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SOLUTION OF THE TWO-GROUP NEUTRON TRANSPORT EQUATION

PART I

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ABSTRACT

The two-group one-dimensional neutron transport equation with isotropic scattering is studied. No analytical solution is found, but the equations are cast in a form which is convenient for numerical computation. This computation involves the solution of two coupled singular integral equations. The explicit form of these equations is obtained for two half-space problems--the Milne problem and the constant isotropic source problem.

I. INTRODUCTION

In recent years, a great deal of effort has gone into the solution of the one-speed neutron transport equation. Most of this work is reviewed in a recent book.[1] Attempts to deal with the energy dependent transport equation have been less successful, although a certain amount of progress has been made. These attempts have more or less followed two lines, a multigroup approach and a continuous energy treatment. In the multigroup method, energy effects are treated in considerably grosser fashion than in the continuous energy scheme but in the latter spatial effects have been quite difficult to include in much detail. However, in a recent paper,[2] it was shown that the multigroup scheme is equivalent to a continuous energy scheme with a particular degenerate scattering kernel. A good review of the continuous schemes has been given by Kuščer,[3,4] to which we refer the reader for references.

In the present paper we consider the multigroup approach, in particular the two-group case. As usual, we limit ourselves to a single space dimension and to isotropic scattering. Our method should be readily generalizable to more than two groups and to anisotropic scattering.

The two-group case has been studied previously by Żelazny and Kuszell.[5] They succeeded in proving a "completeness theorem" which in effect tells us that the normal modes which we develop are adequate to solve all infinite and semi-infinite medium problems. We depend upon this theorem, but because the results are not presented in a form convenient for numerical computation much of their analysis is repeated. Analytical solutions have been obtained in two recent papers[6] for a very special problem, but the restrictions upon the parameters involved are invalid in the neutron transport case. (The work was applied to radiative transport.) Siewert and Shieh[7] have carried out some preliminary work on the more general problem considered here, but their results can be applied directly only to infinite medium problems. Recently some unpublished work[8] along similar lines to that presented here has come to our attention.

In Section II the eigenvalues and eigenfunctions for the two-group transport equation are discussed. Essentially these same eigenvalues and eigenfunctions were obtained by Żelazny and Kuszell,[5] although, as in Ref. 6, we select certain convenient linear combinations of these eigenfunctions.

In Section III we follow the Case[1] approach in deriving a convenient pair of coupled equations for the expansion coefficients of an arbitrary function. Some important simplifications are made in these equations by the judicious use of a two-group X-function identity. These coupled equations are easily solved by an iteration technique using a computer program. (No analytical solution has been found and in fact it appears unlikely that an analytical solution exists.) The numerical techniques and explicit results are presented in a companion paper.[9]

The application to typical half-space problems is made in Section IV. In particular, the solutions to the two-group Milne and constant source problems are presented.

II. EIGENVALUES AND EIGENFUNCTIONS

The one-dimensional, two-group transport equation for isotropic scattering can be written as[1]

$$\mu \frac{\partial \underline{\psi}(z,\mu)}{\partial z} + \underline{\underline{\Sigma}}\, \underline{\psi}(z,\mu) = \underline{\underline{C}} \int_{-1}^{1} \underline{\psi}(z,\mu)d\mu \tag{1}$$

Here

$$\underline{\psi}(z,\mu) = \begin{pmatrix} \psi_1(z,\mu) \\ \psi_2(z,\mu) \end{pmatrix}, \quad \underline{\underline{C}} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

and

$$\underline{\underline{\Sigma}} = \begin{pmatrix} \sigma & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma > 1$$

where

$$C_{ij} = \sigma_{ij}/2\sigma_2 \quad \text{and} \quad \sigma = \sigma_1/\sigma_2 .$$

$\psi_1(z,\mu)$ and $\psi_2(z,\mu)$ are the angular fluxes in groups one and two, respectively; distance is measured in units of mean free path of the second group. We have (without loss of generality) ordered the groups such that $\sigma_2 < \sigma_1$ where σ_1 and σ_2 are the macroscopic total cross sections of the respective groups and σ_{ij}'s are the macroscopic scattering transfer cross sections which describe the scattering from group j to group i. As usual, we assume solutions of the form

$$\underline{\psi}(z,\mu) = e^{-z/\eta}\, \underline{F}(\eta,\mu). \tag{2}$$

This ansatz when substituted into Eq. (1), and after cancellation of the space dependence, yields

$$\underline{\underline{A}}\, \underline{F}(\eta,\mu) = \eta \underline{\underline{C}} \int_{-1}^{1} \underline{F}(\eta,\mu)d\mu \tag{3}$$

where

$$\underline{\underline{A}} = \begin{pmatrix} \sigma\eta-\mu & 0 \\ 0 & \eta-\mu \end{pmatrix}. \tag{4}$$

The eigenvalues (η is the eigenvalue) and the associated eigenfunctions of Eq. (3) are discussed in some detail in Ref. 6. From the matrix $\underline{\underline{A}}$ we note that three separate regions of the eigenvalue spectrum must be considered depending on the vanishing or nonvanishing of the respective terms in $\underline{\underline{A}}$. We list here the three regions of the eigenvalue spectrum with their associated eigenfunctions as obtained in Ref. 6.

Region 1: $\eta\epsilon[-1/\sigma, 1/\sigma]$ with degenerate continuum eigenvectors

$$\underline{F}_1^{(1)}(\eta,\mu) = \begin{pmatrix} \dfrac{C_{11}\eta P}{\sigma\eta-\mu} + [1-2\eta C_{11}\tau(\sigma\eta)]\delta(\sigma\eta-\mu) \\ \\ \dfrac{C_{21}\eta P}{\eta-\mu} - 2C_{21}\eta\tau(\eta)\delta(\eta-\mu) \end{pmatrix} \tag{5a}$$

and

$$\underline{F}_2^{(1)}(\eta,\mu) = \begin{pmatrix} \dfrac{C_{12}\eta P}{\sigma\eta-\mu} - 2C_{12}\eta\tau(\sigma\eta)\delta(\sigma\eta-\mu) \\ \\ \dfrac{C_{22}\eta P}{\eta-\mu} + [1-2\eta C_{22}\tau(\eta)]\delta(\eta-\mu) \end{pmatrix}. \tag{5b}$$

Here $\tau(\eta) = \tanh^{-1}\eta$.

Region 2: $\eta\epsilon[-1, -1/\sigma]$ and $[1/\sigma, 1]$: The continuum eigenfunction is

$$\underline{F}^{(2)}(\eta,\mu) = \begin{pmatrix} \dfrac{C_{12}\eta}{\sigma\eta-\mu} \\ \\ \dfrac{\eta t(\eta)P}{\eta-\mu} + \lambda(\eta)\delta(\eta-\mu) \end{pmatrix} \tag{6}$$

where

$$t(\eta) = C_{22} - 2\eta C\tau(1/\sigma\eta) \qquad (7)$$

and

$$\lambda(\eta) = 1 - 2\eta C_{22}\tau(\eta) - 2\eta C_{11}\tau(1/\sigma\eta) + 4C\eta^2\tau(\eta)\tau(1/\sigma\eta) \ . \qquad (8)$$

We have introduced the abbreviation

$$C = \det \underline{\underline{C}} \ .$$

The symbol P in the above equations indicates as usual that integrals involving these eigenfunctions are to be interpreted in the sense of the Cauchy principal value.

