y,\delta_G,\lambda\rangle$ is a DEIS induced by a legitimate DEVS $M = \langle X,S,Y,t,\delta,\lambda\rangle$, then

$\forall(s,e) \in Q,\ \forall\tau > 0$,

$\delta_G((s,e),\phi_\tau) = (\overline{\delta}_\phi(s,m((s,e),\tau)),\ e + \tau - \sigma(s,m((s,e),\tau))$.

Proof

See [Z1] Ch. IX Sec. 9.13. □

Legitimacy of DEVSs is equivalently formulated as follows.

Lemma 1.2.2

A DEVS $M = \langle X,S,Y,t,\delta,\lambda\rangle$ is legitimate iff $\forall s \in S,\ \sigma(s,n) \xrightarrow[n\to\infty]{} \infty$.

Proof

See [Z1] Ch. IX Sec. 9.13. □

A special case of illegitimacy may be caused by the class of transitory sequential states (recall that $s \in S$ is transitory if $t(s) = 0$). A DEVS can never remain in a transitory state for a time interval of positive length, as the δ_ϕ function is invoked immediately on entering such states. We see that a legitimate DEVS M cannot have a sequential state s such that $\overline{\delta}_\phi(s,n)$ is a transitory sequential state for every $n \geq 0$.

Notice also, that transitory full states never appear in G(M) as the outcome of an application of its transition function, so that they can practically be eliminated from the state set of G(M).

In order to complete our hierarchy of discrete event systems, it only remains to introduce the mathematical system induced by a legitimate DEVS.

Definition 1.2.6

The *mathematical system induced by a legitimate DEVS* $M = \langle X,S,Y,t,\delta,\lambda \rangle$ is the time invariant system $S_{G(M)} = \langle T,X_G,\overline{\Omega}_G^+,Q,Y,\overline{\delta}_G,\lambda \rangle$ induced by the DEIS $G(M) = \langle T,X_G,\Omega_G,Q,Y,\delta_G,\lambda \rangle$ according to Theorem A.2.1 in Appendix A. A mathematical system thus derived is called a *discrete event mathematical system* (abbreviated *DEMS*). □

Our main interest in a DEMS $S_{G(M)}$ lies in the state and output trajectories that it engenders (see Definition A.1.5 in Appendix A). These concepts reflect on the mathematical operating conventions of discrete event systems.

Figure 1.2.1 depicts these conventions pictorially. It superimposes on the same time scale an input segment and the resulting state and output trajectories, in a DEMS $S_{G(M)}$. The full state trajectory is broken down into two component trajectories - the sequential state trajectory and the clock trajectory. The input segment is a pulse-like function whose spikes represent external events while the sets of constancy separating them correspond to nonevent periods. By definition there are only finitely many spikes in each finite time interval.

The definition of $\overline{\delta}_G$ in $S_{G(M)}$ implies that the full state trajectory is right-continuous due to Definition 1.2.1. In terms of metric spaces, the full state space metric is derived from those of its components, say the zero-one metric† on the sequential state space and the natural metric on the elapsed time space.

This means that at jump instants the full state of the system consists of the new sequential state and a zero clock reading. The sequential state trajectory is a right-continuous step function, while the elapsed time trajectory is a right-continuous jig-saw function ascending linearly at 45°. The output trajectory records some observable aspect of system behavior.

Notice that transitory states never appear in state trajectories at the DEMS level, because they have already been removed at the DEIS level.

---

†The zero-one metric d on a set X is defined by

$$\forall x,y \in X \ , \quad d(x,y) = \begin{cases} 0, & \text{if } x = y \\ 1, & \text{if } x \neq y \end{cases}$$

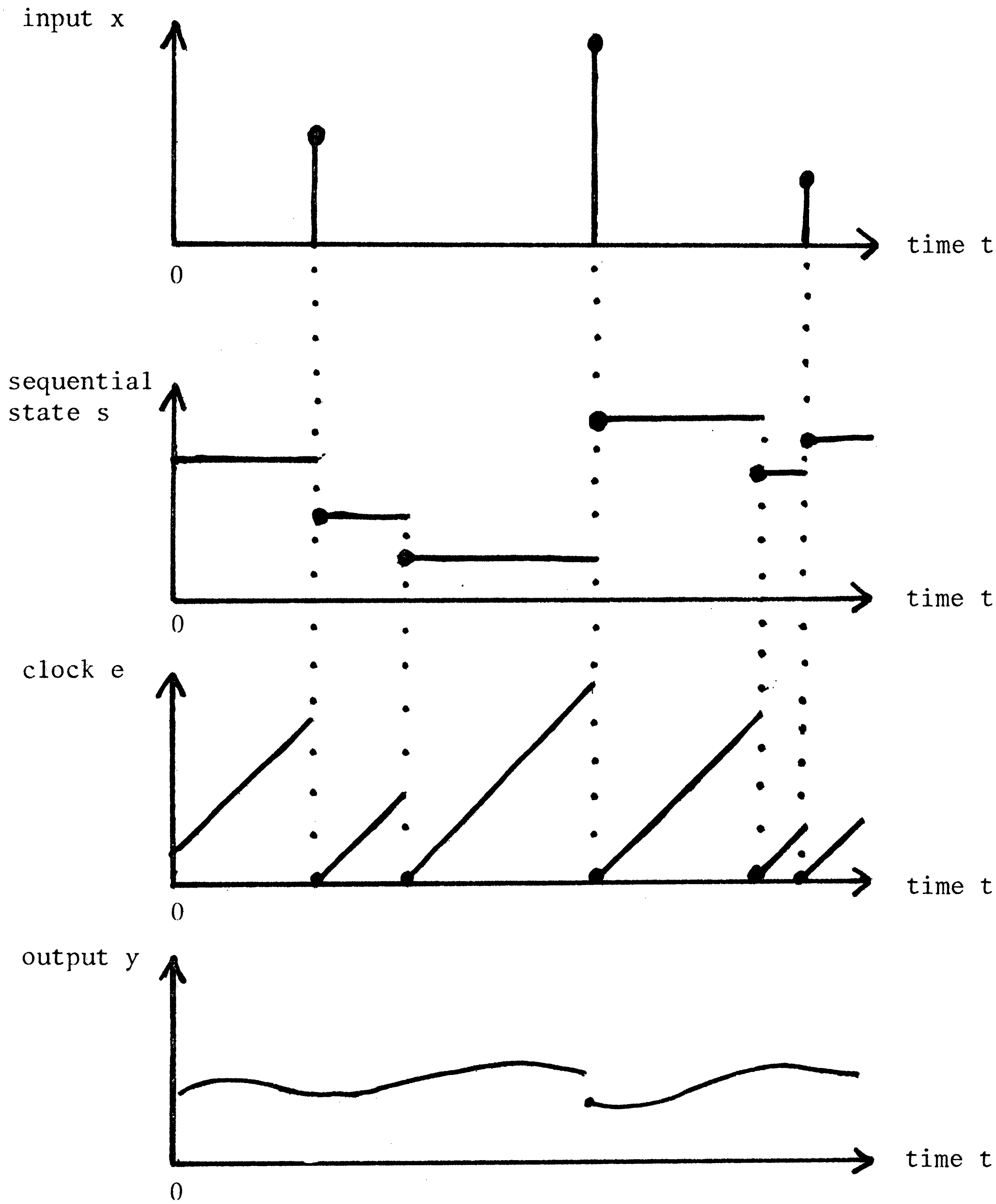

Figure 1.2.1: A Typical Input Segment and the Resulting State and Output Trajectories in a DEMS.

Although transitory states provide a means of describing composition type tie-breaking rules, it is often desirable to deal with DEVSs without such states.

Definition 1.2.7

A DEVS $M = \langle X,S,Y,t,\delta,\lambda\rangle$ is called *regular* if M is legitimate and every $s \in S$ is a regular sequential state. □

It is easy to see that each legitimate DEVS gives rise to a regular one with the same induced DEIS and DEMS.

Henceforth, we shall deal only with regular discrete event systems i.e. with those paradigms $M \mapsto G(M) \mapsto S_{G(M)}$ in which M is a regular DEVS.

In the forthcoming treatment, we shall usually refer to DEVSs as specifying a discrete event system. However, all related concepts in terms of the induced DEISs and DEMSs, and especially the functions δ_G and $\overline{\delta}_G$, will be used freely in the discussion, as if belonging to a DEVS rather than to its induced DEIS or DEMS. The tie-breaking rule adopted from now on for doubly scheduled events is $\delta_M \circ \delta_\phi$.

1.3 Input Matching DEIS Morphisms

In this section we define and investigate a class of DEIS morphisms - the so-called input matching DEIS morphisms. Working our way up from the DEIS level to the DEVS level, our eventual goal will be to derive a DEVS morphism (in the next section), by adding a level of detail to the DEIS morphism concept via input matching DEIS morphisms.

Throughout this chapter, the following notation will be adopted. Unless otherwise specified, a reference to M means a DEVS $M = \langle X,S,Y,t,\delta,\lambda \rangle$ and a reference to G(M) means the DEIS $G(M) = \langle T,X_G,\Omega_G,Q,Y,\delta_G,\lambda \rangle$ induced by M. A reference to M′ and G(M′) refers to a DEVS $M' = \langle X',S',Y',t',\delta',\lambda' \rangle$ and the DEIS $G(M') = \langle T,X'_G,\Omega'_G,Q',Y',\delta'_G,\lambda' \rangle$ induced by it respectively, and similarly for M″ and G(M″), M* and G(M*), $\hat{M}$ and $G(\hat{M})$.

Whenever f is a morphism from structure S to structure S′, then S and S′ are referred to as the *morphic preimage* and the *morphic image* respectively, under f.

Definition 1.3.1

Let (g,h,k) be a specification morphism (see Definition A.2.3 in Appendix A) from G(M) to G(M′).

Then (g,h,k) is called an *input matching DEIS morphism* (abbreviated *IM-DEIS* morphism), if there is a function $g_e : X'_G \rightarrow X_G$ such that

a) $\forall x'_\tau \in \Omega'_X \, , \; g(x'_\tau) = g_e(x')_\tau$

b) $\forall \phi_\tau \in \Omega_\phi \, , \; g(\phi_\tau) = \phi_\tau$

In this case we say that g *matches inputs* via g_e and that ω' and $g(\omega')$ are *matching inputs*. □

Thus in a IM-DEIS morphism (g,h,k) from G(M) to G(M′), the function g preserves generators and length of generators.

Lemma 1.3.1

In a IM-DEIS morphism (g,h,k) from G(M) to G(M′), the function h satisfies

a) $\forall q \in \overline{Q}$, $\forall x' \in X'$, $\forall \tau > 0$

a.1) $h(\delta_G(q,\phi_\tau)) = \delta'_G(h(q),\phi_\tau)$

a.2) $h(\delta_G(q,g_e(x')_\tau)) = \delta'_G(h(q),x'_\tau)$

Proof

Follows immediately from Definitions A.2.3 and 1.3.1. □

Next expand G(M) and G(M′) into their respective DEMSs $S_{G(M)}$ and $S_{G(M')}$.

Definition 1.3.2

If (g,h,k) is a IM-DEIS morphism form G(M) to G(M′), then $(\tilde{g},h,k)$ is a *IM-DEMS morphism* from $S_{G(M)}$ to $S_{G(M')}$ provided $\tilde{g}$ satisfies

$$\tilde{g}(\omega'_1 \odot \omega'_2 \odot \ldots \odot \omega'_n) = g(\omega'_1) \odot g(\omega'_2) \odot \ldots \odot g(\omega'_n) =$$

$$= \begin{cases} g_e(x'_1)_{\tau_1} \odot g_e(x'_2)_{\tau_2} \odot \ldots \odot g_e(x'_n)_{\tau_n}, & \text{if } \omega'_n = (x'_n)_{\tau_n} \\ g_e(x'_1)_{\tau_1} \odot g_e(x'_2)_{\tau_2} \odot \ldots \odot \phi_{\tau_n}, & \text{if } \omega'_n = \phi_{\tau_n} \end{cases}$$

for any $\omega' \in \Omega_G'^{+}$ with m.l.s decomposition $\omega' = \omega_1' \odot \omega_2' \odot \ldots \odot \omega_n'$. □

Thus, in IM-DEIS morphisms $(\tilde{g},h,k)$, the input segment is a pulse train and $\tilde{g}$ merely relables the pulses via g_e.

Definition 1.3.3

An input matching DEIS state-morphism (abbreviated *IM-DEIS state-morphism*) from a DEIS G(M) to a DEIS G(M′) is a pair (g,h) subject to the same restrictions as in Definition 1.3.1. □

The impending discussion of various morphisms will always extend to state-morphisms, as the definition of the latter is properly contained in that of the former. Consequently, we state now once and for all, that all definitions and theorems concerning various morphisms will henceforth extend to their respective state-morphisms.

We now proceed to put an algebra-like structure on the class of IM-DEIS morphisms. The operations considered are composition and inversion.

Theorem 1.3.1

Let (g,h,k) be a IM-DEIS morphism from G(M) to G(M′) and let (g′,h′,k′) be a IM-DEIS morphism from G(M′) to G(M″).

Then there is a IM-DEIS morphism (g″,h″,k″) from G(M) to G(M″).

Proof

Define $(g'',h'',k'') \triangleq (g \circ g', h' \circ h, k' \circ k)$ where the circle operation denotes function composition. Then

a) $g'':\Omega''_G \rightarrow \Omega_G$ matches inputs via

$g''_e : X'' \rightarrow X$, where $g''_e = g_e \circ g'_e$.

b) $h'':h^{-1}(\overline{Q}') \rightarrow Q''$ is onto Q'', since $h|h^{-1}(\overline{Q}')$ is onto $\overline{Q}'$ and $h':\overline{Q}' \rightarrow Q''$ is onto Q''. Clearly $h^{-1}(\overline{Q}') \subset \overline{Q} \subset Q$.

c) $k'':Y \rightarrow Y''$ is onto, since $k:Y \rightarrow Y'$ is onto Y' and $k':Y' \rightarrow Y''$ is onto Y''.

d) $\forall (s,e) \in h^{-1}(\overline{Q}')$, $\forall x'' \in X''$, $\forall \tau > 0$ (see Lemma 1.3.1)

d.1) $h''(\delta_G((s,e),\phi_\tau)) = h'(h(\delta_G(s,e),\phi_\tau))) =$

$h'(\delta'_G(h(s,e),\phi_\tau)) = \delta''_G(h'(h(s,e)),\phi_\tau) = \delta''_G(h''(s,e),\phi_\tau)$

d.2) $h''(\delta_G((s,e),g''_e(x'')_\tau)) = h'(h(\delta_G((s,e),g_e(g'_e(x''))_\tau))) =$

$h'(\delta'_G(h(s,e),g'_e(x'')_\tau)) = \delta''_G(h'(h(s,e)),x''_\tau) =$

$\delta''_G(h''(s,e),x''_\tau)$.

e) $\forall (s,e) \in h^{-1}(\overline{Q}')$

$k''(\lambda(s,e)) = k'(k(\lambda(s,e))) =$

$k'(\lambda'(h(s,e))) = \lambda''(h'(h(s,e))) = \lambda''(h''(s,e))$. □

Theorem 1.3.1 asserts that the IM-DEIS morphism relation is transitive in the sense that the IM-DEIS relation is preserved under composition.

Theorem 1.3.2

Let (g,h,k) be a IM-DEIS morphism from G(M) to G(M´) such that the maps g, h and k are all bijective and $\overline{Q} = Q$.

Then there is a IM-DEIS morphism (g´,h´,k´) from G(M´) to G(M).

Proof

Define $(g',h',k') \triangleq (g^{-1},h^{-1},k^{-1})$ where all inverse maps exist by assumption. Then

a) $g':X \to X'$ matches inputs via $g'_e = g_e^{-1}$

b) $h':Q' \to Q$ is surjective

c) $k':Y' \to Y$ is surjective

d) $\forall (s',e') \in Q'$, $\forall x \in X$, $\forall \tau > 0$ (see Lemma 1.3.1)

d.1) $h'(\delta'_G((s',e'),\phi_\tau)) = h'(\delta'_G(h(h'(s',e')),\phi_\tau)) =$

$h'(h(\delta_G(h'(s',e'),\phi_\tau))) = \delta_G(h'(s',e'),\phi_\tau)$.

d.2) $h'(\delta'_G((s',e'),g'_e(x)_\tau)) = h'(\delta'_G(h(h'(s',e')),g'_e(x)_\tau)) =$

$h'(h(\delta_G(h'(s',e'),g_e(g'_e(x))_\tau))) = \delta_G(h'(s',e'),x_\tau)$.

e) $\forall (s',e') \in Q'$

$k'(\lambda'(s',e')) = k'(\lambda'(h(h'(s',e')))) =$

$k'(k(\lambda(h'(s',e')))) = \lambda(h'(s',e'))$. □

Notice that an invertible IM-DEIS morphism from G(M) to G(M′) merely provides a relabeling of G(M) in terms of G(M′) and vice versa. This relabeling is consistent vis-a-vis full states transitions and output values.

We can now formally define

Definition 1.3.4

Let (g,h,k) be a IM-DEIS morphism from G(M) to G(M′) and let (g′,h′,k′) be a IM-DEIS morphism from G(M′) to G(M″).

The *composition* of (g,h,k) and (g′,h′,k′) is a IM-DEIS morphism from G(M) to G(M″) denoted by (g,h,k)o(g′,h′,k′) and defined by

$(g,h,k)\circ(g',h',k') \triangleq (g\circ g',h'\circ h,k'\circ k)$. □

Definition 1.3.5

Let (g,h,k) be a IM-DEIS morphism from $G(M)$ to $G(M')$.

The *inverse* of (g,h,k) is a IM-DEIS morphism from $G(M')$ to $G(M)$ denoted by $(g,h,k)^{-1}$ and defined, whenever g^{-1}, h^{-1} and k^{-1} exist, by $(g,h,k)^{-1} \triangleq (g^{-1},h^{-1},k^{-1})$. □

It is not difficult to see that this algebra-like structure can be defined analogously on system morphisms at any structural level.

In general, the transitivity of a morphism relation on a class of systems, imposes an obvious hierarchy which is almost a partial order. The *invertibility* relation among systems (i.e. the existence of an invertible morphism that connects them) is easily seen to be an equivalence relation. Thus it partitions the underlying class of systems into equivalence classes. This remark holds true for DEISs and IM-DEIS morphisms in particular.

We now give a standard specialization of IM-DEIS morphisms (cf. Appendix A).

Definition 1.3.6

$G(M) = \langle T,X_G,\Omega_G,Q,Y,\delta_G,\lambda\rangle$ and $G(M') = \langle T,X'_G,\Omega_G,Q',Y',\delta'_G,\lambda'\rangle$ are called *compatible* if

a) $X_G = X'_G$

b) $Y = Y'$

□

In the following definitions, i denotes the identity map.

Definition 1.3.7

Let G(M) and G(M′) be compatible DEISs, and let (i,h,i) be a IM-DEIS morphism from G(M) to G(M′).

a) If $\overline{Q} = Q$ in G(M), then (i,h,i) is called a *IM-DEIS homomorphism* from G(M) to G(M′).

b) If (i,h,i) is a IM-DEIS homomorphism from G(M) to G(M′) and in addition h is bijective, then (i,h,i) is called a *IM-DEIS isomorphism* from G(M) to G(M′). □

Lemma 1.3.2

The IM-DEIS homomorphism relation is preserved under composition. The IM-DEIS isomorphism relation is preserved under inversion.

Proof

Follows immediately from Definition 1.3.7 and Theorems 1.3.1 and 1.3.2. □

IM-morphisms at the DEMS level are analogously defined. In particular

Definition 1.3.8

Let $S_{G(M)}$ and $S_{G(M')}$ be DEMSs.

A trajectory morphism (MATCH,h,k) from TRAJ(q,ω) to TRAJ(q′,ω′) (see Definition A.1.8 in Appendix A) is called a *IM-trajectory* morphism if

a) ω and ω' are matching input segments.

b) MATCH = i. □

1.4 Transitional Covering

This section develops a DEVS morphism concept, the so-called transition covering DEVS morphism, based on the so-called transitional covering relation. The essence of this relation is the ability to perform a partial matching of sequential state jumps in two DEVSs.

Our preoccupation with jumps is motivated by their fundamental importance in discrete event systems. In discrete event modeling situations, sequential state jumps constitute system responses to significant events during system evolution. In contrast, during the time intervals separating jumps, the system is considered quiescent, since its state remains fixed throughout such intervals.

We start by formalizing the transitional covering relation concept.

Definition 1.4.1

Let (g,h,k) be a IM-DEIS morphism from G(M) to G(M').

We say that G(M) is a *transitional covering* of G(M') (or simply, that G(M) *covers* G(M')) if

a) $h(s,e) = (s',0) \Rightarrow e = 0$

In this case (g,h,k) is called a *transition covering DEIS morphism* (abbreviated *TC-DEIS morphism*). The transitional covering relation is denoted $G(M) \sqsupset G(M')$. □

Thus $G(M) \sqsupset G(M')$, if whenever started from h-matching states, under g-matching inputs, a jump occurs in G(M´) only if a jump occurs in G(M) at the same instant.

Consequently, every jump in G(M´) can be matched in time by a jump in G(M), but not necessarily vice versa. In particular this means that all sequential states of M´ can be matched by sequential states in M. This observation motivates the following definition.

Definition 1.4.2

A *transition covering DEVS morphism* (abbreviated *TC-DEVS morphism*) from M to M´ is a quadruple (g, L, h, k), subject to the following restrictions:

a) g is a function $g: X' \to X$ called the *external event encoding function.*

b) L is a function $L: \hat{S} \to N \cup \{0\}$ called the *transition counting function* where $\hat{S} \subset S$ and N is the set of natural numbers.

c) h is a surjective function $h: \hat{S} \to S'$ called the *sequential state decoding function.*

d) k is a surjective function $k: Y \to Y'$ called the *output decoding function.*

e) Let $\hat{Q} \triangleq \{(s,e) \in Q:\ s = \overline{\delta}_\phi(\hat{s}, m((\hat{s},0),\tau)),\ e = \tau - \sigma(\hat{s}, m((\hat{s},0),\tau))$ for some $\hat{s} \in \hat{S}$ and $0 \leq \tau < t'(h(\hat{s}))\}$. If $(s,e) \in \hat{Q}$ is associated with $(\hat{s}_1, \tau_1)$ and $(\hat{s}_2, \tau_2)$, then $h(\hat{s}_1) = h(\hat{s}_2)$ and $\tau_1 = \tau_2$.

f) For any $\hat{s} \in \hat{S}$

f.1) $t'(h(\hat{s})) = \sum_{i=0}^{L(\hat{s})} t(\overline{\delta}_\phi(\hat{s}, i))$

f.2) $\overline{\delta}_\phi(\hat{s}, L(\hat{s})+1) \in \hat{S}$

f.3) $h(\overline{\delta}_\phi(\hat{s}, L(\hat{s})+1)) = \delta'_\phi(h(\hat{s}))$

g) For any $\hat{s} \in \hat{S}$, $0 \leq \tau < \text{ŧ}'(\text{ћ}(\hat{s}))$, let $s \triangleq \overline{\delta}_\phi(\hat{s}, m((\hat{s},0),\tau))$ and

$e \triangleq \tau - \sigma(\hat{s}, m((\hat{s},0),\tau))$. Then for any $x' \in X'$

g.1) $\delta_M((s,e), g(x')) \in \hat{S}$

g.2) $\text{ћ}(\delta_M((s,e), g(x'))) = \delta'_M((\text{ћ}(\hat{s}),\tau), x')$

g.3) $k(\lambda(s,e)) = \lambda'(\text{ћ}(\hat{s}),\tau)$

In this case, we say that M is a *transitional covering* of M′ (or simply that M *covers* M′), and denote $M \sqsupset M'$. □

We need of course to show that this terminology is consistent. To do this, we will need

Lemma 1.4.1

For elements of $\hat{Q}$ in Definition 1.4.2, the representation

a) $(s,e) = (\overline{\delta}_\phi(\hat{s}, m((\hat{s},0),\tau)),\ \tau - \sigma(\hat{s}, m((\hat{s},0),\tau)))$

is equivalent to the representation

b) $(s,e) = \delta_G((\hat{s},0), \phi_\tau)$

Proof

Follows immediately from Lemma 1.2.1. □

We now show that TC-DEVS morphisms induce TC-DEIS morphisms in a natural way.

Theorem 1.4.1

If $M \sqsupset M'$ via a TC-DEVS morphism $(g, L, \text{ћ}, k)$, then $G(M) \sqsupset G(M')$ via some TC-DEIS morphism (g,h,k).

Proof

Define g to be input matching by setting $g_e \triangleq \mathfrak{g}$. Next define $\overline{Q}$ to be the set $\hat{Q}$ of Definition 1.4.2. In view of Lemma 1.4.1, every $q \in \overline{Q}$ has the form

(1) $q = (s,e) = \delta_G((\hat{s},0),\phi_\tau)$ for some $\hat{s} \in \hat{S}$ and $0 \leq$

Now, define $h:\overline{Q} \to Q'$ by

(2) $h(s,e) = h(\delta_G(\hat{s},0),\phi_\tau) \triangleq (\hbar(\hat{s}),\tau) \in Q'$.

h is well-defined due to e) in Definition 1.4.2. It is surjective since $\hbar$ is surjective and since $0 \leq \tau < \mathfrak{t}'(\hbar(\hat{s}))$.

Finally define $k \triangleq \mathfrak{k}$ from Y onto Y'.

It follows from (2) that

(3) $h(s,e) = (s',0) \Rightarrow (s,e) = (\hat{s},0) \Rightarrow e = 0$.

Thus h is transition covering.

Now, for any $\hat{s} \in \hat{S}$ and $0 \leq \tau < \mathfrak{t}'(h(\hat{s}))$

(4) $h(\delta_G((\hat{s},0),\phi_\tau)) = (\hbar(\hat{s}),\tau) =$

$\delta_G'((\hbar(\hat{s}),0),\phi_\tau) = \delta_G'(h(\hat{s},0),\phi_\tau)$

due to (1) and (2).

For any $\hat{s} \in \hat{S}$ and $\tau = \mathfrak{t}'(\hbar(\hat{s}))$

(5) $h(\delta_G((\hat{s},0),\phi_\tau)) = h(\overline{\delta}_\phi(\hat{s},L(\hat{s})+1),0) =$

$(\hbar(\overline{\delta}_\phi(\hat{s},L(\hat{s})+1)),0) = (\delta_\phi'(\hbar(\hat{s})),0) =$

$\delta_G'((\hbar(\hat{s}),0),\phi_\tau) = \delta_G'(h(\hat{s},0),\phi_\tau)$

by f.1), f.2) and f.3) in Definition 1.4.2, (1) and (2).

In view of (1), (4) and (5), we conclude by induction on $m((\hat{s},0),\tau)$ that

(6) $h(\delta_G((s,e),\phi_\tau)) = \delta_G'(h(s,e),\phi_\tau)$, $\forall (s,e) \in \overline{Q}$, $\forall \phi_\tau \in \Omega_\phi'$

due to the composition property of δ_G and δ_G'.

Next, for any $\hat{s} \in \hat{S}$, $0 \leq \tau < \bar{t}'(h(\hat{s}))$ and $x' \in X'$

(7) $h(\delta_G((\hat{s},0),g(x'_\tau))) = h(\delta_G((\hat{s},0),\bar{g}(x')_\tau)) =$

$h(\delta_M(\delta_G((\hat{s},0),\phi_\tau),\bar{g}(x')),0) = (\bar{h}(\delta_M(\delta_G((\hat{s},0),\phi_\tau),\bar{g}(x'))),0) =$

$(\delta'_M((\bar{h}(\hat{s}),\tau),x'),0) = (\delta'_M(\delta'_G((\bar{h}(\hat{s}),0),\phi_\tau),x'),0) =$

$\delta'_G((\bar{h}(\hat{s}),0),x'_\tau) = \delta'_G(h(\hat{s},0),x'_\tau)$

by the definitions of g and h above, the definitions of δ_G and δ'_G (see Definition 1.2.4), Lemma 1.4.1, g.1) and g.2) in Definition 1.4.2, (1) and (2).

In view of (1) and (7) we conclude by induction on $m((\hat{s},0),\tau)$ that

(8) $h(\delta_G((s,e),g(x'_\tau))) = \delta'_G(h(s,e),x'_\tau), \quad \forall (s,e) \in \overline{Q}, \quad \forall x'_\tau \in \Omega'_X$

again due to the composition property of δ_G and δ'_G.

Finally, for any $(s,e) \in \overline{Q}$

(9) $k(\lambda(s,e)) = \bar{k}(\lambda(\delta_G((\hat{s},0),\phi_\tau))) =$

$\lambda'(\bar{h}(\hat{s}),\tau) = \lambda'(h(s,e))$

by (1) and (2), Lemma 1.4.1 and g.3) in Definition 1.4.2.

We conclude from Definition 1.4.1 that (g,h,k) is a TC-DEIS morphism as required. □

Next, we prove that TC-DEIS morphisms induce TC-DEVS morphisms in a natural way.

Theorem 1.4.2

Let $G(M) \sqsupset G(M')$ via a TC-DEIS morphism (g,h,k). Then $M \sqsupset M'$ via some TC-DEVS morphism $(\bar{g},L,\bar{h},\bar{k})$.

Proof

Consider the set $\hat{S} \stackrel{\Delta}{=} \{s \in S\colon (s,0) \in \overline{Q}$ and $h(s,0) = (s',0)\}$.

Decompose h into $h = (h_1,h_2)$ and define $\hbar\colon \hat{S} \to S'$ by

(1) $\hbar(\hat{s}) \stackrel{\Delta}{=} h_1(\hat{s},0)$.

Next, let $g \stackrel{\Delta}{=} g_e$ and $k \stackrel{\Delta}{=} k$.

Let (s,e) be in $\hat{Q}$ of e) in Definition 1.4.2.

Suppose (s,e) is associated with $(\hat{s}_1,\tau_1)$ and $(\hat{s}_2,\tau_2)$. Then

(2) $\delta_G((\hat{s}_1,0),\phi_{\tau_1}) = (s,e) = \delta_G((\hat{s}_2,0),\phi_{\tau_2}) \Rightarrow h(\delta_G((\hat{s}_1,0),\phi_{\tau_1})) =$

$h(\delta_G((\hat{s}_2,0),\phi_{\tau_2})) \Rightarrow \delta'_G(h(\hat{s}_1,0),\phi_{\tau_1}) = \delta'_G(h((\hat{s}_2,0),\phi_{\tau_2})) \Rightarrow$

$\Rightarrow \delta'_G((\hbar(\hat{s}_1),0),\phi_{\tau_1}) = \delta'_G((\hbar(\hat{s}_2),0),\phi_{\tau_2}) \Rightarrow$

$\Rightarrow (\hbar(\hat{s}_1),\tau_1) = (\hbar(\hat{s}_2),\tau_2)$

due to the definitions of $\hat{S}$, $\hbar$ and $\hat{Q}$. Hence

(3) $\hbar(\hat{s}_1) = \hbar(\hat{s}_2)$ and $\tau_1 = \tau_2$

Let $\hat{s} \in \hat{S}$. Then there is $(s',0) \in Q'$ such that $h(\hat{s},0) = (s',0)$.

Now,

(4) $\delta'_G((s',0),\phi_{t'(s')}) = \delta'_G(h(\hat{s},0),\phi_{t'(s')}) =$

$h(\delta_G((\hat{s},0),\phi_{t'(s')})$

On the other hand

(5) $\delta'_G((s',0),\phi_{t'(s')}) = (\delta'_\phi(s'),0)$

Hence, (4) and (5) imply

(6) $h(\delta_G((\hat{s},0),\phi_{t'(s')})) = (\delta'_\phi(s'),0)$

By the transitional covering property, we deduce from (6) that

$\delta_G((\hat{s},0),\phi_{t'(s')})$ has the form

(7) $\delta_G((\hat{s},0),\phi_{\not t'(s')}) = (\overline{\delta}_\phi(\hat{s},L+1),0)$ for some $L = L(\hat{s}) \geq 0$

Thus, (6) and (7) imply that during the interval $(0,\not t'(s')]$ we had one autonomous transition in $G(M')$ from state $(s',0)$ to state $(\delta_\phi'(s'),0)$, while in $G(M)$ we had $L+1$ transitions from state $(\hat{s},0)$ to state $(\overline{\delta}_\phi(\hat{s},L+1),0)$ during the same time interval. In view of (7) we can define a surjective map $L:\hat{S} \twoheadrightarrow N \cup \{0\}$ such that

$$(8) \quad \not t'(\hbar(\hat{s})) = \sum_{i=0}^{L(\hat{s})} \not t(\overline{\delta}_\phi(\hat{s},i)), \quad \forall \hat{s} \in \hat{S}$$

Now, for any $\hat{s} \in \hat{S}$

$$(9) \quad h(\overline{\delta}_\phi(\hat{s},L(\hat{s})+1),0) = h(\delta_G((\hat{s},0),\phi_{\not t'(\hbar(\hat{s}))}) =$$

$$\delta_G'(h(\hat{s},0),\phi_{\not t'(\hbar(\hat{s}))}) = \delta_G'((\hbar(\hat{s}),0),\phi_{\not t'(\hbar(\hat{s}))}) =$$

$$(\delta_\phi'(\hbar(\hat{s})),0)$$

due to (8), and the definitions of $\hat{S}$ and $\hbar$.

Thus, (9) shows that

$$(10) \quad \overline{\delta}_\phi(\hat{s},L(\hat{s})+1) \in \hat{S}$$

by definition of $\hat{S}$. Moreover, from (9) we deduce

$$(11) \quad \hbar(\overline{\delta}_\phi(\hat{s},L(\hat{s})+1)) = \delta_\phi'(\hbar(\hat{s}))$$

by definition of $\hbar$.

Next, for any $\hat{s} \in \hat{S}$ and $0 \leq \tau < \not t'(\hbar(\hat{s}))$, let $s \triangleq \overline{\delta}_\phi(\hat{s},m((\hat{s},0),\tau))$ and $e \triangleq \tau - \sigma(\hat{s},m((\hat{s},0),\tau))$. Then for any $x' \in X'$

$$(12) \quad h(\delta_M((s,e),g(x')),0) = h(\delta_M(\delta_G((\hat{s},0),\phi_\tau),g(x'))) =$$

$$h(\delta_G((\hat{s},0),g(x')_\tau)) = h(\delta_G((\hat{s},0),g(x_\tau'))) =$$

$$\delta_G'(h(\hat{s},0),x_\tau') = (\delta_M'(\delta_G'(h(\hat{s},0),\phi_\tau),x'),0) =$$

$$(\delta_M'(\delta_G'((\hbar(\hat{s}),0),\phi_\tau),x'),0) = (\delta_M'((\hbar(\hat{s}),\tau),x'),0)$$

due to Lemma 1.4.1, by definitions of δ_G and δ_G' (see Definition 1.2.4), and by definition of $\not{g}$, $\not{h}$ and $\hat{S}$.

Thus, (12) shows that

(13) $\delta_M((s,e),\not{g}(x')) \in \hat{S}$

by definition of $\hat{S}$. Moreover, from (12) we deduce that

(14) $\not{h}(\delta_M((s,e),\not{g}(x'))) = \delta_M'((\not{h}(\hat{s}),\tau),x')$

by definition of $\not{h}$.

Finally, by Lemma 1.4.1 and the definition of $\not{h}$ we have

(15) $\not{k}(\lambda(s,e)) = k(\lambda(s,e)) =$

$\lambda'(h(s,e)) = \lambda'(h(\delta_G(\hat{s},0),\phi_\tau)) =$

$\lambda'(\delta_G'(h(\hat{s},0),\phi_\tau)) = \lambda'(h(\hat{s},0),\tau) =$

$\lambda'(\not{h}(\hat{s}),\tau)$

We conclude from Definition 1.4.2 that $(\not{g},L,\not{h},\not{k})$ is a TC-DEVS morphism as required. □

Corollary 1.4.1

(g,h,k) is a TC-DEIS morphism from $G(M)$ to $G(M')$ iff $(\not{g},L,\not{h},\not{k})$ is a TC-DEVS morphism from M to M'.

Moreover, in this case $\not{g} = g_e$, $\not{h} = h_1|\hat{S}\times\{0\}$ and $\not{k} = k$.

Furthermore, $h_1(s,e) \equiv \not{h}(\hat{s})$ whenever $(s,e) = \delta_G((\hat{s},0),\phi_\tau) \in \hat{Q}$. □

Theorem 1.4.2 shows that the essence of a TC-DEVS morphism from M to M' is the ability to define a IM-DEIS morphism (g,h,k) from $G(M)$ to $G(M')$ such that $h = (h_1,h_2)$ satisfies

$h_1(s,e) \equiv \not{h}(s), \ \forall 0 \le e < \not{t}(s)$

for some map $\not{h}$ on $\overline{S} \triangleq \{s\colon (s,e) \in \overline{Q} \text{ for some } e\}$.

In other words $\bar{h}$ is definable whenever $h_1(s,e)$ does not depend on e. This is, of course, possible iff all the jumps of G(M´) can be matched in time with jumps of G(M), i.e. iff (g,h,k) is a TC-DEIS morphism.

We now show that morphisms of the transitional covering type are transitive.

<u>Theorem 1.4.3</u>

If (g,h,k) is a TC-DEIS morphism from G(M) to G(M´) and (g´,h´,k´) is a TC-DEIS morphism from G(M´) to G(M″), then $(g'',h'',k'') \triangleq (g,h,k)\circ(g',h',k')$ is a TC-DEIS morphism from G(M) to G(M″).

<u>Proof</u>

We already know that (g″,h″,k″) is a IM-DEIS morphism from G(M) to G(M″) by Theorem 1.3.1. It remains to show that

(1) $h''(s,e) = (s'',0) \Rightarrow e = 0$

Now, by definition

(2) $h''(s,e) = h'(h(s,e)) = (s'',0)$

Since (g´,h´,k´) is a TC-DEIS morphism, (2) implies

(3) $h(s,e) = (s',0)$ for some $s' \in S'$.

But (g,h,k) is also a TC-DEIS morphism. Hence (3) implies

(4) $e = 0$

which was to be proved. □

One can similarly show that TC-DEVS morphisms are transitive, provided composition of TC-DEVSs is appropriately defined, viz.

$(\bar{g}'',L'',\bar{h}'',\bar{k}'') \triangleq (\bar{g},L,\bar{h},\bar{k})\circ(\bar{g}',L',\bar{h}',\bar{k}')$ where

1. $g'': X'' \rightarrow X$ is defined by

   $g'' \triangleq g \circ g'$

2. $L'': h^{-1}(\hat{S}') \rightarrow N \cup \{0\}$ is defined by

   $$L''(\hat{s}) = \sum_{i=0}^{L'(h(\hat{s}))} L(\overline{\delta}_{\phi}(\hat{s},i))$$

3. $h'': h^{-1}(\hat{S}') \rightarrow S''$ is defined by

   $h'' \triangleq h' \circ h$

4. $k'': Y \rightarrow Y''$ is defined by

   $k'' \triangleq k \circ k'$

The proof is omitted, since it is quite tedious and does not provide additional insight into transition covering morphisms.

We shall, however, proceed to define the standard hierarchy of TC-DEVS morphisms. In the following definitions i denotes the identity function.

Definition 1.4.3

Two DEVSs $M = \langle X,S,Y,t,\delta,\lambda \rangle$ and $M' = \langle X',S',Y',t',\delta',\lambda' \rangle$ are called *compatible* if

a) $X = X'$

b) $Y = Y'$ □

Definition 1.4.4

Let (i,L,h,i) be a TC-DEVS morphism between compatible DEVSs M and M'.

Then (i,L,h,i) is called a *TC-DEVS homomorphism*, if $\hat{Q} = Q$.

A TC-DEVS homomorphism is called a *TC-DEVS isomorphism*, if in addition h is injective. □

The concepts of *TC-DEIS homomorphism* and *TC-DEIS isomorphism* are defined in the obvious way, similarly to the hierarchy of IM-DEIS morphisms.

We conclude this section by carrying over the TC morphisms to the DEMS level.

Definition 1.4.5

Let (g,h,k) be a TC-DEIS morphism from G(M) to G(M´). Let ($\tilde{g}$,h,k) be the induced IM-DEMS morphism from $S_{G(M)}$ to $S_{G(M')}$. (See Definition 1.3.2).

Then ($\tilde{g}$,h,k) is called a *TC-DEMS morphism* from $S_{G(M)}$ to $S_{G(M')}$. In this case we say that $S_{G(M)}$ *covers* $S_{G(M')}$, and denote $S_{G(M)}$ $S_{G(M')}$. □

Conclusion 1.4.1

Definition 1.4.5 requires that (g,h,k) be a TC-DEIS morphism from G(M) to G(M´) iff ($\tilde{g}$,h,k) is a TC-DEMS morphism from $S_{G(M)}$ to $S_{G(M')}$. □

Conclusion 1.4.1 and Corollary 1.4.1 give rise to the TC morphism paradigm of Figure 1.4.1.

At the DEMS level, it is useful to restrict transitional covering to particular trajectories as follows.

Definition 1.4.6

Let $S_{G(M)}$ and $S_{G(M')}$ be DEMSs, and let (i,h,k) be a IM-trajectory morphism from TRAJ(q,ω) to TRAJ(q´,ω´). (See Definition 1.3.8).

We say that (i,h,k) is a *TC-trajectory morphism* if

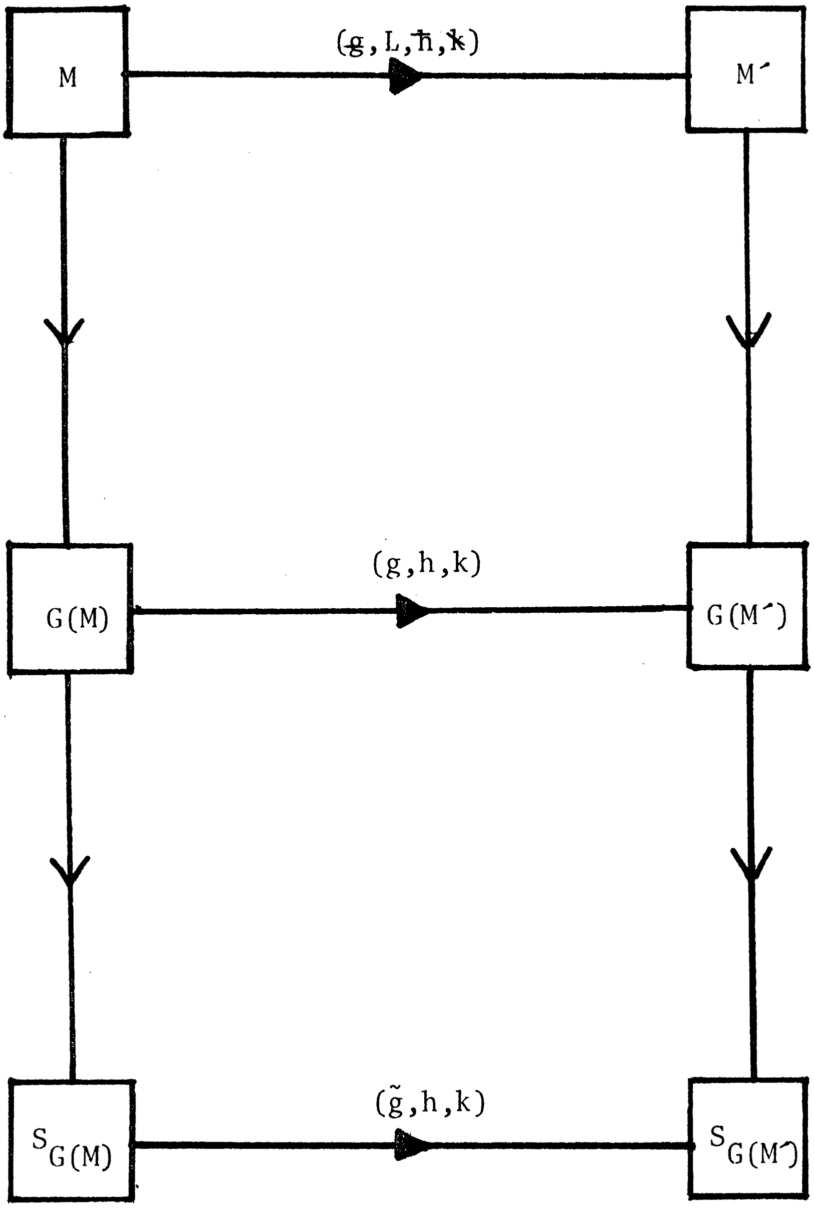

Figure 1.4.1: Relations among Discrete Event Structures and the Associated Transitional Covering Morphisms.

a) $h(STRAJ_{q,\omega}(t)) = (s',0) \Rightarrow STRAJ_{q,\omega}(t) = (s,0)$.

In this case we say that TRAJ(q,ω) is a *transitional covering* of TRAJ(q′,ω′) (or simply that TRAJ(q,ω) covers TRAJ(q′,ω′)), and denote TRAJ(q,ω) ⊐ TRAJ(q′,ω′). □

From Definition 1.4.6 we have the immediate

Conclusion 1.4.2

Let $(\mathbf{g},L,\mathbf{h},\mathbf{k})$ be a TC-DEVS morphism from M to M′ and let (g,h,k) be the TC-DEIS morphism induced by it according to Theorem 1.4.1. Then

a) $\forall \omega' \in \bar{\Omega}_G'^{+}, \quad \forall q \in \bar{Q}, \quad STRAJ_{q,\tilde{g}(\omega')} \sqsupset STRAJ_{h(q),\omega'}$

where $\tilde{g}$ is defined in Definition 1.4.5. □

1.5 Transitional Matching

Transitional covering allows us to match in time all jumps of a morphic image, with some of the jumps of its morphic preimage. In addition this matching is consistent by virtue of the underlying discrete event morphism.

Thus, transitional covering is a situation whereby the morphic preimage undergoes jumps at a "rate" which is higher than in its morphic image. The natural way to specialize covering morphisms is to require those "rates" to equal, so that all the jumps in both the morphic preimage and its morphic image can completely be matched in time. In accordance with the foregoing discussion, this situation will be called transitional matching.

We start by formally defining it at the DEIS level.

Definition 1.5.1

Let (g,h,k) be a IM-DEIS morphism from G(M) to G(M´). We say that G(M) and G(M´) are *transitionally matching* if

a) $h(s,e) = (s',0) \Rightarrow e = 0$

b) $h(s,0) = (s',e') \Rightarrow e' = 0$

In this case (g,h,k) is called a *transition matching DEIS morphism* (abbreviated *IM-DEIS morphism*). □

Definition 1.5.1 shows that a TM-DEIS morphism is a TC-DEIS morphism satisfying condition b) in the above. This means that if G(M) and G(M´) are started from h-matching states under g-matching inputs, then G(M) undergoes a jump iff G(M´) undergoes a jump at that very instant.

The following theorems give necessary conditions for transitional matching.

Theorem 1.5.1

Let (g,h,k) be a TM-DEIS morphism from G(M) to G(M´). Then

a) $h(s,e) = (s',e') \Rightarrow ŧ(s) - e = ŧ'(s') - e'$

Proof

Suppose

(1) $h(s,e) = (s',e')$

Since $(s,e) = \delta_G((s,0),\phi_e)$, we have by the composition property of δ_G that

(2) $\delta_G((s,e),\phi_{\text{ŧ}(s)-e}) = \delta_G(\delta_G((s,0),\phi_e),\phi_{\text{ŧ}(s)-e}) =$

$\delta_G((s,0),\phi_{e+\text{ŧ}(s)-e}) = \delta_G((s,0),\phi_{\text{ŧ}(s)}) =$

$(\delta_\phi(s),0) = (\bar{\delta}_\phi(s,1),0)$

Therefore, using (2)

(3) $\delta'_G((s',e'),\phi_{\text{ŧ}(s)-e}) = \delta'_G(h(s,e),\phi_{\text{ŧ}(s)-e}) =$

$h(\delta_G((s,e),\phi_{\text{ŧ}(s)-e})) = h(\delta_\phi(s),0)$

By transitional matching, we conclude from (3) that

(4) $\delta'_G((s',e'),\phi_{\text{ŧ}(s)-e}) = h(\delta_\phi(s),0) = (\bar{\delta}'_\phi(s',i),0)$

for some $i \geq 1$.

But using the same line of reasoning as in (2)

(5) $\delta'_G((s',e'),\phi_{\text{ŧ}'(s')-e'}) = (\delta'_\phi(s'),0) =$

$(\bar{\delta}'_\phi(s',1),0)$.

Comparing (4) and (5), we conclude that

(6) $\text{ŧ}(s) - e \geq \text{ŧ}'(s') - e'$

Now, applying transition function preservation to (5) yields

(7) $(\delta'_\phi(s'),0) = \delta'_G((s',e'),\phi_{\text{ŧ}'(s')-e'}) =$

$\delta'_G(h(s,e),\phi_{\text{ŧ}'(s')-e'}) = h(\delta_G((s,e),\phi_{\text{ŧ}'(s')-e'}))$.

By transitional matching, we conclude from (7) that

(8) $\delta_G((s,e),\phi_{\text{ŧ}'(s')-e'}) = (\bar{\delta}_\phi(s,j),0)$

for some $j \geq 1$.

Comparing (2) and (8), we conclude that

(9) $\text{ŧ}(s) - e \leq \text{ŧ}'(s') - e'$

Finally, a) follows from (6) and (9). □

Lemma 1.5.1

Under the conditions of Theorem 1.5.1 we have, in particular, that if $h(s,e) = (s',e')$, then

a) $(s,0) \in \overline{Q} \Rightarrow t(s) = t'(s')$.

Proof

By transitional matching

(1) $(s,0) \in \overline{Q} \Rightarrow h(s,0) = (s',0)$

Hence we may set $e = e' = 0$ in condition a) of Theorem 1.5.1, and condition a) of this lemma follows immediately. □

Theorem 1.5.1 states that for TM-DEIS morphisms (g,h,k), the residual times to the next jump of h-matching states, are always equal. However, this is not true for the respective time advance functions, unless as asserted in Lemma 1.5.1, the state in the morphic preimage is such that the jump to its sequential component is in the morphism domain. In this case we have the following characterization of TM-DEIS morphisms.

Theorem 1.5.2

Let (g,h,k) be a IM-DEIS morphism from G(M) to G(M') and define $\overline{S} \triangleq \{s \in S: (s,e) \in \overline{Q}$ for some $0 \le e < t(s)\}$. Suppose $\overline{Q}$ satisfies

(a) $(s,e) \in \overline{Q} \Rightarrow (s,0) \in \overline{Q}$

Then (g,h,k) is a TM-DEIS morphism iff

there is a surjective map $\hbar: \overline{S} \rightarrow S$, such that

(b) $\forall (s,e) \in \overline{Q}, \quad h(s,e) = (\hbar(s),e)$

Proof

($\Leftarrow$) Suppose there is $\hbar$ satisfying (b).

Now, whenever $h(s,e) = (s',e')$, then by (b)

(1) $(s',e') = (\hbar(s),e)$

and hence

(2) $h(s,e) = (s',e') \Rightarrow (e = 0 \text{ iff } e' = 0)$

which is equivalent to the transitional matching property.

($\Rightarrow$) Suppose (g,h,k) is a TM-DEIS morphism.

Define $\hbar:\overline{S} \to S$ by

(3) $\hbar(s) \overset{\Delta}{=} h_1(s,0)$ where $h = (h_1,h_2)$.

If $s' \in S'$, take any $s \in \overline{S}$ such that $h(s,0) = (s',0)$. Such $s \in S$ exists by transitional matching and surjectiveness of h. Clearly, $\hbar(s) = s'$ so that $\hbar$ is surjective. Moreover, by definition of $\hbar$

(4) $\forall s \in \overline{S},\ h(s,0) = (\hbar(s),0)$

But by Lemma 1.5.1 and in view of (a)

(5) $\forall s \in \overline{S},\ t(s) = t'(\hbar(s))$

Finally, taking note of (4) and (5) and using transition function preservation, we have for any $(s,e) \in \overline{Q}$

(6) $h(s,e) = h(\delta_G((s,0),\phi_e)) =$

$\delta'_G(h(s,0),\phi_e) = \delta'_G((\hbar(s),0),\phi_e) =$

$(\hbar(s),e)$

where (6) is identical to (b). □

When condition (a) in Theorem 1.5.2 does not hold, we have a modified version of this theorem.

Corollary 1.5.1

Let (g,h,k) be a IM-DEIS morphism from G(M) to G(M′). Define $\hat{S} \triangleq \{s \in \overline{S}: (s,e) \in \overline{Q} \Rightarrow (s,0) \in \overline{Q}\}$ where $\overline{S}$ is defined in Theorem 1.5.2, and let $\hat{Q} \triangleq \{(s,e) \in \overline{Q}: s \in \hat{S}$ and $0 \leq e < t(s)\}$.

Then

a) $(g,h|\hat{Q},k)$ is a TM-DEIS morphism

iff

b) there is a surjective map $\hbar: \hat{S} \to S'$ such that

$\forall (s,e) \in \hat{Q}, \quad h(s,e) = (\hbar(s),e)$ □

We remark in passing that Theorem 1.5.2 and Corollary 1.5.1 constitute a sharpening of Theorem 6 in [Z1] Ch. X Sec. 10.5.

The concept of TM-DEIS morphisms motivates the following definition of TM-DEVS morphisms.

Definition 1.5.2

Let $(g, L, \hbar, k)$ be a TC-DEVS morphism (see Definition 1.4.2) from M to M′.

We say that M and M′ are *transitionally matching* if

a) $L(\hat{s}) \equiv 0, \quad \forall \hat{s} \in \hat{S}$

In this case, $(g, L, \hbar, k)$ is called a *transition matching DEVS morphism* (abbreviated *TM-DEVS morphism*). □

We again need to show that this definition is consistent. First we show that TM-DEVS morphisms induce TM-DEIS morphisms according to Theorem 1.4.1.

Theorem 1.5.3

Let $(\bar{g}, L, \bar{h}, \bar{k})$ be a TM-DEVS morphism from M to M′. Then there is a TM-DEIS morphism from G(M) to G(M′).

Proof

Since $(\bar{g}, L, \bar{h}, \bar{k})$ is a TC-DEVS morphism, we have that (g,h,k) of Theorem 1.4.1 is a TC-DEIS morphism from G(M) to G(M′). We show that (g,h,k) is a TM-DEIS morphism.

By definition of TM-DEVS morphisms, it follows that for any $\hat{s} \in \hat{S}$

(1) $\bar{t}'(\bar{h}(\hat{s})) = \sum_{i=0}^{L(\hat{s})} \bar{t}(\overline{\delta}_{\phi}(\hat{s},i)) =$

$\bar{t}(\overline{\delta}_{\phi}(\hat{s},0)) = \bar{t}(\hat{s})$

due to f.1) in Definition 1.4.2. Furthermore,

(2) $\overline{\delta}_{\phi}(\hat{s},L(\hat{s})+1) = \overline{\delta}_{\phi}(\hat{s},1) = \delta_{\phi}(\hat{s}) \in \hat{S}$

by f.2) in Definition 1.4.2.

From (1) and (2) we conclude that

(3) $\overline{Q} = \{(s,e): s \in \hat{S},\ 0 \leq e < \bar{t}(\overline{s})\}$

so that in particular

(4) $(s,e) \in \overline{Q} \Rightarrow (s,0) \in \overline{Q}$

Clearly, every $(s,e) \in \overline{Q}$ has the representation

(5) $(s,e) = \delta_G((\hat{s},0),\phi_\tau)$, for $\tau = e$ and $\hat{s} = s \in \hat{S}$

Hence by (2) of Theorem 1.4.1, for any $(s,e) \in \overline{Q}$

(6) $h(s,e) = h(\delta_G((s,0),\phi_e)) = (\bar{h}(s),e)$

where $\bar{h}$ is surjective by definition.

We conclude from Theorem 1.5.2 that (g,h,k) is a TM-DEIS morphism. □

Next, we show that TM-DEIS morphisms induce TM-DEVS morphisms according to Theorem 1.4.2.

Theorem 1.5.4

Let (g,h,k) be a TM-DEIS morphism from G(M) to G(M´). Then there is a TM-DEVS morphism from M to M´.

Proof

Since (g,h,k) is a TM-DEIS morphism, we have that $(\mathfrak{g},L,\mathfrak{h},\mathfrak{k})$ of Theorem 1.4.2 is a TC-DEVS morphism from M to M´. We show that $(\mathfrak{g},L,\mathfrak{h},\mathfrak{k})$ is a TM-DEVS morphism.

It remains to show that

(1) $L(\hat{s}) \equiv 0, \quad \forall \hat{s} \in \hat{S}$

Consider any $\hat{s} \in \hat{S}$. By definition of $\hat{S}$ in Theorem 1.4.2 we have that $(\hat{s},0) \in \overline{Q}$, whence by Lemma 1.5.1

(2) $\mathfrak{t}(\hat{s}) = \mathfrak{t}'(h_1(\hat{s},0))$

where $h = (h_1,h_2)$. But $h_1(\hat{s},0) = \mathfrak{h}(\hat{s})$ by (1) of Theorem 1.4.2.

Hence (2) implies

(3) $\mathfrak{t}(\hat{s}) = \mathfrak{t}'(\mathfrak{h}(\hat{s}))$

By f.1) in Definition 1.4.2

(4) $$\mathfrak{t}'(\mathfrak{h}(\hat{s})) = \sum_{i=0}^{L(\hat{s})} \mathfrak{t}(\overline{\delta}_\phi(\hat{s},i))$$

Comparing (3) and (4) gives us

(5) $$\sum_{i=0}^{L(\hat{s})} \mathfrak{t}(\overline{\delta}_\phi(\hat{s},i)) = \mathfrak{t}(\hat{s})$$

Since M is regular we conclude that $L(\hat{s}) = 0$ which was to be proved. □

The hierarchies of TM morphisms and their variants is analogous to the hierarchies of TC morphisms in the previous section.

We now show that the transitional matching relation is transitive.

Theorem 1.5.5

If (g,h,k) is a TM-DEIS morphism from G(M) to G(M′) and (g′,h′,k′) is a TM-DEIS morphism from G(M′) to G(M″), then

$(g'',h'',k'') \triangleq (g,h,k) \circ (g',h',k')$ is a TM-DEIS morphism from G(M) to G(M″).

Proof

We already know by Theorem 1.4.3 that (g″,h″,k″) is a TC-DEIS morphism from G(M′) to G(M″).

It remains to show

(1) $h''(s,0) = (s'',e'') \Rightarrow e'' = 0$

By definition of h″ we may rewrite the antecedent of (1) as

(2) $h''(s,0) = h'(h(s,0)) = (s'',e'')$

Denote $h(s,0) = (s',e')$. By transitional matching of G(M) and G(M′) via (g,h,k)

(3) $h(s,0) = (s',e') \Rightarrow e' = 0$

Setting (3) in (2) yields

(4) $h''(s,0) = h'(s',0) = (s'',e'')$

But by transitional matching of G(M′) and G(M″) via (g′,h′,k′), (4) implies

(5) $e'' = 0$

which was to be proved. □

The proof at the level of TM-DEVS morphisms is analogous and will be omitted.

A sufficient condition ensuring an invertible TC-DEIS morphism to be a TM-DEIS morphism is given in

Theorem 1.5.6

Let $G(M) \sqsupset G(M')$ via an invertible TC-DEIS state-morphism (g,h). Suppose that in addition $G(M') \sqsupset G(M')$ via the inverse TC-DEIS state-morphism $(g,h)^{-1}$.

Then G(M) and G(M') are transitionally matching.

Proof

By definition $h:Q \to Q'$ is bijective and $h^{-1}:Q' \to Q$ is bijective. Since G(M) covers G(M') it follows that

(1) $h(s,e) = (s',0) \Rightarrow e = 0$

Now, assume

(2) $h(s,0) = (s',e')$

Applying h^{-1} on both sides of (2) gives

(3) $(s,0) = h^{-1}(s',e')$

But since G(M') covers G(M) via $(g,h)^{-1}$ we have

(4) $h^{-1}(s',e') = (s,0) \Rightarrow e' = 0$

From (2), (3) and (4) we conclude

(5) $h(s,0) = (s',e') \Rightarrow e' = 0$

Finally (1) and (5) show that G(M) and G(M') are transitionally matching. $\square$

While a chain of TC morphisms produces a sequence of discrete event systems with a "decreasing rate" of jumps, a chain of TM morphisms keeps the jump "rate" in the sequence "fixed".

TM isomorphisms are easily seen to partition a class of discrete event systems into equivalence classes of mutually TM isomorphic systems. In each such class, whenever the members are started in h-matching states and evolve under g-matching input segments, they will always undergo simultaneous jumps throughout the evolution.

A TM morphism paradigm can be derived analogously to the TC morphism paradigm depicted in Figure 1.4.1.

1.6 The Completion and Parallel Composition Operations

This section discusses two operations on discrete event systems: the so-called completion operation defined on IM morphic pairs of DEVSs, and the parallel composition operation defined for every pair of DEVSs.

The completion operation is motivated by the heuristic observation that every input matching morphism can be strengthened to a transition covering one, in a canonical manner. This is achieved by a "completion" procedure of the morphic preimage, relative to the morphic image. The operation is carried out by the transitional completion algorithm, which is embedded in the following procedure.

Our starting point is any pair of DEVSs $M = \langle X,S,Y,t,\delta,\lambda\rangle$ and $M' = \langle X',S',Y',t',\delta',\lambda'\rangle$, provided there is a IM-DEIS morphism (g,h,k) from G(M) to G(M'). The procedure produces a "complete" DEVS $\tilde{M} = \langle X,\tilde{S},Y,\tilde{t},\tilde{\delta},\tilde{\lambda}\rangle$ in which $\tilde{S}$, $\tilde{t}$, $\tilde{\delta}$ and $\tilde{\lambda}$ are constructively defined by the Transitional Completion Algorithm, to be described later.

This description relies heavily on the definitions of two auxiliary functions which we now proceed to introduce.

First define a function $e_J: S \to [0,\infty]$ by

$$e_J(s) \stackrel{\Delta}{=} \begin{cases} \min\{0 \le e < t(s):\ \exists s' \in S' \ni h(s,e) = (s',0)\}, & \text{if the minimum exists} \\ t(s), & \text{otherwise} \end{cases}$$

Intuitively, $e_J(s)$ gives the time to the first jump in either M or M' when started autonomously from states (s,0) and h(s,0) respectively.

Next, denote $h(s,0) = (s',e')$ whenever $(s,0) \in \overline{Q}$, and define a function $J: S \to N \cup \{0\} \cup \{\infty\}$ by

$$J(s) \triangleq \begin{cases} 0, \text{ if } e_J(s) = t(s) \\ m'((s',e'),t(s)), \text{ if } e_J(s) < t(s) \text{ and } h_2(\delta_\phi(s),0) \neq 0 \\ m'((s',e'),t(s)) - 1, \text{ if } e_J(s) < t(s) \text{ and } h_2(\delta_\phi(s),0) = 0 \end{cases}$$

where m' is the jump counter function (see Definition 1.2.3) in M' and h_2 is the clock coordinate in $h = (h_1,h_2)$. Intuitively, $J(s)$ gives the number of jumps that M' undergoes autonomously from state $h(s,0)$ during the time interval $(0,t(s))$.

We are now ready to describe the Transitional Completion Algorithm.

<u>Algorithm 1.6.1</u> (Transitional Completion Algorithm)

For any $(s,0) \in \overline{Q}$ denote $h(s,0) = (s',e')$. Then perform for any $s \in S$ the following:

1) Put the sequence $\{\tilde{s}_i\}_{i=0}^{J(s)}$ in $\tilde{S}$, where $\tilde{s}_i \triangleq (i,s)$, $0 \leq i \leq J(s)$.

   We assume, without loss of generality, that $S \cap S' = \Phi$, so that S, S' and $\tilde{S}$ are mutually disjoint.

2) Define $\tilde{t}:\tilde{S} \to (0,\infty]$ by

$$\tilde{t}(\tilde{s}_i) \triangleq \begin{cases} e_J(s), \text{ if } i = 0 \text{ and } e_J(s) > 0 \\ t'(s'), \text{ if } i = 0 \text{ and } e_J(s) = 0 \\ t'(\overline{\delta}_\phi(s',i)), \text{ if } 0 < i < J(s) \\ t(s) - \sum_{i=0}^{J(s)-1} \tilde{t}(\tilde{s}_i), \text{ if } i = J(s) \end{cases}$$

3) Define $\tilde{\delta}:\tilde{Q}\times(X \cup \{\phi\}) \to \tilde{S}$ by

$$\tilde{\delta}_\phi(\tilde{s}_i) \triangleq \begin{cases} \tilde{s}_{i+1} = (i+1,s), \text{ if } 0 \leq i < J(s) \\ (0,\delta_\phi(s)), \text{ if } i = J(s) \end{cases}$$

   and

$$\tilde{\delta}_M((\tilde{s}_i,\tilde{e}),x) \stackrel{\Delta}{=} (0,\delta_M((s,\sum_{j=0}^{i-1}\tilde{t}(\tilde{s}_j)+\tilde{e}),x))$$

4) Define $\tilde{\lambda}:\tilde{Q} \rightarrow Y$ by

$$\tilde{\lambda}(\tilde{s}_i,\tilde{e}) \stackrel{\Delta}{=} \lambda(s,\sum_{j=0}^{i-1}\tilde{t}(\tilde{s}_j)+\tilde{e})$$

This completes the Transitional Completion Algorithm. □

Notice that whenever $s \in S - \overline{S}$, where $\overline{S} = \{s \in S: (s,e) \in \overline{Q}$ for some $0 \leq e < t(s)\}$, then the minimum is undefined and we always have $e_J(s) = t(s)$. Consequently, in this case $J(s) = 0$, always. This fact renders steps 1) - 4) meaningful for all $s \in S$.

