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WAVE RESISTANCE SOLUTION OF MICHELL'S INTEGRAL FOR POLYNOMIAL SHIP FORMS

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LIST OF SYMBOLS

a_{mnp}, a_{mn}	Fourier expansion coefficients
c	constant velocity of the ship
f	Froude Number
$f(x,z)$	ship-surface slope function
g	acceleration due to gravitational field
$g(x,z)$	ship-surface equation
$h(u,v)$, $\overline{h}(u,v)$	non-dimensional ship-surface slope function
$\delta h(u,v)$	small change in $h(u,v)$
m, n, p	Eigenvalues
p	fluid pressure
δp	fluid pressure on ship surface caused by waves
p_o	atmospheric surface pressure
s, t	variables in the Michell Function
u, v, w	perturbation velocity components of fluid
$\overline{u}$, $\overline{v}$, $\overline{w}$	fluid velocity components with respect to a fixed coordinate system
$u,\hat{u},w,\hat{w},w'$	non-dimensional variables of integration
x, y, z	moving coordinate system
$\hat{x}$, $\hat{z}$	variables of integration
$z(x,y)$	Free surface equation
A_{rn}, B_{rn}	Fourier expansion coefficients

$A^{I}_{\alpha\beta}$, $A^{II}_{\alpha\beta}$	Hull Function polynomial coefficients
B	Half-breadth of ship
$C(s,t)$	Michell Function
C_w	wave resistance coefficient
δC_w	differential wave resistance coefficient
$[\Delta^{\alpha\beta}C_w]_I$, $[\Delta^{\alpha\beta}C_w]_{II}$	contribution to wave resistance coefficient from terms of Hull Function polynomials of power α and β
D	Draft of ship
F	inverse square of Froude Number
${}_0F_1\left[\begin{matrix} - \;; \\ 1+\nu\;; \end{matrix} -\frac{x^2(t+1)}{4}\right]$	a generalized hypergeometric function
$G(u,v)$	non-dimensional ship surface equation
$H(\xi,\zeta)$, $H(\xi,\zeta)$	Hull Function
$H_\nu(x)$	Struve Function
$\delta H(\xi,\zeta)$	small change in $H(\xi,\zeta)$
$[H(\xi,\zeta]_I$, $[H(\xi,\zeta)]_{II}$	Hull Function
I	symmetric wavemaking integral
I_1, I_2, I_3	auxiliary functions
J	unsymmetric wavemaking integral
$J_\nu(z)$	Bessel Function of the first kind
K_o, K_1	modified Bessel Functions of the second kind
L	Total length of ship
L'	Half length of ship

$M, N, \overline{M}, \overline{N}$	Highest power of the variables in the Hull Function polynomials
$M^{(\beta)}(\xi)$	special function
R_W	Wave resistance
X, Y, Z	Stationary coordinate system
Y_0	Bessel Function of the second kind
α, β	non-negative integers in Hull Function polynomials
$\overline{\alpha}, \overline{\beta}$	Phase angles
γ	Euler's constant
ϵ	Phase angle
$\zeta, \xi, \overline{\zeta}, \overline{\xi}$	non-dimensional variables
θ', λ	variables of integration
ρ	Density of water
$\varphi(x,y,z)$	Steady state perturbation velocity potential
$\varphi_1(X,Y,Z,t)$	Time dependent velocity potential
$\Gamma(a)$	Gamma Function
$\Delta, \Delta_1, \Delta_2$	Differential areas on hull surface
$\Phi(a,c,t)$	Confluent Hypergeometric Function
$\Psi(a,c,t)$	Confluent Hypergeometric Function

CHAPTER I

INTRODUCTION

Statement of the Problem

The determination of wave resistance of ship hulls is one of the most important and interesting subjects in ship theory. Model testing facilities were primarily built for the purpose of investigations of this part of ship resistance. Since Froude's time, an immense amount of experimental results have been published. It can, however, be safely stated that the aim to represent the resistance of a ship in terms of its form has not been solved by experimental methods in a general manner. The appreciable differences in resistance observed at times between ship models of seemingly insignificant variations of shape can, for example, not be satisfactorily explained on the basis of our experimental experience.

Considering the complexity of even the simpler cases of wave phenomena, such as a sphere moving at constant speed and fully submerged, this state of affairs is not surprising. To predict ship wave resistance phenomena in a rational manner, an analytic formulation should become available, and this formulation should preferably be simple enough so that basic deductions can easily be made from it.

This investigation has primarily concerned itself with the problem of providing a simple method for the evaluation of the ship wave resistance. In doing so, it is believed that further insight into the more general problem of interpretation of results has been achieved.

Michell's paper(1) on the wave resistance of ships moving at constant speed in smooth water of infinite depth was the first attempt to treat the wave resistance analytically. The basic assumptions made by Michell in his investigation were as follows:

1. The wave heights are small compared to wave lengths. Thus particle velocities due to wave motion are so small compared to the ship's speed that second order terms in velocities can be neglected.
2. The effects of trim and sinkage are not sufficient to affect the wave motion appreciably.
3. The angles made by the hull surface with the center line plane (longitudinal plane of symmetry) are everywhere small.
4. The motion has persisted long enough so that a steady state has been reached.
5. The fluid is non-viscous and the motion has started from rest. Thus the flow is considered irrotational.
6. The free surface conditions are to be satisfied at the undisturbed water level ($z = 0$).
7. The boundary conditions to be satisfied on the hull surface are assumed to hold at the center line plane, and only the velocity component perpendicular to this plane is accounted

for. The vertical slope of the hull surface is neglected.

To this list, one should add that it is explicitly assumed that the total resistance can be broken down into three major components:

(i) Wave resistance

(ii) Frictional resistance due to viscosity of the water

(iii) Eddy-making or viscous form resistance

and that there are no interaction effects between these components. It is thus clear that Michell's theory of wave resistance is a linear theory and is theoretically valid only for an infinitesimally thin ship.

During the past thirty years, much effort has been exerted in determining the applicability of Michell's integral expression for wave resistance of ships. In principle, two major questions have been asked:

(a) Does the Michell Integral represent the wave resistance of common ship forms with reasonable accuracy?

(b) Can the integral be evaluated for real hull forms in a reasonable time?

Question (a) can be said to have been answered in the affirmative by several investigators such as Havelock[3,4,5,8,9], Wigley[10,11,12,14,16], Weinblum[20], Lunde [18,24], Shearer[31]. It may be argued that quantitatively the theory does not give sufficiently accurate results. A great part of the discrepancies between theory and experimental results has been shown to be due to neglect of viscous effects.

The effects of viscosity may be summarized into the following two items:

1. In a viscous fluid the wave amplitudes decrease as the waves propogate.
2. Due to the presence of the boundary layer the effective wave making form differs from the actual form of the ship. Furthermore, behind a ship, a portion of the regular free waves propogate in the ship's wake, and their speed of advance is slowed as much as the wake's speed. Their wave lengths, are consequently shortened to some degree.

Inui[(43)] has pointed out that if the Michell theory is extended to ships of finite beam, the slope of the hull in the integrand of the Michell Integral and the limits of integration must be changed or the integral will represent the wave resistance of a somewhat different hull form. Having obtained several such modified forms, he performed a series of tests and compared resulss with predictions from theory. The correlation between theory and experiments was extremely good. In his calculations he included corrections for viscous, finite wave amplitude and hull interference effects.

Inui's results show that the mathematical theory is far better than had been anticipated on the basis of works by previous investigators. They also emphasize the shortcomings of the standard methods used in obtaining the wave resistance from experimental measurements, i.e., deducting the frictional resistance of an equivalent flat plate from the total resistance of the model. To obtain a more realistic value of the frictional resistance Inui used data from submerged double model tests.

As previously stated, the present work is concerned with the answers to question (b). It is safe to say that the complexity of the evaluation of Michell's Integral has been the main obstacle to the application of this theory to practical ship design problems. In making actual computations, investigators have in most cases been forced to consider simple mathematical shapes which often bear only a vague resemblance to usual ship forms. Even so, calculations have been lengthy, involving numerical integration.

As soon as numerical calculations are initiated, further analytic evaluation of wave resistance is only possible through systematic variations of parameters and plots of numerical results. It should therefore be the aim of the theory to obtain expressions for the wave-resistance of a ship in terms of its hull form and known mathematical functions. As far as this author has been able to establish, the present work presents for the first time such functional relationships for the wave resistance coefficient for any ship form whose surface can be represented by a polynomial of integral powers of coordinates in the longitudinal plane of symmetry.

For each term of such a polynomial, these expressions could be evaluated by high speed computers and tabulated. The calculation of wave resistance has thus, once such tables become available, been reduced to a minimum of labor, involving only a few multiplications and additions. Indeed, the mathematical wave theory could become a powerful tool in ship design.

The assumption that the ship's surface be represented in a polynomial form is not a serious restriction. In fact, several researchers in the field are strong advocates of such a representation which dates back to D. W. Taylor's work on his famous Standard Series Ships. The many problems in connection with the layout of ship dimensions in the yards also seem to favor the polynomial representation.

Historical Background

Several excellent reviews of the development of the mathematical theory of wave resistance can be found in the literature, some of which include extensive lists of references. Lunde[26], for instance, gives a total of 185 references published before and including 1953. The Transactions of the Institution of Naval Architects, vol. 100, 1958, gives a complete list of papers published by T. H. Havelock on hydrodynamics during the years 1908-1958.

A detailed account of the development of the wave resistance theory is beyond the scope of this work. If some investigators are not mentioned here, it is not because their contributions are considered less important but rather that their works have no direct bearing on the results of this thesis.

Lord Kelvin in 1887 was probably the first physicist to investigate three-dimensional waves. His picture of the pattern generated by a moving pressure point, revealing the existence of diverging and transverse waves, is well known. In presenting his classical paper, however, he made it a condition to the Council that no practical results were to be expected from it.

Thus Michell's paper, published in 1898, marks the beginning of the theory of wave resistance of floating bodies. For many years, this paper was unfortunately overlooked and forgotten. In 1923, Havelock rediscovered Michell's work, and a few years later Wigley put the theory to test by initiating his series of papers on the comparison between theoretical and experimental results. Weinblum, in Germany, started his work on the Michell wave theory around 1930. His first concern was the determination of ship forms of minimum wave resistance. It may be of interest to mention that von Kàrmàn also has contributed to our knowledge on this facet of wave resistance theory. In 1936 Dr. Weinblum published a paper on the theory of bulbous bows and later became interested in the systematic evaluation of the Michell Integral. During his stay at the David Taylor

Model Basin from 1948 to 1952, a partial computation program was sponsored under his supervision. This program has later been continued in Germany. In 1939 Guilloton published his thesis on wave resistance in France. Guilloton used the Michell potential to calculate the pressure disturbance of a simplified geometric body and by adding these disturbances was able to obtain the wave resistance of actual ship forms, and in addition he was able to trace the streamlines of fine hulls.

After having published many papers on wave profiles and wave resistances for pressure points, sphere, etc., Havelock developed a wave resistance theory from a somewhat different approach. By considering a distribution of singularities (sources or doublets), he was able to simulate the presence of the ship, presenting formulae for the velocity potential and the wave resistance. In principle, Havelock's theory is capable of satisfying the boundary conditions on the surface of the ship exactly. In practice, however, the same assumption about form has been made as in Michell's theory, placing the singularities on the center line plane of symmetry. Wigley(13) demonstrated that, under these conditions, Havelock's and Michell's formulae for the wave resistance are identical when applied to the same hull form. In 1953, Timman and Vossers(33) were able to demonstrate the complete agreement between the two theories by means of Fourier Transform techniques, and a source of argument was removed.

Birkhoff, Korvin-Kroukovsky, and Kotik presented an excellent study of the significance of the wave resistance theory[34] in 1954. In it, they also proposed two new transformations of the Michell Integral. The first of these transformations form the basis for this thesis.

A number of authors have extensively exploited the linearized theory of wave resistance during the last 35 years, notable Wigley, Weinblum, Lunde, and Guilloton. A considerable amount of work has also been done in Japan, notably by Inui[32,43,44]. Reference[44] lists over seventy papers on the subject, many of which are unfortunately not translated from Japanese. No real basic modification of Michell's theory has been made, however, and our efforts are even today directed toward the application of his analysis.