Region 3: $\eta \notin [-1,1]$. Here we have the discrete eigenvectors

$$\underline{F}_{i\pm}(\mu) = \begin{pmatrix} C_{12}\eta_i/\sigma\eta_i \mp \mu \\ \eta_i t(\eta_i)/\eta_i \mp \mu \end{pmatrix} = \begin{pmatrix} F_{i\pm,1}(\mu) \\ F_{i\pm,2}(\mu) \end{pmatrix} \qquad (9)$$

where $t(\eta_i)$ is defined by Eq. (7) and the eigenvalue η_i is the positive root of the dispersion equation,

$$\Omega(z) = 1 - 2C_{11}z\tau(1/\sigma z) - 2C_{22}z\tau(1/z) + 4Cz^2\tau(1/z)\tau(1/\sigma z) = 0 \ . \qquad (10)$$

For the problems we have studied there are only two real roots to Eq. (10) which we denote by $\pm\eta_1$. In general, there are either two or four roots which are either real and/or pure imaginary, and thus occur in ± pairs. These roots have been studied in considerable detail by Baran[8] and we reproduce in Table l his analysis of the eigenvalues. We note that $\Omega(z)$ is analytic in the complex plane cut along the real axis from -1 to 1. Introducing the notation

$$\begin{aligned} \Theta_i(\eta) &= 1, \quad \eta\epsilon\text{Region } i \\ &= C, \quad \eta\notin i , \end{aligned} \tag{11}$$

we easily find

$$\begin{aligned} \Omega^{\pm}(\mu) = {} & 1 - 2C_{22}\mu\tau(\mu) - 2C_{11}\mu\tau(\sigma\mu)\Theta_1(\mu) - 2C_{11}\mu\tau(1/\sigma\mu)\Theta_2(\mu) + 4C\mu^2\tau(\mu)\tau(\sigma\mu)\Theta_1(\mu) \\ & + 4C\mu^2\tau(\mu)\tau(1/\sigma\mu)\Theta_2(\mu) - \pi^2\mu^2 C\Theta_1(\mu) \pm \pi i\mu[C_{22}+C_{11}\Theta_1(\mu)-2C\mu\tau(\sigma\mu)\Theta_1(\mu) \\ & - 2C\mu\tau(\mu)\Theta_1(\mu) - 2C\mu\tau(1/\sigma\mu)\Theta_2(\mu)] . \end{aligned} \tag{12}$$

Here $\Omega^{\pm}$ represent the boundary values of $\Omega(z)$ as the branch cut is approached from $\binom{\text{above}}{\text{below}}$, i.e., $\Omega^{\pm}(\mu) = \lim_{\epsilon\to 0} \Omega(\mu\pm i\epsilon)$. It is convenient to use the following linear combination of Region 1 eigenvectors:

$$\underline{\phi}_1(\eta,\mu) = \frac{1}{C_{11}} \underline{F}_1^{(1)}(\eta,\mu) - \frac{1}{C_{12}} \underline{F}_2^{(1)}(\eta,\mu). \tag{13a}$$

Explicitly,

$$\underline{\phi}_1(\eta,\mu) = \begin{pmatrix} \frac{1}{C_{11}} \delta(\sigma\eta-\mu) \\ - \frac{C\eta P}{C_{11}C_{12}(\eta-\mu)} - \frac{\delta(\eta-\mu)}{C_{12}C_{11}} [C_{11}-2C\eta\tau(\eta)] \end{pmatrix}, \tag{13b}$$

$$\eta\epsilon[-1/\sigma,1/\sigma] .$$

The following linear combination is a convenient eigenvector valid over the full range, $\eta\epsilon[-1,1]$:

$$\begin{aligned} \underline{\phi}_2(\eta,\mu) = {} & 2C_{12}\eta\tau(\sigma\eta)\Theta_1(\eta)\underline{F}_1^{(1)}(\eta,\mu) + [1-2C_{11}\eta\tau(\sigma\eta)]\Theta_1(\eta)\underline{F}_2^{(1)}(\eta,\mu) \\ & + \underline{F}^{(2)}(\eta,\mu)\Theta_2(\eta). \end{aligned} \tag{14a}$$

Again the explicit form is

$$\underline{\phi}_2(\eta,\mu) = \begin{pmatrix} C_{12}\eta P/(\sigma\eta-\mu) \\ \\ \dfrac{\eta g(\eta) P}{\eta-\mu} + \lambda(\eta)\delta(\eta-\mu) \end{pmatrix} . \tag{14b}$$

Here we have defined

$$g(\eta) = C_{22} - 2C\eta\tau(\sigma\eta)\Theta_1(\eta) - 2C\eta\tau(1/\sigma\eta)\Theta_2(\eta) \tag{15}$$

and

$$\lambda(\eta) = 1 - 2C_{22}\eta\tau(\eta) - 2C_{11}\eta\tau(\sigma\eta)\Theta_1(\eta) - 2C_{11}\eta\tau(1/\sigma\eta)\Theta_2(\eta) + 4C\eta^2\tau(\eta)$$
$$\times\ \tau(1/\sigma\eta)\Theta_2(\eta) + 4C\eta^2\tau(\eta)\tau(\sigma\eta)\Theta_1(\eta) = \frac{\Omega^+(\eta)+\Omega^-(\eta)}{2} + \pi^2\eta^2 C\Theta_1(\eta) \tag{16}$$

Thus, the eigenvectors consists of two continuum modes $\underline{\phi}_1(\eta,\mu)$ [Eq. (13b)] $\eta\epsilon[-1/\sigma,1/\sigma]$; $\underline{\phi}_2(\eta,\mu)$ [Eq. (14b)], $\eta\epsilon[-1,1]$; and the discrete modes $\underline{F}_{i\pm}(\mu)$ [Eq. (9)]. A half-range completeness theorem for these modes has been proved in Ref. 5. In the next section we reduce the basic equation to one convenient for numerical analysis.

III. TWO-GROUP EQUATIONS

The half-range completeness theorem[1] states that an arbitrary (two component) function $\underline{\psi}(\mu)$ can be expanded in terms of half the eigenvectors if we consider only half of the range, e.g., $\mu > 0$. Specifically,

$$\underline{\psi}(\mu) = \begin{pmatrix} \psi_1(\mu) \\ \psi_2(\mu) \end{pmatrix} = \sum_i A_{i+}\,\underline{F}_{i+}(\mu) + \int_0^{1/\sigma} \alpha_1(\mu)\underline{\phi}_1(\eta,\mu)d\eta$$
$$+ \int_0^1 \alpha_2(\eta)\underline{\phi}_2(\eta,\mu)d\eta, \quad \mu > 0 . \tag{17}$$

Typical proofs involve the solution of this equation for the unknown coefficients A_{i+} and $\alpha_j(\eta)$. This procedure also has the virtue of providing the required answer, i.e., the expansion coefficients. In the case considered here (as is stated in Section I) no analytical solution has been found. However, in Ref. 5 the existence of a unique solution has been proved. Thus, we develop a numerical solution to Eq. (17) by a method that parallels closely typical half-range completeness proofs. By appropriate use of half-space X-function identities, we obtain a form which is found to be most convenient for numerical analysis.