We are now ready to define the transitional completion operation on morphic DEVSs.

Definition 1.6.1

Let (g,h,k) be a IM-DEIS morphism from G(M) to G(M′). The *transitional completion* of M relative to M′ (denoted $\tilde{M}(M')$) is a DEVS $\tilde{M} = \langle \tilde{X},\tilde{S},\tilde{Y},\tilde{t},\tilde{\delta},\tilde{\lambda}\rangle$ where

a) $\tilde{X} = X$

b) $\tilde{Y} = Y$

c) $\tilde{S}$, $\tilde{t}$, $\tilde{\delta}$ and $\tilde{\lambda}$ are defined by applying Algorithm 1.6.1 (the Transitional Completion Algorithm) to each $s \in S$.

In this case we also say that $G(\tilde{M})$ is the *transitional completion* of G(M′) relative to G(M′). □

The construction of a transitional completion is illustrated in

Example 1.6.1

Let $M = \langle X,S,Y,\mathfrak{t},\delta,\lambda\rangle$ be defined by

$X = \Phi$

$S = \{s_0,s_1\}$

$Y = [0,1/2\tau)$

$\mathfrak{t}(s_0) = \mathfrak{t}(s_1) = \tau$

$\delta_\phi(s_i) = s_{1-i}$

$\lambda(s_i,e) = e \pmod{1/2\tau}$

Let $M' = \langle X',S',Y',\mathfrak{t}',\delta',\lambda'\rangle$ be defined by

$X' = \Phi$

$S' = \{s_0',s_1'\}$

$Y' = [0,1/2\tau)$

$\mathfrak{t}'(s_0') = 3/2\tau, \quad \mathfrak{t}'(s_1') = 1/2\tau$

$\delta_\phi'(s_i') = s_{1-i}'$

$\lambda(s_i,e) = e \pmod{1/2\tau}$

Define a IM-DEIS morphism (g,h,k) from G(M) to G(M') where

g is the empty function

$h:Q \rightarrow Q'$ is defined by

$$h(s_i,e) = \begin{cases} (s_0',e), & \text{if } i = 0 \\ (s_0',\tau + e), & \text{if } i = 1 \text{ and } 0 \leq e < 1/2\tau \\ (s_1', e - 1/2\tau), & \text{otherwise} \end{cases}$$

$k:Y \rightarrow Y'$ is the identity function

Now,

$$e_J(s_i) = \begin{cases} \mathfrak{t}(s_0), \text{ if } i = 0 \\ 1/2\tau, \text{ if } i = 1 \end{cases}$$

whence $J(s_i) = i, \quad i = 0,1.$

Define $\tilde{M} = \langle \tilde{X}, \tilde{S}, \tilde{Y}, \tilde{\mathfrak{t}}, \tilde{\delta}, \tilde{\lambda} \rangle$ by

$$\tilde{X} = \Phi$$

$$\tilde{S} = \{(0,s_0), (0,s_1), (1,s_1)\}$$

$$\tilde{Y} = [0,1/2\tau)$$

$$\tilde{\mathfrak{t}}(0,s_0) = \mathfrak{t}(s_0) = \tau, \; \tilde{\mathfrak{t}}(0,s_1) = \tilde{\mathfrak{t}}(1,s_1) = 1/2\tau$$

$$\tilde{\delta}_\phi(0,s_0) = (0,s_1), \; \tilde{\delta}_\phi(0,s_1) = (1,s_1), \; \tilde{\delta}_\phi(1,s_1) = (0,s_0)$$

$$\tilde{\lambda}((j,s_i),e) = e \pmod{1/2\tau}$$

Then $\tilde{M}$ is the transitional completion of M relative to M′. □

From the completion algorithm we derive the following conclusions.

<u>Conclusion 1.6.1</u>

The autonomous operation of $\tilde{M}(M')$ is periodic in the following sense. If $\tilde{M}$ is started in state $((0,s),0) \in \tilde{Q}$, then it evolves autonomously through the sequential state sequence

$$\begin{cases} \{(0,s),(1,s),\ldots,(J(s),s), \; (0,\delta_\phi(s)),\ldots\}, \quad \text{if } J(s) < \infty \\ \{(0,s),(1,s),\ldots\}, \quad \text{if } J(s) = \infty \end{cases}$$

The general scheme is

$$(0,s) \;---\; (0,\overline{\delta}_\phi(s,1)) \;---\; (0,\overline{\delta}_\phi(s,2)) \;---\; \ldots$$
$$---\; (0,\overline{\delta}_\phi(s,n)) \;---\; \ldots$$

where the dashes together with their leftmost sequential states stand for some periodic sequence. □

<u>Conclusion 1.6.2</u>

It follows from 2) and 3) in Algorithm 1.6.1 that for every $(s,e) \in \overline{Q}$, the clock e has the representation

a) $e = \tilde{\sigma}((0,s),i) + \tilde{e}$ for some $0 \leq i \leq J(s)$ and $0 \leq \tilde{e} < \tilde{t}(i,s)$

where $\tilde{\sigma}$ is the total time advance function (see Definition 1.2.2) of $\tilde{M}$.

Moreover, this representation is unique. □

<u>Conclusion 1.6.3</u>

It follows from 2) and 3) in Algorithm 1.6.1 that

(a) $h(s,e) = (s',0)$

iff

(b) $e_J(s) < t(s)$

and

(c) $$e = \begin{cases} e_J(s) + \sum_{j=1}^{i} t'(\overline{\delta}'_{\phi}(s'_0,j)), & \text{if } e_J(s) > 0 \\ \sum_{j=0}^{i} t'(\overline{\delta}'_{\phi}(s'_0,j)), & \text{if } e_J(s) = 0 \end{cases}$$

for some $0 \leq i < J(s)$, where $s'_0 \triangleq h_1(s,0)$. □

The semantics of the transitional completion operation are suggested by the terminology. Intuitively, it amounts to adding jumps to the morphic preimage, which correspond to all jumps in the morphic image. This is done by adding sequential states to the former and redefining its time advance function, transition function and the output function, in a consistent manner. In other words, the transitional completion operation takes any morphic preimage and completes it into a transitional cover of its morphic image.

Formally, we have

Theorem 1.6.1

Let (g,h,k) be a IM-DEIS morphism from G(M′) to G(M′). Let $\tilde{M} \triangleq \tilde{M}(M')$ be the transitional completion of M′ relative to M′. Then

a) $G(\tilde{M}) \sqsupset G(M)$ via a TC-DEIS isomorphism $(i,\tilde{h},i)$

b) $G(\tilde{M}) \sqsupset G(M')$ via a TC-DEIS morphism $(g,\tilde{\tilde{h}},k)$.

Proof

Define a IM-DEIS morphism $(i,\tilde{h},i)$ from $G(\tilde{M})$ to G(M), where $\tilde{h}$ is given by

(1) $\forall((i,s),\tilde{e}) \in \tilde{Q}, \quad \tilde{h}((i,s),\tilde{e}) \triangleq (s,\tilde{\sigma}((0,s),i) + \tilde{e})$

and $\tilde{\sigma}$ is the total time advance function of $\tilde{M}$. Observe that $0 \leq \tilde{\sigma}((0,s),i) + \tilde{e} < t(s)$, by 2) in Algorithm 1.6.1.

For any $(s,e) \in Q$, represent e as $e = \tilde{\sigma}((0,s),i) + \tilde{e}$ for some $0 \leq i \leq J(s)$ and $0 \leq \tilde{e} < \tilde{t}(i,s)$, according to Conclusion 1.6.2. Now, take $((i,s),\tilde{e}) \in \tilde{Q}$. Then $\tilde{h}((i,s),\tilde{e}) = (s,e)$ by (1), and $\tilde{h}$ is shown to be from $\tilde{Q}$ onto Q. Next, suppose

(2) $\tilde{h}((i_1,s_1),\tilde{e}_1) = \tilde{h}((i_2,s_2),\tilde{e}_2)$

Then necessarily $s_1 = s_2 \triangleq s$ by (1). Hence we can rewrite (2) as

(3) $(s,\tilde{\sigma}((0,s),i_1) + \tilde{e}_1) = (s,\tilde{\sigma}((0,s),i_2) + \tilde{e}_2)$

where the representations of the clocks in (3) are unique by Conclusion 1.6.2.

Conclude that $i_1 = i_2$ and $\tilde{e}_1 = \tilde{e}_2$ from which follows

(4) $((i_1,s_1),\tilde{e}_1) = ((i_2,s_2),\tilde{e}_2)$

We have that (2) implies (4), i.e. $\tilde{h}$ is injective.

Now, in view of Conclusion 1.6.1 it suffices to show that transition function preservation holds only within the indicated periods. This is true due to the composition property of transition functions. More accurately, it suffices to show

(5) $\tilde{h}(\tilde{\delta}_G(((0,s),0),\phi_\tau)) = \delta_G(\tilde{h}((0,s),0),\phi_\tau)$

only for all $((0,s),0) \in \tilde{Q}$, and $0 \leq \tau < \mathfrak{t}(s)$, and

(6) $\tilde{h}(\tilde{\delta}_M(((i,s),\tilde{e}),x),0) = (\delta_M(\tilde{h}((i,s),\tilde{e}),x),0)$

for any $((i,s),\tilde{e}) \in \tilde{Q}$ and $x \in X$.

Now,

(7) $\tilde{\delta}_G(((0,s),0),\phi_\tau) = ((i,s),\tilde{e})$

for some $0 \leq i \leq J(s)$ such that

(8) $\tau = \tilde{\sigma}((0,s),i) + \tilde{e}$

Using (7), (1) and (8) we have

(9) $\tilde{h}(\tilde{\delta}_G(((0,s),0),\phi_\tau)) = \tilde{h}((i,s),\tilde{e}) =$

$(s,\tilde{\sigma}((0,s),i) + \tilde{e}) = (s,\tau)$

while using (1) we obtain

(10) $\delta_G(\tilde{h}((0,s),0),\phi_\tau) = \delta_G((s,0),\phi_\tau) = (s,\tau)$

Thus, (9) and (10) show that (5) holds. In view of 3) in Algorithm 1.6.1, (9) and (1), we find that

(11) $\tilde{h}(\tilde{\delta}_M(((i,s),\tilde{e}),x),0) = \tilde{h}((0,\delta_M((s,\tilde{\sigma}((0,s),i) + \tilde{e}),x)),0) =$

$(\delta_M((s,\tilde{\sigma}((0,s),i) + \tilde{e}),x),0) = (\delta_M(\tilde{h}((i,s),\tilde{e}),x),0)$

and (11) shows that (6) holds.

Next, we show preservation of output function. From 4) in Algorithm 1.6.1 we immediately deduce that for any $((i,s),\tilde{e}) \in \tilde{Q}$

(12) $\tilde{\lambda}((i,s),\tilde{e}) = \lambda(s,\tilde{\sigma}((0,s),i) + \tilde{e}) = \lambda(\tilde{h}((i,s),\tilde{e}))$

We have shown that $(i,\tilde{h},i)$ is a IM-DEIS isomorphism from $G(\tilde{M})$ to $G(M)$ and it remains to show that $G(\tilde{M}) \sqsupset G(M)$ via $(i,\tilde{h},i)$.

Suppose

(13) $\tilde{h}((i,s),\tilde{e}) = (s,0)$

By the representation of Conclusion 1.6.2, (13) implies

(14) $\tilde{\sigma}((0,s),i) + \tilde{e} = 0$

which shows in particular that

(15) $\tilde{e} = 0$

Thus, from (13) and (15) we conclude that $G(\tilde{M}) \sqsupset G(M)$ as required, and the proof of a) is concluded.

Next define $(\tilde{\tilde{g}},\tilde{\tilde{h}},\tilde{\tilde{k}}) \triangleq (i,\tilde{h},i)\circ(g,h,k) = (g,h\circ\tilde{h},k)$.

Then $(\tilde{\tilde{g}},\tilde{\tilde{h}},\tilde{\tilde{k}})$ is a IM-DEIS morphism from $G(\tilde{M})$ to $G(M')$ by Theorem 1.3.1. It remains to show that $G(\tilde{M}) \sqsupset G(M')$ via $(\tilde{\tilde{g}},\tilde{\tilde{h}},\tilde{\tilde{k}})$.

Suppose

(16) $\tilde{\tilde{h}}((i,s),\tilde{e}) = (s',0)$

Then

(17) $h(\tilde{h}((i,s),\tilde{e})) = (s',0)$.

Denoting $\tilde{h}((i,s),\tilde{e}) \triangleq (s,e)$, (17) becomes

(18) $h(s,e) = (s',0)$

By Conclusion 1.6.3 and due to (18) we may represent e as follows:

$$(19) \quad e = \begin{cases} e_J(s) + \sum_{j=1}^{i} t'(\overline{\delta}_{\phi}'(s',j)), & \text{if } e_J(s) > 0 \\ \sum_{j=0}^{i} t'(\overline{\delta}_{\phi}'(s',j)), & \text{if } e_J(s) = 0 \end{cases}$$

where $s' \triangleq h_1(s,0)$.

But by 2) in Algorithm 1.6.1, (19) implies that

(20) $e = \tilde{\sigma}((0,s),i)$

On the other hand, from Conclusion 1.6.2 we have that e has a unique representation

(21) $e = \tilde{\sigma}((0,s),i) + \tilde{e}$

Equating (20) to (21) finally gives

(22) $\tilde{e} = 0$

We conclude that (16) implies (22), and the proof of b) is completed.

□

Theorem 1.6.1 shows that every transition in M and M′ can be matched by a transition in $\tilde{M}(M')$. The following theorem shows that conversely, every transition in $\tilde{M}(M')$ can be matched by a transition in either M or M′.

<u>Theorem 1.6.2</u>

Let (g,h,k) be a IM-DEIS morphism from G(M) to G(M′) and let $\tilde{M} \triangleq \tilde{M}(M')$. Let further $\tilde{h}$ and $\tilde{\tilde{h}}$ be as in Theorem 1.6.1.

Suppose $\tilde{q} \in \tilde{Q}$. Then

a) $\tilde{h}(\tilde{q})$ has the form (s,0)

or

b) $\tilde{\tilde{h}}(\tilde{q})$ has the form (s′,0).

<u>Proof</u>

Let $\tilde{q} = ((i,s),0)$ where $s \in S$, $0 \leq i \leq J(s)$, and denote $h(s,0) \triangleq (s',e')$.

Suppose i = 0. Then by definition of $\tilde{h}$

(1) $\tilde{h}(\tilde{q}) = \tilde{h}((0,s),0) = (s,0)$

Suppose $0 < i \leq J(s)$. Then by definition of $\tilde{\tilde{h}}$

(2) $\tilde{\tilde{h}}(\tilde{q}) = h(\tilde{h}((i,s),0)) = h(s,\tilde{\sigma}((0,s),i))$

But by the definition of $\tilde{t}$ in 2) of Algorithm 1.6.1

$$\text{(3)}\quad \tilde{\sigma}((0,s),i) = \begin{cases} e_J(s) + \sum_{j=1}^{i} t'(\overline{\delta}'_\phi(s',j)), & \text{if } e_J(s) > 0 \\ \sum_{j=0}^{i} t'(\overline{\delta}'_\phi(s',j)), & \text{if } e_J(s) = 0 \end{cases}$$

Notice that we also may assume

(4) $e_J(s) < t(s)$

or else $J(s) = 0$ by definition of J.

Now, in view of (2), (3) and (4), Conclusion 1.6.3 implies that

(5) $\tilde{\tilde{h}}(q) = h(s,\tilde{\sigma}((0,s),i)) = (s',0)$.

and the proof is complete. □

Corollary 1.6.1

Let $(\tilde{s},\tilde{e}) \in \tilde{Q}$. Then by Theorems 1.6.1 and 1.6.2 $\tilde{e} = 0$ iff $\tilde{h}(\tilde{s},\tilde{e}) = (s,0)$ or $\tilde{\tilde{h}}(\tilde{s},\tilde{e}) = (s',0)$. □

Corollary 1.6.1 says in fact that the states of M, M′ and $\tilde{M}$(M′) can be matched in such a way that a jump occurs in $\tilde{M}$(M′) iff a jump occurs concurrently in M or in M′. Another way to state it is as follows.

Theorem 1.6.3

Let (g,h,k) be a IM-DEIS morphism from G(M) to G(M′) and let $\tilde{M} \triangleq \tilde{M}(M')$. Let further $\tilde{h}$ and $\tilde{\tilde{h}}$ be as in Theorem 1.6.1.

For any $(\tilde{s},\tilde{e}) \in \tilde{Q}$, denote $h(\tilde{s},\tilde{e}) \triangleq (s,e) \in Q$ and $\tilde{h}(\tilde{s},\tilde{e}) \triangleq (s',e') \in Q'$ whenever $\tilde{\tilde{h}}(\tilde{s},\tilde{e})$ is defined. Then

a) $\tilde{t}(\tilde{s}) - \tilde{e} = t(s) - e$

or

b) $\tilde{t}(\tilde{s}) - \tilde{e} = t'(s') - e'$

Proof

Suppose both a) and b) are false for some $(\tilde{s},\tilde{e}) \in \tilde{Q}$. Let

(1) $\tilde{\delta}_G((\tilde{s},\tilde{e}),\phi_{\tilde{t}(\tilde{s})-\tilde{e}}) = (\tilde{\delta}_\phi(\tilde{s}),0)$

Suppose that

(2) $\tilde{t}(\tilde{s}) - \tilde{e} > t(s) - e$

By transition function preservation

(3) $\tilde{h}(\tilde{\delta}_G((\tilde{s},\tilde{e}),\phi_{t(s)-e})) = \delta_G(\tilde{h}(\tilde{s},\tilde{e}),\phi_{t(s)-e}) =$

$\delta_G((s,e),\phi_{t(s)-e}) = (\delta_\phi(s),0)$

Now, $G(\tilde{M}) \sqsupset G(M)$ by Theorem 1.6.1, so that (3) implies

(4) $\tilde{\delta}_G((\tilde{s},\tilde{e}),\phi_{t(s)-e}) = (\tilde{s}^*,0)$ for some $\tilde{s}^* \in \tilde{S}$

But (4) contradicts (1), in view of (2). Hence we must assume

(5) $\tilde{t}(\tilde{s}) - \tilde{e} < t(s) - e$

Consequently, by transition function preservation

(6) $\tilde{h}(\tilde{\delta}_G((\tilde{s},\tilde{e}),\phi_{\tilde{t}(\tilde{s})-\tilde{e}})) = \delta_G(\tilde{h}(\tilde{s},\tilde{e}),\phi_{\tilde{t}(\tilde{s})-\tilde{e}}) =$

$\delta_G((s,e),\phi_{\tilde{t}(\tilde{s})-\tilde{e}}) = (s,e + \tilde{t}(\tilde{s}) - \tilde{e})$

where $e + \tilde{t}(\tilde{s}) - \tilde{e} > 0$.

Next, suppose

(7) $\tilde{t}(\tilde{s}) - \tilde{e} > t'(s') - e'$

and obtain whenever $\tilde{\tilde{h}}$ is defined

(8) $\tilde{\tilde{h}}(\tilde{\delta}_G((\tilde{s},\tilde{e}),\phi_{t'(s')-e'})) = \delta'_G(\tilde{\tilde{h}}(\tilde{s},\tilde{e}),\phi_{t'(s')-e'}) =$

$\delta'_G((s',e'),\phi_{t'(s')-e'}) = (\delta'_\phi(s'),0)$

Now $G(\tilde{M}) \supset G(M')$ by Theorem 1.6.1, so that (8) implies

(9) $\tilde{\delta}_G((\tilde{s},\tilde{e}),\phi_{t'(s')-e'}) = (\tilde{s}^*,0)$ for some $\tilde{s}^* \in \tilde{S}$.

But (9) contradicts (1) in view of (7). Hence we must assume

(10) $\tilde{t}(\tilde{s}) - \tilde{e} < t'(s') - e'$

Consequently, by transition function preservation

(11) $\tilde{\tilde{h}}(\tilde{\delta}_G((\tilde{s},\tilde{e}),\phi_{\tilde{t}(\tilde{s})-\tilde{e}})) = \delta'_G(\tilde{\tilde{h}}(\tilde{s},\tilde{e}),\phi_{\tilde{t}(\tilde{s})-\tilde{e}}) =$

$\delta'_G((s',e'),\phi_{\tilde{t}(\tilde{s})-\tilde{e}}) = (s',e' + \tilde{t}(\tilde{s}) - \tilde{e})$

where $e' + \tilde{t}(\tilde{s}) - \tilde{e} > 0$.

Combining (6) and (11) and using (1) we see that

(12) $\tilde{h}(\delta_\phi(\tilde{s}),0) = (s,e + \tilde{t}(\tilde{s}) - \tilde{e})$

and

(13) $\tilde{\tilde{h}}(\delta_\phi(\tilde{s}),0) = (s',e' + \tilde{t}(\tilde{s}) - \tilde{e})$

whenever $\tilde{\tilde{h}}$ is defined.

Observe that (12) and (13) contradict Corollary 1.6.1. Hence, a) and b) cannot be both false; i.e. a) or b) must be true. □

We now turn our attention back to the transitional completion algorithm, in the light of the above corollaries and theorems.

Essentially, the algorithm "takes" M and "superimposes" M′ on it, so as to obtain $\tilde{M}(M')$. The process of "taking" M is formalized by mapping each sequential state $s \in S$ into the sequential state $(0,s) \in \tilde{S}$. The process of "superimposing" M′ on M is formalized by generating the sequence of sequential states $\{(i,s)\}_{i=1}^{J(s)}$, to be added to the state (0,s).

This sequence corresponds to jumps in M′ that are not matched by corresponding ones in M.

All in all, each state $s \in S$ induces a sequence of states $\{(i,s)\}_{i=0}^{J(s)} \in \tilde{S}$ where the map $s \mapsto \{(i,s)\}_{i=0}^{J(s)}$ is obviously injective. If $M \sqsupset M'$ to begin with, then all jumps in M′ are already matched by jumps in M, and the transitional completion algorithm reduces to relabeling M to $\tilde{M}$ via the map $s \mapsto (0,s)$. This fact is formalized as follows.

Theorem 1.6.4

Suppose $G(M) \sqsupset G(M')$ and let $\tilde{M} \stackrel{\Delta}{=} \tilde{M}(M')$. Then $G(\tilde{M})$ is TM-DEIS isomorphic to G(M).

Proof

Consider the TC-DEIS isomorphism $(i,\tilde{h},i)$ of Theorem 1.6.1. Since $G(M) \sqsupset G(M')$, it follows from Algorithm 1.6.1 that $e_J(s) = 0$ for all $s \in S$. Hence, $J(s) = 0$ for all $s \in S$. Consequently, $\tilde{S} = \{(0,s): s \in S\}$. Furthermore, $\tilde{t}(0,s) = t(s)$ for any $(0,s) \in \tilde{S}$. The map $\tilde{h}$ reduces then to

(1) $\tilde{h}((0,s),\tilde{e}) = (s,\tilde{e}), \quad \forall (0,s) \in \tilde{S}, \quad \forall 0 \leq \tilde{e} < \tilde{t}(0,s)$

Thus we can define $\hbar: \tilde{S} \to S$ by

(2) $\hbar(0,s) \stackrel{\Delta}{=} s$

and $\hbar$ is clearly surjective. Moreover,

(3) $\tilde{h}((0,s),\tilde{e}) = (\hbar(0,s),\tilde{e}), \quad \forall (0,s) \in \tilde{S}, \quad \forall 0 \leq \tilde{e} < \tilde{t}(0,s)$

Hence by Theorem 1.5.2, (3) implies that $(i,\tilde{h},i)$ is a TM-DEIS isomorphism as was to be proved.

□

The transitional completion operation can be viewed as a "binary operation" acting on pairs of IM morphic DEVSs. As such, it is inherently asymmetric, because it depends on the direction of the underlying IM-DEVS morphism. Actually, if we commute the operands, the operation could be undefined, since the existence of a IM-DEVS morphism in one direction does not guarantee its existence in the other direction. Consequently, we speak about the completion operation as being performed on the morphic preimage with respect to its morphic image.

It is possible, however, to generalize the completion operation to a full-fledged binary operation on the class of DEVSs, by disposing altogether of the dependence on an underlying IM-DEVS morphism. We use the term "parallel composition" to suggest the heuristic content of the operation. Intuitively, the two operand DEVSs give rise to their "parallel composition DEVS" by letting them run concurrently from any two initial states under any two input segments of equal length. The resulting DEVS undergoes a jump whenever either of its operands does, so that the state trajectory, as far as jumps are concerned, is a superposition of the operands' trajectories.

Formally, we define

<u>Definition 1.6.2</u>

Let $M = \langle X,S,Y,t,\delta,\lambda \rangle$ and $M' = \langle X',S',Y',t',\delta',\lambda' \rangle$ be any two DEVSs.

The *parallel composition* of M and M′ (denoted $M \oplus M'$) is a DEVS $M^* = \langle X^*,S^*,Y^*,t^*,\delta^*,\lambda^* \rangle$ where

a) $X^* \triangleq ((X \cup \{\phi\}) \times (X' \cup \{\phi\})) - \{(\phi,\phi)\}$

b) $S^* \triangleq Q \times Q'$

c) $Y^* \triangleq Y \times Y'$

d) $\forall((s,e),(s',e')) \in S^*$,

$t^*((s,e),(s',e')) \triangleq \min\{t(s)-e, t'(s')-e'\}$

e) δ^* is defined as follows.

e.1) $\forall s^* = ((s,e),(s',e')) \in S^*$,

$$\delta^*_\phi((s,e),(s',e')) \triangleq \begin{cases} ((\delta_\phi(s),0),(s',e' + t(s))), & \text{if } t(s)-e < t'(s')-e' \\ ((s,e + t'(s')),(\delta'_\phi(s'),0)), & \text{if } t(s)-e > t'(s')-e' \\ ((\delta_\phi(s),0),(\delta'_\phi(s'),0)), & \text{if } t(s)-e = t'(s')-e' \end{cases}$$

e.2) $\forall s^* = ((s,e),(s',e')) \in S^*$, $\forall 0 \leq e^* < t^*(s^*)$,

$\forall x^* = (\bar{x},\bar{x}') \in X^*$,

$$\delta_M((s^*,e^*),x^*) \triangleq \begin{cases} ((\delta_M((s,e + e^*),\bar{x}),0),(s',e' + e^*)), & \text{if } x^* \in X\times\{\phi\} \\ ((s,e + e^*),(\delta'_M((s',e' + e^*),\bar{x}'),0)), & \text{if } x^* \in \{\phi\}\times X' \\ ((\delta_M((s,e + e^*),\bar{x}),0),(\delta'_M((s',e' + e^*),\bar{x}'),0)), & \text{if } x^* \in X\times X' \end{cases}$$

f) $\forall(s^*,e^*) = (((s,e),(s',e')),e^*) \in Q^*$,

$\lambda^*(s^*,e^*) = (\lambda(s,e + e^*),\lambda'(s',e' + e^*))$ □

It is not difficult to see that if G(M) and G(M´) are IM-DEIS morphic, then $G(M \oplus M')$ subsumes $G(\tilde{M}(M'))$ in the sense that there is a TM-DEIS morphism from the former to the latter. The difference between $\tilde{M}(M')$ and $M \oplus M'$, in this case, is simply a matter of viewing the same phenomenon from different angles. In the process of creating $\tilde{M}(M')$, M is viewed as operating on M´ via the completion operation. This asymmetry is not required to obtain $M \oplus M'$, and both DEVSs are considered as operands.

Although, we shall not engage here in a detailed discussion of the properties of the parallel composition operation, we point out a number of observations.

First, the $\oplus$ operation is associative, provided equality of M and M′ is defined as the existence of an invertible TM-DEIS morphism between G(M) and G(M′).

Second, the $\oplus$ operation is commutative in the same sense of equality.

Third, if $M^* = M \oplus M'$, then $M^* \sqsupset M$ and $M^* \sqsupset M'$.

Finally, we point out that a finite parallel composition $\underset{\alpha \in D}{\oplus} M_\alpha$ is a special case of a DEVN whose components $\{M_\alpha\}_{\alpha \in D}$ do not interact. In other words, the "topology" (influence graph) of a parallel composition reduces to a collection of isolated nodes.

The ability to describe a DEVS M as a DEVN, and in particular the ability to represent M as a parallel composition, entails a conceptual simplification of the system under investigation.

1.7 Standard Covers

In this section we specialize the concept of transitional covers and discuss some of the resulting properties. The specialized transitional covers considered are the so-called standard covers, exhaustive covers, minimal standard covers and exhaustive standard covers. These involve covering relations between a DEIS G(M*) and two other IM-DEIS morphic DEISs G(M) and G(M′). It will be shown that the exhaustive standard cover and the minimal standard cover are equivalent concepts which are embodied in a canonical manner by $G(\tilde{M})$, where $\tilde{M} \triangleq \tilde{M}(M')$. In the sequel, we think of the specialized covers as running not only between DEISs but also between the underlying DEVSs.

Our starting point is

Definition 1.7.1

Let (g,h) be a IM-DEIS state-morphism from G(M) to G(M′) and let $M^* = \langle X^*, S^*, \cdot, t^*, \delta^*, \cdot \rangle$ be a state-DEVS.

G(M*) is called a *standard cover* (abbreviated *SC*) of G(M) and G(M′) if G(M*) satisfies the following:

a) $G(M^*) \sqsupset G(M)$ via a TC-DEIS state-isomorphism (i,h^*).

b) $G(M^*) \sqsupset G(M')$ via the TC-DEIS state-morphism

$$(g^{**},h^{**}) \triangleq (i,h^*)\circ(g,h) = (g,h\circ h^*) \ . \qquad \square$$

The relations among the maps of Definition 1.7.1 are depicted in Figure 1.7.1.

Conclusion 1.7.1

If (g,h) is a IM-DEIS state-morphism from G(M) to G(M′)

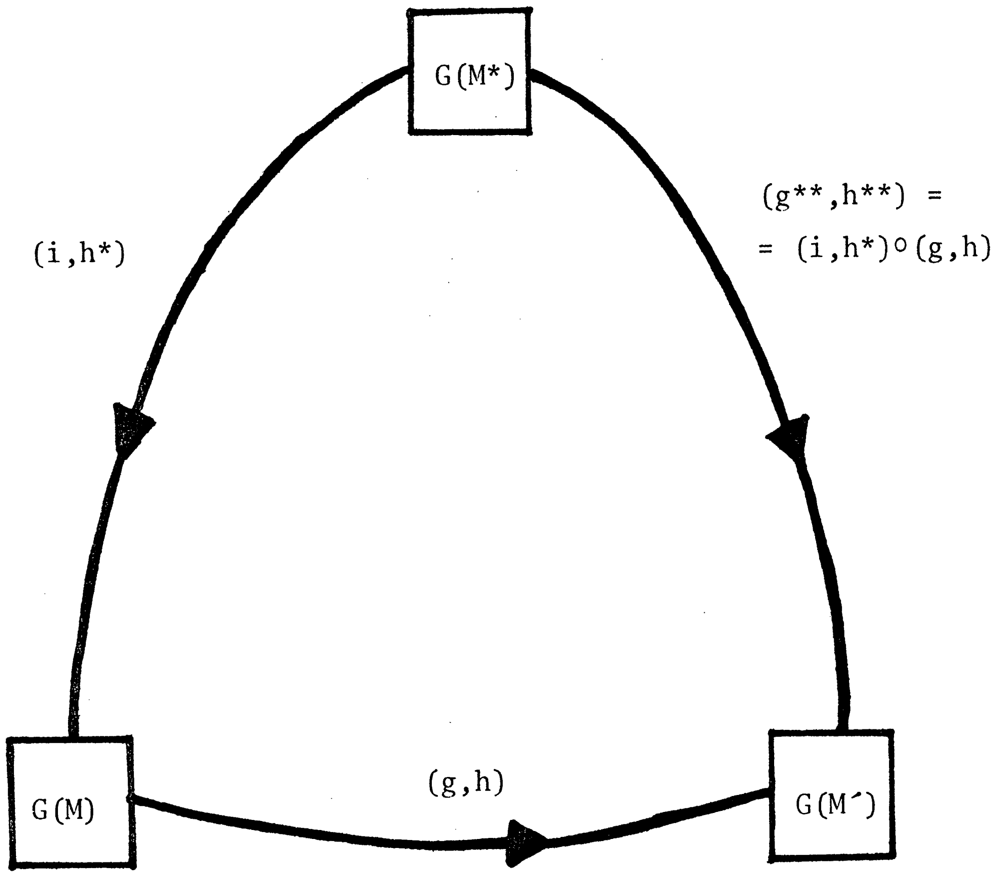

Figure 1.7.1: Relations among the Maps of the Standard Cover Concept.

and $M^* \triangleq \tilde{M}(M')$, then by Theorem 1.6.1 $G(M^*)$ is a SC of M and M′ via $(i,h^*) \triangleq (i,\tilde{h})$ and $(g^{**},h^{**}) \triangleq (i,\tilde{h})\circ(g,h)$

where $\tilde{h}$ is defined in the proof of Theorem 1.6.1. □

Definition 1.7.2

Let $M^* = \langle X^*,S^*,\cdot,t^*,\delta^*,\cdot\rangle$ be a state-DEVS. Let (g,h) be a IM-DEIS state-morphism from $G(M^*)$ to G(M) with domain $\overline{Q}_1^*$, and let (g′,h′) be a IM-DEIS state-morphism from $G(M^*)$ to G(M′) with domain $\overline{Q}_2^*$.

$G(M^*)$ is called an *exhaustive cover* (abbreviated *EC*) of G(M) and G(M′) if

a) $\overline{Q}_1^* \cup \overline{Q}_2^* = Q^*$

b) $h(s^*,e) = (s,0) \in Q$ or $h'(s^*,e) = (s',0) \in Q' \Leftrightarrow e = 0$

for any $(s^*,e) \in Q^*$ □

If $h(s^*,0)$ or $h'(s^*,0)$ are undefined, then the logical value of the esponding disjunct in b) is 'false'. Notice, however, that a) guarantees that there is no $q^* \in Q^*$ for which both $h(s^*,0)$ and $h'(s^*,0)$ are undefined. Consequently, every jump in M^* can be matched by a jump in M or M′, so that the jump matching is exhaustive. Conversely, M^* covers both M and M′ due to condition b).

Definition 1.7.3

A DEIS $G(M^*)$ is an *exhaustive standard cover* (abbreviated *ESC*) of two DEISs G(M) and G(M′), if $G(M^*)$ is both a SC and a EC of G(M) and G(M′) via the same maps. (See Figure 1.7.1). □

Conclusion 1.7.2

If (g,h) is a IM-DEIS state-morphism from G(M) to G(M´) and $M^* \triangleq \tilde{M}(M')$, then G(M*) is a ESC of G(M) and G(M´) by Theorem 1.6.2 and Corollary 1.6.1. □

Theorem 1.7.1

Let (g,h) be a IM-DEIS state-morphism from G(M) to G(M´). Suppose that G(M*) is a ESC of G(M) and G(M´) via a TC-DEIS state-isomorphism (i,h*), and the TC-DEIS state-morphism $(g^{**},h^{**}) \triangleq (i,h^*)\circ(g,h)$ respectively. Let $\hat{M} = \langle \hat{X},\hat{S},\cdot,\hat{t},\hat{\delta},\cdot\rangle$ be a state-DEVS and suppose that $(i,\hat{h}^*)$ is a TM-DEIS state-isomorphism from G(M*) to $G(\hat{M})$.

Then $G(\hat{M})$ is a ESC of G(M) and G(M´).

Proof

Refer to Figure 1.7.2. Define a IM-DEIS state-isomorphism from $G(\hat{M})$ to G(M) by

(1) $(i,\hat{h}) \triangleq (i,\hat{h}^*)\circ(i,h^*) = (i,h^*\circ\hat{h}^*)$

Next, define a IM-DEIS state-morphism from $G(\hat{M})$ to G(M´) by

(2) $(i,\hat{\hat{h}}) \triangleq (i,\hat{h})\circ(g,h) = (i,\hat{h}^*)\circ(i,h^*)\circ(g,h) = (i,\hat{h}^*)\circ(g^{**},h^{**})$

Clearly, $(i,\hat{h})$ is a TC-DEIS state-isomorphism as a composition of two TC-DEIS state-isomorphisms, and $(i,\hat{\hat{h}})$ is a TC-DEIS state-morphism as a composition of two TC-DEIS state-morphisms (see Theorem 1.4.3).

We conclude that $G(\hat{M})$ is a SC of G(M) and G(M´), and it remains to show that $G(\hat{M})$ is exhaustive.

Obviously $\hat{h}^{-1}(Q)\cup\hat{\hat{h}}^{-1}(Q') = \hat{Q}$, since $\hat{h}^{-1}(Q) = \hat{Q}$.

From the TM-DEIS state-isomorphism $(i,\hat{h}^*)$ we have (see Theorem 1.5.2)

(3) $(\hat{s},e)\in\hat{Q} \iff \hat{h}^*(\hat{s},e) = (s^*,e)\in Q^*$ for some $s^*\in S^*$.

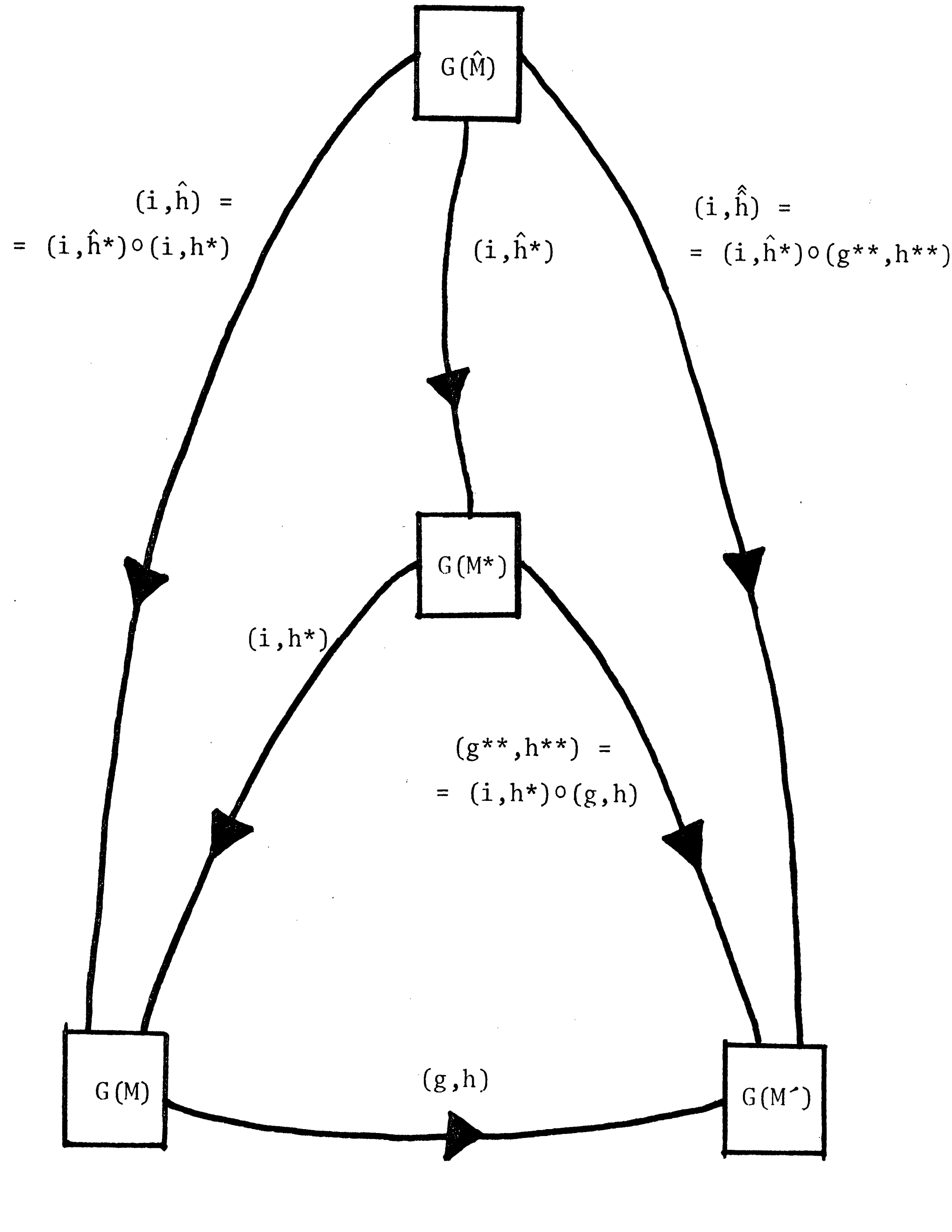

Figure 1.7.2: Relations among the Maps of Theorem 1.7.1.

Since $G(M^*)$ is a ESC of $G(M)$ and $G(M')$, for any $(s^*,e) \in Q^*$

(4) $h^*(s^*,e) = (s,0) \in Q$ or $h^{**}(s^*,e) = (s',0) \in Q' \Leftrightarrow e = 0$

Setting (3) in (4) yields for any $(\hat{s},e) \in \hat{Q}$

(5) $h^*(\hat{h}^*(\hat{s},e)) = (s,0) \in Q$ or $h^{**}(\hat{h}^*(\hat{s},e)) = (s',0) \in Q' \Leftrightarrow e = 0$

Finally, (5) is equivalent by (1) and (2) to

(6) $\hat{h}(\hat{s},0) = (s,0) \in Q$ or $\hat{\hat{h}}(\hat{s},0) = (s',0) \in Q' \Leftrightarrow e = 0$ □

Definition 1.7.4

Let (g,h) be a IM-DEIS morphism from $G(M)$ to $G(M')$. Let $G(M^*)$ be a SC of $G(M)$ and $G(M')$ via a TC-DEIS state-isomorphism (i,h^*) and a TC-DEIS state-morphism $(g^{**},h^{**}) \triangleq (i,h^*)\circ(g,h)$ respectively.

We say that $G(M^*)$ is a *minimal standard cover* (abbreviated *MSC*) of $G(M)$ and $G(M')$, if for any state-DEVS $\hat{M} = \langle \hat{X},\hat{S},\cdot,\hat{t},\hat{\delta},\cdot \rangle$ such that $G(\hat{M})$ is a SC of $G(M)$ and $G(M')$ via any TC-DEIS state-isomorphism $(i,\hat{h})$ and the TC-DEIS state-morphism $(\hat{\hat{g}},\hat{\hat{h}}) \triangleq (i,\hat{h})\circ(g,h)$, we have that $G(\hat{M})$ covers $G(M^*)$ via the TC-DEIS state-morphism $(\hat{g}^*,\hat{h}^*) = (i,\hat{h})\circ(i,h^*)^{-1}$. □

The relations among the morphisms of Definition 1.7.4 are depicted in Figure 1.7.3.

Theorem 1.7.2

Let (g,h) be a IM-DEIS state-morphism from $G(M)$ to $G(M')$. Let $G(M^*)$ be a ESC of $G(M)$ and $G(M')$ via a TC-DEIS state-isomorphism (i,h^*) and the TC-DEIS state-morphism $(g^{**},h^{**}) = (i,h^*)\circ(g,h)$ respectively.

Then $G(M^*)$ is a MSC of $G(M)$ and $G(M')$.

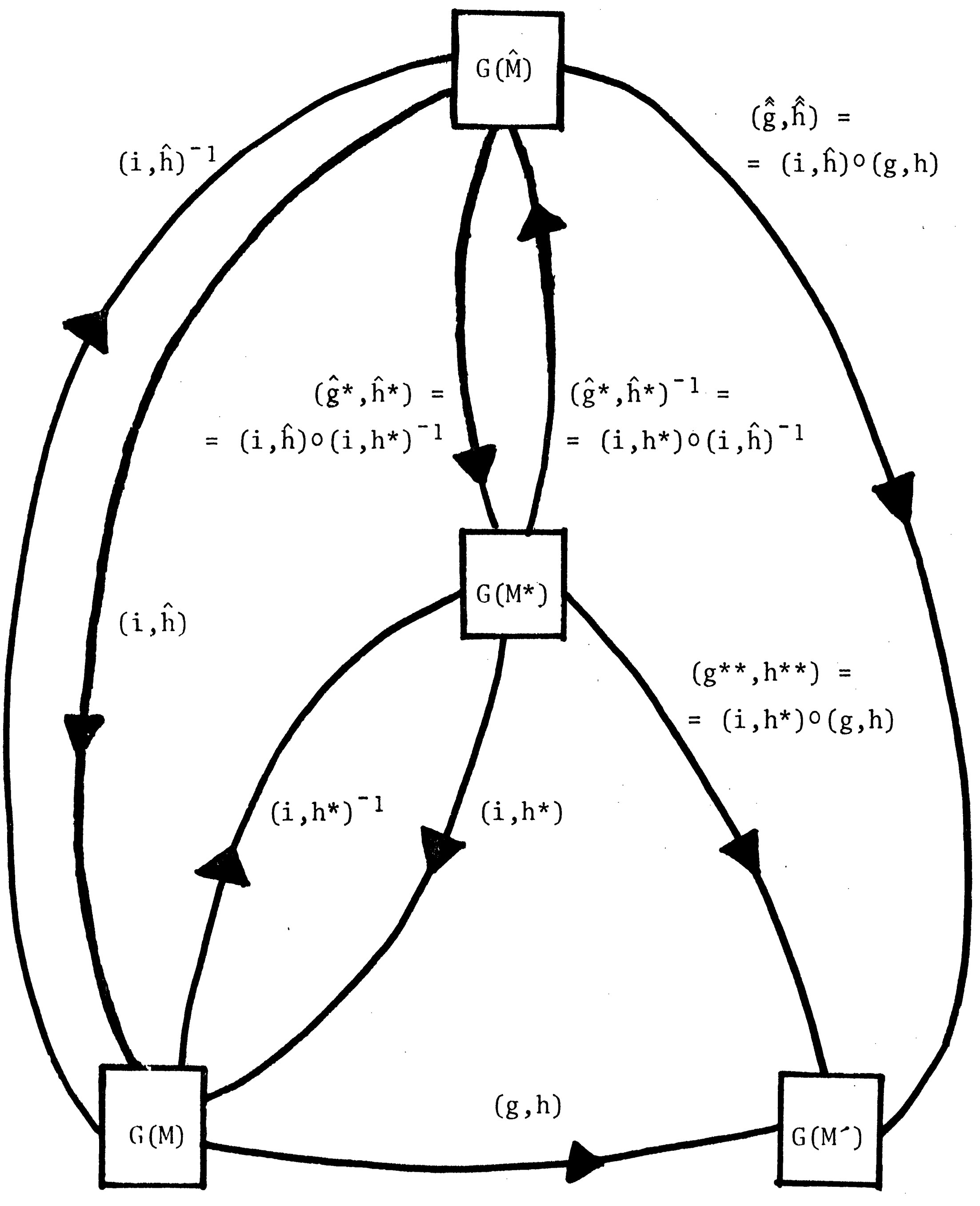

Figure 1.7.3: Relations among the Maps of the Minimal Standard Cover Concept.

Proof

Obviously $G(M^*)$ is a SC of $G(M)$ and $G(M')$ via (i,h^*) and $(g,h \circ h^*)$ respectively, by definition of ESC.

It remains to show that $G(M^*)$ is minimal. Following Figure 1.7.3, let $\hat{M} = \langle \hat{X},\hat{S},\cdot,\hat{t},\hat{\delta},\cdot \rangle$ be any state-DEVS such that $G(\hat{M})$ is also a SC of $G(M)$ and $G(M')$ as follows.

a) $G(\hat{M}) \sqsupset G(M)$ via a TC-DEIS state-isomorphism $(i,\hat{h})$

b) $G(\hat{M}) \sqsupset G(M')$ via the TC-DEIS state-morphism

$(\hat{g},\hat{h}) \triangleq (i,\hat{h}) \circ (g,h) = (i,h \circ \hat{h})$.

We show that $G(\hat{M}) \sqsupset G(M^*)$ via the TC-DEIS state-morphism

$(\hat{g}^*,\hat{h}^*) \triangleq (i,\hat{h}) \circ (i,h^*)^{-1} = (i,h^{*-1} \circ \hat{h})$.

Now, $(\hat{g}^*,\hat{h}^*)$ is a TC-DEIS state-isomorphism as a composition of TC-DEIS state-isomorphisms.

Suppose that

(1) $\hat{h}^*(\hat{s},\hat{e}) = (s^*,0) \in Q^*$.

Equivalently

(2) $h^{*-1}(\hat{h}(\hat{s},\hat{e})) = (s^*,0) \in Q^*$.

Since $G(M^*)$ is a ESC of $G(M)$ and $G(M')$ we have by definition that

(3) $h^*(s^*,0) = (s,0)$ for some $s \in S$

or

(4) $h^{**}(s^*,0) = (s',0)$ for some $s' \in S'$.

must hold.

Suppose that (3) holds. Then premultiply (2) by h^*. For the left side of (2) this gives

(5) $h^*(h^{*-1}(\hat{h}(\hat{s},\hat{e}))) = \hat{h}(\hat{s},\hat{e})$

while the right side of (2) becomes

(6) $h^*(s^*,0)$.

Equating (5) and (6) and applying (3) yields

(7) $\hat{h}(\hat{s},\hat{e}) = h^*(s^*,0) = (s,0) \in Q$ for some $s \in S$.

But $G(\hat{M}) \sqsupset G(M)$ via $(i,\hat{h})$ so that

(8) $\hat{h}(\hat{s},\hat{e}) = (s,0) \Rightarrow \hat{e} = 0$

Suppose that (4) holds. Then premultiply (2) by h^{**}. For the left side of (2) this gives

(9) $h^{**}(h^{*-1}(\hat{h}(\hat{s},\hat{e}))) = h(h^*(h^{*-1}(\hat{h}(\hat{s},\hat{e})))) =$

$h(\hat{h}(\hat{s},\hat{e})) = \hat{\hat{h}}(\hat{s},\hat{e})$

while the right side of (2) becomes

(10) $h^{**}(s^*,0)$.

Equating (9) and (10) and applying (4) yields

(11) $\hat{\hat{h}}(\hat{s},\hat{e}) = h^{**}(s^*,0) = (s',0) \in Q'$ for some $s' \in S'$

But $G(\hat{M}) \sqsupset G(M')$ via $(\hat{\hat{g}},\hat{\hat{h}})$ so that

(12) $\hat{\hat{h}}(\hat{s},\hat{e}) = (s',0) \Rightarrow \hat{e} = 0$

We conclude from (8) and (12) that $G(\hat{M}) \sqsupset G(M^*)$ as required. □

Corollary 1.7.1

If (g,h) is a IM-DEIS state-morphism from G(M) to G(M′) and $M^* \triangleq \tilde{M}(M')$, then G(M*) is a MSC of G(M) and G(M′), since by Conclusion 1.7.2 G(M*) is a ESC of G(M) and G(M′). □

Furthermore, the following theorem shows that $\tilde{M}(M')$ is canonical in the following sense.

Theorem 1.7.3

Let (g,h) be a IM-DEIS state-morphism from G(M) to G(M′), and let $M^* \triangleq \tilde{M}(M')$.

Then G(M*) is a unique MSC of G(M) and G(M′) up to a TM-DEIS state-isomorphism.

Proof

Refer to Figure 1.7.3 assuming that $M^* \stackrel{\Delta}{=} \tilde{M}(M')$, and that $G(\hat{M})$ is an arbitrary MSC of G(M) and G(M′).

Since G(M*) is a MSC of G(M) and G(M′) by Corollary 1.7.1, it follows from Definition 1.7.4 that $G(\hat{M}) \sqsupset G(M^*)$ via the TC-DEIS state-isomorphism $(\hat{g}^*,\hat{h}^*) \stackrel{\Delta}{=} (i,\hat{h})\circ(i,h^*)^{-1}$.

Since $G(\hat{M})$ is a MSC of G(M) and G(M′), it follows from Definition 1.7.4 that $G(M^*) \sqsupset G(\hat{M})$ via the TC-DEIS state-isomorphism $(\hat{g}^*,\hat{h}^*)^{-1} \stackrel{\Delta}{=} (i,h^*)\circ(i,\hat{h})^{-1}$.

Consequently, M* and $\hat{M}$ satisfy the conditions of Theorem 1.5.6, from which it follows that G(M*) and $G(\hat{M})$ are TM-DEIS state-isomorphic. □

Finally, we prove the following equivalence.

Theorem 1.7.4

Let (g,h) be a TM-DEIS state-morphism from G(M) to G(M′). Let G(M*) be a SC of G(M) and G(M′) via a TC-DEIS state-isomorphism (i,h*) and the TC-DEIS state-morphism $(g^{**},h^{**}) \stackrel{\Delta}{=} (i,h^*)\circ(g,h)$ respectively. Then

a) G(M*) is a MSC of G(M) and G(M′)

iff

b) G(M*) is a ESC of G(M) and G(M′).

Proof

($\Rightarrow$) Assume that G(M*) is a MSC of G(M) and G(M′).

Define $\tilde{M} = \tilde{M}(M')$ to be the completion of M relative to M′. By Theorem 1.7.3, G(M*) is TM-DEIS state-isomorphic to $G(\tilde{M})$. But the latter is known to be a ESC of G(M) and G(M′) by Conclusion 1.7.2. Furthermore, by Theorem 1.7.1 the ESC property is invariant under TM-DEIS state-isomorphisms. Consequently G(M*) is an ESC of G(M) and G(M′).

($\Leftarrow$) Assume that G(M*) is a ESC of G(M) and G(M′).

It immediately follows from Theorem 1.7.2 that G(M*) is a MSC of G(M) and G(M′). □

Conclusion 1.7.3

Theorem 1.7.4 shows that the concepts of ESC and MSC are equivalent.

Moreover, each of these concepts is equivalent to $\tilde{M}(M')$ up to TM-DEIS state-isomorphism. □

Conclusion 1.7.3 asserts that ESC and MSC are two equivalent properties that characterize $\tilde{M} \triangleq \tilde{M}(M')$. Thus $G(\tilde{M})$ is a canonical ESC and MSC of any IM-DEIS state-morphic G(M) and G(M′), since all their ESCs and MSCs are mutually TM-DEIS state-isomorphic, and in particular $\tilde{M}(M')$ is one of them.

We can also think of $G(\tilde{M})$ as the representative of the set of all ESCs or MSCs of G(M) and G(M′), whenever G(M) and G(M′) are IM-DEIS morphic. This is so, because the IM-DEIS state-isomorphism relation is clearly an equivalence relation.

Finally, the concepts of ESC or MSC induce on the class of DEISs a lattice-like structure in the sense that for each pair of IM-DEIS morphic DEISs G(M) and G(M´), the DEIS G($\tilde{M}$) provides a l.u.b[†]-like concept.

[†] l.u.b is an abbreviation for least upper bound.

CHAPTER 2

STOCHASTIC DISCRETE EVENT SYSTEMS

2.0 Introduction

The stochastic counterparts of deterministic discrete event systems are stochastic jump processes. In a jump process, the system evolves continuously in time and changes states discretely in time. But while in the deterministic case the time spent in a state and the transition to a next state are deterministic functions, in the stochastic case these are random variables obeying stochastic laws.

Our approach would lead us from the deterministic case to the stochastic one by adding a statistical-theoretic level on top of the existing system-theoretic foundations. In the process we extend our conceptual framework from deterministic systems to stochastic ones by identifying the stochastic counterparts of the deterministic case concepts, and by interpreting the statistical-theoretic objects from a system-theoretic viewpoint. A general procedure that takes us from the deterministic case to the stochastic one may be outlined for discrete event systems as follows:

In Section 2.1 we start with a deterministic discrete event system specified say at the state-DEVS level. More detailed specifications are also admissible provided they can be translated into the state-DEVS level. Next, we render it stochastic by informally describing its stochastic rules of operation.

In Section 2.2 we construct a formal probability space - the so-called coordinate space - which we take to be the statistical representation of our stochastic discrete event system. A connection between the original state-DEVS which was our starting point, and the resulting probability space is pointed out in Section 2.3. It involves system-theoretic representation of the sample space of the coordinate space. Each sample point is associated with a deterministic state-DEVS, from which we derive at the DEMS level, a deterministic state trajectory that models a particular sample realization of the stochastic DEVS.

The merit of this representation stems from the fact that it yields sample points which are considerably structured. Consequently, the definition of random variables becomes natural and intuitive, since it reduces to choosing behavioral frames for each state-DEVS representing a sample point. This is discussed in Section 2.4.

Moreover, relations among a variety of stochastic DEVSs become more transparent at the sample space level. Such sample point relations could induce statistical relations among the corresponding σ-algebras and probability measures. When this happens, one may correctly deduce properties of one stochastic DEVS from those of a related one, via statistical morphisms. Later on we shall take advantage of such situations in a queuing network context, through the formal tool of stochastic simplifications (of probability spaces), to be described in the next chapter, and by using the examples of Section 2.5.

The discussion in this chapter assumes familiarity with the basic concepts of Probability Theory. The reader is referred to standard texts such as [D1], [F1], [F2], [H1], [L1] and [W1] for the relevant background.

2.1 Informal Description of Stochastic DEVSs

A stochastic DEVS is a nondeterministic DEVS whose operation obeys statistical laws. The stochastic aspects of operation will be later on cast in terms of stochastic processes, while the deterministic ones will be described from a system-theoretic standpoint.

We start with a deterministic state-DEVS $M = \langle X,S,\cdot,t,\delta,\cdot\rangle$ and the discrete event paradigm $M \rightarrow G(M) \rightarrow S_{G(M)}$ generated by it. (See Ch. 1 Sec. 1.2). This paradigm gives rise to stochastic discrete event systems formulated as stochastic DEVSs, which we now proceed to describe informally.

We think of a stochastic DEVS as starting its operation at time 0 from some stochastic full state (s,0) under some stochastic input segment which is a stochastic composition of generators in Ω_X. It is convenient to give an informal description of the operation of a stochastic DEVS from a simulation oriented standpoint.

a) When the system is in an initial state and whenever an external event occurs, the next external event is scheduled by a stochastic choice of a generator in Ω_X.

b) When the system is in an initial state (s,0) and whenever it jumps to a new sequential state s, a time advance value $t(s)$ is sampled stochastically to determine the duration that the system will remain in sequential state s.

c) Finally, whenever the system is about to jump to a new sequential state, a stochastic decision is made to determine this new sequential state.

For the moment we can think of the stochastic decision makers as appropriately related random number generators. Mathematically, these would be random variables over the same underlying probability space with prescribed joint distributions. These random variables play analogous roles to the next generator in the m.l.s decomposition (see Appendix A Sec. A.2) of an input segment, the time advance function t, and the transition function δ respectively. They also generate the underlying probability space, so that all observations of a stochastic DEVS become random functions over that space. We assume, however, that the generating random variables are all real valued. This requires all sequential states and all external events to be coded by real numbers.

In the next section we shall map the underlying probability space above into a probabilistically equivalent one, in a canonical manner. The term "probabilistic equivalence" of probability spaces has here the following meaning.

Definition 2.1.1

Let $S = \langle \Omega, A, P \rangle$ and $S' = \langle \Omega', A', P' \rangle$ be probability spaces. We say that S and S' are *(probabilistically) equivalent* if there is a bijective map $h: A \rightarrow A'$ such that

$\forall A \in A,\ P'(h(A)) = P(A).$ □

The aforesaid mapping procedure will yield a constructively specified probability space called the coordinate probability space. This new probability space will constitute the formal statistical representation of our informal DEVS, or for that matter, of any stochastic discrete event system, at any level of informal description. The procedure is sufficiently general to be extended to general stochastic systems, so that the forthcoming discussion need not be restricted to stochastic discrete event ones.

In going from deterministic systems to stochastic ones, we shall retain our original system-theoretic orientation. However, the deterministic case definitions will have to be modified for the stochastic case, and recast in probabilistic terminology. We now outline how our underlying conceptual framework may be extended from deterministic systems to encompass stochastic ones.

Stochastic systems model those systems whose governing laws are "uncertain" to the modeler. This uncertainty results from fragmentary knowledge which is insufficient to determine those rules. The missing factors needed to account for the system's operation are aggregated as "uncertainty", "randomness" or "nondeterminism" and quantified as probabilities.

In other situations, the laws governing the system's operation are too complex to describe or compute, and a stochastic model is chosen to describe a simplified version of the system at the cost of a certain loss of information.

In any event, a stochastic system is formally represented by a probability space $S = \langle \Omega, A, P \rangle$ which captures its stochastic state structure. The objects in S have the following interpretation:

1. The sample space Ω is a set of outcomes. Each outcome $\omega \in \Omega$ represents a particular deterministic sample history obtained from some simulation run of the system. Ω stands for all conceivable outcomes of such runs. Any specification of $\omega \in \Omega$ is admissible provided all specifications are mutually related in a one-one manner. In many cases, ω can be represented by a

deterministic system or an appropriate state trajectory which model the system history ω, and Ω thus becomes a set of systems or state trajectories respectively.

2. The σ-algebra A is a set of events (ensembles of outcomes) for which probabilistic information is available. Since information regarding stochastic systems is cast in probabilistic terms, A describes the scope of such available information.

3. The probability measure P is a set function from A into [0,1] which quantifies the uncertainty of events in A. P(E) is interpretted as the chance that the ensemble of histories E, will indeed occur.

Statements about behavioral aspects of stochastic systems are cast in terms of events describable by random variables over S. The probabilistic information embedded in A allows us to quantify the uncertainty of such statements. In particular, observations of a stochastic system in a certain behavioral frame emerge as stochastic processes over S.

Finally, morphisms among stochastic systems become measure preserving transformations between pairs of probability spaces. In the next chapter, this approach would allow us to extend the concept of system simplification from the deterministic case to the stochastic one.

2.2 The Coordinate Probability Space

The construction of the coordinate probability space is a standard procedure in Probability Theory (cf. [CL1], [D1] and [W1]).

The starting point is a family of finite dimensional distributions

$$\mathcal{F} = \{F_{\theta_1,\ldots,\theta_n}(y_1,\ldots,y_n): \ \theta_1,\ldots,\theta_n \in \Theta \text{ and } n \in N\}$$

where Θ is some index set and N is the set of natural numbers.
We require $\mathcal{F}$ to satisfy two regularity conditions:

1) Consistency viz.

$$\lim_{y_n \to \infty} F_{\theta_1,\ldots,\theta_{n-1},\theta_n}(y_1,\ldots,y_{n-1},y_n) = F_{\theta_1,\ldots,\theta_{n-1}}(y_1,\ldots y_{n-1})$$

and

2) Symmetry viz.

$$F_{\theta_1,\ldots,\theta_n}(y_1,\ldots,y_n) = F_{\theta_{i_1},\ldots,\theta_{i_n}}(y_{i_1},\ldots,y_{i_n})$$

where $(\theta_{i_1},\ldots,\theta_{i_n})$ is an arbitrary permutation of $(\theta_1,\ldots,\theta_n)$.

In this case Kolmogorov showed (see [CL1] Sec. 3.3) that there is a probability space $\mathcal{S} = \langle \Omega, A, P \rangle$ and a stochastic process $\mathcal{Y} = \{Y_\theta\}_{\theta \in \Theta}$ over $\mathcal{S}$ such that for any $\theta_1,\ldots,\theta_n \in \Theta$ and any $n \in N$

$$F_{Y_{\theta_1},\ldots,Y_{\theta_n}}(y_1,\ldots,y_n) = F_{\theta_1,\ldots,\theta_n}(y_1,\ldots,y_n) .$$

Following [D1] we term a probability space $\mathcal{S}$ thus constructed - *the coordinate probability space* induced by $\mathcal{F}$.

In our case, the family of finite dimensional distributions $\mathcal{F}$ will be given a priori semantics in terms of the informal stochastic DEVS of the previous section. This would make the construction procedure of the coordinate probability space a rather intuitive one.

For our case we require that $\Theta = \{1,2,3,4\} \times N$. We distinguish in $\mathcal{F}$ the following types of distributions:

a) A sequence $\{F_{1,j}\}_{j=1}^{\infty}$, later on the distributions of the j-th external event.

b) A sequence $\{F_{2,j}\}_{j=1}^{\infty}$, later on the distributions of the length of the j-th time interval between the j-th and j+1st external events.

c) A sequence $\{F_{3,j}\}_{j=1}^{\infty}$, later on the distributions of the j-th sequential state into which the system evolves.

d) A sequence $\{F_{4,j}\}_{j=1}^{\infty}$, later on the distributions of the j-th value of the time advance function.

We remark that the above interpretation reflects mostly modeling situations where a distinction between the "stochastic system" and its "stochastic environment" is essential. In many cases the "stochastic environment" can be lumped into the state structure to yield an "autonomous stochastic system" thus eliminating distribution types a) and b).

When a higher level description of a stochastic DEVS is given, the semantics of the distribution functions in $\mathcal{F}$ should be assigned in terms of the description employed. Indeed, when we particularize to queues and queuing networks, the two comments above will be invoked. However, the mathematical construction of the coordinate probability space is free of any interpretations of $\mathcal{F}$, and moreover, the procedure we are about to describe is sufficiently general and representative to serve as a prototype or guide lines for the class of stochastic discrete event systems.

We now proceed to describe in some detail the construction of the coordinate probability space $S = \langle \Omega, A, P \rangle$ of an informal stochastic DEVS.

