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Technical Report

A STUDY OF THE TAYLOR-COUETTE STABILITY OF VISCOELASTIC FLUIDS

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NOMENCLATURE

The principal symbols are presented below. Those quantities having only temporary or limited significance are not mentioned here, but are clearly defined in the text. The dyadic notation is discussed in Section 1.3.

Symbol	Definition
a	$a = r_1/d$
CMC	Carboxymethylcellulose
D	Differential operator = d/dX
D/Dt	Material time derivative, (1.16)
$\mathcal{D}/\mathcal{D}t$	Jaumann derivative, (1.17)
d	$d = r_2 - r_1$
$\underset{\sim}{E}$	Rate of strain tensor, (1.14)
$E\ell$	Elasticity number = $\tau\eta_o/\rho d^2$
HEC	Hydroxyethylcellulose
$\underset{\sim}{H}_t(t')$	Deformation history tensor, (1.32)
$\underset{\sim}{I}$	Identity tensor
MW	Molecular weight
$\underset{\sim}{0}$	Zero tensor
p	Pressure
R	Reynolds number = $\rho d^2\Omega_1/\eta$
R_m	Modified Reynolds number = R/β_o
$\underset{\sim}{R}_t(t')$	Rotation tensor, (1.33) and (1.34)

NOMENCLATURE (Continued)

Symbol	Definition
r	Radial coordinate
r_1 and r_2	Radii of inner and outer cylinders respectively
$\underset{\sim}{S}$	Stress tensor, (1.18)
T	Taylor number, (1.1) for large-gap case and (1.60) for small-gap case
T_c	Critical Taylor number
$T_{c,N}$	Critical Taylor number in Newtonian case
T_m	Modified Taylor number $= T/\beta_o^2$
t	Present time
t'	Time previous to present time $(t' < t)$
u	Perturbation on radial velocity component
u^*	$u^* = u/\Omega_1 d$
V	Eigenfunction, (4.40)
v	Perturbation on Θ-velocity component
v^*	$v^* = v/\Omega_1 d$
$\underset{\sim}{v}$	Velocity vector
w	Perturbation on axial velocity component
w^*	$w^* = w/\Omega_1 d$
X	$X = (r-r_1)/d$
Z	$Z = z/d$
z	Axial coordinate

Greek Symbol	Definition
α	$\alpha = (\Omega_2 - \Omega_1)/\Omega_1$
β_o	$\beta_o = \eta_o/\eta$
β_i	Rheological parameters, (4.47) - (4.55)
$\underset{\sim}{\Gamma}$	Transpose of velocity gradient tensor
γ	Shear rate
Δ	Rheogoniometer tangential stress armature displacement
ϵ	dimensionless wave number = $\pi d/\lambda$
ϵ_c	Critical wave number
η	Viscosity
η_o	Zero-shear viscosity
η_1, η_2, η_3, and η_4	New material functions defined in (4.30)
Θ	Angular coordinate
Θ_c	Rheogoniometer cone angle
λ	Height of vortex cells
ξ	Perturbation variable
ρ	Density
σ_1 and σ_2	First and second normal stress differences respectively
σ_3 and σ_4	New material functions defined in (4.30)
τ	Characteristic time for fluid
ψ	Eigenfunction, (4.40)

NOMENCLATURE (Concluded)

Greek Symbol	Definition
Ω_1 and Ω_2	Angular velocities of inner and outer cylinders respectively
$\underset{\sim}{\Omega}$	Vorticity tensor
ω	Angular velocity of rheogoniometer platen

The superscript (o) refers always to quantities associated with the primary motion, while the superscript (1) denotes secondary flow parameters.

ABSTRACT

Theoretical and experimental consideration is given to the stability of viscoelastic flow between concentric rotating cylinders.

In the mathematical analysis, the general Coleman-and-Noll simple fluid with fading memory is employed as the rheological model. Since all previous treatments of this stability problem have dealt with special cases of the simple fluid model, the earlier works are shown to represent special cases of the current investigation. General techniques are developed for treating small perturbations on "viscometric flows" of simple fluids, and these are subsequently applied in analyzing the pertinent stability behavior, under the usual assumption of neutral, axisymmetric disturbances. It is found that there are exactly eight material functions of the rate of shear which are necessary to define this problem. Three of these functions can be determined from existing laboratory tests, but the other five are entirely new, and their form can be deduced theoretically only in the limit of vanishing shear rates.

Another important result of the theoretical investigation concerns the definition of one of the stability criteria, the Taylor number. It is shown that this quantity should be calculated using the apparent fluid viscosity, as evaluated at the average shearing conditions in the apparatus.

Stability tests and viscosity measurements were conducted on aqueous solutions of cellulose-derivative polymers whose three determinate material functions have been reported in the literature at specific concentrations. In all instances, it is found experimentally that the flow is less stable than in the Newtonian case and that the wavelength of the disturbance is greater. By comparing these findings with the results of numerical stability calculations, one concludes that certain well-known approximate models do not suffice to describe the observed stability behavior.

CHAPTER 1

INTRODUCTION

1.1 PRELIMINARY COMMENTS

The tangential flow of any fluid between two long coaxially rotating cylinders (Couette flow) represents a curved motion which is inherently unstable due to centrifugal effects. In 1923, G. I. Taylor[39] first considered this stability phenomenon for the case of incompressible Newtonian liquids, and showed both theoretically and experimentally, that at a certain critical speed of the inner cylinder, a steady, laminar disturbance appears in the flow. This disturbance takes the form of a set of cellular toroidal vortices spaced regularly along the cylindrical axis.

In recent years, two dimensionless parameters, the Taylor number T and the dimensionless wave number ϵ, have come to be associated with the above effect. The Taylor number, which is a measure of the ratio of centrifugal forces to viscous forces, is given by

$$T = -4\left(\frac{\alpha\, a}{2 + \frac{1}{a}} + (\alpha+1)\right)R^2$$

where

$$\alpha = (\Omega_2 - \Omega_1)/\Omega_1$$

$$a = r_1/d \qquad (1.1)$$

$$d = r_2 - r_1$$

$$R \text{ (a Reynolds number)} = \rho d^2 \Omega_1/\eta$$

and where Ω_1 and Ω_2 are respectively the angular velocities of the inner and outer cylinders, r_1 and r_2 are the corresponding radii, ρ is the fluid density, and η is the viscosity. The dimensionless wave number is defined by

$$\epsilon = \pi d/\lambda \tag{1.2}$$

where λ is the height of the vortex cells. Couette flow remains stable as long as the Taylor number is maintained below a critical value T_c. At $T = T_c$ the characteristic vortex pattern develops, and the critical wave number ϵ_c then describes the vortex spacing.

In the case of viscoelastic fluids, non-Newtonian effects such as shear thinning, normal stresses, elasticity, and perhaps other phenomena which have not as yet been discovered may influence the values of T_c and ϵ_c determined for the Newtonian case. These possibilities are considered herein. The theoretical portion of the present study has been devoted to treating the Taylor-Couette stability of a Coleman-and-Noll[9] *simple fluid with fading memory*. This is the most general mechanical constitutive equation currently available and has had remarkable success in predicting all known aspects of viscoelastic behavior. Experimental consideration has been given here to the stability of aqueous solutions of three different polymers; for the systems investigated, the critical Taylor number and dimensionless wave number are presented as functions of polymer concentration.

The remainder of this first chapter will be devoted to familiarizing the reader with the properties of viscoelastic fluids, presenting the preliminary theory, and discussing previous work. In Chapter 2, the experimental work of the current investigation is presented. A theory of infinitesimal secondary flows of "simple fluids" is developed in Chapter 3. This is then applied in Chapter 4 to the Taylor-Couette stability problem; here the disturbance equations are derived and solved, and the theoretical and experimental results are compared. Finally, the conclusions of this study are given in Chapter 5.

1.2 SUMMARY OF VISCOELASTIC FLUID PROPERTIES

The following is a brief summary of some of the unusual effects exhibited by viscoelastic fluids. It is presented in order to acquaint the reader with the type of material being considered and also to establish some nomenclature.

Stress Relaxation and Recoil.—Stress relaxation and recoil are the principal phenomena distinguishing viscoelastic fluids from viscoinelastic fluids. Stress relaxation is characterized by the gradual, as opposed to the instantaneous, decay of stress in a fluid which is held at fixed strain after experiencing a rapid deformation. Recoil, on the other hand, is the partial recovery of strain occurring in a deforming viscoelastic fluid when the driving force causing the motion is suddenly removed.

Shear Thinning (Viscosity) and Normal Stress Effects.—Consider the motion of a viscoelastic fluid between two horizontal parallel plates which are everywhere separated by a distance d. Let the lower plate be held fixed while the upper one is moved at constant speed, $V = \gamma d$ (Fig. 1.1).