We delete the discrete mode from Eq. (17) the discrete modes will be introduced later), substitute the explicit form of $\phi_1(\eta,\mu)$ and $\phi_2(\eta,\mu)$, Eqs. (13b) and (14b), into Eq. (17) and perform the integrations over the delta functions to obtain

$$\psi_1(\mu) = \frac{\alpha_1(\mu/\sigma)}{\sigma C_{11}} + C_{12} P \int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\sigma\eta-\mu} \tag{18}$$

and

$$\psi_2(\mu) = -\frac{C}{C_{11}C_{12}} P \int_0^{1/\sigma} \frac{\eta\alpha_1(\eta)d\eta}{\eta-\mu} - \frac{1}{C_{11}C_{12}} [C_{11}-2C\mu\tau(\mu)]\alpha_1(\mu)\Theta_1(\mu)$$
$$+ P \int_0^1 \frac{\eta\alpha_2(\eta)g(\eta)d\eta}{\eta-\mu} + \lambda(\mu)\alpha_2(\mu). \tag{19}$$

Making the change of variable $\mu \rightarrow \mu\sigma$ in Eq. (18) and solving for $\alpha_1(\mu)$, we obtain

$$\alpha_1(\mu) = \sigma C_{11}\psi_1(\sigma\mu) - C_{11}C_{12} P \int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-\mu} . \tag{20}$$

This equation will determine $\alpha_1(\mu)$ once $\alpha_2(\mu)$ is known. We now obtain the equation for $\alpha_2(\mu)$ by inserting Eq. (20) into Eq. (19) to yield

$$\psi_2(\mu) + \frac{C\sigma}{C_{12}} P\int_0^1 \frac{\eta\psi_1(\sigma\eta)d\eta}{\eta-\mu} + \frac{\sigma}{C_{12}}[C_{11}-2C\mu\tau(\mu)]\psi_1(\sigma\mu)\Theta_1(\mu)$$

$$= CP\int_0^{1/\sigma}\frac{\eta' d\eta'}{\eta'-\mu} P\int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-\eta'} + [C_{11}-2C\mu\tau(\mu)]\Theta_1(\mu)\, P\int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-\mu}$$

$$+ P\int_0^1 \frac{\eta\alpha_2(\eta)g(\eta)d\eta}{\eta-\mu} + \lambda(\mu)\alpha_2(\mu) . \tag{21}$$

Using the Poincaré-Bertrand formula[10] and the partial fraction decomposition

$$\frac{\eta'}{(\eta'-\mu)(\eta-\eta')} = \frac{1}{\eta-\mu}\left(\frac{\mu}{\eta'-\mu} + \frac{\eta}{\eta-\eta'}\right) , \tag{22}$$

we perform an integration over $d\eta'$ in the first term on the right-hand side of Eq. (21) to obtain

$$CP\int_0^{1/\sigma}\frac{\eta' d\eta'}{\eta'-\mu} P\int_0^1 \frac{\eta\alpha_2(\eta)d\eta'}{\eta-\eta'} = P\int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-\mu}\left[\Theta_1(\mu)C\mu\ln\left(\frac{1}{\sigma\mu}-1\right)\right.$$

$$+ \Theta_2(\mu)C\mu\ln\left(1-\frac{1}{\sigma\mu}\right) - \Theta_1(\eta)C\eta\ln\left(\frac{1}{\sigma\eta}-1\right)$$

$$\left. - \Theta_2(\eta)C\eta\ln\left(1-\frac{1}{\sigma\eta}\right)\right] - C\pi^2\mu^2\alpha_2(\mu)\Theta_1(\mu). \tag{23}$$

We can write Eq. (15) as

$$g(\eta) = C_{22} - C\eta\ln\left(1+\frac{1}{\sigma\eta}\right) + C\Theta_1(\eta)\eta\ln\left(\frac{1}{\sigma\eta}-1\right) + C\Theta_2(\eta)\eta\ln\left(1-\frac{1}{\sigma\eta}\right). \tag{24}$$

The insertion of Eqs. (23) and (24) into Eq. (21) yields (after some cancellation and rearrangement), the compact form

$$\psi'(\mu) + C\int_0^1 \frac{\eta\alpha_2(\eta)k(\eta,\mu)d\eta}{\eta-\mu} = \frac{\Omega^+(\mu)-\Omega^-(\mu)}{2\pi i\mu} P\int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-\mu} + \frac{\Omega^+(\mu)+\Omega^-(\mu)}{2}\alpha_2(\mu). \tag{25}$$

Here we have defined

$$\psi'(\mu) = \psi_2(\mu) + \frac{C\sigma}{C_{12}} P\int_0^1 \frac{\eta\psi_1(\sigma\eta)d\eta}{\eta-\mu} + \frac{\sigma}{C_{12}}\psi_1(\sigma\mu)\Theta_1(\mu)[C_{11}-2C\mu\tau(\mu)] \tag{26}$$

and

$$k(\eta,\mu) = \eta\ell n\left(1+\frac{1}{\sigma\eta}\right) - \mu\ell n\left(1+\frac{1}{\sigma\mu}\right) . \tag{27}$$

We note that $\psi'(\mu)$ is a known function.

The second term on the left-hand side of Eq. (25) is a nondegenerate Fredholm term. If this term were absent Eq. (25) could be solved directly by standard procedures.[1] If it were degenerate, the method of Shure and Natelson[11] could be used. However, neither of these conditions obtain and no closed form solution of Eq. (25) has been found. Thus, we shall describe an iterative procedure in Part II of this work[9] similar to that used by Mitsis[1,12] to solve the critical problem and by McCormick and Mendelson[13] in treating the slab albedo problem.

We define

$$\psi''(\mu) = \psi'(\mu) + C\int_0^1 \frac{\eta\alpha_2(\eta)k(\eta,\mu)d\eta}{\eta-\mu} . \tag{28}$$

Next we introduce a function N(z) defined by

$$N(z) = \frac{1}{2\pi i}\int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-z} . \tag{29}$$

If $\alpha_2(\eta)$ of "class G" exists then $N(z)$ has the following properties[1]:

1. $N(z)$ is analytic in the complex plane cut along the real axis from 0 to 1.

2. $N(z) \sim 1/z$ as $z \to \infty$.

3. $N^{\pm}(\mu) = \frac{1}{2\pi i} P \int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-\mu} \pm \frac{1}{2} \mu\alpha_2(\mu)$.