I) Construction of the coordinate sample space Ω:

Intuitively, a sample point represents the outcome of a statistical "experiment". In our case, the experiment is "simulating" an informal stochastic DEVS, and the outcome is the resulting sample history obtained by such a "simulation run". In order to capture the intuitive content of the sample point concept, we define a sample point $\omega \in \Omega$ as a countable aggregate $\omega = \{\omega_{i,j}\}_{i=1}^{4}{}_{j=1}^{\infty}$ where

$$\{\omega_{1,j}\}_{j=1}^{\infty} \triangleq \{a_j\}_{j=1}^{\infty}$$

$$\{\omega_{2,j}\}_{j=1}^{\infty} \triangleq \{b_j\}_{j=1}^{\infty}$$

$$\{\omega_{3,j}\}_{j=1}^{\infty} \triangleq \{c_j\}_{j=1}^{\infty}$$

$$\{\omega_{4,j}\}_{j=1}^{\infty} \triangleq \{d_j\}_{j=1}^{\infty} .$$

Each of the sequences $\{a_j\}$, $\{b_j\}$, $\{c_j\}$ and $\{d_j\}$ is a real sequence representing a certain realization compatible with the interpretations given in a), b), c) and d) respectively. Thus, $\{a_j\}$ represents a particular sequence of external events to occur in a particular sample history of our informal stochastic DEVS, and $\{b_j\}$ represents a particular sequence of inter-event intervals. Consequently, $(\{a_j\},\{b_j\})$ stands for a particular realization of the stochastic input segment; graphically, $(\{a_j\},\{b_j\})$ defines some infinite pulse train starting at the origin. In a similar manner $(\{c_j\},\{d_j\})$ represents a particular trajectory of the sequential state; graphically, $(\{c_j\},\{d_j\})$ defines some infinite step function starting at the origin.

Note that $\{a_j\}$, $\{b_j\}$, $\{c_j\}$ and $\{d_j\}$ jointly (i.e. the sample point ω represented by the above) do indeed specify a sample history that could conceivably be obtained from a trial run of a stochastic DEVS. At this juncture we repeat our previous remark that in many cases, the "stochastic input" is lumped into the "stochastic state" and the sample point ω reduces to the aggregate $(\{c_j\},\{d_j\})$, i.e. to specifications of autonomous sequential state trajectories.

We point out again that sample histories of a stochastic DEVS can be specified at other levels, provided the input and state trajectories are derivable from them. This point will be later illustrated in a queuing context.

Generally, in order to qualify for a sample space of a stochastic DEVS, Ω has to consist of sample points ω, each being a countable aggregate of real numbers that is adequate to specify a particular sample history.

II) Construction of the σ-algebra $\mathcal{A}$:

Let $\mathcal{B}$ be the *Borel field* on the real line $\mathbb{R}$, i.e. the minimal σ-algebra generated by the intervals of $\mathbb{R}$. Likewise, let $\mathcal{B}^n$ be the *Borel field* on the n-dimensional Euclidean space $\mathbb{R}^n$.

A set $C \subset \Omega$ is called a *cylinder set* if C has the form

$$C = \{\omega \in \Omega: (\omega_{i_1,j_1}, \omega_{i_2,j_2}, \ldots, \omega_{i_n,j_n}) \in B\}$$

for any $n \in N$ and any $B \in \mathcal{B}^n$.

Consider the collection $\mathcal{C}$ of all cylinder sets in Ω, and let $\sigma(\mathcal{C})$ be the minimal σ-algebra generated by $\mathcal{C}$.

III) Construction of the probability measure P:

Recall that $\Theta = \{1,2,3,4\} \times N$. For any $\theta_1, \ldots \theta_n \in \Theta$ let $P_{\theta_1, \ldots, \theta_n}$ be the probability measure induced on $\mathcal{B}^n$ by the joint distribution $F_{\theta_1, \ldots, \theta_n}$ in $\mathcal{F}$. Since the map $F_{\theta_1, \ldots, \theta_n} \mapsto P_{\theta_1, \ldots, \theta_n}$ is injective (see [W1] p. 7), it follows that $\mathcal{F}$ induces an equivalent family of probability measures $\mathcal{P} = \{P_{\theta_1, \ldots, \theta_n} : \theta_1, \ldots, \theta_n \in \Theta,\ n \in N\}$.

Now, the cylinder sets in $\mathcal{C}$ constitute an algebra. Moreover, this is the minimal algebra generated by the cylinder sets (see [W1] p. 7). Define a probability measure μ on the σ-algebra $\sigma(\mathcal{C})$ as follows. Let $C = \{\omega \in \Omega : (\omega_{i_1, j_1}, \ldots, \omega_{i_n, j_n}) \in B\}$ be a cylinder set and take $P_{(i_1, j_1), \ldots, (i_n, j_n)} \in \mathcal{P}$. Define $\mu(C) \triangleq \int_B dP_{(i_1, j_1), \ldots, (i_n, j_n)}$.

Now, by Carathiodory's extension theorem, μ can be extended from $\mathcal{C}$ to $\sigma(\mathcal{C})$ in a unique way (see [W1] p. 3).

Finally, let $A \triangleq \overline{\sigma(\mathcal{C})}$ be the *completion* of $\sigma(\mathcal{C})$ with respect to μ, and let $\mathbb{P}$ be the completed version of μ.

Definition 2.2.1

The *statistical representation* of an informal stochastic DEVS is the coordinate probability space $\mathcal{S} = \langle \Omega, A, P \rangle$ induced by the informal stochastic DEVS and whose construction is outlined in I), II) and III) above.

□

The term *stochastic discrete event system* will refer to any informal description of a system that is modellable as an informal stochastic

DEVS, along with the coordinate probability space induced by it. A further justification for this terminology is provided in the next section.

2.3 System-Theoretic Representations of Coordinate Sample Points

Our next step is to make the system-theoretic aspects of the coordinate sample space more direct and more explicit. To do this we first associate with each $\omega \in \Omega$ a state-DEVS $M(\omega)$. Finally, we derive from it an infinite state trajectory $STRAJ_{q,\eta}$ that serves as a system-theoretic representation of the sample queuing history ω. The derivation follows the paradigm $\omega \mapsto M(\omega) \mapsto G(M(\omega)) \mapsto S_{G(M(\omega))} \mapsto STRAJ_{q_\omega,\eta_\omega}$ (See Ch. 1 for relevant background).

Let $\omega = (\{a_j\},\{b_j\},\{c_j\},\{d_j\})$ be any sample point in Ω. Define the associated state-DEVS $M(\omega) = \langle X_\omega, S_\omega, \cdot, t_\omega, \delta_\omega, \cdot \rangle$ as follows:

$X_\omega \triangleq \{a_j: j = 1,2,\ldots\}$

$S_\omega \triangleq \{(j,c_j): j = 1,2,\ldots\}$

$t_\omega : S_\omega \to (0,\infty]$ is defined by

$t_\omega(j,c_j) \triangleq d_j$

$\delta_\omega : Q_\omega \times (X_\omega \cup \{\phi\}) \to S_\omega$ is defined by

$\delta_{\omega,\phi}(j,c_j) \triangleq (j+1,c_{j+1})$

and

$\delta_{\omega,M}(((j,c_j),e),a_k) \triangleq (j+1,c_{j+1})$

Let η_ω be the infinite input segment $\eta_\omega \triangleq \bigodot_{j=1}^{\infty} (a_j)_{b_j}$ and let $q_\omega \triangleq ((1,c_1),0)$. Following the aforesaid paradigm all the way to the

DEMS level, it is then possible to define the infinite state trajectory $STRAJ_{q_\omega,\eta_\omega}$ in an obvious way. It is easy to see that the map $\omega \mapsto STRAJ_{q_\omega,\eta_\omega}$ is injective, and this fact enables us to replace the aggregate representation of ω by the appropriate (infinite) state trajectory $STRAJ_{q_\omega,\eta_\omega}$.

Sometimes, it is more convenient to represent $\omega \in \Omega$ as a state-DEVS by choosing $M(\omega)$ to be an autonomous state-DEVS whose external input is built into its state structure. To do this we define $M(\omega) = \langle X_\omega, S_\omega, \cdot, t_\omega, \delta_\omega, \cdot \rangle$ as follows:

$$X_\omega \triangleq \phi$$

$$S_\omega \triangleq \{(m,a_m)\}_{m=1}^{\infty} \times (0,\infty] \times \{(n,c_n)\}_{n=1}^{\infty} \times (0,\infty]$$

$t_\omega : S_\omega \to (0,\infty]$ is defined by

$$t_\omega((m,a_m),r_a,(n,c_n),r_c) \triangleq \min\{r_a,r_c\}$$

$\delta_\omega : Q_\omega \times \{\phi\} \to S_\omega$ is defined by

$$\delta_{\omega,\phi}((m,a_m),r_a,(n,c_n),r_c) \triangleq$$

$$\begin{cases} ((m+1,a_{m+1}),b_{m+1},(n,c_n),r_c - r_a), & \text{if } r_a < r_c \\ ((m,a_m),r_a - r_c,(n+1,c_{n+1}),d_{n+1}), & \text{if } r_c < r_a \\ ((m+1,a_{m+1}),b_{m+1},(n+1,c_{n+1}),d_{n+1}), & \text{if } r_a = r_c \end{cases}$$

The state trajectory representing ω is $STRAJ_{q_\omega,\eta_\omega}$ where $q_\omega \triangleq (1,c_1)$ and $\eta_\omega \triangleq \phi_\infty$.

In the sequel we shall interchange the aggregate representation, the autonomous state-DEVS representation and the state trajectory representation of ω as the need arises. We are justified in doing so, because all three representations are mutually related in a one-one manner.

It should be born in mind that although these representations describe time invariant systems in the system-theoretic sense, the state process of the stochastic system as represented by the coordinate probability space is not necessarily time invariant in the statistical sense.

A system-theoretic representation of Ω has additional advantages, aside from making the system-theoretic aspects of stochastic systems more transparent. In a statistical-theoretic analysis of such systems, the major interest lies in some statistical state process of the system. A standard approach would be to define the state space so as to render the statistical state process a Markov process. The statistical state would usually coincide with the system-theoretic state or with parts thereof. Moreover, the Markov property would require in general that the "state of the input" (i.e. recent input symbol and elapsed or residual time) be part of the "state process" under consideration. This fact further makes the autonomous state-DEVS and state trajectory representations of $\omega \in \Omega$ rather intuitive conceptualizations. It also allows us to classify stochastic DEVSs from a system-theoretic viewpoint as follows.

Definition 2.3.1

Let $S = \langle \Omega, A, P \rangle$ be the statistical representation of a stochastic DEVS, and let $M(\omega)$ be the autonomous state-DEVS representation of $\omega \in \Omega$. Then

a) the stochastic DEVS is *legitimate* if the set $\{\omega \in \Omega$: $M(\omega)$ is *not* a legitimate state-DEVS$\}$ is a null set, i.e. almost all $M(\omega)$ are legitimate state-DEVSs.

b) the stochastic DEVS is *regular* if the set $\{\omega \in \Omega: M(\omega)$ is *not* a regular state-DEVS$\}$ is a null set, i.e. almost all $M(\omega)$ are regular state-DEVSs. □

Naturally, we require a stochastic DEVS to be at least legitimate, in order that sample histories be well defined.

Another line of classification is suggested by the intuitive concept of multiple scheduling. Suppose the user partitions DEVS jumps into "types" which are attributable to various "types" of system-theoretic events. A multiple scheduling relative to the underlying partition takes place when a jump is attributed to the simultaneous occurrence of more than one system-theoretic event. Let us define the *event multiplicity* of a deterministic DEVS as the largest number of system-theoretic event "types" involved in any jump. Then the *event multiplicity* of a stochastic *DEVS* is defined as the smallest integer n such that the set $\{\omega \in \Omega: M(\omega)$ has event multiplicity larger than $n\}$ is a null set.

We remark in passing that most queuing systems are modellable by stochastic DEVSs which are regular and whose event multiplicity is 1 relative to the natural partitioning of transitions into arrival and service completion "types".

2.4 Random Variables over a Coordinate Probability Space

We discern two main classes of random variables over a coordinate probability space $S = \langle\Omega,A,P\rangle$ representing a stochastic DEVS. The first class consists of random variables, that generate A. The second class consists of all stochastic processes over S and is identified with the set of all behavioral frames of the underlying stochastic discrete event system.

The generating random variables are formally defined as projection functions on Ω as follows.

Definition 2.4.1

Let $G = \{g_{i,j}\}_{i=1}^{4}\,{}_{j=1}^{\infty}$ be an aggregate of functions over a coordinate probability space $S = \langle\Omega,A,P\rangle$.

Let each $g_{i,j}:\Omega \rightarrow \mathbb{R}$ be defined by

$$\forall\,\omega = \{\omega_{i,j}\}_{i=1}^{4}\,{}_{j=1}^{\infty} \in \Omega,\ g_{i,j}(\omega) = \omega_{i,j}.$$

Then G is called the (statistical) *generator set* of S. □

This terminology is justified by the fact that the σ-algebra $\sigma(G)$ generated by G is precisely the one generated by the cylinder sets C. In other words $A = \overline{\sigma\{G\}}$. (See [W1] p. 39.)

Consequently, the generator set G has a family of finite dimensional distributions which is precisely the one prescribed by F in Sec. 2.2, viz.

$$F_{g_{i_1,j_1},\ldots,g_{i_n,j_n}} \equiv F_{(i_1,j_1),\ldots,(i_n,j_n)}\ .$$

Indeed, the interpretation of the generator set $\mathcal{G}$ is compatible with the interpretation of the joint distributions in $\mathcal{F}$ as given in a), b), c) and d) of Sec. 2.2. That is,

a) $g_{1,j}$, j = 1,2,... is the random variable of the j-th external event.

b) $g_{2,j}$, j = 1,2,... is the random variable of the j-th inter-event time interval.

c) $g_{3,j}$, j = 1,2,... is the random variable of the j-th sequential state into which the system evolves.

d) $g_{4,j}$, j = 1,2,... is the random variable of the time advance assigned to the j-th sequential state.

To sum up, the coordinate probability space was constructed according to Kolmogorov's theorem so as to ensure that $\mathcal{G}$ generates it and has $\mathcal{F}$ as its family of finite dimensional distributions.

The second class of random variables over $\mathcal{S}$ consists of statistical observations pertaining to a certain behavioral aspect of our stochastic discrete event system.

<u>Definition 2.4.2</u>

Let $\mathcal{S} = \langle \Omega, A, P \rangle$ be a coordinate probability space representing some stochastic DEVS. Let $\mathcal{Y} = \{Y_\theta\}_{\theta \in \Theta}$ be a stochastic process over $\mathcal{S}$. Finally, let $\mathcal{S}_\mathcal{Y} = \langle \Omega, A_\mathcal{Y}, P_\mathcal{Y} \rangle$ be the probability space induced by $\mathcal{Y}$ in $\mathcal{S}$ where

$$A_\mathcal{Y} \triangleq \sigma(\{A \in A\colon\ A = Y_\theta^{-1}(B),\ \theta \in \Theta,\ B \in \mathcal{B}\}) \text{ and } P_\mathcal{Y} \triangleq P|A_\mathcal{Y}\ .$$

Then $\mathcal{Y}$ is called a *behavioral frame* of $\mathcal{S}$, and $\mathcal{S}_\mathcal{Y}$ is called the *probabilistic frame* induced by $\mathcal{Y}$ on $\mathcal{S}$.

□

Definition 2.4.3

Let $\mathcal{Y} = \{Y_\theta\}_{\theta\in\Theta}$ and $\mathcal{Y}' = \{Y'_\theta\}_{\theta\in\Theta}$ be two behavioral frames over $S = \langle\Omega,A,P\rangle$ and $S' = \langle\Omega',A',P'\rangle$ respectively, with the same index set Θ. Then $(\mathcal{Y},\mathcal{Y}')$ is called a *behavioral pair* of S and S'. We say that $\mathcal{Y}$ and $\mathcal{Y}'$ are *distribution equivalent* if they have the same family $F_{\mathcal{Y}} = F_{\mathcal{Y}'}$ of finite dimensional distributions. □

Clearly, if $(\mathcal{Y},\mathcal{Y}')$ is a distribution equivalent behavioral pair, then the probabilistic frames $S_{\mathcal{Y}}$ and $S_{\mathcal{Y}'}$ induced by them are probabilistically equivalent up to null sets in the sense of Definition 2.1.1.

Notice how the above definitions fit into our conceptual framework. As stated before, the totality of information carried by a stochastic system is embedded in the probability space representing it. By the same token, a behavioral frame should focus on a certain behavioral aspect by reducing that totality of information to the relevant part. Indeed, the σ-algebra $A_{\mathcal{Y}}$ coarsens the underlying σ-algebra A, as $A_{\mathcal{Y}} \subset A$. The desired effect is achieved because in $S_{\mathcal{Y}}$ we are left with a less extensive σ-algebra which can give us probabilistic information concerning only the stochastic observations of interest.

The most important behavioral frames are the "full state" process and the "sequential state" process. Whenever they are measurable, they define continuous parameter stochastic processes whose parameter is interpreted as time. Most behavioral frames of interest would be functions of the stochastic state, much as in the deterministic case.

The behavioral frame "initial state of the system" is especially important when a stochastic DEVS is specified through a stochastic transition structure. In this case, the "initial state" random variable is essential in specifying a sample history, while subsequent states are not, and can be removed from the generator set. This situation is typical of queuing systems as will be seen later.

The definition of functions over Ω becomes especially intuitive when the state trajectory representation or the autonomous state-DEVS representation of Ω are used. Such definitions involve a conceptual "simulation run" of $M(\omega)$ and observation of a particular aspect of the trajectories generated by it.

The problems of measurability of such functions (i.e. showing them to be random variables over *S*) are basically unchanged. When the problem arises, a typical technique amounts to showing that the prospective sample space functions can be obtained from the generating random variables via "measurable" operations. Loosely speaking, one must show that the "simulation" and "observation" operations, alluded to above, preserve the measurability of the generator set elements which are used in the process.

We point out that the scope of behavioral frames, definable on Ω, depends crucially on the representation chosen for Ω. While the aggregate representation contains maximum information, an alternative representation may incur a loss of information. For example, in queuing context, if $M(\omega)$ is a state-DEVS representation of ω whose sequential states keep track of queue length rather than of queue configuration, then behavioral frames concerning individual customers (e.g. waiting times) cannot be described, as the necessary information is lost in the course of the mapping $\omega \mapsto M(\omega)$. In order to recover such behavioral frames, we need a more elaborate state-DEVS model that keeps track of queue configuration and consequently of individual customers. Indeed, customer oriented behavioral frames are statistically harder to compute, a fact which has an obvious system-theoretic explanation in view of the increased complexity of the generic $M(\omega)$ required for the task. These points will be revisited and demonstrated in the examples of the next section and in Chapter 5.

We conclude this section by providing a standard reference frame for the class of behavioral frames.

Definition 2.4.4

Let $S = \langle \Omega, A, P \rangle$ be a coordinate probability space representing a stochastic DEVS. Let $S_{\mathcal{Y}} = \langle \Omega, A_{\mathcal{Y}}, P_{\mathcal{Y}} \rangle$ be the probabilistic frame induced on S by a behavioral frame $\mathcal{Y} = \{Y_\theta\}_{\theta \in \Theta}$

For any finite subset $L = \{i_1, \ldots, i_{|L|}\} \in \Theta$ define†

$\beta(L) = \langle \mathbb{R}^{|L|}, \mathcal{B}^{|L|}, P_L \rangle$ where $P_L(B) \triangleq P(\{\omega \in \Omega: (Y_{i_1}(\omega), \ldots, Y_{i_{|L|}}(\omega)) \in \mathcal{B}\})$ for any $B \in \mathcal{B}^{|L|}$. Then the collection $\beta(\mathcal{Y}) = \{\beta(L): L \subset \Theta$ is a finite subset$\}$ is called the *Borel frame* induced by $\mathcal{Y}$. □

The concept of a Borel frame merely maps the probabilistic frames induced by each finite subset of random variables in $\mathcal{Y}$, into equivalent frames whose sample space is always Euclidean and its σ-algebra is always the Borel one. Instead of dealing with a variety of sample spaces and σ-algebras of probabilistic frames, we can now deal with their standard counterparts. Thus, the problem of showing a behavioral pair $(\mathcal{Y}, \mathcal{Y}')$ with index set Θ to be distribution equivalent, reduces to one of showing that the P_L and P'_L measures in the corresponding $\beta(L)$ and $\beta'(L)$ are identical measures, for any finite $L \subset \Theta$.

† $|L|$ is the cardinality of L.

2.5 Queuing-Theoretic Examples

In this section we illustrate the construction of the coordinate probability space associated with various queuing systems.

Example 2.5.1 (Single queue)

A) Informal description:

Consider a queuing system composed of one service station with one server in it. Customers arrive randomly and join a waiting line if necessary. The service time given to each customer is of random duration. The line discipline is FIFO (first in first out) and the line itself has infinite capacity.

The j-th inter-arrival time interval is a random variable A_j with distribution function F_{A_j} and the service time given to the j-th customer is a random variable S_j with distribution function F_{S_j}. The initial line length is a random variable L_0 with distribution function F_{L_0}. In addition, assume that there is given a family $\mathcal{F}$ of finite dimensional distributions of the random variables $\{A_j\}_{j=1}^{\infty}$, $\{S_j\}_{j=1}^{\infty}$ and L_0 which is consistent and symmetric.

B) The coordinate probability space:

To determine a sample queuing history we need to know an initial line length of the system, a particular sequence of inter-arrival time intervals, and a particular sequence of service times given to the customers. Consequently, a sample point $\omega \in \Omega$ is an aggregate $\omega = \{\omega_{i,j}\} \triangleq (\ell_0, \{a_j\}_{j=1}^{\infty}, \{s_j\}_{j=1}^{\infty})$ where

0. $\{\omega_{0,j}\}_{j=0}^{0} = \ell_0$ is an initial line length.

1. $\{\omega_{1,j}\}_{j=1}^{\infty} = \{a_j\}_{j=1}^{\infty}$ is a sample sequence of inter-arrival time intervals.

2. $\{\omega_{2,j}\}_{j=1}^{\infty} = \{s_j\}_{j=1}^{\infty}$ is a sample sequence of service times given to the customers.

Next we take $\sigma(C)$ i.e. the minimal σ-algebra generated by all cylinder sets of Ω. The family F of finite dimensional distributions is used to define a measure P on $\sigma(C)$ as in III) of Sec. 2.2. Finally, $\sigma(C)$ is completed with respect to P to yield $\overline{\sigma(C)}$ and P is extended appropriately from $\sigma(C)$ to $\overline{\sigma(C)}$. This completes the construction of the coordinate probability space S associated with the informal description in part A).

C) The generator set:

The generator set G of S is $G = \{L_0, \{A_j\}, \{S_j\}: j=1,2,\ldots\}$ where the elements of G are redefined on S as the appropriate coordinate (projection) functions as follows.

Let $\omega = \{\omega_{i,j}\} = (\{\ell_0, \{a_j\}_{j=1}^{\infty}, \{s_j\}_{j=1}^{\infty})$ be any sample point in Ω. Then

0. $L_0(\omega) \triangleq \ell_0$

1. $A_j(\omega) \triangleq a_j$

2. $S_j(\omega) \triangleq s_j$

The random variables in G retain their interpretations as given in the informal description of part A).

D) System-theoretic representations of Ω:

Two representations of Ω via state-DEVSs will be exemplified. Let $\omega = (\ell_0, \{a_j\}_{j=1}^{\infty}, \{s_j\}_{j=1}^{\infty})$

D.1) Define a state-DEVS $M(\omega) = \langle X_\omega, S_\omega, \cdot, t_\omega, \delta_\omega, \cdot \rangle$ by

$X_\omega \triangleq \{1\}$

$S_\omega \triangleq \{(0,n,\infty): n \in N\} \cup \{(\ell,n,r): \ell > 0, n \in N, 0 \leq r < s_n\}$

$t_\omega(\ell,n,r) \triangleq r$

$$\delta_{\omega,\phi}(\ell,n,r) \triangleq \begin{cases} (0,n+1,\infty), & \text{if } \ell = 1 \\ (\ell-1,n+1,s_{n+1}), & \text{if } \ell > 1 \end{cases}$$

$$\delta_{\omega,M}(((\ell,n,r),e),x) \triangleq \begin{cases} (1,n,s_n), & \text{if } \ell = 0 \\ (\ell+1,n,r-e), & \text{if } \ell > 0 \end{cases}$$

For double scheduling any composition-type rule is applicable. This happens when an exogenous arrival and a service completion occur simultaneously. The state trajectory representation for ω is $STRAJ_{q_\omega,\eta_\omega}$ such that

$q_\omega = (s_0, 0)$ where

$$s_0 \triangleq \begin{cases} (0,1,\infty), & \text{if } \ell_0 = 0 \\ (\ell_0,1,s_1), & \text{if } \ell_0 > 0 \end{cases} \quad \text{and } \eta_\omega \triangleq \mathop{\circledcirc}_{j=1}^{\infty} 1_{a_j} .$$

D.2) Define an autonomous state-DEVS $M(\omega) = \langle X_\omega, S_\omega, \cdot, t_\omega, \delta_\omega, \cdot \rangle$ by

$X_\omega \triangleq \Phi$

$S_\omega \triangleq \{(0,(m,r_a),(n,\infty)): m,n \in N, 0 \leq r_a < a_m\} \cup$

$\{(\ell,(m,r_a),(n,r_s)): \ell,m,n \in N, 0 \leq r_a < a_m, 0 \leq r_s < s_n\}$

$t_\omega(\ell,(m,r_a),(n,r_s)) \triangleq \min\{r_a, r_s\}$

$$\delta_{\omega,\phi}(\ell,(m,r_a),(n,r_s)) \triangleq \begin{cases} (1,(m+1,a_{m+1}),(n,s_n)), & \text{if } r_a < r_s \text{ and } \ell = 0 \\ (\ell+1,(m+1,a_{m+1}),(n,r_s - r_a)), & \text{if } r_a < r_s \text{ and } \ell > 0 \\ (0,(m,r_a - r_s),(n+1,\infty)), & \text{if } r_s < r_a \text{ and } \ell = 1 \\ (\ell-1,(m,r_a - r_s),(n+1,s_{n+1})), & \text{if } r_s < r_a \text{ and } \ell > 1 \\ (\ell,(m+1,a_{m+1}),(n+1,s_{n+1})), & \text{if } r_a = r_s \end{cases}$$

Notice, that in the autonomous state-DEVS representation, the case $r_a = r_s$ corresponds to double scheduling of events in the state-DEVS representation D.1).

The state trajectory representation of ω is $STRAJ_{q_\omega,\eta_\omega}$ such that

$q_\omega = (s_0,0)$ where

$$s_0 \triangleq \begin{cases} (0,(1,a_1),(1,\infty)), & \text{if } \ell_0 = 0 \\ (\ell_0,(1,a_1),(1,s_1)), & \text{if } \ell_0 > 0 \end{cases} \quad \text{and } \eta_\omega \triangleq \phi_\infty .$$

□

Following the discussion in the previous section, we see that the state-DEVS representations D.1) and D.2) for Ω precludes customer-oriented behavioral frames, since $M(\omega)$ does not keep track of line configuration and consequently of individual customer identity. In order to attain such behavioral frames, $M(\omega)$ should be redefined so as to preserve that information.

This comment is also pertinent to the following two examples.

Example 2.5.2 (Single queue with feedback)

A) Informal Description:

Consider a queuing system composed of one service station with one server in it, and waiting line conventions as in Example 2.5.1. Customers arrive at the system randomly, and after service is completed, they instantaneously invoke a random desision maker which we call a decomposition switch. The switch has two readings coded by 0 and 1. If the switch indicates 0, the customer leaves the system altogether. If, however, it indicates 1, the customer is instantaneously fed back to the tail of the line to obtain another service in due time. The inter-arrival times of exogenous customers, the service times and the initial line length are random variables with distribution functions as in the previous example. In addition, the j-th switch reading (at the time of the j-th service completion) is a random variable V_j with distribution function F_{V_j}. Again, assume that an appropriate family $\mathcal{F}$ of finite dimensional distributions is given.

B) The coordinate probability space:

To determine a sample queuing history, we need to know an initial line length, a particular sequence of inter-arrival times of exogenous customers, a particular sequence of service times given to customers and a particular sequence of switch readings encountered by the customers.

Consequently, a sample point $\omega \in \Omega$ is an aggregate $\omega = \{\omega_{i,j}\} \stackrel{\Delta}{=} (\ell_0, \{a_j\}_{j=1}^{\infty}, \{s_j\}_{j=1}^{\infty}, \{v_j\}_{j=1}^{\infty})$ where

0. $\{\omega_{0,j}\}_{j=0}^{0} = \ell_0$ is an initial line length.

1. $\{\omega_{1,j}\}_{j=1}^{\infty} = \{a_j\}_{j=1}^{\infty}$ is a sample sequence of inter-arrival times of exogenous customers.

2. $\{\omega_{2,j}\}_{j=1}^{\infty} = \{s_j\}_{j=1}^{\infty}$ is a sample sequence of service times given to customers.

3. $\{\omega_{3,j}\}_{j=1}^{\infty} = \{v_j\}_{j=1}^{\infty}$ is a sample sequence of switch readings encountered by customers.

The coordinate probability space S is constructed analogously to part B) in Example 2.5.1.

C) The generator set:

The generator set is $G = \{L_0, \{A_j\}_{j=1}^{\infty}, \{S_j\}_{j=1}^{\infty}, \{V_j\}_{j=1}^{\infty}\}$ and its elements are redefined on Ω as the obvious projection functions with the obvious interpretations.

D) System-theoretic representations of Ω:

Let $\omega = (\ell_0, \{a_j\}_{j=1}^{\infty}, \{s_j\}_{j=1}^{\infty}, \{v_j\}_{j=1}^{\infty}) \in \Omega$ and define a sequence of random variables $\{Z_j\}_{j=1}^{\infty}$ almost everywhere on Ω by

$$Z_j(\omega) \triangleq \begin{cases} 0, \text{ if } j = 0 \\ \min\{k: k > Z_{j-1}(\omega) \text{ and } V_k(\omega) = 0\}, \text{ if } j > 0 \text{ and the minimum exists} \\ \text{undefined, otherwise} \end{cases}$$

$Z_j(\omega)$ is the index of the j-th 0 in $\{V_j(\omega)\}_{j=1}^{\infty}$, i.e. in an infinite sequence of Bernoulli trials.

Let $M(\omega) = \langle X_\omega, S_\omega, \cdot, t_\omega, \delta_\omega, \cdot \rangle$ be a state-DEVS given by

$$X_\omega \triangleq \{1\}$$

$$S_\omega \triangleq \{(0,n,v_n,\infty): n = Z_{j-1}(\omega)+1, j \in N\} \cup \{(\ell,n,v_n,r): \ell,n \in N, 0 \leq r < s_n\}$$

$$t_\omega(\ell,n,v_n,r) \triangleq r$$

$$\delta_{\omega,\phi}(\ell,n,v_n,r) \triangleq \begin{cases} (0,n+1,v_{n+1},\infty), \text{ if } \ell = 1 \text{ and } v_n = 0 \\ (\ell-1,n+1,v_{n+1},s_{n+1}), \text{ if } \ell > 1 \text{ and } v_n = 0 \\ (\ell,n+1,v_{n+1},s_{n+1}), \text{ if } v_n = 1 \end{cases}$$

$$\delta_{\omega,M}(((\ell,n,v_n,r),e),1) \stackrel{\Delta}{=} \begin{cases} (1,n,v_n,s_n), & \text{if } \ell = 0 \\ (\ell + 1,n,v_n,r - e), & \text{if } \ell > 0 \end{cases}$$

The tie-breaking rule for double scheduling is of the composition type. The state-trajectory representation of ω is $STRAJ_{q_\omega,\eta_\omega}$ such that

$q_\omega = (s_0,0)$ where

$$s_0 \stackrel{\Delta}{=} \begin{cases} (0,1,v_1,\infty), & \text{if } \ell_0 = 0 \\ (\ell_0,1,v_1,s_1), & \text{if } \ell_0 > 0 \end{cases} \quad \text{and } \eta_\omega \stackrel{\Delta}{=} \mathop{\circledcirc}_{j=1}^{\infty} 1_{a_j} \ .$$

□

Example 2.5.3 (Queuing network)

A) Informal description:

Consider a queuing system composed of m service stations labeled 1,2,...,m each housing a single server, and an infinite capacity waiting line with FIFO discipline. The initial line length at service station i is a random variable $L_{i,0}$ with distribution function $F_{L_{i,0}}$. Each service station can have a random input stream of customers from an exogenous source. The j-th inter-arrival time interval to service station i is a random variable $A_{i,j}$ with distribution function $F_{A_{i,j}}$. Customers are served at the service stations for random time periods. The j-th service time given in service station i is a random variable $S_{i,j}$ with distribution function $F_{S_{i,j}}$. When service is done, each customer enters a decomposition switch and a random decision is made regarding the next destination (switching) of that customer. The j-th switching decision at service station i is a discrete random variable $V_{i,j}$ with distribution function $F_{V_{i,j}}$. Each $V_{i,j}$ can assume a switching value from the set {0,1,...,m} where a value 0 means that the customer leaves the system altogether and all other values stand for service stations in the system. The topology of a queuing network may be described by a directed graph whose nodes represent service stations and whose arcs stand for permissible flow paths (switchings) of customers. It is often convenient to add to such a graph a fictitious node 0 which represents the "environment". The "environment" can be viewed both as the source of all exogenous customer streams as well as the sink of all customer streams that leave the system altogether.

In the sequel we shall often discuss the network in terms of its associated graph. As a matter of fact, we use the terms "nodes" and "service stations" interchangeably, and similarly for the terms "arcs" and "switching decisions".

As usual we assume that there is given a consistent and symmetric family F of finite dimensional distributions for the random variables $L_{i,0}$, $A_{i,j}$, $S_{i,j}$ and $V_{i,j}$ above.

B) The coordinate probability space:

To determine a sample queuing history we need to know an initial line length at each node, a particular sequence of exogenous interarrival time intervals at each node, a particular sequence of service times awarded at each node and a particular sequence of switching decisions made at the decomposition switch of each node. Consequently, a sample point $\omega \in \Omega$ is an aggregate

$$\omega = \{\omega_{k,i,j}\} \stackrel{\Delta}{=} (\{\ell_{i,0}\},\{a_{i,j}\}_{j=1}^{\infty},\{s_{i,j}\}_{j=1}^{\infty},\{v_{i,j}\}_{j=1}^{\infty},\ i = 1,2,...m)$$

where for every $i = 1,2,...,m$

0. $\{\omega_{0,i,j}\}_{j=1}^{\infty} = \ell_{i,0}$ is an initial line length at service station i.

1. $\{\omega_{1,i,j}\}_{j=1}^{\infty} = \{a_{i,j}\}_{j=1}^{\infty}$ is a sample sequence of inter-arrival times of exogenous customers at service station i.

2. $\{\omega_{2,i,j}\}_{j=1}^{\infty} = \{s_{i,j}\}_{j=1}^{\infty}$ is a sample sequence of service times given to customers in service station i.

3. $\{\omega_{3,i,j}\}_{j=1}^{\infty} = \{v_{i,j}\}_{j=1}^{\infty}$ is a sample sequence of switching decisions made at the decomposition switch of service station i.

The coordinate probability space is now constructed analogously to part B) of Example 2.5.1.

C) The generator set:

$$G = \{\{L_{i,0}\},\{A_{i,j}\}_{j=1}^{\infty},\{S_{i,j}\}_{j=1}^{\infty},\{V_{i,j}\}_{j=1}^{\infty}: i = 1,2,\ldots m\}$$

is the generator set and its elements are redefined on Ω as the obvious projection functions, with the obvious interpretations.

D) System-theoretic representation of Ω:

A natural way of representing $\omega \in \Omega$ as a DEVN is as follows. Let $\omega = (\{\ell_{i,0}\},\{a_{i,j}\}_{j=1}^{\infty},\{s_{i,j}\}_{j=1}^{\infty},\{v_{i,j}\}_{j=1}^{\infty}, i = 1,2,\ldots m)$ be a sample point, and define a sequence $\{Z_{i,j}\}_{j=1}^{\infty}$, $1 \le i \le m$, of random variables almost everywhere on Ω by

$$Z_{i,j}(\omega) \triangleq \begin{cases} 0, \text{ if } j = 0 \\ \min\{k: k > Z_{i,j-1}(\omega) \text{ and } V_{i,k}(\omega) \neq i\}, \text{ if } j > 0 \text{ and the minimum exists} \\ \text{undefined, otherwise} \end{cases}$$

Let $N(\omega) = \left\langle D, \{M_\alpha(\omega)\}_{\alpha \in D}, \{I_\alpha\}_{\alpha \in D}, \{Z_{\alpha,\beta}\}_{\substack{\alpha \in D \\ \beta \in I_\alpha}}, \{J_\alpha\}_{\alpha \in D} \right\rangle$ be the DEVN associated with ω. N(ω) is defined as follows:

$D \triangleq \{1,2,\ldots,m\}$

For $1 \leq \alpha \leq m$, $I_\alpha \triangleq \{$nodes β: there is an arc (α,β) in the associated graph$\}$

For $1 \leq \alpha \leq m$, $M_\alpha(\omega) = \left\langle X_{\alpha,\omega}, S_{\alpha,\omega}, \cdot, t_{\alpha,\omega}, \delta_{\alpha,\omega}, \cdot \right\rangle$ is a state-DEVS given by

$$X_{\alpha,\omega} \triangleq \{1_{\alpha,\beta}: \beta \in I_\alpha \text{ and } \beta \neq \alpha\} \cup \{1_\alpha\}$$

$$S_{\alpha,\omega} \triangleq \{(0,n,v_{\alpha,n},\infty): n = Z_{\alpha,j-1}(\omega) + 1, j \in N\} \cup \{(\ell,n,v_{\alpha,n},r): \ell,n \in N, 0 \leq r < s_{\alpha,n}\}$$

$$t_{\alpha,\omega}(\ell,n,v_{\alpha,n},r) \triangleq r$$

$$\delta_{\alpha,\omega,\phi}(\ell,n,v_{\alpha,n},r) \triangleq \begin{cases} (0,n+1,v_{\alpha,n+1},\infty), & \text{if } \ell = 1 \text{ and } v_{\alpha,n} \neq \alpha \\ (\ell-1,n+1,v_{\alpha,n+1},s_{\alpha,n+1}), & \text{if } \ell > 1 \text{ and } v_{\alpha,n} \neq \alpha \\ (\ell,n+1,v_{\alpha,n+1},s_{\alpha,n+1}), & \text{if } v_{\alpha,n} = \alpha \end{cases}$$

$$\delta_{\alpha,\omega,M}(((\ell,n,v_{\alpha,n},r),e),x) \triangleq \begin{cases} (1,n,v_{\alpha,n},s_{\alpha,n}), & \text{if } \ell = 0 \\ (\ell+1,n,v_{\alpha,n},r-e), & \text{if } \ell > 0 \end{cases}$$

For $1 \leq \alpha,\beta \leq m$,

$$Z_{\alpha,\beta}(\ell,n,v_{\alpha,n},r) \triangleq \begin{cases} 1_{\beta,\alpha}, & \text{if } v_{\alpha,n} = \beta \neq \alpha \\ \text{undefined}, & \text{otherwise} \end{cases}$$

For $1 \le \alpha \le m$, $J_{\alpha,\omega}: Q_{\alpha,\omega} \times 2^{\tilde{X}_{\alpha,\omega}} \rightarrow S_{\alpha,\omega}$ is defined by

$$J_{\alpha,\omega}(((\ell,n,v_{\alpha,n},r),e),E_\alpha) \triangleq$$

$$\begin{cases} \delta_{\alpha,\omega,\phi}(\ell + |E_\alpha| - 1,n,v_{\alpha,n},r), & \text{if } \phi_\alpha \in E_\alpha \\ \delta_{\alpha,\omega,M}(((\ell + |E_\alpha| - 1,n,v_{\alpha,n},r),e),x), & \text{if } \phi_\alpha \notin E_\alpha \text{ and } E_\alpha \neq \Phi \\ (\ell,n,v_{\alpha,n},r - e), & \text{if } E_\alpha = \Phi \end{cases}$$

Finally, we expand the DEVN $N(\omega)$ into the state-DEVS $M_N(\omega)$ associated with it (see Ch. 1, Sec. 1.1) and we derive the DEMS $S_{G(M_N(\omega))}$ (see Ch. 1, Sec. 1.2).

The state trajectory representation of ω is $STRAJ_{q_\omega,\eta_\omega}$ such that

$q_\omega = (s_0,0)$ where

$s_0 = (s_{1,0},\ldots,s_{m,0})$ is defined by

$$s_{\alpha,0} \triangleq \begin{cases} (0,1,v_{\alpha,1},\infty), & \text{if } \ell_{\alpha,0} = 0 \\ (\ell_{\alpha,0},1,v_{\alpha,1},s_{\alpha,1}), & \text{if } \ell_{\alpha,0} > 0 \end{cases} \qquad \text{for } 1 \le \alpha \le m$$

and $\eta_\omega = (\overset{\infty}{\underset{j=1}{\circledcirc}} a_{1,j}, \overset{\infty}{\underset{j=1}{\circledcirc}} a_{2,j}, \ldots, \overset{\infty}{\underset{j=1}{\circledcirc}} a_{m,j})$.

□

CHAPTER 3

STOCHASTIC MORPHISMS AND SIMPLIFICATIONS

3.0 Introduction

In Chapter 2, a conceptual framework for stochastic discrete event systems was set forth. In particular, Chapter 2 exemplified how a stochastic discrete event system may be canonically represented in coordinate probability space.

In this chapter, we extend this underlying conceptual framework to relations among stochastic systems in probability space representation. In accordance with Appendix B, these relations will be collectively referred to as *stochastic morphisms*; these will give rise to *stochastic simplifications*. Formally, stochastic morphisms are described as measure preserving relations between probability spaces. Such relations are employed, for example, in [Col] in a modeling context. Since the treatment in this chapter is at the probability space level, the extension alluded to above goes beyond stochastic systems, as interpretations of probability spaces are not restricted to stochastic systems in the sense of Chapter 2.

The organization of this chapter is as follows.

Section 3.1 introduces a class of stochastic morphisms of the measure preserving transformation type (cf. [D1] Ch. X and [H1] Ch. VIII), - the so-called *measure preserving point morphisms*.

Section 3.2 fits stochastic simplifications into the broader conceptual framework of Appendix B. This section treats the so-called *point simplifications*, brought about by measure preserving morphisms.

In addition, Theorems 3.1.1 and 3.2.1 provide sufficient conditions that establish a point simplification and guarantee it to preserve the probability law of behavioral frames. These theorems supply a basis for reducing the problem of "stochastic" preservation of stochastic processes to that of "deterministic" preservation of their sample functions.

Finally, Section 3.3 discusses the effect exerted by point simplifications on behavioral frames.

As in the previous chapter, the discussion in this chapter assumes familiarity with the basic concepts of Probability Theory. The reader is referred to standard texts such as [D1], [F1], [F2], [H1], [L1] and [W1] for the relevant background.

3.1 Stochastic Morphisms

Throughout this chapter we shall always assume, without loss of generality, that all probability spaces under consideration are complete.

The following definition isolates a class of stochastic morphisms.

Definition 3.1.1

Let $S = \langle \Omega, A, P \rangle$ and $S' = \langle \Omega', A', P' \rangle$ be probability spaces. Let $H: \Omega \to \Omega'$ be a surjective point mapping satisfying:

a) $\forall E' \in A', \quad H^{-1}(E') \in A$

i.e. *preservation of events* .

b) $\forall E' \in A', \quad P'(E') = P(H^{-1}(E'))$

i.e. *preservation of measure* .

Then H is called a *measure preserving point morphism* (abbreviated *m.p.p.m*), or simply a *point morphism* from S to S'. □

Measure preserving point morphisms are variants of measure preserving point transformations (cf. [D1] p. 453 and Sec. 3 of Supplement; [H1] Ch. VIII). Observe the simplification effect implicit in this class of stochastic morphisms (see Appendix B for a formal discussion of the simplification concept). A complexity reduction is achieved at two levels. At the sample space level, H lumps sample points, due to the fact that H is surjective but not necessarily injective. At the σ-algebra level, $H^{-1}(A')$ is a sub-σ-algebra of A by a) in Definition 3.1.1, so that the original event information in A may be reduoed. The preservation effect is described by condition b) as a measure preserving effect.

Note also that conditions a) and b) of Definition 3.1.1 do not generally hold in the other direction. For one thing if $E \in A$ then the m.p.p.m definition does not guarantee that $H(E) \in A'$. Even so, a probability preservation relation $P(E) = P'(H(E))$ does not necessarily follow, due to the inclusion $E \subset H^{-1}(H(E))$. To illustrate this point consider

Example 3.1.1

Take $\Omega \triangleq [0,2]$, $\Omega' \triangleq [0,1]$, and let A and A' be their respective Lebesque measurable sets.

$$\text{Define } P(E) \triangleq \begin{cases} 0, \text{ if } E \subset [0,1] \\ \text{Lebesque measure of } E, \text{ otherwise} \end{cases}$$

and define P' to be the Lebesque measure. Finally define a m.p.p.m $H:[0,2] \longrightarrow [0,1]$ by $H(\omega) \triangleq \omega \bmod 1$.

Take $N = [0,1]$. Then, $P(N) = 0$, but $P'(H(N)) = P'([0,1]) = 1$. □

The following definition gives a standard hierarchy of point morphisms.

Definition 3.1.2

Let H be a m.p.p.m from $S = \langle \Omega, A, P \rangle$ to $S' = \langle \Omega', A', P' \rangle$.

a) If $H^{-1}(A') = A$, then H is called a *measure preserving point homomorphism* (abbreviated *m.p.p.h*), or simply a *point homomorphism* from S to S'.

b) If in addition H is bijective, then H is called a *measure preserving point isomorphism* (abbreviated *m.p.p.i*), or simply a *point isomorphism* from S to S'. □

A simple instance of a measure preserving point morphism is a real random variable with a Borel measurable range.

Example 3.1.2

Let Y be a (real) random variable over a probability space $S = \langle \Omega, A, P \rangle$, such that the range of Y is Borel measurable. Define a probability space $S' = \langle \Omega', A', P' \rangle$ where

$\Omega' \triangleq Y(\Omega)$

A' is the Borel field over Ω'

$P' \triangleq P_Y$ is the probability measure induced on A' by the distribution of Y.

Define $H:\Omega \to \Omega'$ by $H \triangleq Y$. H is surjective by definition of Ω'. Moreover,

a) $\forall E' \in A'$, $H^{-1}(E') = \{\omega: Y(\omega) \in E'\} \in A$

b) $\forall E' \in A'$, $P'(E') = P_Y(E') = P\{\omega: Y(\omega) \in E'\} = P(H^{-1}(E'))$.

We conclude that Y is a m.p.p.m from S to S'. □

Two more examples of point morphisms follow.

Example 3.1.3

Let $S = \langle \Omega, A, P \rangle$ be a probability space and let $\overline{A}$ be a sub-σ-algebra of A. Take $S' = \langle \Omega', A', P' \rangle$ where $\Omega' \triangleq \Omega$, $A' \triangleq \overline{A}$ and $P' \triangleq P|\overline{A}$. Finally, $H:\Omega \to \Omega'$, defined as the identity function is a m.p.p.m from S to S'. □

Example 3.1.4

Let $S_1 = \langle \Omega_1, A_1, P_1 \rangle$ and $S_2 = \langle \Omega_2, A_2, P_2 \rangle$ be probability spaces. Let $S = \langle \Omega, A, P \rangle$ be the product space of S_1 and S_2, i.e. $\Omega \triangleq \Omega_1 \times \Omega_2$, A is the minimal σ-algebra generated by $A_1 \times A_2$, and P is the product measure. Finally take $S' \triangleq S_1$. Define $H:\Omega \to \Omega'$ to be the projection function $H(\omega_1, \omega_2) \triangleq \omega_1$. Then H is a surjective map satisfying:

a) $\forall E' \in A'$, $H^{-1}(E') = E' \times \Omega_2 \in A$

b) $\forall E' \in A'$, $P'(E') = P_1(E')P_2(\Omega_2) = P(E' \times \Omega_2) = P(H^{-1}(E'))$.

Hence, H is a m.p.p.m from S to S_1. □

The following theorem characterizes the class of measure preserving point morphisms.

Theorem 3.1.1

Let $S = \langle \Omega, A, P \rangle$ and $S' = \langle \Omega', A', P' \rangle$ be probability spaces and let $H: \Omega \to \Omega'$ be a surjective map.

Then H is a m.p.p.m from S to S' *iff* there are stochastic processes $Y = \{Y_\theta\}_{\theta \in \Theta}$ and $Y' = \{Y'_\theta\}_{\theta \in \Theta}$ over S and S' respectively, such that

a) Y' generates A' up to completion.

b) Y and Y' are distribution equivalent.

c) For every $\theta \in \Theta$, there is a null set $N_\theta \in A$ such that

(c.1) $\forall \omega \notin N_\theta,\ Y_\theta(\omega) = Y'_\theta(H(\omega))$.

Proof

($\Leftarrow$) Assume that there are Y and Y' satisfying conditions a) - c). Fix any finite $L \triangleq \{\theta_1, \dots, \theta_n\} \subset \Theta$ and any $B \in B^{|L|} = B^n$, where $|L| = n$ is the cardinality of L. Consider the sets

(1) $E \triangleq \{\omega: (Y_{\theta_1}(\omega), \dots, Y_{\theta_n}(\omega)) \in B\} \in A$

(2) $E' \triangleq \{\omega': (Y'_{\theta_1}(\omega'), \dots, Y'_{\theta_n}(\omega')) \in B\} \in A'$

and the Borel spaces

(3) $\beta(L) \triangleq \langle \mathbb{R}^{|L|}, B^{|L|}, P_L \rangle$

(4) $\beta'(L) \triangleq \langle \mathbb{R}^{|L|}, B^{|L|}, P'_L \rangle$.

Since Y and Y' are distribution equivalent, it follows from Definition 2.4.4 that

(5) $P_L \equiv P'_L$

In particular, it follows from (5) that

(6) $P(E) = P_L(B) = P'_L(B) = P'(E')$.

Next, let $\mathcal{N}$ be the collection of all null sets in $\mathcal{A}$. Then clearly

(7) $$N_L \triangleq \bigcup_{\theta \in L} N_\theta \in \mathcal{N}$$

Furthermore, in view of (c.1), (1), (2) and (7)

(8) $$H^{-1}(E') - N_L \subset E \subset H^{-1}(E') \quad N_L \; .$$

Since $\mathcal{A}$ is complete we can conclude from (8) that there is a null set $\hat{N} \subset N_L$ such that

(9) $$H^{-1}(E') = E \triangleright \hat{N} \in \mathcal{A}$$

whence, due to (6),

(10) $$P(H^{-1}(E')) = P(E) = P'(E').$$

As L ranges over all finite subsets of Θ and B ranges over all events in $\mathcal{B}^{|L|}$, the resulting sets E' in (2) range over the minimal algebra $\alpha(\mathcal{Y}')$ generated by $\mathcal{Y}'$. Thus, from (9) and (10) we conclude that conditions a) and b) of Definition 3.1.2 hold for any event $E' \in \alpha(\mathcal{Y}')$.

A standard application of the Caratheodory Theorem (see [L1] p. 87) extends the validity of (9) and (10) from the minimal algebra $\alpha(\mathcal{Y}')$ to the minimal σ-algebra $\sigma(\mathcal{Y}')$ generated by $\mathcal{Y}'$. It then readily follows that (9) and (10) are also true for every E' in the completion $\overline{\sigma(\mathcal{Y}')} = \mathcal{A}$. Hence H is a m.p.p.m from $\mathcal{S}$ to $\mathcal{S}'$.

($\Rightarrow$) Assume that H is a m.p.p.m from $\mathcal{S}$ to $\mathcal{S}'$.

Define $\Theta \triangleq \mathcal{A}'$, $Y' \triangleq \{I_{E'}\}_{E' \in \mathcal{A}'}$ and $Y \triangleq \{I_{H^{-1}(E')}\}_{E' \in \mathcal{A}'}$ where I_A is the indicator function of the set A. It follows that $\mathcal{Y}$ and $\mathcal{Y}'$ thus defined trivially satisfy conditions a) - c).

□

3.2 Stochastic Simplifications

Stochastic simplifications are defined via stochastic morphisms whose simplificational effect is discussed in the preceeding section and in Appendix B. For measure preserving point morphisms we make

Definition 3.2.1

Let $S = \langle \Omega, A, P \rangle$ and $S' = \langle \Omega', A', P' \rangle$ be probability spaces. We say that the ordered pair (S,S') is a *simplification pair*, if there is a m.p.p.m H running from S to S'. In this case we write $S \overset{H}{\mapsto} S'$ and refer to it as a (stochastic) *point simplification*. In this context, S will be termed the *base space*, and S' the *lumped space* of the point simplification $S \overset{H}{\mapsto} S'$. □

We note in passing that the point simplification relation among probability spaces is transitive.

The complexity reduction effect of a point simplification is that of lumping, since the map H and the set transformation h induced by H may be thought of as coarsening the base space's sample space and σ-algebra respectively.

The preservation effect of a stochastic simplification $S \mapsto S'$ on a behavioral pair $(\mathcal{Y}, \mathcal{Y}')$ should naturally be a statistical one. The most important preservation notion from an analytical standpoint is that of distribution equivalence of $\mathcal{Y}$ and $\mathcal{Y}'$. This situation will be referred to as *preservation in distribution*.

Weaker notions of preservation include preservation of one dimensional distributions, preservation of means and of higher moments of the respective random variables in $\mathcal{Y}$ and $\mathcal{Y}$.

For preservation in distribution we have the following sufficient condition.

Theorem 3.2.1

Let $S \overset{H}{\mapsto} S'$ be a point simplification where $S = \langle \Omega, A, P \rangle$ and $S' = \langle \Omega', A', P' \rangle$. Let $\mathcal{Y} = \{Y_\theta\}_{\theta \in \Theta}$ and $\mathcal{Y}' = \{Y'_\theta\}_{\theta \in \Theta}$ be stochastic processes over S and S' respectively. Suppose that the m.p.p.m H satisfies

a) for every $\theta \in \Theta$ there is a null set $N_\theta \in A$ such that

(a.1) $\forall \omega \notin N_\theta,\ Y_\theta(\omega) = Y'_\theta(H(\omega))$.

Then $\mathcal{Y}$ and $\mathcal{Y}'$ are distribution equivalent.

Proof

Take any finite $L \triangleq \{\theta_1, \ldots, \theta_n\} \subset \Theta$ and any $B \in \mathcal{B}^n$. Define $N_L \triangleq \bigcup_{\theta \in L} N_\theta$; then N_L is a null set of A. Consider the events

(1) $E \triangleq \{\omega\colon (Y_{\theta_1}(\omega), \ldots, Y_{\theta_n}(\omega)) \in B\} \in A.$

(2) $E' \triangleq \{\omega'\colon (Y'_{\theta_1}(\omega'), \ldots, Y'_{\theta_n}(\omega')) \in B\} \in A'.$

It follows from (1), (2) and a) that

(3) $H^{-1}(E') - N_L \subset E \subset H^{-1}(E') \cup N_L.$

From (3) we conclude that there is a null set $\hat{N} \in A$ such that

(4) $E = H^{-1}(E') \triangleright \hat{N}$.

But since H is a m.p.p.m, (4) implies

(5) $P(E) = P(H^{-1}(E') \triangleright \hat{N}) = P(H^{-1}(E')) = P'(E').$

Consequently, the Borel spaces $\beta(L) \triangleq \langle \mathbb{R}^{|L|}, \mathcal{B}^{|L|}, P_L \rangle$ and $\beta'(L) \triangleq \langle \mathbb{R}^{|L|}, \mathcal{B}^{|L|}, P'_L \rangle$ (see Definition 2.4.4) satisfy

(6) $P_L \equiv P'_L$ for any finite $L \subset \Theta$.

whence $\mathcal{Y}$ and $\mathcal{Y}'$ are distribution equivalent, as was to be proved.

□

Corollary 3.2.1

Replace condition a) in Theorem 3.2.1 by the following one:

a) for every $\theta \in \Theta$, there is a null set $N'_\theta \in \mathcal{A}'$ such that

(a.1) $\forall \omega' \notin N'_\theta$, $Y_\theta(\omega) = Y'_\theta(\omega')$

where ω is any inverse image of ω' by H, i.e. $H(\omega) = \omega'$.

Then Theorem 3.2.1 still holds.

□

3.3 The Effect of Point Simplifications on Behavioral Frames

In this section we shall investigate the simplification effect exercised by a point simplification on the behavioral frames of the base space. Our goal is to elucidate the nature of this effect and to derive an interpretation that would properly fit the underlying conceptual framework of Appendix B and Chapter 2.

We start with an interpretation based on the characterization of measure preserving point morphisms in Theorem 3.1.1. Loosely speaking, the theorem states that the existence of a m.p.p.m between probability spaces is equivalent to preservation in distribution of certain distinguished and comprehensive behavioral pairs.

Under our conceptual framework, this interplay between "structure" and "behavior" is hardly surprising. It coincides with our general view that structure is the totality of behavior and that the two notions are dual. Thus, in Theorem 3.1.1, point simplifications which are structure lumping at the sample space level and measure preserving at the σ-algebra level, emerge as equivalent to preservation in distribution of certain superframes.

Now, Theorem 3.2.1 provides a natural way of matching random variables over point morphic probability spaces. In the sequel, if $S = \langle \Omega, A, P \rangle$ is some underlying probability space, then $M(S)$ will denote the set of random variables over S.

Definition 3.3.1

Let $S \overset{H}{\mapsto} S'$ be a point simplification from $S = \langle \Omega, A, P \rangle$ to $S' = \langle \Omega', A', P' \rangle$. The *matching operator* from S to S' associated with

$S \overset{H}{\longmapsto} S'$ is $H{:}M(S') \to M(S)$ defined by

$H(Y') = Y$, where for every $\omega \in \Omega$ $Y(\omega) \overset{\Delta}{=} Y'(H(\omega))$. □

The matching operator is investigated in [D1] (see Sec. 3 of Supplement). Some properties of H are given in

Lemma 3.3.1

Let H be the matching operator associated with a point simplification $S \overset{H}{\longmapsto} S'$. Then

a) H satisfies RV_1, RV_2, RV_3 and RV_4 in [D1] pp. 453-454.

b) The range of H (denoted $R(H)$) satisfies

$R(H) \subset \{Y\colon Y \in M(S)$, and Y is constant on $H^{-1}(\omega'),\ \forall \omega' \in \Omega'\}$.

c) Every pair of random variables $Y' \in M(S')$ and $H(Y') \in M(S)$ is distribution equivalent.

Proof

$H^{-1}(A') \overset{\Delta}{=} \{A\colon A = H^{-1}(A')$ for some $A' \in A'\}$ is a sub-σ-algebra of A. Consider the set transformation $h\colon H^{-1}(A') \to A$ induced by H where

(1) $h(A) \overset{\Delta}{=} \bigcup_{\omega \in A} \{H(\omega)\}\ ,\quad \forall A \in H^{-1}(A')$.

It can be verified that h is bijective, and furthermore, that both h and h^{-1} satisfy MP_1, MP_2 and MP_3 in [D1] pp. 452-453. From [D1] p. 454 it now follows that H is the unique transformation satisfying a).

Condition b) follows from the fact that if $Y = H(Y')$, then

(2) $\forall \omega \in \Omega,\quad Y(\omega) = Y'(H(\omega))$

by definition of H. Equation (2) further implies that

(3) $\{\omega\colon Y(\omega) \in B\} = H^{-1}(\{\omega'\colon Y'(\omega') \in B\})$

for any Borel set B, whence c) follows by the measure preserving property of H^{-1}. □

A full characterization of the range of H is given in the following theorem.

Theorem 3.3.1

Let H be the matching operator associated with a point simplification $S \overset{H}{\mapsto} S'$ where $S = \langle \Omega, A, P \rangle$ and $S' = \langle \Omega', A', P' \rangle$. Then

(a) $R(H) = \{Y\colon Y \text{ is measurable on } H^{-1}(A')\}$.

Proof

If $Y \in R(H)$, then Y is measurable on $H^{-1}(A')$ by (3) in Lemma 3.3.1. Conversely, suppose that Y is measurable on $H^{-1}(A')$. Suppose first that $Y = I_A$ for some $A \in H^{-1}(A')$. Then $Y \in R(H)$ because $H(I_{H(A)}) = I_A$ by RV_2. The proof for an arbitrary Y measurable on $H^{-1}(A')$ follows from RV_3 and RV_4. □

Loosely speaking, Theorem 3.3.1 shows that the effect of H on $M(S')$ is to match it with a subset of $M(S)$, whose elements have restricted measurability. Moreover, Y' is a distribution equivalent lumped version of $H(Y')$, due to c) and b) respectively in Lemma 3.3.1. Y' is also seen to be a coarser version of $H(Y')$ by (a) in Theorem 3.3.1.

Later on, we shall argue that this restrictional effect may be viewed as the effect of the point simplification $S \overset{H}{\mapsto} S'$ on the set of behavioral frames of S. To clarify this view we shall consider point simplifications which are substantive in the following sense.

Definition 3.3.2

Let $S = \langle \Omega, A, P \rangle$ and $S' = \langle \Omega', A', P' \rangle$ be probability spaces. A point simplification $S \overset{H}{\mapsto} S'$ is called *strict* if

a) $\exists E_0 \in A$ such that $\forall E' \in A'$, $P(E_0 \triangleright H^{-1}(E')) > 0$. □

Observe that non-strictness means that S and S' are point homomorphic (i.e. $H^{-1}(A')$ equals A) up to null sets.

In the sequel, we shall use the equivalence relation $\overset{a.s.}{=}$ (equality almost surely) on $M(S)$, where $S = \langle \Omega, A, P \rangle$ is some underlying probability space. This relation is defined by

$Y_1 \overset{a.s.}{=} Y_2$ iff $P(\{\omega: Y_1(\omega) \neq Y_2(\omega)\}) = 0$.

An equivalence class under $\overset{a.s.}{=}$ is denoted $[Y]$ for any representative $Y \in M(S)$, and will be referred to as the set of *versions* of Y.

Next, we characterize strict point simplifications in terms of its matching operator H.

Theorem 3.3.2

Let $S \overset{H}{\mapsto} S'$ be point simplification. Then

$S \overset{H}{\mapsto} S'$ is strict iff

H is not surjective in the sense that there is $Y \in M(S)$ such that

(a) $[Y] \cap R(H) = \Phi$.

Proof

($\Rightarrow$) Assume $S \overset{H}{\mapsto} S'$ is strict. Let $E_0 \in A$ be the event satisfying (a) of Definition 3.3.2. Consider the indicator function I_{E_0} of E_0. Suppose that for some null set $N \in A$ there is a version $I_{E_0 \triangleright N}$ of I_{E_0}

such that $I_{E_0 \triangleright N} \in \mathcal{R}(H)$. But from Theorem 3.3.1 it follows that $I_{E_0 \triangleright N}$ is measurable on $H^{-1}(A')$. In particular, $E_0 \triangleright N \in H^{-1}(A')$ viz.

(1) $H^{-1}(E_0') = E_0 \triangleright N$ for some $E_0' \in A'$.

Taking note of (1) we have

$P(E_0 \triangleright H^{-1}(E_0')) = P(E_0 \triangleright (E_0 \triangleright N)) = 0$

which contradicts (a) of Definition 3.3.1. We conclude that

$[I_{E_0}] \cap \mathcal{R}(H) = \Phi$.

($\Leftarrow$) Assume that $S \overset{H}{\mapsto} S'$ is not strict. Then $A = \sigma(\{H^{-1}(A') \cup N\})$ where N is the class of null sets in A. Consequently, if Y is A-measurable, there is a version $Y^* \in [Y]$ which is $H^{-1}(A')$-measurable. By Theorem 3.3.1, $Y^* \in \mathcal{R}(H)$; so that $[Y] \cap \mathcal{R}(H) \neq \Phi$ as required. □

From Definition 3.3.2 we see that in order to render a strict point simplification a nonstrict one, one needs to coarsen the σ-algebra of the base space. Now, Theorem 3.3.2 asserts that this is equivalent to limiting the scope of random variables over it to those which have a version in $\mathcal{R}(H)$, and by Theorem 3.3.1 these are $H^{-1}(A')$-measurable. We then proceed to claim that this can be viewed as the effect of a point simplification on the behavioral frames of its base space. To do this we argue that the underlying point simplification may be replaced by an equivalent one as follows.

Let $S \overset{H}{\mapsto} S'$ be a strict point simplification where $S = \langle \Omega, A, P \rangle$ and $S' = \langle \Omega', A', P' \rangle$. The alleged equivalent point simplification is $S \overset{I}{\mapsto} \hat{S}$ where I is the m.p.p.m of Example 3.1.3; that is, $\hat{S} = \langle \hat{\Omega}, \hat{A}, \hat{P} \rangle$ where $\hat{\Omega} \triangleq \Omega$, $\hat{A} \triangleq H^{-1}(A')$, $\hat{P} \triangleq P|\hat{A}$, and I is the identity map. (See Figure 3.3.1.)