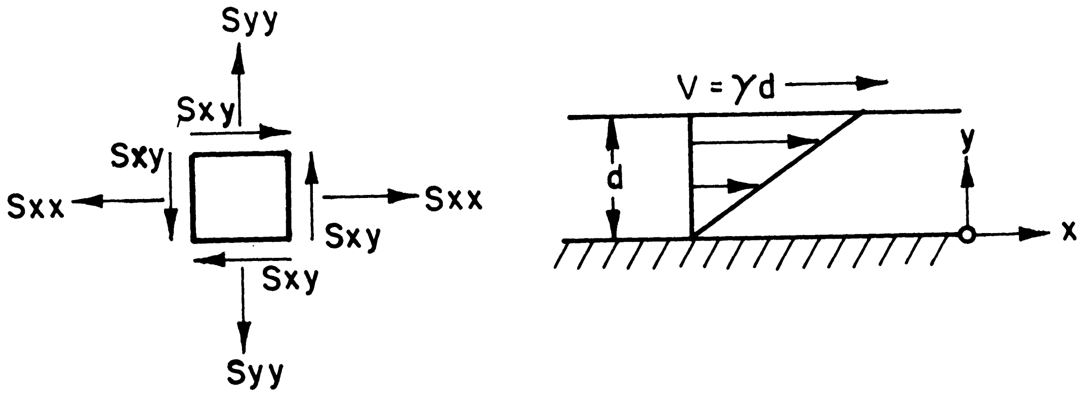

Fig. 1.1. Schematic diagram of simple shear flow.

At steady state, the velocity profile will be linear and given by

$$\begin{aligned} v_y &= 0 \\ v_x &= \gamma y \end{aligned} \tag{1.3}$$

where the quantity γ is referred to as the shear rate.

Viscoelastic fluids are found to yield the following stress pattern for simple shear between parallel plates[7]:

$$\begin{aligned} S_{xy} &= \gamma\eta(\gamma) \\ S_{xx} - S_{zz} &= \sigma_1(\gamma) \\ S_{yy} - S_{zz} &= \sigma_2(\gamma) \\ S_{yz} = S_{xz} &= 0 \end{aligned} \tag{1.4}$$

where η is the fluid viscosity, and σ_1 and σ_2 are the so-called first and second normal stress differences, respectively. In a Newtonian fluid, there are no normal stresses and, in addition, the viscosity is independent of the shear rate. (The reader should note that, with regard to the subscripts on the σ's, a notation opposite to that of Coleman, Markovitz, and Noll[7] has been employed here; it was felt that the quantity commonly defined as the second normal stress difference (S_{yy}-S_{zz}) should be named σ_2 rather than σ_1.)

A typical viscosity curve for a viscoelastic fluid is shown in Fig. 1.2 (which, as a rule, involves several decades of magnitude of η and γ).

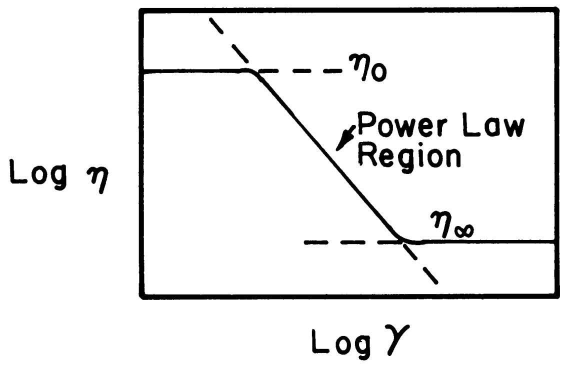

Fig. 1.2. Schematic of a viscosity curve for a viscoelastic fluid.

As can be seen from this figure, three distinct regions are present. At very low rates of shear the viscosity approaches η_o the "zero-shear viscosity." For extremely high shear rates, the "upper limiting viscosity," η_∞, is approached. The intermediate region is characterized by a viscosity which decreases with shear rate, a phenomenon referred to as shear thinning. The familiar "power law" viscosity model is frequently found to apply in this region.

The two normal stress differences σ_1 and σ_2 have been studied extensively for a great many molten polymers and polymer solutions. It has been found that σ_1 is always positive in sign and considerably larger in magnitude than σ_2, which usually assumes a value quite close to zero. Weissenberg,[45] in 1948, set forth the so-called "Weissenberg hypothesis" which claims that $\sigma_2 \equiv 0$ for polymer solutions in general. Today, this hypothesis is no longer accepted as fact by most rheologists, since recent experimental evidence points to the conclusion that $\sigma_2 > 0$. We therefore may write

$$\sigma_1 \gg \sigma_2 > 0 \quad . \tag{1.5}$$

Weissenberg Climbing Effect.—When a stirring rod, placed vertically in a beaker containing a viscoelastic fluid, is rotated about its axis, the fluid will begin to climb the rod. This is very different from what is observed for a Newtonian fluid.

Swelling of a Jet.—When a viscoelastic fluid emerges from a tube, the resulting jet of liquid increases in diameter indicating, to some extent, the action of the normal stresses present.

Phase Lag in Oscillatory Shear.—When a Newtonian fluid is placed between two infinite parallel plates and the upper plate is oscillated, it is found (in the case where inertial effects are negligible) that the shear stress is in phase with the rate of oscillation. However, for a viscoelastic fluid, the stress will lag behind the rate of oscillation.

1.3 KINEMATICS

Differences in the response of various materials under identically imposed boundary conditions are the result of differences in the stress-strain laws, or constitutive equations, for the materials. In general, these laws can involve all the kinematic quantities which define the states of the same particle of material during its previous history of motion through space. Therefore, in our discussion of the preliminary theory, we shall first consider, briefly, the kinematics of fluid motion.

The Gibbs dyadic notation is employed throughout the present work to represent vectors and second-order tensors. (For a good summary of this notation, the reader is referred to the appendices of either Transport Phenomena by Bird, Stewart, and Lightfoot[2] or Principles and Applications of Rheology by Fredrickson.[17] A more comprehensive treatment may be found in Vector Analysis with an Introduction to Tensor Analysis by Wills.[47])

The following conventions will hold here:

(1) Vectors and second-order tensors will be denoted by lower and upper case letters, respectively, with a tilde beneath each.

(2) The transpose or conjugate of a second-order tensor $\underset{\sim}{A}$ will be given by $\underset{\sim}{A}^{\dagger}$ while its inverse, if it exists, will be $\underset{\sim}{A}^{-1}$.

(3) $\underset{\sim}{I}$ will denote the unit tensor or idemfactor (identity tensor):

$$\underset{\sim}{A} \cdot \underset{\sim}{I} = \underset{\sim}{A} \tag{1.6}$$

(4) $\underset{\sim}{0}$ will denote the zero tensor:

$$\underset{\sim}{A} \cdot \underset{\sim}{0} = \underset{\sim}{0} \tag{1.7}$$

(5) The matrix of the physical components of a tensor will be indicated by employing the symbol $\Longleftrightarrow$. Thus in cartesian coordinates, for

example,

$$\underset{\sim}{A} \Longleftrightarrow \begin{bmatrix} A_{xx} & A_{xy} & A_{xz} \\ A_{yx} & A_{yy} & A_{yz} \\ A_{zx} & A_{zy} & A_{zz} \end{bmatrix} \tag{1.8}$$

while, in cylindrical coordinates,

$$\underset{\sim}{A} \Longleftrightarrow \begin{bmatrix} A_{rr} & A_{r\theta} & A_{rz} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta z} \\ A_{zr} & A_{z\theta} & A_{zz} \end{bmatrix} \tag{1.9}$$

Further,

(6) $\underset{\sim}{A} = \underset{\sim}{A}^{\dagger}$ implies that $\underset{\sim}{A}$ is a symmetric tensor.

(7) $\underset{\sim}{A} = -\underset{\sim}{A}^{\dagger}$ implies that $\underset{\sim}{A}$ is an antisymmetric tensor.

(8) $\underset{\sim}{A}^{-1} = \underset{\sim}{A}^{\dagger}$ implies that $\underset{\sim}{A}$ is orthogonal.

(9) $\underset{\sim}{\nabla}$ denotes the gradient (vector) operator.

(10) tr $\underset{\sim}{A}$ shall denote the trace of $\underset{\sim}{A}$, a scalar: tr $\underset{\sim}{A} = (\underset{\sim}{A} : \underset{\sim}{I})$.

(11) $\underset{\sim}{A} \cdot \underset{\sim}{A}$ is given by $\underset{\sim}{A}^2$.

Now that we have established this notation, let us proceed with the discussion of kinematics.

The kinematic state of a fluid at some time t (taken to be the present time) is determined if we know the velocity, acceleration, etc., of every particle of the fluid at that time. We shall assume that the fluid is a continuum and that the kinematic variables associated with a point fixed in space may be regarded as continuous functions of the spatial coordinates.

Of primary concern here is the motion which occurs relative to a particular fluid particle since the rheological behavior of a material is presumably independent of the motion as a whole (the "principle of local action"[42]).