We note from Property 3 that

$$\alpha_2(\mu) = \frac{N^+(\mu)-N^-(\mu)}{\mu} . \qquad (30)$$

Inserting Eqs. (28) and using Property 3 in Eq. (25) we find (after some rearrangement) that

$$\mu\psi''(\mu) = N^+(\mu)\Omega^+(\mu) - \Omega^-(\mu)N^-(\mu). \qquad (31)$$

We recall that $\Omega(z)$ is analytic in the complex plane cut from -1 to 1. From Property 1, $N(z)$ is analytic in the complex plane cut from 0 to 1. This requires (as in one-speed theory)[1] the introduction of a function $X(z)$ such that

$$\frac{X^+(\mu)}{X^-(\mu)} = \frac{\Omega^+(\mu)}{\Omega^-(\mu)} , \quad \mu > 0. \qquad (32)$$

The $X(z)$ function satisfying the necessary restrictions[1] for the half-range and for one pair of discrete roots only is

$$X(z) = \frac{1}{1-z} \exp \frac{1}{\pi} \int_0^1 \frac{\text{Arg } \Omega^+(\mu)d\mu}{\mu-z} . \qquad (33)$$

The ratio condition, Eq. (32), is inserted into Eq. (31) to yield

$$\gamma(\mu)\psi''(\mu) = N^+(\mu)X^+(\mu) - N^-(\mu)X^-(\mu) \qquad (34)$$

where

$$\gamma(\mu) = \mu \frac{X^+(\mu)}{\Omega^+(\mu)} . \tag{35}$$

Assuming that the left-hand side of Eq. (34) is known, we can write the solution as

$$N(z) = \frac{1}{2\pi i X(z)} \int_0^1 \frac{\gamma(\mu)\psi''(\mu)d\mu}{\mu - z} . \tag{36}$$

The Plemelj formulae[10] give the boundary values of N(z) from Eq. (36). We insert these boundary values into Eq. (30) to obtain

$$\alpha_2(\mu) = \frac{1}{2\pi i\mu}\left(\frac{1}{X^+(\mu)} - \frac{1}{X^-(\mu)}\right) P \int_0^1 \frac{\gamma(\mu')\psi''(\mu')d\mu'}{\mu' - \mu}$$

$$+ \frac{1}{2\mu}\left(\frac{1}{X^+(\mu)} + \frac{1}{X^-(\mu)}\right) \gamma(\mu)\psi''(\mu) . \tag{37}$$

(However, this is not a solution because we note from Eq. (28) that the unknown $\alpha_2(\mu)$ is still contained in $\psi''(\mu)$.)

We next define

$$\ell_1(\mu) = -\frac{1}{2\pi i\mu}\left(\frac{1}{X^+(\mu)} - \frac{1}{X^-(\mu)}\right) = \frac{\Omega^+(\mu) - \Omega^-(\mu)}{2\pi i\gamma(\mu)\Omega^+(\mu)\Omega^-(\mu)} \tag{38a}$$

and

$$\ell_2(\mu) = \frac{1}{2\mu}\left(\frac{1}{X^+(\mu)} + \frac{1}{X^-(\mu)}\right)\gamma(\mu) = \frac{\Omega^+(\mu) + \Omega^-(\mu)}{2\Omega^+(\mu)\Omega^-(\mu)} . \tag{38b}$$

The last form of Eqs. (38) is derived from Eqs. (32) and (35). By defining the singular integral operator $O(\mu)$ by

$$O(\mu)\psi(\mu) = \ell_1(\mu) P \int_0^1 \frac{\gamma(\mu')\psi(\mu')d\mu'}{\mu - \mu'} + \ell_2(\mu)\psi(\mu) \tag{39}$$

we can rewrite Eq. (37) in the compact form

$$\alpha_2(\mu) = O(\mu)\psi''(\mu) . \tag{40}$$

Property 2 requires that $N(z) \sim 1/z$ as $|z| \to \infty$. We note from Eq. (33) that $X(z) \sim -1/z$ as $z \to \infty$. Thus, from Eq. (36), we must require

$$\int_0^1 \gamma(\mu) \left[\psi'(\mu) + C \int_0^1 \frac{d\eta\,\eta\alpha_2(\eta)k(\eta,\mu)}{\eta-\mu}\right] d\mu = 0 \tag{41}$$

where we have used Eq. (28). For the case of one pair of discrete roots we have the discrete eigenfunctions available to satisfy Eq. (41). We make the replacement of $\underline{\psi}(\mu)$ in Eq. (41) by

$$\underline{\psi}(\mu) - A_+ \underline{F}_{1+}(\mu) . \tag{42}$$

We recall that $\psi'(\mu)$ is defined by Eq. (26) and is a functional of the components of $\underline{\psi}(\mu)$, i.e., $\psi_1(\mu)$ and $\psi_2(\mu)$. We define $\phi_+(\mu)$ as the corresponding functional of the components of $\underline{F}_{1+}(\mu)$. Thus the replacement given by the expression (42) is equivalent to replacing $\psi'(\mu)$ in Eq. (41) by

$$\psi'(\mu) - A_+\phi_+(\mu) . \tag{43}$$

With this replacement made in Eq. (41) we then solve for A_+ to yield

$$A_+ = \frac{\int_0^1 \gamma(\mu)\psi'(\mu)d\mu + C\int_0^1 \gamma(\mu)d\mu \int_0^1 \frac{\eta\alpha_2(\eta)k(\eta,\mu)d\eta}{\eta-\mu}}{\int_0^1 \gamma(\mu)\phi_+(\mu)d\mu} . \tag{44}$$

Likewise with (43) inserted into (28) and subsequently into Eq. (40) we obtain

$$\alpha_2(\mu) = O(\mu)[\psi'(\mu)-A_+\phi_+(\mu)] + CO(\mu)\int_0^1 \frac{\eta\alpha_2(\eta)k(\eta,\mu)d\eta}{\eta-\mu} . \tag{45}$$

Finally, the replacement of $\psi_1(\sigma\mu)$ by $\psi_1(\sigma\mu)-A_+F_{1+,1}(\mu)$ in Eq. (20) yields

$$\alpha_1(\mu) = \sigma C_{11}[\psi_1(\sigma\mu)-A_+F_{1+,1}(\sigma\mu)] - C_{11}C_{12}P\int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-\mu} . \quad (46)$$

An iteration procedure for solving Eqs. (44) and (45) for A_+ and $\alpha_2(\mu)$ is discussed in Ref. 9. Providing this procedure converges, we can then insert A_+ and $\alpha_2(\mu)$ into Eq. (46) and solve for $\alpha_1(\mu)$ completely determining all expansion coefficients.

A number of important simplifications are now made for the terms containing $\phi_+(\mu)$ in Eqs. (44) and (45). First, from Eq. (26) we can write the explicit form of $\phi_+(\mu)$ as

$$\phi_+(\mu) = F_{1+,2}(\mu) + \frac{C\sigma}{C_{12}} P\int_0^1 \frac{\eta F_{1+,1}(\sigma\mu)}{\eta-\mu} + \frac{\sigma}{C_{12}} F_{1+,1}(\sigma\mu)\Theta_1(\mu)[C_{11}-2C\mu\tau(\mu)]. \quad (47)$$

Next we insert the discrete modes as given by Eq. (9) into Eq. (47) and perform the integration to obtain

$$\phi_+(\mu) = \frac{\eta_1}{\eta_1-\mu}\left[C_{22}+C_{11}\Theta_1(\mu)+C\mu\Theta_1(\mu)\ln\left(\frac{1}{\sigma\mu}-1\right) + C\mu\Theta_2(\mu)\ln\left(1-\frac{1}{\sigma\mu}\right)\right.$$
$$\left. - 2C\Theta_1(\mu)\mu\tau(\mu) - C\eta_1\ln\left(1+\frac{1}{\sigma\eta_1}\right)\right] . \quad (48)$$