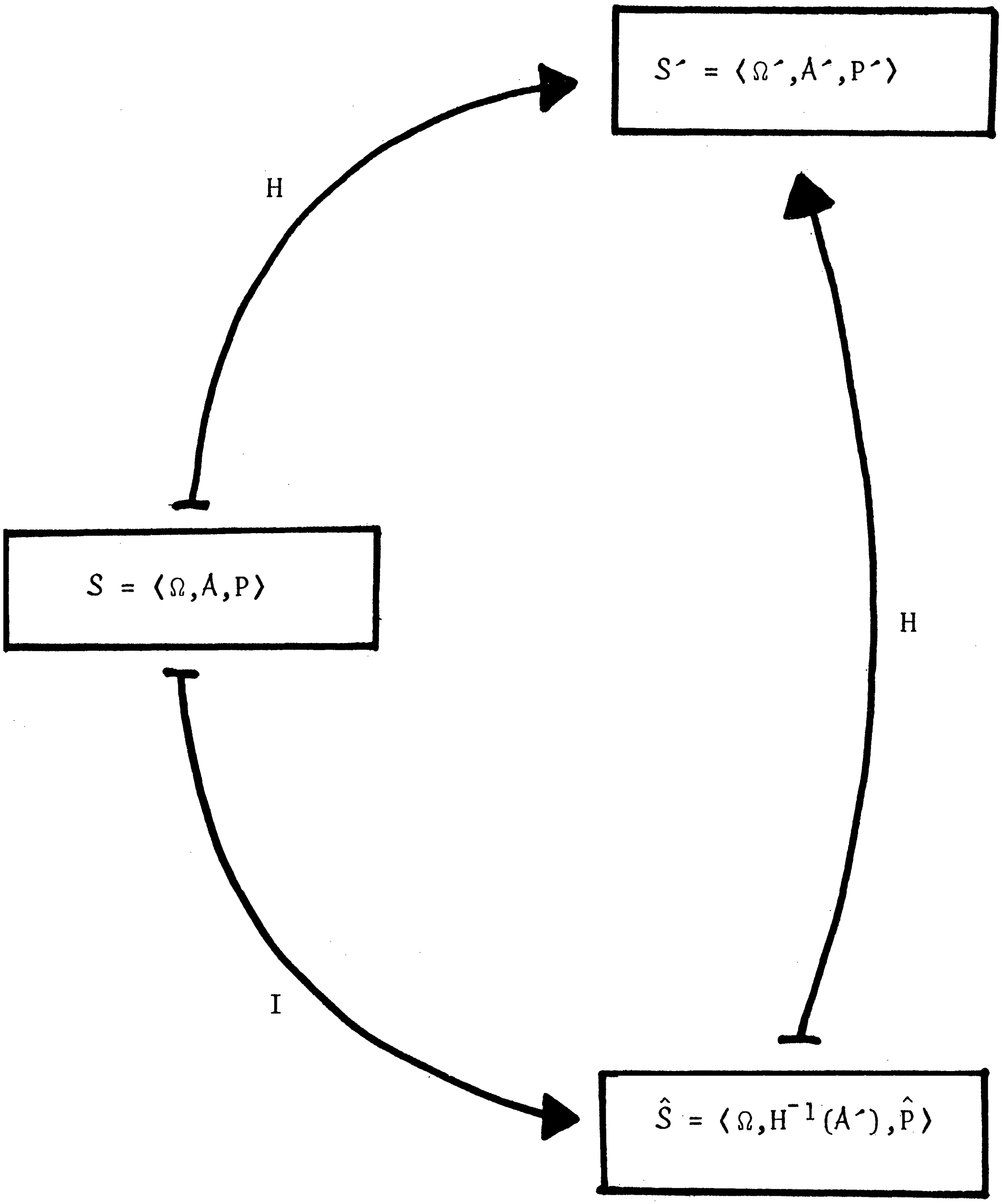

Figure 3.3.1: Relations among the Equivalent Point Simplifications $S \overset{H}{\mapsto} S'$ and $S \overset{I}{\mapsto} \hat{S}$.

To qualify the equivalence claim above we merely point out that S and S' are point homomorphic, so that $\hat{S} \overset{H}{\mapsto} S'$ is a nonstrict point simplification. Consequently, $S \overset{H}{\mapsto} S'$ and $S \overset{I}{\mapsto} \hat{S}$ can be viewed as equivalent simplifications, since their base spaces are identical, and their lumped spaces are point homomorphic and therefore probabilistically equivalent in the sense of Definition 2.1.1.

In particular, Theorem 3.3.2 guarantees that all random variables in $M(S')$ and $M(\hat{S})$ can be exhaustively matched (up to equality almost surely) in a distribution preserving manner via the matching operator H associated with the non-strict point simplification $S' \overset{H}{\mapsto} \hat{S}$.

Thus, we are justified in trying to determine the simplification effect of the point simplification $S \overset{H}{\mapsto} S'$ from the equivalent point simplification $S \overset{I}{\mapsto} \hat{S}$, especially as regards the behavioral frames.

The lumping effect of $S \overset{I}{\mapsto} \hat{S}$, as far as "structure" is concerned, is evident, since $H^{-1}(A')$ is a coarsening of A in the sense that the former is a sub-σ-algebra of the latter. In particular, every atom† of $H^{-1}(A')$ is a union of atoms of A.

The simplification effect of $S \overset{I}{\mapsto} \hat{S}$ as far as "behavior" is concerned can now be described as a reduction in the scope of the behavioral frames of the base space S. For one thing, $M(S) \supset M(\hat{S})$. Furthermore, the random variables in $M(\hat{S})$ are coarser than those in $M(S)$, because restricted measurability of random variables increases their sets of constancy.

This simplification effect can be seen even more clearly when one examines random variables in $M(S)$ and $M(\hat{S})$ that have mathematical

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†An event A is an atom if every measurable subset of it is either A or Φ.

expectations. It is easily seen that the class of such random variables over $\hat{S}$ can be obtained as conditional expectations of random variables in $M(S)$ with respect to the σ-algebra $H^{-1}(A')$. The smoothing effect of conditional expectations is well known (see [L1] p. 349); loosely speaking, random variables (with expectation) in $M(S)$ are averaged to constants over the non-null atoms of $H^{-1}(A')$, thus yielding a random variable in $M(S')$.

CHAPTER 4

JACKSON QUEUING NETWORKS

4.0 Introduction

Jackson queuing networks are a generalization of M/M/s queues, and as such they provide the simplest generalization from single queues to networks of queues.

Thus, their study constitutes an essential step in the study of queuing networks. However, the apparent simplicity, alluded to above, is rather deceptive. Actually, one witnesses a steep increase of conceptual and analytical complexities (see Appendix B Sec. B.3), when going from M/M/s queues to Jackson networks. This increase is characteristic of the difficulties presented by queuing networks as compared to single queues.

The term "Jackson networks" was chosen to acknowledge the pioneering work of R. R. P. Jackson and J. R. Jackson during the 50's and 60's. In [JR1] and [JR2] R. R. P. Jackson initiates the study of tandem Jackson networks, with the main result being a now-classical derivation of the equilibrium line lengths distributions.

The work of J. R. Jackson in [JJ1] and [JJ2] subsumes the previous work, by extending the line length results to arbitrarily connected Jackson networks (which are called by him Jobshop-like networks).

Jackson networks provide an analytical stochastic model for a variety of real life systems. Typical applications are: computer operating systems, communication networks, and industrial manufacturing and repair processes. In this chapter we investigate various operating characteristics of arbitrarily connected Jackson networks. The discussion will be restricted to Jackson networks whose stations consist of single servers, unless otherwise specified.

4.1 Informal Description of Jackson Networks

A *Jackson network* is composed of finitely many stations, each housing a finite number of identical independent servers operating in parallel. The service stations are arbitrarily interconnected by directed arcs, indicating permissible paths of customer flow.

A typical service station is depicted in Figure 4.1.1. A customer may arrive at a service station either from an exogenous source or from other service stations. Exogenous customers arrive according to independent Poisson processes. Each service station has a recomposition switch that superposes all incoming customer streams. An arriving customer is directed into a FIFO† (first in first out) waiting line with infinite capacity. Consequently, customers are never lost at the recomposition switch on account of lack of waiting room. When a customer's turn comes to be served, he samples an exponentially distributed service time. When service is done, the customer enters a decomposition switch which is a stochastic decision maker whose task is to route a customer to his next destination. At this point the customer may leave the system altogether for an exogenous sink, or he may be directed to any other service station. Each routing decision is obtained according to a multinominal Bernoulli trial. Such decomposition switches are called *Bernoulli*

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†Actually, almost all results are independent of queue discipline.

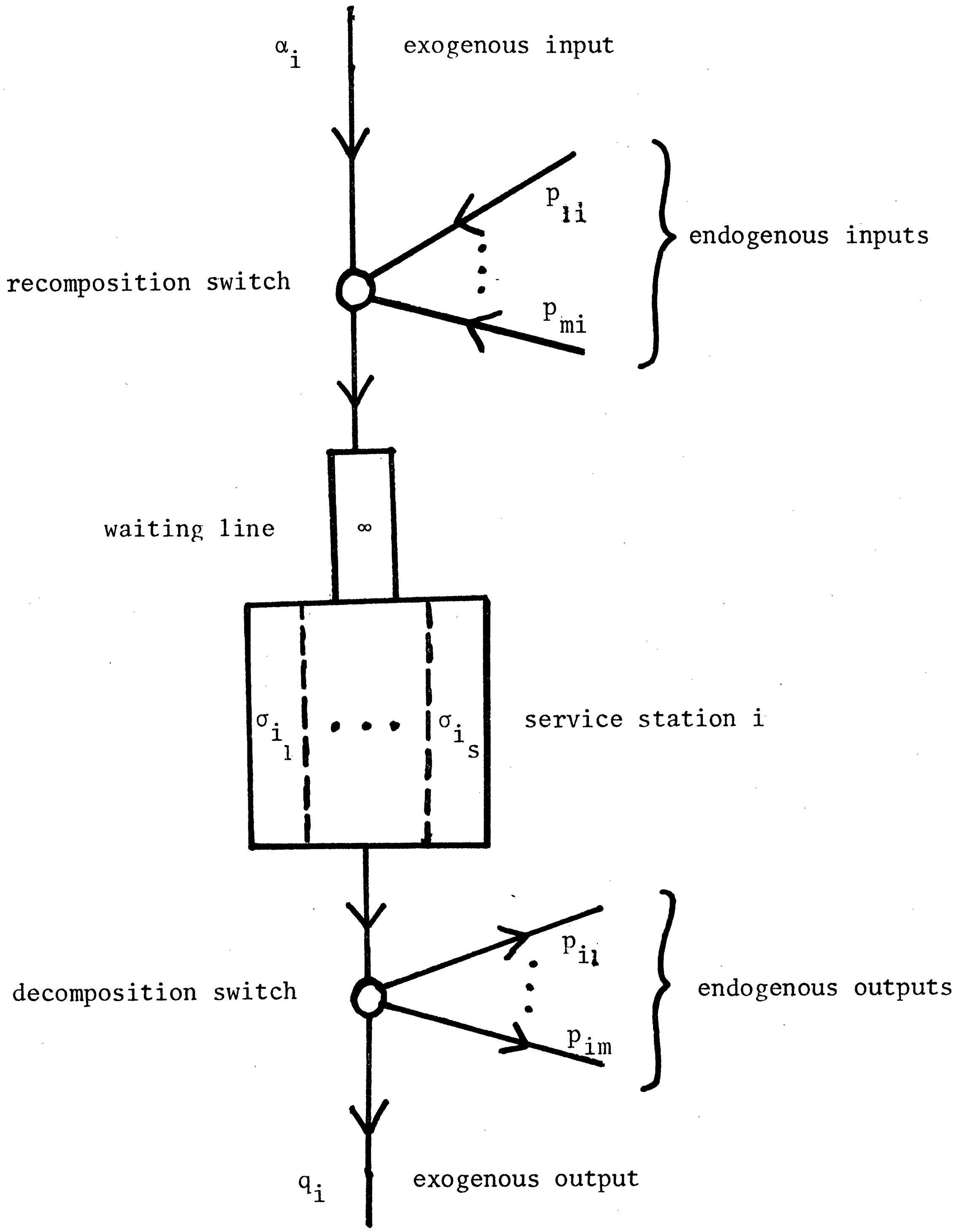

Figure 4.1.1: Typical Node i in a Jackson Network with Multiple Server Nodes.

switches (see [DC1]). All switchings are instantaneous operations.

Finally, all exogenous arrival processes, all service processes and all routing decisions consist of mutually independent random variables.

This completes the informal description (in the sense of Chapter 2) of Jackson queuing networks. A formal representation of a Jackson network as a coordinate probability space follows Example 2.5.3. Notice that a family of finite dimensional distributions has been specified via one-dimensional ones, due to the mutual independence of the subsequent generator set, viz. exogenous arrivals, services and routings. The coordinate space representation will be used in the next chapter.

It is convenient to associate with a Jackson network a directed graph, to describe its "stochastic" topology. The nodes of the graph are numbered 1,2,...m and stand for service stations. The *node set* of a Jackson network is denoted $M \triangleq \{1,2,\ldots m\}$; the arcs are denoted (i,j), $0 \le 1,j \le m$, in the natural way. Node 0 denotes a fictitious service station interpreted as the "environment" (i.e. both the exogenous "source" and the exogenous "sink"). Each arc (i,j), $1 \le i,j \le m$, is labeled with the routing probability p_{ij} associated with it. The resulting substochastic matrix $P_{m\times m} = [p_{ij}]$ is called the *switching matrix* of the network. The probability p_{i0} of quitting the network at node i is denoted by $q_i \triangleq 1 - \sum_{j=1}^{m} p_{ij} \triangleq p_{i0}$.

The arcs leading to the environment sink and those originating at the environment source are called, respectively, *outlets* and *inlets* of the network.

The parameter of the Poisson arrival process to node i is denoted by α_i. The vector of input parameters $\alpha \triangleq (\alpha_1, \alpha_2, \ldots \alpha_m)$ is called an

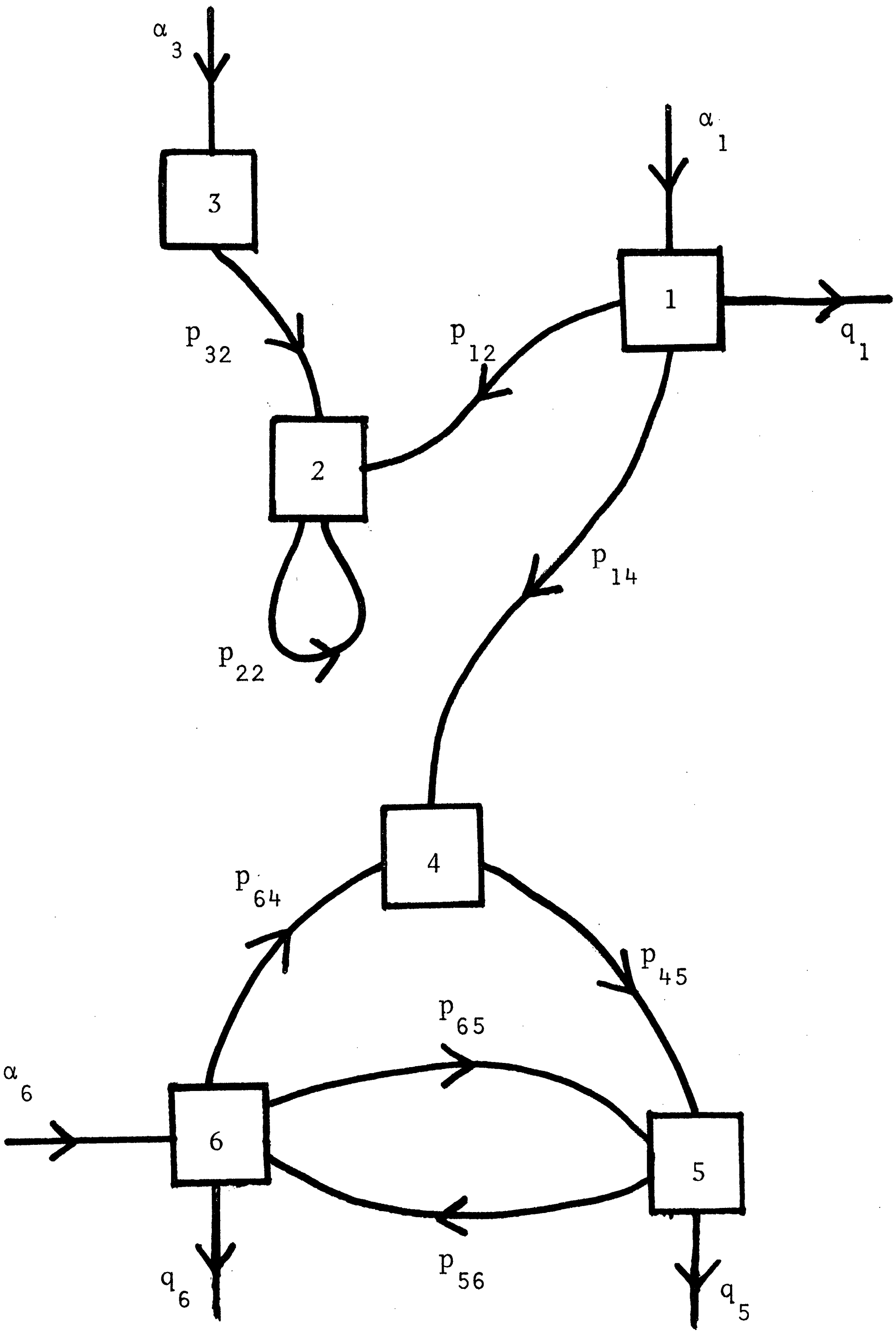

Figure 4.1.2: Graph Representation of the Topology of a Jackson Network.

arrival vector if $\alpha \geq 0$. Observe that $p_{0i} = \alpha_i / \sum_{j=1}^{m} \alpha_j$. We shall label inlets of the network by α_i rather than by p_{0i}. Likewise, the *service vector* is $\sigma \triangleq (\sigma_1, \ldots, \sigma_m)$, provided $\sigma > 0$.

The graph associated with a Jackson network is depicted in Figure 4.1.2. Note that arcs labeled by $p_{ij} = 0$ or $\alpha_i = 0$ are simply deleted. Graph terminology is extensively used throughout Chapters 4 and 5.

Definition 4.1.1

Let $M = \{1,2,\ldots m\}$ be a finite node set. Let $\alpha = (\alpha_1, \ldots \alpha_m)$ be an arrival vector, $\sigma = (\sigma_1, \ldots, \sigma_m)$ - a service vector, and $P_{m \times m}$ - a switching matrix. Then the quadruple $JN = (M,\alpha,\sigma,P)$ is called a *Jackson network specification.* □

Once the background conventions of Jackson networks are understood, a Jackson network specification JN is an economical way to describe a particular network by specifying its parameters.

4.2 A Stochastic Queuing Model

In this section we develop a formal stochastic model for the queuing process described informally in the previous section.

Let $JN = (M,\alpha,\sigma,P)$ be a Jackson network specification. We begin with $m(m+2)$ mutually independent right-continuous Poisson processes denoted $\{A_i^{ex}(t)\}_{t \geq 0}$ (each with intensity α_i) and $\{S_{ij}(t)\}_{t \geq 0}$ (each with intensity $\sigma_i p_{ij}$), for $1 \leq i \leq m$, $0 \leq j \leq m$. The $\{A_i^{ex}(t)\}_{t \geq 0}$ model the incoming streams of exogenous customers at the respective nodes. Each $\{A_i^{ex}(t)\}_{t \geq 0}$ is called the *exogenous arrival process* at node i. The $\{S_{ij}(t)\}_{t \geq 0}$, $1 \leq i \leq m$, $0 \leq j \leq m$, will later on aid us in modeling the traffic processes on the arcs (i,j).

The service mechanism at each node i is modeled by the process $\{S_i(t)\}_{t\geq 0}$ where $S_i(t) \triangleq \sum_{j=0}^{m} S_{ij}(t)$. The $\{S_i(t)\}_{t\geq 0}$ are mutually independent Poisson processes (see [Cil], p. 87) with respective intensities σ_i. Each $\{S_i(t)\}_{t\geq 0}$ will be referred to as the *service process* at node i.

Finally, the associated *underlying jump process* $\{J(t)\}_{t\geq 0}$ is defined by $J(t) \triangleq \sum_{i=1}^{m} A_i^{ex}(t) + \sum_{i=1}^{m} \sum_{j=0}^{m} S_{ij}(t) = \sum_{i=1}^{m} (A_i^{ex}(t)+S_i(t))$.

The queuing model to be described is motivated by the following observation.

Consider the stream of customers emerging from service station i at its decomposition switch. Given that throughout some time interval the queue was nonempty (busy period), the customer stream in that interval accords with a Poisson process with intensity σ_i. When the queue becomes empty (idle period), the customer stream dries up. Thus, this customer stream is, loosely speaking, a periodically suspended (intermittent) Poisson process.

Another way of saying it is that this customer stream is a filtered Poisson process whose count in idle periods is masked out, so that only counts taken during busy periods are registered.

A similar observation is valid for the customer stream on the arcs (i,j). During busy periods, these streams are obtained from a Poisson process with intensity σ_i, acted upon by a Bernoulli switch, with probability p_{ij} for choosing arc (i,j). It follows that during busy periods, these traffic streams are mutually independent Poisson processes with intensities $\sigma_i p_{ij}$ (see [Cil], p. 89). The $\{S_{ij}(t)\}_{t\geq 0}$ defined before will play the role of the background Poisson processes whose appropriate filtering will later on yield the traffic processes on the arcs (i,j).

We now proceed to define two sets of stochastic processes. First some preliminaries. A process in the first set is called the *traffic process* on arc (i,j) and denoted $\{A_{ij}(t)\}_{t\geq 0}$, $1 \leq i \leq m$, $0 \leq j \leq m$. $A_{ij}(t)$ is the traffic count on arc (i,j) in the time interval $[0,t]$.

A process in the second set is called the (local) *state process* at node i, and denoted $\{Q_i(t)\}_{t\geq 0}$, $1 \leq i \leq m$. $Q_i(t)$ is the line size at node i (including any customer in service at time t). We shall also need the following auxiliary processes derived from the above whenever they are defined.

The *endogenous arrival process* $\{A_i^{en}(t)\}_{t\geq 0}$ at node i is defined by

$$A_i^{en}(t) \triangleq \sum_{j=1}^{m} A_{ji}(t) \quad , \qquad 1 \leq i \leq m .$$

The *departure process* $\{D_i(t)\}_{t\geq 0}$ at node i is defined by

$$D_i(t) \triangleq \sum_{j=0}^{m} A_{ij}(t) \ , \qquad 1 \leq i \leq m .$$

The *state indicator process* $\{B_i(t)\}_{t\geq 0}$ at node i is defined by

$$B_i(t) \triangleq \begin{cases} 0, & \text{if } Q_i(t) = 0 \\ 1, & \text{if } Q_i(t) > 0 \end{cases} \quad , 1 \leq i \leq m .$$

We assume that there are given random variables $Q_i(0)$, $1 \leq i \leq m$, such that $Q(0) \triangleq (Q_1(0),\ldots,Q_m(0))$, the $A_i^{ex}(t)$, and the $S_{ij}(t)$ are mutually independent. $Q_i(0)$ is called the *initial state* of node i.

Let $\{\tau_n\}_{n=0}^{\infty}$ be a sequence of random variables where τ_n is the n-th jump instant of $\{J(t)\}_{t\geq 0}$ and $\tau_0 \triangleq 0$. Then, almost surely $0 = \tau_0 < \tau_1 < \ldots < \tau_n < \ldots$. Recall also that almost surely $A_i^{ex}(0) = 0$ and $S_{ij}(0) = 0$, $1 \leq i \leq m$, $0 \leq j \leq m$.

The definition of the traffic processes and local state processes is carried out in two steps.

In the first step we define simultaneously the sequences of random variables $\{A_{ij}(\tau_n)\}_{n=0}^{\infty}$ and $\{Q_i(\tau_n)\}_{n=0}^{\infty}$, $1 \le i \le m$, $0 \le j \le m$, by

$$\text{(A.1)} \quad A_{ij}(\tau_n) \triangleq \sum_{k=1}^{n} B_i(\tau_{k-1})[S_{ij}(\tau_k) - S_{ij}(\tau_{k-1})]$$

$$\text{(A.2)} \quad Q_i(\tau_n) \triangleq Q_i(0) + A_i^{ex}(\tau_n) + A_i^{en}(\tau_n) - D_i(\tau_n) .$$

<u>Lemma 4.2.1</u>

The sequences $\{A_{ij}(\tau_n)\}_{n=0}^{\infty}$ and $\{Q_i(\tau_n)\}_{n=0}^{\infty}$, $1 \le i \le m$, $0 \le j \le m$, are well-defined.

<u>Proof</u>

The proof follows from the fact that the $A_{ij}(\tau_n)$ have a recursive representation

$$(1) \quad A_{ij}(\tau_n) = \begin{cases} 0, \text{ if } n = 0 \\ A_{ij}(\tau_{n-1}) + B_i(\tau_{n-1})[S_{ij}(\tau_n) - S_{ij}(\tau_{n-1})], \text{ if } n > 0 \end{cases} \square$$

In the second step, we extend the $\{A_{ij}(\tau_n)\}_{n=0}^{\infty}$ and the $\{Q_i(\tau_n)\}_{n=0}^{\infty}$ to the respective continuous parameter stochastic processes $\{A_{ij}(t)\}_{t\ge 0}$ and $\{Q_i(t)\}_{t\ge 0}$, $1 \le i \le m$, $0 \le j \le m$, by setting for any $t \ge 0$

$$\text{(B.1)} \quad A_{ij}(t) \triangleq \sum_{k=1}^{J(t)} B_i(\tau_{k-1})[S_{ij}(\tau_k) - S_{ij}(\tau_{k-1})]$$

$$\text{(B.2)} \quad Q_i(t) \triangleq Q_i(0) + A_i^{ex}(t) + A_i^{en}(t) - D_i(t)$$

We point out that the processes defined above do indeed comply with their informal description in the previous section.

First, note that the sum in (B.1) is taken over all possible jumps of $\{J(t)\}_{t\geq 0}$ in the interval [0,t]. Each term $S_{ij}(\tau_k) - S_{ij}(\tau_{k-1})$, almost surely equals 0 or 1. Only when both $B_i(\tau_{k-1})$ and $S_{ij}(\tau_k) - S_{ij}(\tau_{k-1})$ evaluate to one, does the respective term contribute to the sum in (B.1).

Next consider $D_i(t) \triangleq \sum_{k=1}^{J(t)} B_i(\tau_{k-1})[S_i(\tau_k) - S_i(\tau_{k-1})]$ which counts service completions at node i in the time interval [0,t]. Clearly, the time interval separating any two consecutive jumps of $\{S_i(t)\}_{t\geq 0}$ is exponentially distributed with parameter σ_i. We argue that any time interval separating any jump of $\{A_i^{ex}(t)\}_{t\geq 0}$ and the very next jump of $\{S_i(t)\}_{t\geq 0}$ is also exponentially distributed with parameter σ_i, due to the forgetfulness property of Poisson processes. Furthermore, all such time intervals are mutually independent. Consequently, exponential services rendered are correctly modeled.

Finally, (B.2) is a stochastic balance equation that keeps track of the line length at time t, in terms of its initial value and the traffic through the respective node during the time interval [0,t].

In order to facilitate the investigation of the processes above, we shall rewrite (B.1) in equivalent intergral representation

(C.1) $$A_{ij}(t) = \int_0^t B_i(x-)dS_{ij}(x) \qquad \text{(almost surely)}$$

by which we mean that the sample functions are Riemann-Stieltjes integrals

(C.2) $$A_{ij}(\omega,t) = \int_0^t B_i(\omega,x-)dS_{ij}(\omega,x), \text{ for almost every } \omega.$$

(C.2) is almost surely well-defined, because for almost every ω, $B_i(\omega,t)$ and $S_{ij}(\omega,t)$ are step functions with finitely many jumps. It can now be directly verified that the integral representation (C.1) reduces to the random sum representation (B.1).

Henceforth, $\sigma(Y)$ will denote the σ-algebra generated by a set Y of random variables.

Theorem 4.2.1

For any $1 \leq i \leq m$, $0 \leq j \leq m$ and $t \geq 0$

(a) $E(A_{ij}(t)) = \sigma_i p_{ij} \int_0^t \Pr(B_i(x)=1)dx.$

Proof

Consider the integral representation

(1) $A_{ij}(t) = \int_0^t B_i(x-)dS_{ij}(x).$

For any fixed t, let $\Pi_n : \ 0 = t_0 < t_1^{(n)} < \ldots < t_{\ell_n}^{(n)} = t$ be a sequence of partitions of the time interval [0,t] such that

$$\Delta_n \triangleq \max_{0 \leq k < \ell_n} \left\{ \left| t_{k+1}^{(n)} - t_k^{(n)} \right| \right\} \xrightarrow[n\to\infty]{} 0 \ .$$

For the same fixed t, define a sequence $\left\{A_{ij}^{(n)}(t)\right\}_{n=1}^{\infty}$ of random variables where

(2) $A_{ij}^{(n)}(t) \triangleq \sum_{k=1}^{\ell_n} B_i(t_{k-1}^{(n)})[S_{ij}(t_k^{(n)}) - S_{ij}(t_{k-1}^{(n)})].$

Next, we show that

(3) $A_{ij}^{(n)}(t) \xrightarrow[n\to\infty]{} A_{ij}(t)$ (almost surely).

Let ω be a sample point such that

(4) $\Delta(\omega) \triangleq \inf\{t'-t'': J(\omega,t') - J(\omega,t'') > 0 \text{ and } t',t''\varepsilon[0,t]\} > 0.$

Observe that $\Delta(\omega) > 0$ almost surely.

Next, let $n_0 = n_0(\omega)$ be an integer such that

(5) $\Delta_n < \Delta(\omega)$, $\forall n \geq n_0$

Then

(6) $A_{ij}^{(n)}(t) = A_{ij}(t)$, $\forall n \geq n_0$

whence (3) follows.

Next, we deduce from (1) that $E(A_{ij}(t))$ exists and is finite because

(7) $0 \le E(A_{ij}(t)) \le E(S_{ij}(t)) = \sigma_i p_{ij}$.

Hence

(8) $E(A_{ij}^{(n)}(t)) \xrightarrow[n\to\infty]{} E(A_{ij}(t))$.

We now proceed to compute this limit. Since each $B_i(t_{k-1}^{(n)})$ is measurable on

$\sigma(\{Q_i(0),\ A_i^{ex}(t),\ S_{ij}(t):\ 1 \le i \le m,\ 0 \le j \le m,\ 0 \le t \le t_{k-1}^{(n)}\})$

(cf. Lemma 4.2.1), each $B_i(t_{k-1}^{(n)})$ is independent of $S_{ij}(t_k^{(n)}) - S_{ij}(t_{k-1}^{(n)})$. Consequently, from (2)

$$(9)\quad E(A_{ij}^{(n)}(t)) = \sum_{k=1}^{\ell_n} E(B_i(t_{k-1}^{(n)}))E(S_{ij}(t_k^{(n)}) - S_{ij}(t_{k-1}^{(n)})) = \sum_{k=1}^{\ell_n} \Pr(B_i(t_{k-1}^{(n)}) = 1)\sigma_i p_{ij}(t_k^{(n)} - t_{k-1}^{(n)}) \ .$$

(9) Is a Riemann sum whose integrand $\Pr(B_i(t)=1)$ is continuous in t. To see this, note that $\{B_i(t)\}_{t\ge 0}$ is stochastically continuous, viz.

$$(10)\quad \Pr(|B_i(t+\varepsilon) - B_i(t-\varepsilon)| \ne 0) \le \Pr(|J(t+\varepsilon) - J(t-\varepsilon)|) = 1 - e^{-(\sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{m} \sigma_i)\cdot 2\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0$$

which implies convergence in distribution (see [W1], p. 23). Thus, (a) is obtained from (9) by passage to the limit as $n\to\infty$. □

We now direct our attention to the *state process* $\{Q(t)\}_{t\ge 0}$ where $Q(t) \triangleq (Q_1(t),\ldots,Q_m(t))$ is the vector of line sizes in the network at time t. We shall likewise denote $B(t) \triangleq (B_1(t),\ldots,B_m(t))$, $A^{ex}(t) \triangleq (A_1^{ex}(t),\ldots,A_m^{ex}(t))$, $A^{en}(t) \triangleq (A_1^{en}(t),\ldots,A_m^{en}(t))$, $D(t) = (D_1(t),\ldots,D_m(t))$, etc. We shall also denote for any $s \le t$, $A^{ex}(s,t] \triangleq A^{ex}(t) - A^{ex}(s)$, $A^{en}(s,t] \triangleq A^{en}(t) - A^{en}(s)$, $D(s,t] \triangleq D(t) - D(s)$, etc.

Theorem 4.2.2

The state process $\{Q(t)\}_{t\geq 0}$ is a Markov process with stationary transition probabilities. Moreover, $\{Q_t\}_{t\geq 0}$ is *conservative*.†

Proof

Consider the stochastic equation

(1) $Q(t) = Q(0) + A^{ex}(t) + A^{en}(t) - D(t) \quad , \quad t \geq 0$

derived from (B.2).

For any $s \leq t$, (1) can be rewritten as

(2) $Q(t) = Q(s) + A^{ex}(s,t] + A^{en}(s,t] - D(s,t]$

From this representation and by tracing back the definitions of $A^{ex}(s,t]$, $A^{en}(s,t]$ and $D(s,t]$, we deduce (with the aid of the recursive representation in Lemma 4.2.1) that $Q(t)$ is measurable on

(3) $\sigma(\{Q_i(s), A_i^{ex}(u) - A_i^{ex}(s), S_{ij}(u) - S_{ij}(s): \ u\varepsilon(s,t], \ 1 \leq i \leq m, \ 0 \leq j \leq m\})$

for any $s \leq t$.

Since $Q(s)$ is measurable on

(4) $\sigma(\{Q_i(0), A_i^{ex}(r), S_{ij}(r): \ r \leq s, 1 \leq i \leq m, 0 \leq j \leq m\})$,

it follows that $\sigma(Q(s))$ is independent of the σ-algebra $\sigma(\{A_i^{ex}(u) - A_i^{ex}(s), S_{ij}(u) - S_{ij}(s): \ u\varepsilon(s,t], 1 \leq i \leq m, 0 \leq j \leq m\})$. The Markov property of $\{Q(t)\}_{t\geq 0}$ now follows from Theorem C.1.1 in Appendix C, in view of (2) and (4).

Next, it follows from (4) that $\{Q(t)\}_{t\geq 0}$ has stationary transition probabilities, because the $\{A_i^{ex}(t)\}_{t\geq 0}$ and $\{S_{ij}(t)\}_{t\geq 0}$ have independent increments with stationary distributions.

Finally, $\{Q(t)\}_{t\geq 0}$ is conservative because its jumps are contained

† Has finite number of jumps in every finite interval with probability 1.

in those of the conservative process $\{J(t)\}_{t\geq 0}$. □

In view of Theorem 4.2.2, the discussion in Sec. C.1 of Appendix C applies to the state process $\{Q(t)\}_{t\geq 0}$. Accordingly, we denote the probability vector of Q(t) by q(t). In particular, the Kolmogorov forward equation of the state process (see ibid.) is equivalent to the system of integral equations

$$\text{(D)} \quad q_\nu(t) = q_\nu(0)e^{-c_\nu t} + \sum_\mu \int_0^t q_\mu(x)\ c_\mu r_{\mu\nu} e^{-c_\nu(t-x)} dx$$

for every state $\nu = (n_1,\ldots,n_m) \geq 0$. The summation in (D) is over all m-dimensional non-negative integer vectors μ.

In our case

$$\text{(E)} \quad c_\nu = \sum_{i=1}^m \alpha_i + \sum_{i=1}^m \sigma_i(1-p_{ii})\cdot b(n_i), \text{ for any } \nu = (n_1,\ldots,n_m) \geq 0$$

where

$$b(n_i) = \begin{cases} 0, & \text{if } n_i = 0 \\ 1, & \text{if } n_i > 0 \end{cases}$$

Furthermore, the quantity $e^{-c_\nu t}$ giving the probability that no event capable of altering the state ν occurs during (0,t], also satisfies

$$e^{-c_\nu t} = \Pr\left(\bigcap_{i=0}^m \bigcap_{\substack{j=0 \\ j\neq i}}^m (A_{ij}(t) = 0)\right)$$

Hence, the probability of a jump from state ν in the interval (t,t+h] is $c_\nu h + o(h)$.

Since a transition between non-adjacent states (see Definition C.2.2 in Appendix C) requires more than one jump, it follows that the time derivatives of the respective transition functions satisfy

$$\dot{p}_{\nu\mu}(t,t) = \dot{p}_{\mu\nu}(t,t) \equiv 0 \quad , \quad t \geq 0 .$$

Thus, $\{Q(t)\}_{t\geq 0}$ is an m-dimensional birth-and-death process (see Definition C.2.3, ibid.).

Denoting $P_t(n_1,\ldots,n_m) \triangleq Pr(Q_1(t) = n_1,\ldots,Q_m(t) = n_m)$ and with a dot to denote derivative with respect to t, the birth-and-death equations for a Jackson network JN = (M,α,σ,P) with single server nodes are

(F) $\dot{P}_t(n_1,n_2,\ldots,n_m) =$

(F.1) $\sum_{i=1}^{m} P_t(n_1,\ldots,n_i-1,\ldots,n_m)\alpha_i \cdot b(n_i) +$

(F.2) $\sum_{j=1}^{m} P_t(n_1,\ldots,n_j+1,\ldots,n_m)\sigma_j q_j +$

(F.3) $\sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq 1}}^{m} P_t(n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_m)\sigma_j p_{ji} \cdot b(n_i) -$

(F.4) $P_t(n_1,\ldots,n_m)[\sum_{i=1}^{m} \sigma_i + \sum_{j=1}^{m} \sigma_j q_j \cdot b(n_j) + \sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq 1}}^{m} \sigma_j p_{ji} \cdot b(n_j)]$

$\nu = (n\ ,\ldots,n_m) \geq 0.$

Note that lines (F.1) - (F.3) give transition rate into state ν; (F.1) is due to exogenous arrivals, (F.2) is due to exogenous departures, and (F.3) is due to departures from node i resulting in an endogenous arrival at node j. Observe that traffic on feedback arcs (i,i) does not change the state of the system.

Line (F.4) gives the transition rate out of state ν.

It turns out that equilibrium solutions (see Definition C.3.1 in Appendix C) for (F) depend crucially on the so-called traffic equation. This equation will be investigated in Sec. 4.4.

4.3 Notational Conventions and Terminology

In the sequel we shall occasionally discuss convergence of matrices and vectors. Usually we deal with finite dimensional matrices and vectors. In this case it is not necessary to specify the underlying norm, because on finite dimensional linear spaces all norms are equivalent.†

However, whenever norm evaluation is required, it will allude to the norm

(A) $$||A|| = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|$$

for any $m \times n$ matrix $A \triangleq [a_{ij}]$.

Observe that for a vector $v \triangleq (v_1, \ldots, v_m)$, this convention implies

(B) $$||v|| = \sum_{i=1}^{m} |v_i|$$

All arithmetical relations involving vectors and matrices are pointwise relations; e.g. if $A = [a_{ij}]$ is a matrix, then $A \geq 0$ means that $a_{ij} \geq 0$ for all i and j. The transpose of a matrix A is denoted by A^T.

If S is a subset of a universal set U, then $\bar{S}$ will denote the complement U-S of S in U. The cardinality of a set S is always denoted by $|S|$.

To designate submatrices and subvectors, we introduce the following notation. If v is a vector with index set K and $S \subset K$, then v_S denotes the partial vector obtained from v by deleting all coordinates v_i, $i \in \bar{S}$. Similarly, if Q is a square matric with index set $K \times K$ and $S \subset K$, then Q_S denotes the partial matrix obtained from Q by deleting all rows and columns, indexed by $\bar{S}$.

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†Two norms over the same normed space are equivalent, if they give rise to the same set of convergent sequences over the space.

Further preliminary comments and additional notation, concerning stochastic processes, may be found in Appendix C.

We now proceed to establish a classificatory terminology for Jackson networks and related conventions. Consider again the graph representation of a Jackson network (see e.g. Figure 4.1.2). Suppose a customer arrives along an inlet at some node i. Then, the *path*† traced by him thereafter constitutes a finite Markov chain whose states are the nodes of the graph, and whose transition probabilities are those labeling the arcs of the graph. Thus, the graph may be used to represent the transition probabilities of this process, provided all outlets are understood to lead into a fictitious node 0 (the "environment sink"). This node corresponds to an absorbing state.

The transition matrix $\tilde{P}$ of this Markov chain is obtained from P by adjoining an absorbing state 0 as follows.

$$\text{(C)} \quad \tilde{P} \triangleq \begin{cases} \left[\begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline p_{10} & & & \\ \vdots & & P & \\ p_{m0} & & & \end{array}\right] & \text{, if } P \text{ is not stochastic} \\ P, & \text{otherwise} \end{cases}$$

where $p_{i0} \triangleq q_i = 1 - \sum_{j=1}^{m} p_{ij}$.

The Markov chain induced by $\tilde{P}$ will be seen to play an important role in determining system and customer behavior. In discussing it, we shall adopt the usual Markov chain terminology and notation. The reader is referred to Chapters XV and XVI in [F1] for the pertinent

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†A path in the associated graph is any sequence of nodes connected by arcs which are labeled by positive probabilities.

theory. In particular $p_{ij}^{(n)}$ designates the *n-step transition probability* from i to j.

Definition 4.3.1

Let $P_{m\times m}$ be a substochastic matrix. Then:

a) j is *accessible* from i (denoted $i \rightsquigarrow j$), if $p_{ij}^{(n)} > 0$ for some $n \geq 0$.

b) i *communicates* with j (denoted $i \leftrightsquigarrow j$), if $i \rightsquigarrow j$ and $j \rightsquigarrow i$. □

In the sequel, we shall make it a habit to interchange the terms "state" and "node", so as to take advantage of the intuitive content of the graph representation.

The forthcoming classification of Jackson networks is based on their probabilistic topology, and cast in terms of $\tilde{P}$ and its associated graph. First, we give a node classification.

Definition 4.3.2

Let $\tilde{P}$ be the stochastic matrix associated with a Jackson network $JN = (M, \alpha, \sigma, P)$. Let i be any node in M. Then

a) i is called *open* if $\lim_{n\to\infty} p_{i0}^{(n)} > 0$.

The set of all open nodes is denoted O.

b) i is called *completely open* if $\lim_{n\to\infty} p_{i0}^{(n)} = 1$.

The set of all completely open nodes is denoted A.

c) i is called *partially open* if $\lim_{n\to\infty} p_{i0}^{(n)} < 1$.

The set of all partially open nodes is denoted B.

d) i is called *closed* if $\lim_{n\to\infty} p_{i0}^{(n)} = 0$.

The set of all closed nodes is denoted C. □

Figure 4.3.1 depicts a decomposition of a switching matrix P which illustrates the relations among the node sets of Definition 4.3.2.

Henceforth, the sets R and T denote the *recurrent* and *transient* node sets, respectively, in a Jackson network.

We make the following remarks concerning the "random walk" of a customer in the network. This "random walk" is described implicitly in the informal description of Jackson networks in Sec. 4.1.

Remark 4.3.1

a) An open node i has a path leading from it to an outlet of the network (i.e. $i \rightsquigarrow 0$). Thus, a customer in node i will eventually leave the network with positive probability.

b) A customer at a completely open node i will eventually leave the network with probability 1. In particular, A is an open set closed under $\rightsquigarrow$, that is

$$i \in A \text{ and } i \rightsquigarrow j \implies j \in A$$

c) A partially open node i must have a path leading from i to the sink 0, and another path leading from i to a closed node. That is,

$i \rightsquigarrow 0$ and $i \rightsquigarrow j$ for some $j \in C$.

d) A closed node i has no path leading from it to the sink 0 (i.e. $i \not\rightsquigarrow 0$). Any customer in it is trapped in the sense that he leaves the network with zero probability. C is closed under $\rightsquigarrow$ in the same sense as A. □

$$P = \begin{array}{c} \overbrace{\hphantom{A \quad B \quad C\text{-}R}}^{T} \\ \begin{array}{cccc} A & B & C\text{-}R & R \end{array} \\ \left[\begin{array}{c|c|c|c} P_1 & 0 & 0 & 0 \\ \hline P_2 & P_3 & P_4 & P_5 \\ \hline 0 & 0 & P_6 & P_7 \\ \hline 0 & 0 & 0 & P_8 \end{array}\right] \end{array} \begin{array}{l} \left.\begin{array}{l} \}\,A \\ \}\,B \end{array}\right\} O \\ \left.\begin{array}{l} \}\,C\text{-}R \\ \}\,R \end{array}\right\} C \end{array}$$

<u>Figure 4.3.1</u> Decomposition of a Switching Matrix

Remark 4.3.2

a) $M = O \cup C$ where $O \cap C = \Phi$

b) $O = A \cup B$ where $A \cap B = \Phi$. All the nodes in O are transient.

c) C contains a nonempty finite collection $R = \{R_k\}_{k \varepsilon K}$ of recurrent equivalence classes of nodes (under the communication relation of Definition 4.3.1), where each R_k is irreducible.

d) The set of all transient nodes is $T = O \cup (C-R)$. □

We now introduce a Jackson network classification which follows the pattern of the node classification.

Definition 4.3.3

Let $JN = (M,\alpha,\sigma,P)$ specify a Jackson network.

Then

a) JN is called *open*, if O = M.

b) JN is called *closed*, if C = M.

c) JN is called *mixed*, if it is neither open nor closed.

d) A subnetwork of JN is called *autonomous*, if it is not accessible from any inlet of the network. □

Remark 4.3.3

a) If a Jackson network is open then all its nodes are completely open, because in this case $B = \Phi$; thus, $O = A$ by part b) of Remark 4.3.2.

b) If a Jackson network is not open then it contains a collection of mutually non-communicating closed sets. This collection is $\{R_k\}_{k\varepsilon K}$ in c) of Remark 4.3.2. □

We now demonstrate our classification in

Example 4.3.1

Consider the Jackson network of Figure 4.1.2. We have

$A = \{4,5,6\}$

$B = \{1\}$

$O = A \cup B = \{1,4,5,6\}$

$C = \{2,3\}$

$R = \{2\}$

$T = O \cup C - R = \{1,4,5,6,3\}$

The network is clearly a mixed one. □

4.4 The Traffic Equation

The traffic equation is a formal expression of a flow conservation relation, which plays a crucial role in determining the equilibrium behavior of a Jackson network.

Definition 4.4.1

Let JN = (M,α,σ,P) be a Jackson network specification. The *traffic equation* associated with it is

(A) $\delta = \alpha + \delta P$

in the unknowns $\delta = (\delta_1,\ldots,\delta_m)$.

A solution $\delta \geq 0$ for (A) is called a *traffic solution.* □

The intuitive content of (A) is best seen when we rewrite it as a system of linear equations.

(B) $\delta_i = \alpha_i + \sum_{j=1}^{m} \delta_j p_{ji} \quad , \quad 1 \leq i \leq m$

Now, if one interprets each δ_i as the traffic intensity of customers through node i in equilibrium, then (B) merely states that the total input intensity to node i equals the output intensity from it when the system is in equilibrium.

J. R. Jackson used this intuition in [JJ1] to give sufficient conditions, for open Jackson networks to evolve into equilibrium, in terms of the traffic solutions of (A). However, he does not investigate (A) and its solutions nor does he justify the intuitive interpretation above of δ.

In this section we shall investigate the formal equation (A). The results will be later on tied to a discussion of state equilibrium in the next section.

Lemma 4.4.1 (cf. [BMZ1], Theorem 4.1)

Let JN = (M,α,σ,P) specify a Jackson network, and let T be its set of transient nodes. Then there is always a unique traffic solution for T, given by

(a) $\delta_T = \alpha_T \sum_{n=0}^{\infty} P_T^n$.

Proof

It follows from our definitions that

(1) $i \varepsilon T$ and $j \varepsilon R \implies j \not\rightarrow i \implies p_{ji} = 0$.

Now, (B) and (1) enable us to write

(2) $\delta_T = \alpha_T + \delta_T P_T$.

From the transience of T we have (see [KS1], p. 22) that $\sum_{n=0}^{\infty} P_T^n$ is finite.

Furthermore,

(3) $(I - P_T)^{-1} = \sum_{n=0}^{\infty} P_T^n$

where I is the identity matrix.

Next, rewrite (2) as

(4) $\delta_T(I - P_T) = \alpha_T$

In view of (3) we immediately conclude that (a) is a traffic solution for T. Moreover, this solution is guaranteed to be unique by the existence of the inverse $(I - P_T)^{-1}$. □

Lemma 4.4.2

Let JN = (M,α,σ,P) be a Jackson network specification, and define $D \triangleq B \cup (C - R)$. Assume that $\alpha_D = 0$.

Then $\delta_D = 0$ is the unique traffic solution for D.

Proof

It follows from our definitions that

(1) $i \varepsilon D$ and $j \varepsilon \bar{D} \Longrightarrow j \not\rightsquigarrow i \Longrightarrow p_{ji} = 0$.

Again, (B) and (1) permit us to write

(2) $\delta_D = \alpha_D + \delta_D P_D$

and since $\alpha_D = 0$, (2) reduces to

(3) $\delta_D = \delta_D P_D$.

Since $D \subset T$, it follows from Lemma 4.4.1 that there is a unique solution for (3). Clearly $\delta_D \triangleq 0$ is a solution for (3). Hence, this must be the unique traffic solution for D. □

We are now in a position to characterize the existence of a traffic solution.

Theorem 4.4.1

Let JN = (M,α,σ,P) be a Jackson network specification and let A be its set of completely open nodes.

Then, a traffic solution exists iff $\alpha_{\bar{A}} = 0$.

Proof

Partition P, α and δ as follows:

$$P = \left[\begin{array}{c|c} P_A & 0 \\ \hline Q & P_{\bar{A}} \end{array}\right]$$

$\alpha = (\alpha_A \ ; \ \alpha_{\bar{A}})$ and $\delta = (\delta_A \ ; \ \delta_{\bar{A}})$.

Consequently, the traffic equation (A) is equivalent to the following two equations:

(1) $\delta_A = \alpha_A + \delta_A P_A + \delta_{\bar{A}} Q$

(2) $\delta_{\bar{A}} = \alpha_{\bar{A}} + \delta_{\bar{A}} P_{\bar{A}}$.

Note that

(3) $\bar{A} = B \cup C = B \cup (C - R) \cup R = D \cup R$.

If $\bar{A} = \Phi$ the theorem holds trivially by Lemma 4.4.1. Therefore we may assume that $\bar{A} \neq \Phi$ in the sequel.

($\Longleftarrow$) Assume that

(4) $\alpha_{\bar{A}} = 0$.

Since $A \subset T$, there exists a unique solution for δ_A due to Lemma 4.4.1. Therefore, the existence of a traffic solution is equivalent to the existence of a solution for $\delta_{\bar{A}}$ in (2).

But by assumption (4), Equation (2) reduces to

(5) $\delta_{\bar{A}} = \delta_{\bar{A}} P_{\bar{A}}$

which has a traffic solution $\delta_{\bar{A}} = 0$.

($\Longrightarrow$) Suppose $\alpha_{\bar{A}} \neq 0$, and let $i \epsilon \bar{A}$ have

(6) $\alpha_i > 0$.

By (3), $i \epsilon B \cup C$. If $i \epsilon C$, then there is clearly a node $r \epsilon R$ such that $i \rightsquigarrow r$. If $i \epsilon B$, then by c) of Remark 4.3.1 there is a node $c \epsilon C$ such that $i \rightsquigarrow c$. Consequently, for each $i \epsilon \bar{A}$

(7) $\exists r \epsilon R$ such that $i \rightsquigarrow r$, i.e. $\exists n_0$ such that $p_{ir}^{(n_0)} > 0$.

It follows from Markov Chain Theory ([F1], p. 389) that

(8) $$\sum_{n=0}^{k} p_{ir}^{(n)} \geq p_{ir}^{(n_0)} \sum_{n=n_0}^{k} p_{rr}^{(n-n_0)} \xrightarrow[k\to\infty]{} \infty$$

Next, substituting $\delta_{\bar{A}}$ repeatedly on the right hand side of (2) k times yields

$$(9) \quad \delta_{\bar{A}} = \alpha_{\bar{A}} \cdot \sum_{n=0}^{k} (P^n)_{\bar{A}} + \delta_{\bar{A}}(P^{k+1})_{\bar{A}} > \alpha_{\bar{A}} \cdot \sum_{n=0}^{k} (P^n)_{\bar{A}} .$$

For node $r\varepsilon\bar{A}$ we have from (9)

$$(10) \quad \delta_r \geq \sum_{j\varepsilon\bar{A}} \alpha_j \sum_{n=0}^{k} p_{jr}^{(n)} \geq \alpha_i \sum_{n=0}^{k} p_{ir}^{(n)} \xrightarrow[k\to\infty]{} \infty$$

since due to (8), $\sum_{n=0}^{k} p_{rr}^{(n)} \xrightarrow[k\to\infty]{} \infty$.

Hence, δ_r has no bounded solution, and consequently, $\delta_{\bar{A}}$ has no solution either. □

Corollary 4.4.1

$\alpha_{\bar{A}} = 0$ iff no closed node is accessible from an inlet. Moreover, $\alpha_{\bar{A}} = 0 \implies \delta_{\bar{A}}Q = 0$, where Q is a partial matrix of P given in the Theorem above. □

Although Corollary 4.4.1 follows from Theorem 4.4.1, the following direct proof of $\alpha_{\bar{A}} = 0 \implies \delta_{\bar{A}}Q = 0$ sheds more light on the situation. Refer to Theorem 4.4.1, assuming that $\alpha_{\bar{A}} = 0$.

Notice that each coordinate in $\delta_{\bar{A}}Q$ has the form $\sum_{j\varepsilon\bar{A}} \delta_j p_{ji}$ for some $i\varepsilon A$.

In view of (3) in Theorem 4.4.1, either $j\varepsilon D$ or $j\varepsilon R$.

If $j\varepsilon D$, then $\delta_j = 0$ by Lemma 4.4.2, since in particular $\alpha_D = 0$.

If, $j\varepsilon R$, then $j \not\rightarrow i$, since $i\varepsilon A$. This implies that $p_{ji} = 0$.

Consequently, either $\delta_j = 0$ or $p_{ji} = 0$.

Hence, in any event, $\delta_j p_{ji} = 0$ for any $j\varepsilon\bar{A}$ and any $i\varepsilon A$, whence $\sum_{j\varepsilon\bar{A}} \delta_j p_{ji} = 0$.

We now proceed to characterize the uniqueness of the traffic solution.

Theorem 4.4.2

Let JN = (M,α,σ,P) be a Jackson network specification for which a traffic solution exists.

Then, the traffic solution is unique iff the network is open.

Proof

($\Longleftarrow$) Assume that the network is open. Then by a) in Remark 4.3.3, $\bar{A} = \Phi$, so that $M = T$. The uniqueness of the traffic equation now follows from Lemma 4.4.1.

($\Longrightarrow$) Assume that the network is not open. Then $C \neq \Phi$ and consequently $\bar{A} \neq \Phi$.

By Theorem 4.4.1, for a traffic solution to exist, it is necessary that

(1) $\alpha_{\bar{A}} = 0$

so that Equation (2) of Theorem 4.4.1 reduces to

(2) $\delta_{\bar{A}} = \delta_{\bar{A}} P_{\bar{A}}$.

By Lemma 4.4.2 $\delta_D = 0$, where $D \triangleq B \cup (C - R)$, since (1) implies $\alpha_D = 0$.

Hence, in particular,

(3) $\delta_{C-R} = 0$.

Now, (3) and (B) allow us to deduce from (2) that

(4) $\delta_R = \delta_R P_R$

where $R = \{R_k\}_{k\varepsilon K}$ is the set of recurrent nodes and each R_k is irreducible. Observe that P_R is a stochastic matrix. Due to the nature of R, Equation (4) is equivalent to the system of equations

(5) $\delta_{R_k} = \delta_{R_k} P_{R_k}$, $k\varepsilon K$

where each R_k is a stochastic matrix.

It is known from Markov Chain Theory (see [KS1], p. 100), that for each $k \varepsilon K$, there is a probability vector Π_k satisfying the respective equation in (5). It is also clear that for each $k \varepsilon K$, the entire linear space spanned by Π_k solves the respective equation in (5). Hence, (4) does not have a unique solution in δ_R. We conclude that the traffic solution is not unique. □

Corollary 4.4.2

If JN = (M,α,σ,P) specifies an open Jackson network, then the unique traffic solution is

$$\delta = \alpha(I - P)^{-1} = \alpha \sum_{n=0}^{\infty} P^n .$$ □

Since the traffic solution may not be unique, it is of interest to determine the dimensionality of the traffic solution space.

Theorem 4.4.3

Let JN = (M,α,σ,P) specify a Jackson network for which there is a non-unique traffic solution. Let $\delta^* \triangleq (\delta^*_T; \delta^*_{R_1}; \ldots ; \delta^*_{R_{|K|}})$ be such a solution, where K is the index set of the irreducible classes $\{R_k\}_{k \varepsilon K}$.

Then, the traffic solution space is $|K|$-dimensional in the sense that any traffic solution δ has a representation $\delta = (\delta^*_T; \gamma_1 \delta^*_R; \ldots ; \gamma_{|K|}\delta^*_{R_{|K|}})$ in terms of δ^* and some scalars $\gamma_1, \ldots, \gamma_{|K|} \geq 0$.

Proof

Since the traffic solution for T is unique and in view of Theorem 4.4.2, it suffices to show that for every $k \varepsilon K$ the respective equation

(1) $\delta_{R_k} = \delta_{R_k} P_{R_k}$

has a 1-dimensional traffic solution space in the sense above. But this follows immediately from the fact that (1) has a unique probability solution (see [KS1] p. 100). □

The conservation aspects of the traffic equation is illustrated in

Theorem 4.4.4

Let JN = (M,α,σ,P) specify a Jackson network for which there exists a traffic solution δ. Then

(a) $\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \delta_i q_i$.

Proof

From the traffic equation (A) we have

(1) $\alpha = \delta(I - P)$.

Let $u \triangleq (1,1,\ldots,1)$ be the m-dimensional row vector of 1's. Post-multiplying both sides of (1) by u^T gives

(2) $\alpha u^T = \delta(I - P)u^T$.

A direct computation shows that $\alpha u^T = \sum_{i=1}^{m} \alpha_i$ and $(I - P)u^T = q^T$, where $q \triangleq (q_1,\ldots,q_m)$. Hence (2) becomes

(3) $\sum_{i=1}^{m} \alpha_i = \delta q^T = \sum_{i=1}^{m} \delta_i q_i$

which was to be proved. □

Intuitively, Theorem 4.4.4 asserts that the total influx intensity of customers into a Jackson network equals the total outflux of customers from it, when the network is in equilibrium.

On the basis of the facts accumulated thus far, we can now give an intuitive interpretation summarizing the investigation of the traffic equation.

First, by Theorem 4.4.1, the existence of a traffic solution is equivalent to the fact that only completely open nodes may have inlets. If, however, any other node had an inlet, then there would perforce be a path from an inlet to a recurrent subset of nodes. Such subsets are closed by definition, and hence are customer trapping. Intuitively, this means that customers would pile up indefinitely in a trapping sub-network. Clearly, this subnetwork would be out of balance as regards the rates of customer flow through it; the influx of customers into it would be positive but the outflux would be zero, in contradiction with the intuitive interpretation of the traffic equation as describing a balanced flow rate of customers through each node and each subnetwork for that matter.

Indeed, the transient node set has always a traffic solution by Lemma 4.4.1, since customers will never be trapped in them and flow rate balance can be always maintained. When a traffic solution is guaranteed to exist, it follows that, in particular, nodes in T-A cannot have access from an inlet. Consequently, they eventually lose their customers due to their transient nature, without being replenished with new ones. Eventually, customer traffic in them would die out and this part of the network would come to a standstill. Indeed, Lemma 4.4.2 shows that the equilibrium traffic rates through them is zero.

However, customers that drain out of this set and into the recurrent nodes of the network would cycle there forever. Since this set is autonomous, it neither gains nor loses customers. Eventually, the number of customers in each irreducible class will reach a fixed level, and a balanced flow rate through its nodes will be attained.

Indeed, even though a traffic solution exists for the recurrent part, by Theorem 4.4.3 it cannot be unique. It depends on the total (fixed) number of customers that cycle in each of its irreducible classes. Intuitively, this depends on the initial configuration of customers in the network and how they drain into the recurrent node set from the non-completely open part of the transient node set. Since there are $|K|$ irreducible classes, the solution has $|K|$ degrees of freedom in accordance with Theorem 4.4.3. Each degree of freedom corresponds to a choice of total number of customers in each irreducible subnetwork (in equilibrium), and the resulting traffic solution is proportional to this total number.

Our discussion has several important ramifications, provided that δ may be interpreted as equilibrium flow rates of customers through nodes, and that the existence of a traffic solution is necessary for equilibrium. Since these will be shown to be true in the next section, we shall defer the discussion of this issue until this intuition can be formally justified.

4.5 The State Process in Equilibrium

In this section we study equilibrium properties of the state process and equilibrium related aspects. The reader is referred to Appendix C, Sec. C.3 for some relevant background. Accordingly, a probability vector of the state process $\{Q(t)\}_{t\geq 0}$ will be denoted by $q(t)$, and an equilibrium vector by q^o.

State equilibrium results may be found in the literature for open Jackson networks and autonomous ones. These we now proceed to cite; the reader is reminded that all Jackson networks alluded to have single server nodes.

The following classical result for open Jackson networks is due to J. R. Jackson (See [JJ1]).

Theorem 4.5.1 (Jackson's Theorem†)

Let JN = (M,α,σ,P) specify an open Jackson network. Suppose that for each $1 \leq i \leq m$,

(a) $\rho_i \triangleq \frac{\delta_i}{\sigma_i} < 1$

where $\delta = (\delta_1,\ldots,\delta_m)$ is the (unique) traffic solution of JN.

Then, the birth-and-death equations of the state process $\{Q(t)\}_{t\geq 0}$ have an equilibrium solution vector q^o, which for any $\nu = (n_1,\ldots,n_m) \geq 0$ is given by

(b) $q^o_\nu \triangleq Pr(Q_1(t) = n_1,\ldots,Q_m(t) = n_m) = \prod_{i=1}^{m} (1-\rho_i)\rho_i^{n_i}$.

Proof

By direct substitution into the birth-and-death equation (F) in Sec. 4.2 (see [JJ1]). □

†Originally, this theorem was proved for open networks with arbitrary number of servers in the nodes.

An analogous result was proved by Gordon and Newell for autonomous closed Jackson networks (see [GN1]).

Theorem 4.5.2 (Gordon-Newell Theorem†)

Let JN = (M,α,σ,P) specify an autonomous closed Jackson network with communicating nodes. Let #M be the total number of customers in the network, such that

(a) $\Pr(\#M = n) = 1$.

Then, the birth-and-death equations of the state process have an equilibrium solution vector $q^o = q^o(n)$ (depending on n), which for any $\nu = (n_1,\ldots,n_m) \geq 0$ is given by

(b) $q^o_\nu(n) \stackrel{\Delta}{=} \Pr(Q_1(t) = n_1,\ldots,Q_m(t) = n_m \mid \#M = n) =$

$$\begin{cases} 0, \text{ if } \|\nu\| \neq n \\ \dfrac{1}{g(n)} \displaystyle\prod_{i=1}^{m} \rho_i^{n_i} \end{cases}$$

where g(n) is a normalization factor, and $\rho_i \stackrel{\Delta}{=} \dfrac{\delta_i}{\sigma_i}$ where $\delta = (\delta_1,\ldots,\delta_m)$ is any traffic solution.

Proof

By direct substitution into the birth-and-death equations (F) in Sec. 4.2 (see [GN1]). □

†Originally, this theorem was proved for autonomous closed networks with arbitrary number of servers in the nodes.

The results cited above reveal a remarkable property of the state process; they exemplify an equilibrium solution for the state process $\{Q(t)\}_{t \geq 0}$, whereby the local states $Q_i(t)$, $1 \leq i \leq m$, are mutually independent for every fixed $t \geq 0$. Moreover, each local state process $\{Q_i(t)\}_{t \geq 0}$ in the open network behaves *as if* node i were a M/M/1 queue with exogenous input parameter δ_i (see [JJ1] pp. 378-379).

J. R. Jackson points out (see [JJ2] pp. 135-136) that for open networks, the recurrent state set of the global state process is irreducible, and thus the equilibrium solution vector of Theorem 4.5.1 is a long run vector (see Definition C.3.1 in Appendix C). However, only an outline of a proof is given by him. We shall now prove this fact in detail.

Theorem 4.5.3

Let $JN = (M,\alpha,\sigma,P)$ specify an open Jackson network that satisfies the conditions of Theorem 4.5.1.

Then, its equilibrium solution vector q^o is a long run vector.

Proof

It suffices to show that the recurrent states of the global state process are irreducible (see [Cil] p. 264). To do this we show that the zero state $\theta \triangleq (0,\ldots,0)$ is accessible from every state $\nu = (n_1,\ldots,n_m) \geq 0$. It suffices to show that $\nu' \rightsquigarrow \nu$ for every pair of adjacent states ν' and ν such that $\nu' = \nu + e_i$ for some $1 \leq i \leq m$ (e_i is the unit vector with 1 in the i-th coordinate). The desired result $\nu' \rightsquigarrow \theta$ then follows by an immediate induction on $||\nu'||$.

Now, since the network is open there is a sequence of distinct nodes $j_1, j_2, \ldots, j_k$ such that

(1) $p_{ij_1} p_{j_1 j_2} \cdots p_{j_k 0} > 0$

Consider the sequence of adjacent states $\{\nu_\ell\}_{\ell=0}^{k+1}$, where $\nu_0 = \nu'$, $\nu_{k+1} = \nu$ and $\nu_\ell = \nu_{\ell-1} - e_{j_{\ell-1}} + e_{j_\ell}$, $1 \le \ell \le k$. We show that $p_{\nu_\ell \nu_{\ell+1}}(t) > 0$ for $0 \le \ell \le k$ and $t > 0$, by using the integral representation (G) in Sec. C.1 of Appendix C for the functions $p_{\nu_\ell \nu_{\ell+1}}(t)$ viz.