Let P and P' be two material points of a fluid, which at time t are located at points $\underset{\sim}{x}$ and $\underset{\sim}{x}+d\underset{\sim}{x}$ with respect to the same frame of reference. The velocity at point P relative to point P' may be written as

$$d\underset{\sim}{v} = \underset{\sim}{\Gamma} \cdot d\underset{\sim}{x} \tag{1.10}$$

where $\underset{\sim}{\Gamma}$ is the transpose of the velocity gradient tensor:

$$\underset{\sim}{\Gamma} = (\underset{\sim}{\nabla}\underset{\sim}{v})^{\dagger} \quad . \tag{1.11}$$

In cartesian coordinates,

$$\underset{\sim}{\Gamma} \Longleftrightarrow \begin{bmatrix} \frac{\partial v_x}{\partial x} & \frac{\partial v_x}{\partial y} & \frac{\partial v_x}{\partial z} \\ \frac{\partial v_y}{\partial x} & \frac{\partial v_y}{\partial y} & \frac{\partial v_y}{\partial z} \\ \frac{\partial v_z}{\partial x} & \frac{\partial v_z}{\partial y} & \frac{\partial v_z}{\partial z} \end{bmatrix} \quad . \tag{1.12}$$

Let us now express $\underset{\sim}{\Gamma}$ as the sum of a symmetric and an antisymmetric part:

$$\underset{\sim}{\Gamma} = \frac{1}{2}(\underset{\sim}{\Gamma}+\underset{\sim}{\Gamma}^{\dagger}) + \frac{1}{2}(\underset{\sim}{\Gamma}-\underset{\sim}{\Gamma}^{\dagger}) \quad . \tag{1.13}$$

For the symmetric part, we write

$$\underset{\sim}{E} = \frac{1}{2}(\underset{\sim}{\Gamma}+\underset{\sim}{\Gamma}^{\dagger}) \tag{1.14}$$

and for the antisymmetric part,

$$\underset{\sim}{\Omega} = \frac{1}{2} (\underset{\sim}{\Gamma} - \underset{\sim}{\Gamma}^{\dagger}) . \qquad (1.15)$$

The tensor $\underset{\sim}{E}$ is a measure of the rate at which strain occurs; it is called, indeed, the rate of strain tensor. $\underset{\sim}{\Omega}$ represents the average rate of material rotation near a fluid particle and for this reason, it is referred to as the <u>vorticity tensor</u>.

There are two time derivatives which are also frequently employed in considering the rheological behavior of a fluid. These are defined as follows:

(1) The <u>material derivative</u>, $D\underset{\sim}{A}/Dt$ of a tensor $\underset{\sim}{A}$ is the derivative of $\underset{\sim}{A}$ as seen by an observer who is "translating with a given fluid particle." From the laws of partial differentiation we have

$$\frac{D\underset{\sim}{A}}{Dt} = \frac{\partial \underset{\sim}{A}}{\partial t} + \underset{\sim}{v} \cdot \underset{\sim}{\nabla} \underset{\sim}{A} . \qquad (1.16)$$

(2) The <u>Jaumann derivative</u>, $\mathcal{D}\underset{\sim}{A}/\mathcal{D}t$ of a tensor $\underset{\sim}{A}$ is the derivative of $\underset{\sim}{A}$ as seen by an observer who is not only translating with the fluid, but also rotating with its angular velocity. This derivative is given by

$$\frac{\mathcal{D}\underset{\sim}{A}}{\mathcal{D}t} = \frac{D\underset{\sim}{A}}{Dt} - \underset{\sim}{\Omega} \cdot \underset{\sim}{A} + \underset{\sim}{A} \cdot \underset{\sim}{\Omega} . \qquad (1.17)$$

This completes our discussion of some of the basic aspects of the kinematics of fluid flow.

1.4 CONSTITUTIVE EQUATIONS

The mechanical constitutive equation or rheological equation of state for a material is an expression of the relation between the state of stress present within the material and the kinematics of the motion causing this stress.

We shall begin our discussion of constitutive equations by first defining the stress tensor $\underset{\sim}{S}$, which enters into the determination of the dynamic state of a body. Next, the various types of "fluid models" currently being considered as constitutive equations for viscoelastic fluids will be presented. A definition of the general simple fluid of Noll[30] will then be given. This definition will be augmented by a discussion of the principle of fading memory. Finally, the special class of fluid motions called "viscometric flows" will be considered.

1.4.1 Definition of the Stress Tensor

Let a differential element of surface dA be located in the interior of a deforming body of material and let the orientation of dA be specified by giving the unit normal vector $\underset{\sim}{n}$ (Fig. 1.3). Furthermore, let the resultant contact force, acting on the side of dA towards which $\underset{\sim}{n}$ is directed, be denoted by $\underset{\sim}{s}$dA where $\underset{\sim}{s}$ is the stress vector.

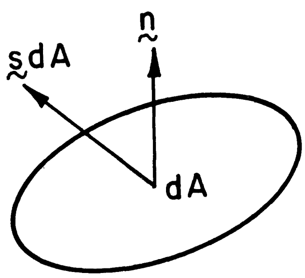

Fig. 1.3. Geometrical specification of contact forces.

In general, $\underset{\sim}{s}$ will depend on $\underset{\sim}{n}$. One can show by a force balance that this dependence may be expressed in the form

$$\underset{\sim}{s} = \underset{\sim}{S} \cdot \underset{\sim}{n} \tag{1.18}$$

where $\underset{\sim}{S}$ is defined as the stress tensor. Equation (1.18) expresses the fact that the stress tensor $\underset{\sim}{S}$ transforms any unit vector into the stress vector on a surface normal to that unit vector.

By a balance of moments, it is furthermore possible to show that the stress tensor $\underset{\sim}{S}$ is symmetric. (More precisely, this condition, which is often attributed to Cauchy, defines a "non-polar" material.[42])

1.4.2 Fluid Models

As understood here, "fluid models" are particular, as opposed to general mathematical statements of the constitutive equations for certain idealized materials. Some models, such as that of the Newtonian fluid, are of special importance in that they can describe the rheological behavior of real materials over a wide range of flow conditions. However, most models, due to their limited nature and sometimes arbitrary method of formulation, represent nothing more than mathematical curiosities.

All previous Taylor-Couette stability analyses on viscoelastic fluids have involved consideration of one type or another of special fluid model. Hence, we discuss fluid models here in order to put these analyses into the proper perspective with regard to the present more general treatment.

The rheological equation of state for an incompressible Newtonian fluid is given by

$$\underset{\sim}{S} + p\underset{\sim}{I} = 2\eta_0\underset{\sim}{E} \qquad (1.19)$$

where p is a pressure and η_0 is the viscosity, a material constant.[7] It is a well-established fact that this equation gives an excellent description of the mechanics of a great many fluids; unfortunately there are also numerous examples of real fluids for which it is quite inadequate.

In 1945, Reiner[35] attempted to account for more complicated material effects in fluids by assuming that the stress tensor $\underset{\sim}{S}$ could be expressed as a polynomial in the rate of strain tensor $\underset{\sim}{E}$. By making use of the characteristic equation for $\underset{\sim}{E}$, he was able to show that the most general equation obtainable for an incompressible fluid would then be

$$\underset{\sim}{S} + p\underset{\sim}{I} = 2\alpha_1\underset{\sim}{E} + 4\alpha_2\underset{\sim}{E}^2 \qquad (1.20)$$

where α_1 and α_2 are scalar functions of the invariants of $\underset{\sim}{E}$. Because of the later work of Rivlin,[36] Eq. (1.20) has come to be known as the Reiner-Rivlin model.

The Reiner-Rivlin fluid gives a definite improvement in some respects over the Newtonian model. A fluid obeying (1.20) would, for example, not only exhibit a shear-dependent viscosity but also normal-stress effects. However, the two normal-stress functions σ_1 and σ_2 resulting from the Reiner-Rivlin equation are found to be equal to one another, a situation contrary to what is observed experimentally on most polymer solutions and melts. Furthermore, Eq. (1.20) is incapable of representing fluids which are elastic in nature since such materials exhibit stress relaxation; in contrast, this model predicts an instantaneous decay of stress (to a hydrostatic pressure) when motion ceases ($\underset{\sim}{E} = \underset{\sim}{0}$). As Fredrickson[17] points out, there are no known substances, other than Newtonian fluids, which are described by the Reiner-Rivlin theory.

During the past twenty years, considerable progress has been made in the development of fluid models which presumably can better represent viscoelastic fluid behavior. There currently appear to be four types which are enjoying prominence: differential models, integral models, Rivlin-Ericksen models, and "anisotropic fluid" models.