We define the function

$$f(\mu) = \frac{\Omega^+(\mu)-\Omega^-(\mu)}{2\pi i\mu} = C_{22} + C_{11}\Theta_1(\mu) - 2C\Theta_1(\mu)\mu\tau(\mu) - C\mu\Theta_1(\mu)\ln\left(1+\frac{1}{\sigma\mu}\right)$$
$$+ C\mu\Theta_1(\mu)\ln\left(1-\frac{1}{\sigma\mu}\right) - C\mu\Theta_2(\mu)\ln\left(1+\frac{1}{\sigma\mu}\right) + C\mu\Theta_2(\mu)\ln\left(1-\frac{1}{\sigma\mu}\right) . \quad (49)$$

We insert Eq. (49) into Eq. (48) [recalling Eq. (27)] to write the compact form

$$\phi_+(\mu) = \frac{\eta_1}{\eta_1-\mu}[f(\mu)-Ck(\eta_1,\mu)] . \tag{50}$$

The substitution of Eq. (50) into the integrand of the denominator of Eq. (44) yields

$$\int_0^1 \gamma(\mu)\phi_+(\mu)d\mu = -\eta_1 X(\eta_1) - C\int_0^1 \frac{\gamma(\mu)\eta_1 k(\eta_1,\mu)d\mu}{\eta_1-\mu} \tag{51}$$

where we have used the X-function identity

$$X(z) = \int_0^1 \frac{\gamma(\mu)f(\mu)d\mu}{\mu-z} . \tag{52}$$

The proof of this identity parallels closely that for the one-group case.[1]

Next we consider the term $O(\mu)\phi_+(\mu)$ in Eq. (45). With $O(\mu)$ given by Eq. (39) and $\phi_+(\mu)$ by Eq. (50), we write

$$O(\mu)\phi_+(\mu) = \ell_1(\mu)P\int_0^1 \frac{\gamma(\mu')\eta_1 f(\mu')d\mu'}{(\mu-\mu')(\eta_1-\mu')} + \frac{\ell_2(\mu)\eta_1 f(\mu)}{\eta_1-\mu} - CO(\mu)\left[\frac{\eta_1 k(\eta_1,\mu)}{\eta_1-\mu}\right] . \tag{53}$$

The partial fraction decomposition

$$\frac{\eta_1}{(\mu-\mu')(\eta_1-\mu')} = \frac{\eta_1}{\eta_1-\mu}\left(\frac{1}{\mu-\mu'} - \frac{1}{\eta_1-\mu'}\right)$$

inserted into the first term on the right-hand side of Eq. (53) yields

$$\ell_1(\mu)P\int_0^1 \frac{\gamma(\mu')\eta_1 f(\mu')d\mu'}{(\mu-\mu')(\eta_1-\mu')} = \ell_1(\mu)\frac{\eta_1}{\eta_1-\mu}\left[P\int_0^1 \frac{\gamma(\mu')f(\mu')d\mu'}{\mu-\mu'} - \int_0^1 \frac{\gamma(\mu')f(\mu')d\mu'}{\eta_1-\mu'}\right] . \tag{54}$$

The last term in brackets in Eq. (54) is $-X(\eta_1)$ and the first term can be written in terms of the boundary values of X(z). Explicitly,

$$P \int_0^1 \frac{\gamma(\mu')f(\mu')d\mu'}{\mu-\mu'} = -\frac{X^+(\mu)+X^-(\mu)}{2} = -\frac{\gamma(\mu)}{2\mu}\left[\Omega^+(\mu) + \Omega^-(\mu)\right] . \qquad (55)$$

The last form in Eq. (55) was derived by using Eqs. (32) and (35). We insert Eq. (55) into Eq. (54) to obtain

$$\ell_1(\mu) P \int_0^1 \frac{\gamma(\mu')\eta_1 f(\mu')d\mu'}{(\mu-\mu')(\eta_1-\mu')} = \ell_1(\mu) \frac{\eta_1}{\eta_1-\mu} \left[X(\eta_1) - \frac{\gamma(\mu)}{2\mu}(\Omega^+(\mu)+\Omega^-(\mu))\right] . \qquad (56)$$

We insert Eqs. (38), (56), and (49) into Eq. (53) to obtain (after some cancellation)

$$O(\mu)\phi_+(\mu) = \frac{\ell_1(\mu)\eta_1 X(\eta_1)}{\eta_1-\mu} - CO(\mu)\left[\frac{\eta_1 k(\eta_1,\mu)}{\eta_1-\mu}\right] . \qquad (57)$$

Now Eqs. (51) and (57) can be substituted into Eqs. (44) and (45) to obtain a somewhat simpler form,

$$A_+ = \frac{\int_0^1 \gamma(\mu)\psi'(\mu)d\mu + C\int_0^1 \gamma(\mu)d\mu \int_0^1 \frac{\eta\alpha_2(\eta)k(\eta,\mu)d\eta}{\eta-\mu}}{-\eta_1 X(\eta_1) - C\int_0^1 \frac{\gamma(\mu)k(\eta_1,\mu)\eta_1 d\mu}{\eta_1-\mu}} \qquad (58a)$$

$$\alpha_2(\mu) = O(\mu)\psi'(\mu) + CO(\mu)\int_0^1 \frac{\eta\alpha_2(\eta)k(\eta,\mu)d\eta}{\eta-\mu}$$
$$-A_+\left\{\frac{\ell_1(\mu)\eta_1 X(\eta_1)}{\eta_1-\mu} - CO(\mu)\left[\frac{\eta_1 k(\eta_1,\mu)}{\eta_1-\mu}\right]\right\} . \qquad (58b)$$

We recall that the kernel $k(\eta,\mu)$ is defined by Eq. (27). The operator $O(\mu)$ is given by Eq. (39) and $\ell_1(\mu)$ by (38a). Equations (58a) and (58b) are the set which will be treated numerically,[9] as simultaneous equations for the expansion coefficients A_+ and $\alpha_2(\eta)$. After A^+ and $\alpha_2(\mu)$ are obtained from the numerical solution of Eqs. (58a) and (58b), $\alpha_1(\mu)$ is computed from

$$\alpha_1(\mu) = \sigma C_{11}\left[\psi_1(\sigma\mu) - A_+ \frac{C_{12}\eta_1}{\sigma(\eta_1-\mu)}\right] - C_{11}C_{12}P\int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-\mu} . \quad (58c)$$

This completes the reduction of the general two-group expansion to a form convenient for numerical analysis. In the next section, we shall find some specific forms for $\psi'(\mu)$ in Eqs. (58a) and (58b) and for $\psi_1(\sigma\mu)$ in Eq. (58c). We are then able to simplify the terms involving $\psi'(\mu)$ in Eqs. (58a) and (58b).