CHAPTER II

THE MICHELL INTEGRAL

Boundary Conditions

As a ship moves with constant velocity on the surface of an infinitely deep, incompressible, and inviscous fluid, it gives rise to a perturbation velocity field. If the fluid is assumed initially at rest, this field will be irrotational and it follows that a velocity potential φ_1 exists. Using a rectangular coordinate system with the XY-plane located in the undisturbed water surface and Z-axis vertically downward, the velocity components $\bar{u}$, $\bar{v}$, $\bar{w}$, which satisfy the equation of continuity

$$\frac{\partial \bar{u}}{\partial X} + \frac{\partial \bar{v}}{\partial Y} + \frac{\partial \bar{w}}{\partial Z} = 0$$

are given by

$$\bar{u} = -\frac{\partial \phi_1}{\partial X} \quad ; \quad \bar{v} = -\frac{\partial \phi_1}{\partial Y} \quad ; \quad \bar{w} = -\frac{\partial \phi_1}{\partial Z}$$

Here φ_1 is a solution of Laplace's equation

$$\frac{\partial^2 \phi_1}{\partial X^2} + \frac{\partial^2 \phi_1}{\partial Y^2} + \frac{\partial^2 \phi_1}{\partial Z^2} = 0$$

The Bernoulli's equation for unsteady irrotational flow is

$$-\frac{\partial \phi_1}{\partial t} + \frac{1}{2}(\bar{u}^2 + \bar{v}^2 + \bar{w}^2) + \frac{p}{\rho} - gZ = f(t)$$

To reduce the fluid flow to a steady state case, a coordinate system fixed with respect to the ship is introduced. The origin is located amidships in the center line plane and the positive x-axis is in the direction of motion (Fig. 1). Furthermore, the ship is assumed stationary and a uniform flow of velocity c equal to that of the ship is superimposed, in the negative x-direction. It follows then that

$$\overline{u} = -c + u; \ \overline{v} = v; \ \overline{w} = w$$

where u, v and w are the components of the perturbation velocity caused by the presence of the ship in the uniform flow. These components are assumed small - i.e.,

$$u \ll c; \ v \ll c; \ w \ll c$$

Introducing the perturbation potential φ defined by

$$u = -\frac{\partial\phi}{\partial x} \ ; \ v = -\frac{\partial\phi}{\partial y} \ ; \ w = -\frac{\partial\phi}{\partial z}$$

one has that $$\phi_1 = cs + \phi$$

and $$\nabla^2\phi = 0 \qquad (2.1)$$

where ∇^2 is the Laplace operator.

Neglecting small quantities of second order, the Bernoulli's equation now becomes

$$\frac{p}{\rho} + \frac{1}{2}(c^2 + 2c\frac{\partial\phi}{\partial x}) - gz = C_1 \qquad (2.2)$$

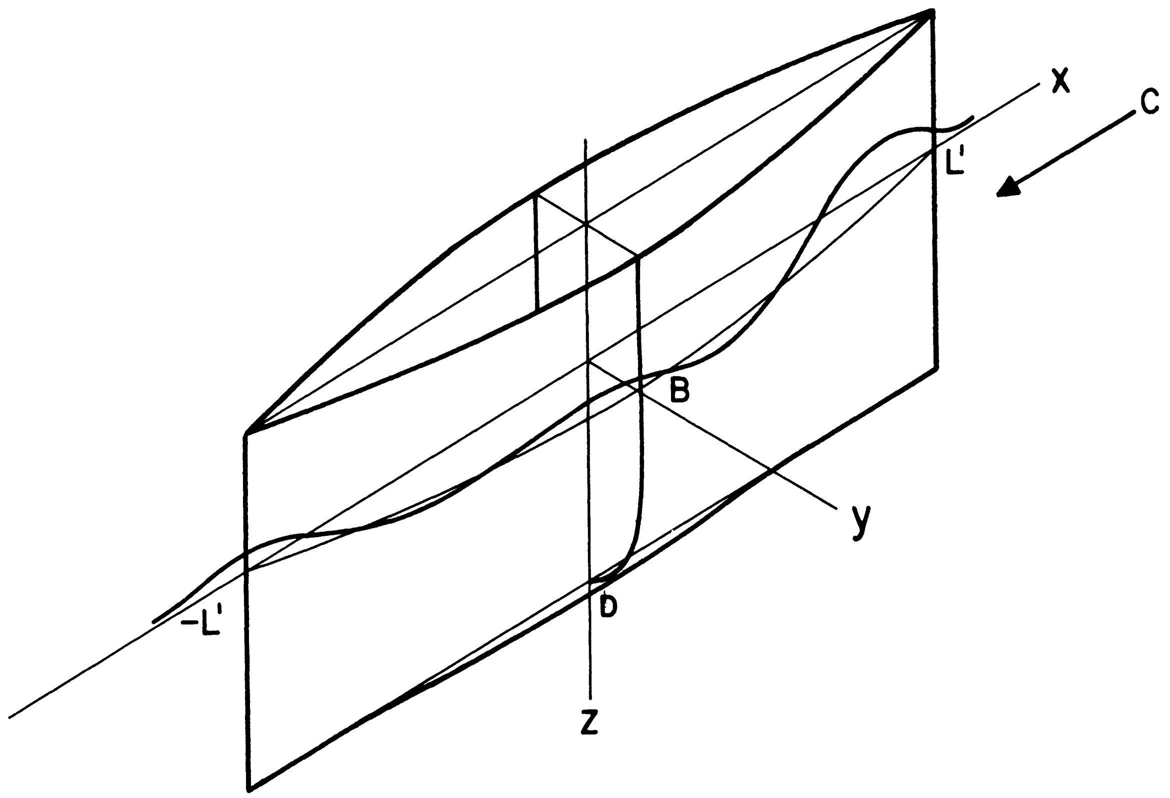

Figure 1. Coordinate System Fixed In Stationary Ship In A Uniform Flow

The constant C_1 can be determined from the condition that far upstream the surface elevation must be equal to zero. Thus

$$C_1 = \frac{1}{2}c^2 + \frac{p_o}{\rho}$$

where p_o is the atmospheric pressure. Letting the free surface be defined by

$$z = -\bar{z}(x,y)$$

the condition of constant pressure on the surface becomes

$$c\frac{\partial\phi}{\partial x} + g\bar{z} = 0 \tag{2.3}$$

The kinematical boundary condition to be satisfied by an inviscous fluid is that the velocity of a particle on a bounding surface must be tangential to it - i.e.,

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \tag{2.4}$$

where $F(x,y,z,t) = 0$ is the equation of the surface. For the free surface

$$F(x,y,z) = z + \bar{z}(x,y) = 0$$

Applying Equation (2.4), the kinematical free surface condition becomes

$$(-c+u)\frac{\partial\bar{z}}{\partial x} + v\frac{\partial\bar{z}}{\partial y} + w = 0$$

Neglecting terms of higher order, this expression reduces to

$$c \frac{\partial \bar{z}}{\partial x} + \frac{\partial \phi}{\partial z} = 0 \quad ; \quad z = -\bar{z} \tag{2.5}$$

Eliminating $\bar{z}$ from (2.3) and (2.5) gives

$$\frac{\partial^2 \phi}{\partial x^2} - \frac{g}{c^2} \frac{\partial \phi}{\partial z} = 0 \tag{2.6}$$

Since perturbation velocities are small, Equation (2.6) is assumed to hold at $z = 0$.

The kinematical boundary condition of (2.4) must also hold at the surface of the ship. If that surface is represented by

$$y = g(x,z) \tag{2.7}$$

the boundary condition then is, by (2.4)

$$-(-c+u)\frac{\partial g}{\partial x} + v - w\frac{\partial g}{\partial z} = 0$$

If one introduces the restriction that the tangent plane of the ship surface makes a small angle with the xz-plane, i.e.,

$$\frac{\partial g}{\partial x} \ll 1 \quad ; \quad \frac{\partial g}{\partial z} \ll 1$$

then the boundary conditions on the surface of the ship simplifies to

$$\frac{\partial \phi}{\partial y} = c\frac{\partial g}{\partial x} = c f(x,z) \quad ; \quad y = g(x,z)$$

or

$$\frac{\partial \phi}{\partial y} = c f(x,z) \quad ; \quad y = 0 \tag{2.8}$$

The boundary conditions given by (2.6) and (2.8) are necessary but not sufficient. To make the solution unique additional restrictions must be introduced. Since the ship is assumed to advance into still water it will be required that the waves are trailing aft. Furthermore, the perturbation velocities are zero at infinite depth. The velocity potential φ must therefore satisfy the following requirements:

1. $\nabla^2\varphi = 0; \quad -\infty < x < \infty; \quad 0 \leq y < \infty; \quad 0 \leq z < \infty$

2. $\dfrac{\partial^2\varphi}{\partial x^2} - \dfrac{g}{c^2}\dfrac{\partial\varphi}{\partial z} = 0; \quad z = 0; \quad -\infty < x < \infty; \quad 0 < y < \infty$

3. $\dfrac{\partial\varphi}{\partial y} = cf(x,z); \quad y = 0; \quad -\infty < x < \infty; \quad z \geq 0$

4. $z \to \infty; \quad \varphi = 0$

5. $x \to +\infty; \quad \varphi = 0$

6. $y \to \pm\infty; \quad \varphi = 0; \quad -\infty < x < \infty; \quad z \geq 0$

<u>The Wave Resistance Integral</u>

The derivation of the wave resistance integral which follows is due to Michell[1]. It is included in this thesis for the sake of continuity and completemess.

It is at first assumed that depth of water is finite and equal to h. Hence the velocity potential φ must satisfy

$$\frac{\partial\varphi}{\partial y} = 0; \quad z = h \tag{2.9}$$

A typical term of the solution to Laplace's equation, satisfying (2.9) is

$$a_{mnp} \cos n(z - h) \cos(mx + \bar{\alpha}) \cos(py + \bar{\beta})$$

where $m^2 + n^2 + p^2 = 0$, and m must be taken real since $-\infty \leq x \leq \infty$. n and p may be either real or imaginary, but if p is imaginary, e.g. $p=ip'$, the last term must take the form $e^{-p'y}$, since due to symmetry it is sufficient to consider positive values of y only.

The free surface conditions (2.6) will be satisfied if

$$n \tan nh = - \frac{c^2m^2}{g}. \tag{2.10}$$

For each value of m, this equation has an infinite number of real roots and one pure imaginary root given by

$$n' \tanh n'h = \frac{c^2m^2}{g}; \quad (n = in').$$

It will be shown that the imaginary root alone is responsible for the wave resistance. As for p, it is always imaginary for the real roots of n and so for the one imaginary root of n if $m > n'$. In order to satisfy Equation (2.8), let f(x,z) be expanded in terms of a Fourier series.

$$f(x,z) = \sum_{m,n} a_{mn} \cos n(z - h) \cos(mx + \bar{\alpha}).$$

If it is supposed that f(x,z) is periodic,

$$f(x + L', z) = f(x - L', z)$$

then f(x,z) can be written

$$f(x,z) = \sum_{r}\sum_{n}\left\{A_{rn}\cos\frac{\pi r x}{L'} + B_{rn}\sin\frac{\pi r x}{L'}\right\}(\cos n(z-h)) \qquad (2.11)$$

The orthogonality properties of the trigonometric functions now imply that

$$\int_{-L'}^{L'} f(x,z)\cos\frac{\pi r x}{L'}\,dx = L'\sum_{n} A_{rn}\cos n(z-h)$$

$$\int_{-L'}^{L'} f(x,z)\sin\frac{\pi r x}{L'}\,dx = L'\sum B_{rn}\cos n(z-h)$$

and also that

$$\int_{0}^{h}\int_{-L'}^{L'} f(x,z)\cos\frac{\pi r x}{L'}\cos n(z-h)\,dx\,dz = L'A_{rn}\int_{0}^{h}\cos^2 n(z-h)\,dz$$

$$= L'A_{rn}\frac{1}{4n}(2nh + \sin 2nh) \qquad (2.12)$$

$$\int_{0}^{h}\int_{-L'}^{L'} f(x,z)\sin\frac{\pi r x}{L'}\cos n(z-h)\,dx\,dz = L'B_{rn}\int_{0}^{h}\cos^2 n(z-h)\,dz$$

$$= L'B_{rn}\frac{1}{4n}(2nh + \sin 2nh)$$

When $n = in'$, Equation (2.12) becomes

$$\int_0^h\int_{-L'}^{L'} f(x,z)\cos\frac{\pi r x}{L'}\cosh n'(z-h)\,dx\,dz = L'A_{rn'}\frac{1}{4n'}(2n'h+\sinh 2n'h)$$

$$\int_0^h\int_{-L'}^{L'} f(x,z)\sin\frac{\pi r x}{L'}\cosh n'(z-h)\,dx\,dz = L'B_{rn'}\frac{1}{4n'}(2n'h+\sinh 2n'h) \tag{2.13}$$