(2) $$p_{\nu_\ell \nu_{\ell+1}}(t) = \sum_\mu \int_0^t p_{\nu_\ell \mu}(x) c_\mu r_{\mu \nu_{\ell+1}} e^{-c_{\nu_{\ell+1}}(t-x)} dx$$

because $\nu_\ell \neq \nu_{\ell+1}$, $0 \le \ell \le k$. However (see ibid.),

(3) $$p_{\nu_\ell \nu_\ell}(t) \ge e^{-c_{\nu_\ell} t} > 0.$$

Substituting (3) in the term $\mu = \nu_\ell$ on the right hand side of (2) yields for $t > 0$,

(4) $$p_{\nu_\ell \nu_{\ell+1}}(t) \ge \int_0^t p_{\nu_\ell \nu_\ell}(x)\, c_{\nu_\ell} r_{\nu_\ell \nu_{\ell+1}} e^{-c_{\nu_{\ell+1}}(t-x)} dx \ge$$

$$\int_0^t e^{-c_{\nu_\ell} x} c_{\nu_\ell} r_{\nu_\ell \nu_{\ell+1}} e^{-c_{\nu_{\ell+1}}(t-x)} dx.$$

Observe that $r_{\nu_\ell \nu_{\ell+1}} > 0$ for all $0 \le \ell \le k$ since

$$r_{\nu_0 \nu_1} = r_{\nu' \nu_1} = \frac{\sigma_i p_{ij_1}}{c_{\nu_0}} > 0$$

and

$$r_{\nu_\ell \nu_{\ell+1}} = \frac{\sigma_{j_\ell} p_{j_\ell j_{\ell+1}}}{c_{\nu_\ell}} > 0 \quad \text{for} \quad 1 \le \ell \le k,$$

due to (1).† Hence,

(5) $p_{\nu_\ell \nu_{\ell+1}}(t_\ell) > 0$ for some $t_\ell > 0$.

---

†The $r_{\nu_\ell \nu_{\ell+1}}$ are the conditional probabilities that the state will jump from ν_ℓ to $\nu_{\ell+1}$ given that a jump has taken place.

Now, applying repeatedly the Chapman-Kolmogorov equations gives us

(6) $p_{\nu'\nu}(\sum_{\ell=0}^{k} t_\ell) = \prod_{\ell=0}^{k} p_{\nu_\ell \nu_{\ell+1}}(t_\ell) > 0.$

Consequently, $\nu' \rightsquigarrow \nu$ as required. □

We note in passing, that Theorem 4.5.3 holds for the networks of Theorem 4.5.2, but it does not hold, in general, for arbitrary ones. Also, the state process need not be recurrent in mixed networks. For example, transient states are engendered by putting $n_i > 0$ in $\nu = (n_1,\ldots,n_m)$ for any open node i with $\delta_i = 0$.

Even if a mixed network has an equilibrium vector q^o, it may not be unique, because the asymptotic state of closed subnetworks does depend on the initial conditions.

Our next step is to exhibit to what extent condition (a) in Theorem 4.5.1 is necessary for existence of equilibrium in mixed Jackson networks.

A requirement of the form $\frac{\delta_i}{\sigma_i} < 1$ is an "obvious" necessary condition for equilibrium, provided δ coincides with the vector of equilibrium traffic rates through nodes.

We shall now show that this intuition is largely justified. Formally, we prove

Theorem 4.5.4

Let JN = (M,α,σ,P) specify a mixed Jackson network that possesses an equilibrium vector q^o for its state process. For any $t \geq 0$, let $E(D(t,t+1]) \triangleq (E(D_1(t,t+1]),\ldots,E(D_m(t,t+1]))$.

Then, $E(D(t,t+1])$ satisfies the traffic equation, provided $q(0) = q^o$.

Proof

Since $D_i(t) = \sum_{j=0}^{\infty} A_{ij}(t)$, it follows from Theorem 4.2.1

(1) $E(D_i(t)) = \sum_{j=0}^{m} \sigma_i p_{ij} \int_0^t Pr(B_i(x)=1)dx = \sigma_i \cdot \int_0^t Pr(B_i(x)=1)dx.$

In particular,

(2) $E(D_i(t,t+1]) = E(D_i(t+1)) - E(D_i(t)) = \sigma_i \cdot \int_t^{t+1} Pr(B_i(x)=1)dx.$

By the equilibrium assumption on $\{Q(t)\}_{t\geq 0}$ it follows that each $\{B_i(t)\}_{t\geq 0}$ is in equilibrium and thus $Pr(B_i(x)=1)$ = const. for all $x\varepsilon(t,t+1]$. Hence, (2) becomes

(3) $E(D_i(t,t+1]) = \sigma_i Pr(B_i(t)=1) = \sigma_i Pr(Q_i(t) > 0).$

Next, we apply generating function methods to the birth-and-death equation (F) in Sec. 4.2. The generating function of q^o is defined by

(4) $\Phi(z_1,\ldots,z_m) \stackrel{\Delta}{=} \sum_{\nu=(n_1,\ldots,n_m)\geq 0} q_\nu^o \cdot \prod_{i=1}^{m} z_i^{n_i}$

and it exists in the domain $\{z = (z_1,\ldots,z_m): |z_i| \leq 1, 1 \leq i \leq m\}$, provided we define $0^0 \stackrel{\Delta}{=} 1$.

For each $\nu = (n_1,\ldots,n_m)$, multiply both sides of (F) in Sec. 4.2 by $\prod_{i=1}^{m} z_i^{n_i}$, and sum the outcome over $\nu = (n_1,\ldots,n_m) \geq 0$. Since the left hand side of (F) in Sec. 4.2 is always zero for q^o (see Theorem C.3.1 in Appendix C), we obtain after some algebraic manipulation

(5) $$0 = \sum_{i=1}^{m} \Phi(z_1,\ldots,z_m)\alpha_i[z_i-1] + \sum_{i=1}^{m} [\Phi(z_1,\ldots,z_m) - \Phi(z_1,\ldots,0_i,\ldots,z_m)]\sigma_i q_i[\frac{1}{z_i} - 1] + \sum_{i=1}^{m} \sum_{\substack{j=1\\ j\neq i}}^{m} [\Phi(z_1,\ldots,z_m) - \Phi(z_1,\ldots,0_j,\ldots,z_m)]\sigma_j p_{ji}[\frac{z_i}{z_j} - 1]$$

$0 < |z_i| \leq 1 \quad , \quad 1 \leq i \leq m .$

Here and in the sequel 0_i indicates a zero in the i-th coordinate and similarly for 1_i.

Observe that by setting $z_k = 1$ in $\Phi(z_1,\ldots,z_m)$, the resulting function $\Phi(z_1,\ldots,1_k,z_m)$ is precisely the generating function of the process $(Q_1(t),\ldots,Q_{k-1}(t),Q_{k+1}(t),\ldots,Q_m(t))$, subject to $q(0) = q^o$; thus, $\Phi(z_1,\ldots,1_k,\ldots,z_m) = \Phi(z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_m)$. Consequently, whenever we set $z_k = 1$, $k \neq i$ for any fixed $1 \leq i \leq m$, (5) reduces to

$$(6)\quad 0 = \Phi(z_i)\alpha_i[z_i-1] + [\Phi(z_i) - \Phi(0_i)]\sigma_i q_i[\frac{1}{z_i} - 1] + \sum_{\substack{j=1\\ j\neq i}}^{m}[\Phi(z_i) - \Phi(z_i,0_j)]\sigma_j p_{ji}[z_i-1] + \sum_{\substack{j=1\\ j\neq i}}^{m}[\Phi(z_i) - \Phi(0_i)]\sigma_i p_{ij}[\frac{1}{z_i} - 1]$$

$$0 < |z_i| \leq 1.$$

After collecting terms, (6) becomes

$$(7)\quad 0 = (\phi(z_i)\alpha_i + \sum_{\substack{j=1\\ j\neq i}}^{m}[\Phi(z_i) - \Phi(z_i,0_j)]\sigma_j p_{ji})[z_i-1] + [\Phi(z_i) - \Phi(0_i)](\sigma_i q_i + \sum_{\substack{j=1\\ j\neq i}}^{m}\sigma_i p_{ij})[\frac{1}{z_i} - 1]$$

$$0 < |z_i| \leq 1,$$

and a further simplification of (7) yields

$$(8)\quad 0 = (\Phi(z_i)\alpha_i + \sum_{\substack{j=1\\ j\neq i}}^{m}[\Phi(z_i) - \Phi(z_i,0_j)]\sigma_j p_{ji})[z_i-1] + [\Phi(z_i) - \Phi(0_i)]\sigma_i(1-p_{ii})[\frac{1}{z_i} - 1]$$

$$0 < |z_i| \leq 1.$$

For $0 < |z_i| < 1$ we may divide both sides of (8) by $z_i - 1$, whence (8) becomes

(9) $$[\Phi(z_i) - \Phi(0_i)]\sigma_i(1-p_{ii}) = (\Phi(z_i)\alpha_i + \sum_{\substack{j=1\\ j\neq i}}^{m} [\Phi(z_i) - \Phi(z_i,0_j)]\sigma_j p_{ji})z_i$$

$0 < |z_i| < 1.$

Equating coefficients of the $z_i^{n_i}$ on both sides of (9) gives us

(10) $$Pr(Q_i(t) = n_i)\sigma_i(1-p_{ii}) =$$

$$Pr(Q_i(t) = n_i-1)\alpha_i +$$

$$\sum_{\substack{j=1\\ j\neq i}}^{m} [Pr(Q_i(t) = n_i-1) - Pr(Q_i(t) = n_i-1, Q_j(t) = 0)]\sigma_j p_{ji}$$

$1 \leq n_i < \infty$.

For each $1 \leq i \leq m$, sum the system of equations (10) over $1 \leq n_i < \infty$. We get

(11) $$Pr(Q_i(t) > 0)\sigma_i(1-p_{ii}) = \alpha_i + \sum_{\substack{j=1\\ j\neq i}}^{m} Pr(Q_j(t) > 0)\sigma_j p_{ji}$$

$1 \leq i \leq m$.

Substituting (3) in (11) and rearranging its terms gives us

(12) $$E(D_i(t,t+1]) = \alpha_i + \sum_{j=1}^{m} E(D_j(t,t+1])p_{ji} \quad , \quad 1 \leq i \leq m$$

Comparing (12) with (B) in Sec. 4.4 shows that $E(D(t,t+1])$ does indeed satisfy the traffic equation, for any $t \geq 0$. □

Corollary 4.5.1

If $JN = (M,\alpha,\sigma,P)$ is a mixed Jackson network in any equilibrium q^o, then

a) the associated traffic equation always has a solution δ^* defined by

$\delta^* \triangleq E(D(t,t+1])$

where δ^* depends on the equilibrium vector q^o.

b) $E(D(s,u]) = \delta^*\cdot(u-s)$ for any $s \le u$.

c) $\delta_i^* = \sigma_i \cdot \int_t^{t+1} Pr(B_i(x)=1)dx \le \sigma_i$, $1 \le i \le m$, for any $t \ge 0$. □

We now refine part (c) in the above corollary, as follows.

Theorem 4.5.5

Let $JN = (M,\alpha,\sigma,P)$ specify a mixed Jackson network. If there is an equilibrium vector q^o for $\{Q(t)\}_{t\ge 0}$, then

(a) $\rho_i < 1$, for every node i with $p_{ii} < 1$

where $\rho_i \triangleq \frac{\delta_i^*}{\sigma_i}$ and $\delta_i^* \triangleq E(D_i(t,t+1])$ for any $t \ge 0$.

Proof

By c) of Corollary 4.5.1 the traffic solution $\delta^* \triangleq E(D(t,t+1])$ satisfies

(1) $\delta_i^* \le \sigma_i$, $1 \le i \le m$.

Suppose, however, that there is $i\varepsilon M$ with $p_{ii} < 1$, but $\delta_i = \sigma_i$.

Then by c) of Corollary 4.5.1 we get $Pr(Q_i(t) > 0) = 1$ whence

(2) $Pr(Q_i(t)=0) = 0$

We proceed by induction. Suppose that

(3) $Pr(Q_i(t)=n_i-1) = 0$.

Setting (3) in (10) of Theorem 4.5.4 yields

(4) $Pr(Q_i(t)=n_i)\sigma_i(1-p_{ii}) = 0$.

By assumption $p_{ii} < 1$, whence from (4)

(5) $Pr(Q_i(t)=n_i) = 0$

as $\sigma_i > 0$ always.

But (5) shows that every state of $\{Q(t)\}_{t\geq 0}$ is transient, which is impossible in view of the fact that it has an equilibrium vector q^o (see [Cil] p. 263).

We conclude that $\delta_i = \sigma_i$ is impossible for nodes i with $p_{ii} < 1$, so that (a) follows from (1). □

We can now sum up our discussion of state equilibria situations of open and autonomous Jackson networks with single server nodes.

The following theorem characterizes existence of equilibrium in mixed networks.

Theorem 4.5.6

Let $JN = (M,\alpha,\sigma,P)$ specify a mixed Jackson network. Then, the network has a state equilibrium vector q^o iff the following two conditions hold:

a) the associated traffic equation has a traffic solution $\delta^* \triangleq E(D(t,t+1])$, $t \geq 0$;

b) $\rho_i \triangleq \frac{\delta_i^*}{\sigma_i} < 1$ for any completely open node.

Proof

($\Longrightarrow$) Suppose the network has an equilibrium vector q^o. Then a) holds due to part a) of Corollary 4.5.1, and b) is implied by (a) of Theorem 4.5.5 (observe that in equilibrium every completely open node i always has $p_{ii} < 1$).

($\Longleftarrow$) Suppose a) and b) hold.

Denote by $J(\nu_A)$ the Jackson solution (see (b) in Theorem 4.5.1) for the completely open part A.

Denote by $q^o(||\nu_{R_k}||)$ the Gordon-Newell solution (see (b) in Theorem 4.5.2), obtained for each recurrent irreducible node set R_k, $k\varepsilon K$. Having chosen a distribution for each $\#R_k$, $k\varepsilon K$ (recall that $\#R_k$ is the total number of customers in R_k in equilibrium), we define

(1) $$G(\nu_{R_k}) \stackrel{\Delta}{=} \sum_{n=0}^{\infty} q^o_{\nu_{R_k}}(n)\Pr(\#R_k=n) = q^o_{\nu_{R_k}}(||\nu_{R_k}||)\Pr(\#R_k=||\nu_{R_k}||).$$

Lemma 4.4.2 guarantees that the remaining node set D has a unique traffic solution $\delta_D = 0$. Denote $Z(\nu_D) \stackrel{\Delta}{=} \begin{cases} 1, & \text{if } \nu_D = 0 \\ 0, & \text{otherwise} \end{cases}$.

Finally, it can be verified by direct substitution into the birth-and-death equations (F) in Sec. 4.2 that

(2) $$q^o_\nu \stackrel{\Delta}{=} J(\nu_A)\cdot Z(\nu_D)\cdot \prod_{k\varepsilon K} G(\nu_{R_k}), \quad \nu = (n_1,\ldots,n_m)$$

is an equilibrium vector of these equations. □

Condition a) of Theorem 4.5.6 agrees with the heuristic observation that a network containing a closed subnetwork, which is accessible from an inlet, cannot have an equilibrium vector.

Intuitively, in this case, customers would be "trapped" in that closed subnetwork, and their number would grow indefinitely. Indeed, Theorem 4.4.1 guarantees that this does not happen, because existence of a traffic solution δ is equivalent to the requirement $\alpha_{\bar{A}} = 0$.

Condition a) of Theorem 4.5.5 agrees with the intuition that in equilibrium each node i, excluding the trivial case $P_{ii}=1$, must have service rate σ_i which exceeds the influx rate δ^*_i of customers into i. Otherwise, customers would "pile up" in that node and its line would grow indefinitely.

We now proceed to characterize uniqueness of an equilibrium vector for mixed Jackson networks.

Theorem 4.5.7

Under the conditions of Theorem 4.4.6

a) there is a unique equilibrium vector q^o

iff

b) the traffic equation has a unique solution δ.

Otherwise, every initial condition $q(0)$ determines an equilibrium vector q^o such that $q(t) \xrightarrow[t\to\infty]{} q^o$.

Proof

Condition a) holds iff the equilibrium vector defined in (2) of Theorem 4.5.6 has no G factors.

Now, this happens iff the network has no closed nodes (i.e. iff the network is open). But then we know by Theorem 4.4.2 that a Jackson network is open iff it has a unique traffic solution.

Next observe that $q(0)$ determines the asymptotic distribution of total number of customers in each R_k and hence of $\#R_k$, $k\varepsilon K$ (see, e.g. [KS1] p. 52 for absorbing probabilities of single customers). This in turn determines the choice of the $G(\nu_{R_k})$ in (2) of Theorem 4.5.6. □

The foregoing discussion shows that in equilibrium, the state process of a mixed Jackson network can be studied separately for the completely open part A and each irreducible recurrent part R_k, $k\varepsilon K$. The remaining node set D is devoid of customers with probability 1, and for all practical purposes can be removed from the network.

It is also interesting to note that the equilibrium state behavior can be completely determined from a simple algebraic equation--the traffic equation--as far as existence, uniqueness and form of equilibria solutions are concerned.

4.6 Total Service Times and Number of Visits to Nodes

In this section we investigate two customer-oriented behavioral frames: total service time and number of visits to individual nodes.

The *total service time* $\tilde{S}$ of a customer is the sum of all service times the customer receives at the various nodes of a Jackson network from the instant of his arrival at the network until he exits the system at some outlet. The *total service time of a customer in the network, given that his entry node to the network was i,* is denoted here $\tilde{S}_i$.

Our main tool of analysis will be generating functions--in this case the Laplace-Stietjes transform (abbreviated LS transform)--of the relevant distribution functions.

The LS transform of the distribution of $\tilde{S}$ is defined by

$$\text{(A)} \quad g(\zeta) \triangleq \int_{-\infty}^{\infty} e^{-\zeta x} dF_{\tilde{S}}(x)$$

where $dF_{\tilde{S}}(x)$ designates the Laplace-Stieljes measure induced by the distribution $F_{\tilde{S}}$ of $\tilde{S}$. Likewise, $f_i(\zeta)$ denotes the LS transform of the distribution of $\tilde{S}_i$, and $v_i(\zeta)$ denotes the LS transform of the distribution of the *service time* S_i at node i. Observe that

$$\text{(B)} \quad v_i(\zeta) = \frac{\sigma_i}{\zeta+\sigma_i} \quad , \quad \zeta \geq 0$$

because node i accomodates exponential servers.

Next, let f be the column vector $f \triangleq (f_1,\ldots,f_m)^T$ and q the column vector $q \triangleq (q_1,\ldots,q_m)^T$ where $q_i \triangleq p_{i0}$. Finally, let Γ be the *diagonal matrix whose i-th diagonal entry is* σ_i.

Theorem 4.6.1 (cf. [BMZ1], Theorem 5.1)

Let JN = (M,α,σ,P) specify a Jackson network. Then for $\zeta > 0$

(1) $f(\zeta) = U^{-1}(\zeta)q$

where $U(\zeta) = [u_{ij}(\zeta)]$ is an $m \times m$ matrix defined by

(2) $u_{ij}(\zeta) \triangleq (\frac{\zeta}{\sigma_i} + 1)\zeta_{ij} - p_{ij}$

where δ_{ij} is Kronecker's delta.

Proof

A customer arriving at node i receives a service time S_i with LS transform $v_i(\zeta)$. Then, he either exits the network (with probability q_i) or is routed to node j (with probability p_{ij}), whereby his residual total service time is $\tilde{S}_j$ with LS transform $f_j(\zeta)$.

Since all individual service times are mutually independent, we are led to the renewal-like equation

(3) $f_i(\zeta) = [q_i + \sum_{j=1}^{m} p_{ij} f_j(\zeta)] v_i(\zeta)$, $1 \le i \le m$.

Substituting $v_i(\zeta)$ into (3) from (B) and switching to vector notation gives us

(4) $(\zeta I + \Gamma) f(\zeta) = \Gamma(q + Pf(\zeta))$.

Premultiplying (4) by Γ^{-1} and factoring out $f(\zeta)$ yields

(5) $(\zeta\Gamma^{-1} + I - P) f(\zeta) = q$.

Now, define

(6) $U(\zeta) \triangleq \zeta\Gamma^{-1} + I - P$.

Then $U(\zeta)$ coincides with (2), and (5) becomes

(7) $U(\zeta) f(\zeta) = q$.

It remains to show that $U(\zeta)$ is invertible for any $\zeta > 0$.

Now, $U(\zeta)$ can be written as

(8) $U(\zeta) = (I - P(\zeta\Gamma^{-1} + I)^{-1}) \cdot (\zeta\Gamma^{-1} + I)$.

Clearly, $\zeta\Gamma^{-1} + I$ is invertible. Moreover, $||(\zeta\Gamma^{-1} + I)^{-1}|| < 1$ for any $\zeta > 0$, and $||P|| \le 1$. From these we immediately have $||P(\zeta\Gamma^{-1} + I)^{-1}|| \le ||P|| \cdot ||(\zeta\Gamma^{-1} + I)^{-1}|| < 1$, and we conclude that $(I - P(\zeta\Gamma^{-1} + I)^{-1})^{-1}$ exists (see [KS1] p. 22). Hence,

(9) $U^{-1}(\zeta) = (\zeta\Gamma^{-1} + I)^{-1} \cdot (I - P(\zeta\Gamma^{-1} + I)^{-1})^{-1}$, $\zeta < 0$

exists since each of its factors exists.

We now show that $f(\zeta)$ as defined by (1) is indeed a generating function.

Define a sequence of vector functions $\{f^{(n)}(\zeta)\}_{n=0}^{\infty}$ as follows. Let $f^{(0)}(\zeta) \stackrel{\Delta}{\equiv} 0$, and let $f^{(n+1)}(\zeta)$ be recursively defined by

(10) $(\zeta I + \Gamma) f^{(n+1)}(\zeta) = \Gamma(q + Pf^{(n)}(\zeta))$, $\zeta \ge 0$

It can now be shown by induction on n (using (3) and (10)) that for any $1 \le i \le m$ and $n=0,1,\ldots$, the following hold:

(11.a) $0 \le f_i^{(n)}(\zeta) \le 1$, $\zeta \ge 0$.

(11.b) $f_i^{(n)}(\zeta)$ is a possibly defective generating function (shown by sending $\zeta \longrightarrow 0+$).

(11.c) $f_i^{(n+1)}(\zeta) = [q_i + \sum_{j=1}^{m} p_{ij} f_j^{(n)}(\zeta)] v_i(\zeta) \ge$
$[q_i + \sum_{j=1}^{m} p_{ij} f_j^{(n-1)}(\zeta)] v_i(\zeta) = f^{(n)}(\zeta)$, $\zeta \ge 0$.

We conclude that the vector function sequence $\{f^{(n)}(\zeta)\}_{n=0}^{\infty}$ is monotone and bounded in each coordinate i for every $\zeta \ge 0$. Hence this sequence converges pointwise such that

(12) $f^{(n)} \xrightarrow[n\to\infty]{} f^{(\infty)}$.

Moreover, the limit function $f^{(\infty)}$ is itself a possibly defective generating function (see [F2], XIII.1, Theorem 2). Taking limits as $n\to\infty$ in (10) shows that $f^{(\infty)}(\zeta)$ solves (4).

By uniqueness of the solution $f(\zeta)$ of (4) we conclude that

(13) $f = f^{(\infty)} \overset{\Delta}{=} \lim_{n\to\infty} f^{(n)}$ (pointwise limit)

so that f is indeed a possibly defective generating function. □

The defect of each $f_i(\zeta)$ is interpreted as $\Pr(\tilde{S}_i = \infty)$. To compute the defect we employ

Theorem 4.6.2 (cf. [BMZ1], Theorem 5.1)

Let JN = (M,α,σ,P) specify a Jackson network. Then, for any $1 \leq i \leq m$, $\Pr(\tilde{S}_i < \infty) = \lim_{n\to\infty} p_{i0}^{(n)}$. Moreover, $\lim_{n\to\infty} p_{i0}^{(n)} = \lim_{\zeta\to 0+} f_i(\zeta)$ or alternatively $\lim_{n\to\infty} p_{i0}^{(n)}$, $1 \leq i \leq m$, constitutes the minimal solution of the equation

(1) $u = q + Pu$

in the column vector of unknowns $u = (u_1 \ldots u_m)^T$.

Proof

The defect of each $f_i(\zeta)$ is

(2) $1 - \lim_{\zeta\to 0+} f_i(\zeta) = \Pr(S_i = \infty)$.

By (11.c) in Theorem 4.6.1 we know that for any $\zeta \geq 0$, $f^{(n)}(\zeta)\uparrow f(\zeta)$ as $n\to\infty$. Furthermore, since $v_i(\zeta)\uparrow v_i(0)$ as $\zeta\to 0+$ for each i (see (A)), it follows from (10) in Theorem 4.6.1 that for every n = 0,1,... $f^{(n)}(\zeta)\uparrow f^{(n)}(0)$ as $\zeta\to 0+$, by induction on n.

Using these facts we obtain from (2)

(3) $\Pr(\tilde{S}_i < \infty) = \lim_{\zeta\to 0+} f_i(\zeta) = \lim_{\zeta\to 0+} \lim_{n\to\infty} f_i^{(n)}(\zeta) =$

$\lim_{n\to\infty} \lim_{\zeta\to 0+} f_i^{(n)}(\zeta) = \lim_{n\to\infty} f_i^{(n)}(0) = f_i^{(\infty)}(0) = f_i(0)$

because monotone limits are interchangeable.

Next, denote $f^{(n)}(0) \triangleq u^{(n)}$ and set $\zeta = 0$ in equation (10) of Theorem 4.6.1. Premultiplying the outcome by Γ^{-1} for each $n = 0,1,\ldots$ gives us

(4) $u^{(0)} = 0$ and $u^{(n+1)} = q + Pu^{(n)}$, $n = 0,1,\ldots$.

Now, $\{u^{(n)}\}_{n=0}^{\infty}$ is a monotone and bounded sequence and thus its pointwise limit u exists. Sending $n\to\infty$ in (4) shows that u satisfies (1), and by induction on n one can show u to be the minimal non-negative solution of (1). Hence

(5) $\Pr(\tilde{S}_i < \infty) = f_i(0) \triangleq u_i$, $1 \le i \le m$

and it remains to show

(6) $u_i = \lim_{n\to\infty} p_{i0}^{(n)}$, $1 \le i \le m$.

Expanding (1) by components and writing p_{i0} for q_i gives us

(7) $u_i = p_{i0} + \sum_{j=1}^{m} p_{ij}u_j$, $1 \le i \le m$.

A standard result in Markov chain theory (see [F1], Sec. XV.8, Theorem 2) shows that the minimal non-negative solution of (7) is precisely the probability of eventually being absorbed in node 0, given that the initial node is i. Thus, (6) holds as required. □

The defect of $f(\zeta)$ can now be characterized in terms of the topology of the network as follows.

<u>Corollary 4.6.1</u>

For any Jackson network

a) $\Pr(\tilde{S}_i < \infty) = 0$, if $i\in C$.

b) $\Pr(\tilde{S}_i < \infty) = 1$, if $i\in A$.

c) $0 < Pr(S_i < \infty) < 1$, if $i \in O-A$. □

It is interesting to note that $f(\zeta)$ has a relatively simple form. The representation $f(\zeta) = U^{-1}(\zeta)q$ shows that each $f_i(\zeta)$ is a rational function whose denominator is a polynomial in ζ of degree m at most. This is so because the denominator of each entry in $U^{-1}(\zeta)$ is the determinant of $U(\zeta)$, and by definition of $U(\zeta)$ it is seen to be such a polynomial. Consequently, $f(\zeta)$ is a transform of mixed exponentials.

The comments above are also pertinent to the unconditional total service time $\tilde{S}$ due to the following.

Lemma 4.6.1 (cf. [BMZ1]), Sec. V)

Let JN = (M,α,σ,P) specify a Jackson network. Then the generating function of $\tilde{S}$ is $g(\zeta) = r\cdot f(\zeta)$ where $r \triangleq \frac{\alpha}{||\alpha||}$.

Proof

The probability that a customer enters the network at node i is

$\frac{\alpha_i}{\sum_{j=1}^{m} \alpha_j} = \frac{\alpha_i}{||\alpha||}$. Hence the generating function of $\tilde{S}$ is

$$g(\zeta) = \sum_{i=1}^{m} r_i f_i(\zeta) = r\cdot f(\zeta) \text{ as required.}$$ □

Corollary 4.6.2 (cf. [BMZ1], Sec. V)

$$E(\tilde{S}) = \sum_{i=1}^{m} r_i E(\tilde{S}_i) .$$

Consequently $E(\tilde{S}) < \infty$ iff $\alpha_{\bar{A}} = 0$, i.e. no closed node is accessible from an inlet. This is equivalent to existence of a traffic solution by Theorem 4.4.1. □

We now proceed to compute $E(\widetilde{S})$ when it is finite. Clearly, in this case, it suffices to compute $E(\widetilde{S})$ for open Jackson networks.

Theorem 4.6.3 (cf. [BMZ1], Theorem 5.2)

Let JN = (M,α,σ,P) be an open Jackson network. Then $E(\widetilde{S}) = \frac{||\rho||}{||\alpha||}$ where $\rho \triangleq (\frac{\delta_1}{\sigma_1},\ldots,\frac{\delta_m}{\sigma_m})$.

Proof

By Lemma 4.6.1, the generating function of $\widetilde{S}$ is

(1) $g(\zeta) = r\cdot f(\zeta)$.

Using in (1) the moment property of generating functions (in our case LS transform) we get

(2) $E(S) = -\lim_{\zeta\to 0+} r\cdot f'(\zeta)$

where the prime indicates differentiation with respect to ζ.

Next, differentiate both sides of

(3) $U(\zeta)\cdot f(\zeta) = q \quad , \quad \zeta > 0$.

(Cf. Equation (1) of Theorem 4.6.1.)

We obtain

(4) $U(\zeta)\cdot f'(\zeta) + U'(\zeta)\cdot f(\zeta) = 0 \quad , \quad \xi > 0$

and since $U^{-1}(\zeta)$ exists by Theorem 4.6.1 for $\zeta > 0$, (4) becomes

(5) $-f'(\zeta) = U^{-1}(\zeta)\cdot U'(\zeta)\cdot f(\zeta) \quad , \quad \zeta > 0$.

Now, for open networks, the defect of each $f_i(\zeta)$ is

(6) $1 - Pr(\widetilde{S}_i < \infty) = 1 - \lim_{n\to\infty} p_{i0}^{(n)} = 1 - 1 = 0$

because every node is transient in $\widetilde{P}$, save 0 which is an absorbing node. Therefore

(7) $f(\zeta) \xrightarrow[\zeta\to 0]{} u^T$

where u is the row vector of 1's.

Next, since $U(\zeta) = \zeta\Gamma^{-1} + I - P$ (see (6) in Theorem 4.6.1), we have

(8) $U'(\zeta) \equiv \Gamma^{-1}$.

Finally, we assert that

(9) $U^{-1}(\zeta) \xrightarrow[\zeta\to 0+]{} (I - P)^{-1} = \sum_{n=0}^{\infty} P^n$.

To see this observe that $(U(\zeta)\cdot\Gamma)^{-1} = R_\zeta$ where R_ζ is defined as the resolvent operator for $(I - P)\Gamma$ at the value ζ (see [T1] Ch. 5, Sec. 5.1). The origin belongs to the resolvent set as $R_0 = (I - P)^{-1}$ exists. Hence, $R_\zeta \longrightarrow R_0$ as $\zeta\to 0+$; i.e. $\Gamma^{-1}U^{-1}(\zeta) \xrightarrow[\zeta\to 0+]{} \Gamma^{-1}U^{-1}(0)$ (see ibid.). (9) now follows, since $U^{-1}(0) = (I - P)^{-1}$ by (6) in Theorem 4.6.1.

Substituting (5) into (2) and using (7), (8) and (9) gives us

(10) $E(\tilde{S}) = \lim_{\zeta\to 0+} r\cdot U^{-1}(\zeta)\cdot U'(\zeta)\cdot f(\zeta) = r\cdot(I - P)^{-1}\cdot\Gamma^{-1}\cdot u^T$.

Substituting $r \triangleq \frac{\alpha}{||\alpha||}$ and denoting $\sigma^{-1} \triangleq (\frac{1}{\sigma_1},\dots,\frac{1}{\sigma_m})$, we see that (10) becomes

(11) $E(\tilde{S}) = \frac{\alpha}{||\alpha||}(I - P)^{-1}(\sigma^{-1})^T = \frac{\delta}{||\alpha||}(\sigma^{-1})^T = \frac{||\rho||}{||\alpha||}$

since $\delta = \alpha(I - P)^{-1}$ from the definition of the traffic equation. □

We now proceed to investigate the number of times K_i, that a customer visits node i during his stay in the network. Let K denote the vector $K \triangleq (K_1,\dots,K_m)$.

Theorem 4.6.4 (cf. [BMZ1], Theorem 6.1)

Let JN = (M,α,σ,P) specify a Jackson network. Then

(1) $E(K) = r(I - P)^{-1}$

where $r \triangleq \frac{\alpha}{||\alpha||}$.

Proof

Let K_{ij} be the number of visits of a customer to node j, given that his entry node to the network was i. It is a well known result (derived from the finite Markov chain with transition matrix $\tilde{P}$) that $E(K_{ij}) = \sum_{n=0}^{\infty} p_{ij}^{(n)}$ (see [KS1] p. 46).

Hence the unconditional number of visits K_j to node j satisfies $E(K_j) = \sum_{i=1}^{m} r_i \sum_{n=0}^{\infty} p_{ij}^{(n)}$. In matrix notation this is expressed as (1), which was to be proved. □

Remark 4.6.1

Since $r \triangleq \frac{\alpha}{||\alpha||}$, we have $E(K) = \frac{\alpha}{||\alpha||}(I - P)^{-1}$. Consequently, $E(K) < \infty$ iff the traffic equation has a traffic solution δ. □

Since in this case $\alpha_{\bar{A}} = 0$, we have that $E(K_i) = 0$ for any $i \in \bar{A}$. Taking $\delta^* \triangleq (\delta_A; 0)$ we can write $E(K) = \frac{\delta^*}{||\delta||}$. Thus, E(K) then satisfies the normalized version $E(K) = r + E(K)P$ of the traffic equation $\delta = \alpha + \delta P$, obtained by dividing the forcing term α--and hence the solution δ--by $||\alpha||$. If no traffic solution exists, then $E(K_i) < \infty$ for $i \in A$, $E(K_i) = 0$ for every i which is not accessible from any inlet, and $E(K_i) = \infty$ for all the other nodes.

Remark 4.6.2

The expected total number of visits to nodes by an arbitrary customer is $||E(K)||$.

If $E(K) < \infty$, this becomes $\frac{||\delta||}{||\alpha||}$. □

4.7 Traffic Processes on Arcs

In this section we investigate traffic processes on the arcs of Jackson networks. Recall that $\{A_{ij}(t)\}_{t\geq 0}$ is the traffic process on arc (i,j), where $A_{ij}(t)$ is the customer count on it during the time interval (0,t].

In the process, we isolate a class of arcs whose traffic processes will be shown to be Poisson processes, when the network is in equilibrium. This result may be viewed as a generalization of Burke's Theorem (see [B1]) which states that the traffic process on the outlet of a M/M/1 queue, in equilibrium, is a Poisson process. The generalization, however, is stronger in that we show that the set of arcs to which it applies includes the outlets of Jackson networks. Moreover, for certain sets of such arcs, we will show that the Poisson traffic processes on them are mutually independent processes.

The treatment relies heavily on the switching matrix P, or equivalently, on topological properties of the graph associated with the underlying Jackson network. Recall that the associated graph can be viewed as a representation of the accessibility relation $\rightsquigarrow$ (see Definition 4.3.1) among the network's nodes. The communication relation $\leftrightsquigarrow$ (see ibid.) is easily seen to be an equivalence relation; as such it induces a partition into equivalence classes, each consisting of mutually communicating nodes.

Let us call each such equivalence class a *component* of the network, and for every node i, let [i] denote the component C such that $i \in C$.

The accessibility relation $\rightsquigarrow$ on the node set of a Jackson network induces a partial ordering on its set of components. This partial order will also be denoted by $\rightsquigarrow$; namely $C_1 \rightsquigarrow C_2$ iff there exist $i \epsilon C_1$ and $j \epsilon C_2$ such that $i \rightsquigarrow j$.

It follows that if $C_1 \rightsquigarrow C_2$ then $i \rightsquigarrow j$ and $j \not\rightsquigarrow i$ for any $i \epsilon C_1$ and $j \epsilon C_2$. The partial ordering of network nodes and components induces a hierarchy of Jackson subnetworks as follows.

Definition 4.7.1

Let $JN = (M,\alpha,\sigma,P)$ specify a Jackson network and let $L \subset M$.

We say that $JN(L) \triangleq (L,\alpha_L,\sigma_L,P_L)$ is a *partial network* of JN if

a) $i \epsilon L$ and $j \rightsquigarrow i \implies j \epsilon L$. □

Remark 4.7.1

Equivalently, L in Definition 4.7.1 satisfies

$i \epsilon L$ and $j \notin L \implies p_{ji} = 0$ □

Observe that a partial network of some underlying Jackson network is self-contained in the sense that it can be analyzed as a full-fledged Jackson network. As a matter of fact its complement can be completely ignored because a partial network is not accessible from its complement.

Definition 4.7.2

Let $L \subset M$ be a subset of nodes in a Jackson network $JN = (M,\alpha,\sigma,P)$. Let $\tilde{L} \triangleq \{i\colon i \epsilon M$ and $\exists j \epsilon L$ such that $i \rightsquigarrow j\}$.

Then $JN(\tilde{L}) \triangleq (\tilde{L},\alpha_{\tilde{L}},\sigma_{\tilde{L}},P_{\tilde{L}})$ is called the *partial network generated by L*. □

Notice that the partial network generated by a subset of nodes L, contains all the components [i] such that $i \in L$ together with all the components from which L is accessible.

In addition to the hierarchy of partial networks, the accessibility relation $\rightsquigarrow$ may be used to partition the arcs of a Jackson network into two claases as follows.

If C_1 and C_2 are components such that $C_1 \rightsquigarrow C_2$, then $C_2 \not\rightsquigarrow C_1$. Thus, arcs fall into two disjoint categories: those between components and those within components. The former set may be characterized as follows.

<u>Definition 4.7.3</u> (cf. [BM1], Definition 3.1)

An arc (i,j), $1 \leq i \leq m$, $0 \leq j \leq m$, in a Jackson network $JN = (M,\alpha,\sigma,P)$ is called an *exit arc* if

a) $p_{ij} > 0$ but $j \not\rightsquigarrow i$. □

Intuitively, an exit arc (i,j) is characterized by the fact that a customer that takes it will never return to i for further services. In this respect, an exit arc behaves much like an outlet. Indeed, every exit arc is an outlet of some partial network, and this fact provides the basis for the aforesaid generalization of Burke's Theorem.

<u>Theorem 4.7.1</u>

Let $JN = (M,\alpha,\sigma,P)$ be a Jackson network, and let $\hat{A}(t)$ be any subset of the traffic processes $\{A_{ij}(t)\}_{t \geq 0}$, $1 \leq i \leq m$, $0 \leq j \leq m$, on some subset of arcs.

Then $(Q(t); \hat{A}(t))_{t \geq 0}$ is a conservative Markov process with stationary transition probabilities.

Proof

For any $s \leq t$, the stochastic equation

(1) $(Q(t);\hat{A}(t)) = (Q(s);\hat{A}(s)) + (A(s,t];\hat{A}(s,t]) - (D(s,t];\theta)$

holds almost surely, where θ is a vector of zeros.

The rest of the argument is analogous to the one used in the proof of Theorem 4.2.2. □

Let $JN(L) = (L,\alpha_L,\sigma_L,P_L)$ be a partial network of $JN = (M,\alpha,\sigma,P)$, where without loss of generality $L \triangleq \{1,2,\ldots,\ell\}$. Denote $\bar{L} \triangleq \{0\} \cup (M-L)$.

If (i,j) is an exit arc, then we write $E_{ij}(t)$ for $A_{ij}(t)$. The vector of traffic processes on the outlets of JN(L) (which are all exit arcs) is denoted by $E_L(t)$. Thus,

$E_L(t) \triangleq (E_{10}(t),E_{1\ell+1}(t),\ldots,E_{1m}(t);\ldots,E_{\ell 0}(t),E_{\ell\ell+1}(t),\ldots,E_{\ell m}(t))$.

By virtue of Theorem 4.7.1, $(Q_L(t); E_L(t))$ is a Markov process, and in view of Appendix C we may proceed to treat the appropriate birth-and-death equations.

Denote $P_t(n_1,\ldots,n_\ell;\ k_{10},k_{1\ell+1},\ldots,k_{1m},\ldots,k_{\ell 0},k_{\ell\ell+1},\ldots,k_{\ell m}) \triangleq$

$$P_r\Big(\bigcap_{i\varepsilon L}\ \bigcap_{j\varepsilon\bar{L}} [(Q_i(t)=n_i) \cap (E_{ij}(t)=k_{ij})]\Big)$$

for any $t \geq 0$ and any vector of non-negative integers $(n_1,\ldots,n_m;k_{10},k_{1\ell+1},\ldots,k_{1m},\ldots,k_{\ell 0},k_{\ell\ell+1},\ldots,k_{\ell m})$.

With a dot to denote a derivative with respect to t, the birth-and-death equations of the process $(Q_L(t);E_L(t)$ are

(A) $\dot{P}_t(n_1,\ldots,n_\ell;\ k_{10},k_{1\ell+1},\ldots,k_{\ell m}) =$

$$\sum_{i\in L} P_t(n_1,\ldots,n_i-1,\ldots,n_\ell;\ k_{10},k_{1\ell+1},\ldots,k_{\ell m})\sigma_i b(n_i) +$$

$$\sum_{i\in L}\sum_{j\in \bar{L}} P_t(n_1,\ldots,n_i+1,\ldots,n_\ell;\ k_{10},k_{1\ell+1},\ldots,k_{ij}-1,\ldots,k_{\ell m})\sigma_i p_{ij} b(k_{ij})+$$

$$\sum_{i\in L}\sum_{\substack{j\in L\\ j\neq i}} P_t(n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_\ell;\ k_{10},k_{1\ell+1},\ldots,k_{\ell m})\sigma_j p_{ji} b(n_i) -$$

$$P_t(n_1,\ldots,n_\ell;\ k_{10},k_{1\ell+1},\ldots,k_{\ell m})[\sum_{i\in L}\alpha_i + \sum_{i\in L}\sum_{j\in \bar{L}}\sigma_i p_{ij} b(n_i) + \sum_{i\in L}\sum_{\substack{j\in L\\ j\neq i}}\sigma_j p_{ji} b(n_j)]$$

for any $(n_1,\ldots,n_\ell;\ k_{10},k_{1\ell+1},\ldots,k_{\ell m}) \geq 0$.

Recall that $b(n_i) \triangleq \begin{cases} 0, & \text{if } n_i = 0 \\ 1, & \text{if } n_i > 0 \end{cases}$

Theorem 4.7.2 (cf. [BM1], Theorem 3.1)

Let JN(L) be a partial network of JN = (M,α,σ,P) in equilibrium.

Then, the random variables in the set $(Q_L(t);\ E_L(t))$ are mutually independent for each fixed $t \geq 0$. Moreover, each $E_{ij}(t)$ in $E_L(t)$ is Poisson distributed with parameter $\delta_i p_{ij} t$.

Proof

We may assume that the network is open, because the closed part C has no outlets except for trivial ones on which the traffic process in equilibrium is zero almost surely.

In view of the birth-and-death equations (A), the equilibrium assumption is

(1) $$P_0(n_1,\ldots,n_\ell,k_{10},k_{1\ell+1},\ldots,k_{\ell m}) \triangleq \begin{cases} \prod_{i=1}^{\ell} (1-\rho_i)\rho_i^{n_i} & , \text{ if } k_{ij} = 0 \ \forall i\varepsilon L, \forall j\varepsilon \bar{L} \\ 0, \text{ otherwise} \end{cases}$$

due to Jackson's Theorem (see Theorem 4.5.1). We shall analyze (A) by means of generating functions, similarly to the equilibrium analysis of the state process. In our case, the generating function is the $\ell(m - \ell + 2)$-dimensional z-transform defined by

(2) $$\Phi(z_1,\ldots,z_\ell;\ y_{10},y_{1\ell+1},\ldots,y_{1m},\ldots,y_{\ell 0},y_{\ell\ell+1},\ldots,y_{\ell m}) \triangleq$$
$$\sum_{\nu} P_t(\nu) \prod_{i\varepsilon L} \prod_{j\varepsilon \bar{L}} z_i^{n_i} \cdot y_{ij}^{k_{ij}} \quad , \quad |z_i| \le 1,\ |y_{ij}| \le 1, \quad i\varepsilon L, \quad j\varepsilon \bar{L} ,$$

where the sum ranges over all non-negative integer vectors $\nu = (n_1,\ldots,n_\ell;\ k_{10},k_{1\ell+1},\ldots,k_{\ell m})$.

The z-transformed version of (A) is obtained by multiplying each equation corresponding to each ν by $\prod_{i\varepsilon L} \prod_{j\varepsilon \bar{L}} z_i^{n_i} \cdot y_{ij}^{k_{ij}}$, $|z_i| \le 1$, $|y_{ij}| \le 1$, with the convention $0^0 \triangleq 1$, and then summing the resulting equations over all integer vectors $\nu = (n_1,\ldots,n_\ell;\ k_{10},k_{1\ell+1},\ldots,k_{\ell m}) \ge 0$.

After manipulating the above summation and collecting terms, analogously to the procedure in Theorem 4.5.4, we obtain

(3) $$\dot{\Phi}_t(z_1,\ldots,z_\ell;\ y_{10},y_{1\ell+1},\ldots,y_{\ell m}) =$$
$$\sum_{i\varepsilon L} \Phi_t(z_1,\ldots,z_\ell;\ y_{10},y_{1\ell+1},\ldots,y_{\ell m})\alpha_i[z_i-1] +$$
$$\sum_{i\varepsilon L} \sum_{j\varepsilon \bar{L}} [\Phi_t(z_1,\ldots,z_\ell;\ y_{10},y_{1\ell+1},\ldots,y_{\ell m}) -$$
$$\Phi_t(z_1,\ldots,0_i,\ldots,z_\ell;\ y_{10},y_{1\ell+1},\ldots,y_{\ell m})]\sigma_i p_{ij}[\frac{y_{ij}}{z_i} - 1] +$$
$$\sum_{i\varepsilon L} \sum_{\substack{j\varepsilon L \\ j\neq i}} [\Phi_t(z_1,\ldots,z_\ell;\ y_{10},y_{1\ell+1},\ldots,y_{\ell m}) -$$
$$\Phi_t(z_1,\ldots,0_j,\ldots,z_\ell;\ y_{10},y_{1\ell+1},\ldots,y_{\ell m})]\sigma_j p_{ji}[\frac{z_i}{z_j} - 1]$$
$$0 < |z_i| \le 1 \quad , \quad |y_{ij}| \le 1 \quad ; \quad i\varepsilon L \quad , \quad j\varepsilon \bar{L} .$$

Here 0_i stands for $z_i = 0$.

The initial condition (1) is z-transformed into

(4) $$\Phi_0(z_1,\ldots,z_\ell;y_{10},y_{1\ell+1},\ldots,y_{\ell m}) = \prod_{i\varepsilon L}\frac{1-\rho_i}{1-\rho_i z_i}$$

We shall now show that equation (A) and initial condition (1) are satisfied by

(5) $$P_t^*(n_1,\ldots,n_\ell;k_{10},k_{1\ell+1},\ldots,k_{\ell m}) \triangleq$$
$$\prod_{i\varepsilon L}(1-\rho_i)\rho_i^{n_i} \prod_{j\varepsilon \bar{L}} e^{-\delta_i p_{ij} t}\frac{(\delta_i p_{ij} t)^{k_{ij}}}{k_{ij}!}$$

Equivalently, one has to show that (3) and (4) are satisfied by the z-transformed version of (5), namely by

(6) $$\Phi_t^*(z_1,\ldots,z_\ell;y_{10},y_{1\ell+1},\ldots,y_{\ell m}) = \prod_{i\varepsilon L}\frac{1-\rho_i}{1-\rho_i z_i}\prod_{j\varepsilon \bar{L}} e^{\delta_i p_{ij} t(y_{ij}-1)}$$

To prove this we use the following identities :

(7) $$\Phi_0^*(z_1,\ldots,z_\ell;y_{10},y_{1\ell+1},\ldots,y_{\ell m}) = \prod_{i\varepsilon L}\frac{1-\rho_i}{1-\rho_i z_i} \quad .$$

(8) $$\dot{\Phi}_t^*(z_1,\ldots,z_\ell;y_{10},y_{1\ell+1},\ldots,y_{\ell m}) =$$
$$\Phi_t^*(z_1,\ldots,z_\ell;y_{10},y_{1\ell+1},\ldots,y_{\ell m})\sum_{i\varepsilon L}\sum_{j\varepsilon \bar{L}}\delta_i p_{ij}(y_{ij}-1) \,.$$

(9) $$\Phi_t^*(z_1\ldots z_\ell;y_{10},y_{1\ell+1}\ldots y_{\ell m}) - \Phi_t^*(z_1\ldots 0_i\ldots z_\ell;y_{10},y_{1\ell+1}\ldots y_{\ell m}) =$$
$$\Phi_t^*(z_1\ldots z_\ell;y_{10},y_{1\ell+1}\ldots y_{\ell m})\cdot\rho_i z_i \qquad , \qquad 1 \leq i \leq m \,.$$

for any i=1,2,... m.

Equations (7) - (9) may be verified by direct calculation.

Now, (7) shows that that $\Phi_t^*(z_1,\ldots,z_\ell;\ y_{10},y_{1\ell+1}\ldots y_{\ell m})$ satisfies initial condition (4).

Setting identities (8) and (9) in (3) and writing Φ_t^* for $\Phi_t^*(z_1,\ldots,z_\ell;\ y_{10},y_{1\ell+1},\ldots,y_{\ell m})$ gives us

$$(10)\quad \Phi_t^* \sum_{i\varepsilon L}\sum_{j\varepsilon \bar{L}} \delta_i p_{ij}(y_{ij}-1) = \Phi_t^* \sum_{i\varepsilon L}\alpha_i(z_i-1) +$$

$$\Phi_t^* \sum_{i\varepsilon L}\sum_{j\varepsilon \bar{L}} \rho_i z_i \cdot \sigma_i p_{ij}\left(\frac{y_{ij}}{z_i} - 1\right) +$$

$$\Phi_t^* \sum_{i\varepsilon L}\sum_{\substack{j\varepsilon L\\ j\neq i}} \rho_j z_j \cdot \sigma_j p_{ji}\left(\frac{z_i}{z_j} - 1\right)$$

$$0 < |z_i| \le 1 \quad , \quad |y_{ij}| \le 1 \quad ; \quad i\varepsilon L \quad , \quad j\varepsilon \bar{L} \; .$$

Since $\Phi_t^* > 0$ for all $t \ge 0$, whenever $0 < |z_i| \le 1$, $|y_{ij}| \le 1$, we may divide both sides of (10) by Φ_t^*. Noting that $\sigma_i\rho_i = \delta_i$, $1 \le i \le m$, we can further simplify (10) to

$$(11)\quad \sum_{i\varepsilon L}\sum_{j\varepsilon \bar{L}} \delta_i p_{ij}(y_{ij} - 1) =$$

$$\sum_{i\varepsilon L}\alpha_i(z_i-1) + \sum_{i\varepsilon L}\sum_{j\varepsilon \bar{L}} \delta_i p_{ij}(y_{ij}-z_i) + \sum_{i\varepsilon L}\sum_{\substack{j\varepsilon L\\ j\neq i}} \delta_j p_{ij}(z_i-z_j)$$

$$0 < |z_i| \le 1 \quad , \quad |y_{ij}| \le 1 \quad ; \quad i\varepsilon L \quad , \quad j\varepsilon \bar{L} \; .$$

After some manipulation and regrouping of terms in (11) we obtain

$$(12)\quad \sum_{i\varepsilon L}\sum_{j\varepsilon \bar{L}} \delta_i p_{ij}(y_{ij}-y_{ij}) - \left(\sum_{i\varepsilon L}\sum_{j\varepsilon \bar{L}} \delta_i p_{ij} - \sum_{i\varepsilon L}\alpha_i\right) =$$

$$\sum_{i\varepsilon L}\left(\alpha_i + \sum_{\substack{j\varepsilon L\\ j\neq i}} \delta_j p_{ji}\right) z_i - \sum_{i\varepsilon L}\left(\sum_{j\varepsilon \bar{L}} \delta_i p_{ij} + \sum_{\substack{j\varepsilon L\\ j\neq i}} \delta_i p_{ij}\right) z_i$$

$$0 < |z_i| \le 1 \quad , \quad |y_{ij}| \le 1 \quad ; \quad i\varepsilon L \quad , \quad j\varepsilon \bar{L} \; .$$

Now, from (B) in Sec. 4.4 it follows that

$$(13.1)\quad \alpha_i + \sum_{\substack{j\varepsilon L\\ j\neq i}} \delta_j p_{ji} = \delta_i(1-p_{ii}) \quad , \quad 1 \le i \le m \; .$$

$$(13.2)\quad \sum_{j\varepsilon \bar{L}} \delta_i p_{ij} + \sum_{\substack{j\varepsilon L\\ j\neq i}} \delta_i p_{ij} = \sum_{\substack{j=0\\ j\neq i}} \delta_i p_{ij} = \delta_i(1-p_{ii}) \quad , \quad 1 \le i \le m$$

and by Theorem 4.4.4

$$(13.3)\quad \sum_{i\varepsilon L}\sum_{j\varepsilon \bar{L}} \delta_i p_{ij} = \sum_{i\varepsilon L}\alpha_i \; .$$

In view of the identities given in (13.1) - (13.3), Equation (12) is seen to reduce to an identity.

This completes the proof of the theorem. □

<u>Lemma 4.7.1</u> (cf. [BM1, Corollary 3.1)

Under the conditions of Theorem 4.7.2, the random variables in the process $\{(Q_L(t); E_L(t) - E_L(s))\}_{t \geq s}$ are mutually independent for every fixed s, $s \leq t$.

Moreover, in this case, for any fixed s, each $E_{ij}(t) - E_{ij}(s)$ in $E_L(t) - E_L(s)$ is Poisson distributed with parameter $\delta_i p_{ij}(t-s)$.

<u>Proof</u>

The process $\{(Q_L(t); E_L(t) - E_L(s))\}_{t \geq s}$ is Markovian by an argument identical to the one in Theorem 4.7.1. In view of the time invariance of the birth-and-death equations (A), they still hold when t is replaced by $u \triangleq t - s$. In particular, the initial condition (1) in Theorem 4.7.2 holds for $u \triangleq 0$. Consequently, we obtain the required independence. It also follows that each $E_{ij}(t) - E_{ij}(s)$ is Poisson distributed with parameter $\delta_i p_{ij} u = \delta_i p_{ij}(t-s)$, for every fixed $u \geq 0$. □

Notice that the mutual independence, alluded to in Theorem 4.7.2, applies to each fixed t. We can, however, prove a stronger independence result with the aid of Lemma 4.7.1.

<u>Lemma 4.7.2</u> (cf. [BM1], Theorem 3.2)

Let JN = (M,α,σ,P) and JN(L) be as in Theorem 4.7.2. If JN is in equilibrium, then for any fixed s and t such that $0 < s \leq t$ we have that every event $\Lambda\varepsilon\sigma(\{(Q_L(u);\ E_L(u) - E_L(t)):\ u \geq t\})$ is independent of every event in $\sigma(E_L(t) - E_L(s))$.

Proof

We show first that for every $n_L = (n_1,\ldots,n_\ell) \geq 0$ and $k_L = (k_{10},k_{1\ell+1},\ldots,k_{\ell m}) \geq 0$ we have

(1) $Pr(\Lambda|Q_L(t)=n_L,\ E_L(t)-E_L(s)=k_L) = Pr(\Lambda|Q_L(t)=n_L)$.

First, observe that for every interval $(s,u]$

(2) $\sigma(\{Q(t):\ t\varepsilon(s,u]\}) \supset \sigma(\{E_L(t)-E_L(s):\ t\varepsilon(s,u]\})$

because the jumps of the process $\{E_L(t)-E_L(s)\}_{t\varepsilon(s,u]}$ are determined by the jumps of the process $\{Q_L(t)\}_{t\varepsilon(s,u]}$.

Therefore, it follows from (2) that

(3) $\Lambda\varepsilon\sigma(\{Q_L(u):\ u \geq t\})$

and (1) is true in view of (3) and the Markov property of $\{Q_L(t)\}_{t\geq 0}$ (see Appendix C, Sec. C.1, Equation (C)).

Taking advantage of (1) and of Lemma 4.7.1, we compute

(4) $Pr(\Lambda,Q_L(t)=n_L,E_L(t)-E_L(s)=k_L) =$

$Pr(\Lambda|Q_L(t)=n_L,E_L(t)-E_L(s)=k_L)\cdot Pr(Q_L(t)=n_L,E_L(t)-E_L(s)=k_L) =$

$Pr(\Lambda|Q_L(t)=n_L)\cdot P\ (Q_L(t)=n_L)\cdot Pr(E_L(t)-E_L(s)=k_L) =$

$Pr(\Lambda,Q_L(t)=n_L)\cdot Pr(E_L(t)-E_L(s)=k_L)$.

Summing (4) over all integer vectors $n_L \geq 0$ gives us

(5) $Pr(\Lambda,E_L(t)-E_L(s)=k_L) = Pr(\Lambda)\cdot Pr(E_L(t)-E_L(s)=k_L)$

which was to be proved. □

Intuitively, Lemma 4.7.2 asserts that the instantaneous independence of the state $Q_L(t)$ and the count $E_L(t)$ engender a stronger independence whereby every increment of a past count $E_L(t)-E_L(s)$, $s \le t$, is independent of the future evolution of $\{Q_L(u)\}_{u\ge t}$. But since the σ-algebra generated by every future increment $\{E_L(u)-E_L(t)\}_{u\ge t}$ is contained in the σ-algebra generated by the future state $\{Q_L(u)\}_{u\ge t}$, we have in particular

Corollary 4.7.1

The process $\{E_L(t)\}_{t\ge 0}$ has independent increments in equilibrium.

Consequently, for any $i\varepsilon L$ and $j\varepsilon\bar{L}$, $\{E_{ij}(t)\}_{t\ge 0}$ is a Poisson process in equilibrium. □

We now prove an even stronger independence property of $\{E_L(t)\}_{t\ge 0}$.

Theorem 4.7.3 (cf. [BM1], Theorem 3.3)

Let $JN = (M,\alpha,\sigma,P)$ and $JN(L)$ be as in Theorem 4.7.2. If JN is in equilibrium, then the traffic processes in $\{E_L(t)\}_{t\ge 0}$ are mutually independent Poisson processes with respective parameters $\delta_i p_{ij} t$.†

Proof

We already know that each traffic process $E_{ij}(t)$ in $E_L(t)$ is a Poisson process by Corollary 4.7.1.

Let Π: $0 = t_0 < t_1 < \ldots < t_r = t$ be any partition of the time interval $[0,t]$. Define a set of events

†This result agrees with more general results due to F. P. Kelly (see [Kl] p. 553).

(1) $C_{ij}^{(n)} \triangleq (E_{ij}(t_n) - E_{ij}(t_{n-1}) = k_{ij}^{(n)}),\ i\varepsilon L,\ j\varepsilon \bar{L}$

for any choice of integers $k_{ij}^{(n)}$.

It suffices to show that the events $C_{ij}^{(n)}$ are mutually independent. Proof is by induction on r. For r = 1 we have

$$(2)\quad Pr(\bigcap_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} C_{ij}^{(1)}) = \prod_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} Pr(C_{ij}^{(1)})$$

by Theorem 4.7.2.

Assume that the $C_{ij}^{(n)}$ are mutually independent for every partition Π with r-1 division points, and show that this is true for every partition Π with r division points.

By Corollary 4.7.1 we have

$$(3)\quad Pr(\bigcap_{n=1}^{r} \bigcap_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} C_{ij}^{(n)}) = Pr((\bigcap_{n=1}^{r-1} \bigcap_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} C_{ij}^{(n)}) \cap (\bigcap_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} C_{ij}^{(r)})) =$$

$$Pr(\bigcap_{n=1}^{r-1} \bigcap_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} C_{ij}^{(n)})\cdot Pr(\bigcap_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} C_{ij}^{(r)}) \ .$$

By the induction hypothesis

$$(4)\quad Pr(\bigcap_{n=1}^{r-1} \bigcap_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} C_{ij}^{(n)}) = \prod_{n=1}^{r-1} \prod_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} Pr(C_{ij}^{(n)}) \ .$$

Furthermore, by Lemma 4.7.1

$$(5)\quad Pr(\bigcap_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} C_{ij}^{(r)}) = \prod_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} Pr(C_{ij}^{(r)}) \ .$$

On substituting (4) and (5) on the right side of (3) we get

$$(6)\quad Pr(\bigcap_{n=1}^{r} \bigcap_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} C_{ij}^{(n)}) = \prod_{n=1}^{r} \prod_{\substack{i\varepsilon L\\ j\varepsilon \bar{L}}} Pr(C_{ij}^{(n)})$$

which shows the induction step to be valid. □

Corollary 4.7.2

In particular, the traffic processes $(E_{10}(t),\ldots,E_{m0}(t))$, on the outlets of a Jackson network JN = (M,α,σ,P) in equilibrium, are mutually independent Poisson processes with respective intensities $\delta_i q_i$. □

The foregoing discussion allows us to identify the traffic processes on exit arcs of a Jackson network in equilibrium.

Corollary 4.7.3

Let JN = (M,α,σ,P) specify a Jackson network. If the network is in equilibrium, then the traffic process $E_{ij}(t)$ on each exit arc (i,j) is a Poisson process with intensity $\delta_i p_{ij}$.

Proof

Consider the partial network $JN(\{\tilde{i}\})$, generated by i. It is easy to see that (i,j) is an outlet of $JN(\{\tilde{i}\})$. The required result follows immediately from Corollary 4.7.1. □

The Poisson nature of traffic processes on exit arcs of a Jackson network in equilibrium has interesting ramifications as regards the decomposition of the network into components.

J. R. Jackson's cautious statement, that every node i in a Jackson network JN = (M,α,σ,P) in equilibrium behaves *as if* it were a single M/M/1 queue in equilibrium, can now be strengthened. The italicized reservation in the above statement stems from the fact that it was not known whether the arrival process $\{A_i(t)\}_{t\geq 0}$ is Poisson, or equivalently, whether the traffic processes in $\{(A_{0i}(t),A_{1i}(t),\ldots,A_{mi}(t))\}_{t\geq 0}$ are mutually independent Poisson processes which are in addition independent of the service and switching processes of node i (as is the case in the M/M/1 queue).

As a matter of fact, this is *not* the case in general, and we shall qualify this statement in the sequel. Nevertheless, certain subnetworks which are not partial networks do more than behave *as if* they were Jackson networks; it can be shown that in equilibrium they indeed *are* Jackson networks.

Formally we have

Theorem 4.7.4

Let JN = (M,α,σ,P) be a Jackson network. Then, in equilibrium, every component C is a Jackson network $JN_C = (C,\gamma_C,\sigma_C,P_C)$ where $\gamma_i \triangleq \alpha_i + \sum_{j \notin C} \delta_j p_{ji}$, for any $i \in C$.

Proof

Let I(C) be the set of inlets of C, not including inlets of the network JN; that is, $I(C) \triangleq \{\text{arcs } (i,j)\colon i \in M-C,\ j \in C \text{ and } p_{ij} > 0\}$.

Now, every arc in I(C) is an exit arc since it runs between disjoint components. Consider the partial network generated by the set $L \triangleq \{i\colon i \in M \text{ and } \exists j \in C \text{ such that } (i,j) \in I(C)\}$. Clearly the exist arcs in I(C) are a subset of the outlets of this partial network. Furthermore, the traffic process $\{E_{ij}(t)\}_{t \ge 0}$, $(i,j) \in I(C)$, are mutually independent Poisson processes with respective intensities $\delta_i p_{ij}$, due to Theorem 4.7.3.

Observe that $\sigma(\{E_{ij}(t)\colon (i,j) \in I(C),\ t \ge 0\})$ is independent of $\sigma(\{Q_i(0), A_i^{ex}(t), S_{ij}(t)\colon i \in C,\ 0 \le j \le m,\ t \ge 0\})$, because the former is contained in $\sigma(\{Q_i(0), A_i^{ex}(t), S_{ij}(t): i \notin C, 0 \leq j \leq m, t \geq 0\})$ (see Sec. 4.2).

In particular, for each $i \in C$ we may group (superpose) the independent Poisson processes $\{A_i^{ex}(t)\}_{t\geq 0}$ and $\{E_{ji}(t)\}_{t\geq 0}$, $j \notin C$, into a Poisson process $\{A_i^{ex}(t) + \sum_{j\notin C} E_{ji}(t)\}_{t\geq 0}$ which has intensity γ_i as required. Furthermore, the $\{A_i^{ex}(t) + \sum_{j\notin C} E_{ji}(t)\}_{t\geq 0}$ $(i \in C)$, the $\{S_{ij}(t)\}_{t\geq 0}$ $(i \in C, 0 \leq j \leq m)$, and $Q_C(0)$ are all mutually independent. We conclude that, in equilibrium, JN_C is a Jackson network by definition. □

Theorem 4.7.4 shows that every Jackson network may be decomposed into components such that, in equilibrium, each is a full-fledged Jackson network, which can be treated separately.

We remark that the results have been obtained for Jackson networks with single server nodes.

We are, however, prepared to make the following

Conjecture 4.7.1

The results obtained thus far hold true for Jackson networks with arbitrary number of servers in each node. □

In order to validate Conjecture 4.7.1, one has to modify the birth-and-death equations (A) and attempt to verify that the alleged solution still holds.