IV. APPLICATION TO MILNE AND CONSTANT SOURCE PROBLEMS

(a) Milne Problem

We define the two-group Milne problem[14] in a manner similar to the one-group case.[1] The solution must satisfy the conditions

$$\underline{\psi}(o,\mu) = 0, \quad \mu > 0 \quad (59a)$$

and

$$\underline{\psi}(z,\mu) \underset{z\to\infty}{\sim} \underline{F}_{1-}(\mu)e^{z/\eta_1} . \quad (59b)$$

The solution which obeys (59b) is expanded in the two-group normal modes of the transport equation as

$$\underline{\psi}(z,\mu) = A_- \underline{F}_{1-}(\mu)e^{z/\eta_1} + A_+ \underline{F}_{1+}(\mu)e^{-z/\eta_1} + \int_0^{1/\sigma} \alpha_1(\eta)\underline{\phi}_1(\eta,\mu)e^{-z/\eta}d\eta$$
$$+ \int_0^1 \alpha_2(\eta)\underline{\phi}_2(\eta,\mu)e^{-z/\eta} d\eta . \quad (60)$$

We use the boundary condition given by Eq. (59a) (normalize by setting $A_- = 1$) to obtain

$$\underline{\psi}_m(\mu) = -\underline{F}_{1-}(\mu) = A_+ \underline{F}_{1+}(\mu) + \int_0^{1/\sigma}\alpha_1(\eta)\underline{\phi}_1(\eta,\mu)d\eta + \int_0^1 \alpha_2(\eta)\underline{\phi}_2(\eta,\mu)d\eta. \quad (61)$$

The appropriate $\psi_m'(\mu)$ for the Milne problem [see Eq. (26)] is closely related to the $\phi_+(\mu)$ as defined by Eq. (47). In fact

$$\psi_m'(\mu) = -\phi_+(\mu_1-\eta_1) . \tag{62}$$

This means that [see Eq. (50)]

$$\psi_m'(\mu) = -\frac{\eta_1}{\eta_1+\mu}[f(\mu)-Ck(-\eta_1,\mu)]. \tag{63}$$

Thus we have from Eqs. (51) and (57) the result that

$$\int_0^1 \gamma(\mu)\psi_m'(\mu)d\mu = -\eta_1 X(-\eta_1) + C\int_0^1 \frac{\gamma(\mu')\eta_1 k(-\eta_1,\mu')d\mu'}{\eta_1+\mu} \tag{64}$$

and

$$O(\mu)\psi_m'(\mu) = -\frac{\ell_1(\mu)\eta_1 X(-\eta_1)}{\eta_1+\mu} + CO(\mu)\left[\frac{\eta_1 k(-\eta_1,\mu)}{\eta_1+\mu}\right] . \tag{65}$$

We insert Eqs. (64) and (65) into Eqs. (58a) and (58b) to yield

$$A_+ = \frac{-\eta_1 X(-\eta_1) + C\int_0^1 \frac{\gamma(\mu')\eta_1 k(-\eta_1,\mu')d\mu'}{\eta_1+\mu} + C\int_0^1 \gamma(\mu)d\mu\int_0^1 \frac{\eta\alpha_2(\eta)k(\eta,\mu)d\eta}{\eta-\mu}}{-\eta_1 X(\eta_1) - C\int_0^1 \frac{\gamma(\mu)\eta_1 k(\eta_1,\mu)d\mu}{\eta_1-\mu}} \tag{66}$$

and

$$\alpha_2(\mu) = -\frac{\ell_1(\mu)\eta_1 X(-\eta_1)}{\eta_1+\mu} + CO(\mu)\left[\frac{\eta_1 k(-\eta_1,\mu)}{\eta_1+\mu}\right] + CO(\mu)\int_0^1 \frac{\eta\alpha_2(\eta)k(\eta,\mu)d\eta}{\eta-\mu}$$
$$- A_+\left\{\frac{\ell_1(\mu)\eta_1 X(\eta_1)}{\eta_1-\mu} - CO(\mu)\left[\frac{\eta_1 k(\eta_1,\mu)}{\eta_1-\mu}\right]\right\} . \tag{67}$$

Since

$$\psi_1(\sigma\mu) = -\frac{C_{12}\eta_1}{\sigma(\eta_1+\mu)} , \tag{68}$$

we have from Eq. (46) that

$$\alpha_1(\mu) = -C_{11}C_{12}\eta_1\left(\frac{1}{\eta_1+\mu} + \frac{A_+}{\eta_1-\mu}\right) - C_{11}C_{12}P\int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-\mu} . \qquad (69)$$

Equations (66), (67), and (69) are the final reduced equations for the Milne problem which are solved by numerical methods. We note that Eqs. (66) and (67) are two coupled equations for the expansion coefficients A_+ and $\alpha_2(\mu)$. The operator $O(\mu)$ in Eq. (67) is a singular integral operator which requires special treatment for numerical analysis,[9] otherwise the solution is quite straightforward. Finally with A_+ and $\alpha_2(\mu)$ known, Eq. (69) provides the solution for $\alpha_1(\mu)$.

We can easily prove from the form of the operator $O(\mu)$ and Eq. (58b) that

$$\alpha_2(\mu) \to 0, \quad \mu = 1/\sigma \text{ and } \quad \mu = 1. \qquad (70)$$

This result is important because otherwise the angular flux would be singular at these values of μ.

With the expansion known from the computer solution of Eqs. (66), (67), and (69), the two-group angular fluxes are calculated from Eq. (60). The eigenvectors given by Eqs. (13b), (14b), and (9) are substituted into Eq. (60) and the integration performed where possible to derive

$$\psi_1(z,\mu) = \frac{C_{12}\eta_1 e^{z/\eta_1}}{\sigma\eta_1+\mu} + \frac{C_{12}\eta_1 A_+ e^{-z/\eta_1}}{\sigma\eta_1-\mu} + \frac{1}{C_{11}\sigma}\alpha_1(\mu/\sigma)e^{-\sigma z/\mu}[\Theta_1(\mu)+\Theta_2(\mu)]$$

$$+ C_{12}P\int_0^1 \frac{\eta\alpha_2(\eta)e^{-z/\eta}d\eta}{\sigma\eta-\mu} \qquad (71)$$

where

$$-1 \leq \mu \leq 1,$$

and

$$\psi_2(z,\mu) = \frac{\eta_1 t(\eta_1)e^{z/\eta_1}}{\eta_1+\mu} + \frac{\eta_1 t(\eta_1)A_+e^{-z/\eta_1}}{\eta_1-\mu} - \frac{C}{C_{11}C_{12}} P\int_0^{1/\sigma} \frac{\eta\alpha_1(\eta)e^{-z/\eta}d\eta}{\eta-\mu}$$

$$- \frac{\alpha_1(\mu)}{C_{12}C_{11}} [C_{11}-2C\mu\tau(\mu)]\Theta_1(\mu)e^{-z/\mu} + P\int_0^1 \frac{\eta g(\eta)\alpha_2(\eta)e^{-z/\eta}d\eta}{\eta-\mu}$$

$$+ \lambda(\mu)\alpha_2(\mu)e^{-z/\mu} [\Theta_1(\mu)+\Theta_2(\mu)], \quad \mu\epsilon[-1,1]. \tag{72}$$