Substituting Equations (2.12) and (2.13) into Equation (2.11) one has

$$f(x,z) = \sum_r\sum_n \frac{4n\cos n(z-h)}{L'(2nh+\sin 2nh)}\int_{-L'}^{L'}\int_0^h f(\hat{x},\hat{z})\cos\frac{\pi r}{L'}(\hat{x}-x)\cos n(\hat{z}-h)\,d\hat{z}\,d\hat{x}$$

$$+\sum_r \frac{4n'\cosh n'(z-h)}{L'(2n'h+\sinh 2n'h)}\int_{-L'}^{L'}\int_0^h f(\hat{x},\hat{z})\cos\frac{\pi r}{L'}(\hat{x}-x)\cosh n'(\hat{z}-h)\,d\hat{z}\,d\hat{x}$$

If now $L' \to \infty$ with

$$\frac{\pi r}{L'} = m \quad ; \quad \frac{\pi}{L'} = dm$$

the infinite summation on r is transformed into an integral of the form

$$f(x,z) = \frac{4}{\pi}\sum_n \int_0^{\infty}\int_{-\infty}^{\infty}\int_0^h f(\hat{x},\hat{z})\,\frac{n\cos n(z-h)\cos n(\hat{z}-h)}{2nh+\sin 2nh}\times \cos m(\hat{x}-x)\,d\hat{z}\,d\hat{x}\,dm$$

$$+\frac{4}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty}\int_0^h f(\hat{x},\hat{z})\,\frac{n'\cosh n'(z-h)\cosh n'(\hat{z}-h)}{2n'h+\sinh 2n'h}\times \cos m(\hat{x}-x)\,d\hat{z}\,d\hat{x}\,dm \tag{2.14}$$

If furthermore, the depth of water is allowed to become infinite, the following relationships are obtained:

$$nh = \bar{n}\pi + \epsilon \quad ; \quad dn = \frac{\pi}{h}$$

$$\tan nh = \tan \epsilon$$

$$\cos n(z-h) = (-1)^{\bar{n}} \cos(nz - \epsilon)$$

$$\lim_{n\to\infty} \left(2n + \frac{\sin 2nh}{n}\right) = 2n$$

$$n \tan \epsilon = -km^2 \quad ; \quad k = \frac{c^2}{g}$$

$$\lim_{n'\to\infty} \tanh n'h = 1$$

$$n' = km^2$$

$$\lim_{n'\to\infty} \frac{\cosh n'(z-h)\cosh n'(\hat{z}-h)}{2n'h + \sinh 2n'h} = \frac{1}{2} e^{-n'(z+\hat{z})}$$

Substituting these relationships into Equation (2.14), one finally has

$$\begin{aligned} f(x,z) = {} & \frac{2}{\pi^2} \int_0^\infty \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\hat{x},\hat{z}) \cos(nz-\epsilon)\cos(n\hat{z}-\epsilon) \times \\ & \qquad \cos m(\hat{x}-x)\, d\hat{x}\, dm\, d\hat{z}\, dn \\ & + \frac{2c^2}{\pi g} \int_0^\infty \int_0^\infty \int_{-\infty}^\infty f(\hat{x},\hat{z})\, m^2 e^{-\frac{c^2}{g} m^2 (z+\hat{z})} \times \\ & \qquad \cos m(\hat{x}-x)\, d\hat{x}\, d\hat{z}\, dm \end{aligned} \tag{2.15}$$

The velocity potential satisfying all boundary conditions is therefore

$$\phi = -\frac{2c}{\pi^2}\int_0^\infty\int_0^\infty\int_0^\infty\int_{-\infty}^\infty f(\hat{x},\hat{z})\,\frac{\cos(nz-\epsilon)\cos(n\hat{z}-\epsilon)}{\sqrt{m^2+n^2}}\times$$

$$\cos m(\hat{x}-x)\,e^{-y\sqrt{m^2+n^2}}\,d\hat{x}\,d\hat{z}\,dm\,dn$$

$$+\frac{2c^3}{\pi g}\int_{\frac{g}{c^2}}^\infty\int_0^\infty\int_{-\infty}^\infty f(\hat{x},\hat{z})\,\frac{m\,e^{-\frac{m^2c^2(z+\hat{z})}{g}}}{\sqrt{\frac{m^2c^4}{g^2}-1}}\times \tag{2.16}$$

$$\sin\left\{m(x-\hat{x})+m\sqrt{\frac{m^2c^4}{g^2}-1}\;y\right\}d\hat{x}\,d\hat{z}\,dm$$

$$-\frac{2c^3}{\pi g}\int_0^{\frac{g}{c^2}}\int_0^\infty\int_{-\infty}^\infty f(\hat{x},\hat{z})\,\frac{m\,e^{-\frac{m^2c^2(z+\hat{z})}{g}}}{\sqrt{1-\frac{m^2c^4}{g^2}}}\cos m(\hat{x}-x)\times$$

$$e^{-my\sqrt{1-\frac{m^2c^4}{g^2}}}\,d\hat{x}\,d\hat{z}\,dm$$

The term $\sin\left\{m(x-x)+m\sqrt{\frac{m^2c^4}{g^2}-1}\;y\right\}$ was introduced by Michell to make, in his words, the waves trail aft. Timmon and Vossers[33] have obtained Equation (2.16) by means of a Fourier Transform technique and have shown that the velocity potential is uniquely determined by the boundary conditions given on page 15. If one lets δp be the hydrodynamic pressure due to the wave disturbance, the wave resistance will be given by

$$R_w = -2\iint \delta p\; f(x,z)\,dx\,dz$$

where the integral is taken over the center line plane of the ship. Neglecting terms of higher order in the Bernouilli's equation the hydrodynamic pressure due to waves becomes

$$\delta p = -\rho c \frac{\partial \phi}{\partial x}.$$

Thus one has that

$$R_w = 2\rho c \iint \frac{\partial \phi}{\partial x} f(x,z)\,dx\,dz.$$

The first and third integrals of Equation (2.16) make no contribution to the wave resistance since

$$\iiiint f(\hat{x},\hat{z}) f(x,z) \sin m(\hat{x}-x)\,d\hat{x}\,d\hat{z}\,dx\,dz = 0.$$

Hence it follows that

$$R_W = \frac{4\rho c^4}{\pi g} \int_{\frac{g}{c^2}}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} \int_{-\infty}^{\infty} f(x,z) f(\hat{x},\hat{z}) \frac{m^2 e^{-\frac{m^2 c^2 (z+\hat{z})}{g}}}{\sqrt{\frac{m^2 c^4}{g^2} - 1}} \times \cos m(x-\hat{x})\,dx\,dz\,d\hat{x}\,d\hat{z}\,dm \tag{2.17}$$

$$= \frac{4\rho c^4}{\pi g} \int_{\frac{g}{c^2}}^{\infty} (\bar{I}^2 + \bar{J}^2) \frac{m^2\,dm}{\sqrt{\frac{m^2 c^4}{g^2} - 1}}$$

Let $\lambda = \frac{mc^2}{g}$, so that the wave resistance is given by

$$R_W = \frac{4\rho g^2}{\pi c^2} \int_1^{\infty} (I^2 + J^2) \frac{\lambda^2\,d\lambda}{\sqrt{\lambda^2 - 1}} \tag{2.18}$$

where

$$I = \int_0^{\infty} \int_{-\infty}^{\infty} f(x,z)\, e^{-\frac{\lambda^2 g z}{c^2}} \cos \frac{\lambda g x}{c^2}\,dx\,dz$$

$$J = \int_0^{\infty} \int_{-\infty}^{\infty} f(x,z)\, e^{-\frac{\lambda^2 g z}{c^2}} \sin \frac{\lambda g x}{c^2}\,dx\,dz$$

Equation (2.18) is called the Michell Wave Resistance Integral. It is observed that for a ship symmetrically fore and aft I = 0. Furthermore theory predicts the wave resistance to be the same for motion in either direction along the x-axis.

For real ships, f(x,z) is defined as non-zero only on a domain

$$S = \left\{ -\frac{L}{2} \leq x \leq \frac{L}{2} \; ; \; 0 \leq z \leq D \right\}$$

where L = Length of ship

D = Draft of ship

It will be found convenient to express the multiple integral of Equation (2.17) in a non-dimensional form. For this reason, the following variables are introduced at this point:

$$u = \frac{x}{L} \; ; \; \hat{u} = \frac{\hat{x}}{L} \; ; \; w = \frac{z}{L} \; ; \; \hat{w} = \frac{\hat{z}}{L}$$

Furthermore, it is convenient to let $\frac{gL}{c^2} = F = f^{-2}$ where f is the Froude Number. Then from (2.17)

$$R_w = \frac{4\rho g^2 L}{\pi c^2} \int_1^\infty \iint_{S'} \iint_{S'} f(Lu, Lw) f(L\hat{u}, L\hat{w}) \frac{\lambda^2}{\sqrt{\lambda^2 - 1}} e^{-\lambda^2 F(w+\hat{w})}$$

$$\times \cos[\lambda F(u-\hat{u})] \, L^2 du\, dw\, d\hat{u}\, d\hat{w}\, d\lambda$$

$$S' = \left\{ -\frac{1}{2} \leq u \leq \frac{1}{2} \; ; \; 0 \leq w \leq \frac{D}{L} \right\}$$

Defining the maximum ship beam by 2B, the hull surface may be described by

$$\frac{y}{B} = G\left(\frac{x}{L}, \frac{z}{L}\right) = G(u,\ w) \tag{2.19}$$

From the dimensionless slope function defined by

$$h(u,\ w) = \frac{\partial G}{\partial u} \tag{2.20}$$

one has that $\qquad Bh(u,w) = Lf(x,z).$

Thus, if follows that

$$R_w = \frac{4\rho g^2 L^2}{\pi c^2} \int_1^{\infty} \iint_{S'} \iint_{S'} B^2 h(u,w)\, h(\hat{u},\hat{w})\, \frac{\lambda^2}{\sqrt{\lambda^2-1}}\, e^{-\lambda^2 F(w+\hat{w})}$$

$$\times \cos[\lambda F(u-\hat{u})]\, du\, dw\, d\hat{u}\, d\hat{w}\, d\lambda$$

The wave resistance coefficient defined by

$$C_w = \frac{R_w}{\frac{1}{2}\rho c^2 B^2}$$

is therefore given by

$$C_w = \frac{8F^2}{\pi} \int_1^{\infty} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^{\frac{D}{L}} \int_0^{\frac{D}{L}} h(u,w)\, h(\hat{u},\hat{w})\, \frac{\lambda^2}{\sqrt{\lambda^2-1}}\, e^{-\lambda^2 F(w+\hat{w})}$$

$$\times \cos[\lambda F(u+\hat{u})]\, du\, d\hat{u}\, dw\, d\hat{w}\, d\lambda \tag{2.21}$$

CHAPTER III

SOLUTION OF MICHELL INTEGRAL FOR POLYNOMIAL SHIP FORMS

Introduction

The transformation of the Michell Integral proposed by G. Birkhoff and J. Kotik in 1954[34,35] led to a separation of the parameters contributing to the wave resistance of ships. By introducing auxiliary functions, they were able to divide the integrand in the Michell Integral into one part containing all the properties of the ship form and another describing the influence of ship speed.

In making actual use of the method, it was suggested that numerical integration be employed. This led to two basic difficulties. First, the integrand possessed an irregular singularity on the boundary of the domain over which it had to be integrated. Second, the part of the integrand which is a function of ship speed was given in an integral form only. This integral, called the Michell Function, has been investigated in parts by Birkhoff, Kotik and Parikh[36,37,38].