The differential models are often founded on the assumption that there is some analogy between the stress-strain behavior of dilute solutions of randomly coiling macromolecules (polymer solutions) and the behavior of dilute suspensions of elastic spheres in Newtonian fluids (which have been studied theoretically). Characteristically, the differential constitutive laws are written in the form of ordinary differential equations in time which relate the stress tensor and its time derivatives to the rate of strain tensor and its time derivatives. Unfortunately, terms are all too often added in a rather arbitrary manner to these equations, in order to include certain higher order viscoelastic effects; for this reason, they cannot be expected to give more than a qualitative description of any existing fluids. One example of a differential type model is the equation

$$\underset{\sim}{S} + \tau_1 \left[\frac{\mathcal{D}\underset{\sim}{S}}{\mathcal{D}t} - \mu_1(\underset{\sim}{S}\cdot\underset{\sim}{E}+\underset{\sim}{E}\cdot\underset{\sim}{S}) + \nu_1(\underset{\sim}{E}:\underset{\sim}{S})\underset{\sim}{I} + \mu_0(\underset{\sim}{S}:\underset{\sim}{I})\underset{\sim}{I}\right]$$

$$= 2\eta_0\left[\underset{\sim}{E} + \tau_2\left[\frac{\mathcal{D}\underset{\sim}{E}}{\mathcal{D}t} - 2\mu_2\underset{\sim}{E}^2 + \nu_2(\underset{\sim}{E}:\underset{\sim}{E})\underset{\sim}{I}\right]\right] \qquad (1.21)$$

where η_0, μ_0, μ_1, μ_2, ν_1, ν_2, τ_1, and τ_2 are assumed to be constants. This is the "Oldroyd[31] 8-constant model."

The so-called Boltzmann principle of superposition (see e.g., Fredrickson[17]) provides the basis for a certain class of integral models of viscoelastic fluids. In deriving these models, it is assumed that, for a particular fluid particle, a small strain at one instant of time t' results in a small stress at some later time t and that the two are related through an influence function $\phi(t-t')$. The relation between stress and strain is then generalized to a "convolution" integral of the form

$$\text{stress } (t) = \int_{t'=-\infty}^{t} \phi(t-t')d[\text{strain } (t')] \qquad (1.22)$$

by further assuming that stresses are linearly superposible.

When expressed in the properly invariant form (relating the stress tensor $\underset{\sim}{S}$ to integrals of kinematic quantities), the integral models can exhibit normal stresses, shear dependent viscosity, and stress relaxation. However, there is no guarantee that any existing fluids will conform to these models.

Fredrickson[17] cites two examples of integral models which, in the present dyadic notation, take the forms:

$$\underset{\sim}{S} + p\underset{\sim}{I} = 2\int_{-\infty}^{t} \phi(t-t')\underset{\sim}{F}_t^{\dagger}(t') \cdot \underset{\sim}{E}(t') \cdot \underset{\sim}{F}_t(t')dt' \qquad (1.23)$$

and

$$\underset{\sim}{S} + p\underset{\sim}{I} = 2\int_{-\infty}^{t} \phi(t-t')\underset{\sim}{F}_t^{-1}(t') \cdot \underset{\sim}{E}(t') \cdot \underset{\sim}{F}_t^{-\dagger}(t')dt' \qquad (1.24)$$

In these equations, $\underset{\sim}{F}_t(t')$ represents the <u>relative deformation gradient</u> tensor,[7] a quantity which has been shown[20] to satisfy the differential equation

$$\frac{D\underset{\sim}{F}_t(t')}{Dt'} = \underset{\sim}{\Gamma}(t') \cdot \underset{\sim}{F}_t(t') \qquad (1.25)$$

subject to the condition

$$\underset{\sim}{F}_t(t) = \underset{\sim}{I} \qquad (1.26)$$

Equations (1.23) and (1.24) have been discussed by Walters,[44] who refers to them as liquids A' and B', respectively.

In 1955, Rivlin and Ericksen[37] developed a new theory of nonlinear viscoelasticity. By assuming that the stress within a fluid at a given time depends on the first-order spatial gradients of the displacement, velocity, acceleration, etc., at that time and by making use of certain invariance conditions, these authors were able to derive a constitutive equation which, for incompressible fluids, is expressible in the form

$$\underset{\sim}{S} + p\underset{\sim}{I} = \underset{\sim}{f}[\underset{\sim}{A}_1,\underset{\sim}{A}_2,\ldots,\underset{\sim}{A}_n] \qquad (1.27)$$

where

$$\underset{\sim}{A}_1 = 2\underset{\sim}{E} \qquad (1.28)$$

$$A_{n+1} = \frac{\mathcal{D}A_n}{\mathcal{D}t} + E \cdot A_n + A_n \cdot E \qquad (1.29)$$

and where the tensors A_n are commonly referred to as the Rivlin-Ericksen tensors. As with the Reiner-Rivlin theory, the Rivlin-Ericksen approach is insufficient to account for complex "memory" effects, such as stress relaxation, in viscoelastic fluids.

The Ericksen[14] anisotropic fluids represent the last type of fluid models to be mentioned here. Briefly, these have been founded on the following three assumptions:

(1) There is a preferred direction, defined by a vector n, associated with each particle of a given fluid.

(2) Variations in the magnitude and direction of n are governed by the fluid motion.

(3) The stress at any point in the fluid is a function of the rate of strain tensor E and the preferred direction n.

A typical constitutive equation for an Ericksen anisotropic fluid is given by

$$S + pI = 2\mu E + [\mu_1 + \mu_2(E:nn)]nn + 2\mu_3(E \cdot nn + nn \cdot E)$$

where (1.30)

$$\frac{Dn}{Dt} = \Omega \cdot n + \mu_o[E - (E:nn)I] \cdot n$$

and where μ, μ_o, μ_1, μ_2, and μ_3 are material constants. The anisotropic fluid models might be expected to give good qualitative descriptions of certain substances (such as suspensions of rod-like or ellipsoidal particles in a Newtonian fluid) which retain a flow-induced orientation long after motion has ceased. However, for polymer solutions and melts, they show little advantage over any of the other types of models discussed.

1.4.3 Definition of a Noll Simple Fluid

Noll[30] has recently developed a very general theory of fluid behavior which appears capable of describing the stress patterns in viscoelastic fluids for all motions that have been studied experimentally. The essence of this theory is embodied in the definition of an <u>incompressible simple fluid</u>,* a substance which satisfies the following two postulates:

(1) The present stress at a material point in a <u>simple fluid</u> is determined, to within a hydrostatic pressure, by the history of the motion in the immediate neighborhood of that point.

(2) No preferred configurations exist in a <u>simple fluid</u>.

All the types of fluid models (except the Ericksen[14] anisotropic fluids**) discussed in the previous section are seen to satisfy these two postulates, and thus represent special cases of simple fluids.

There are numerous ways of expressing statements (1) and (2) in symbolic form; for the present purposes we choose the constitutive relation which has been suggested by Goddard[19]:

$$\underset{\sim}{S} + p\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\underset{\sim}{\mathcal{F}}}} [\underset{\sim}{H}_t(t')] \qquad (1.31)$$

*Henceforth, the term "simple fluid" shall imply "incompressible simple fluid."

**One can show that, for the present analysis, anisotropic fluids may also be regarded as simple fluids.

where $\underset{\sim}{H}_t(t')$ gives a measure of the history of the rate of strain. It is defined, for $-\infty < t' < t$, by

$$\underset{\sim}{H}_t(t') \equiv \underset{\sim}{R}_t^{\dagger}(t') \cdot \underset{\sim}{E}(t') \cdot \underset{\sim}{R}_t(t') \tag{1.32}$$

where $\underset{\sim}{R}_t(t')$, an orthogonal tensor, satisfies the differential equation

$$\frac{D\underset{\sim}{R}_t(t')}{Dt'} = \underset{\sim}{\Omega}(t') \cdot \underset{\sim}{R}_t(t') \tag{1.33}$$

subject to

$$\underset{\sim}{R}_t(t) = \underset{\sim}{I} \tag{1.34}$$

Physical interpretation may be given to both $\underset{\sim}{R}_t(t')$ and $\underset{\sim}{H}_t(t')$. $\underset{\sim}{R}_t(t')$ represents the average rotation suffered by material in the neighborhood of a given fluid particle during the time interval (t',t); $\underset{\sim}{H}_t(t')$ is the rate of strain tensor, at time t', as seen by an observer in a coordinate frame that is both translating and rotating with a particular fluid particle.