The total flux and current for each group are derived by appropriate integrals over $d\mu$ and $\mu d\mu$, respectively. We shall use superscripts on the ρ's to indicate group number. Thus,

$$\rho_0^{(1)}(z) = \int_{-1}^1 \psi_1(z,\mu)d\mu = 2C_{12}\eta_1\tau(1/\sigma\eta_1)[e^{z/\eta_1}+A_+e^{-z/\eta_1}] + \frac{1}{C_{11}}\int_0^{1/\sigma}$$

$$\times \alpha_1(\eta)e^{-z/\eta} d\eta - C_{12}\int_0^1 \eta\alpha_2(\eta)e^{-z/\eta} [g(\eta)-C_{22}]d\eta \tag{73a}$$

$$\rho_1^{(1)}(z) = \int_{-1}^1 \mu\psi_1(z,\mu)d\mu = 2C_{12}\eta_1 e^{z/\eta_1}[1-\sigma\eta_1\tau(1/\sigma\eta_1)] + 2C_{12}\eta_1 e^{-z/\eta_1}A_+$$

$$\times [\sigma\eta_1\tau(1/\sigma\eta_1)-1] + \frac{\sigma}{C_{11}} \int_0^{1/\sigma} \eta\alpha_1(\eta)e^{-z/\eta}d\eta - 2C_{12}\int_0^1 \eta\alpha_2(\eta)e^{-z/\eta}d\eta$$

$$- \sigma C_{12}\int_0^1 \eta^2\alpha_2(\eta)e^{-z/\eta} [g(\eta)-C_{22}]d\eta \tag{73b}$$

$$\rho_0^{(2)}(z) = \int_{-1}^1 \psi_2(z,\mu)d\mu = 2\eta_1 t(\eta_1)\tau(1/\eta_1)[e^{z/\eta_1}+A_+e^{-z/\eta_1}] - \frac{2C}{C_{11}C_{12}}$$

$$\times \int_0^{1/\sigma} \eta\alpha_1(\eta)\tau(\eta)e^{-z/\eta}d\eta - \frac{1}{C_{11}C_{12}} \int_0^{1/\sigma}[C_{11}-2C\eta\tau(\eta)]\alpha_1(\eta)e^{-z/\eta}d\eta$$

$$+ 2\int_0^1 \eta g(\eta)\alpha_2(\eta)\tau(\eta)e^{-z/\eta} d\eta + \int_0^1 \lambda(\eta)\alpha_2(\eta)e^{-z/\eta} d\eta \tag{73c}$$

$$\rho_1^{(2)}(z) = \int_{-1}^{1} \mu\psi_2(z,\mu)d\mu = 2\eta_1 t(\eta_1)e^{z/\eta_1}[1-\eta_1\tau(1/\eta_1)] + 2\eta_1 t(\eta_1)e^{-z/\eta_1}$$

$$\times\ A_+[\eta_1\tau(1/\eta_1)-1] - \frac{2C}{C_{11}C_{12}}\int_0^{1/\sigma} \eta\alpha_1(\eta)[\eta\tau(\eta)-1]e^{-z/\eta}\,d\eta$$

$$- \frac{1}{C_{11}C_{12}}\int_0^{1/\sigma} \eta\alpha_2(\eta)[C_{11}-2C\eta\tau(\eta)]e^{-z/\eta}\,d\eta + 2\int_0^{1} \eta g(\eta)\alpha_2(\eta)$$

$$\times\ [\eta\tau(\eta)-1]e^{-z/\eta}\,d\eta + \int_0^{1} \eta\lambda(\eta)\alpha_2(\eta)e^{-z/\eta}\,d\eta. \tag{73d}$$

The extrapolation distance is given by either Eq. (73a) or (73c). Thus we wish to determine z_o such that

$$e^{z_o/\eta_1} + A_+e^{-z_o/\eta_1} = 0\ . \tag{74}$$

The solution for z_o from this equation gives the same result as in the one-speed case,[1]

$$z_o = -\frac{\eta_1}{2}\,\ell n\left(-\frac{1}{A_+}\right)\ . \tag{75}$$

Again we emphasize that for one pair of discrete roots the extrapolation distances are equal for each group. For four discrete roots we would calculate a different extrapolation distance for each group. Baran[8] has studied this problem in detail.

(b) Constant Isotropic Source Problem

Assume an isotropic source S_{o2} in Group 2. (The associated problem with a source in group one is virtually identical.) We now seek solutions to the inhomogeneous transport equation which vanish at $+\infty$ subject to the condition,

$$\underline{\psi}(o,\mu) = 0 \quad \text{for} \quad \mu > 0\ . \tag{76}$$

The solution consists of a particular integral $\underline{\psi}_p(z,\mu)$ plus a solution of the homogeneous equations, $\underline{\psi}_h(z,\mu)$. The latter consists only of those modes which vanish as $+\infty$, i.e.,

$$\underline{\psi}_h(z,\mu) = A_+ e^{-z/\eta_1} \underline{F}_{1+}(\mu) + \int_0^{1/\sigma} \alpha_1(\eta)\underline{\phi}_1(\eta,\mu)e^{-z/\eta}\, d\eta + \int_0^1 \alpha_2(\eta)\underline{\phi}_2(\eta,\mu)e^{-z/\eta} d\eta. \tag{77}$$

The particular integral can be found from the two-group transport equation [Eq. (1)] in the limit as $z\to\infty$. In this case, $\underline{\psi}(z,\mu)$ approaches a constant value denoted for each group by ψ_{1S} and ψ_{2S}. With $\underline{\psi}(z,\mu)$ constant, Eq. (1) reduces to a pair of simultaneous equations which are easily solved to yield

$$\psi_{1S} = \frac{2SC_{12}}{(1-2C_{22})(\sigma-2C_{11})-4C_{12}C_{21}} = S_1 \tag{78}$$

and

$$\psi_{2S} = \frac{S(\sigma-2C_{11})}{(1-2C_{22})(\sigma-2C_{11})-4C_{12}C_{21}} = S_2 \tag{79}$$

where

$$S = \frac{S_{20}}{\sigma_2} .$$

The complete solution is written as

$$\underline{\psi}(z,\mu) = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} + A_+ e^{-z/\eta_1} \underline{F}_{1+}(\mu) + \int_0^{1/\sigma} \alpha_1(\eta)\underline{\phi}_1(\eta,\mu)e^{-z/\eta}\, d\eta$$

$$+ \int_0^1 \alpha_2(\eta)\underline{\phi}_2(\eta,\mu)e^{-z/\eta}\, d\eta . \tag{80}$$

By setting z = 0 in Eq. (80) and applying Eq. (76), we obtain

$$-\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \int_0^{1/\sigma} \alpha_1(\eta)\underline{\phi}_1(\eta,\mu)d\eta + \int_0^1 \alpha_2(\eta)\underline{\phi}_2(\eta,\mu)d\eta + A_+ \underline{F}_{1+}(\mu). \quad (81)$$

We note from Eq. (81) that

$$\underline{\psi}(\mu) = \begin{pmatrix} \psi_1(\mu) \\ \psi_2(\mu) \end{pmatrix} = -\begin{pmatrix} S_1 \\ S_2 \end{pmatrix} . \quad (82)$$