This thesis presents for the first time a series solution of the Michell Function. A solution of the Michell Integral for wave resistance has also been obtained under the sole assumption that the function containing the properties of the ship's surface is of a polynomial form.

A Transformation of Michell's Integral

The following transformation is the first of two proposed by Birkhoff[34].

In Equation (2.21), two new variables ξ and ζ are introduced by

$$\xi = \hat{u} - u; \quad \zeta = \hat{w} + w. \tag{3.1}$$

For u and w constant, it follows that

$$d\xi = d\hat{u}; \quad d\zeta = d\hat{w}$$

and (2.21) becomes

$$C_W = \frac{8F^2}{\pi} \int_{-\frac{1}{2}}^{\frac{1}{2}} du \int_0^{\frac{D}{L}} dw\, h(u,w) \int_1^{\infty} d\lambda \int_{-\frac{1}{2}-u}^{\frac{1}{2}-u} d\xi \int_w^{w+\frac{D}{L}} d\zeta\, h(\xi+u, \zeta-w)$$

$$\times\, e^{-\lambda^2 F\zeta} \frac{\lambda^2}{\sqrt{\lambda^2-1}} \cos(\lambda F\xi)$$

If the order of integration is interchanged, C_W can be written

$$C_W = \frac{8F^2}{\pi} \int_{+1}^{-1} d\xi \int_0^{\frac{2D}{L}} d\zeta \left\{ \int_{-\xi-\frac{1}{2}}^{\frac{1}{2}-\xi} du \int_{\zeta}^{\zeta-\frac{D}{L}} dw\, h(u,w)\, h(\xi+u, \zeta-w) \right\}$$

$$\left\{ \int_1^{\infty} e^{-\lambda^2 F\zeta} \cos(\lambda F\xi) \frac{\lambda^2}{\sqrt{\lambda^2-1}}\, d\lambda \right\}$$

or in a more compact form

$$C_W = \frac{8F^2}{\pi} \int_{-1}^{1} d\xi \int_0^{\frac{2D}{L}} d\zeta\, H(\xi,\zeta)\, C(F\xi, F\zeta) \tag{3.2}$$

where

$$H(\xi,\zeta) = \int_{-\xi-\frac{1}{2}}^{\frac{1}{2}-\xi} du \int_{\zeta-\frac{D}{L}}^{\zeta} dw\, h(u,w)\, h(\xi+u,\zeta-w) \tag{3.3}$$

$$C(s,t) = \int_1^{\infty} e^{-t\lambda^2} \cos(s\lambda)\, \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}} \tag{3.4}$$

Here $H(\xi,\zeta)$ is called the Hull Function and $C(s,t)$ the Michell Function.

In order to show that Equations (2.21) and (3.2) are equivalent, it is necessary to prove the following:

Theorem I.

Suppose $\int_{-\frac{1}{2}}^{\frac{1}{2}} |h(u,w)|\, du < M$ for some finite M.

Then (2.21) is an absolutely convergent multiple integral.

Proof.

Since the integrand of

$$I = \int_1^{\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\int_0^{\frac{D}{L}}\int_0^{\frac{D}{L}} \frac{\lambda^2}{\sqrt{\lambda^2-1}}\, e^{-\lambda^2 F(w+\hat{w})}\, |h(u,w)| \times |h(\hat{u},\hat{w})|\, d\hat{w}\, dw\, d\hat{u}\, du\, d\lambda$$

is positive, measurable, and larger in absolute value than the original integrand, it is, by the Fubini theorem[41], sufficient to show that one iterated integral is finite. By hypothesis, one has

$$I \le \int_1^{\infty} \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}} \int_0^{\frac{D}{L}}\int_0^{\frac{D}{L}} M^2 e^{-\lambda^2 F w} e^{-\lambda^2 F \hat{w}}\, dw\, d\hat{w}$$

$$\le \frac{M^2}{F^2} \int_1^{\infty} \frac{d\lambda}{\lambda^2\sqrt{\lambda^2-1}} = \frac{M^2}{F^2} < \infty \;;\; F \neq 0$$

The equivalence of (2.21) and (3.2) is now a corollary to Theorem I.

The Hull Function

From the form of Equation (2.12), it follows that

$$H(-\xi,\zeta) = H(\xi,\zeta)$$

Obviously, the Michell Function is symmetric with respect to s. It suffices therefore to consider positive values of ξ only. Thus (3.2) can be replaced by

$$C_W = \frac{16F^2}{\pi}\int_0^1 d\xi \int_0^{\frac{2D}{L}} d\zeta \; H(\xi,\zeta)\, C(F\xi, F\zeta) \tag{3.5}$$

Since $h(u,w) = 0$ everywhere outside the region S', the limits of integration of Equation (3.3) can be reduced somewhat. If $\xi \geq 0$, then $h(u,w) = 0$ for $u < -\frac{1}{2}$, and the lower limit on u in (3.3) becomes $(-\frac{1}{2})$. In regard to the limits on w, two cases must be considered.

(i) $\zeta = \frac{D}{L}$; the limit $\zeta - \frac{D}{L}$ can be replaced by zero, since $h(u,w) = 0$, $w < 0$.

(ii) $\frac{D}{L} \leq \zeta \leq \frac{2D}{L}$; the upper limit on w can be replaced by $w = \frac{D}{L}$, since $h(u,w) = 0$, $w > \frac{D}{L}$.

The Hull Function may therefore be defined as follows:

$$H(\xi,\zeta) = [H(\xi,\zeta)]_I = \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} du \int_0^{\zeta} dw \; h(u,w)\, h(\xi+u, \zeta-w)$$

$$\xi \geq 0 \; ; \quad 0 \leq \zeta \leq \frac{D}{L} \tag{3.6}$$

$$H(\xi,\zeta) = [H(\xi,\zeta)]_{II} = \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} du \int_{\zeta-\frac{D}{L}}^{\frac{D}{L}} dw \; h(u,w)\, h(\xi+u, \zeta-w)$$

$$\xi \geq 0 \; ; \quad \frac{D}{L} \leq \zeta \leq \frac{2D}{L}$$

These expression show immediately that

$H(\xi,\zeta) = 0$ along the boundaries of S".

Furthermore, it is noted that for small values of ζ, $H(\xi,\zeta)$ can be approximated by

$$H(\xi,\zeta) \simeq \zeta \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} h(u,0)\, h(\xi+u,0)\, du$$

Thus $H(\xi,\zeta)$ vanishes at least as fast as a linear function of ξ on $\zeta = 0$. By means of similar approximations, it can be shown that the Hull Function vanishes like a linear function (or faster) on the complete boundary of the region S". $S'' = \{-1 \leq \xi \leq 1 ;\ 0 \leq \zeta \leq \frac{2D}{L}\}$

Now by Leibniz rule

$$\frac{\partial H(\xi,\zeta)}{\partial \xi} = \int_0^{\zeta} dw \frac{\partial}{\partial \xi} \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} h(\xi+u,\zeta-w)\, h(u,w)\, du$$

$$= \int_0^{\zeta} dw \left[-h(\tfrac{1}{2},\zeta-w)\, h(\tfrac{1}{2}-\xi,w) + \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} h(u,w) \frac{\partial}{\partial \xi} h(\xi+u,\zeta-w)\, du\right]$$

Also since

$$\frac{\partial}{\partial \xi} h(u+\xi,\zeta-w) = \frac{\partial}{\partial u} h(u+\xi,\zeta-w)$$

it follows that

$$\left.\frac{\partial H(\xi,\zeta)}{\partial \xi}\right|_{\xi=0^+} = \int_0^{\zeta} dw \left[-h(\tfrac{1}{2},\zeta-w)\, h(\tfrac{1}{2},w) + \int_{-\frac{1}{2}}^{\frac{1}{2}} h(u,w) \frac{\partial}{\partial u} h(u,\zeta-w)\, du\right] \qquad (3.7)$$

If one lets $w' = \zeta - w$, (3.7) may be written

$$\left.\frac{\partial H(\xi,\zeta)}{\partial \xi}\right|_{\xi=0^+} = \int_0^{\zeta} dw' \left[-h(\tfrac{1}{2},w')\, h(\tfrac{1}{2},\zeta-w') + \int_{-\frac{1}{2}}^{\frac{1}{2}} h(u,\zeta-w') \frac{\partial}{\partial u} h(u,w')\, du\right] \qquad (3.8)$$

Dropping the prime, one obtains from (3.7) and (3.8)

$$2\left.\frac{\partial H(\xi,\zeta)}{\partial \xi}\right|_{\xi=0^+} = \int_0^{\zeta} dw \Big[-2h(\tfrac{1}{2},w)\,h(\tfrac{1}{2},\zeta-w)$$

$$+\int_{-\frac{1}{2}}^{\frac{1}{2}} h(u,\zeta-w)\frac{\partial}{\partial u}h(u,w)\,du + \int_{-\frac{1}{2}}^{\frac{1}{2}} h(u,w)\frac{\partial}{\partial u}h(u,\zeta-w)\,du\Big]$$

$$= \int_0^{\zeta} dw \Big\{-2[h(\tfrac{1}{2},w)\,h(\tfrac{1}{2},\zeta-w) + \int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\partial}{\partial u}[h(u,w)\,h(u,\zeta-w)]\,du\Big\}$$

$$= -\int_0^{\zeta} dw \Big\{h(\tfrac{1}{2},w)\,h(\tfrac{1}{2},\zeta-w) + h(-\tfrac{1}{2},w)\,h(-\tfrac{1}{2},\zeta-w)\Big\}$$

If the waterline angles at the bow and stern are zero, then

$$\frac{\partial H(\xi,\zeta)}{\partial \xi} = 0 \text{ at } \xi = 0^+$$

However if the slope is different from zero, the slope of the Hull Function has a jump along the line $\xi = 0$ since

$$H(\xi,\zeta) = H(-\xi,\zeta).$$

The Hull Function for a simple mathematical ship surface is derived in Appendix I.

To gain insight into the behavior of the Hull Function, it is useful to consider the effects of variation in h(u,w) doe to changes in the ship's form. Let

$$\bar{h}(u,w) = h(u,w) + \delta h(u,w) \tag{3.9}$$

be the new slope function after alterations. The corresponding Hull Function then becomes

$$\bar{H}(\xi,\zeta) = \int du \int dw [h(u,w)+\delta h(u,w)][h(\xi+u,\zeta-w)+\delta h(\xi+u,\zeta-w)]$$

Defining the change in the Hull Function by $\delta H(\xi,\zeta)$, it follows that

$$\begin{aligned} \delta H(\xi,\zeta) &= \bar{H}(\xi,\zeta) - H(\xi,\zeta) \\ &= \int du \int dw \; [h(u,w)\delta h(\xi+u,\zeta-w) + h(\xi+u,\zeta-w)\delta h(u,w) \\ &+ \int du \int dw \; [\delta h(u,w)\delta h(\xi+u,\zeta-w)] \end{aligned} \tag{3.10}$$

The first integral of (3.10) represents interaction effects between original hull form and the change $\delta h(u,w)$, whereas the second integral represents the Hull Function due to $\delta h(u,w)$ when this is derived from the change as a separate body. The latter integral will always lead to a positive contribution to the wave resistance. Thus, for a change in hull form to be beneficial, the contribution of the first integral must be negative and of greater magnitude than that of the second integral. Equation (3.10) should prove to be valuable in the analysis of bulbous bow designs.

So far, the general functional behavior of the Hull Function has been studied. To get the physical significance of the function, however, it is necessary to proceed in a somewhat different manner.

Consider two elementary areas Δ_1 and Δ_2 ($\Delta_1 = \Delta_2 = \Delta$) located on the hull surface at (u_1,w_1) and (u_2,w_2) respectively, and let

$$h(u_1,w_1) = h_1, \; h(u_2,w_2) = h_2.$$

Since $\quad H(\xi,\zeta) = \int du \int dw \; h(u,w)h(\xi+u,\zeta-w)$

the Hull Function will be different from zero only for specific values of ξ and ζ. For example,

$$H(0,\ w_1) = \Delta\, h_1{}^2$$

$$H(0,\ w_2) = \Delta\, h_2{}^2$$

$$H(u_1-u_2,\ w_1+w_2) = \Delta\, h_2 h_1$$

$$H(u_2-u_1,\ w_1+w_2) = \Delta\, h_1 h_2$$

From these expressions, one concludes that $H(0,\zeta)$ represent the wave resistance of all the elements of area considered separately and that $H(\xi,\zeta)$, $\xi > 0$, represents interaction effects.