The symbol $\underset{\sim}{\mathfrak{F}}$ in Eq. (1.31) stands for a functional, that is, an operator which maps a tensor-valued function into a tensor. It merely relates the present stress $\underset{\sim}{S}$ to all values assumed by the deformation history $\underset{\sim}{H}_t(t')$ as t' varies between $-\infty$ and t (the present time). The form of the functional $\mathfrak{F}$ determines the rheological characteristics of each particular simple fluid; thus, $\mathfrak{F}$ will vary from one fluid to another. Consider, as one example, the case of a Newtonian fluid for which

$$\underset{\sim}{S} + p\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\mathcal{Z}}} [\underset{\sim}{H}_t(t')] = 2\eta_o \underset{\sim}{H}_t(t')]_{t'=t} = 2\eta_o \underset{\sim}{E}(t) \quad (1.35)$$

There is an important restriction which must be placed on Eq. (1.31) due to the so-called "principle of material objectivity."[7] Briefly, this principle requires that the state of stress within a deforming material be independent of the motion of the observer. For the functional of (1.31), Goddard[19] gives as the required condition,

$$\underset{t'=-\infty}{\overset{t}{\mathcal{Z}}} [\underset{\sim}{Q} \cdot \underset{\sim}{H}_t(t') \cdot \underset{\sim}{Q}^{\dagger}] = \underset{\sim}{Q} \cdot \underset{t'=-\infty}{\overset{t}{\mathcal{Z}}} [\underset{\sim}{H}_t(t')] \cdot \underset{\sim}{Q}^{\dagger} \quad (1.36)$$

for all constant orthogonal tensors $\underset{\sim}{Q}$ and for each function $\underset{\sim}{H}_t(t')$.

Before concluding the present section, it is perhaps appropriate to introduce a special property of the deformation history $\underset{\sim}{H}_t(t')$ which we shall have occasion to use in the forthcoming analysis. Let us begin by taking the material derivative of $\underset{\sim}{H}_t(t')$:

$$\begin{aligned}\frac{D\underset{\sim}{H}_t(t')}{Dt'} &= \frac{D}{Dt'} [\underset{\sim}{R}_t^{\dagger}(t') \cdot \underset{\sim}{E}(t') \cdot \underset{\sim}{R}_t(t')] \\ &= \left[\frac{D\underset{\sim}{R}_t^{\dagger}(t')}{Dt'}\right] \cdot \underset{\sim}{E}(t') \cdot \underset{\sim}{R}_t(t') + \underset{\sim}{R}_t^{\dagger}(t') \cdot \left[\frac{D\underset{\sim}{E}(t')}{Dt'}\right] \cdot \underset{\sim}{R}_t(t') \\ &\quad + \underset{\sim}{R}_t^{\dagger}(t') \cdot \underset{\sim}{E}(t') \cdot \left[\frac{D\underset{\sim}{R}_t(t')}{Dt'}\right] \quad . \end{aligned} \quad (1.37)$$

Substituting (1.33) into this equation and combining terms, we then have

$$\frac{D\underset{\sim}{H}_t(t')}{Dt'} = \underset{\sim}{R}_t^{\dagger}(t') \cdot \left[\frac{D\underset{\sim}{E}(t')}{Dt'} - \underset{\sim}{\Omega}(t') \cdot \underset{\sim}{E}(t') + \underset{\sim}{E}(t') \cdot \underset{\sim}{\Omega}(t')\right] \cdot \underset{\sim}{R}_t(t') \ . \quad (1.38)$$

But from the definition of the Jaumann derivative (Eq. (1.17)), this expression becomes

$$\frac{D\underset{\sim}{H}_t(t')}{Dt'} = \underset{\sim}{R}_t^{\dagger}(t') \cdot \frac{\mathcal{D}\underset{\sim}{E}(t')}{\mathcal{D}t'} \cdot \underset{\sim}{R}_t(t') . \tag{1.39}$$

Thus, the material derivative of $\underset{\sim}{H}_t(t')$ is related to the Jaumann derivative of $\underset{\sim}{E}$. By taking successively higher order material derivatives of $\underset{\sim}{H}_t(t')$, it is furthermore possible to show that, in general,

$$\frac{D^n\underset{\sim}{H}_t(t')}{Dt'^n} = \underset{\sim}{R}_t^{\dagger}(t') \cdot \left[\frac{\mathcal{D}^n\underset{\sim}{E}(t')}{\mathcal{D}t'^n}\right] \cdot \underset{\sim}{R}_t(t') . \tag{1.40}$$

1.4.4 Principle of Fading Memory

The principle of fading memory has been developed by Coleman and Noll[9] in order to allow for stress relaxation and other memory effects in simple fluids. Essentially, this principle implies that, for a simple fluid with fading memory, deformations which occurred in the distant past have less effect on the present stress than deformations which occurred in the recent past.

In order to put these intuitive notions of fluid behavior on firmer mathematical grounds, Coleman and Noll have made use of the "norm" (or magnitude) of a deformation history which may be defined as follows (for the present analysis):

$$\text{norm}[\underset{\sim}{H}_t(t')] = \|\underset{\sim}{H}_t(t')\| = \left[\int_{-\infty}^{t} h^2(t-t')\,\text{tr}\underset{\sim}{H}_t^2(t')dt'\right]^{1/2} \tag{1.41}$$

Here, $h(t-t')$ is a "weighting" or "influence function" which varies with the fluid under consideration; it is positive, continuous, and real-valued and goes to zero rapidly as $t-t'$ becomes large.

The norm is interpreted as the "distance" of a given deformation history $\underset{\sim}{H}_t(t')$ from the "zero-history," $\underset{\sim}{H}_t(t') \equiv \underset{\sim}{0}$. It possesses the important property that the weighting function $h(t-t')$ places greater emphasis on values of $\underset{\sim}{H}_t(t')$ for small $t-t'$ (recent past) than for large $t-t'$ (distant past). Thus, a deformation history with a "small norm" is expected to give rise to only a small contribution to the present stress.

Coleman and Noll[9] have made use of the idea of a norm of a history in conjunction with the operation of <u>Fréchet differentiation</u> of constitutive functionals: If $\underset{\sim}{\mathcal{F}}$ represents the constitutive *functional* for a simple fluid, it is Frèchet differentiable at the function $H_t(t')$,

if there exists a functional $\delta\underset{\sim}{\mathcal{F}}$ (which depends on $\underset{\sim}{\mathcal{F}}$) such that the following equation holds for all $\Delta\underset{\sim}{H}_t(t')$ with finite norm:

$$\underset{t'=-\infty}{\overset{t}{\underset{\sim}{\mathcal{F}}}}[\underset{\sim}{H}_t(t')+\Delta\underset{\sim}{H}_t(t')] = \underset{t'=-\infty}{\overset{t}{\underset{\sim}{\mathcal{F}}}}[\underset{\sim}{H}_t(t')] + \underset{t'=-\infty}{\overset{t}{\delta\underset{\sim}{\mathcal{F}}}}[\underset{\sim}{H}_t(t'),\Delta\underset{\sim}{H}_t(t')]$$

$$+ \mathcal{O}[\|\Delta\underset{\sim}{H}_t(t')\|] \qquad (1.42)$$

where $\delta\underset{\sim}{\mathcal{F}}$ is both <u>linear</u> and <u>continuous</u> in $\Delta\underset{\sim}{H}_t(t')$ and where $\underset{\sim}{\mathcal{O}}[\|\Delta\underset{\sim}{H}_t(t')\|]$ satisfies the equation

$$\lim_{\|\Delta\underset{\sim}{H}_t(t')\|\to 0} \frac{\underset{\sim}{\mathcal{O}}[\|\Delta\underset{\sim}{H}_t(t')\|]}{\|\Delta\underset{\sim}{H}_t(t')\|} = \underset{\sim}{0} \ . \qquad (1.43)$$

That is, $\underset{\sim}{\mathcal{F}}$ has a Fréchet derivative at $\underset{\sim}{H}_t(t')$ if the difference between the values of $\underset{\sim}{\mathcal{F}}$ at $\underset{\sim}{H}_t(t')$ and $\underset{\sim}{H}_t(t')+\Delta\underset{\sim}{H}_t(t')$ is given to a good approximation by a continuous, linear functional *of* $\Delta\underset{\sim}{H}_t(t')$ whenever $\Delta\underset{\sim}{H}_t(t')$ is small in norm.

Let us now record, for later use, one additional condition which must be met by $\delta\underset{\sim}{\mathcal{F}}$ as a consequence of Eq. (1.36). This is

$$\underset{t'=-\infty}{\overset{t}{\delta \mathcal{F}}} [\underset{\sim}{Q}\cdot\underset{\sim}{H}_t(t')\cdot\underset{\sim}{Q}^{\dagger}, \underset{\sim}{Q}\cdot\Delta\underset{\sim}{H}_t(t') \cdot \underset{\sim}{Q}^{\dagger}] = \underset{\sim}{Q} \cdot \underset{t'=-\infty}{\overset{t}{\delta \mathcal{F}}} [\underset{\sim}{H}_t(t'),\Delta\underset{\sim}{H}_t(t')] \cdot \underset{\sim}{Q}^{\dagger} \quad (1.44)$$

for all orthogonal tensors $\underset{\sim}{Q}$ and for all functions $\underset{\sim}{H}_t(t')$ and $\Delta\underset{\sim}{H}_t(t')$ which satisfy (1.42).

1.4.5 Viscometric Flows

There exists a special class of fluid motions, called viscometric flows, which possess extremely simple deformation histories. Their common characteristic is that they are all kinematically equivalent to simple shear between parallel plates, except for a time dependent rigid rotation of each material particle. Viscometric flows are of interest in the present study mainly because the primary motion in the Taylor-Couette stability analysis is viscometric.