The rest of the argument is virtually unchanged. We shall not undertake to prove or disprove Conjecture 4.7.1 in this work.

The intuitive basis for making Conjecture 4.7.1 is the topological properties of exit arcs. We observe that this class of arcs is amenable to a generalization of Burke's Theorem, because exit arcs behave as outlets in the sense that they don't "affect" the component from which they originate. Heuristically, this "effect" is carried by customer traffic, and the lack of "effect" means here that customers that take an exit arc will never visit it again.

Thus, the independent increments of the Poisson counts on exit arcs in equilibrium may be intuitively attributed to this inherent lack of future effect.

Quite naturally, this situation begs the question whether on non-exit arcs (i.e. arcs within components), the equilibrium counting process is no longer Poisson. If this is true for every non-exit arc, then the intuitive explication for the Poisson counts on exit arcs would gain increased credibility. This would also lead to a characterization of equilibrium traffic processes on arcs, and considerable insight into them will be gained.

The salient feature of non-exit arcs is, of course, that customers taking these arcs may revisit them with positive probability. In terms of the associated graph, there is a cycle (closed path) that begins and ends with each non-exit arc. This is due to the fact that non-exit arcs are within components and these consist of mutually communicating nodes. Thus, in contrast with exit arcs, customers that travel on non-exit arcs *do carry* future "effect" on them. As a matter of fact, part of the customers in a past count increment on a non-exit arc will revisit it and contribute to future increment counts on the very same non-exit arc. Thus, we cannot intuitively expect to have there independent count increments in non-overlapping time intervals, even in equilibrium.

The foregoing discussion leads us to state

Conjecture 4.7.2

Excluding the trivial case $p_{ii} = 1$, the traffic processes on a non-exit arc (i,j) can never be a Poisson process or even have independent increments. □

While at this juncture we are unable to prove Conjecture 4.7.2 for every non-exit, we can, however, show that the traffic processes on certain subsets of non-exit arcs are not Poisson processes in equilibrium.

We know from Theorem 4.7.1 that every process $(Q(t); A_{vw}(t)$ is a Markov process for any arc (v,w). However, in writing the relevant birth-and-death equation, one has to distinguish between two cases.

Using the previous notation we have :

Case 1: $v \neq w$, $1 \leq v, w \leq m$.

In this case we have

$$
\begin{aligned}
(1.a)\quad \dot{P}_t(n_1,\ldots,n_m;k_{vw}) = {} & \sum_{i=1}^{m} P_t(n_1,\ldots,n_i-1,\ldots,n_m;k_{vw})\alpha_i \cdot b(n_i) + \\
& \sum_{j=1}^{m} P_t(n_1,\ldots,n_j+1,\ldots,n_m;k_{vw})\sigma_j q_j + \\
& \sum_{\substack{i=1 \\ (j,i)\neq(v,w)}}^{m} \sum_{\substack{j=1 \\ j\neq i}}^{m} P_t(n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_m;k_{vw})\sigma_j p_{ji} \cdot b(n_i) + \\
& P_t(n_1,\ldots,n_w-1,\ldots,n_v+1,\ldots,n_m;k_{vw}-1)\sigma_v p_{vw} \cdot b(n_w) b(k_{vw}) - \\
& P_t(n_1,\ldots,n_m;k_{vw})\Big[\sum_{i=1}^{m}\alpha_i + \sum_{j=1}^{m}\sigma_j q_j b(n_j) + \sum_{i=1}^{m}\sum_{\substack{j=1 \\ j\neq i}}^{m} \sigma_j p_{ji} b(n_j)\Big] \\
& (n_1,\ldots,n_m;k_{vw}) \geq 0 \ .
\end{aligned}
$$

The z-transformed version of (1.a) is

$$(1.b)\quad \dot{\Phi}_t(z_1,\ldots,z_m;y_{vw}) = \sum_{i=1}^{m} \Phi_t(z_1,\ldots,z_m;y_{vw})\alpha_i[z_i-1] +$$

$$\sum_{j=1}^{m} [\phi_t(z_1,\ldots,z_m;y_{vw}) - \phi_t(z_1,\ldots,0_j,\ldots,z_m;y_{vw})]\sigma_j q_j [\frac{1}{z_j} - 1] +$$

$$\sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq i \\ (j,i)\neq(v,w)}}^{m} [\Phi_t(z_1,\ldots,z_m;y_{vw}) - \Phi_t(z_1,\ldots,0_j,\ldots,z_m;y_{vw})]\sigma_j p_{ji} [\frac{z_i}{z_j} - 1] +$$

$$[\Phi_t(z_1,\ldots,z_m;y_{vw}) - \Phi_t(z_1,\ldots,0_v,\ldots,z_m;y_{vw})]\sigma_v p_{vw} [\frac{z_w y_{vw}}{z_v} - 1]$$

$$0 < |z_i| \leq 1, \quad 1 \leq i \leq m \quad ; \quad |y_{vw}| \leq 1 . \qquad \square$$

Case 2: $v = w$, $1 \leq v \leq m$

In this case we have

$$(2.a)\quad \dot{P}_t(n_1,\ldots,n_m;k_{vv}) = \sum_{i=1}^{m} P_t(n_1,\ldots,n_i-1,\ldots,n_m;k_{vv})\alpha_i \cdot b(n_i) +$$

$$\sum_{j=1}^{m} P_t(n_1,\ldots,n_j+1,\ldots,n_m;k_{vv})\sigma_j q_j +$$

$$\sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq i}}^{m} P_t(n_1,\ldots,n_i-1,\ldots,n_j+1,\ldots,n_m;k_{vv})\sigma_j p_{ji} \cdot b(n_i) +$$

$$P_t(n_1,\ldots,n_m;k_{vv}-1)\sigma_v p_{vv} \cdot b(n_v) -$$

$$P_t(n_1,\ldots,n_m;k_{vv})[\sum_{i=1}^{m}\alpha_i + \sum_{j=1}^{m}\sigma_j q_j b(n_j) +$$

$$\sum_{i=1} \sum_{\substack{j=1 \\ j\neq i}} \sigma_j p_{ji} b(n_j) + \sigma_v p_{vv} b(n_v)]$$

$$(n_1,\ldots,n_m;k_{vv}) \geq 0 .$$

The z-transformed version of (2.a) is

$$\text{(2.b)}\quad \dot{\Phi}_t(z_1,\ldots,z_m;y_{vv}) = \sum_{i=1}^{m} \Phi_t(z_1,\ldots,z_m;y_{vv})\alpha_i[z_i-1] +$$

$$\sum_{j=1}^{m} [\Phi_t(z_1,\ldots,z_m;y_{vv}) - \Phi_t(z_1,\ldots,0_j,\ldots,z_m;y_{vv})]\sigma_j q_j[\frac{1}{z_j} - 1] +$$

$$\sum_{i=1}^{m} \sum_{\substack{j=1 \\ j\neq i}}^{m} [\Phi_t(z_1,\ldots,z_m;y_{vv}) - \Phi_t(z_1,\ldots,0_j,\ldots,z_m;y_{vv})]\sigma_j p_{ji}[\frac{z_i}{z_j} - 1] +$$

$$[\Phi_t(z_1,\ldots,z_m;y_{vv}) - \Phi_t(z_1,\ldots,0_v,\ldots,z_m;y_{vv})]\sigma_v p_{vv}[y_{vv}-1]$$

$$0 < |z_i| \le 1 \ , \ 1 \le i \le m \quad ; \quad |y_{vv}| \le 1 . \qquad \Box$$

We now proceed to characterize conditions under which a traffic process is Poisson distributed. This will later on aid us in showing the non-Poisson character of certain non-exit arcs.

Theorem 4.7.5

Let JN = (M,α,σ,P) specify a Jackson network. Then

a) every traffic process $\{A_{vw}(t)\}_{t\ge 0}$, $1 \le v,w \le m$, with $p_{vw} > 0$ is Poisson distributed

iff

b) $B_v(t)$ and $A_{vw}(t)$ are independent for every fixed $t \ge 0$.

Proof

Set $z_i = 1$, $1 \le i \le m$, in (1.b) and (2.b). In both cases the equations reduce to

$$\text{(1)}\quad \dot{\Phi}_t(y_{vw}) = [\Phi_t(y_{vw}) - \Phi_t(0_v;y_{vw})]\sigma_v p_{vw}[y_{vw}-1] \quad , \quad |y_{vw}| \le 1 .$$

($\Longrightarrow$) Assume that $A_{vw}(t)$ is Poisson distributed. Then its parameter must be (see Theorem 4.2.1) $E(A_{vw}(t)) = \sigma_v p_{vw} \cdot \int_0^t \Pr(B_v(x)=1)dx$. Hence,

$$\text{(2)}\quad \phi_t(y_{vw}) = e^{E(A_{vw}(t))(y_{vw}-1)} .$$

$$\text{(3)}\quad \dot{\Phi}_t(y_{vw}) = \Phi_t(y_{vw})\sigma_v p_{vw} \Pr(B_v(t)=1)[y_{vw}-1] .$$

Substituting (2) and (3) in (1) yields

$$(4)\quad \Phi_t(y_{vw})\sigma_v p_{vw} \Pr(B_v(t)=1)[y_{vw}-1] = [\Phi_t(y_{vw}) - \Phi_t(0_v;y_{vw})]\sigma_v p_{vw}[y_{vw}-1]$$

$$|y_{vw}| \leq 1 .$$

Dividing both sides of (4) by $\sigma_v p_{vw}[y_{vw}-1]$ for $y_{vw} \neq 1$ gives us (recall that $\sigma_v p_{vw} > 0$)

$$(5)\quad \Phi_t(y_{vw})\Pr(B_v(t)=1) = \Phi_t(y_{vw}) - \Phi_t(0_v;y_{vw})$$

$$|y_{vw}| < 1 .$$

Equating coefficients on both sides of (5) results in the system of equations

$$(6)\quad \Pr(A_{vw}(t)=k_{vw})\cdot\Pr(B_v(t)=1) = \Pr(A_{vw}(t)=k_{vw}) - \Pr(Q_v(t)=0,A_{vw}(t)=k_{vw})$$

$$k_{vw} = 0,1,\ldots$$

The required independence now follows, since (6) is equivalent to

$$(7)\quad \Pr(A_{vw}(t)=k_{vw})\cdot\Pr(B_v(t)=1) = \Pr(A_{vw}(t)=k_{vw},B_v(t)=1)$$

$$k_{vw} = 0,1,\ldots$$

as $B_v(t)$ is a zero-one random variable.

($\Leftarrow$) Assume that $B_v(t)$ and $A_{vw}(t)$ are independent for every fixed $t \geq 0$. Then (1) may be rewritten as

$$(8)\quad \dot{\Phi}_t(y_{vw}) = [\Phi_t(y_{vw}) - \Phi_t(0_v)\Phi_t(y_{vw})]\sigma_v p_{vw}[y_{vw}-1],$$

$$|y_{vw}| \leq 1$$

which reduces to

$$(9)\quad \dot{\Phi}(y_{vw}) = \Phi_t(y_{vw})[1-\Phi_t(0_v)]\sigma_v p_{vw}[y_{vw}-1]$$

$$|y_{vw}| \leq 1 .$$

But

$$(10)\quad 1 - \Phi_t(0_v) = 1 - \Pr(Q_v(t)=0) = \Pr(Q_v(t)>0) = \Pr(B_v(t)=1).$$

Hence, (9) becomes

(11) $\dot{\Phi}_t(y_{vw}) = \Phi_t(y_{vw})\Pr(B_v(t)=1)\sigma_v p_{vw}[y_{vw}-1]$

and (11) can be recognized as the z-transform of a Poisson distributed random variable with intensity $\Pr(B_v(t)=1)\sigma_v p_{vv}$. □

Corollary 4.7.4

Under the conditions of Theorem 4.7.5, $\{A_{vw}(t)\}_{t\geq 0}$ has fixed intensity iff $\{B_v(t)\}_{t\geq 0}$ is in equilibrium. Moreover, in this case, the intensity is $\delta_v^* p_{vw}$, where for a recurrent node v $\delta_v^* \triangleq E(D_v(t,t+1])$ depends on the initial condition $q(0) = q^o$. □

Remark 4.7.2

Theorem 4.7.5 and Remark 4.7.2 remain true for each departure process $D_i(t) \triangleq \sum_{j=0}^{m} A_{ij}(t)$.

It can be shown that the proofs of Theorem 4.7.5 and Corollary 4.7.4 go through for the $\{D_i(t)\}_{t\geq 0}$. This is so, because for each $i\varepsilon M$ the birth-and-death equations for $\{(Q(t);D_i(t))\}_{t\geq 0}$ constitute a combination of Case 1 and Case 2. □

We are now prepared to point out a subclass of non-exit arcs on which the traffic process is not Poisson in equilibrium.

Theorem 4.7.6

Let JN = (M,α,σ,P) specify a Jackson network in equilibrium. Let v be a node satisfying :

a) $\delta_v^* \triangleq E(D_v(t,t+1]) > 0$.

b) $0 < p_{vv} < 1$.

Then $\{A_{vv}(t)\}_{t\geq 0}$ is not a Poisson process.

Proof

Set $z_i = 1$, $i \varepsilon M-\{v\}$, in equation (2.b) of Case 2. We obtain

(1) $\dot{\Phi}_t(z_v;y_{vv}) =$

$$(\Phi_t(z_v;y_{vv})\alpha_v + \sum_{\substack{j=1\\ j\neq v}}^{m}[\Phi_t(z_v;y_{vv}) - \Phi_t(z_v,0_j;y_{vv})]\sigma_j p_{jv})[z_v-1] +$$

$$[\Phi_t(z_v;y_{vv}) - \Phi_t(0_v;y_{vv})]\sigma_v(1-p_{vv})[\frac{1}{z_v} - 1] +$$

$$[\Phi_t(z_v;y_{vv}) - \Phi_t(0_v;y_{vv})]\sigma_v p_{vv}[y_{vv}-1]$$

$$0 < |z_v| \leq 1 \quad , \quad |y_{vv}| \leq 1 .$$

Equating the free coefficients on both sides of (1) yields

(2) $\frac{\partial}{\partial t} Pr(Q_v(t)=0,A_{vv}(t)=0) =$

$$-Pr(Q_v(t)=0,A_{vv}(t)=0)\alpha_v -$$

$$\sum_{\substack{j=1\\ j\neq v}}^{m}[Pr(Q_v(t)=0,A_{vv}(t)=0) - Pr(Q_v(t)=0,Q_j(t)=0,A_{vv}(t)=0)]\sigma_j p_{jv} +$$

$$Pr(Q_v(t)=1,A_{vv}(t)=0)\sigma_v(1-p_{vv}) ,$$

$$t \geq 0 .$$

If we assume that $A_{vv}(t)$ is Poisson distributed, its intensity must be $\delta_v^* p_{vv}$, by Corollary 4.7.4.

Moreover, by Theorem 4.7.5, $A_{vv}(t)$ is independent of $B_v(t)$ for every fixed $t \geq 0$. Therefore,

(3) $Pr(Q_v(t)=0,A_{vv}(t)=0) = Pr(B_v(t)=0,A_{vv}(t)=0) = Pr(B_v(t)=0)e^{-\delta_v^* p_{vv} t}$

(4) $\frac{\partial}{\partial t} Pr(Q_v(t)=0,A_{vv}(t)=0) = Pr(B_v(t)=0)e^{-\delta_v^* p_{vv} t}(-\delta_v^* p_{vv})$

since $Pr(B_v(t)=0)$ is constant in t.

Next send $t \to 0+$ on both sides of (2).

By continuity in t of all functions in (2), we may set $t = 0$ on both sides of (2). In view of (4), the left-hand side (LHS) of (2) becomes

(5) $\lim_{t\to 0+}$ (LHS of (2)) = $\lim_{t\to 0+} Pr(B_v(t)=0)e^{-\delta_v^* p_{vv} t}(-\delta_v^* p_{vv})$ =

$Pr(B_v(0)=0)(-\delta_v^* p_{vv}) < 0$

because in equilibrium, $Pr(B_v(t)=0) > 0$, due to c) in Corollary 4.5.1.

The right-hand side (RHS) of (2) becomes

(6) $\lim_{t\to 0+}$ (RHS of (2)) = $-Pr(Q_v(0)=0, A_{vv}(0)=0)\alpha_v$ -

$$\sum_{\substack{j=1\\ j\neq v}}^{m} [Pr(Q_v(0)=0, A_{vv}(0)=0) - Pr(Q_v(0)=0, Q_j(0)=0, A_{vv}(0)=0)]\sigma_j p_{jv} +$$

$Pr(Q_v(0)=1, A_{vv}(0)=0)\sigma_v(1-p_{vv})$ =

$$-Pr(Q_v(0)=0)\alpha_v - \sum_{\substack{j=1\\ j\neq v}} [Pr(Q_v(0)=0) - Pr(Q_v(0)=0)\cdot Pr(Q_j(0)=0]\sigma_j p_{jv} +$$

$Pr(Q_v(0)=1)\sigma_v(1=p_{vv})$ =

$$-Pr(Q_v(0)=0)[\alpha_v + \sum_{\substack{j=1\\ j\neq v}}^{m} Pr(Q_j(0)>0)\sigma_j p_{jv}] + Pr(Q_v(0)=1)\sigma_v(1-p_{vv}) =$$

$-Pr(Q_v(0)=0)\delta_v^*(1-p_{vv}) + Pr(Q_v(0)=1)\sigma_v(1-p_{vv})$.

In the calculation above we used the mutual independence of $Q_1(0),\ldots,Q_m(0),A_{vv}(0)$ since $Q_1(0),\ldots,Q_m(0)$ are mutually independent in equilibrium and $Pr(A_{vv}(0)=0) = 1$.

We now proceed to argue that in equilibrium

(7) $Pr(Q_v(0)=1) = \frac{\delta_v^*}{\sigma_v} Pr(Q_v(0)=0)$

To see this, observe that v is either completely open or recurrent. If v is completely open, then (7) follows from Theorem 4.5.6 and Theorem 4.5.1. Otherwise, v is in an irreducible set R_k. From Theorem 4.5.6 and Theorem 4.5.2 we see that for every $t \geq 0$

(8) $Pr(Q_v(t)=1|\#R_k=n) = \frac{\delta_v^*}{\sigma_v} Pr(Q_v(t)=0|\#R_k=n)$, n = 1, 2, ...

where $\#R_k$ is the equilibrium total customers in R_k. Using (8) we deduce

(9) $$Pr(Q_v(t)=1) = \sum_{n=1}^{\infty} Pr(Q_v(t)=1 \mid \#R_k=n) Pr(\#R_k=n) =$$

$$\sum_{n=1}^{\infty} \frac{\delta_v^*}{\sigma_v} Pr(Q_v(t)=0 \mid \#R_k=n) Pr(\#R_k=n) = \frac{\delta_v^*}{\sigma_v} Pr(Q_v(t)=0, \#R_k > 0),$$

$t \geq 0$.

But the assumption $\delta^* = E(D(t,t+1]) > 0$ implies that $Pr(\#R_k>0) = 1$, whence (7) follows. Substituting (7) into (6) yields

(10) $$\lim_{t\to 0+} (\text{RHS of } (2)) =$$

$$-Pr(Q_v(0)=0)\delta_v^*(1-p_{vv}) + Pr(Q_v(0)=0)\frac{\delta^*}{\sigma_v} \sigma_v(1-p_{vv}) = 0$$

in contradiction with (5).

We conclude that $A_{vv}(t)$ cannot be Poisson distributed and hence is not a Poisson process in equilibrium. □

We are now in a position to make

<u>Remark 4.7.3</u>

Under the conditions of Theorem 4.7.6, the departure process $\{D_v(t)\}_{t\geq 0}$ from node v cannot be Poisson distributed. Otherwise, the Bernoulli switch would render $\{A_{vv}(t)\}_{t\geq 0}$ Poisson distributed, in contradiction with Theorem 4.7.6. □

We remark in passing that the method employed in Theorem 4.7.6 breaks down when attempting to apply it to non-exit arcs which are not feedback arcs. The reason for this phenomenon is that equation (1.a) applies to exit arcs as well as non-exit arcs, so that the topological properties of non-exit arcs are not captured by it. However, equation (2.a) *does capture* the topological properties of feedback arcs (which can never be exit arcs by definition), and the desired contradiction can be demonstrated.

We now proceed to identify another subset of non-exit arcs on which the equilibrium traffic process fails to be Poisson distributed.

Theorem 4.7.7

Let JN = (M,α,σ,P) specify a Jackson network in equilibrium. Let (v,w) be an arc such that $\delta_v^* p_{vw} > ||\alpha||$ where $\delta_v^* \triangleq E(D_v(t,t+1])$.

Then, the traffic process $\{A_{vw}(t)\}_{t\geq 0}$ is not a Poisson process.

Proof

We already know from Theorem 4.7.1 that $(Q(t);A_{vw}(t))_{t\geq 0}$ is a Markov process. By Theorem C.2.1 in Appendix C, the birth and-death equations of this process are equivalent to the integral equations

$$(1)\quad P_t(\nu) = P_0(\nu)e^{-c_\nu t} + \int_0^t \sum_\mu P_x(\mu)c_\mu r_{\mu\nu} e^{-c_\nu(t-x)}dx,$$

$$\nu = (n_1,\ldots,n_m;k_{vw}) \geq 0,$$

where μ ranges over the state space of $\{(Q(t);A_{vw}(t))\}_{t\geq 0}$.

From (1) we conclude

$$(2)\quad P_t(\nu) \geq P_0(\nu)e^{-c_\nu t}.$$

Next, set $\nu_0 \triangleq (0_1,\ldots,0_m;0_{vw})$ in (2). Observe that $c_{\nu_0} = \sum_{i=1}^m \alpha_i = ||\alpha||$; also $P_0(\nu_0) \triangleq K > 0$, because $P_0(\nu_0) \triangleq Pr(Q(0) = \nu_0) = \prod_{i=1}^m (1-\rho_i) \triangleq K$. Hence,

$$(3)\quad P_t(\nu_0) \geq K\cdot e^{-||\alpha||t} > 0 \quad , \quad t \geq 0$$

or equivalently

$$(4)\quad Pr(Q(t)=0;A_{vw}(t)=0) \geq K\cdot e^{-||\alpha||t} \quad , \quad t \geq 0 .$$

Now, assume that $A_{vw}(t)$ is Poisson distributed in equilibrium.

In view of Corollary 4.7.4, we have in particular

(5) $Pr(A_{vw}(t)=0) = e^{-\delta_v^* p_{vw} t} > 0 \quad , \quad t \geq 0.$

Dividing both sides of (4) by (5) yields

(6) $Pr(Q(t)=0 | A_{vw}(t)=0) \geq K \cdot e^{-||\alpha|| t + \delta_v^* p_{vw} t} =$

$$K \cdot e^{(\delta_v^* p_{vw} - ||\alpha||)t} \xrightarrow[t\to\infty]{} \infty$$

since we assumed $\delta_v^* p_{vw} - ||\alpha|| > 0$.

This is a contradiction, since (6) must be bounded by 1. We conclude that $\{A_{vw}(t)\}_{t\geq 0}$ cannot be Poisson distributed, and thus is not a Poisson process in equilibrium. □

Remark 4.7.4

An identical argument shows that any departure process $\{D_v(t)\}_{t\geq 0}$ or arrival process $\{A_v(t)\}_{t\geq 0}$, with $\delta_v^* > ||\alpha||$, cannot be Poisson distributed in equilibrium. □

We now demonstrate by an example that the class of arcs satisfying Theorem 4.7.7 and Remark 4.7.4 is not a trivial one.

Example 4.7.1

Consider the Jackson network in Figure 4.7.1. We have that $||\alpha|| = \alpha_1$ and the traffic equation is

(1) $\begin{cases} \delta_1 = \alpha_1 + \delta_2 \\ \delta_2 = p_{12}\delta_1 \end{cases}$

The traffic solution is

$$(\delta_1, \delta_2) = \left(\frac{\alpha_1}{q_1}, \frac{p_{12}\alpha_1}{q_1}\right).$$

Clearly, $\delta_1 = \frac{\alpha_1}{q_1} > \alpha_1 = ||\alpha||$ whenever $0 < q_1 < 1$.

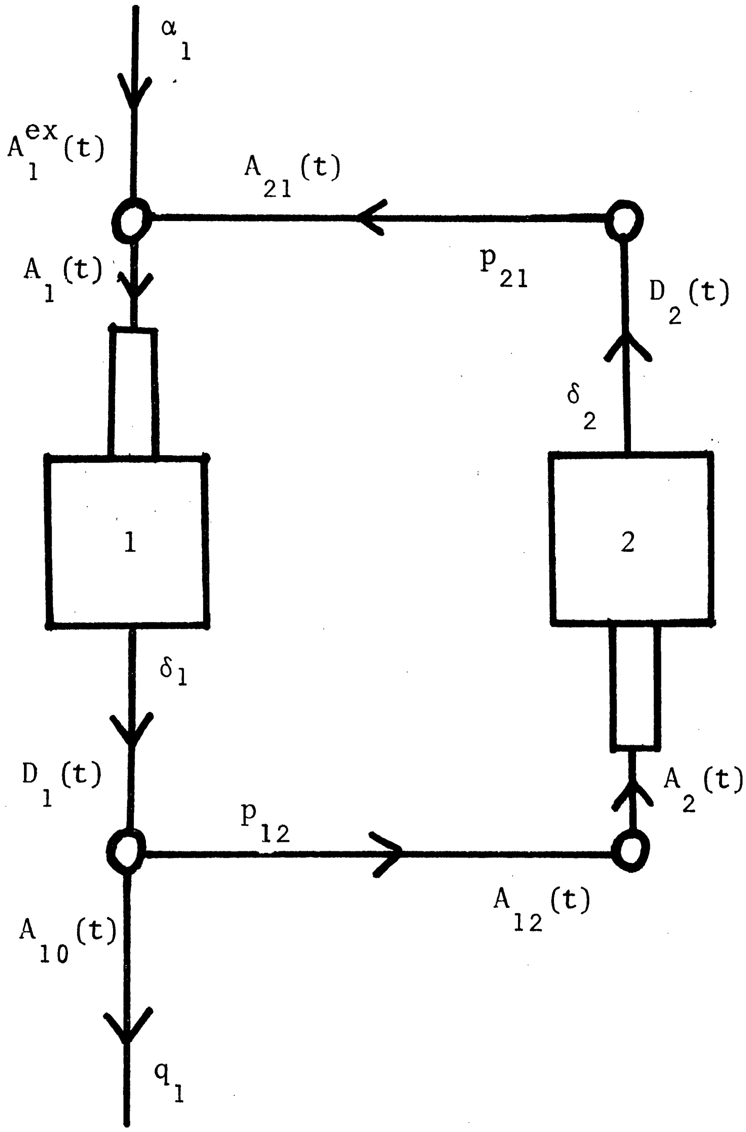

Figure 4.7.1: A Jackson Network with Arcs Satisfying Theorem 4.7.7 and Remark 4.7.4.

Thus, by Remark 4.7.4, $\{D_1(t)\}$ is not a Poisson process in equilibrium nor is the arrival process $\{A_1(t)\}_{t\geq 0}$, where $A_1(t) \triangleq A_1^{ex}(t) + A_{21}(t)$. Furthermore, if $p_{12} > q_1$, then by Theorem 4.7.7, neither $\{A_{12}(t)\}_{t\geq 0}$ nor $\{A_{21}(t)\}_{t\geq 0}$ can be a Poisson process in equilibrium.

Indeed, all the above are non-exit arcs. The only exit arcs are the inlet arc (0,1) and the outlet arc (1,0), on which the traffic processes are Poisson in equilibrium, due to Corollary 4.7.3. □

We note in passing that an exit arc (i,j) never satisfies $\delta_i p_{ij} > ||\alpha||$. It suffices to show this for outlets of the network since every exit arc is an outlet of some partial network. But this follows immediately from the identity $\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \delta_i q_i \triangleq \sum_{i=1}^{m} \delta_i p_{i0}$ (see Theorem 4.4.4).

As a closing remark we conjecture that Theorem 4.7.6 and Theorem 4.7.7 can be extended to arbitrary Jackson networks in accordance with Conjecture 4.7.2.

CHAPTER 5

SIMPLIFICATIONS OF JACKSON QUEUING NETWORKS

5.0 Introduction

Simplifications of queuing networks fall within the scope of the general conceptual framework outlined in Appendix B.

Simplifications of queuing networks are motivated by the considerable analytical complexity frequently encountered by the investigator. As a matter of fact, in trying to extract stochastic properties of queuing networks, one often finds the problem to be analytically intractable. Consequently, it becomes necessary to resort to computer simulation. However, the computer complexity of such simulations (i.e. the requisite computer resources) could often render a simulation prohibitively costly or even impossible.

Thus, conditions for simplifications that reduce the conceptual complexity, simulation complexity, etc. are of interest at both the theoretical and applied level.

The organization of this chapter is as follows.

Sections 5.1 - 5.3 investigate three classes of simplifications that take Jackson networks into Jackson networks (recall that all the networks alluded to are always assumed to have single server nodes). These are the so-called *F-simplifications* (which remove feedback arcs from nodes), *A-simplifications* (which remove all arcs among a subset of nodes), and *L-simplifications* (which lump a subset of nodes into a single node).

Section 5.4 discusses simulation complexities of Jackson networks. Two types of such complexities are treated: time complexities and space complexities. Finally, we compare the effect on such complexities under the three classes of simplifications above.

The reader is referred to Appendix B for a description of the underlying framework and for further orientation.

5.1 F-Simplifications

An *F-simplification* (feedback simplification) of a node i takes a Jackson network JN = (M,α,σ,P) into a Jackson network JN' = (M,α',σ',P'), subject to

1) $p'_{kj} = p_{kj}$ for any $k \in M-\{i\}$ and $0 \leq j \leq m$.

2) If $p_{ii} < 1$, then $$p'_{ij} = \begin{cases} 0, & \text{if } i = j \\ \dfrac{p_{ij}}{1-p_{ii}}, & \text{if } i \neq j. \end{cases}$$

3) If $p_{ii} = 1$, then $p'_{ij} = p_{ij}$, $\quad 0 \leq j \leq m$.

In other words, F-simplifications eliminate feedback arcs in Jackson networks (see Figure 5.1.1), excluding the trivial case $p_{ii} = 1$.

It will be shown that certain F-simplifications preserve the distributions of the state and traffic processes. To do this we use measure preserving point morphisms (see Ch. 3) in coordinate probability space (see Ch. 2). We are justified in taking a coordinate space representation because the probabilistic structure in terms of distributions does not depend on the sample space representation.

The fact that enables us to use system-theoretic models for coordinate sample points is contained in

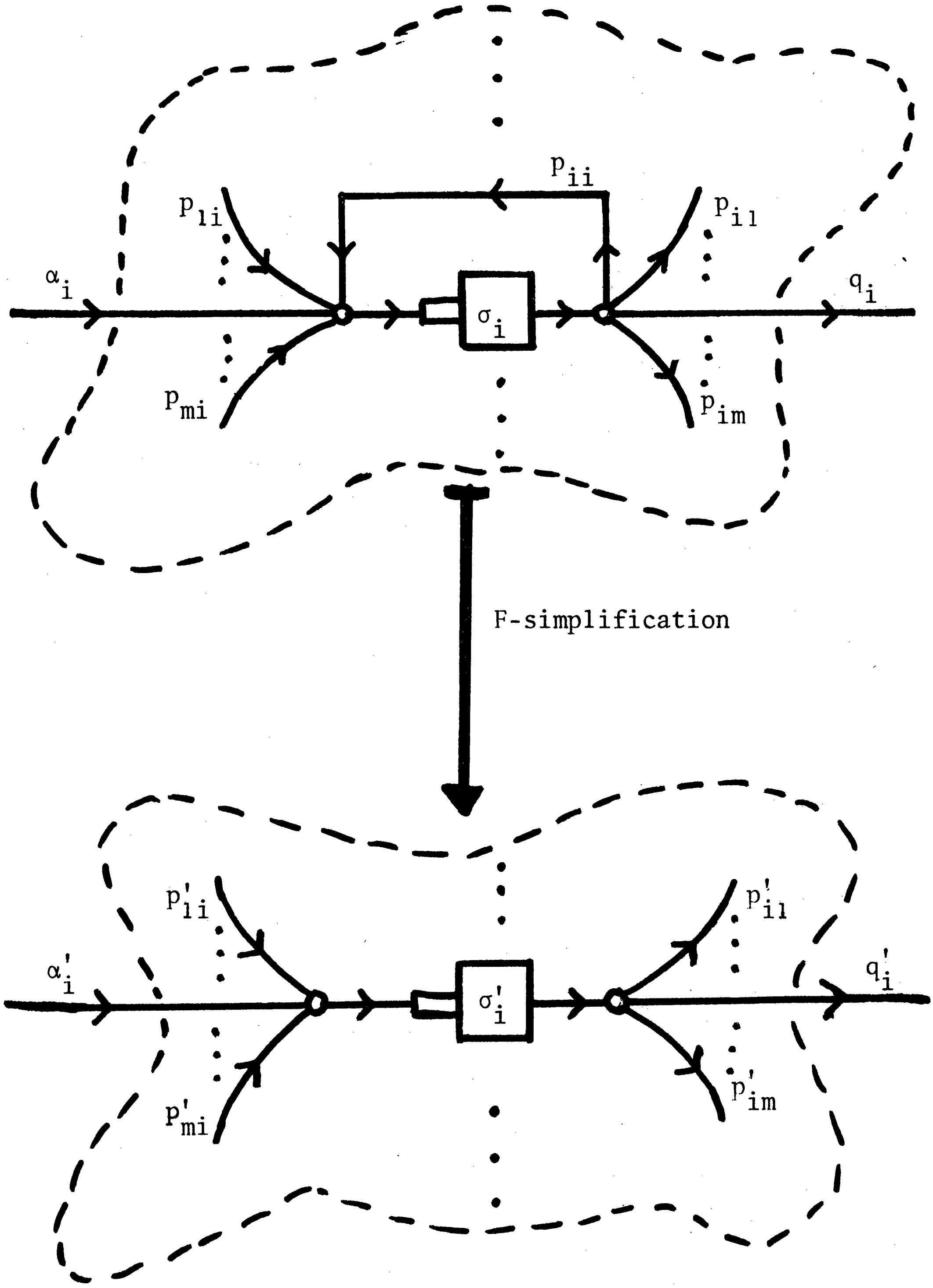

Figure 5.1.1: The Effect of a F-Simplification on a Typical Node i in a Jackson Network.

Lemma 5.1.1

Let $S = \langle \Omega, A, P \rangle$ be the coordinate probability space of Example 2.5.3 corresponding to a Jackson network. Let $N(\omega)$ be the state-DEVN associated with a coordinate sample point $\omega \varepsilon \Omega$. Then, for almost every $\omega \varepsilon \Omega$, the state-DEVS $M_{N(\omega)}$ of Example 2.5.3 is regular.†

Proof

The underlying stochastic processes of a Jackson network are finitely many and mutually independent Poisson processes. The Lemma follows (see [D1] p. 401) because each of these processes is conservative (has almost surely finite number of jumps in every finite interval). □

We start with a F-simplification of a M/M/1 queue with feedback.

Consider the F-simplification in Figure 5.1.2. This F-simplification takes the M/M/1 queue with feedback and maps it into a M/M/1 queue. The arrival parameter is unchanged but the new service parameter is $\sigma' = q\sigma$, where σ is the old service parameter and q is the probability of leaving the system. The quantity p is the feedback probability and it assumed that $p + q = 1$.

Consider the coordinate probability space $S = \langle \Omega, A, P \rangle$ of Example 2.5.2 for the base queue, and the coordinate probability space $S' = \langle \Omega', A', P' \rangle$ of Example 2.5.1 for the lumped queue, both in Figure 5.1.2.

Let us define a map $H:\Omega \longrightarrow \Omega'$ as follows:

Let $\omega = (\ell_0, \{a_j\}_{j=1}^{\infty}, \{s_j\}_{j=1}^{\infty}, \{v_j\}_{j=1}^{\infty}) \varepsilon \Omega$,

and define a sequence of random variables $\{Z_j\}_{j=0}^{\infty}$ almost everywhere on Ω by

†See Definition 1.2.7.

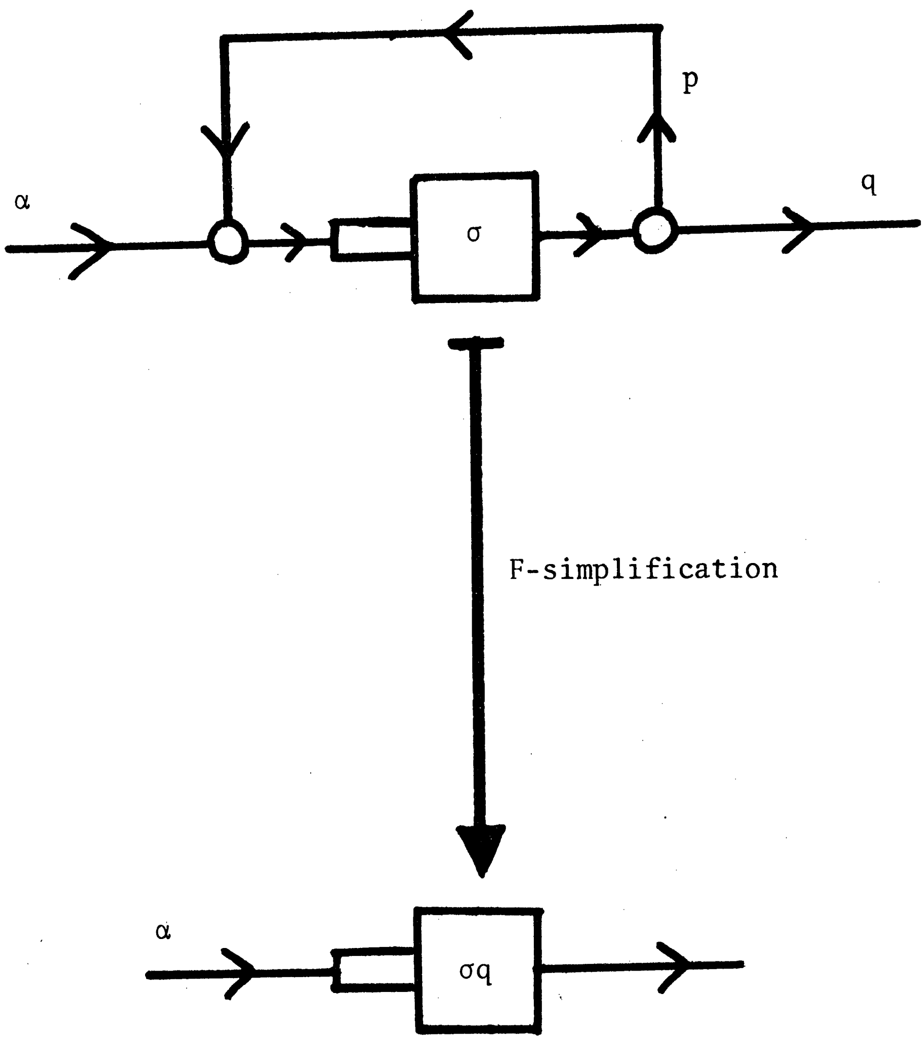

Figure 5.1.2: A F-Simplification of a M/M/1 Queue with Feedback.

$$(A) \quad Z_j(\omega) \triangleq \begin{cases} 0, & \text{if } j = 0 \\ \min\{k: k > Z_{j-1}(\omega) \text{ and } V_k(\omega)=0\}, & \text{if the minimum exists} \\ \text{undefined, otherwise} \end{cases}$$

Z_j is the index of the j-th zero in $\{V_j\}_{j=1}^{\infty}$, i.e. in an infinite sequence of Bernoulli trials.

Now, define $H(\omega) = \omega'\varepsilon\Omega'$ such that

$$(B) \quad \omega' \triangleq (\ell_0, \{a_j\}_{j=1}^{\infty}, \{s_j'\}_{j=1}^{\infty}) \quad \text{where } s_j' = \sum_{i=Z_{j-1}(\omega)+1}^{Z_j(\omega)} s_i \ .$$

On the null set of Ω for which $\{Z_j\}_{j=0}^{\infty}$ is undefined, we define H arbitrarily.

Theorem 5.1.1

The map H above is a measure preserving point morphism (m.p.p.m.).

Proof

We show that the sufficient conditions of Theorem 3.1.1 in Chapter 3 are satisfied for H.

H is clearly surjective (but not injective) because for every $\omega'\varepsilon\Omega'$ there is at least one $\omega\varepsilon\Omega$ such that $\{s_j'\}_{j=1}^{\infty}$ has the representation $s_j' = \sum_{i=Z_{j-1}(\omega)+1}^{Z_j(\omega)} s_i$.

Let $\mathcal{V}' \triangleq \{L_0', A_j', S_j': j=1,2,\ldots\}$ be the obvious projection functions on the coordinates of $\omega'\varepsilon\Omega'$ (see Ch. 2, Sec. 2.4). Then $\mathcal{V}'$ generates $\mathcal{A}'$ in $\mathcal{S}'$ by definition of $\mathcal{S}'$ (see Ch. 2, Sec. 2.2), and condition a) of Theorem 3.1.1 is satisfied. Likewise, let $\mathcal{G} \triangleq \{L_0, A_j, S_j, V_j: j=1,2,\ldots\}$ be the generator set of $\mathcal{S}$ (see Definition 2.4.1).

By definition of the sequence $\{Z_j\}_{j=0}^{\infty}$, the $Z_j - Z_{j-1}$ are mutually independent, identically and geometrically distributed with common parameter q.

Define a sequence of random variables $\{\bar{S}_j\}_{j=1}^{\infty}$ over S by

$$\bar{S}_j(\omega) \overset{\Delta}{=} \sum_{i=Z_{j-1}(\omega)+1}^{Z_j(\omega)} S_i(\omega) \qquad \text{almost surely.}$$

Then the $\bar{S}_j$ are mutually independent and identically distributed with a common Laplace-Stieltjes (LS) transform $f(\zeta)$. Moreover, if we let $g(\zeta) \overset{\Delta}{=} \frac{\sigma}{\zeta+\sigma}$ be the common LS transform of the service times $\{S_j\}_{j=1}^{\infty}$, we can write

(1) $$f(\zeta) = qg(\zeta) + qpg^2(\zeta) + qp^2g^3(\zeta) + \ldots =$$

$$qg(\zeta)\sum_{n=0}^{\infty}(pg(\zeta))^n = qg(\zeta)\frac{1}{1-pg(\zeta)} = \frac{\sigma q}{\zeta+\sigma q}.$$

Consequently, each $\bar{S}_j$ is exponentially distributed with parameter $q\sigma$.

It follows that $Y \overset{\Delta}{=} \{L_0, A_j, S_j : 1,2,\ldots\}$ and $Y' \overset{\Delta}{=} \{L_0', A_j', S_j' : j=1,2\ldots\}$ are distribution equivalent since the one-dimensional distribution functions are identical, and both sets consist of mutually independent random variables. Thus Condition b) in Theorem 3.1.1 is satisfied.

To verify that condition c) of this theorem also holds we note that

(2) $L_0(\omega) = \ell_0 = L_0'(H(\omega))$ almost surely

(3) $A_j(\omega) = a_j = A_j'(H(\omega))$ almost surely

and finally from (B)

(4) $$\bar{S}_j(\omega) = \sum_{i=Z_{j-1}(\omega)+1}^{Z_j(\omega)} s_i = s_j' = S_j'(H(\omega)) \qquad \text{almost surely.}$$

We conclude from Theorem 3.1.1 that H is a m.p.p.m. as required.

□

A few comments regarding the simplification aspects of H are warranted at this point.

In the terminology and the conceptual framework of Chapter 3 (Sec. 3.2), $S \overset{H}{\longmapsto} S'$ is a stochastic point simplification, since H was shown to be a m.p.p.m. The lumping effect of H at the sample space level is evident. The map H eliminates the last component of ω by lumping the service sequence $\{s_j\}_{j=1}^{\infty}$ and the switching sequence $\{v_j\}_{j=1}^{\infty}$ into the new service sequence $\{s_j'\}_{j=1}^{\infty}$ as given in (B). The matching operator $\mathcal{H}$ of Definition 3.3.1 sends the set $\mathcal{Y}'$ to the set $\mathcal{Y}$ in Theorem 5.1.1. The essence of H is captured by the observation that the service times $\{\bar{S}_j\}_{j=1}^{\infty}$ and $\{S_j'\}_{j=1}^{\infty}$ are identically distributed, because the total service time awarded between departures in the base queue is distributed as a single service time in its lumped version. We also have the following relation at the sample point level.

Theorem 5.1.2

Let $M(\omega)$ be the state-DEVS associated with $\omega \varepsilon \Omega$ in D) of Example 2.5.2, and let $M(H(\omega))$ be the state-DEVS associated with $H(\omega) \varepsilon \Omega'$ in D.1) of Example 2.5.1. Then for almost every $\omega \varepsilon \Omega$, $M(\omega) \sqsupset M(H(\omega))$† via a TC-DEVS state-homomorphism $(i, L, \hbar)$.††

Proof

Let ω be such that $M(\omega)$ and $M(H(\omega))$ are regular and $\{Z_j(\omega)\}_{j=1}^{\infty}$ is well-defined. For such fixed ω, denote $Z_j(\omega) \triangleq z_j$, $j = 1,2,\ldots$

For any $s = (\ell, n, v_n, r) \varepsilon S_\omega$, there is a (unique) $j=j(n)$, such that $z_{j-1} < n \le z_j$. Taking $\hat{S}_\omega \triangleq S_\omega$, we define $\hbar : \hat{S}_\omega \longrightarrow S_{H(\omega)}$ by

---

† $\sqsupset$ is the transitional covering relation (see Definition 1.4.2).
††See Definition 1.4.4.

$$(1)\quad \hbar(\ell,n,v_n,r) \triangleq \begin{cases} (0,\ j,\ \infty), \text{ if } \ell = 0 \\ (\ell,\ j,\ r + \sum_{i=n+1}^{z_j} s_i), \text{ if } \ell > 0 \end{cases}$$

where $j \triangleq j(n)$.

Note that in $s = (\ell,n,v_n,r)\varepsilon S_\omega$, ℓ represents line size, n the current customer, v_n the current switch position, and r the residual current service time. The interpretation of components in $s' = (\ell',n',r')\varepsilon S'_{H(\omega)}$ is analogous.

Note that the map $\hbar$ is surjective because every $0 \le r' \le s'_j$ has a unique representation

$$(2)\quad r' = r + \sum_{i=n+1}^{z_j} s_i \quad \text{for some n satisfying } z_{j-1} < n \le z_j.$$

Next, define for any $s \triangleq (\ell,n,v_n,r)\varepsilon S_\omega$

$$L(s) \triangleq \begin{cases} 0, \text{ if } \ell = 0 \\ z_j - n, \text{ if } \ell > 0 \end{cases}$$

where $j \triangleq j(n)$. It follows that

$$(3)\quad t_{H(\omega)}(\hbar(s)) = \begin{cases} \infty, \text{ if } \ell = 0 \\ r + \sum_{i=n+1}^{z_j} s_i, \text{ if } \ell > 0 \end{cases} = \sum_{i=0}^{L(s)} t_\omega(\bar{\delta}_{\omega,\phi}(s,i)).$$

Also, for $s \triangleq (\ell,n,v_n,r)\varepsilon S_\omega$ with $\ell > 0$, and denoting $j \triangleq j(n)$,

$$(4)\quad \hbar(\bar{\delta}_{\omega,\phi}(s,L(s)+1)) = \hbar(\bar{\delta}_{\omega,\phi}(s,z_j-n+1)) =$$

$$\begin{cases} \hbar(0,z_j+1,v_{z_j+1},\infty), \text{ if } \ell = 1 \\ \hbar(\ell-1,z_j+1,v_{z_j+1},s_{z_j+1}), \text{ if } \ell > 1 \end{cases} =$$

$$\begin{cases} (0,j+1,\infty), \text{ if } \ell = 0 \\ (\ell-1,j+1,s'_{j+1}), \text{ if } \ell > 0 \end{cases} = \delta_{H(\omega),\phi}(\hbar(s)) .$$

Finally, for any $\hat{s}\varepsilon\hat{S}_\omega$ and $0 \le \tau < t_{H(\omega)}(\hbar(\hat{s}))$, represent $\hat{s} = (\ell, n-k, v_{n-k}, \hat{r})$ and $\tau = \hat{r} + \sum_{i=n-k+1}^{n} s_i - d$, provided $j(n-k) = j(n) \triangleq j$. Let $(s,e) \triangleq \delta_G((\hat{s},0),\phi_\tau)$ where δ_G is the transition function of $G(M(\omega))$ (see Lemma 1.4.1); hence $s = (\ell, n, v_n, r)$ and $e = r-d$. Then,

$$(5)\quad \hbar(\delta_{\omega,M}((s,e),1)) = \begin{cases} \hbar(1,n,v_n,s_n), \text{ where } n = z_{j-1}+1, & \text{if } \ell = 0 \\ \hbar(\ell+1,n,v_n,r-e), & \text{if } \ell > 0 \end{cases} =$$

$$\begin{cases} (1,j,\sum_{i=z_{j-1}+1}^{z_j} s_i), & \text{if } \ell = 0 \\ (\ell+1,j,r-e+\sum_{i=n+1}^{z_j} s_i), & \text{if } \ell > 0 \end{cases} =$$

$$\begin{cases} (1,j,s_j'), & \text{if } \ell = 0 \\ (\ell+1, j, d+\sum_{i=n+1}^{z_j} s_i), & \text{if } \ell > 0 \end{cases} =$$

$$\begin{cases} \delta_{H(\omega),M}(((0,j,\infty),\tau),1), & \text{if } \ell = 0 \\ \delta_{H(\omega),M}(((\ell,j,\hat{r}+\sum_{i=n-k+1}^{z_j} s_i),\tau),1), & \text{if } \ell > 0 \end{cases} = \delta_{H(\omega),M}((\hbar(\hat{s}),\tau),1).$$

From (3), (4) and (5) we conclude (see Definition 1.4.2) that $M(\omega) \sqsupset M(H(\omega))$, and this is true almost surely due to Lemma 5.1.1 and by definition of $\{Z_j\}_{j=0}^{\infty}$. □

<u>Comment 5.1.1</u>

In particular $STRAJ_{q_\omega,\eta_\omega} \sqsupset STRAJ_{q_{H(\omega)},\eta_{H(\omega)}}$ via (i,h) where $h(s,e) \triangleq (\hbar(s),h_2(s,e))$, by Conclusion 1.4.2. To see this we note that $\eta_\omega = \eta_{H(\omega)} \triangleq \bigodot_{j=1}^{\infty} 1_{a_j}$.

Furthermore, $q_\omega = (s_0,0)$ where $s_0 = \begin{cases} (\ell_0,1,v_1,\infty), & \text{if } \ell = 0 \\ (\ell_0,1,v_1,s_1), & \text{if } \ell > 0 \end{cases}$

$$\text{and } q_{H(\omega)} = (s_0',0) \text{ where } s_0' = \begin{cases} (\ell_0,1,\infty), & \text{if } \ell = 0 \\ (\ell_0,1,s_1'), & \text{if } \ell > 0 . \end{cases}$$

Finally, $h(q_\omega) = q_{H(\omega)}$, because $\hbar(s_0) = s_0'$ and $h_2(s_0,0) = 0$. □

Having established the fact that H is a m.p.p.m. such that $M(\omega) \sqsupseteq M(H(\omega))$ almost surely, the next step is to check the scope of preservation of behavioral frames under the point simplification $S \overset{H}{\longmapsto} S'$.

Theorem 3.2.1 is used as a sufficiency criterion for preservation in the sense of distribution equivalence, in the following theorems.

<u>Theorem 5.1.3</u>

Suppose an L-simplification of an M/M/1 queue with feedback $JN = (\{1\},\alpha,\sigma,p)$ yielded a M/M/1 queue $JN' = (\{1\},\alpha,\sigma q,0)$ where $q = 1 - p$ (see Figure 5.1.2). Then the L-simplification $JN \longmapsto JN'$ preserves the state process, provided the initial states are distribution equivalent.

<u>Proof</u>

Let $\{Q_t\}_{t\geq 0}$ and $\{Q_t'\}_{t\geq 0}$ be the state processes in the base queue and lumped queue, respectively, of Figure 5.1.2.

For $\omega \in \Omega$ such that $M(\omega)$ and $M(H(\omega))$ are regular and $\{Z_j\}_{j=0}^{\infty}$ is well-defined, let $OTRAJ_{q_\omega,\eta_\omega}$ and $OTRAJ_{q_{H(\omega)},\eta_{H(\omega)}}$ be the "line size" output trajectories of $STRAJ_{q_\omega,\eta_\omega}$ and $STRAJ_{q_{H(\omega)},\eta_{H(\omega)}}$ respectively. That is

(1) $OTRAJ_{q_\omega,\eta_\omega}(t) = \lambda(STRAJ_{q_\omega,\eta_\omega}(t)) = \lambda((\ell,n,v_n,r),e) \overset{\Delta}{=} \ell$

(2) $OTRAJ_{q_{H(\omega)},\eta_{H(\omega)}}(t) = \lambda'(STRAJ_{q_{H(\omega)},\eta_{H(\omega)}}(t)) = \lambda'((\ell',n',r'),e') \overset{\Delta}{=} \ell'$.

Now, for any $t \geq 0$

(3) $Q_t(\omega) = \mathrm{OTRAJ}_{q_\omega,\eta_\omega}(t)$ and $Q'_t(H(\omega)) = \mathrm{OTRAJ}_{q_{H(\omega)},\eta_{H(\omega)}}(t)$.

Since $\mathrm{STRAJ}_{q_\omega,\eta_\omega} \sqsupset \mathrm{STRAJ}_{q_{H(\omega)},\eta_{H(\omega)}}$ via (i,h) by Comment 5.1.1, it follows that in particular

(4) $h(\mathrm{STRAJ}_{q_\omega,\eta_\omega}(t)) = \mathrm{STRAJ}_{q_{H(\omega)},\eta_{H(\omega)}}(t) \quad , \quad t \geq 0$

where $h = (h_1,h_2) = (\hbar,h_2)$ and $\hbar$ is defined in (1) of Theorem 5.1.2. But by (1) in Theorem 5.1.2

(5) $\hbar(\ell,n,v_n,r) = (\ell',n',r') \Longrightarrow \ell = \ell'$

whence by (1) and (2)

(6) $\mathrm{OTRAJ}_{q_\omega,\eta_\omega}(t) = \mathrm{OTRAJ}_{q_{H(\omega)},\eta_{H(\omega)}}(t) \quad , \quad t \geq 0.$

From Lemma 5.1.1 and by definition of $\{Z_j\}_{j=0}^{\infty}$ we conclude that for almost every $\omega\varepsilon\Omega$

(7) $Q_t(\omega) = Q'_t(H(\omega)) \quad , \quad t \geq 0$ almost surely

whence by Theorem 3.2.1, $\{Q_t\}_{t\geq 0}$ and $\{Q'_t\}_{t\geq 0}$ are distribution equivalent. □

Corollary 5.1.1

The busy and idle period processes are also preserved under the F-simplification of Theorem 5.1.3. □

Theorem 5.1.4

The departure process is preserved under the F-simplification of Theorem 5.1.3, provided the initial states are distribution equivalent.

Proof

Let $\{D_t\}_{t\geq 0}$ and $\{D'_t\}_{t\geq 0}$ be the departure counting processes in the base queue and in the lumped queue, respectively, of Figure 5.1.2. For $\omega\varepsilon\Omega$ be such that $M(\omega)$ and $M(H(\omega))$ are regular and $\{Z_j\}_{j=0}^{\infty}$ is well-defined, let $OTRAJ_{q_\omega,\eta_\omega}$ and $OTRAJ_{q_{H(\omega)},\eta_{H(\omega)}}$ be the "departure count" output trajectories of $STRAJ_{q_\omega,\eta_\omega}$ and $STRAJ_{q_{H(\omega)},\eta_{H(\omega)}}$ respectively.

That is,

(1) $OTRAJ_{q_\omega,\eta_\omega}(t) = \lambda(STRAJ_{q_\omega,\eta_\omega}(t)) = \lambda((\ell,n,v_n,r),e) \stackrel{\Delta}{=} j-1$

where $j = j(n)$ satisfies $Z_{j-1}(\omega) < n \leq Z_j(\omega)$, and

(2) $OTRAJ_{q_{H(\omega)},\eta_{H(\omega)}}(t) = \lambda(STRAJ_{q_{H(\omega)},\eta_{H(\omega)}}(t)) =$
$\lambda'((\ell',n',r'),e') \stackrel{\Delta}{=} n'-1.$

Now, for any $t \geq 0$

(3) $D_t(\omega) = OTRAJ_{q_\omega,\eta_\omega}(t)$ and $D_t'(H(\omega)) = OTRAJ_{q_{H(\omega)},\eta_{H(\omega)}}(t)$.

By Comment 5.1.1, it follows that in particular

(4) $h(STRAJ_{q_\omega,\eta_\omega}(t)) = STRAJ_{q_{H(\omega)},\eta_{H(\omega)}}(t) \quad , \quad t \geq 0$

where $h = (h_1,h_2) = (\hbar,h_2)$ and $\hbar$ is defined in (1) of Theorem 5.1.2.

But by (1) in Theorem 5.1.2

(5) $\hbar(\ell,n,v_n,r) = (\ell',n',r') \Longrightarrow n' = j(n) \stackrel{\Delta}{=} j$

where $Z_{j-1}(\omega) < n \leq Z_j(\omega)$.

Hence, by (1) and (2)

(6) $OTRAJ_{q_\omega,\eta_\omega}(t) = OTRAJ_{q_{H(\omega)},\eta_{H(\omega)}}(t) \quad , \quad t \geq 0$

From Lemma 5.1.1 and by definition of $\{Z_j\}_{j=0}^{\infty}$ we conclude that for almost every $\omega\varepsilon\Omega$

(7) $D_t(\omega) = D_t'(H(\omega)) \quad , \quad t \geq 0$

whence by Theorem 3.2.1, $\{D_t\}_{t\geq 0}$ and $\{D_t'\}_{t\geq 0}$ are distribution equivalent. ☐

Theorems 5.1.3, 5.1.4 and Corollary 5.1.1 agree with related results in [Dal] where F-simplifications of a large class of single queues with feedback are investigated. It is possible, however, to extend these results in a different direction, namely to arbitrarily connected Jackson networks with single server nodes.

Theorem 5.1.5

Let JN = (M,α,σ,P) be any Jackson network with single server nodes. Suppose an F-simplification is performed only on each node $i \varepsilon M$ with $0 < p_{ii} < 1$, such that the resulting lumped network JN' = (M,α,σ',P') satisfies (see Figure 5.1.1):

a) $\sigma_i' = \sigma_i(1-p_{ii})$, $1 \leq i \leq m$.

b) $p_{ij}' = \begin{cases} \dfrac{p_{ij}}{1-p_{ii}} & , \text{ if } i \neq j,\ 1 \leq i \leq m,\ 0 \leq j \leq m \\ 0 & , \text{ if } i = j\ ,\ 1 \leq i \leq m\ . \end{cases}$

Then, the state process and each traffic process on a non-feedback arc are preserved, provided the initial states are distribution equivalent.

Proof

Since the proof is analogous to the one for the F-simplification of Theorem 5.1.3 (see Figure 5.1.2), only an outline will be given.

Consider part D) of Example 2.5.3. As usual, $S = \langle\Omega,A,P\rangle$ and $S' = \langle\Omega',A',P'\rangle$ denote the coordinate probability space of the base network and the lumped network of this theorem. First, we define a m.p.p.m. $H:\Omega \longrightarrow \Omega'$ as follows. Let

(1) $\omega = (\ell_{0,i}, \{a_{i,j}\}_{j=1}^{\infty}, \{s_{i,j}\}_{j=1}^{\infty}, \{v_{i,j}\}_{j=1}^{\infty} : i=1,2,\ldots,m) \varepsilon \Omega$

For every $i = 1,2,\ldots m$, define a sequence $\{Z_{i,j}\}_{j=1}^{\infty}$ of random variables by

$$(2)\quad Z_{i,j}(\omega) \overset{\Delta}{=} \begin{cases} 0, \text{ if } j = 0 \\ \min\{k: k > Z_{i,j-1}(\omega) \text{ and } V_{i,k}(\omega) \neq i\}, \text{ if the minimum exists.} \\ \text{undefined, otherwise} \end{cases}$$

Note that the $Z_{i,j} - Z_{i,j-1}$ are mutually independent, and for every fixed $i=1,2,\ldots,m$ they are identically and geometrically distributed with common parameter $1 - p_{ii}$. This is so, because $Z_{i,j}$ is the index of the j-th non-feedback switching decision at node i, where the sequence of switching decisions constitutes an infinite sequence of multinomial Bernoulli trials. Now, the sequences $\{Z_{i,j}(\omega)\}_{j=1}^{\infty}$ are almost surely simultaneously defined.

For such $\omega\varepsilon\Omega$, define $H(\omega) = \omega'$ where

$$(3)\quad \omega' \overset{\Delta}{=} (\ell_{i,0}, \{a_{i,j}\}_{j=1}^{\infty}, \{s'_{i,j}\}_{j=1}^{\infty}, \{v'_{i,j}\}_{j=1}^{\infty} : i = 1,2,\ldots,m)$$

where for any $j = 1,2,\ldots$

$$(4)\quad s'_{i,j} = \sum_{n=Z_{i,j-1}(\omega)+1}^{Z_{i,j}(\omega)} s_{i,n} \quad \text{and} \quad v'_{i,j} = v_{i,Z_{i,j}(\omega)} .$$

On the null set of Ω for which the $\{Z_{i,j}\}_{j=1}^{\infty}$ are undefined, H is defined arbitrarily. Thus, H is surjective because every sequence $\{s'_{i,j}\}_{j=1}^{\infty}$ and $\{v'_{i,j}\}_{j=1}^{\infty}$ has at least one representation as in (4).

Let $\mathcal{Y}' \overset{\Delta}{=} \{L'_{i,0}, A_{i,j}, S_{i,j}, V_{i,j} : i \leq i \leq m, j = 1,2,\ldots\}$ be the obvious projection functions on Ω'. Then $\mathcal{Y}'$ generates $\mathcal{A}'$ in $\mathcal{S}'$ (condition a) of Theorem 3.1.1).

Finally, let $\mathcal{G} \overset{\Delta}{=} \{L_{i,0}, A_{i,j}, S_{i,j}, V_{i,j} : 1 \leq i \leq m, j = 1,2,\ldots\}$ be the generator set of $\mathcal{S}$, and define a set of random variables $\mathcal{Y} \overset{\Delta}{=} \{L_{i,0}, A_{i,j}, \bar{S}_{i,j}, \bar{V}_{i,j} : 1 \leq i \leq m, j = 1,2,\ldots\}$ over $\mathcal{S}$ where

$$(5)\quad \bar{S}_{i,j} \overset{\Delta}{=} \sum_{n=Z_{j-1}+1}^{Z_j} S_n \quad \text{and} \quad \bar{V}_{i,j} \overset{\Delta}{=} V_{i,Z_{i,j}} \qquad \text{almost surely} .$$

A calculation similar to (1) of Theorem 5.1.1 reveals that each pair $\bar{S}_{i,j}$ and $S'_{i,j}$ has the same exponential distribution with parameter $\sigma_i(1-p_{ii})$. Moreover, each pair $\bar{V}_{i,j}$ and $V'_{i,j}$ has the same distribution, as for every fixed $1 \le i \le m$, both $\bar{V}_{i,j}$ and $V'_{i,j}$ correspond to a multinomial Bernoulli trial that assumes values in the set $\{n: n\varepsilon M-\{i\}\}$ with probabilities

(6) $\Pr(\bar{V}_{i,j}=n) = \Pr(V'_{i,j}=n) = p'_{in} = \dfrac{p_{in}}{1-p_{ii}}$

We can now conclude that $\mathcal{V}$ and $\mathcal{V}'$ are distribution equivalent (condition b) of Theorem 3.1.1), because they consist of mutually independent random variables. Finally, condition c) of Theorem 3.1.1 is verifiable as in Theorem 5.1.1.

This establishes the fact that H is a m.p.p.m. from $\mathcal{S}$ to $\mathcal{S}'$ according to Theorem 3.1.1.

Next, we expand each $N(\omega)$ into $M_{N(\omega)}$ and each $N(H(\omega))$ into $M_{N(H(\omega))}$ (see Ch. 1, Sec. 1.1), and compare the state trajectory representations of ω and $H(\omega)$ at the DEMS level. It again follows that for almost every $\omega\varepsilon\Omega$, $M_{N(\omega)}$ and $M_{N(H(\omega))}$ are regular state-DEVSs.

To verify that the state and traffic processes are preserved we merely make the following observations.

First, it can be shown that for each α, $1 \le \alpha \le m$, we have $M_\alpha(\omega) \sqsupset M_\alpha(H(\omega))$ almost surely, as in Theroem 5.1.2. Thus, departures from each component $M_\alpha(\omega)$ are concurrent with those of $M_\alpha(H(\omega))$ almost surely. Furthermore, the switchings in $M_\alpha(H(\omega))$ were set up so that departures along each non-feedback arc are also concurrent, almost surely.

Consequently, $M_{N(\omega)} \sqsupset M_{N(H(\omega))}$ in such a way that line sizes and traffic along each non-feedback arc are identical for almost every $\omega\varepsilon\Omega$, because the initial line sizes are identical.

This completes the outline of proof for this theorem. □

We remark that Theorem 5.1.5 can be verified directly by writing the birth-and-death equations of the state process augmented by any subset of traffic processes, and observing that the same equations ensue.