The contribution to wave resistance by the two elementary areas, considering interference effect only, is given by (3.2). Thus it follows that

$$\delta_1 C_W = \frac{8F^2}{\pi}\Delta^2 h_1 h_2 \int_1^\infty e^{-\lambda^2 F(w_1+w_2)}\left[\cos\lambda F(u_1-u_2)+\cos\lambda F(u_2-u_1)\right]\times \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}$$

$$= \frac{16F^2}{\pi}\Delta^2 h_1 h_2 \int_1^\infty e^{-\lambda^2 F(w_1+w_2)} \cos\lambda F(u_1-u_2)\,\frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}$$

This expression is the same as that obtained by Michell from the original form of the Michell Integral.

If the two elementary areas are located on the same vertical line, $u_1 = u_2$ or $\xi = 0$, the interference effect becomes

$$\delta_1 C_w \Big|_{\xi=0} = \frac{16F^2}{\pi}\Delta^2 h_1 h_2 \int_1^\infty e^{-\lambda^2 F(w_1+w_2)} \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}$$

Now Michell[1] has shown that

$$\int_1^\infty e^{-a\lambda^2} \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}} = \frac{1}{4} e^{-\frac{a}{2}} \left[K_0\left(\frac{a}{2}\right) + K_1\left(\frac{a}{2}\right)\right] \qquad (3.11)$$

where $K_0\left(\frac{a}{2}\right)$ and $K_1\left(\frac{a}{2}\right)$ are the modified Bessel functions of the second kind. From this it follows that

$$\delta_1 C_w \Big|_{\xi=0} = \frac{4F^2}{\pi}\Delta^2 h_1 h_2 e^{-F\frac{w_1+w_2}{2}}\left[K_0\left(F\frac{w_1+w_2}{2}\right) + K_1\left(F\frac{w_1+w_2}{2}\right)\right]$$

Now if h_1 and h_2 have the same sign, $\delta_1 C_w(\xi = 0)$ is always positive. The favorable effect of a bulbous bow in the area immediately above the bulb is therefore due, to a large extent, to the reduction of hull slope over the portion where the bulb is located.

The contribution to wave resistance by the two elementary areas considered separately is given by

$$\delta_2 C_w = \frac{8F^2}{\pi}\Delta^2 \left[h_1^2 \int_1^\infty e^{-2\lambda^2 F w_1} \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}} + h_2^2 \int_1^\infty e^{-2\lambda^2 F w_2} \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}\right]$$

$$= \frac{2F^2}{\pi}\Delta^2 \Big\{ h_1^2 e^{-Fw_1} \left[K_0(Fw_1) + K_1(Fw_1)\right]$$

$$+ h_2^2 e^{-Fw_2} \left[K_0(Fw_2) + K_1(Fw_2)\right] \Big\}$$

This expression shows that $\delta_2 C_W$ is always positive, and its magnitude depends upon how far the elementary areas are located below the undisturbed surface and upon the absolute value of the hull slopes.

From what has been said, it becomes apparent that the wave resistance of a ship is made up of two parts, namely:

a) The sum of the effects of all elementary areas taken separately; and

b) The sum of interference effects between any two elementary areas taken in pairs.

It has been shown that the interference effects depend upon the horizontal distance between the areas and the sum of vertical distances to the undisturbed surface. The horizontal distance is represented by the variable ξ and the sum of vertical distances by the variable ζ. It follows then that for a given value of ξ and ζ, the Hull Function $H(\xi,\zeta)$ represents the total interference effects of all areas, taken in pairs, having a horizontal spacing equal to ξ and for which the sum of vertical coordinates is equal to ζ.

The wave resistance is determined by the products of the functions $H(\xi,\zeta)$ and $C(F\xi, F\zeta)$. Over any region in the (ξ, ζ)-plane where these functions are of equal sign, one may say that interaction effects are detrimental, whereas opposite signs indicate favorable conditions. Since

$C(F\xi, F\zeta)$ is completely determined for a specified F (a given Froude number), the Hull Function represents all the wave resistance characteristics of the ship at any given speed of advance. This function should therefore prove, within the limits of the theory, to be an invaluable analytic tool in determining ship forms of minimum wave resistance.

The Michell Function

The Michell Function is defined by (3.4) as

$$C(s,t) = \int_1^\infty e^{-t\lambda^2} \cos s\lambda \, \frac{\lambda^2 d\lambda}{\sqrt{\lambda^2-1}}$$

Differentiating under the integral sign, one observes that C(s,t) satisfies the homogeneous heat equation:

$$\frac{\partial^2 u}{\partial s^2} = \frac{\partial u}{\partial t} \, ; \; t > 0. \tag{3.12}$$

It is known, by reference 40 that

$$\int_1^\infty \cos x\lambda \, \frac{d\lambda}{\sqrt{\lambda^2-1}} = -\frac{\pi}{2} Y_o(x) \tag{3.13}$$

where $Y_o(x)$ is the Bessel function of the second kind. Furthermore, it can be verified that

$$\lim_{t\to 0^+} \int_1^\infty e^{-\lambda^2 t} \cos s\lambda \, \frac{d\lambda}{\sqrt{\lambda^2-1}} = \int_1^\infty \cos s\lambda \, \frac{d\lambda}{\sqrt{\lambda^2-1}}$$

Hence it follows that

$$\lim_{t\to 0^+} C(s,t) = \frac{\pi}{2} \frac{\partial^2}{\partial s^2} Y_o(s) \tag{3.14}$$

An expression for C(0, t) is obtained by making use of the relationship (3.11), namely

$$C(0,\ t) = \frac{e^{-\frac{t}{2}}}{4}\left[K_0\left(\tfrac{t}{2}\right) + K_1\left(\tfrac{t}{2}\right)\right] \tag{3.15}$$

An asymptotic expression for large values of s and fixed t, obtained by J. Kotic and rederived in reference 37, is given by

$$C(s,t) = \sqrt{\frac{\pi}{2s}}\, e^{-t}\Big\{\cos\left(s+\tfrac{\pi}{4}\right) - \tfrac{1}{s}\left(\tfrac{7}{8}-t\right)\sin\left(s+\tfrac{\pi}{4}\right) - \tfrac{1}{s^2}\left(\tfrac{3}{2}t^2 - \tfrac{27}{8}t + \tfrac{57}{128}\right)\cos\left(s+\tfrac{\pi}{4}\right) + O(s^{-3})\Big\} \tag{3.16}$$

It follows from this formula that C(s, t) tends to zero for large values of s and t.

Birkhoff(37) outlines methods and procedures for obtaining numerical values of the Michell Function. If a formal integration of the product of Hull Function and Michell Function over the region S" is contemplated, however, where $S''\{0 \leqq \xi \leqq 1;\ 0 \leq \xi \leqq \frac{2D}{L}\}$, numerical values are not sufficient. A general expression of the Michell Function must be found. One method of obtaining such an expression is as follows:

In Equation (3.4) let

$$\lambda^2 = \bar{\lambda} + 1 \qquad ; \qquad d\lambda = \frac{d\bar{\lambda}}{2\sqrt{\bar{\lambda}+1}}$$

so that the Michell Function becomes

$$C(s,t) = \tfrac{1}{2}e^{-t}\int_0^\infty e^{-t\bar{\lambda}} \cos s\sqrt{\bar{\lambda}+1}\ \frac{\sqrt{\bar{\lambda}+1}}{\bar{\lambda}^{\frac{1}{2}}}\, d\bar{\lambda} \tag{3.17}$$

$$= \tfrac{1}{2}e^{-t} L\left\{\cos s\sqrt{\bar{\lambda}+1}\ \frac{\sqrt{\bar{\lambda}+1}}{\bar{\lambda}^{\frac{1}{2}}}\right\}$$

where $L\{\ \}$ is the Laplace Transform operator. By reference 42, the transform of (3.17) exists.

From the well known relationship

$$J_{-\frac{1}{2}}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \cos z$$

one has that

$$\cos s\sqrt{\bar{\lambda}+1} = \left(\frac{\pi}{2} s\sqrt{\bar{\lambda}+1}\right)^{\frac{1}{2}} J_{-\frac{1}{2}}(s\sqrt{\bar{\lambda}+1})$$

Furthermore, it can be shown that (see Appendix II)

$$\left(\frac{s}{2}\right)^{\nu}\left(\frac{1}{2}s\sqrt{\bar{\lambda}+1}\right)^{-\nu} J_{\nu}(s\sqrt{\bar{\lambda}+1}) = \sum_{n=0}^{\infty} \frac{J_{\nu+n}(s)\left(\frac{-s\bar{\lambda}}{2}\right)^{n}}{n!}$$

Thus

$$\cos s\sqrt{\bar{\lambda}+1} = \left(\frac{\pi s}{2}\right)^{\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^{n} J_{-\frac{1}{2}+n}(s)\,(s\bar{\lambda})^{2}}{2^{n}\, n!} \tag{3.18}$$

Substituting into Equation (3.17)

$$C(s,t) = \frac{1}{2}\left(\frac{\pi s}{2}\right)^{\frac{1}{2}} e^{-t} \sum_{n=0}^{\infty} \frac{(-1)^{n} J_{-\frac{1}{2}+n}(s)\, s^{n}}{2^{n}\, n!} \int_{0}^{\infty} e^{-t\bar{\lambda}}\, \bar{\lambda}^{\,n-\frac{1}{2}} (1+\bar{\lambda})^{\frac{1}{2}} d\lambda \tag{3.19}$$

By reference 39, p. 269

$$\int_{0}^{\infty} e^{-t\bar{\lambda}}\, \bar{\lambda}^{\,n-\frac{1}{2}} (1+\bar{\lambda})^{\frac{1}{2}} d\bar{\lambda} = \Gamma(n+\tfrac{1}{2})\, \Psi(n+\tfrac{1}{2}, n+2; t) \tag{3.20}$$

where $\Psi(n+\frac{1}{2}, n+2; t)$ is the Confluent Hypergeometric Function. Since $(n+2)$ is always a positive integer, $\Psi(n+\frac{1}{2}, n+2; t)$ is logarithmic near origin. (See Appendix III.)

Using the relationship

$$\Gamma(n+\tfrac{1}{2}) = \sqrt{\pi}\,(\tfrac{1}{2})_{n}$$

where

$$\left(\tfrac{1}{2}\right)_n = \tfrac{1}{2}\left(\tfrac{1}{2} + 1\right)\left(\tfrac{1}{2} + 2\right)\cdots\cdots\left(\tfrac{1}{2} + n - 1\right)$$

and substituting (3.20) in (3.19), the expression for the Michell Function becomes

$$C(s,t) = \frac{\pi}{2}\left(\frac{s}{2}\right)^{\frac{1}{2}} e^{-t} \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{1}{2}\right)_n s^n}{2^n n!} \left[J_{-\frac{1}{2}+n}(s)\right]\left[\Psi\left(n+\tfrac{1}{2},\ n+2;\ t\right)\right] \tag{3.21}$$

Wave Resistance

The wave resistance of a surface ship moving at constant speed on a straight course in water of infinite depth is given by Equation (3.5) with the Hull Function as defined by (3.6) and the Michell Function by (3.21)

It will now be assumed that the Hull Function can in general be expressed as polynomials--i.e.,

$$\begin{aligned} \left[H(\xi,\zeta)\right]_{I} &= \sum_{\alpha,\beta}^{M,N} A^{I}_{\alpha\beta}\, \xi^{\alpha} \zeta^{\beta} \\ \left[H(\xi,\zeta)\right]_{II} &= \sum_{\alpha,\beta}^{\bar{M},\bar{N}} A^{II}_{\alpha\beta}\, \xi^{\alpha} \zeta^{\beta} \end{aligned} \tag{3.22}$$

where $A^{I}_{\alpha\beta}$ and $A^{II}_{\alpha\beta}$ are constants and α and β are non-negative integers. Since $H(\xi, \zeta)$ vanishes at least as fast as a linear function as $\zeta \rightarrow 0$, one concludes that in $[H(\xi, \zeta)]_I \beta \geqq 1$.