Fredrickson,[17] for one, has cited some well-known examples of viscometric flows:

(1) Simple shear between parallel plates.

(2) Flow in a pipe of circular cross section.

(3) Tangential flow between concentric rotating cylinders (Couette flow).

(4) Flow between a plate and cone.

(5) Combined tangential and axial flow (helical flow) in an annulus.

It has been shown by Goddard and Miller[20] that the kinematics of all viscometric flows may be described by the following set of equations: The characteristic equation for $\underset{\sim}{E}$ is

$$\underset{\sim}{E}^3 = \frac{\gamma^2}{4} \underset{\sim}{E} \tag{1.45}$$

where

$$\gamma = \sqrt{2(\underset{\sim}{E}:\underset{\sim}{E})} \tag{1.46}$$

is the shear rate. Furthermore,

$$\frac{\mathcal{D}^2 \underset{\sim}{E}}{\mathcal{D}t^2} + \gamma^2 \underset{\sim}{E} = \underset{\sim}{0} \tag{1.47}$$

$$\frac{\mathcal{D}\underset{\sim}{E}^2}{\mathcal{D}t} = \underset{\sim}{0} \tag{1.48}$$

and

$$\frac{D\gamma}{Dt} = 0 \quad . \tag{1.49}$$

These relations can be put in a more convenient form (involving the deformation history $\underset{\sim}{H}_t(t')$) by employing Eq. (1.40) and recalling that the tensor $\underset{\sim}{R}_t(t')$ is orthogonal. The final result is

$$\underset{\sim}{H}_t^3(t') = \frac{\gamma^2}{4} \underset{\sim}{H}_t(t') \tag{1.50}$$

$$\gamma = \sqrt{2(\underset{\sim}{H}_t(t'):\underset{\sim}{H}_t(t'))} \tag{1.51}$$

$$\frac{D^2 \underset{\sim}{H}_t(t')}{Dt'^2} + \gamma^2 \underset{\sim}{H}_t(t') = \underset{\sim}{0} \tag{1.52}$$

$$\frac{D[\underset{\sim}{H}_t^2(t')]}{Dt'} = \underset{\sim}{0} \quad . \tag{1.53}$$

Let us now solve Eq. (1.52), an ordinary, linear, second-order differential equation, for $\underset{\sim}{H}_t(t')$. We obtain

$$\underset{\sim}{H}_t(t') = \underset{\sim}{E}(t)\cos \gamma(t-t') - \frac{1}{\gamma} \frac{\mathcal{D}\underset{\sim}{E}}{\mathcal{D}t} \sin \gamma(t-t') \tag{1.54}$$

which holds for general viscometric flows.

Coleman, Markovitz, and Noll[7] have considered theoretically the mechanical behavior of <u>simple fluids</u> undergoing viscometric flows and have shown that, for all such motions, the state of stress at a material particle is determined by three material functions of the shear rate γ. These "viscometric functions" are the viscosity $\eta(\gamma)$ and two normal stresses $\sigma_1(\gamma)$ and $\sigma_2(\gamma)$, all of which are even functions of γ. With particular regard to steady laminar shear between parallel plates, this means that a simple fluid will give a stress pattern identical with that represented by Eq. (1.4) for real viscoelastic fluids.

Now, for flows obeying (1.50)-(1.53), it is also possible to show that the constitutive functional $\mathcal{F}$ for a simple fluid may be reduced to any one of the following equivalent forms:

$$\underset{\sim}{S} + p\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\mathcal{F}}} [\underset{\sim}{H}_t(t')] = 2\eta(\gamma)\underset{\sim}{H}_t(t')\Big]_{t'=t} + \frac{2(\sigma_1+\sigma_2)}{\gamma^2}\underset{\sim}{H}_t^2(t')\Big]_{t'=t} + \frac{(\sigma_2-\sigma_1)}{\gamma^2}\frac{D\underset{\sim}{H}_t(t')}{Dt'}\Big]_{t'=t} \tag{1.55}$$

$$= 2\eta(\gamma)\underset{\sim}{E} + \frac{2(\sigma_1+\sigma_2)}{\gamma^2}\underset{\sim}{E}^2 + \frac{(\sigma_2-\sigma_1)}{\gamma^2}\frac{\mathcal{D}\underset{\sim}{E}}{\mathcal{D}t} \tag{1.56}$$

$$= \eta(\gamma)\underset{\sim}{A}_1 + \frac{\sigma_1}{\gamma^2}\underset{\sim}{A}_1^2 + \frac{(\sigma_2-\sigma_1)}{2\gamma^2}\underset{\sim}{A}_2 \tag{1.57}$$

where we recall that $\underset{\sim}{A}_n$ are the Rivlin-Ericksen tensors (see Eqs. (1.28) and (1.29)).

As was mentioned above, Coleman, Markovitz, and Noll[7] have determined the minimum number of material functions necessary for describing viscometric flows of simple fluids. One of the objectives of the present study is to do the same for the Taylor-Couette stability problem.

1.5 DISCUSSION OF PREVIOUS WORK

1.5.1 Theoretical

The stability of fluid flow between two long concentric rotating cylinders was first considered by Lord Rayleigh[34] in 1916 for the inviscid case ($\underset{\sim}{S}+p\underset{\sim}{I} = \underset{\sim}{0}$). He derived a simple condition for instability with respect to rotationally symmetric disturbances based on energy considerations. Rayleigh's criterion is as follows: if the magnitude of the circulation increases outwards, the flow is stable, whereas, if it decreases outwards, the flow is unstable.

Assuming that, in real fluids, viscosity serves to maintain the steady flow but does not affect the occurrence of instability, Rayleigh's criterion leads one to conclude that, for cylinders rotating in the same direction, the flow is stable if $\Omega_2 r_2^2 > \Omega_1 r_1^2$, where Ω_1 and Ω_2 are respectively the angular velocities of the inner and outer cylinders, and r_1 and r_2 are the corresponding radii. For cylinders rotating in opposite directions the circulation decreases outwards in at least part of the field of flow and the flow is always unstable.

In 1923, G. I. Taylor[39] published his now-famous work on the Couette stability of incompressible Newtonian fluids. This theoretical treatment has since served as a model for many later stability analyses. Taylor began by assuming that, superimposed upon the primary Couette motion

$\underset{\sim}{v}^{(o)}$, there is a small secondary velocity perturbation $\underset{\sim}{v}^{(1)}$ which is a function of the radial and axial coordinates but not of annular position (axisymmetric disturbance). He employed this assumption in the Navier-Stokes and continuity equations, dropped those terms involving products of secondary quantities, and obtained a set of "disturbance equations" which are linear and homogeneous in the components of $\underset{\sim}{v}^{(1)}$. When combined with the homogeneous boundary conditions on $\underset{\sim}{v}^{(1)}$, these disturbance equations defined a characteristic-value problem for Ω_1, the angular velocity of the inner cylinder. The minimum value of Ω_1, over all allowable eigenvectors $\underset{\sim}{v}^{(1)}$ then gives the critical conditions for the onset of instability. In solving the disturbance equations, Taylor confined attention to the neutral mode of stability where $\underset{\sim}{v}^{(1)}$ neither grows, decays, nor oscillates with time, and further assumed that $\underset{\sim}{v}^{(1)}$ is periodic in the axial direction (which amounts to resolving the disturbance into its normal spatial modes). From Taylor's calculations, it is found that the criterion for the onset of instability may be expressed in the following form:

$$T_c = \frac{2\pi^4}{\alpha[0.0571\ P+0.00056/P]} \qquad (1.58)$$

where

$$P = \left[\frac{2+\alpha}{\alpha} + \frac{0.652}{a}\right] \qquad (1.59)$$

and where α and a are defined in Eqs. (1.1). Equations (1.58) and (1.59) give an excellent approximation for T_c, to terms of order 1/a and for $\alpha \geq -1$. Taylor also determined that, for the case of cylinders rotating in the same direction ($\alpha \geq -1$), the critical wave number ϵ_c will have a value approximately equal to π. From Eq. (1.2), one therefore notes that the vortex cells will have roughly the same height as width, and will thus be nearly square in cross section.

Chandrasekhar[5] has recently (1954) reconsidered the Newtonian stability problem for the case where the radii of the cylinders are extremely large compared to the distance between them ($1/a \ll 1$). This represents the so-called small-gap approximation and gives rise to considerable simplification in the mathematical treatment of the problem. In particular, for a small gap, it is found that:

(1) The primary Couette motion approximates simple shear between parallel plates with

$$\gamma \doteq (\Omega_2-\Omega_1)r_1/d, \text{ a constant.}$$

(2) The definition of the Taylor number reduces in form to

$$T = -2\alpha a R^2 \quad . \tag{1.60}$$

(3) The form of the disturbance equations and their solution are greatly simplified.