However, the merit of stochastic simplifications via measure preserving point morphisms is twofold. First, it provides considerable intuition and insight into simplifications because it enables the user to employ system-theoretic tools and principles which are inherent in queuing systems. Second, Theorems 3.1.1 and 3.2.1 (which provide the basis for stochastic point simplifications) are rather general and are not restricted a priori to a certain class of stochastic processes.

It should also be pointed out that once a m.p.p.m. is found, one may test its scope of preservation via the sufficiency conditions of Theorem 3.2.1. Furthermore, using system-theoretic tools, these conditions can be readily tested by comparing queuing histories and observing the behavioral frame of interest. In our case, we saw that the existence of H allowed us to conclude that behavioral frames such as line sizes, traffic process, busy and idle periods etc., which require no information concerning customer identity, are all preserved.

It is natural to ask whether customer-oriented behavioral frames such as waiting times and transit times are also preserved. First, we point out that such behavioral frames cannot be defined on representations of sample points ω which are derived from the associated DEVS $M(\omega)$. The reason is that the $M(\omega)$ model does not contain information regarding individual customers, because the ℓ components of its sequential states retain line size rather than line configuration

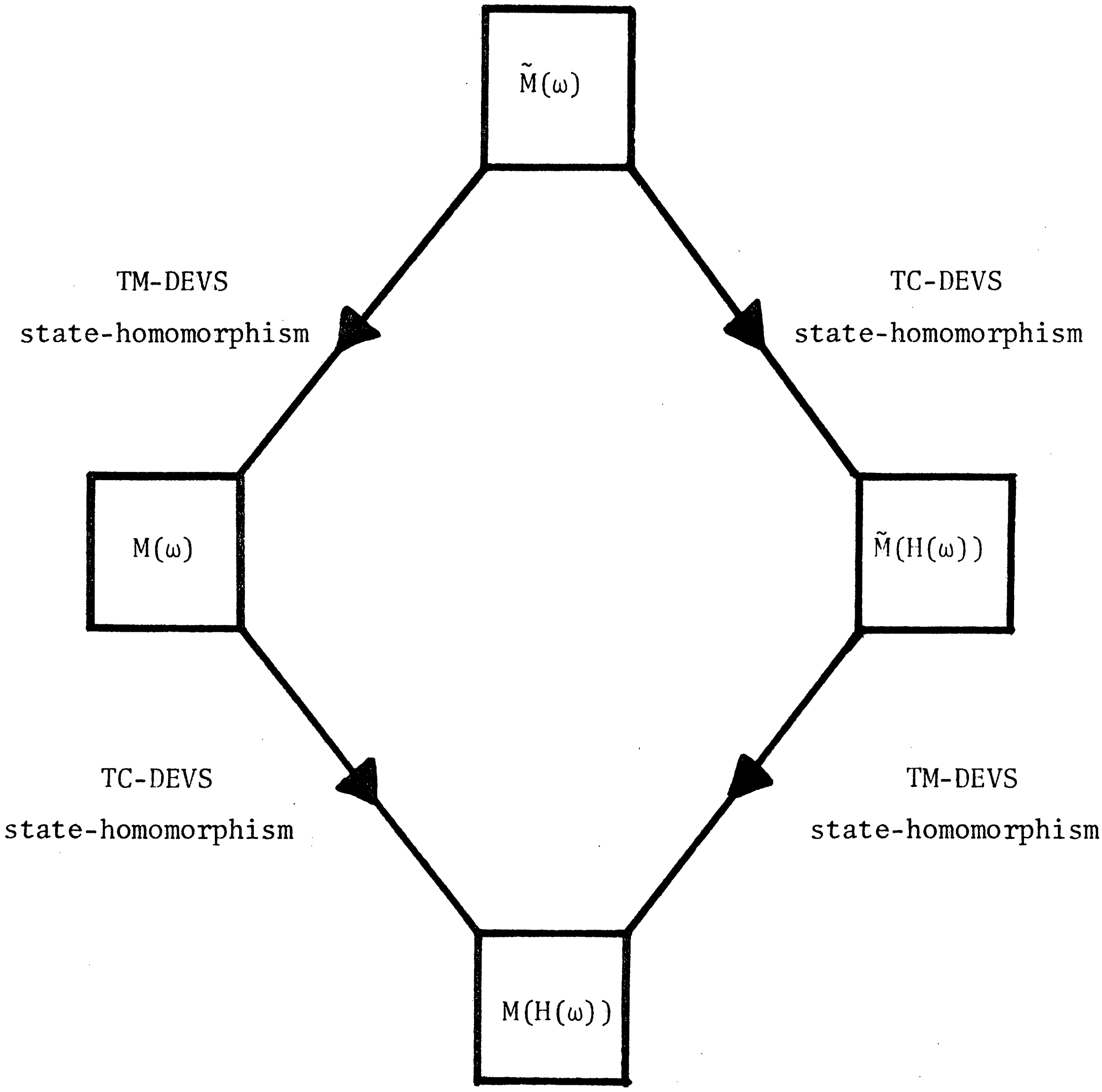

Figure 5.1.3: System-Theoretic Relations Engendered by the F-Simplifications of Figures 5.1.1 and 5.1.2.

(see Examples 2.5.1 - 2.5.3). Consequently, a more elaborate model $\tilde{M}(\omega)$ has to be associated with every ω, whereby the ℓ component is replaced by a c component where c is an ordered string of customer tags which describes the line configuration (see Examples 1.1.1 and 1.1.2).

On comparing $\tilde{M}(\omega)$ with $\tilde{M}(H(\omega))$ and the state trajectories that they engender under the F-simplification of Theorem 5.1.3, one observes that condition a) of Theorem 3.2.1 cannot be verified for waiting and transit times. This stands in agreement with the facts found in [Dal]. There is no reason to believe that customer-oriented behavioral frames are preserved in the F-simplification of Theorem 5.1.5 either.

We conclude this section with some system-theoretic remarks.

The DEVSs $\tilde{M}(\omega)$ and $\tilde{M}(H(\omega))$ are more complex and contain more information than $M(\omega)$ and $M(H(\omega))$ respectively. This explains why customer-oriented behavioral frames are relatively difficult to derive. It can be shown that there is a TM-DEVS state-morphism $(g;\hbar)$ from $\tilde{M}(\omega)$ to $M(\omega)$ such that $\hat{S} = S$ (see Definition 1.5.2). The effect of the map $\hbar$ on the sequential states of $\tilde{M}(\omega)$ is to lump the c-component into a ℓ-component such that $|c| = \ell$ where $|c|$ is the length of the string c.

Figure 5.1.3 summarizes the system-theoretic properties of H and the relations among the DEVS models associated with coordinate sample points engendered by the F-simplification of Theorems 5.1.3 and 5.1.5.

5.2 A-Simplifications

In the rest of the chapter we shall adopt the following notation.

Let JN = (M,α,σ,P) be a Jackson network with single server nodes. As usual we write $p_{i0} \triangleq 1 - \sum_{j=1}^{m} p_{ij} \triangleq q_i$ and $p_{0i} \triangleq \frac{\alpha_i}{||\alpha||}$. $||\cdot||$ is the norm of (A) in Sec. 4.3, while $|\cdot|$ is used as the cardinality symbol.

For any $i,j \varepsilon M$ and $C,D \subset M \cup \{0\}$ we write $p(i,C) \triangleq \sum_{j\varepsilon C} p_{ij}$ and $p(C,i) \triangleq \sum_{j\varepsilon C} p_{ji}$ for the switching probabilities from i to C and from C to i, respectively. We shall also use the notation $\delta(i,j) \triangleq \delta_i p_{ij}$ for the expected equilibrium traffic rate on arc (i,j). Likewise, we shall use $\delta(i,C) \triangleq \sum_{j\varepsilon C} \delta(i,j)$, $\delta(C,i) \triangleq \sum_{j\varepsilon C} \delta(j,i)$ and $\delta(C,D) \triangleq \sum_{i\varepsilon C} \sum_{j\varepsilon D} \delta(i,j)$ for the expected equilibrium traffic rates from i to C, from C to i, and from C to D, respectively.

Finally, complements of $C \subset M$ will always mean complements of $M \cup \{0\}$, i.e. $\bar{C} \triangleq (M \cup \{0\}) - C$. We shall often deal with partitions $\Pi = \{C_\ell\}_{\ell \varepsilon L}$ of the node set M, in which case the C_ℓ will be referred to as *blocks* of the partition Π.

A *A-simplification* (arc simplification) of a Jackson network operates on a subset of nodes C, to the effect of removing all arcs among all nodes in C (see Figure 5.2.1). Formally, it takes JN = (M,α,σ,P) into JN' = (M,α',σ',P') such that

1) $p'_{ij} = p_{ij}$ for any $i \notin C$ and $0 \le j \le m$.

2) $p'_{ij} = 0$ for any $i \varepsilon C$ and $j \varepsilon C$.

In this section, we shall be interested in A-simplifications that preserve distributions of equilibrium line sizes and the total service time obtained by a customer in a subset of nodes C.

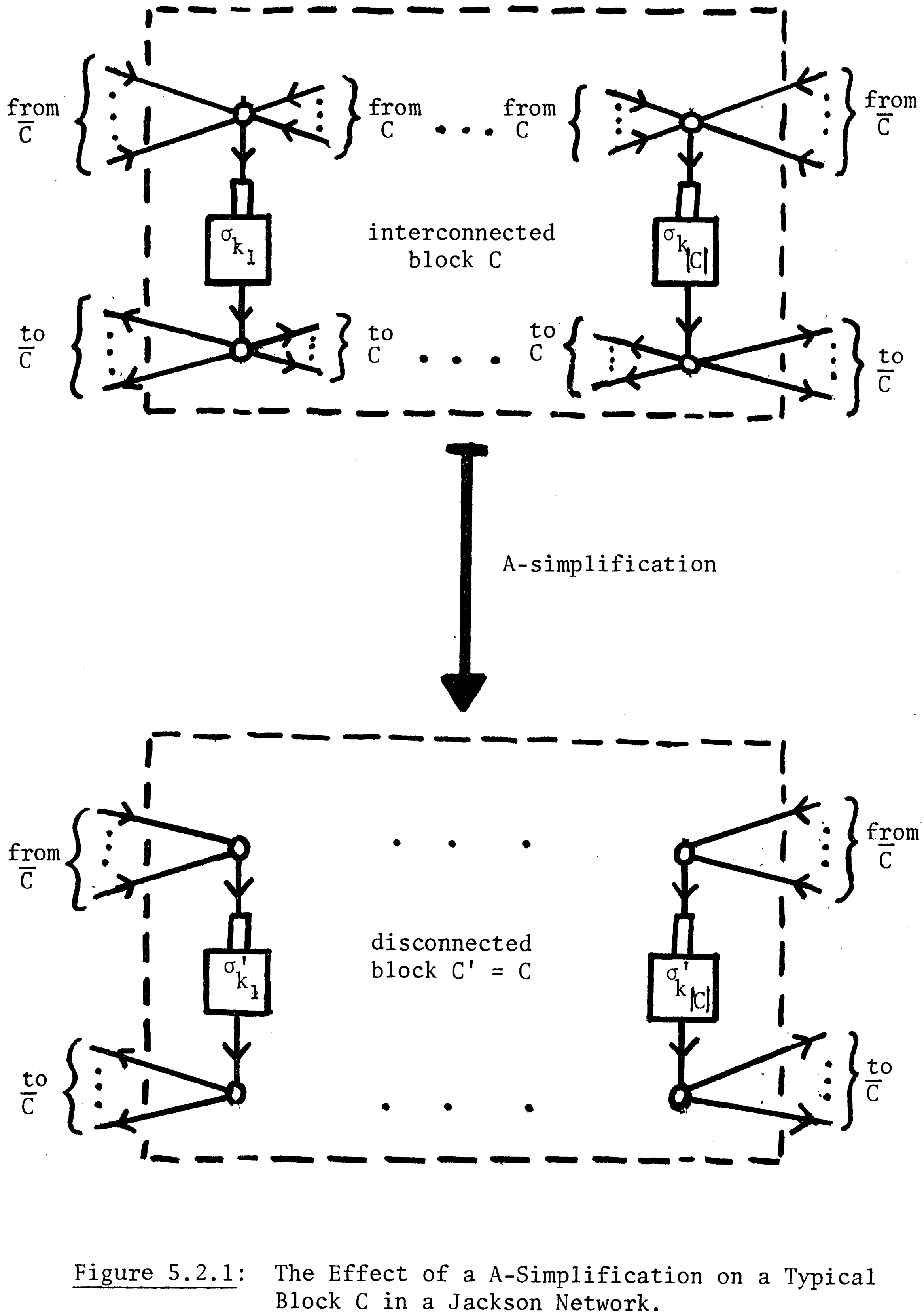

Figure 5.2.1: The Effect of a A-Simplification on a Typical Block C in a Jackson Network.

Theorem 5.2.1

Let JN = (M,α,σ,P) be an open Jackson network. Let $C \subset M$ satisfy $p(k,\bar{C}) > 0$ for all $k \in C$. Suppose an A-simplification was performed on C (see Figure 5.2.1), yielding a Jackson network JN' = (M,α,σ',P') where

a) for any $i,j \in M$

$$\text{(a.1)} \quad \sigma'_i \triangleq \begin{cases} \sigma_i, & \text{if } i \notin C \\ \sigma_i p(i,\bar{C}), & \text{if } i \in C \end{cases}$$

$$\text{(a.2)} \quad p'_{ij} \triangleq \begin{cases} p_{ij}, & \text{if } i \notin C \\ 0, & \text{if } i \in C \text{ and } j \in C \\ \dfrac{p_{ij}}{p(i,\bar{C})}, & \text{if } i \in C \text{ and } j \notin C \end{cases}$$

Then, the A-simplification above gives rise to a new traffic solution δ' satisfying

$$\text{(b)} \quad \delta'_i = \begin{cases} \delta_i, & \text{if } i \notin C \\ \delta(\bar{C},i), & \text{if } i \in C \end{cases} \quad , \quad \forall i \in M$$

and

$$\text{(c)} \quad \rho'_i = \rho_i \quad , \quad \forall i \in M$$

iff

$$\text{(d)} \quad \delta(\bar{C},k) = \delta_k p(k,\bar{C}) \quad , \quad \forall k \in C \, .$$

Proof

($\Longrightarrow$) Suppose that (b) and (c) hold.

From (a.1) and (b), it follows that

$$\text{(1)} \quad \rho'_k = \frac{\delta'_k}{\sigma'_k} = \frac{\delta(\bar{C},k)}{\sigma_k p(k,\bar{C})} \, , \qquad \forall k \in C.$$

Applying (1) to (c) gives us

$$\text{(2)} \quad \frac{\delta_k}{\sigma_k} = \frac{\delta(\bar{C},k)}{\sigma_k p(k,\bar{C})} \, , \qquad \forall k \in C$$

whence (d) immediately follows.

($\Leftarrow$) Suppose that (d) holds.

We show first that δ' as given by (b) satisfies the traffic equation of JN'. Taking note of (a.2) and (d) we have:

(3) for $i \epsilon C$,

$$\delta'_i = \alpha_i + \sum_{j=1}^{m} \delta'_j p'_{ji} = \alpha_i + \sum_{j \notin C} \delta_j p_{ji} = \delta(\bar{C},i) .$$

(4) For $i \notin C$,

$$\delta'_i = \alpha_i + \sum_{j \notin C} \delta'_j p'_{ji} + \sum_{j \epsilon C} \delta'_j p'_{ji} = \alpha_i + \sum_{j \notin C} \delta_j p_{ji} + \sum_{j \epsilon C} \delta(\bar{C},j) \frac{p_{ji}}{p(j,\bar{C})} =$$

$$\alpha_i + \sum_{j \notin C} \delta_j p_{ji} + \sum_{j \epsilon C} \delta_j p(j,\bar{C}) \frac{p_{ji}}{p(j,\bar{C})} = \alpha_i + \sum_{j=1}^{m} \delta_j p_{ji} = \delta_i .$$

Since the traffic solution is unique for an open Jackson network, we conclude from (3) and (4) that Equation (b) is perforce the traffic solution of JN'. Finally, using (a.1), (b) and (d) gives us for every $i \epsilon M$,

(5) $$\rho'_i = \frac{\delta'_i}{\sigma'_i} = \begin{cases} \dfrac{\delta_i}{\sigma_i}, & \text{if } i \notin C \\ \dfrac{\delta(\bar{C},i)}{\sigma_i p(i,\bar{C})}, & \text{if } i \epsilon C \end{cases} =$$

$$= \begin{cases} \dfrac{\delta_i}{\sigma_i}, & \text{if } i \notin C \\ \dfrac{\delta_i p(i,\bar{C})}{\sigma_i p(i,\bar{C})}, & \text{if } i \epsilon C \end{cases} = \frac{\delta_i}{\sigma_i} = \rho_i$$

as required. □

Corollary 5.2.1

If the A-simplification JN $\longmapsto$ JN' of Theorem 5.2.1 satisfies condition (d), then JN has a state equilibrium iff JN' does. Moreover, in this case the equilibrium state distributions are identical. □

Condition (d) of Theorem 5.2.1 can be equivalently stated in terms of node parameters in C only as follows.

Lemma 5.2.1

The condition

(a) $\delta(\bar{C},k) = \delta_k p(k,\bar{C})$, $\forall k \varepsilon C$

is equivalent to the condition

(b) $\delta(C,k) = \delta_k p(k,C)$, $\forall k \varepsilon C$.

Proof

By definition we can decompose for every $k \varepsilon C$

(1) $\delta_k = \delta(C,k) + \delta(\bar{C},k)$

(2) $1 = p(k,C) + p(k,\bar{C})$.

Hence, for every $k \varepsilon C$

(3) $\delta_k = \delta_k[p(k,C) + p(k,\bar{C})] = \delta_k p(k,C) + \delta_k p(k,\bar{C})$.

From (1) and (3) we conclude that (a) holds iff (b) holds. □

The equivalent conditions (a) and (b) of Lemma 5.2.1 are conservation equations which assert that in equilibrium, the expected traffic rates through the nodes of C are balanced with respect to C. In other words, condition (a) requires that in equilibrium, the expected traffic rate into each node $k \varepsilon C$ from the nodes *outside* C (including the exogenous input) equals the expected traffic rate from $k \varepsilon C$ to the nodes outside C. Likewise, condition (b) requires that, in equilibrium, the expected traffi- rate into each node $k \varepsilon C$ from the nodes *inside* C equals the expected traffic rate from $k\varepsilon C$ to the nodes *inside* C. This is a stronger balance condition as compared to the balance condition postulated by the traffic equation, whereby the expected traffic rate into a node equals the expected traffic rate out of it, in equilibrium.

Theorem 5.2.1 also enables us to make the following extension.

Theorem 5.2.2

Let JN = (M,α,σ,P) be an open Jackson network. Let $\Pi = \{C_\ell\}_{\ell\varepsilon L}$ be a partition of M, such that

(a) $p(k,\bar{C}_\ell) > 0$, whenever $k\varepsilon C_\ell$

Suppose the A-simplification of Theorem 5.2.1 was performed on each $C_\ell\varepsilon\Pi$.

Then, the A-simplification above gives rise to a new traffic solution δ' where

(b) $\delta'_k = \delta(\bar{C}_\ell,k)$, whenever $k\varepsilon C_\ell$

and

(c) $\rho'_i = \rho_i$, $\forall i\varepsilon M$

iff

(d) $\delta(\bar{C}_\ell,k) = \delta_k p(k,\bar{C}_\ell)$, whenever $k\varepsilon C_\ell$.

Proof

($\Longrightarrow$) Necessity is proven exactly as in Theorem 5.2.1, as every $k\varepsilon M$ is in some $C_\ell\varepsilon\Pi$.

($\Longleftarrow$) The A-simplification of this theorem can be obtained by successive A-simplifications of the C_ℓ (simplification procedure) as follows:

$$JN^{(0)} = JN \longmapsto JN^{(1)} \longmapsto \ldots \longmapsto JN^{(|L|)} = JN'.$$

It follows from Theorem 5.2.1 that at each stage we obtain a Jackson network $JN^{(n)}$ whose traffic solution $\delta^{(n)}$ satisfies for every $i\varepsilon M$,

$$(1)\quad \delta_i^{(n)} = \begin{cases} \delta_i^{(n-1)}, & \text{if } i \varepsilon C_\ell \\ \delta^{(n-1)}(C_\ell, i) & \text{if } i \varepsilon C_\ell . \end{cases}$$

It is easy to see by induction that on setting $n = |L|$ in (1), condition (b) follows. Condition (c) holds, because the ρ parameters remain unchanged at each simplification stage $JN^{(n-1)} \longmapsto JN^{(n)}$ due to Theorem 5.2.1. □

Since the preservation effect of the last theorem depends on condition (d) of Theorem 5.2.2 (which is derived from the base network specification), we now proceed to give a set of structural conditions that imply the behavioral condition (d) above.

Theorem 5.2.3

Let $JN = (M,\alpha,\sigma,P)$ be an open Jackson network. Let $\Pi = \{C_\ell\}_{\ell \varepsilon L}$ be a partition of M, such that for each $\ell \varepsilon L$,

(a) $p(C_\ell,k) = p(k,C_\ell)$, $\forall k \varepsilon C_\ell$.

(b) $p(C_\ell,k) = \text{const.}$, $\forall k \varepsilon C_n$, $\forall C_n \varepsilon \Pi$.

(c) $\alpha_k = \text{const.}$, $\forall k \varepsilon C$.

Then, for each $\ell \varepsilon L$,

(d) $\delta(\bar{C}_\ell,k) = \delta_k p(k,\bar{C}_\ell)$, $\forall k \varepsilon C_\ell$.

Proof

We show first that conditions (b) and (c) imply that for each $\ell \varepsilon L$,

(1) $\delta_k = \text{const.}$, $\forall k \varepsilon C_\ell$.

By Corollary 4.4.2

$$(2)\quad \delta = \alpha \sum_{n=0}^{\infty} P^n .$$

Now, the set

(3) $K(\Pi) \overset{\Delta}{=} \{v = (v_1,\ldots,v_m) \in \mathbb{R}^m\colon$ for each $\ell \in L$, $v_k = \text{const.}$, $\forall k \in C_\ell\}$

is a linear subspace of $\mathbb{R}^m$. It can be directly verified that $K(\Pi)$ is invariant under linear transformations whose matrix representation is a $m\times m$ matrix satisfying condition (b).

Hence, since $\alpha \in K(\Pi)$, it follows that

(4) $\alpha P^n \in K(\Pi)$, $n = 0,1,\ldots$

and from (2) we conclude that (1) holds, as $K(\Pi)$ is complete.

Now, condition (a) can be written for each $\ell \in L$ as

(5) $$\sum_{j\in C_\ell} p_{jk} = \sum_{j\in C_\ell} p_{kj} , \qquad \forall k \in C_\ell .$$

In view of (5), Equation (1) allows us to write for each $\ell \in L$,

(6) $$\sum_{j\in C_\ell} \delta_j p_{jk} = \sum_{j\in C_\ell} \delta_k p_{kj} , \qquad \forall k \in C_\ell .$$

But Equation (6) is by definition for each $\ell \in L$,

(7) $\delta(C_\ell,k) = \delta_k p(k,C_\ell)$, $\forall k \in C_\ell$

and (7) is equivalent to condition (d) by Lemma 5.2.1. □

Finally, we observe

<u>Corollary 5.2.2</u>

Let $\Pi = \{C_\ell\}_{\ell \in L}$ be a partition of the node set of an open Jackson network $JN = (M,\alpha,\sigma,P)$. If for each $\ell \in L$,

a) $p_{kj} = p_{jk}$, $\forall j,k \in C_\ell$

b) $p_{ij} = p_{ik}$, $\forall i \in M$, $\forall j,k \in C_\ell$

c) $\alpha_k = \text{const.}$, $\forall k \in C_\ell$

then conditions (a), (b) and (c), respectively, of Theorem 5.2.3 hold; hence, condition (d) of Theorem 5.2.3 is also satisfied. □

We now proceed to discuss A-simplifications that preserve the total service time obtained by a customer from a subset of nodes. More precisely, $\tilde{S}_{k,C}$ will denote the sum of service times obtained by a customer that enters a given subset of nodes C at node $k\varepsilon C$, till the first departure from C (cf. Ch. 4, Sec. 4.6, where the case C = M was investigated).

Theorem 5.2.4

Let JN = (M,α,σ,P) be a Jackson network and let $C \subset M$. Suppose the A-simplification of Theorem 5.2.1 was performed on C.

Then the total service time in C is preserved (in distribution) iff

(a) $\sigma_k p(k,\bar{C}) = \text{const.}, \qquad k\varepsilon C.$

In this case, the $\tilde{S}_{k,C}$ are exponentially distributed with common parameter $\sigma_k p(k,\bar{C})$.

Proof

Let $f_k(\zeta)$ be the Laplace-Stieltjes (LS) transform of $\tilde{S}_{k,C}$, $k\varepsilon C$. Let $v_k(\zeta)$ be the LS transform of the service time S_k at node $k\varepsilon C$, viz.

(1) $$v_k(\zeta) \triangleq \frac{\sigma_k}{\zeta+\sigma_k}, \qquad k\varepsilon C.$$

The $f_k(\zeta)$ satisfy the equation (cf. (3) in Theorem 4.6.1)

(2) $$f_k(\zeta) = p(k,\bar{C})v_k(\zeta) + \sum_{j\varepsilon C} p_{kj} v_k(\zeta) f_j(\zeta) .$$

We show first that the $\tilde{S}_{k,C}$ are identically distributed iff (a) holds. Suppose the $\tilde{S}_{k,C}$ are identically distributed with the same LS transform

(3) $$f(\zeta) = f_k(\zeta), \qquad k\varepsilon C.$$

Setting (3) and (1) into (2) gives us

(4) $$f(\zeta) = p(k,\bar{C})\frac{\sigma_k}{\zeta+\sigma_k} + \sum_{j\varepsilon C} p_{kj} \frac{\sigma_k}{\zeta+\sigma_k} f(\zeta) \qquad , \qquad k\varepsilon C .$$

Solving (4) for $f(\zeta)$ yields

$$(5) \quad f(\zeta) = \frac{p(k,\bar{C})\sigma_k}{\zeta+p(k,\bar{C})\sigma_k} \quad , \quad k\varepsilon C .$$

Thus, whenever $i,j\varepsilon C$,

$$(6) \quad f(\zeta) = \frac{p(i,\bar{C})\sigma_i}{\zeta+p(i,\bar{C})\sigma_i} = \frac{p(j,\bar{C})\sigma_j}{\zeta+p(j,\bar{C})\sigma_j} \quad , \quad \zeta \geq 0$$

whence $p(i,\bar{C})\sigma_i = p(j,\bar{C})\sigma_j$ for any $i,j\varepsilon C$, so that (a) follows.

Conversely, suppose $\sigma_k p(k,\bar{C}) = \text{const.}$, for all $k\varepsilon C$. Define

$$(7) \quad f_k(\zeta) \triangleq \frac{\sigma_k p(k,\bar{C})}{\zeta+\sigma_k p(k,\bar{C})} \equiv f(\zeta) \quad , \quad k\varepsilon C .$$

It is easy to verify by direct substitution, that the $f_k(\zeta)$, $k\varepsilon C$, in (7) satisfy Equation (2). Moreover, Theorem 4.6.1 implies that this is the unique solution for (2), so that $f_k(\zeta) \equiv f(\zeta)$ for all $k\varepsilon C$.

We note that, in particular, (a) ensures the $\tilde{S}_{k,C}$, $k\varepsilon C$, to be exponentially distributed with the common parameter $\sigma_k p(k,\bar{C})$, $k\varepsilon C$. The theorem follows from the observation that the nodes in C in the simplified network are disconnected, so that the new $\tilde{S}'_{k,C}$ coincide with the new service times S'_k, for all $k\varepsilon C$.

5.3 L-Simplifications

A *L-simplification* (lumping simplification) of a Jackson network operates on a subset of nodes and lumps it into a single node (e.g. Figure 5.3.1). Typically, one partitions the node set of a Jackson network $JN = (M,\alpha,\sigma,P)$ via some partition $\Pi = \{C_\ell\}_{\ell\varepsilon L}$, and then one proceeds to lump each block C_ℓ into a single node ℓ, thus obtaining a new Jackson network $JN'' = (L,\alpha'',\sigma'',P'')$. This situation will be referred to as a

L-simplification of JN with respect to Π.

Now, suppose that the matrix P is *strongly lumpable* with respect to Π, or equivalently (see [KS1], p. 124)

(A) $p(i,C_n) = p(j,C_n) \quad , \quad \forall i,j \in C_\ell, \quad \forall C_\ell, C_n \in \Pi$.

In this case, one can define the switching probability from C_ℓ to C_n as the common value above, viz.

(B) $p(C_\ell,C_n) \triangleq p(k,C_n) \qquad C_\ell, C_n \in \Pi$

where k is any representative node in C_ℓ.

If a partition Π gives rise to condition (A), then Π will be called a *strongly lumpable partition* of P.

In this section we shall be interested in L-simplification of Jackson networks with respect to strongly lumpable partitions Π.

We first investigate the effect of such L-simplifications on equilibrium operating characteristics and especially on the traffic equation.

Theorem 5.3.1

Let JN = (M,α,σ,P) be an open Jackson network. Let $\Pi = \{C_\ell\}_{\ell \in L}$ be a partition of M which is strongly lumpable with respect to P.

Let JN'' = $(L,\alpha'',\sigma'',P'')$ be obtained from JN by means of a L-simplification with respect to Π such that

a) for every $\ell, n \in L$

(a.1) $\alpha''_\ell \triangleq \sum_{k \in C_\ell} \sigma_k$

(a.2) $\sigma''_\ell \triangleq \sum_{k \in C_\ell} \sigma_k$

(a.3) $p''_{\ell n} \triangleq p(C_\ell,C_n)$.

Then

(b) $\delta''_\ell = \sum_{k \in C_\ell} \delta_k \quad , \quad \forall \ell \in L$.

(c) $\delta''(\ell,n) = \delta(C_\ell,C_n) \qquad \forall \ell,n\varepsilon L$.

Proof

For every $k\varepsilon M$,

(1) $$\delta_k = \alpha_k + \sum_{j=1}^{m} \delta_j p_{jk}$$

Summing (1) over $k\varepsilon C$, for any $\ell\varepsilon L$, yields

(2) $$\sum_{k\varepsilon C_\ell} \delta_k = \sum_{k\varepsilon C_\ell} \alpha_k + \sum_{k\varepsilon C_\ell} \sum_{j=1}^{m} \delta_j p_{jk} =$$

$$\alpha''_\ell + \sum_{j=1}^{m} \delta_j \sum_{k\varepsilon C_\ell} p_{jk} = \alpha''_\ell + \sum_{j=1}^{m} \delta_j p(j,C_\ell)$$

due to (a.1) and (a.3). But

(3) $$\sum_{j=1}^{m} \delta_j p(j,C_\ell) = \sum_{n\varepsilon L} \sum_{j\varepsilon C_n} \delta_j p(j,C_\ell) =$$

$$\sum_{n\varepsilon L} \sum_{j\varepsilon C_n} \delta_j p(C_n,C_\ell) = \sum_{n\varepsilon L} p(C_n,C_\ell) \sum_{j\varepsilon C_n} \delta_j$$

due to strong lumpability of Π.

Since $p(C_n,C_\ell) \triangleq p''_{n\ell}$ and in view of (3), Equation (2) becomes

(4) $$\sum_{k\varepsilon C_\ell} \delta_k = \alpha''_\ell + \sum_{n\varepsilon L} p'_{n\ell} \sum_{j\varepsilon C_n} \delta_j \qquad , \qquad \forall \ell\varepsilon L .$$

Thus, (4) shows that (b) satisfies the traffic equation of JN'', and therefore must be the (unique) traffic solution of JN''.

Next, we compute for every $\ell,n\varepsilon L$,

(5) $$\delta(C_\ell,C_n) = \sum_{k\varepsilon C_\ell} \sum_{j\varepsilon C_n} \delta_k p_{kj} = \sum_{k\varepsilon C_\ell} \delta_k \sum_{j\varepsilon C_n} p_{kj} =$$

$$\sum_{k\varepsilon C_\ell} \delta_k p(C_\ell,C_n) = p''_{\ell n} \sum_{k\varepsilon C_\ell} \delta_k$$

Equation (c) now follows by substituting (b) into the right-hand side of (5). □

Corollary 5.3.1

Under the simplification JN $\longmapsto$ JN'' of Theorem 5.3.1, JN'' evolves into equilibrium if JN does. □

An interpretation of Equations (a.3), (b) and (c) in Theorem 5.3.1 results in

Corollary 5.3.2

Let the partition Π of Theorem 5.3.1 induce a partition Π' of L into singletone blocks, i.e. $\{\ell\}\varepsilon\Pi'$ iff $C_\ell \varepsilon \Pi$. Then, the L-simplification of Theorem 5.3.1 leaves the following quantities unchanged:

a) the switching probabilities between blocks of Π and the respective blocks in Π';

b) the expected equilibrium traffic rates through blocks of Π and the respective blocks in Π';

c) the expected equilibrium traffic rates among blocks of Π and the respective blocks in Π'. □

The next theorem exemplifies how a simplification procedure (see Appendix B) may simplify the investigation of complex simplifications. Here, a L-simplification is decomposed into two simplification stages: a A-simplification followed by a L-simplification (see Figure 5.3.1).

Theorem 5.3.2

Let JN = (M,α,σ,P) be an open Jackson network which possesses a state equilibrium. Let $\Pi = \{C_\ell\}_{\ell \varepsilon L}$ be a strongly lumpable partition of P such that

a) for each $\ell \varepsilon L$,

(a.1) $p(k,\bar{C}_\ell) > 0$, $\forall k \varepsilon C_\ell$

(a.2) $\sigma_k = \text{const.}$, $\forall k \varepsilon C_\ell$

(a.3) $\delta_k = \text{const.}$, $\forall k \varepsilon C_\ell$

(a.4) $\delta(\bar{C}_\ell,k) = \delta_k p(k,\bar{C}_\ell)$. , $\forall k \varepsilon C_\ell$.

Next, let JN'' = $(L,\alpha'',\sigma'',P'')$ be obtained from JN by a L-simplification with respect to Π (see Figure 5.3.1) such that

b) for every $\ell,n \varepsilon L$,

(b.1) $\alpha''_\ell \triangleq \sum_{k \varepsilon C_\ell} \alpha_k$

(b.2) $\sigma''_\ell \triangleq \sum_{k \varepsilon C_\ell} \sigma_k p(k,\bar{C}_\ell)$

(b.3) $p''_{\ell n} \triangleq \begin{cases} 0 \ , & \text{if } \ell = n \\ \dfrac{p(C_\ell,C_n)}{p(k,\bar{C}_\ell)} \text{ for any } k \varepsilon C_\ell, & \text{if } \ell \neq n \ . \end{cases}$

Then, the L-simplification JN $\longmapsto$ JN'' above possesses a state equilibrium, and it further gives rise to the following relations between behavioral frames of JN and JN'':

c) for any blocks $C_\ell,C_n \varepsilon \Pi$ in JN and the respective nodes $\ell,n \varepsilon L$ in JN'', we have that

(c.1) the equilibrium line distribution of any node $k \varepsilon C_\ell$ equals that of node ℓ;

(c.2) the switching probability from C_ℓ to C_n and from ℓ to n are related by $p''_{\ell n} \cdot p(k,\bar{C}_\ell) = (1-\delta_{\ell n}) \cdot p(C_\ell,C_n)$, where $\delta_{\ell n}$ is Kronecker's delta and k is any node in C_ℓ;

(c.3) the expected equilibrium traffic rates from C_ℓ to C_n and from ℓ to n are related by $\delta''(\ell,n) = (1-\delta_{\ell n}) \cdot \delta(C_\ell,C_n)$;

(c.4) the ratio of expected total service time in C_ℓ to expected

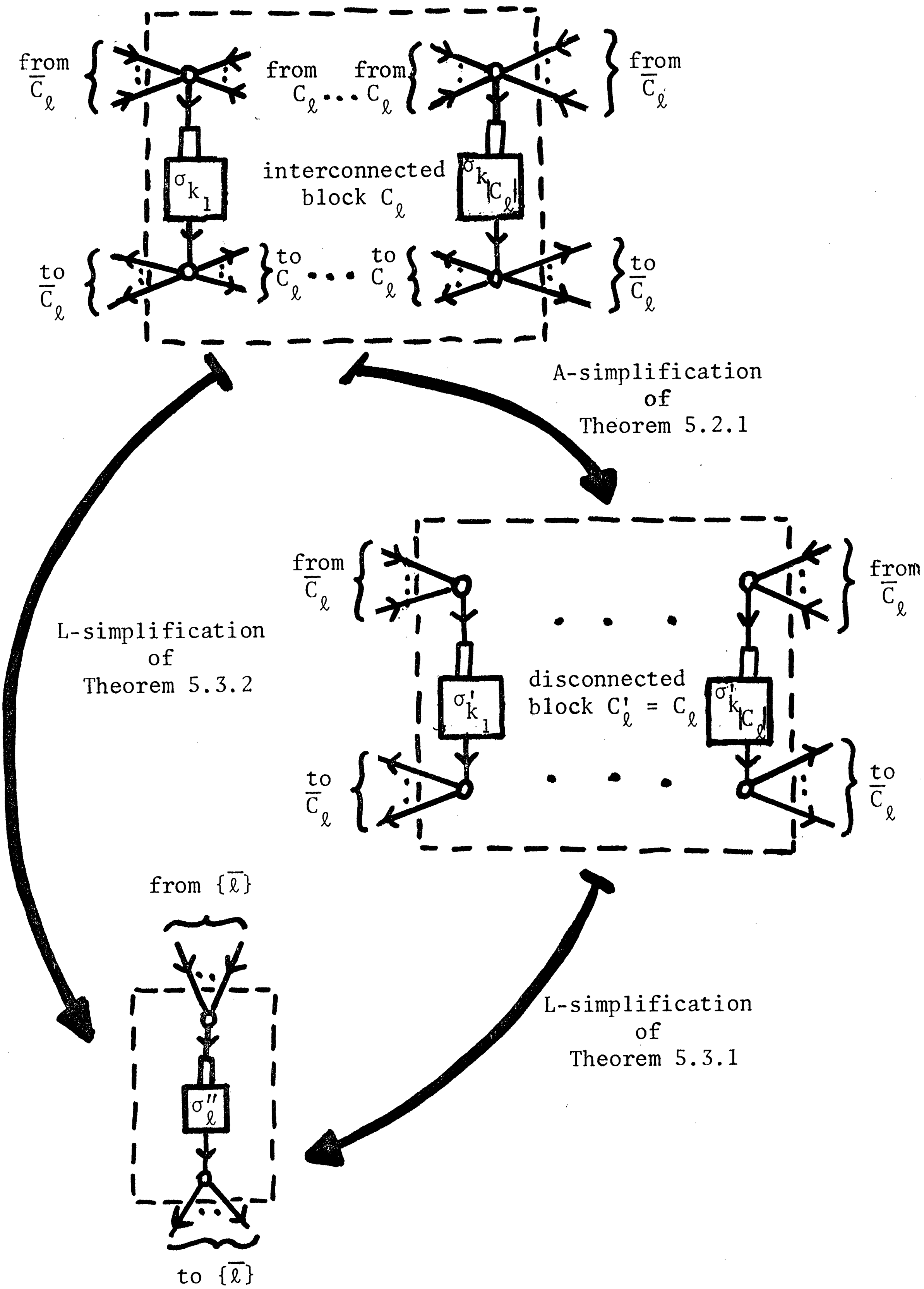

Figure 5.3.1: A Decomposition of the L-Simplification of Theorem 5.3.2 When Operating on a Typical Block C_ℓ.

service time at ℓ is $|C_\ell| : 1$;

(c.5) the ratio of the expected total number of customers in C_ℓ in equilibrium, to the expected line length at node ℓ in equilibrium is $|C_\ell| : 1$.

Proof

By strong lumpability of P with respect to Π, we have

(1) $p(k,C_n) = \text{const.} \quad , \quad \forall k \in C_\ell$

for any $\ell, n \in L$. It follows that for each $\ell \in L$,

(2) $p(k,\bar{C}_\ell) = \text{const.} \quad , \quad \forall k \in C_\ell$

Combining (2), (a.3) and (a.4) yields for each $\ell \in L$,

(3) $\delta(\bar{C}_\ell,k) = \delta_k p(k,\bar{C}_\ell) = \text{const.} \quad , \quad \forall k \in C_\ell$.

We now show that the L-simplification JN $\longmapsto$ JN" can be decomposed into two simplification stages JN $\longmapsto$ JN' $\longmapsto$ JN" (see Figure 5.3.1), where the first one is the A-simplification of Theorem 5.2.1 and the second one is the L-simplification of Theorem 5.3.1.

More specifically, JN $\longmapsto$ JN' is the first stage A-simplification, where JN' = (M,α,σ',P') such that

(4) for every $i,j \in M$,

(4.1) $\sigma_i' \triangleq \sigma_i p(i,\bar{C}_\ell)$ whenever $i \in C_\ell$

(4.2) $$p_{ij}' \triangleq \begin{cases} 0, & \text{if } i \in C_\ell \text{ and } j \in C_\ell \text{ for some } \ell \in L \\ \dfrac{p_{ij}}{p(i,\bar{C}_\ell)}, & \text{if } i \in C_\ell \text{ and } j \notin C_\ell \text{ for some } \ell \in L. \end{cases}$$

Thus, JN $\longmapsto$ JN' is the A-simplification of Theorem 5.2.1, and by virtue of Theorem 5.2.2 and (a.4)

(5) $\delta_i' = \delta(\bar{C}_\ell, i) = \delta_i p(i,\bar{C}_\ell)$ whenever $i \varepsilon C_\ell$.

Next, show that Π is a strongly lumpable partition of P'. For every $k\varepsilon M$, $C_n \varepsilon \Pi$,

$$(6) \quad p'(k,C_n) \triangleq \sum_{j\varepsilon C_n} p'_{kj} = \begin{cases} 0, & \text{if } k\varepsilon C_n \\ \sum_{j\varepsilon C_n} \dfrac{p_{kj}}{p(k,\bar{C}_\ell)}, & \text{if } k\varepsilon C_\ell,\ \ell \neq n \end{cases} =$$

$$\begin{cases} 0, & \text{if } k\varepsilon C_n \\ \dfrac{p(k,C_n)}{p(k,\bar{C}_\ell)}, & \text{if } k\varepsilon C_\ell,\ \ell \neq n \end{cases} = \begin{cases} 0, & \text{if } k\varepsilon C_n \\ \dfrac{p(C_\ell,C_n)}{p(k,\bar{C}_\ell)}, & \text{if } k\varepsilon C_\ell,\ \ell \neq n. \end{cases}$$

From (6) it is seen that for every $\ell, n\varepsilon L$

(7) $p'(k,C_n) = \text{const.}, \quad \forall k\varepsilon C_\ell$

i.e. Π is a lumpable partition of P'. Consequently, we may proceed to perform the second stage L-simplification JN' $\longmapsto$ JN", where JN" = $(L,\alpha'',\sigma'',P'')$ such that

(8) for every $\ell, n\varepsilon L$,

$$(8.1) \quad \alpha''_\ell \triangleq \sum_{k\varepsilon C_\ell} \alpha'_k = \sum_{k\varepsilon C_\ell} \alpha_k$$

$$(8.2) \quad \sigma''_\ell \triangleq \sum_{k\varepsilon C_\ell} \sigma'_k = \sum_{k\varepsilon C_\ell} \sigma_k p(k,\bar{C}_\ell)$$

$$(8.3) \quad p''_{\ell n} \triangleq p'(C_\ell,C_n) = \begin{cases} 0, & \text{if } \ell = n \\ \dfrac{p(C_\ell,C_n)}{p(k,\bar{C}_\ell)} \text{ for any } k\varepsilon C_\ell, & \text{if } \ell \neq n \end{cases}$$

Thus, JN' $\longmapsto$ JN" is the L-simplification of Theorem 5.3.1. In view of Theorem 5.3.1 and (5)

$$(9) \quad \delta''_\ell = \sum_{k\varepsilon C_\ell} \delta'_k = \sum_{k\varepsilon C_\ell} \delta(\bar{C}_\ell,k) = \sum_{k\varepsilon C_\ell} \delta_k p(k,\bar{C}_\ell), \quad \forall \ell\varepsilon L.$$

A comparison of (8.1) - (8.3) with (b.1) - (b.3) shows that the simplification procedure resulted correctly in JN".

Since JN was assumed to possess a state equilibrium, it follows from Corollary 5.2.1 that JN' possesses a state equilibrium, as JN $\longmapsto$ JN' was defined to be the A-simplification of Theorem 5.2.1. It now follows from Corollary 5.3.1 that JN" also possesses a state equilibrium, as JN' $\longmapsto$ JN" was defined to be the L-simplification of Theorem 5.3.1.

We now proceed to prove assertions (c.1) - (c.5).

Proof of (c.1):

From (a.2) and (a.3) it follows that for each $\ell\varepsilon L$,

$$(10)\quad \rho_k = \frac{\delta_k}{\sigma_k} = \text{const.} \quad , \quad \forall k\varepsilon C_\ell .$$

Moreover, in view of (9), (a.3) and (2)

$$(11)\quad \delta''_\ell = |C_\ell|\delta_k p(k,\bar{C}_\ell) \quad \text{for any } k\varepsilon C_\ell, \quad \forall \ell\varepsilon L .$$

Finally, in view of (8.2), (a.2) and (2)

$$(12)\quad \sigma''_\ell = |C_\ell|\sigma_k p(k,\bar{C}_\ell) \quad \text{for any } k\varepsilon C_\ell, \quad \forall \ell\varepsilon L .$$

Hence, from (10), (11) and (12)

$$(13)\quad \rho_k \triangleq \frac{\delta_k}{\sigma_k} = \frac{|C_\ell|\delta_k p(k,\bar{C}_\ell)}{|C_\ell|\sigma_k p(k,\bar{C}_\ell)} = \frac{\delta''_\ell}{\sigma''_\ell} \triangleq \rho''_\ell \quad \text{for any } k\varepsilon C_\ell, \quad \forall \ell\varepsilon L .$$

Assertion (c.1) now follows, since the equilibrium line distributions are determined by the ρ parameters.

Proof of (c.2):

Follows directly from (8.3).

Proof of (c.3):

For any $C_\ell, C_n \varepsilon \Pi$,

$$(14)\quad \delta'(C_\ell,C_n) \triangleq \sum_{k\varepsilon C_\ell}\sum_{j\varepsilon C_n}\delta'_k p'_{kj} = \begin{cases} 0\ , & \text{if} \quad \ell = n \\ \sum\limits_{k\varepsilon C_\ell}\sum\limits_{j\varepsilon C_n}\delta_k p(k,\bar{C}_\ell)\dfrac{p_{kj}}{p(k,\bar{C}_\ell)}\ , & \text{if}\ \ell \neq n \end{cases} =$$

$$\begin{cases} 0\ , & \text{if} \quad \ell = n \\ \sum\limits_{k\varepsilon C_\ell}\sum\limits_{j\varepsilon C_n}\delta_k p_{kj}\ , & \text{if} \quad \ell \neq n \end{cases} = \begin{cases} 0\ , & \text{if}\ \ell = n \\ \delta(C_\ell,C_n)\ , & \text{if} \quad \ell \neq n \end{cases}$$

by virtue of (5) and (4.2).

Hence, for any $C_\ell, C_n \varepsilon \Pi$,

$$(15)\quad \delta'(C_\ell,C_n) = (1-\delta_{\ell n})\cdot\delta(C_\ell,C_n)$$

But by Theorem 5.3.1

$$(16)\quad \delta'(C_\ell,C_n) = \delta''(\ell,n)\ , \quad \forall \ell,n\varepsilon L$$

whence assertion (c.3) follows.

Proof of (c.4):

By Theorem 5.2.4 and in view of (a.2) and (2), the expected total service time in each $C_\ell \varepsilon \Pi$ is $\dfrac{1}{\sigma_k p(k,\bar{C}_\ell)}$, where k is any node in C_ℓ.

From (12) we conclude that the expected service time at the respective node $\ell \varepsilon L$ is $\dfrac{1}{|C_\ell|\sigma_k p(k,\bar{C}_\ell)}$, where k is any node in C_ℓ.

The requisite ratio is, thus, seen to be $|C_\ell| : 1$.

Proof of (c.5):

Since the equilibrium distribution of the line size at node i is geometrical with parameter ρ_i, the respective expectation is $\dfrac{\rho_i}{1-\rho_i}$.

Consequently, the expected total number of customers, in equilibrium, in each block $C_\ell \varepsilon \Pi$ is

$$(17) \quad \sum_{k\varepsilon C_\ell} \frac{\rho_k}{1-\rho_k} = \frac{|C_\ell|\rho_k}{1-\rho_k} \qquad \text{for any } k\varepsilon C_\ell$$

due to (10).

We already know from (13) that $\rho''_\ell = \rho_k$ where k is any node in C_ℓ. Hence, the expected line length, in equilibrium, at each node $\ell\varepsilon L$ is

$$(18) \quad \frac{\rho''_\ell}{1-\rho''_\ell} = \frac{\rho_k}{1-\rho_k} \quad , \qquad \text{for any } k\varepsilon C_\ell \ .$$

From (17) and (18) it follows that the requisite ratio is $|C_\ell| : 1$. □

We note in passing that Theorem 5.2.3 and Corollary 5.2.2 may be used to give structural conditions that imply the behavioral conditions (a.3) and (a.4) of Theorem 5.3.2.

In conclusion, we remark that Theorem 5.3.2 illustrates a heuristic principle involving simplifications of the lumping type. In such situations, a network of components is partitioned into blocks and then each of them is lumped into a simpler component.

Heuristically speaking, we can expect a considerable preservation of behavioral frames, when the block in the base model consists of components which are similar or uniform in some sense. Consonant with this view, the base network of Theorem 5.3.2 was partitioned into blocks with "similar" components, and then each block was lumped into a single node.

The resulting lumped network turned out to be a scaled down version of the base network with a variety of remarkably related operating characteristics.

5.4 Simulation Complexities of Jackson Networks

A *simulation complexity* is a measure of computer resources required to run a computer simulation of a model. In practice, the model may be run for some interval of simulation time until a sufficient number of customers are simulated, or until some other stopping criterion is met. A good simulation complexity should not only allow a user to compare the simulation costs of various models but should also aid him in obtaining a reasonable estimate of computer resources, e.g. total CPU time and average memory space needed for the simulation. Other measures such as maximum requisite memory, time-space product, etc. are of considerable interest in estimating simulation cost, though hard to compute.

In this section, we shall discuss some *time complexities* and *space complexities* as described above. In what follows we have in mind a discrete simulation language which is of the transaction flow type (e.g. GPSS; see [Sch1]), or of the event scanning type (e.g. GASP; see [PK1]). Such a discrete simulation language makes it easy for a user to simulate a discrete event system, say a DEVS.

The language software handles the queuing up of future sequential state transitions (jumps) in an ordered list (called the *future event list*) according to their time of occurrence. It then processes the jumps by computing the new sequential state, again in order of occurrence. In this context, the jumps alluded to above, are called "events" (not to be confused with probabilistic events).

In queuing-theoretic context, the probabilistic analogue of a system-theoretic event is, loosely speaking, a discontinuity in the sample functions of the state process $\{Q(t)\}_{t \geq 0}$. To avoid ambiguities we shall refer to system-theoretic events as *simulation events*.

Since a particular computer simulation pertains to a particular though "random" queuing history, it is often reasonable, as an intermediate step, to define first random time complexities and random storage complexities. These random complexities should be random variables whose realizations measure the cost of CPU effort and memory space required by the respective sample simulation run.

The resulting time and space complexities would then be defined as deterministic quantities in terms of the respective expectations, time averages, etc. Throughout the impending discussion, we shall assume that $JN = (M,\alpha,\sigma,P)$ is an underlying Jackson network with which those complexity measures are associated.

We begin with a discussion of simulation time complexities (denoted C_T). Consider the time complexity

(1) $C_T^{(1)} \triangleq |M| = m$.

$C_T^{(1)}$ is a measure of network size, and it reflects on the rate of simulation events in the network, since every node is a location of "activity". (The number of arcs is irrelevant in this respect, because only arrivals and departures at nodes generate such events.) $C_T^{(1)}$ is a crude measure because it does not take note of the probabilistic topology of JN.

$C_T^{(2)}$ is similarly crude;

(2) $C_T^{(2)} \triangleq ||\alpha|| + ||\sigma|| = \sum_{i=1}^{m}\alpha_i + \sum_{i=1}^{m}\sigma_i$.

It is defined as the sum of expected arrival rates and potential departure rates in the network. $C_T^{(2)}$ has the additional disadvantage of being dependent on the time units, in which α and σ are measured; e.g. if the time unit changes from seconds to minutes, $C_T^{(2)}$ also changes.

A better time complexity is provided by network "closedness."

Loosely speaking, network "closedness" measures how "difficult" it is to leave the network. Its validity as a time complexity stems from the fact that the harder it is for a customer to leave the network, the more simulation events are going to be induced by him.

Denoting $q_i \triangleq 1 - \sum_{j=1}^{m} p_{ij}$, we define for open networks

(3) $C_T^{(3)} \triangleq (\frac{1}{m} \sum_{i=1}^{m} q_i)^{-1}$.

Observe that $0 < C_T^{(3)} \leq 1$; thus, the larger $C_T^{(3)}$, the larger the number of visits paid by a customer to nodes in the network. For closed networks $C_T^{(3)} = \infty$ as it should be.

$C_T^{(3)}$ is defined as the "average closedness", but it does not take note of how likely a customer is to arrive at node i. Thus, if we add nodes which are never reached by any customer, $C_T^{(3)}$ will still be affected.

To remedy this deficiency, consider the number of visits of an incoming customer at node i, during his stay in an open Jackson network. Denoting this random variable by K_i, we define

(4) $C_T^{(4)} \triangleq E(\sum_{i=1}^{m} K_i)$.

$C_T^{(4)}$ is the expected total number of visits to nodes made by an incoming customer during his stay in the network. This time complexity comes closer to CPU effort than any of the above. $C_T^{(4)}$ has the additional advantage of being computable for open networks (see Remark 4.6.2)

(5) $C_T^{(4)} = \frac{||\delta||}{||\alpha||}$

which also shows that $C_T^{(4)}$ takes full account of the network topology. Recall that δ is the traffic solution of the network, and that in equilibrium it coincides with the expected rate of service completions at network nodes.

Thus, (5) gives rise to another interpretation; namely, that $C_T^{(4)}$ is the equilibrium ratio of expected service completion rate to expected exogenous arrival rate. This ratio can be viewed as the internal load ($= ||\delta||$) induced by a unit of external load ($= \alpha$) in equilibrium, and it makes sense to compare these quantities for networks with fixed external load. However, $C_T^{(4)}$ ignores simulation events induced by customers that were initially in the network, as well as those whose stay in the network has not been completed.

$C_T^{(4)}$ has yet another interpretation as a measure of "closedness". To see this note that $\sum_{i=1}^{m} \alpha_i = \sum_{i=1}^{m} \delta_i q_i$ (see Theorem 4.4.4) whence

$$(6) \quad \frac{1}{\max\{q_i:\ 1 \le i \le m\}} \le C_T^{(4)} \le \frac{1}{\min\{q_i:\ 1 \le i \le m\}} .$$

Thus $C_T^{(4)}$ is approximately equal to $C_T^{(3)}$, and this approximation becomes exact as the q_i approach a common value. Indeed, it can be shown that $C_T^{(4)} \longrightarrow \infty$ as $C_T^{(3)} \longrightarrow 0$.

$C_T^{(4)}$ is suited for situations when the cost per simulated customer is of interest. For instance, one may wish to simulate a certain number of customers, so as to obtain reasonably reliable statistics. Typically in this case, the number of customers to be simulated is fixed, while the simulation time interval is unspecified.

A different situation arises when the simulation interval is fixed and the number of customers is unspecified. In this case, the total number of simulation events occurring in a simulation interval $[0,t]$ reflects on the requisite CPU effort. Observe that a simulation event occurs iff there was an exogenous arrival or a service completion at some node. This observation gives rise to

$$(7) \quad C_T^{(5)}(t) \triangleq \frac{1}{||\alpha|| t} E(\sum_{i=1}^{m} D_i(t))$$

where $D_i(t)$ is the total number of service completions at node i in the simulation interval [0,t]. $C_T^{(5)}$ is interpreted as the total internal load per unit external load in the simulation interval [0,t].

$C_T^{(5)}$ differs from $C_T^{(4)}$ in the way it accounts for customers initially in the system and those who don't leave it. It has the representation (see Theorem 4.2.1)

$$(8) \quad C_T^{(5)}(t) = \frac{1}{||\alpha|| t} \sum_{i=1}^{m} \sigma_i \cdot \int_0^t \Pr(Q_i(x)>0)\,dx$$

from which we conclude that $C_T^{(5)}(t) \leq \frac{||\sigma||}{||\alpha||}$ for any $t \geq 0$. For open Jackson networks in equilibrium, $C_T^{(5)}$ becomes

$$(9) \quad C_T^{(5)}(t) \equiv \frac{||\delta||}{||\alpha||} .$$

It is interesting to note that in this case $C_T^{(5)}$ and $C_T^{(4)}$ coincide.

$C_T^{(5)}$ may be used to estimate the total number of service completions that occur during a simulation whose stopping rule is the arrival of N exogenous customers. We define

$$(10) \quad C_T^{(6)}(N) \triangleq E(\sum_{i=1}^{m} D_i(\frac{N}{||\alpha||}))$$

which for open networks becomes, in equilibrium,

$$(11) \quad C_T^{(6)}(N) = N \frac{||\delta||}{||\alpha||} .$$

Notice that $\frac{N}{||\alpha||}$ is the expected arrival time of the N-th customer.

Thus, $C_T^{(6)}$ is an estimate of the number of simulation events in the simulation interval $[0, \frac{N}{||\alpha||}]$.

However, $C_T^{(6)}$ is not the exact expectation of the total service completions required to simulate the network until the arrival of the N-th customer, because this time interval and the total number of service completions occurring in it are apparently correlated.

We now consider time complexities that come closer to measuring actual CPU time.

Let T be the processing time of a simulation event. T is a random variable whose randomness is mainly due to the variable length of the future event list at the time of processing. In practice, T might be a constant plus a term proportional to the length of the future event list.

Next, refine $C_T^{(7)}$ by

$$(12) \quad C_T^{(7)} \triangleq E(\sum_{i=1}^{m} K_i T) \ .$$

In other words, $C_T^{(7)}$ is the expected CPU time required for simulating an incoming customer. Unfortunately, it is not readily computable. Even if E(T) is known, we still need to know how K_i and T are correlated. When zero correlation can be assumed, $C_T^{(7)}$ becomes for open networks

$$(13) \quad C_T^{(7)} = E(\sum_{i=1}^{m} K_i) E(T) = \frac{||\delta||}{||\alpha||} E(T) \ .$$

A similar situation arises when an attempt is made to refine $C_T^{(5)}$ and $C_T^{(6)}$ by defining respectively

$$(14) \quad C_T^{(8)}(t) \triangleq \frac{1}{||\alpha|| t} E(\sum_{i=1}^{m} D_i(t) T)$$

and

$$(15) \quad C_T^{(9)}(N) \triangleq E(\sum_{i=1}^{m} D_i(\frac{N}{||\alpha||}) T) \ .$$

$C_T^{(8)}(t)$ is the expected total CPU time required to process service completions (total internal load) in the simulation interval [0,t] per unit external load in the same interval. Likewise, $C_T^{(9)}(N)$ estimates the expected total CPU time required to simulate the network until the N-th exogenous customer arrives.

We now proceed to discuss simulation space complexities (denoted C_S). Space complexities have two components: static and dynamic.

Static space complexities arise from the memory storage required to represent a queuing network topology in a computer, not including waiting lines. Thus, these complexities are essentially a measure of network size in terms of nodes and arcs. For example,

$$(16) \quad C_S^{(1)} \triangleq s_N|M| + \sum_{(i,j):p_{ij}>0} s_A$$

where s_N and s_A denote memory storage required to represent a node and an arc respectively in the computer, exclusive of waiting lines.

Dynamic space complexities, on the other hand, reflect the total length of waiting lines in the network during a simulation. For example,

$$(17) \quad C_S^{(2)}(t) \triangleq E(\sup\{\sum_{i=1}^{m} Q_i(\tau):\ 0\le\tau\le t\}) .$$

$C_S^{(2)}$ estimates the maximal total length of queues in the simulation interval [0,t]. Unfortunately, it is difficult to compute. Consider instead the smaller measures

$$(18) \quad C_S^{(3)}(t) \triangleq \sup\{E(\sum_{i=1}^{m} Q_i(\tau)):\ 0\le\tau\le t\}$$

and

$$(19) \quad C_S^{(4)}(t) \triangleq \frac{1}{t}\int_0^t E(\sum_{i=1}^{m} Q_i(\tau))d\tau .$$

which are more amenable to computation.

For open Jackson networks in equilibrium, both $C_S^{(3)}$ and $C_S^{(4)}$ reduce to the same time independent function

$$(20) \quad C_S^{(3)}(t) = C_S^{(4)}(t) \equiv E(\sum_{i=1}^{m} Q_i(t)) = \sum_{i=1}^{m}\frac{\rho_i}{1-\rho_i} .$$

where $\rho_i = \frac{\delta_i}{\sigma_i}$ (see Theorem 4.5.1).

As the network approaches instability (viz. $\rho_i \longrightarrow 1$ for some i), $C_S^{(3)} \longrightarrow \infty$ as $\frac{1}{1-\rho_i}$.

Notice that the variance of the instantaneous total length of queues $\sum_{i=1}^{m} Q_i(t)$ in equilibrium is

$$(21) \quad V(\sum_{i=1}^{m} Q_i(t)) \equiv \sum_{i=1}^{m} \frac{\rho_i}{(1-\rho_i)^2}$$

due to the independence of the individual queues (see Theorem 4.5.1). Consequently, as the network approaches instability, $V(\sum_{i=1}^{m} Q_i(t)) \longrightarrow \infty$ as $\frac{1}{(1-\rho_i)^2}$, and our confidence in $C_S^{(3)}$ and $C_S^{(4)}$ as dynamic storage estimates decreases very quickly.

$C_S^{(3)}$ and $S_S^{(4)}$ were defined as functions of the simulation interval. When the simulation requires that sufficient number of customers be simulated, the counterparts of $C_S^{(3)}$ and $C_S^{(4)}$ are respectively

$$(22) \quad C_S^{(5)}(N) \triangleq \sup \{E(\sum_{i=1}^{m} Q_i(\tau)) : 0 \leq \tau < \frac{N}{||\alpha||}\}$$

and

$$(23) \quad C_S^{(6)}(N) \triangleq \frac{N}{||\alpha||} \cdot \int_0^{\frac{N}{||\alpha||}} E(\sum_{i=1}^{m} Q_i(\tau))d\tau \quad .$$

In equilibrium, $C_S^{(5)}$ and $C_S^{(6)}$ reduce to the same constant function.

We conclude this chapter by comparing the effect of some of the simplifications in Sections 5.1 - 5.3 on some of the simulation complexities of this section. Figure 5.4.1 summarizes these effects. It employs the following notation.

A simulation complexity can be non-increasing (denoted by ↓), or unchanged (denoted by =). A question mark indicates that the behavior

	F-simplification of Theorem 5.1.5	A-simplification of Theorem 5.2.1	L-simplification of Theorem 5.3.1	L-simplification of Theorem 5.3.2
$C_T^{(1)}$	=	=	↓	↓
$C_T^{(2)}$	↓	↓	=	↓
$C_T^{(3)}$	↓	↓	?	?
$C_T^{(4)}$	↓	↓	=	↓
$C_T^{(5)}$	↓	$\downarrow^{o}$	$=^{o}$	$\downarrow^{o}$
$C_T^{(6)}$	↓	$\downarrow^{o}$	$=^{o}$	$\downarrow^{o}$
$C_S^{(1)}$	↓	↓	↓	↓
$C_S^{(2)}$	=	?	?	?
$C_S^{(3)}$	=	$=^{o}$	?	$\downarrow^{o}$
$C_S^{(4)}$	=	$=^{o}$	?	$\downarrow^{o}$
$C_S^{(5)}$	=	$=^{o}$	?	$\downarrow^{o}$
$C_S^{(6)}$	=	$=^{o}$	?	$\downarrow^{o}$

Figure 5.4.1: A Comparison of the Effect of Various Simplifications of Jackson Networks on Some Simulation Complexities.

is unknown or mixed (depending on simplification parameters). An appended circle means that both the base model and the lumped model of the indicated simplification are assumed to be in state equilibrium.

The results in Figure 5.4.1 follow from the theorems alluded to in the headings of its columns and from the discussion in this section. It should be born in mind that the results presuppose that the conditions of those theorems hold for the simplifications under consideration. For those complexities which are functions of t or N, the comparison is valid for any fixed argument.

CHAPTER 6

CONCLUSION

6.0 Summary

Two lines of research have been pursued. The first line of research concerned analysis and simplifications of discrete event systems. The logic of deterministic discrete event systems was studied, when formalized by DEVS-related concepts. A hierarchy of morphic relations was developed in accordance with the conceptual framework of Appendices A and B. An extension of this framework to stochastic discrete event systems was proposed. In this approach system-theoretic and statistical-theoretic aspects are combined via representation in coordinate probability space. A hierarchy of morphic relations for stochastic systems was then developed in terms of measure preserving transformations. Finally, we derive a methodology that provides sufficient conditions which ensure preservation of behavioral frames under point simplifications.