In the following, expressions for the contribution to wave resistance due to a general term of the polynomials of (3.22) will be presented. Depending upon whether the term belongs to the first or second polynomial, this contribution will be defined by $[\Delta^{\alpha\beta}C_w]_I$ and $[\Delta^{\alpha\beta}C_w]_{II}$ respectively.

From Equations (3.5), (3.21) and (3.22), it follows that

$$[\Delta^{\alpha\beta}C_w]_I = \frac{16F^2}{\pi}\int_0^1 d\xi \int_0^{\frac{D}{L}} d\zeta\, A^I_{\alpha\beta}\ \xi^\alpha \zeta^\beta \frac{\pi}{2}\left(\frac{\xi F}{2}\right)^{\frac{1}{2}} e^{-\zeta F} \left\{\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})_n(F\xi)^n}{2^n\, n!}\left[J_{-\frac{1}{2}+n}(\xi F)\right]\left[\Psi(n+\tfrac{1}{2}, n+2; F\zeta)\right]\right. \tag{3.23}$$

Introducing new variables defined by

$$F\xi = \bar{\xi} \text{ and } F\zeta = \bar{\zeta},$$

Equation (3.23) becomes

$$[\Delta^{\alpha\beta}C_w]_I = \frac{4\sqrt{2}\,A^I_{\alpha\beta}}{F^{\alpha+\beta}}\int_0^F d\bar{\xi}\int_0^{\frac{FD}{L}} d\bar{\zeta}\left\{\bar{\xi}^{\alpha+\frac{1}{2}}\bar{\zeta}^{\beta} e^{-\bar{\zeta}}\right\} \left\{\sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})_n(\bar{\xi})^n}{2^n\, n!}\left[J_{-\frac{1}{2}+n}(\bar{\xi})\right]\left[\Psi(n+\tfrac{1}{2}, n+2; \bar{\zeta}\right]\right. \tag{3.24}$$

Since the logarithmic case of the Confluent Hypergeometric Function has a singularity at $\bar{\zeta} = 0$, consider

$$I_1 = \lim_{\epsilon\to 0^+}\int_{\epsilon}^{\frac{FD}{L}} e^{-\bar{\zeta}}\bar{\zeta}^{\beta}\Psi(n+\tfrac{1}{2}, n+2; \bar{\zeta})\, d\bar{\zeta} \tag{3.25}$$

Now from reference 39

$$\frac{d}{d\bar{\zeta}}\left[e^{-\bar{\zeta}}\Psi(n+\tfrac{1}{2}, n+1; \bar{\zeta})\right] = -e^{-\bar{\zeta}}\Psi(n+\tfrac{1}{2}, n+2; \bar{\zeta}) \tag{3.26}$$

If (3.25) is integrated by parts, one has

$$I_1 = \lim_{\epsilon\to 0^+}\left\{-\bar{\zeta}^{\beta}e^{-\bar{\zeta}}\Psi(n+\tfrac{1}{2}, n+1; \bar{\zeta})\Big|_{\epsilon}^{\frac{FD}{L}} + \beta\int_{\epsilon}^{\frac{FD}{L}}\bar{\zeta}^{\beta-1}e^{-\bar{\zeta}}\Psi(n+\tfrac{1}{2}, n+1; \bar{\zeta})\, d\bar{\zeta}\right\}$$

So that by iteration,

$$I_1 = \lim_{\epsilon \to 0^+} \left\{ -e^{-\bar{\xi}} \sum_{k=0}^{\beta} (-\beta)_k (-1)^k \bar{\xi}^{\beta-k} \Psi(n+\tfrac{1}{2}, n+1-k; \bar{\xi}) \right|_{\epsilon}^{\frac{FD}{L}} \tag{3.27}$$

Substituting (3.27) in (3.24), it follows that

$$\begin{aligned}
[\Delta^{\alpha\beta} C_w]_I = \frac{4\sqrt{2}\, A_{\alpha\beta}^{I}}{F^{\alpha+\beta}} \Bigg[&-\int_0^F d\bar{\xi} \left\{ \bar{\xi}^{\alpha+\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n (\bar{\xi})^n}{2^n\, n!} [J_{-\frac{1}{2}+n}(\bar{\xi})] \right\} \\
&\left\{ e^{-\frac{FD}{L}} \sum_{k=0}^{\beta} (-\beta)_k \left(\tfrac{FD}{L}\right)^{\beta-k} \Psi(n+\tfrac{1}{2}, n+1-k; \tfrac{FD}{L}) \right\} \\
&+ \lim_{\epsilon \to 0^+} \int_0^F d\bar{\xi} \left\{ \bar{\xi}^{\alpha+\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n (\bar{\xi})^n}{2^n\, n!} [J_{-\frac{1}{2}+n}(\bar{\xi})] \right\} \\
&\left\{ e^{-\epsilon} \sum_{k=0}^{\beta} (-\beta)_k (\epsilon)^{\beta-k} \Psi(n+\tfrac{1}{2}, n+1-k; \epsilon) \right\} \Bigg]
\end{aligned} \tag{3.28}$$

If the integral representation of the Confluent Hypergeometric Function and relationship (a) of Appendix II is used, the second integral in (3.28) becomes

$$\begin{aligned}
I_2 = \lim_{\epsilon \to 0^+} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_0^F d\bar{\xi} \Bigg\{ &\bar{\xi}^{\alpha} e^{-\epsilon} \sum_{k=0}^{\beta} (-\beta)_k (-1)^k \epsilon^{\beta-k} \times \\
&\int_0^{\infty} e^{-\epsilon\bar{\lambda}} \cos \bar{\xi}\sqrt{\bar{\lambda}+1}\ \frac{d\bar{\lambda}}{\bar{\lambda}^{\frac{1}{2}} (\bar{\lambda}+1)^{k+\frac{1}{2}}} \Bigg\}
\end{aligned}$$

In the limit as $\epsilon \to 0$, I_2 reduces to

$$\begin{aligned}
I_2 &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \beta! \int_0^F \bar{\xi}^{\alpha} d\bar{\xi} \int_0^{\infty} \cos \bar{\xi}\sqrt{\bar{\lambda}+1}\ \frac{d\bar{\lambda}}{\bar{\lambda}^{\frac{1}{2}} (\bar{\lambda}+1)^{\beta+\frac{1}{2}}} \\
&= \int_0^F \bar{\xi}^{\alpha} M^{(\beta)}(\bar{\xi})\, d\bar{\xi}
\end{aligned} \tag{3.29}$$

where the $M^{(\beta)}(\bar{\xi})$ depends upon the value of β. Now from (3.13),

$$Y_0(\bar{\xi}) = -\frac{1}{\pi}\int_0^\infty \cos \bar{\xi}\sqrt{\bar{\lambda}+1}\ \frac{d\bar{\lambda}}{\bar{\lambda}^{\frac{1}{2}}(\bar{\lambda}+1)^{\frac{1}{2}}}$$

so that

$$\frac{d^{2\beta}}{d\bar{\xi}^{2\beta}}\left[M^{(\beta)}(\bar{\xi})\right] = (-1)^{\beta+1}\beta!\sqrt{2\pi}\,Y_0(\bar{\xi}) \tag{3.30}$$

Moreover, from (3.29) one has

$$M^{(\beta)}(0) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\beta!\int_0^\infty \frac{d\bar{\lambda}}{\bar{\lambda}^{\frac{1}{2}}(\bar{\lambda}+1)^{\beta+\frac{1}{2}}}$$

This may be transformed by

$$\bar{\lambda} = \tan^2\theta \quad ; \quad d\bar{\lambda} = 2\tan\theta\sec^2\theta\, d\theta$$

so that

$$\begin{aligned} M^{(\beta)}(0) &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\beta!\int_0^{\frac{\pi}{2}}\cos^{2\beta-1}\theta\, d\theta \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\beta!\int_0^{\frac{\pi}{2}}\sum_{k=0}^{\beta-1}\frac{(-1)^k(\beta-1)!\sin^{2k}\theta}{k!(\beta-1-k)!}\cos\theta\, d\theta \\ &= \left(\frac{2}{\pi}\right)^{\frac{1}{2}}\beta!\sum_{k=0}^{\beta-1}\frac{(-1)^k(\beta-1)!}{k!(\beta-1-k)!(2k+1)} \end{aligned} \tag{3.31}$$

Also from (3.29) it follow that

$$\frac{d}{d\bar{\xi}}M^{(\beta)}(\bar{\xi}) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\beta!\int_0^\infty \sin\bar{\xi}\sqrt{\bar{\lambda}+1}\ \frac{d\bar{\lambda}}{\bar{\lambda}^{\frac{1}{2}}(\bar{\lambda}+1)^{\beta}}$$

$$\frac{d^2}{d\bar{\xi}^2}M^{(\beta)}(\bar{\xi}) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\beta!\int_0^\infty \cos\bar{\xi}\sqrt{\bar{\lambda}+1}\ \frac{d\bar{\lambda}}{\bar{\lambda}^{\frac{1}{2}}(\bar{\lambda}+1)^{\beta-\frac{1}{2}}}$$

Also, by iteration it can be shown that the derivatives of $M^{(\beta)}(\bar{\xi})$

at $\bar{\xi} = 0$ are

$$\begin{aligned} \left.\frac{d^m}{d\bar{\xi}^m}M^{(\beta)}(\bar{\xi})\right|_{\bar{\xi}=0} &= 0 \quad ; \quad m = 1,3,5\cdots(2\beta-1) \\ &= (-1)^{\frac{m}{2}}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\beta!\sum_{k=0}^{\beta-1-\frac{m}{2}}\frac{(-1)^k(\beta-1-\frac{m}{2})!}{(2k+1)k!(\beta-1-\frac{m}{2}-k)!} \\ & \qquad m = 0,2,4\cdots(2\beta-2) \end{aligned} \tag{3.32}$$

Now the Bessel Function of the second kind and of zero order is given by

$$Y_0(x) = \frac{2}{\pi}\left\{\gamma\, J_0(x) + \log \tfrac{1}{2}x\; J_0(x)\right\}$$
$$- \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n}}{(n!)^2}\left\{1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{n}\right\}$$

where

$$J_0(x) = \sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{2}x\right)^{2n}}{(n!)^2}$$

so that integrating $Y_0(\xi)$ repeatedly and using (3.32) and (3.31), it can be shown that

$$M^{(\beta)}(\xi) = \frac{2}{\pi}(-1)^{\beta+1}\beta!\sqrt{2\pi}\Big[\sum_{n=0}^{\infty}(-1)^n \frac{\left(\frac{1}{2}\right)^{2n}\xi^{2n+2\beta}}{(2n+1)_{2\beta}\,(n!)^2}\left\{\gamma+\log\tfrac{1}{2}\xi-\left(\tfrac{1}{2n+1}+\tfrac{1}{2n+2}+\cdots+\tfrac{1}{2n+2\beta}\right)\right\}$$
$$-\sum_{n=1}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)^{2n}\xi^{2n+2\beta}}{(2n+1)_{2\beta}\,(n!)^2}\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\cdots+\tfrac{1}{n}\right)\Big] \qquad (3.33)$$
$$+\sum_{n=0}^{\beta-1}(-1)^n\frac{1}{(2n)!}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\beta!\left(\sum_{k=0}^{\beta-1-n}\frac{(-1)^k(\beta-1-n)!}{(2k+1)\,k!\,(\beta-1-n-k)!}\right)\xi^{2n}$$

If this is substituted into (3.29), then

$$I_2 = \frac{2}{\pi}(-1)^{\beta+1}\beta!\sqrt{2\pi}\Big[\sum_{n=0}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)^{2n}F^{2n+2\beta+\alpha+1}}{(2n+2\beta+\alpha+1)(2n+1)_{2\beta}(n!)^2}\times$$
$$\left\{\gamma+\log\tfrac{1}{2}F-\left(\tfrac{1}{2n+2\beta+\alpha+1}+\tfrac{1}{2n+1}+\tfrac{1}{2n+2}+\cdots+\tfrac{1}{2n+2\beta}\right)\right\}$$
$$-\sum_{n=1}^{\infty}(-1)^n\frac{\left(\frac{1}{2}\right)^{2n}F^{2n+2\beta+\alpha+1}}{(2n+2\beta+\alpha+1)(2n+1)_{2\beta}(n!)^2}\left(1+\tfrac{1}{2}+\cdots+\tfrac{1}{n}\right)\Big] \qquad (3.34)$$
$$+\sum_{n=0}^{\beta-1}(-1)^n\frac{1}{(2n+\alpha+1)\,(2n)!}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\beta!\left(\sum_{k=0}^{\beta-1-n}\frac{(-1)^k(\beta-1-n)!}{(2k+1)\,k!\,(\beta-1-n-k)!}\right)F^{2n+\alpha+1}$$