Chandrasekhar's results agree quite well with those of Taylor for comparable situations. For the case of a stationary outer cylinder ($\alpha=-1$), he finds that

$$T_c = 3390$$

and

$$\epsilon_c = 3.12 \simeq \pi \; .$$

Interest in the Couette stability of non-Newtonian fluids apparently began in 1961 with the work of Graebel.[21] This author investigated the behavior of Reiner-Rivlin fluids under circumstances where the functions α_1 and α_2 in Eq. (1.20) are constants. Here, the three viscometric functions are given by

$$\begin{aligned} \eta(\gamma) &= \alpha_1 = \text{constant} \\ \sigma_1(\gamma) &= \alpha_2\gamma^2 \\ \sigma_2(\gamma) &= \alpha_2\gamma^2 \; . \end{aligned} \qquad (1.61)$$

By employing the small-gap approximation to simplify the analysis,* Graebel found that the coefficient of cross viscosity α_2 can strongly influence the results obtained for the critical Taylor number and critical wave number. He showed in fact that for $\alpha_2 > 0$, $T_c < T_{c,N}$ while $\epsilon_c > \epsilon_{c,N}$, where $T_{c,N}$ and $\epsilon_{c,N}$ represent the Newtonian values of T_c and ϵ_c. Although Graebel's analysis has been performed for the somewhat unrealistic Reiner-Rivlin model, it does give some general indication of what one might expect from a real non-Newtonian fluid, that is, that the critical parameters may differ from their Newtonian values.

Graebel[22] has also presented a treatment of the Taylor-Couette stability problem for Bingham plastics. According to the standard model,

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*Most non-Newtonian stability studies, including the present, have employed the small-gap approximation, and this fact will be tacitly understood throughout the further discussion.

such materials exhibit no normal stresses in simple shear between parallel plates but do have a viscosity which depends on shear rate:

$$\eta(\gamma) = \mu \left[1 + \frac{\vartheta}{\mu|\gamma|}\right] \qquad (1.62)$$

where μ is a constant, with units of viscosity, and ϑ is the yield stress. Graebel defined the Taylor number as

$$T = -2\alpha a \left[\frac{\rho d^2 \Omega_1}{\mu}\right]^2 \qquad (1.63)$$

and concluded from his solution to the disturbance equations that the flow is more stable than in the Newtonian case. One might suspect, however, that a Taylor number based on the apparent fluid viscosity $\eta(\gamma)$ evaluated at the gap shear rate γ might give a truer measure of the ratio of centrifugal to viscous forces. Had Graebel employed this definition, his above conclusion would have been reversed. This illustrates an important problem which we shall attempt to resolve in the present analysis, namely, that of determining which viscosity, in general, is most appropriate for defining the Taylor number.

Thomas and Walters[40,41] have considered the stability in flow between coaxial rotating cylinders of a Walters B' liquid (Eq. (1.24)), which has for viscometric functions

$$\begin{aligned} \eta(\gamma) &= \eta_o = \text{constant} \\ \sigma_1(\gamma) &= 2K_o\gamma^2 \\ \sigma_2(\gamma) &= 0 \end{aligned} \qquad (1.64)$$

where

$$\eta_o = \int_0^\infty \phi(t)dt$$

and

$$K_o = \int_0^\infty t\phi(t)dt \quad .$$

They found for the physically meaningful case of $K_o > 0$, that $T_c < T_{c,N}$ while $\epsilon_c > \epsilon_{c,N}$.

Chan Man Fong[6] has treated this same stability problem for the related Walters A' model, in order to compare results with those of Thomas and Walters. The A' liquid may be shown to yield

$$\begin{aligned} \eta = \eta_o &= \text{constant} \\ \sigma_1(\gamma) &= 0 \\ \sigma_2(\gamma) &= -2K_o\gamma^2 \end{aligned} \tag{1.65}$$

as viscometric functions. For values of $K_o > 0$, Chan Man Fong determined that $T_c > T_{c,N}$ and $\epsilon_c < \epsilon_{c,N}$, exactly the opposite of what Thomas and Walters obtained. Chan Man Fong summarized:

> "Comparing the conclusions of the present study with those of Thomas and Walters, it is seen that the stability criterion is very dependent on the particular elastico-viscous model considered."

It is not surprising that the results of these two studies should differ so greatly, expecially when one considers that the A' and B' models exhibit extremely different normal-stress patterns. Indeed, the question naturally arises as to whether such diversity in behavior might not almost entirely be attributed to differences in the viscometric functions η, σ_1, and σ_2. It shall become apparent from the results of the present study that this is in fact the case with regard to liquids A' and B'. However, for simple fluids in general, there will be additional material functions which can influence flow stability.

In 1964, Datta[11] published a theoretical analysis on the Taylor-Couette stability of the particular Rivlin-Ericksen fluid represented by

$$\underset{\sim}{S} + p\underset{\sim}{I} = \alpha_1 \underset{\sim}{A}_1 + \alpha_2 \underset{\sim}{A}_1^2 + \alpha_3 \underset{\sim}{A}_2 \tag{1.66}$$

where α_1, α_2, and α_3 are material constants. Coleman and Noll[8] have shown that this model, referred to by Truesdell[42] as a "fluid of second grade," describes the limiting behavior of simple fluids with fading memory at extremely low rates of deformation (or, equivalently, of fluids with very short memories). If we choose for the moment to consider the special case of extremely slow viscometric flows, we conclude from (1.66) and (1.57) that

$$\begin{aligned} \alpha_1 &= \lim_{\gamma \to 0} \eta(\gamma) = \eta_o \\ \alpha_2 &= \lim_{\gamma \to 0} \sigma_1(\gamma)/\gamma^2 \\ \alpha_3 &= \lim_{\gamma \to 0} (\sigma_2 - \sigma_1)/2\gamma^2 \ . \end{aligned} \tag{1.67}$$

Thus, the three viscometric functions for this model are given by

$$\begin{aligned} \eta(\gamma) &= \alpha_1 \\ \sigma_1(\gamma) &= \alpha_2 \gamma^2 \\ \sigma_2(\gamma) &= (\alpha_2 + 2\alpha_3)\gamma^2 \ . \end{aligned} \tag{1.68}$$

Several very important results have come from Datta's treatment. First of all, Datta noted that he could reproduce the disturbance equations derived by Graebel[21] for the Reiner-Rivlin fluid and by Thomas and Walters[40] for their "liquid B'" (for the case of short fluid memory) merely by suitable choice of η, σ_1, and σ_2 corresponding to the viscometric functions in these investigations; now, it is furthermore possible to show that the same could have been done for Chan Man Fong's[6] treatment of the liquid A' (again for the case of short fluid memory). In fact, one can demonstrate that, even for fluids with long memories, the disturbance equations for the liquid A' and liquid B' analyses could be reproduced <u>almost exactly</u> from Datta's equations. These findings are highly suggestive of the possibility that the three viscometric functions η, σ_1, and σ_2 may play a large role in determining the stability criteria.

Datta has also shown that, for a fluid obeying (1.66), the sign and magnitude of the second normal stress coefficient $(\alpha_2+2\alpha_3)$ can drastically affect the values obtained for the critical parameters T_c and ϵ_c. In particular, a small increase in $(\alpha_2+2\alpha_3)$ can result in a relatively large decrease in T_c along with a relatively large increase in ϵ_c.

In 1966, Giesekus[18] reconsidered the disturbance equations developed by Datta for the purpose of treating two special cases; these were, simple shear between parallel plates where $a\to\infty$, and the case of negligible inertia where $T\to 0$. He showed that, in both situations, the parameter $(\alpha_2+2\alpha_3)$ is again of considerable importance. Furthermore, as part of this same work, Giesekus put forth an interesting suggestion which is worthy of note here. He postulated that it might be possible to approximate the stability behavior of certain viscoelastic fluids outside the range of applicability of Eq. (1.66) (i.e., at finite rates of shear) by replacing the material constants α_1, α_2, and α_3 in Datta's disturbance equations by the functions which they approach at $\gamma = 0$. That is

$$\begin{aligned} \alpha_1 &= \eta(\gamma) \\ \alpha_2 &= \sigma_1(\gamma)/\gamma^2 \\ \alpha_3 &= [\sigma_2(\gamma)-\sigma_1(\gamma)]/2\gamma^2 \ . \end{aligned} \tag{1.69}$$

This same author stressed the fact that such a substitution would not yield a strictly valid stability criterion within the framework of the general simple-fluid theory but indicated that it was the best that one could do at that time. Based on the results of the present study, we now know that this was <u>not</u> the case. If Giesekus had instead substituted (1.69) into Datta's original fluid model and <u>then</u> rederived the disturbance equations, his results would have come considerably closer to what is actually obtained for simple fluids with fading memory.