The second line of research concerned analysis and simplifications of Jackson queuing networks with single server nodes. In studying their operating characteristics, especially state equilibrium, a number of theoretical gaps in the extant theory have been closed. Results on open and closed Jackson networks were unified as results for mixed networks. The main result is derived in a study of equilibrium traffic processes on arcs, as an extension of Burke's Theorem (see [B1]) from M/M/s queues to Jackson networks with single server node. This result has applications to decompositions of Jackson networks.

Finally, three types of simplifications of Jackson networks are exemplified, as well as their effect on a number of simulation complexities associated with them.

6.1 Further Research

Several lines of further research emerge from these studies. As regards the area of discrete event systems, the DEVN (discrete event network specification) concept warrants special attention.

The ability to identify components in a DEVS (so that it can be represented as a DEVN) entails a conceptual simplification and better understanding of its operation. A hierarchy of DEVN morphisms, where each morphism can be decomposed into local DEVS morphisms between components, is of interest for similar reasons. This line of study has potential applications to modeling of discrete event systems.

In the study of Jackson networks, the lack of customer-oriented operating characteristics, such as waiting and transit times, is a glaring omission. Little is known about these important problems (see [R1] for a survey of related problems). We remark that their solution is necessary for attaining a balanced set of operating characteristics.

More research is also needed to elucidate the nature of traffic processes on non-exit arcs. An immediate problem is to prove or disprove the conjecture that such arcs cannot have Poisson or even renewal traffic on them (excluding the trivial case $p_{ii}=1$). This line of research has potential applications to decompositions of Jackson networks.

Finally, an attempt should be made to generalize Jackson networks to more realistic queuing network models. The main directions of generalization that have recently emerged are: general servers, general switches and multiple classes of customers.

In addition, we suggest that the simplification methodology (set forth in Theorems 3.1.1 and 3.2.1 in Chapter 3) can be applied to simplifications of the generalized queuing networks alluded to above. As an example, we claim that it readily provides a proof for the following conjecture: in any queuing network, the idle-busy period process is invariant (in distribution) under queuing disciplines such as first come first served, last come first served, time sharing and preemptive resume.

APPENDIX A

SOME BASIC SYSTEM THEORY

A.0 Introduction

This appendix provides some system-theoretic background for readers who are not familiar with the terminology and mathematically oriented approach to System Theory. The entire appendix is a digest of the relevant sections in Part 2 (Chapters IX and X) of [Z1], with rather minor modifications. The latter merely consist of slightly altered conventions and terminology that better conform to the goals of this thesis.

The appendix is intended to be an introduction to Chapter 1. It also outlines the conceptual framework into which Chapters 1, 2 and 3 are fitted.

A.1 Mathematical Systems

The Mathematical System concept is a fundamental formal tool for description and analysis of most real life systems. The central conception is that the system evolves in time through a succession of states, under some external input. It produces an output according to its current state. The following is a standard formal definition of a mathematical system.

Definition A.1.1

A *Mathematical System* (also known as an *Input-Output System*, or *I/O System*) is a structure

$$S = \langle T, X, \Omega, Q, Y, \delta, \lambda \rangle$$

where

T is the *time base set*

X is the *input value set*

Ω is the *input segment set*

Q is the *state set*

Y is the *output value set*

δ is the *state transition function*

λ is the *output function*

subject to the following constraints:

a) T is a well-ordered Abelian group

b) The input segments in Ω are functions $\omega = \omega_{(t_1,t_2]}$ where

$\omega : (t_1, t_2] \rightarrow X \quad t_1, t_2 \in T.$

c) Ω is closed under composition (juxtaposition) of contiguous input segments, viz.

$$\omega_{(t_1,t_2]}, \omega'_{(t_2,t_3]} \in \Omega \quad \Rightarrow \quad \omega_{(t_1,t_2]} \odot \omega'_{(t_2,t_3]} \in \Omega$$

where the function $\omega_{(t_1,t_2]} \odot \omega'_{(t_2,t_3]} = \omega''_{(t_1,t_3]}$ is defined by

$$\omega''_{(t_1,t_3]}(t) \triangleq \begin{cases} \omega_{(t_1,t_2]}(t), & \text{if } t_1 < t \leq t_2 \\ \omega_{(t_2,t_3]}(t), & \text{if } t_2 < t \leq t_3 \end{cases}$$

d) δ is a function $\delta : Q \times \Omega \rightarrow Q$ satisfying the following *composition property*:

$\forall \omega_{(t_1,t_2]}, \omega'_{(t_2,t_3]} \in \Omega, \forall q \in Q,$

$$\delta(q, \omega_{(t_1,t_2]} \odot \omega'_{(t_2,t_3]}) = \delta(\delta(q, \omega_{(t_1,t_2]}), \omega'_{(t_2,t_3]}) .$$

e) λ is a function $\lambda : Q \rightarrow Y$.

□

An important operation on the input segments is described by the *translation operator* TRANS_τ where $\text{TRANS}_\tau(\omega) = \hat{\omega}$ such that if $\omega = \omega_{(t_1,t_2]}$ then $\hat{\omega} = \hat{\omega}_{(t_1+\tau,t_2+\tau]}$ is defined by $\hat{\omega}(t) \triangleq \omega(t - \tau)$.

If Ω is closed under translation, we may extend the composition operation from contiguous input segments to arbitrary input segments $\omega = \omega_{(t_1,t_2]}$ and $\omega' = \omega'_{(t_3,t_4]}$ by

$\omega \odot \omega' = \omega''$ where $\omega'' = \omega''_{(t_1,t_2+t_4-t_3]}$ is defined by

$$\omega''(t) \triangleq \begin{cases} \omega(t), \text{ if } t_1 < t \leq t_2 \\ (\text{TRANS}_{t_2-t_3}(\omega'))(t), \text{ if } t_2 < t \leq t_2 + t_4 - t_3 \end{cases}$$

We now define an important class of systems.

<u>Definition A.1.2</u>

A Mathematical system $S = \langle T,X,\Omega,Q,Y,\delta,\lambda \rangle$ is *time invariant* if

a) Ω is closed under translation viz.

$\omega \in \Omega \Rightarrow \text{TRANS}_\tau(\omega) \in \Omega$, for any $\tau \geq 0$.

b) δ is time invariant viz.

$\forall q \in Q, \forall \omega \in \Omega, \forall \tau \geq 0, \delta(q,\omega) = \delta(q,\text{TRANS}_\tau(\omega))$.

□

Notice that for time invariant systems, it suffices to consider only those input segments that start at the origin.

Our interpretation of the system concept runs as follows.

A system is conceived of as having two elements: an internal element which we call "structure", and an external element we call "behavior". The term "structure" refers to the state space and the state transition function δ. Pictorially, a system is viewed as some black box which undergoes internal changes when stimulated by an input segment. The internal change (the transition function) sends the system into a new state as a function of the initial state and the input segment only. Moreover, this internal state transition is deterministic. On the other hand, "behavior" refers to the external and observable manifestations of the internal processes of state transitions. Pictorially speaking, a behavioral aspect is recorded by inserting a particular probe into our "black box" which can measure a certain aspect of the system's internal state.

Consonant with these views, we introduce the following definitions.

Definition A.1.3

A *mathematical state-system* is a mathematical system $S = \langle T,X,\Omega,Q,\cdot,\delta,\cdot\rangle$ with unspecified output value set Y and output function λ.

□

Definition A.1.4

A *behavioral frame* of a mathematical state-system $S = \langle T,X,\Omega,Q,\cdot,\delta,\cdot\rangle$ is a structure $\Psi = \langle Y,\lambda\rangle$ where the symbols in the angular brackets have the same meaning and constraints as in Definition A.1.1.

□

Our definition of a behavioral frame is a simplified version of the concept of experimental frame in a modeling context (see [Z2] and [Z1] Ch. II). In this context the term experimental frame is used to capture the observational limitations imposed by reality on the modeler. Here, however, we deal with an idealized situation and both terms may be considered coincident. Notice that when we particularize to a certain behavioral frame of a mathematical state-system, we obtain some I/O system of Definition A.1.1.

In other words, a state-system is more fundamental in the sense that it spawns a host of I/O systems which stand in a one-one relation to all possible choices of its behavioral frames.

We regard this collection as an equivalence class induced by a state-system. The symbol $\langle T,X,\Omega,Q,\cdot,\delta,\cdot\rangle$ will also stand for a representative of such a class. This notation will be used in the sequel, whenever we wish to focus on the state structure, whereas the behavioral frame may remain unspecified. Consequently, the dots in the structure $\langle T,X,\Omega,Q,\cdot,\delta,\cdot\rangle$ should be understood as generic variables or "don't care" symbols according to the context. Furthermore, the terms state-system, representative system or simply system will be used interchangeably, whenever the context precludes ambiguities.

Indeed, from a modeling standpoint, "structure" is more fundamental than "behavior". The modeler starts with a set of empirical data ("behavior"), and tries to postulate a model ("structure"), that can account for the data. The process of modeling consists of successive refinements of that model (structure adding) to account for a growing set of empirical data. Theoretically, if the full structure (state-system) is known, then the modeler can predict any system behavior, and modeling is completed. In most cases, this requires infinite time and cannot be accomplished.

Mathematically, the concepts of "structure" and "behavior" are completely dual; "structure" accounts for all "behavior", while given all "behavior" we can always postulate a "structure" to account for it. To clarify our view we point out an analogous situation in the field of formal languages, (see e.g. [AU1] Ch. 2, Sec. 2.1.2). A formal language is the analogue of a mathematical system. It may be dually defined either by a set of transformations (called productions) on some initial strings, or by specifying the set of strings thus generated. Given the set of productions ("structure") we may run (or simulate) the system in various ways to yield various strings ("behavior"). Conversely, the enterprise of modeling becomes that of finding the set of productions that can account for a given set of strings.

Mathematical systems can be described in terms of their state and output trajectories. These trajectories assign a full state and an output value respectively to time points.

<u>Definition A.1.5</u>

Let $S = \langle T,X,\Omega,Q,Y,\delta,\lambda\rangle$ be a mathematical system. Let $q \in Q$ be any state and let $\omega \in \Omega$ be any input segment where $\omega:(t_1,t_2] \to X$. The *trajectory* of (q,ω) is a pair $TRAJ(q,\omega) \triangleq (STRAJ_{q,\omega}, OTRAJ_{q,\omega})$ where

a) $STRAJ_{q,\omega}:[t_1,t_2] \to Q$ is a function defined by

$$STRAJ_{q,\omega}(t) \triangleq \begin{cases} q, \text{ if } t = t_1 \\ \delta(q,\omega|(t_1,t]), \text{ if } t \in (t_1,t] \text{ and } \omega|(t_1,t] \in \Omega \\ \text{undefined, otherwise} \end{cases}$$

and called the *state trajectory* of (q,ω).[†]

b) $OTRAJ_{q,\omega}:(t_1,t_2] \to Y$ is a function defined by

$$OTRAJ_{q,\omega}(t) \triangleq \lambda(STRAJ_{q,\omega}(t))$$

and called the *output trajectory* of (q,ω). □

To wrap up the discussion of mathematical systems we show how to identify subsystems within a mathematical system. Formally, we define

Definition A.1.6

Let $S = \langle T,X,\Omega,Q,Y,\delta,\lambda\rangle$ be a mathematical system. A mathematical system $\hat{S} = \langle T,X,\hat{\Omega},\hat{Q},Y,\hat{\delta},\hat{\lambda}\rangle$ is a *subsystem* of S if

a) $\hat{\Omega} \subset \Omega$

b) $\hat{Q} \subset Q$

c) $\hat{\delta} = \delta|\hat{Q}\times\hat{\Omega}$

d) $\hat{\lambda} = \lambda|\hat{Q}$ □

In other words, a subsystem is a system restricted to a subset of states. Notice that for a subsystem $\hat{S}$ of S to be well-defined, it is necessary and sufficient that $\hat{Q}$ be closed under δ and $\hat{\Omega}$. That is $q \in \hat{Q}$ and $\omega \in \hat{\Omega} \Rightarrow \delta(q,\omega) \in \hat{Q}$.

Next, we turn our attention to relations among mathematical systems and their trajectories. The class of relations, that we consider here is called morphisms. Roughly speaking, morphisms preserve various aspects of system structure and behavior, in a complexity-

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[†]A vertical bar designates restriction of a function domain.

reducing manner.

Definition A.1.7

A *system morphism* from a (mathematical) system $S = \langle T,X,\Omega,Q,Y,\delta,\lambda\rangle$ to a (mathematical) system $S' = \langle T',X',\Omega',Q',Y',\delta',\lambda'\rangle$ is a triple (g,h,k) subject to the following restrictions:

a) g is a function $g:\Omega' \rightarrow \Omega$ called the *input segment encoding function.*

b) h is a surjective (onto) function $h:\overline{Q} \rightarrow Q'$ called the *state decoding function* and $\overline{Q} \subset Q$.

c) k is a surjective function $k:Y \rightarrow Y'$ called the *output decoding function.*

d) $\forall q \in \overline{Q}$, $\forall \omega' \in \Omega'$ we have $h(\delta(q,g(\omega'))) = \delta'(h(q),\omega')$

i.e. *transition function preservation.*

e) $\forall q \in \overline{Q}$ we have $k(\lambda(q)) = \lambda'(h(q))$

i.e. *output function preservation.*

□

The relations among the components of S and S′ are depicted in Figure A.1.1.

The preservation aspects of the functions h and k with respect to δ and λ respectively are described by the commuting diagrams of Figures A.1.2 and A.1.3 respectively.

An important way of viewing morphisms is to regard them as system simplifications (see Appendix B for more details). Informally, a simplification involves reduction of complexity as well as preservation of certain aspects of structure and behavior. Consonant with this view we give the following interpretation.

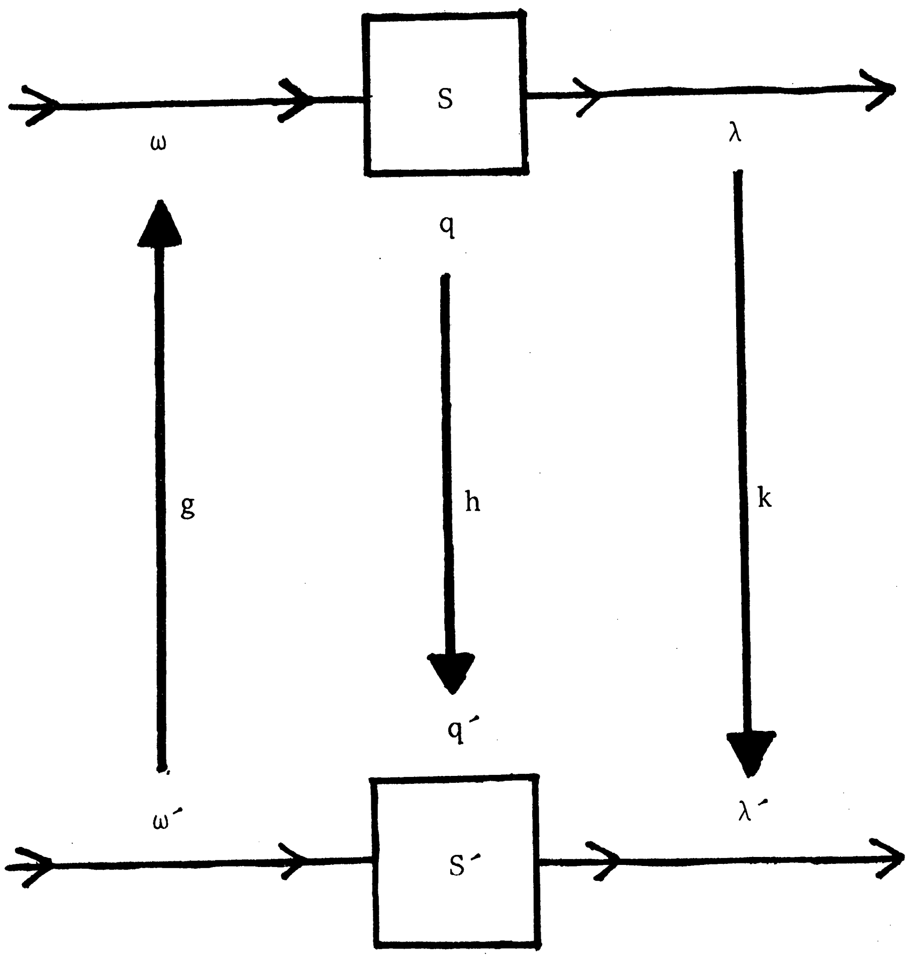

Figure A.1.1: Relations among Components of Morphic Systems.

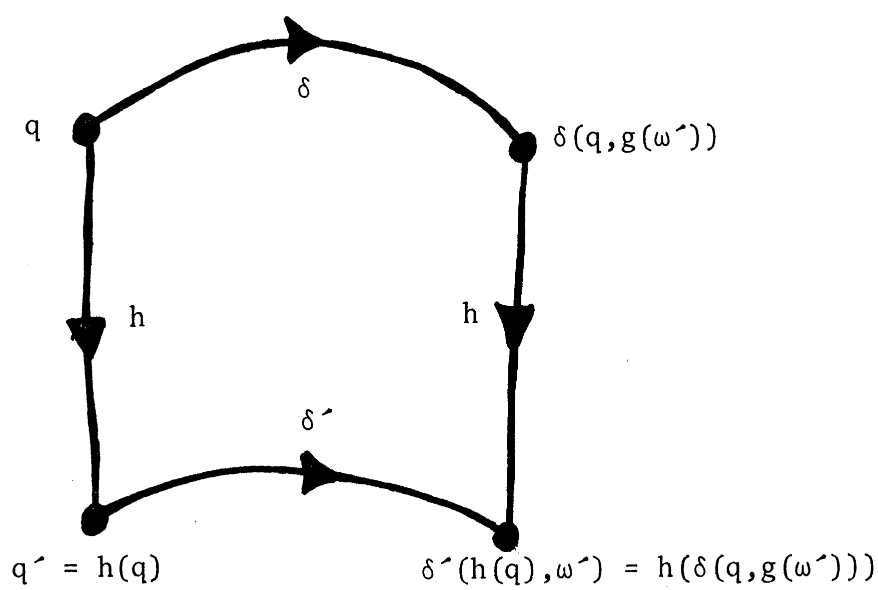

Figure A.1.2: Transition Function Preservation.

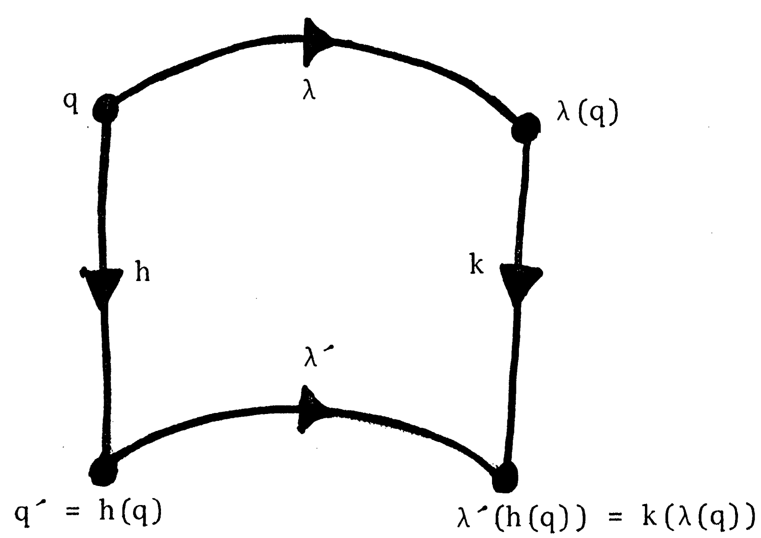

Figure A.1.3: Output Function Preservation.

The encoding function g matches compatible input segments. The decoding functions h and k simplify the system structure and behavior respectively. The simplification aspect of h and k results from the fact that they are surjective (onto) but not necessarily injective (one-one), in view of properties b) and c) in Definition A.1.7. Such maps incur an information loss, when one attempts to deduce the pre-image from its image. This information loss embodies the complexity-reduction effect while properties d) and e) in Definition A.1.7 represent the preservation effect of a simplification.

State and output trajectories are sufficiently important to warrant a separate morphism concept.

Definition A.1.8

Let $\omega_{(t_1,t_2]}$ and $\omega'_{(t_3,t_4]}$ be input segments of two mathematical systems S and S′ respectively. Let q and q′ be states of S and S′ respectively.

A *trajectory morphism* from TRAJ(q,ω) to TRAJ(q′,ω′) is a triple (MATCH,h,k), subject to the following restrictions:

a) $MATCH: [t_1,t_2] \rightarrow [t_3,t_4]$ is a bijective (one-one and onto) function called the *time matching function*.

b) $h: Q_{q,\omega} \rightarrow Q'_{q',\omega'}$ is a surjective function where

$Q_{q,\omega} = \{q \in Q: \exists t \in [t_1,t_2] \ni q = STRAJ_{q,\omega}(t)\}$ and similarly for $Q'_{q',\omega'}$. h is called the *state decoding function.*

c) $k: Y \rightarrow Y'$ is a surjective function called the *output decoding function.*

d) $\forall t,t' \in [t_1,t_2],\ t \leq t' \Rightarrow \text{MATCH}(t) \leq \text{MATCH}(t')$

i.e. MATCH *preserves ordering*.

e) $\forall t \in [t_1,t_2],\ h(\text{STRAJ}_{q,\omega}(t)) = \text{STRAJ}_{q',\omega'}(\text{MATCH}(t))$

i.e. *state trajectory preservation*.

f) $\forall t \in [t_1,t_2],\ k(\text{OTRAJ}_{q,\omega}(t)) = \text{OTRAJ}_{q',\omega'}(\text{MATCH}(t))$

i.e. *output trajectory preservation*.

□

Roughly speaking, a system morphism (g,h,k) is a super trajectory morphism which is uniformly good for any $(q,g(\omega'))$ and $(h(q),\omega')$.

We now define some important cases of specialized system morphisms.

Definition A.1.9

Two systems $S = \langle T,X,\Omega,Q,Y,\delta,\lambda\rangle$ and $S' = \langle T',X',\Omega',Q',Y',\delta',\lambda'\rangle$ are called *compatible* if

a) $T = T'$

b) $X = X'$

c) $\Omega = \Omega'$

d) $Y = Y'$

□

Definition A.1.10

Let S and S′ be compatible systems and let i denote the identity map. Then:

a) A system morphism (i,h,i) from S to S′ with $\overline{Q} = Q$ is called a *system homomorphism*.

b) A system homomorphism (i,h,i) from S to S′ such that h is bijective is called a *system isomorphism*. □

When focusing on the state structure we obtain the following analogue of the system morphism concept.

Definition A.1.11

A *system state-morphism* from a representative system $S = \langle T,X,\Omega,Q,\cdot,\delta,\cdot\rangle$ to a representative system $S' = \langle T',X',\Omega',Q',\cdot,\delta',\cdot\rangle$ is a pair (g,h) with the same meaning and restrictions as in Definition A.1.7.

Likewise, a *state-trajectory morphism* is a pair (MATCH,h) with the same meaning and restrictions as in Definition A.1.8. □

It is now obvious how to define compatibility of representative systems and how to proceed to define the concepts of state-homomorphism (i,h) and state-isomorphism (i,h) among them. The case of state-trajectory morphisms is analogous.

Finally, we note that morphic relations induce a hierarchy on the class of systems, as it is not difficult to see that these relations are transitive. We shall not dwell on this point in this section.

A.2 Iterative Specifications of Mathematical Systems

When dealing with a mathematical system of a special type, it is often more convenient to specify it indirectly at a certain level of detail. A translation process will then furnish the means of going from that particular specification to the normal system specification of Definition A.1.1. An important class of more structured specifications for time invariant systems is the class of iterative system specifications. Essentially, what happens here is that the input segment set is generated by a set of elementary input segments and similarly for the state transition function.

First some background concepts. Let $(X,T)_0$ be the set of functions of the form $\omega:(0,\tau]\rightarrow X$, $\tau \in T$. The composition operation defined on $(X,T)_0$ becomes $\omega_{(0,\tau_1]} \odot \omega'_{(0,\tau_2]} = \omega''_{(0,\tau_1+\tau_2]}$ where

$$\omega''_{(0,\tau_1+\tau_2]}(t) \triangleq \begin{cases} \omega_{(0,\tau_1]}(t), & \text{if } 0< t \leq \tau_1 \\ \omega'_{(0,\tau_2]}(t - \tau_1), & \text{if } \tau_1< t \leq \tau_1 + \tau_2 \end{cases}$$

This renders $(X,T)_0$ and the composition operation a semigroup.

If $\Gamma \subset (X,T)_0$, then the composition closure of Γ is called the *semigroup generated* by Γ and is denoted Γ^+ . If $\Gamma^+ = \Omega$, then Γ is called the *generator set* of Ω. In that case, given $\omega \in \Omega$, we wish to decompose it into generator segments in a canonical manner, via right or left *segmentation*. The term segmentation refers to the operation of restricting an input segment ω to subintervals. More specifically, a *left segment* of $\omega_{(t_1,t_2]}$ at t is defined by $\omega_{t>} \triangleq \omega|(t_1,t]$ for any $t \in (t_1,t_2]$. Similarly, $\omega_{<t} \triangleq \omega|(t,t_2]$ is a *right segment* of $\omega_{(t_1,t_2]}$

for any $t \in (t_1,t_2]$. The canonical decomposition we choose is that obtained by taking successive maximal left segments and then chopping them off the remaining segment repeatedly.

More accurately, $\omega_1,\omega_2,\ldots\omega_n \in \Gamma$ is a *maximal length segment (m.l.s) decomposition* of ω if for each i = 1,2,...n, whenever $\omega' \in \Gamma$ is a left segment of $\omega_i \odot \omega_{i+1} \odot \ldots \odot \omega_n$ then ω' is a left segment of ω_i. The merit of the m.l.s decomposition is the fact that if it exists, then it is unique (see [Z1] Ch. IX Sec. 9.8.1). We say that Γ is an *admissible generator set* for Ω if $\Omega = \Gamma^+$ such that each $\omega \in \Omega$ has a (unique) m.l.s decomposition.

We are now ready for the main definition.

Definition A.2.1

An *iterative specification* (of a mathematical system) is a structure $G = \langle T,X,\Omega,Q,Y,\delta,\lambda \rangle$ where

T is the *time base set*

X is the *input value set*

Ω is the *input generator set*

Q is the *state set*

Y is the *output value set*

δ is the *transition function*

λ is the *output function*

subject to the following restrictions:

a) T is a well ordered Abelian group.

b) Ω is an admissible set of generators of the form $\omega:(0,\tau] \rightarrow X$, $\tau \in T$.

c) δ is a function $\delta: Q\times\Omega \to Q$ satisfying the following *composition property*:

$$\omega_1,\omega_2 \in \Omega \text{ and } \omega_1\odot\omega_2 \in \Omega \;\Rightarrow\; \forall q \in Q,\ \delta(q,\omega_1\odot\omega_2) = \delta(\delta(q,\omega_1),\omega_2)$$

d) λ is a function $\lambda: Q \to Y$. □

The function δ in the above definition can be extended as follows:

<u>Definition A.2.2</u>

Let $\overline{\Omega}^+$ be the translation closure of Ω^+. The *extension of* δ is a function $\overline{\delta}: Q\times\overline{\Omega}^+ \to Q$ defined recursively by:

$$\forall q \in Q,\ \forall\omega = \omega_{(t_1,t_2]} \in \overline{\Omega}^+$$

$$\overline{\delta}(q,\omega) \triangleq \begin{cases} \delta(q,\mathrm{TRANS}_{-t_1}(\omega)), & \text{if } \mathrm{TRANS}_{-t_1}(\omega) \in \Omega \\ \overline{\delta}(\delta(q,\omega_1),\omega_2\odot\ldots\odot\omega_n), & \text{otherwise} \end{cases}$$

where $\omega_1\odot\omega_2\odot\ldots\odot\omega_n$ is the m.l.s decomposition of ω in terms of the translated generators $\mathrm{TRANS}_{t_1}(\Omega)$. □

We now show how an iterative specification is translated into a time invariant system.

<u>Theorem A.2.1</u>

If $G = \langle T,X,\Omega,Q,Y,\delta,\lambda\rangle$ is an iterative specification, then it induces a time invariant (mathematical) system $S_G = \langle T,X,\overline{\Omega}^+,Q,Y,\overline{\delta},\lambda\rangle$.

<u>Proof</u>

See [Z1] Ch. IX Sec. 9.8.2. □

An iterative subspecification $\hat{G}$ of an iterative specification G is defined by restricting δ and λ to $\hat{Q} \subset Q$ and $\hat{\Omega} \subset \Omega$ precisely as in mathematical subsystems (see Definition A.1.6). It is easily seen that if an iterative specification G induces a mathematical system S_G, then any iterative subspecification $\hat{G}$ of G induces a mathematical subsystem $\hat{S}_G$ of S_G.

We now turn our attention to morphic relations among iterative specifications. This will follow the pattern, set up for system morphisms in the previous section.

The basic definition now follows.

Definition A.2.3

A *specification morphism* from an iterative specification $G = \langle T,X,\Omega,Q,Y,\delta,\lambda \rangle$ to an iterative specification $G' = \langle T',X',\Omega',Q',Y',\delta',\lambda' \rangle$ is a triple (g,h,k), subject to the following restrictions:

a) g is a function $g:\Omega' \to \Omega^+$ called the *generator encoding function.*

b) h is a surjective function $h:\overline{Q} \to Q'$ called the *state decoding function* and $\overline{Q} \subset Q$.

c) k is a surjective function $k:Y \to Y'$ called the *output decoding function.*

d) $\forall q \in \overline{Q}$, $\forall \omega' \in \Omega'$ we have $h(\overline{\delta}(q,g(\omega'))) = \delta'(h(q),\omega')$ i.e. *transition function preservation.*

e) $\forall q \in \overline{Q}$ we have $k(\lambda(q)) = \lambda'(h(q))$ i.e. *output function preservation.*

□

Just as iterative specifications translate into mathematical systems, specification morphisms expand into system morphisms.

Theorem A.2.2

Let (g,h,k) be a specification morphism from G to G´. Then $(\tilde{g},\tilde{h},\tilde{k})$ is a system morphism from S_G to $S_{G'}$ where

a) $\tilde{g}$ is the extension of g to $\overline{\Omega}^{+}$ derived as follows:

Let $\omega_1'\odot\omega_2'\odot,\ldots,\odot\omega_n'$ be the m.l.s decomposition of $\omega' = \omega'_{(t_1,t_2]} \in \overline{\Omega}'^{+}$ in terms of the translated generators $TRANS_{t_1}(\Omega')$. Then $\tilde{g}(\omega') = g(\omega_1')\odot g(\omega_2')\odot\ldots\odot g(\omega_n')$.

b) $\tilde{h} = h$

c) $\tilde{k} = k$

Proof

See [Z1] Ch. X Sec. 10.5. □

Specification homomorphisms and isomorphisms as well as the concepts of specification state-morphisms and trajectory morphisms may be defined analogously to those in the previous section.

We will not elaborate on this point.

APPENDIX B

FORMAL SIMPLIFICATIONS

B.0 Introduction

Simplification is a widely used method in the Sciences. Simplifications are applied to such diverse entities as equation systems, networks, system-theoretic models etc. They are extensively employed in modeling and simulation of systems, deterministic as well as stochastic.

In the conceptual framework developed by Zeigler in [Z2], [Z4], [Z5] and [Z6], going from a base model to some lumped model is a typical instance of a simplification process. The simplification effect manifests itself at various levels. When operating on informal descriptions of system-theoretic models, a simplification may aggregate components, simplify assumptions etc. (See e.g. [WZL1], [Z3], [Z7]). On the other hand a simplification of a probability space may be viewed as a measure preserving coarsening of the underlying sample space and σ-algebra. A simplification of an equation system is obvious enough.

In spite of their different appearances, all the simplification notions above have an underlying conceptual similarity demonstrated by two salient features.

1) They all reduce, in some sense, the complexity of the entity to be simplified.
2) They all are meant to preserve some aspects of the entity to be simplified.

The rationale for the enterprise of simplification is what may be termed the "simplification strategy". The essence of this strategy is the ability to take advantage of a simpler entity [due to feature 1)], whose manipulation is easier, yet yields valid conclusions pertaining to the original entity [due to feature 2)]. Thus, a simplification could enable us to use the simplification strategy towards a solution of our problem.

In order to be able to deal uniformly with the diverse manifestations of simplifications in a variety of contexts, it is necessary to formalize the conceptual similarity described by features 1) and 2) above. A formal definition is required to capture these intuitive features, so as to allow us to derive and recognize a broad range of simplification instances by an assignment of the appropriate semantics. In particular, this would provide us a uniform conceptual framework for treating simplifications of deterministic systems and stochastic ones alike.

We proceed to propose such a formalism in the sequel.

B.1 Simplification Predicates

Our discussion employs predicate-like notation similar to [Fol]. Assume that the following are given:

a) A set Σ of "*systems*" (descriptions).

b) A family $\{\Psi_\sigma\}_{\sigma \in \Sigma}$ of "*aspect*" sets for each element in "*systems*".

c) A set C of "*complexity*" functions for "*systems*", where $c \in C$ is a function $c: \Sigma \to K_c$ and K_c is a totally ordered set under an order relation "$\leq_c$".

d) A set Π of "*preservation*" relations between "*aspect*" *pairs* in $\bigcup_{(\sigma_1,\sigma_2)\in\Sigma^2} \Psi_{\sigma_1} \times \Psi_{\sigma_2}$. We write $\psi_1 \ \pi \ \psi_2$ if "aspect" $\psi_1 \in \Psi_{\sigma_1}$ is "preserved" in the sense of $\pi \in \Pi$ by "aspect" $\psi_2 \in \Psi_{\sigma_2}$.

The terms cloaked in double quotes should be understood as semantics free, although they were chosen so as to be suggestive. Instances of the cloaked terms are obtained by interpreting them in some domain of application. Thus, an instance of "systems" can be a system of equations, a set of mathematical systems (see Appendix A), a set of DEVSs (see Ch. 1), or a set of queuing networks. An instance of "aspects" could be a particular set of solutions, behavioral frames (see Ch. 1 and Ch. 2) and functions thereof (e.g. means, time averages etc.).

The complexity functions are devised to capture quantifiable as well as intuitive complexity notions; e.g. computational complexity of algorithmic solutions, conceptual complexity of a mathematical system, size of a queuing network etc. (See also Sec. B.3).

A "preservation" notion can range from outright equality to the existence of a translation process from Ψ_{σ_1} to Ψ_{σ_2} (e.g. as formalized by various morphisms in Ch. 1 and Ch. 3).

Various concepts of approximate preservation, e.g. allowing a tolerance of an ϵ-error such as in approximate morphisms (see [Z1] Ch. XIII) and other relaxed versions of preservation (e.g. in mean rather than in distribution), fall into the category of "preservation" notions.

An ordered pair $(\beta,\lambda) \in \Sigma^2$ is a *simplification over* Σ *relative to* c, if $c(\lambda)$ "$\leq_c$" $c(\beta)$. To stress this fact we shall also write $\beta \underset{c}{\mapsto} \lambda$. Following [Z3] we term β the *base model*, and λ the *lumped model* of the simplification $\beta \underset{c}{\mapsto} \lambda$.

Next, a *simplification procedure over* Σ is a finite chain of pairs (β_1,λ_1), $(\lambda_1,\lambda_2),\ldots,$ $(\lambda_{n-1},\lambda_n)$ over Σ^2, such that $\beta_1 \underset{c}{\mapsto} \lambda_n$ is a simplification for some $c \in C$. In this case we write $\beta_1 \underset{c}{\mapsto} \lambda_1 \mapsto \ldots \mapsto \lambda_n$.

A simplification procedure merely decomposes a simplification operation into a chain of successive stages, each of which may be regarded as an "intermediate simplification". Simplification procedures simplify the analysis of complex formal simplifications. For if a simplification can be broken down into a composition representation in terms of successive application of "elementary simplifications", then its analysis reduces to the examination of the "easier" simplification effect that is brought about in each stage. Thus, simplification procedures provide a means for "simplifying simplifications". An example of a simplification procedure is described in Sec. 5.3 of Ch. 5.

Let us define a *simplification predicate* S on sets of the form

$$\bigcup_{(\beta,\lambda)\in\Sigma^2} \{(\beta,\lambda)\}\times C\times(\Psi_\beta\times\Psi_\lambda)\times\Pi \quad \text{by}$$

$S((\beta,\lambda),c,(\psi_\beta,\psi_\lambda),\pi)$ = 'true' iff $\psi_\beta \pi\, \psi_\lambda$ under $\beta \underset{c}{\mapsto} \lambda$; (that is, iff the "aspect" $\psi_\beta \in \Psi_\beta$ is π-"preserved" by the "aspect" $\psi_\lambda \in \Psi_\lambda$ under the simplification (β,λ) relative to the "complexity" notion c).
In this case we say that the simplification $\beta \underset{c}{\mapsto} \lambda$ is *valid* in the "aspect" pair $(\psi_\beta,\psi_\lambda)$ under the "preservation" notion π.

Simplification predicates enable us to make statements about simplifications in a formal manner. They also embody our intuitive requirements which were preimposed at the outset on any formalism for simplifications. The complexity reduction idea is obviously captured by the "complexity" function concept; the preservation idea is built into the concept of "preservation" relations.

B.2 Simplification Problems and Their Solutions

A *simplification problem* SP is stated in terms of a set of simplification predicates to be evaluated over a *simplification problem domain* $\mathcal{D}(SP)$ such that $\mathcal{D}(\text{SP}) \subset \bigcup_{(\beta,\lambda)\in\Sigma^2} \{(\beta,\lambda)\} \times C \times (\Psi_\beta \times \Psi_\lambda) \times \Pi$.

Some of the frequently encountered simplification problems can be formulated as follows:

SP1: Given a simplification $\beta^* \underset{c^*}{\mapsto} \lambda^*$, characterize all "aspect" pairs $(\psi_\beta,\psi_\lambda) \in \Psi_{\beta^*}\times\Psi_{\lambda^*}$ and "preservation" relations $\pi \in \Pi$, such that $S((\beta^*,\lambda^*),c^*,(\psi_\beta,\psi_\lambda),\pi)$ = 'true'.

The problem domain of SP1 is $\mathcal{D}(\text{SP1}) = \{(\beta^*,\lambda^*)\}\times\{c^*\}\times(\Psi_{\beta^*}\times\Psi_{\lambda^*})\times\Pi$. Intuitively, SP1 is tantamount to taking a particular simplification $\beta^* \underset{c^*}{\mapsto} \lambda^*$ and asking: what "aspects" are "preserved" by it, and in what sense of preservation? More simply, the problem is to find the preservation scope of $\beta^* \underset{c^*}{\mapsto} \lambda^*$.

SP2: Given a subset of "systems" pairs $\Gamma \subset \Sigma^2$, a collection of "aspect" pairs $\{(\psi^*_\beta,\psi^*_\lambda)\}_{(\beta,\lambda)\in\Gamma}$, where $(\psi^*_\beta,\psi^*_\lambda) \in \Psi_\beta\times\Psi_\lambda$, and a preservation relation $\pi^* \in \Pi$ - characterize all simplifications $\beta \underset{c}{\mapsto} \lambda$ over Γ, such that $S((\beta,\lambda),c,(\psi^*_\beta,\psi^*_\lambda),\pi^*)$ = 'true'.

The problem domain of SP2 is $\mathcal{D}(\text{SP2}) = \bigcup_{(\beta,\lambda)\in\Gamma} \{(\beta,\lambda)\}\times C\times\{(\psi^*_\beta,\psi^*_\lambda)\}\times\{\pi^*\}$.

Intuitively, SP2 is tantamount to taking a set of prospective simplifications, then choosing an "aspect" pair for each and a "preservation" notion, and asking: what simplifications would be valid in their respective "aspect" pair, under the predetermined "preservation" relation, and relative to what "complexity" criteria? In other words, we wish to find out the validity scope of our prospective simplifications and the scope of complexity reduction achieved by them.

A solution S_{SP} of a simplification problem SP is a triple $S_{SP} = \langle$'assertion', 'proof', 'algorithm'$\rangle$ where

1. 'assertion' is a statement asserting the scope of truth of a simplification predicate S when evaluated over $\mathcal{D}$(SP).
2. 'proof' is a proof of correctness for 'assertion'.
3. 'algorithm' is a finite decision process that effectively evaluates the simplification predicate S for any argument in $\mathcal{D}$(SP).

The quotes cloaking the elements in S_{SP} merely indicate that they are generic. Usually, only 'assertion' and 'proof' need to be given, whereas 'algorithm' often turns out to be implicit in the condition set of 'assertion' (see Example B.4.3).

B.3 Complexity Notions

Complexity notions, be they formal or intuitive, are used to capture some aspect of difficulty presented to the investigator by the entities under consideration. Formally, a complexity notion for a set of entities Σ is represented by a *complexity function* $c:\Sigma \rightarrow K_c$, where K_c is a totally ordered set under some order relation $\leq_c$. This definition ensures that for any two entities $\sigma_1, \sigma_2 \in \Sigma$, the associated complexities $c(\sigma_1)$ and $c(\sigma_2)$ are comparable under $\leq_c$.

When the complexity notion c is quantifiable (that is, K_c is a subset of the reals), then the complexity function c will be referred to as a *complexity measure.*

Our main interest in the complexity concept will lie in its role as a simplification criterion. Since simplifications are perceived as complexity reducing maps among the entities under consideration, examining the complexities of the prospective base and lumped models is a means of deciding whether or not they constitute a simplification pair. Furthermore, if K_c has sufficient structure, say group structure, then the same process would allow us to determine the extent of a simplification, as well as to compare the complexity reduction effect among simplification pairs.

We now proceed to discuss rather briefly some important classes of complexity notions, both intuitive and quantifiable.

C1) Conceptual Complexities:

Conceptual complexities have to do with the parsimony of system specification. Two main components are involved: structural complexities and behavioral complexities.

If, for example, the system is specified as a DEVS (see Ch. 1) or an informal stochastic DEVS (see Ch. 2), then its structural complexities reside in the size and nature of its state space, while its behavioral complexities reflect the conceptual difficulty of the rules that govern state transitions.

For probability spaces structural complexities are identified with the size or detail level of the underlying sample space and σ-algebra. Deterministic system morphisms as well as stochastic ones give rise to simplifications which are primarily structural complexity reducing (see Chapters 1, 2 and 3).

For systems describable by networks of interacting components, structural complexities can be derived from the topological complexity of the associated graph (e.g. its size in terms of nodes and arcs). In a queuing network, behavioral complexities involve the waiting line discipline, rules of servicing and the method of customer switching.

Conceptual complexities are probably the most important and fundamental notions of complexity. While structural complexities are relatively amenable to quantification, most behavioral complexities remain intuitive notions.

C2) Analytical Complexities:

Analytical complexities bear a close relation to conceptual ones. They have to do with analytical manipulations aimed at finding mathematical solutions for the operating characteristics of a system. It is obvious that analytical complexities are directly linked to conceptual complexities, both structural and behavioral.

In Queuing Theory we find that the analytical complexity (in the intuitive sense) jumps tremendously when passing from single queues to queuing networks. We also find that the equation systems are analytically less complex for exponential servers as compared to Erlangian ones, for FIFO queue discipline as compared to preemptive resume, and for Bernulli switches as compared to non-Markovian ones. If there are algorithmical solutions, then analytical complexities may be quantified as ordinary computational complexities, i.e. as measures of time and space required for finding such solutions.

C3) Simulation Complexities:

Simulation complexities are the analogue of computational complexities when the algorithm is a simulation run of the system, (mainly a stochastic one). Simulations of a stochastic system are used to derive some information, when a complete analytical solution is not within our reach. Simulation complexities are inherently programming oriented and fully quantifiable. They measure computer resources in terms of CPU time and memory storage required to simulate a system under some stopping rule. For stochastic systems, one simulates sample histories (realizations) using random number generators. For such cases, the resources required for a run become random functions of the sample histories to be simulated. When these random functions are measurable, one is typically interested in the respective expectations and variances, as they project the average resources and the fluctuations about it, to result from repeated simulation runs. Some examples of simulation complexity measures of stochastic discrete event systems (specifically, queuing networks) may be found in Sec. 5.4 of Ch. 5.

In general, different complexity notions need not be consistent in the sense that their behavior could involve opposing monotoneity trends. For example, if an Erlangian queuing network admits of a reduction to an exponential network, then this would decrease the behavioral and analytical complexities. On the other hand, the structural complexity would increase considerably, as we add more nodes and arcs. Similar phenomena are pointed out in [Z5] in the domain of structured functions (abstractions of networks).

The choice of a complexity notion is up to the user, and it varies from situation to situation. Therefore, a simplification process as guided by complexity criteria is really in the eye of the beholder.

B.4 Examples

In this section we further exemplify instances of simplifications and demonstrate how our formalism works.

Example B.4.1

For deterministic systems such as mathematical systems, iterative specifications (see Appendix A) and DEVSs or DEVNs (see Ch. 1), the set of "systems" is the corresponding set of state-systems while "aspects" are formalized as behavioral frames. The main vehicle for simplifications over classes of such deterministic systems is the morphism concept (see ibid.). A morphism (g,h,k) has inherent simplification properties of "complexity" reduction, and the "aspect preservation" effect is manifested by the existence of a translation process via h and k between the structure and behavior, respectively, of the base and lumped models. For a more detailed discussion of the simplification effect of morphisms, the reader is referred to Appendix A. Observe that in a hierarchy of morphisms, the more specialized the morphism, the smaller the simplification effect. As we specialize h from a mere morphism to a homomorphism and an isomorphism, the structural modification of the morphic preimage to its morphic image is reduced, the gradient of structural complexities declines, and we can expect to preserve more behavioral frames. □

Example B.4.2

Simplifications of stochastic systems such as stochastic DEVSs (see Ch. 2) follow the basic pattern of Example B.4.1, subject to some modifications.

The set of "systems" is composed of probability spaces. These are usually coordinate probability spaces that represent informal descriptions of stochastic systems. The "aspects" set consists of behavioral frames formalized as stochastic processes (see Ch. 2). Stochastic simplifications are identified with the existence of a stochastic morphism H defined as a variant of the measure preserving transformation concept (see Ch. 3). The simplification effect of a stochastic morphism is analogous to its deterministic counterpart. "Complexity" reduction and "preservation" are attained by lumping sample points and coarsening the base model's σ-algebra in a measure preserving manner. The reader is referred to Ch. 3 for more details.

□

Example B.4.3

In this example, we exemplify how our proposed formalism works in a queuing-theoretic context.

Our set of "systems" is the class of Jackson queuing networks (described in Ch. 4) in their coordinate probability space representation (see Ch. 2).

The "aspects" set of a "system" is the set of all stochastic processes over the associated probability space. Define a "complexity" notion c* as the size of the networks, say the sum of nodes and arcs. The "preservation" notion π^* of any "aspect" pair $(\psi_1,\psi_2) \in \Psi_{\sigma_1} \times \Psi_{\sigma_2}$ is defined as distribution equivalence of ψ_1 and ψ_2 (i.e. as the equality $F_{\psi_1} = F_{\psi_2}$ of their families of finite dimensional distributions).

Next we focus on "aspect" pairs (ψ_1^*,ψ_2^*) where ψ_i^* is the total service time sampled by an arbitrary customer, in the network σ_i. Let us formulate an informal simplification problem as follows: informal SP: "Characterize all A-simplifications over the class of Jackson networks (a A-simplification of a queuing network removes all arcs among the nodes, and therefore is a simplification relative to c*), such that the total time service time sampled by an arbitrary customer in the network is preserved in distribution."

The formal version of SP runs as follows:

formal SP: "Characterize all A-simplifications (β,λ) over the class of Jackson networks such that $S((\beta,\lambda),c^*,(\psi_\beta^*,\psi_\lambda^*),\pi^*)$ = 'true'."

Notice that the "preservation of total service time" alluded to in the informal SP really refers to an "aspect" pair (total service time of arbitrary customer in β; total service time of arbitrary customer in λ). Actually the former is an informal shorthand for the latter. This point is not altogether trivial. For example, if in β the waiting time distribution of arbitrary customer is the same as the transit time distribution of arbitrary customer in λ such that $c(\lambda)$"$\leq_c$" $c(\beta)$, then our formalism would recognize $\beta \underset{c}{\mapsto} \lambda$ to be a valid simplification in the "aspect" pair (waiting time, transit time). There is no intuitive reason why "aspect" pairs, whose components do not play the same intuitive role, should not be regarded as being "preserved" under some otherwise intuitive simplification.

The domain of SP is $\mathcal{D}(SP) = \bigcup_{(\beta,\lambda)\in\Gamma} \{(\beta,\lambda)\}\times\{c^*\}\times\{(\psi^*_\beta,\psi^*_\lambda)\}\times\{\pi^*\}$ where Γ is the set of all A-simplifications over Jackson networks.

A solution of SP is based on Theorem 5.2.4 in Ch. 5. Define $S_{SP} = \langle$'assertion', 'proof', 'algorithm'$\rangle$ where

1. 'assertion' = "a A-simplification (β,λ), whose lumped model is obtained according to Theorem 5.2.4, satisfies

   $S((\beta,\lambda),c^*,(\psi^*_\beta,\psi^*_\lambda),\pi^*)$ = 'true' iff

   every node n in the base model β satisfies the condition

   $q_n\sigma_n$ = const."

   (The quantities q_n and σ_n are structural parameters of a Jackson network, i.e. part of its description.)

2. 'proof' is given in Theorem 5.2.4.

3. 'algorithm' amounts simply to checking the condition $q_n\sigma_n$ = const. directly from the description of β, and verifying whether it holds or not. This clearly is a finite decision process that allows us to decide effectively the validity of S for each simplification $\beta \underset{c^*}{\mapsto} \lambda$ in Γ. □

Further examples of simplifications over the class of Jackson networks may be found in Sections 5.1 - 5.3 of Chapter 5.

APPENDIX C

SOME STOCHASTIC PROCESSES BACKGROUND

C.0 Introduction

This appendix reviews some basic facts pertaining to Markov processes, birth-and-death equations and stochastic equilibrium. Relevant material can be found in standard references such as [Cil] (see Ch. 8), [D1] (see Ch. VI) and [F2] (see Ch. X).

C.1 Markov Processes

If X and $\{Y_\theta: \theta \in \Theta\}$ are random variables over a probability space $S = \langle \Omega, A, P \rangle$ and $E(|X|) < \infty$, then $E(X|Y_\theta, \theta \in \Theta)$ will denote the *conditional expectation* of X with respect to $\sigma(\{Y_\theta: \theta \in \Theta\})^\dagger$ (see [D1], Ch. I). If $\Lambda \in A$, then $P(\Lambda|Y_\theta, \theta\in\Theta)$ will denote the *conditional probability* of Λ with respect to $\sigma(\{Y_\theta: \theta \in \Theta\})$ (see ibid.). Conditional probabilities are special cases of conditional expectations when $X = I_\Lambda$ is the indicator function of Λ.

In the sequel, Θ will denote a subset of the real line.

Definition C.1.1

An n-dimensional stochastic process $Y = \{Y_\theta\}_{\theta\in\Theta}$ over a probability space $S = \langle \Omega, A, P \rangle$ is called a *Markov process* if whenever $s \leq t < u$, the equality

(A) $P(Y_u \in B|Y_\theta, \theta \in (s,t]) = P(Y_u \in B|Y_t)$

---

†the σ-algebra generated by $\{Y_\theta\}_{\theta\in\Theta}$.

holds almost surely for every Borel set $B \in \mathcal{B}^n$, where $\mathcal{B}^n$ is the Borel σ-algebra on the n-dimensional Euclidean space $\mathbb{R}^n$. □

An equivalent statement of (A), called the *Markov property* (see [D1] p. 81), asserts that if $\mathcal{Y}$ is a Markov process, then for every random variable Z measurable on $\sigma(\{Y_\theta: \theta \geq t\})$ with $E(|Z|) < \infty$,

(B) $E(Z|Y_\theta: \theta \leq t) = E(Z|Y_t)$ almost surely.

In particular, if $Z = I_\Lambda$ and $\Gamma \in \sigma(\{Y_\theta: \theta \leq t\})$, then almost surely (cf. [C1] p. 136)

(C) $P(\Lambda|\Gamma, Y_\theta, \theta \leq t) = P(\Lambda|Y_t)$

where Γ above should be understood as I_Γ.

We now exhibit a sufficient condition that guarantees a stochastic process to be a Markov process.

Theorem C.1.1

Let $\mathcal{Y} = \{Y_\theta\}_{\theta \in \Theta}$ be an n-dimensional stochastic process over a probability space $\mathcal{S} = \langle \Omega, A, P \rangle$. Suppose that $\mathcal{Y}$ satisfies a stochastic equation of the form

a) $Y_u = f(Y_s, \{Z_t\}_{s<t\leq u})$ for any $s < u$

where $\{Z_t\}_{s<t\leq u}$ is a set of random variables over $\mathcal{S}$, such that

b) $\sigma(\{Z_t: s < t \leq u\})$ is independent of $\sigma(\{Y_\theta: \theta \leq s\})$ and $f(Y_s, \{Z_t\}_{s<t\leq u})$ is measurable.

Then $\mathcal{Y}$ is a Markov process.

Proof

See e.g. [S1], Ch. 3 pp. 73-75. □

We remark in passing that the term "Markov jump process" is sometimes used, thus reflecting the fact that under mild regularity conditions, the sample functions may be chosen to be step functions almost surely (see [D1] p. 246).

For a Markov process $\mathcal{Y} = \{Y_\theta\}_{\theta\in\Theta}$ with a denumerable state space, Definition C.1.1 may be restated in terms of "ordinary" probabilities as follows:

(D) $\Pr(Y_{\theta_n} = \nu_n | Y_{\theta_i} = \nu_i,\ 1 \le i \le n-1) = \Pr(Y_{\theta_n} = \nu_n | Y_{\theta_{n-1}} = \nu_{n-1})$

for any indices $\theta_1 < \theta_2 < \ldots < \theta_{n-1} < \theta_n$ and states

ν_i, $1 \le i \le n$, and provided $\Pr(Y_{\theta_i} = \nu_i,\ 1 \le i \le n-1) > 0$.

The right hand side of (D) is called a (Markov) *transition function* and denoted $p_{\nu_{n-1}\nu_n}(\theta_{n-1},\theta_n)$. To simplify matters we assume the transition functions to be always defined.

We now restrict the discussion to Markov processes $\mathcal{Y} = \{Y_\theta\}_{\theta\in\Theta}$ with a denumerable state space $\mathcal{R}(\mathcal{Y})$, where $\Theta = [0,\infty)$ and $\mathcal{Y}$ has *stationary transition probabilities* (i.e. the transition functions $p_{\nu_1\nu_2}(\theta_1,\theta_2)$ depend only on ν_1,ν_2 and $t = \theta_2-\theta_1$). In this case, the latter reduce to $p_{\nu_1\nu_2}(t)$, and the *transition matrix* consisting of transition functions becomes $P(t) \triangleq [p_{\nu_1\nu_2}(t)]$. In particular, $P(t)$ satisfies the Chapman-Kolmogorov equations.

(E) $P(s + t) = P(s)P(t),\ \forall s,t \in \Theta$.

If one assumes $P(t) \xrightarrow[t\to 0+]{} I$ (identity matrix), then $\dot{P}(0)$ exists as a right derivative (in t), but may have infinite components (see [C1] p. 126). However, under certain regularity conditions (finiteness of $\dot{p}_{\nu\nu}(0)$, $\nu \in R(Y)$), P(t) is guaranteed to have continuous derivatives everywhere (see [C1] p. 130).

In this case, the Kolmogorov *forward* and *backward* equations can be derived from (E) by differentiating (E) with respect to s and t respectively and setting each variable to 0 (see [D1] p. 240).

With dots denoting derivatives in t, we obtain respectively

(F.1) $\dot{P}(t) = P(t)G$ subject to $P(0) = I$

(F.2) $\dot{P}(t) = GP(t)$ subject to $P(0) = I$

where $G \triangleq \dot{P}(0)$ is called the *infinitesimal generator matrix* of Y (see [F2] p. 456).

Moreover, the boundedness of the $-g_{\nu\nu} \triangleq c_\nu$, $\nu \in R(Y)$, guarantees (see [F1] p. 475) that both (F.1) and (F.2) have a minimal solution P(t) which is honest (i.e. its rows are probability vectors). The quantities c_ν are extremely important, as they hold the key to existence and uniqueness of an honest transition matrix for a Markov process. Each c_ν is interpreted as the rate of transition from state ν, and by stationarity of the transition probabilities, this rate does not depend on t. If the c_ν are unbounded, the minimal solution (which always exists) may not be honest, and the defect is interpreted as probabilities due to infinite number of jumps in finite intervals (see [F2] p. 329). In this case, the solution for P(t) is not unique. However, with bounded c_ν, the Markov process Y is guaranteed to be *conservative*, i.e. to have almost surely a finite number of jumps in each finite interval, and vice versa. In this case, one can show by direct calculation that, say the forward equation (F.1), is equivalent to the system of integral equations (see [F2] p. 484)

$$\text{(G)}\quad p_{\lambda\nu}(t) = \delta_{\lambda\nu}e^{-c_\lambda t} + \sum_{\mu\in\mathcal{R}(\mathcal{Y})}\int_0^t p_{\lambda\mu}(x)c_\mu r_{\mu\nu}e^{-c_\nu(t-x)}dx$$

$\lambda,\nu \in \mathcal{R}(\mathcal{Y})$.

Here, $\delta_{\lambda\nu}$ is Kronecker's delta and each $r_{\mu\nu}$ is the conditional probability that a jump will take place from state μ to state ν, given that a jump has taken place from state μ.

Furthermore, differentiating (G) yields

$$\dot{p}_{\lambda\nu}(0) = \begin{cases} -c_\lambda, & \text{if } \lambda = \nu \\ c_\lambda r_{\lambda\mu}, & \text{if } \lambda \neq \nu \end{cases}$$

It is known that the minimal solution may be obtained as a pointwise limit of the sequence $\{P^{(n)}(t)\}_{n=0}^{\infty}$ defined recursively by

$$\text{(H.1)}\quad p_{\lambda\nu}^{(0)}(t) \triangleq \delta_{\lambda\nu}e^{-c_\lambda t}$$

$$\text{(H.2)}\quad p_{\lambda\nu}^{(n+1)}(t) \triangleq \delta_{\lambda\nu}e^{-c_\lambda t} + \sum_{\mu\in\mathcal{R}(\mathcal{Y})}\int_0^t p_{\lambda\mu}^{(n)}(x)c_\mu r_{\mu\nu}e^{-c_\nu(t-x)}dx$$

(see [F2] p. 485 for a derivation in the Laplace-Stieltjes transform domain).

The treatment for the backward equations is analogous, except that the backward equations might have solutions that do not satisfy the forward equation (see [F1] p. 478).

C.2 Absolute Probabilities of Markov Processes

Although the development of Markov Processes is traditionally carried out via their transition structure, in applications one is mainly interested in the trajectories of the state absolute probabilities.

Formally, we define

Definition C.2.1

Let $\mathcal{Y} = \{Y_\theta\}_{\theta\in\Theta}$ be a stochastic process with a denumerable state space $\mathcal{R}(\mathcal{Y})$. The *probability vector* $y(\theta)$ of Y_θ is a vector whose ν-th coordinate, $\nu \in \mathcal{R}(\mathcal{Y})$, is given by

$$y_\nu(\theta) \triangleq \Pr(Y_\theta = \nu).$$

The probability vector $y(0)$ is called the *initial condition* of $\mathcal{Y}$. The function $y(\theta)$ (in θ) is called the *probability trajectory of* $\mathcal{Y}$ and $y_\nu(\theta)$ is called the *probability trajectory of state* ν. □

The relation between the transition structure given by $P(t)$ and the probability trajectory $y(t)$ is such that the former determines the latter up to an initial condition $y(0)$. In other words

(A) $y(t) = y(0)P(t)$.

If $P(t)$ is everywhere differentiable in t, then

(B) $\dot{y}(t) = y(0)\dot{P}(t)$

Thus, premultiplication by $y(0)$ of the Kolmogorov forward equation (F.1) in Sec. C.1 gives us

(C) $\dot{y}(t) = y(t)G$

which we call the *forward absolute probability equations*. The backward equations are similarly obtained by postmultiplying (F.2) in Sec. C.1 by the transpose of y(0).

We now give conditions under which $\mathcal{Y}$ has almost surely finite number of jumps in each finite interval. In this case, equations (C) above and (F.1) in Sec. C.1 are equivalent in the sense of existence and uniqueness of respective solutions P(t) and y(t) for them.

Theorem C.2.1

Let $\mathcal{Y} = \{Y(t)\}_{t\geq 0}$ be a Markov process with stationary transition probabilities and almost surely finite number of jumps in every finite interval. Let y(0) be an initial condition for $\mathcal{Y}$, and G its infinitesimal generator matrix. Consider the equation

(1) $\dot{u}(t) = u(t)G$ subject to $u(0) = y(0)$

Then (1) has a unique probability solution u(t) which is precisely the probability trajectory y(t) of $\mathcal{Y}$. Moreover, y(t) is obtained as a minimal solution of (1) and each coordinate $y_\nu(t)$ in y(t) satisfies the integral equation

(2) $$y_\nu(t) = y_\nu(0)e^{-c_\nu t} + \sum_{\mu\in\mathcal{R}(\mathcal{Y})} \int_0^t y_\mu(x)c_\mu r_{\mu\nu} e^{-c_\nu(t-x)}dx$$

$\nu \in \mathcal{R}(\mathcal{Y})$.

Proof

See [BM1], Lemma 2.1.

□

An important class of Markov processes with denumerable state space is obtained, when state transitions are restricted to adjacent states in the sense of

Definition C.2.2

Let $\mathcal{Y} = \{Y_t\}_{t\geq 0}$ be an m-dimensional Markov process with $\mathcal{R}(\mathcal{Y}) = \{(n_1,\ldots,n_m): n_i$ is an integer$\}$. Let e_i, $1 \leq i \leq m$ be the m-dimensional unit vector with 1 in the i-th coordinate. Then two states $\nu,\mu \in \mathcal{R}(\mathcal{Y})$ are called *adjacent* if either of the following holds:

a) $\nu = \mu + e_i$ for some $1 \leq i \leq m$

b) $\nu = \mu - e_i$ for some $1 \leq i \leq m$

c) $\nu = \mu + e_i - e_j$ for some $1 \leq i,j \leq m$ □

Thus, adjacent states are "neighboring" lattice points. We now define formally the restrictions on state transitions by

Definition C.2.3

Let $\mathcal{Y} = \{Y_t\}_{t\geq 0}$ be as in definition C.2.2. We say that $\mathcal{Y}$ is an *m-dimensional birth-and-death process*, if whenever μ and ν are *not* adjacent states, we have

a) $\dot{p}_{\nu\mu}(t,t) = \dot{p}_{\mu\nu}(t,t) = 0,\quad t \geq 0$ (derivative with repsect to the second argument)

In this case, Equation (C) will be referred to as the *birth-and-death equation* of $\mathcal{Y}$.

□

Observe that for m = 1, Definition C.2.3 properly reduces to the ordinary definition of (1-dimensional) birth-and-death processes (see e.g. [Fl] p. 454).

C.3 Equilibrium Concepts

When dealing with stochastic processes, one is often interested in its equilibrium properties. Intuitively, an equilibrium situation may be attained asymptotically - when the process has been evolving for a "long time", or immediately - if started with the appropriate initial condition. Formally, we define

Definition C.3.1

Let $\mathcal{Y} = \{Y_\theta\}_{\theta\in\Theta}$ be a stochastic process with a denumberable state space $\mathcal{R}(\mathcal{Y})$. Let $y(\theta)$ be a probability vector of Y_θ.

Then

a) We say that $\mathcal{Y}$ is *in equilibrium* (or in *steady state*) under y^o, if $y(\theta)$ is time invariant in the sense of

$$y(0) = y^o \Rightarrow y(\theta) \equiv y^o, \quad \forall\theta \in \Theta .$$

In this case, y^o is called an *equilibrium vector* of $\mathcal{Y}$.

b) If y^o is an equilibrium vector of $\mathcal{Y}$ such that for any choice of an initial condition $y(0)$, we have

$$y(t) \xrightarrow[t\to\infty]{} y^o \text{ (pointwise convergence)},$$

then y^o is called a *long run vector* of $\mathcal{Y}$. □

Although, in general, $\mathcal{Y}$ may have several equilibrium vectors, it can have at most one long run vector. In this case, the long run vector becomes the unique equilibrium vector of $\mathcal{Y}$.

Statistical equilibrium is a situation whereby the probabilistic behavior of $\mathcal{Y}$ (in terms of probability trajectories) does not fluctuate in time. If the process evolves asymptotically into equilibrium, then in general, the equilibrium situation evolved into depends on the initial condition. However, the existence of a long run vector guarantees $\mathcal{Y}$ to evolve asymptotically into a unique equilibrium situation, regardless of initial conditions.

We point out in passing that an equilibrium vector of a Markov process is a long run vector, whenever the recurrent part of the state space is irreducible (see [Cil] p. 264).

For Markov processes with denumerable state space, we have the following necessary and sufficient condition for a probability vector to be an equilibrium vector.

Theorem C.3.1

Let $\mathcal{Y} = \{Y_t\}_{t\geq 0}$ be a Markov process whose forward absolute probability equation is

(1) $\dot{y}(t) = y(t)G$.

Then y^o is an equilibrium vector of $\mathcal{Y}$ iff y^o is a probability vector satisfying

(2) $0 = y^o G$.

Proof

By Definition C.3.1, y^o is an equilibrium vector of $\mathcal{Y}$ iff y^o satisfies (1) such that $y^o(t) \equiv y^o$ for all $t \geq 0$.

But $y^o(t) \equiv y^o$ iff $y^o(t)G = \dot{y}^o(t) \equiv 0$, i.e. iff (2) holds. □

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