Returning to Equation (3.28), one observes that the first integral is of the type

$$I_3 = \int_0^F \bar{\xi}^{\alpha+\frac{1}{2}+n} J_{-\frac{1}{2}+n}(\bar{\xi})\, d\bar{\xi} \tag{3.35}$$

Let

$$u = \bar{\xi}^{\alpha}\, ; \quad du = \alpha\, \bar{\xi}^{\alpha-1} d\bar{\xi}$$

$$dv = \bar{\xi}^{\frac{1}{2}+n} J_{-\frac{1}{2}+n}(\bar{\xi})\, d\bar{\xi}$$

$$v = \bar{\xi}^{\frac{1}{2}+n} J_{\frac{1}{2}+n}(\bar{\xi})$$

(Ref. 40, p. 192)

so that integrating by parts gives

$$I_3 = \bar{\xi}^{\alpha+\frac{1}{2}+n} J_{\frac{1}{2}+n}(\bar{\xi})\Big|_0^F - \alpha \int_0^F \bar{\xi}^{\alpha-2}\, \bar{\xi}^{\frac{3}{2}+n} J_{\frac{1}{2}+n}(\bar{\xi})\, d\bar{\xi}$$

Continuing by iteration, one obtains

$$I_3 = \sum_{k=0}^{\frac{\alpha}{2}} 2^k \left(-\tfrac{\alpha}{2}\right)_k F^{\alpha+\frac{1}{2}+n-k} J_{\frac{1}{2}+n+k}(F) \tag{3.36}$$

This Equation is valid, however, only for α an even integer. For α an odd integer the last integral of (3.35), after integration by parts $\left[\frac{\alpha}{2}\right]$ times, is of the form

$$2^{k_1}\left(-\tfrac{\alpha}{2}\right)_{k_1} \int_0^F J_{\frac{\alpha}{2}+n}(\bar{\xi})\, \bar{\xi}^{\frac{\alpha}{2}+n} d\bar{\xi}$$

where $k_1 = \left[\frac{\alpha+1}{2}\right]$ --the greatest integer in $\frac{\alpha+1}{2}$.

Thus by reference 40, p. 194, for α an odd integer, I_3 must be replaced by

$$I_3 = \sum_{k=0}^{[\frac{\alpha}{2}]} 2^k(-\tfrac{\alpha}{2})_k F^{\alpha+\frac{1}{2}+n-k} J_{\frac{1}{2}+n+k}(F)$$

$$+ 2^{k_1}(-\tfrac{\alpha}{2})_{k_1} 2^{\frac{\alpha}{2}+n-1} \pi^{\frac{1}{2}} \Gamma(\tfrac{\alpha}{2}+n+\tfrac{1}{2}) \times$$

$$F\left\{ J_{\frac{\alpha}{2}+n+1}(F) H_{\frac{\alpha}{2}+n}(F) - J_{\frac{\alpha}{2}+n}(F) H_{\frac{\alpha}{2}+n+1}(F) \right\} \qquad (3.37)$$

$$+ 2^{k_1}(-\tfrac{\alpha}{2})_{k_1} \frac{F^{\frac{\alpha}{2}+n+1}}{2(\frac{\alpha}{2}+n)+1} J_{\frac{\alpha}{2}+n}(F)$$

where $H_\nu(x)$, the Struve Function, is defined by

$$H_\nu(x) = \sum_{n=0}^{\infty} (-1)^n \frac{(\frac{1}{2}x)^{\nu+2n+1}}{\Gamma(n+\frac{3}{2})\Gamma(\nu+n+\frac{3}{2})}$$

From Equation (3,23), (3.34), (3.36) or (3.37), one finally obtains

$$[\Delta^{\alpha\beta}C_w]_I = \frac{4\sqrt{2}\, A^I_{\alpha\beta}}{F^{\alpha+\beta}} \left\{ I_2 - \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2})_n}{2^n\, n!} I_3 \right. \qquad (3.38)$$

$$\left. [e^{-\frac{FD}{L}} \sum_{k=0}^{\beta} (-\beta)_k (-1)^k (\tfrac{FD}{L})^{\beta-k} \Psi(n+\tfrac{1}{2}, n+1-k; \tfrac{FD}{L})] \right\}$$

By inspection, the corresponding expression for $[\Delta^{\alpha\beta}C_w]_{II}$ becomes

$$[\Delta^{\alpha\beta}C_w]_{II} = \frac{4\sqrt{2}\, A^{II}_{\alpha\beta}}{F^{\alpha+\beta}} \sum_{n=0}^{\infty} \frac{(-1)^n(\frac{1}{2})_n}{2^n\, n!} I_3$$

$$\left\{ [e^{-\frac{FD}{L}} \sum_{k=0}^{\beta} (-\beta)_k (-1)^k (\tfrac{FD}{L})^{\beta-k} \Psi(n+\tfrac{1}{2}, n+1-k; \tfrac{FD}{L})] \right. \qquad (3.39)$$

$$\left. - [e^{-\frac{2FD}{L}} \sum_{k=0}^{\beta} (-\beta)_k (-1)^k (\tfrac{2FD}{L})^{\beta-k} \Psi(n+\tfrac{1}{2}, n+1-k; \tfrac{2FD}{L})] \right\}$$

The total wave resistance coefficient is obtained by adding the contributions from the individual terms of the Hull Function polynomials as given by (3.38) and (3.39)--i.e.,

$$C_W = \sum_{\alpha,\beta}^{M,N} [\Delta^{\alpha\beta} C_W]_{I} + \sum_{\alpha,\beta}^{\bar{M},\bar{N}} [\Delta^{\alpha\beta} C_W]_{II} \tag{3.40}$$

The expressions for the wave resistance as given by (3.38) and (3.39) are convergent. (See Appendix IV.) It should be emphasized that Equation (3.40) is not restricted to ship forms symmetrical fore and aft. The expression is however, only valid for ships with rectangular longitudinal plane of symmetry.

CHAPTER IV

CONCLUSION

A general formula for the wave resistance of polynomial ship forms has been obtained. The result is based on the Michell Integral, and all assumptions made in the linearized wave resistance theory of thin ships apply.

The series form solution is not too complicated in form. It does, however, involve several transcendental functions, some of which are not readily available. Also, one notes that even for the simplest ship form, at least about 20 series expressions will have to be evaluated at each Froude number investigated, a prohibitively long task if it was to be repeated for each individual design.

The real significance of the results obtained lies in the fact that a systematic program of evaluations of the formula can be undertaken with the aid of high speed digital computers. The functions so computed depend upon the Froude Number and the power of the term of the hull polynomial, but not on the individual ship form directly. For each value of α and β of the Hull Function the wave resistance can therefore be tabulated for various values of the Froude Number. Once such tables become available, the wave resistance calculation will have been reduced to a few elementary operations. Model experiments, in conjunction with theoretical calculations, have repeatedly shown that the linearized wave resistance theory is capable of predicting effects of even small variations in hull form. Indeed, it is the opinion of several researchers in the field that the linear theory will determine hull forms of minimum wave resistance with sufficient accuracy. As a result of the formula presented, it should become possible to investigate families of ship forms systematically and to determine such forms. This has always been a principle aim of the ship hydrodynamicist.

The complexity of the formulae given makes it difficult to study the behavior of the functions involved analytically at this time. This task must be left to the computer.

As an extension to the present work, it is hoped that it will become possible to obtain expressions for the complete velocity potential in terms of ship form and speed. It would also be interesting to consider perturbation potentials in order to improve the assumption made on both the ship surface and the free surface of the waves. These are but two of the problems of ship wave resistance still left open.

APPENDIX I

THE HULL FUNCTION FOR A SIMPLE SHIP FORM

The term Simple Ship Forms refers to ships' surfaces whose equations are given by

$$g(x,z) = g_1(x)g_2(z).$$

For these shapes, the Hull Functions can be written

$$H(\xi,\zeta) = H_1(\xi)H_2(\zeta).$$

Consider the particular case

$$g(x,z) = B(1 - \frac{z}{D})(1 - 4\frac{x^2}{L^2})$$

which implies that

$$G(u,w) = (1 - \frac{L}{D}w)(1 - 4u^2)$$

$$\text{and } h(u,w) = \frac{\partial G}{\partial u} = -8u(1 - \frac{L}{D}w).$$

By Equation (3.6)

$$[H(\xi,\zeta)]_I = \int_{-\frac{1}{2}}^{\frac{1}{2}-\xi} \int_0^{\zeta} 8(u+\xi)\, 8u \left[1 - \frac{L}{D}(\zeta - w)\right]\left[1 - \frac{L}{D}w\right] du\, dw$$

$$= 64\left\{\frac{1}{6}\xi^3 - \frac{1}{4}\xi + \frac{1}{12}\right\}\left\{\frac{1}{6}\frac{L^2}{D^2}\zeta^3 - \frac{L}{D}\zeta^2 + \zeta\right\}$$

$$= H_1(\xi)\, H_2(\zeta)$$

where

$$H_1(\xi) = 16\left(\frac{2}{3}\xi^3 - \xi + \frac{1}{3}\right) \; ; \quad \xi \geq 0$$

$$H_2(\zeta) = \zeta - \frac{L}{D}\zeta^2 + \frac{L^2}{6D^2}\zeta^3 ; \quad 0 \leq \zeta \leq \frac{D}{L}$$

Similarly

$$[H(\xi,\zeta)]_{II} = H_1(\xi)\int_{\zeta-\frac{D}{L}}^{\frac{D}{L}} \left[1-\frac{L}{D}(\zeta-w)\right]\left[1-\frac{L}{D}w\right] dw$$

$$= H_1(\xi)\left[\frac{4}{3}\frac{D}{L} - 2\zeta + \frac{L}{D}\zeta^2 - \frac{1}{6}\frac{L^2}{D^2}\zeta^3\right]$$

$$= H_1(\xi)H_2(\zeta) \quad ; \quad \frac{D}{L} \le \zeta \le \frac{2D}{L}$$

A plot of $H_1(\xi)$ and $H_2(\zeta)$ is shown in figure 2.