By inserting Eq. (82) into the appropriate terms of Eq. (26) and then performing the integration on the second term on the right-hand side, we find

$$\psi_S'(\mu) = -S_2 - \frac{\sigma S_1}{C_{12}} \left[\frac{C}{\sigma} + C\mu\Theta_1(\mu)\,\ell n \left(\frac{1}{\sigma\mu} - 1 \right) + C\mu\Theta_2(\mu)\,\ell n \left(1 - \frac{1}{\sigma\mu} \right) \right.$$
$$\left. - 2C\mu\tau(\mu)\Theta_1(\mu) + C_{11}\Theta_1(\mu) \right] . \quad (83)$$

The term in brackets in the second term on the right-hand side of Eq. (83) has terms similar to Eq. (49). In fact we can write

$$\psi_S'(\mu) = -S_2 - \frac{\sigma S_1}{C_{12}} \left[\frac{C}{\sigma} + f(\mu) + C\mu\,\ell n \left(1 + \frac{1}{\sigma\mu} \right) - C_{22} \right] . \quad (84)$$

We recall from Eqs. (57) that

$$O(\mu)\psi_S'(\mu)$$

and

$$\int_0^1 \gamma(\mu)\psi_S'(\mu)d\mu$$

are required in order to calculate the expansion coefficients A_+ and $\alpha_2(\mu)$. By the same method that was used in Section III to simplify $O(\mu)\phi_+(\mu)$, we can easily prove

$$O(\mu)f(\mu) = 0 . \tag{85}$$

Also from Eq. (52) we have that

$$\lim_{z\to\infty} zX(z) = -\int_0^1 \gamma(\mu)f(\mu)d\mu . \tag{86}$$

But from Eq. (33)

$$X(z) \underset{z\to\infty}{\sim} -\frac{1}{z}$$

therefore, from Eq. (86),

$$\int_0^1 \gamma(\mu)f(\mu)d\mu = 1.$$

Thus for constant source problems, we obtain

$$O(\mu)\psi_S'(\mu) = O(\mu)\left[w - \frac{\sigma S_1 C}{C_{12}}\,\mu \ln\left(1 + \frac{1}{\sigma\mu}\right)\right] \tag{87a}$$

and

$$\int_0^1 \gamma(\mu)\psi_S'(\mu)d\mu = -\frac{\sigma S_1}{C_{12}} + \int_0^1 \gamma(\mu)\left[w - \frac{\sigma S_1 C}{C_{12}}\,\mu \ln\left(1 + \frac{1}{\sigma\mu}\right)\right]d\mu \tag{87b}$$

where

$$w = -S_2 - \frac{S_1 C}{C_{12}} + \frac{\sigma S_1 C_{22}}{C_{12}} \tag{88}$$

and S_1 and S_2 are defined by Eqs. (78) and (79), respectively.

By inserting Eqs. (87) into Eqs. (58), we obtain

$$A_+ = \frac{-\frac{\sigma S_1}{C_{12}} + \int_0^1 d\mu\gamma(\mu)\left[w - \frac{\sigma S_1 C}{C_{12}}\,\mu\ln\left(1 + \frac{1}{\sigma\mu}\right)\right] + C\int_0^1 \gamma(\mu)d\mu\int_0^1 \frac{\eta\alpha_2(\eta)k(\eta,\mu)d\eta}{\eta-\mu}}{-\eta_1 X(\eta_1) - C\int_0^1 \frac{\gamma(\mu)k(\eta_1,\mu)d\mu}{\eta_1-\mu}} \tag{89}$$

and

$$\alpha_2(\mu) = O(\mu)\left[w - \frac{\sigma S_1 C}{C_{12}}\,\mu \ell n\left(1 + \frac{1}{\sigma\mu}\right)\right] + CO(\mu)\int_0^1 \frac{\eta\alpha_2(\eta)k(\eta,\mu)d\eta}{\eta-\mu}$$

$$- A_+\left\{\frac{\ell_1(\mu)\eta_1 X(\eta_1)}{\eta_1-\mu} - CO(\mu)\left[\frac{\eta_1 k(\eta_1,\mu)}{\eta_1-\mu}\right]\right\} . \qquad (90)$$

The final equation for $\alpha_1(\mu)$ is obtained from Eq. (58c) where

$$\psi_1(\sigma\mu) = -S_1 .$$

Explicitly,

$$\alpha_1(\mu) = -\sigma C_{11}S_1 - \frac{C_{11}C_{12}\eta_1 A_+}{\eta_1-\mu} - C_{11}C_{12}P\int_0^1 \frac{\eta\alpha_2(\eta)d\eta}{\eta-\mu} . \qquad (91)$$

We note the similarity of Eqs. (89), (90), and (91) to the corresponding equations for the Milne problem. We refer the reader to the earlier comments on solution techniques for the Milne problem [see paragraph following Eq. (69)].

The angular fluxes for the constant source problem are given by Eqs. (71) and (72) where we replace the first term in the right-hand side by S_1 and S_2, respectively. The neutron current in each group is given by Eqs. (73b) and (73d) where we delete the first term (term with positive exponential) on the right-hand side. For the total flux we replace the first term on the right-hand side (i.e., term with positive exponential) of Eqs. (73a) and (73c) by $2S_1$ and $2S_2$, respectively.

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14. This definition is correct only for the two-discrete eigenvalue case.

TABLE 1

THE ZEROS OF THE DISPERSION FUNCTION

Conditions			Roots
$C = 0$		$C_{11} + \sigma C_{22} < \sigma/2$ $C_{11} + \sigma C_{22} > \sigma/2$ $C_{11} + \sigma C_{22} = \sigma/2$	2 Real 2 Imaginary 2 Infinite
$C < 0$		$C_{11} + \sigma C_{22} - 2C < \sigma/2$ $C_{11} + \sigma C_{22} - 2C > \sigma/2$ $C_{11} + \sigma C_{22} - 2C = \sigma/2$	2 Real 2 Imaginary 2 Infinite
$C > 0$ $C_{22} > 2C\tau(1/\sigma)$		$C_{11} + \sigma C_{22} - 2C < \sigma/2$ $C_{11} + \sigma C_{22} - 2C > \sigma/2$ $C_{11} + \sigma C_{22} - 2C = \sigma/2$	2 Real 2 Imaginary 2 Infinite
	$C_{11} \leq \sigma/2$ and $C_{22} \geq 1/2$ or $C_{11} \geq \sigma/2$ and $C_{22} \leq 1/2$		2 Real and 2 Imaginery
$C > 0$ $C_{22} \leq 2C\tau(1/\sigma)$	$C_{11} < \sigma/2$ and $C_{22} < 1/2$	$C_{11} + \sigma C_{22} - 2C < \sigma/2$ $C_{11} + \sigma C_{22} - 2C > \sigma/2$ $C_{11} + \sigma C_{22} - 2C = \sigma/2$	4 Real 2 Real and 2 Imaginary 2 Real and 2 Infinite
	$C_{11} > \sigma/2$ and $C_{22} > 1/2$	$C_{11} + \sigma C_{22} - 2C < \sigma/2$ $C_{11} + \sigma C_{22} - 2C > \sigma/2$ $C_{11} + \sigma C_{22} - 2C = \sigma/2$	4 Imaginary 2 Real and 2 Imaginary 2 Imaginary and 2 Infinite