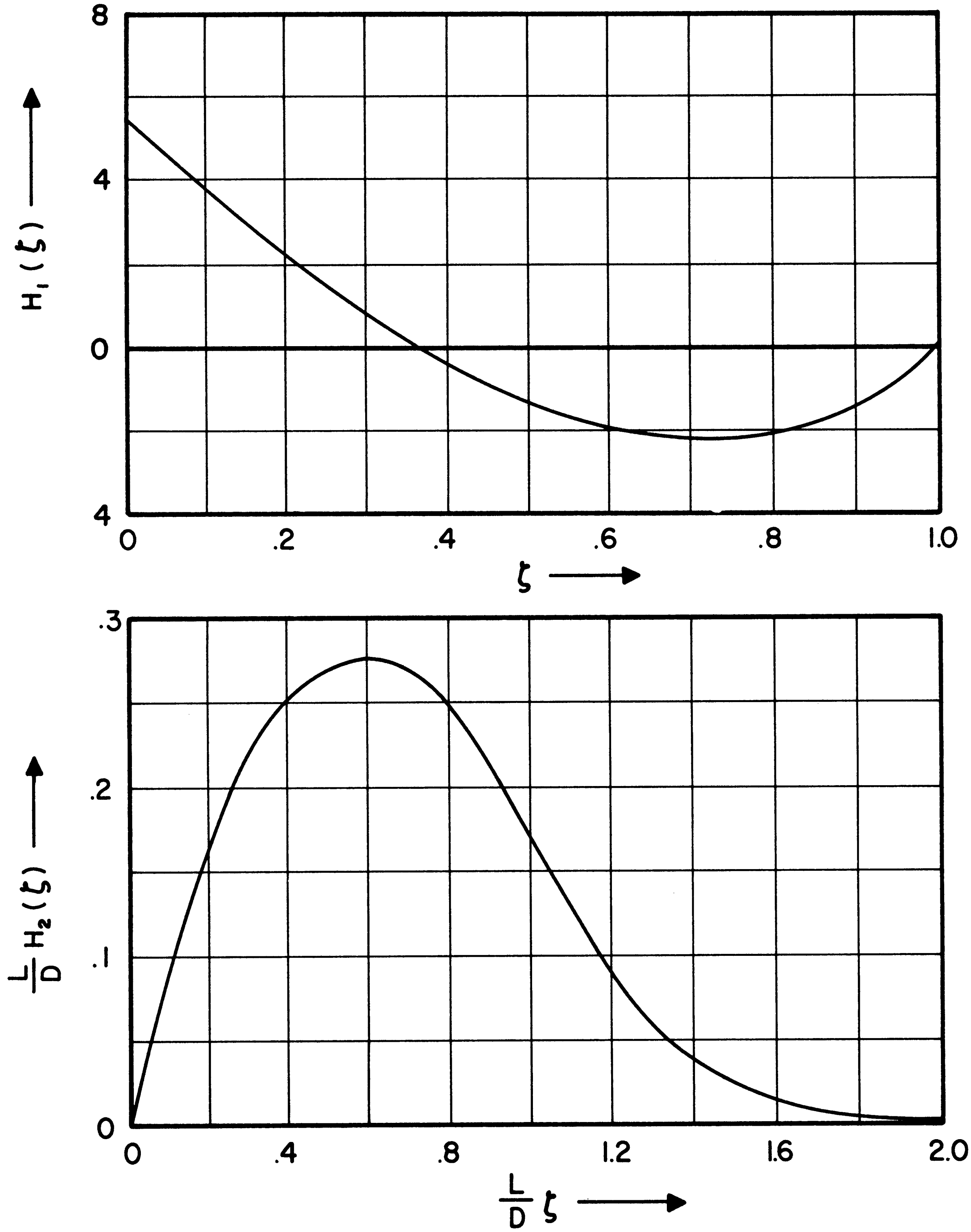

Figure 2. The Hull Function For $G(u.w.) = (1-\frac{L}{D}w)(1-u^2)$

APPENDIX II

A BESSEL FUNCTION RELATIONSHIP

Consider the generalized hypergeometric function

$$ {}_0F_1\left[\begin{matrix} - \\ 1+\nu \end{matrix}\,;\; -\frac{x^2(t+1)}{4}\right] = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}(t+1)^n}{(1+\nu)_n\, 2^{2n}\, n!} $$

$$ = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(-1)^n x^{2n}\, t^{n-k}}{(1+\nu)_n\, 2^{2n}\, k!\,(n-k)!} $$

where the last step follows from the Binomial Theorem. It can be shown that

$$ \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(n,k) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(n+k,k) $$

from which

$$ {}_0F_1\left[\begin{matrix} - \\ 1+\nu \end{matrix}\,;\; -\frac{x^2(t+1)}{4}\right] = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{n+k} x^{2n+2k}\, t^n}{(1+\nu)_{n+k}\, 2^{2n+2k} k!\, n!} $$

$$ = \left(\frac{2}{x}\right)^{\nu} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n+\nu}\left(-\frac{xt}{2}\right)^n}{(1+\nu)_n(1+\nu+n)_k\, k!\, n!\, 2^{n+2k+\nu}} $$

However

$$ (1+\nu)_n = \frac{\Gamma(1+\nu+n)}{\Gamma(1+\nu)} \quad ; \quad (1+\nu+n)_k = \frac{\Gamma(1+\nu+n+k)}{\Gamma(1+\nu)} $$

so that

$$ (1+\nu)_n(1+\nu+n)_k = \frac{\Gamma(1+\nu+n+k)}{\Gamma(1+\nu)} $$

Hence

$${}_0F_1\left[\begin{matrix}-\,;\\1+\nu\,;\end{matrix}\;-\frac{x^2(t+1)}{4}\right]$$

$$= \Gamma(1+\nu)\left(\frac{2}{x}\right)^{\nu}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+n+\nu}\left(-\frac{xt}{2}\right)^n}{\Gamma(1+\nu+n+k)\,k!\,n!\,2^{n+2k+\nu}}$$

$$= \Gamma(1+\nu)\left(\frac{2}{x}\right)^{\nu}\sum_{n=0}^{\infty}\frac{J_{\nu+n}(x)(-1)^n(xt)^n}{2^n\;n!}$$

It is also known that

$${}_0F_1\left[\begin{matrix}-\,;\\1+\nu\,;\end{matrix}\;-\frac{x^2(t+1)}{4}\right] = \frac{\Gamma(1+\nu)}{\left(\frac{1}{2}x\sqrt{t+1}\right)^{\nu}}\,J_{\nu}\left(x\sqrt{t+1}\right)$$

so that from these two relationships, one obtains

(a) $$\left(\frac{x}{2}\right)^{\nu}\left(\frac{1}{2}x\sqrt{t+1}\right)^{-\nu}J_{\nu}\left(x\sqrt{t+1}\right) = \sum_{n=0}^{\infty}\frac{J_{\nu+n}(x)\,(-1)^n(xt)^n}{2^n\;n!}$$

APPENDIX III

NOTES ON THE CONFLUENT HYPERGEOMETRIC FUNCTIONS

In the notation of reference 39, the Confluent Hypergeometric Function is defined by

$$\Phi(a, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(c)_n n!} \tag{III.1}$$

In the notation of generalized hypergeometric series, this is ${}_1F_1(a; c; x)$.

Now $\Phi(a, c, x)$ is a solution of the differential equation

$$x \frac{d^2y}{dx^2} + (c-x) \frac{dy}{dx} - ay = 0 \tag{III.2}$$

and a second solution of (III.2) is given by

$$\Psi(a, c; x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)} \Phi(a, c; x) + \frac{\Gamma(c-1)}{\Gamma(a)} x^{1-c} \Phi(a-c+1, 2-c; x) \tag{III.2}$$

for c not an integer.

For c an integer $\Psi(a, c; x)$ becomes

$$\Psi(a, p+1; x) = \frac{(-1)^{p-1}}{p!\,\Gamma(a-p)} \Big\{ \Phi(a, p+1; x) \log x$$

$$+ \sum_{q=0}^{\infty} \frac{(a)_q}{(p+1)_q} [\psi(a+q) - \psi(1+q) - \psi(1+p+q)] \frac{x^q}{q!}$$

$$+ \frac{(p+1)!}{\Gamma(a)} \sum_{q=0}^{p-1} \frac{(a-p)_q}{(1-p)_q} \frac{x^{q-p}}{q!} \; ; \quad p = 0, 1, 2 \cdots \qquad \text{(III.3)}$$

where $\psi(z)$ is the logarithmic derivation of $\Gamma(z)$.

With $a = n + \frac{1}{2}$; $p = n - k$, this becomes

$$\Psi(n+\tfrac{1}{2}, n+1-k; z) = \frac{(-1)^{n-k-1}}{(n-k)!\,\Gamma(\frac{1}{2}-k)} \Big\{ \Phi(n+\tfrac{1}{2}, n+1-k; z) \log z$$

$$+ \sum_{q=0}^{\infty} \frac{(n+\frac{1}{2})_q}{(n+1-k)_q} [\psi(n+1+q) - \psi(1+q) - \psi(1+n-k+q)] \frac{z^q}{q!} \qquad \text{(III.4)}$$

$$+ \frac{(n-k-1)!}{\Gamma(n+\frac{1}{2})} \sum_{q=0}^{n-k} \frac{(\frac{1}{2}+k)_q}{(1-n+k)_q} \frac{x^{q-n+k}}{q!}$$

Equation (III.3) is valid for $k \leqq n$. When $k > n$--i.e., $p < 0$, an expression for $\Psi(a, p+1; z)$ is obtained by (III.3) from the transformation

$$\Psi(a,c;z) = z^{1-c}\Psi(a-c+1, 2-c; z) \qquad \text{(III.5)}$$

APPENDIX IV

CONVERGENCE OF SOLUTION

From the definition of $M^{(\beta)}(\xi)$ by Equation (3.30) it is readily shown that $\alpha \geq 0$ and $\beta \geq 0$ is a sufficient condition for I_2 given by Equation (3.29) to be finite. This condition on α and β is satisfied by polynomial ship forms. One need therefore consider only the infinite summation on n. Equations (3.36) and (3.37) define I_3 for α an even and α an odd integer respectively. It can be argued that for given values of α and β

$$[\Delta^{\alpha}, \beta C_W]_I < [\Delta^{\alpha+1}, \beta C_W]_I < [\Delta^{\alpha+2}, \beta C_W]_I$$

or

$$[\Delta^{\alpha}, \beta C_W]_I > [\Delta^{\alpha+1}, \beta C_W]_I > [\Delta^{\alpha+2}, \beta C_W]_I.$$

Hence only the cases where α is an even integer need be considered. Furthermore, to prove convertence of Equation (3.38), it is sufficient to consider the terms in the summation on n obtained from the product of a general term of the finite summation of Equation (3.36) and a general term of the finite summation on k of Equation (3.38). The resulting infinite series on n becomes

$$S = \bar{C} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2})_n \bar{z}^n}{2^n n!} J_{\frac{1}{2}+n+k}(F)\, \Psi(n+\tfrac{1}{2}, n+1-k; \tfrac{FD}{L}) \tag{III.1}$$

where $\bar{C} = \left[2^k (-\frac{\alpha}{2})_k F^{\alpha+\frac{1}{2}-k} e^{-\frac{FD}{L}} (-\beta)_{k'} (-1)^{k'} (\frac{FD}{L})^{\beta-k'} \right]$

Since
$$\Psi(a, c; z) = \frac{1}{\Gamma(a)} \int_0^{\infty} e^{-zt} t^{a-1} (1+t)^{c-a-1} dt$$

it follows that

$$\Psi(n+\tfrac{1}{2}, n+1-k; \tfrac{FD}{L}) = \frac{1}{\Gamma(n+\frac{1}{2})} \int_0^{\infty} e^{-\frac{FD}{L}t}\, t^{n-\frac{1}{2}} (1+t)^{k-\frac{3}{2}} dt$$

and then

$$S = \frac{1}{\sqrt{\pi}} \bar{c} \int_0^\infty e^{-\frac{FD}{L}t} t^{-\frac{1}{2}} (1+t)^{k-\frac{3}{2}} \Big[\sum_{n=0}^{\infty} \frac{(-1)^n (Ft)^n}{2^n \, n!} J_{\frac{1}{2}+k+n}(F) \Big] dt \qquad \text{(III.2)}$$

But from Appendix II

$$\sum_{n=0}^{\infty} \frac{(-1)^n J_{\frac{1}{2}+k+n}(F) \; (Ft)^n}{2^n \, n!} = \left(\frac{F}{2}\right)^{\frac{1}{2}+k} \left(\tfrac{1}{2} F \sqrt{t+1}\right)^{-\frac{1}{2}-k} J_{\frac{1}{2}+k}(F\sqrt{t+1}) \qquad \text{(III.3)}$$

This relationship was obtained by a rearrangement of the terms of a ${}_0F_1(-;a;x)$. The infinite series is therefore uniformly convergent and it follows that Equations (III.1) and (III.2) are identical.

Substituting Equation (III.3) into Equation (III.2)

$$S = \frac{1}{\sqrt{\pi}} \bar{c} \int_0^\infty e^{-\frac{FD}{L}t} t^{-\frac{1}{2}} (1+t)^{\frac{k}{2}-\frac{7}{4}} J_{\frac{1}{2}+k}(F\sqrt{t+1}) \, dt$$

Because $\quad |J_{\frac{1}{2}+k}(F\sqrt{t+1})| \leq 1 \;;\; k > 0 \;;\; F\sqrt{t+1} \geq 0$

$$S < \frac{1}{\sqrt{\pi}} \bar{c} \int_0^\infty e^{-\frac{FD}{L}t} t^{-\frac{1}{2}} (1+t)^{\frac{k}{2}-\frac{7}{4}} \, dt \qquad \text{(III.4)}$$

The upper bound for S given by (III.4) is finite by the convergence theorem for the Laplace Transform[42].

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