Leslie[25] has considered the stability in Couette flow of the idealized Ericksen anisotropic fluid represented by Eqs. (1.30) for the case of $\mu_1 = 0$ and $|\mu_0| > 1$. The viscometric functions for such a material are given by Leslie as

$$\eta(\gamma) = \mu + \mu_3 + \mu_2 n_1^2 n_2^2$$

$$\sigma_1(\gamma) = n_1 n_2(\mu_2 n_2^2 + 2\mu_3)\gamma \qquad (1.70)$$

$$\sigma_2(\gamma) = n_1 n_2(\mu_2 n_1^2 + 2\mu_3)\gamma$$

where

$$2n_1^2 = (\mu_0 - 1)/\mu_0$$

and

$$2n_2^2 = (\mu_0 + 1)/\mu_0$$

and where n_1 and n_2 are respectively the x and y components of the preferred-direction vector $\underset{\sim}{n}$ in simple shear between parallel plates (Fig. 1.1). The author defined the Taylor number by

$$T = -2\alpha a \left[\frac{\rho d^2 \Omega_1}{\mu}\right]^2 (1+\alpha/2) \qquad (1.71)$$

and then solved the disturbance equations for the case of $\mu = \mu_2 = \mu_3$, $\mu_0 = 2$, $\alpha = -\frac{1}{2}$, and a = 20. His calculations reveal that when $n_1 > 0$ and $n_2 > 0$ one has $T_c \simeq 9000$ and $\epsilon_c \simeq 3.0$ while for $n_1 > 0$ and $n_2 < 0$ one has $T_c \simeq 5000$ and $\epsilon_c \simeq 5.0$. It is interesting to note that these results would have been quite different had Leslie defined the Taylor number as in (1.60) and, moreover, calculated its value using the apparent fluid viscosity $\eta(\gamma)$ rather than μ. Under such circumstances, he would have obtained $T_c \simeq 2500$ for $n_1 > 0$ and $n_2 > 0$, and $T_c \simeq 1400$ for $n_1 > 0$ and $n_2 < 0$. These findings may be compared with the "exact" value of 2275 given by Thomas and Walters[41] for the Newtonian case (for $\alpha = -\frac{1}{2}$).

Recently, Davies[12] has investigated the Couette stability of the idealized viscoelastic fluid defined by

$$\underset{\sim}{S} + p\underset{\sim}{I} - \mu_o[(\underset{\sim}{S}+p\underset{\sim}{I}):\underset{\sim}{I}]\underset{\sim}{E} = 2\int_{-\infty}^{t} \phi(t-t')\underset{\sim}{F}_t^{-1}(t') \cdot \underset{\sim}{E}(t') \cdot \underset{\sim}{F}_t^{-\dagger}(t')dt' \quad (1.72)$$

which reduces to the Walters B' liquid when $\mu_o = 0$, and which has as viscometric functions

$$\begin{aligned} \eta(\gamma) &= \eta_o[1+\mu_o K_o \gamma^2/\eta_o] \\ \sigma_1(\gamma) &= 2K_o\gamma^2 \\ \sigma_2(\gamma) &= 0 \quad . \end{aligned} \quad (1.73)$$

His purpose was to elucidate the separate effects of (a) elasticity and normal stresses, and (b) shear-dependent viscosity, on flow stability. In order to provide a basis for comparison, he also considered the behavior of the model

$$\underset{\sim}{S} + p\underset{\sim}{I} = 2\eta_o[1+2\mu_o K_o(\underset{\sim}{E}:\underset{\sim}{E})/\eta_o]\underset{\sim}{E} \quad (1.74)$$

which exhibits the same viscosity variation as (1.72), but yields neither normal stresses nor elastic effects. Davies defined the Taylor number in terms of the zero-shear viscosity η_o; however, he recognized the fact that, in stability problems involving a shear-dependent viscosity, two definitions of T are actually possible, one based on a characteristic parameter with units of viscosity (e.g., η_o) and the other based on the true apparent viscosity $\eta(\gamma)$. Unfortunately, the author made no attempt to resolve the question as to which definition, in general, is more meaningful. The results of Davies' analysis may be summarized as follows:

For $K_o > 0$ and $\mu_o < 0$,

(1) A fluid obeying (1.74) yields $T_c < T_{c,N}$ and $\epsilon_c < \epsilon_{c,N}$.

(2) A fluid obeying (1.72) gives a value of T_c which is less than that for (1.74) and a value of ϵ_c which is greater.

These findings are independent of the definition of the Taylor number employed.

All of the stability analyses we have discussed up to this point have involved consideration of various and particular simple-fluid models. As such, they shall henceforth become special cases of the present more comprehensive treatment, in which the disturbance equations for a general simple fluid with fading memory are derived. It will be shown here that there are eight material functions of the shear rate which can influence flow stability, and the special forms of these functions will be listed for each of the models mentioned so far.

In formulating the disturbance equations for the present study, it will be assumed, following Taylor[39] and Chandrasekhar,[5] that the onset of instability may be characterized by an infinitesimal, steady velocity disturbance superimposed upon the primary viscometric Couette flow. We shall thus be dealing with a total flow which is "almost viscometric" in nature, that is, viscometric except for a small perturbation. The general techniques necessary for treating such motions of simple fluids will be developed in Chapter 3.

Metzner, White, and Denn[29] have recently suggested that flows which are sufficiently "close" to viscometric motions might adequately be described by an equation of the form (1.57), which holds in the strictest sense only for perfectly viscometric flows. If this viscometric hypothesis were correct, then knowledge of the three viscometric functions would suffice to determine the stability behavior of a given fluid a priori; furthermore, no new information concerning the fluid could then be obtained from a stability test. It is therefore considered imperative that we determine here the validity of the "viscometric hypothesis."

Beard, Davies, and Walters[1] have recently presented the results of the only Taylor-Couette stability treatment that is not a special case of the present analysis. Consideration was given to the so-called overstable mode of behavior for the Walters B' liquid, where it was assumed that the disturbance, rather than being steady in time (neutral mode), fluctuates with a complex frequency. The authors found that, for certain values of the fluid parameters, the overstable mode can yield a lower critical Taylor number than the neutral mode. However, as Giesekus[18] has pointed out, this type of behavior is not necessarily characteristic of all simple fluids. The experimental results of the present study indicate that, for the polymer solutions tested, the neutral mode of behavior is preferred.

1.5.2 Experimental

As part of his well-known work on the Couette stability of incompressible Newtonian flows, G.I. Taylor[39] performed several experiments designed to test the validity of his theoretical results. The apparatus employed consisted of an opaque inner cylinder contained within an outer cylinder of glass; the cylinders could be rotated in the same or in opposite directions. A colored dye was injected into the region near the inner cylinder in order to allow for observation of the onset of instability which was characterized by the formation of a compartmentalized cellular-vortex pattern. Taylor confined attention to the case of water, and found his experimental results to agree very closely with theory.

In 1928, Lewis[26] extended Taylor's findings to cover a wider range of fluids and conditions. The fluid motion in this study was followed by means of tiny suspended aluminum particles rather than by dye injection. Thus, the problem of diffusion of dye into the fluid was avoided, and each experiment could be repeated several times for the same sample. Lewis' results have, for the most part, confirmed the mathematical relations derived by Taylor.

The first investigators to consider experimentally the stability of viscoelastic flow between coaxially rotating cylinders were Merrill, Mickley, and Ram[28] in 1962. These authors determined the shear stress-shear rate relation for several polymer solutions using a Couette viscometer with a stationary outer cylinder; high rates of shear were employed and the gap between the cylinders was quite small compared to the radii ($a \cong 180$). The onset of instability was observed as an abrupt increase in the slope of the shear-stress vs. shear-rate curve. Unfortunately, the findings of this investigation must be viewed with some skepticism since, from the authors' description of the experimental procedure, helical flow was present within the apparatus to an unknown extent. Merrill et al., have found that

(1) For the Newtonian fluids tested, the critical Taylor number was about 37% higher than predicted by theory.

(2) The polymer solutions studied showed no significant deviation from Newtonian behavior, provided that the Taylor number was calculated from the "thinned," apparent viscosity of the fluid.

Rubin and Elata[38] have recently considered the stability in Couette flow of various dilute polymer solutions and report that, for all the systems examined, the critical Taylor number increases with concentration while the cellular spacing remains relatively constant at its Newtonian value. Here, the viscosity at zero shear rate has been used in calculating T_c. As part of this same study, the authors have compared the above experimental results with the predictions of several rheological models; their attention was focused upon the idealized fluids studied by Thomas and Walters,[40,41] Datta,[11] Chan Man Fong,[6] and Leslie.[25] In addition, Rubin and Elata have performed several new calculations of their own based on Leslie's analysis. They conclude:

> "Ericksen's anisotropic fluid for $n_1 > 0$ and $n_2 > 0$ might be a suitable model for the investigated fluids. Not only can this model predict an increase of T_c, without appropriate changes in ϵ_c, but both P_{11}-P_{22} (σ_1-σ_2) and P_{22}-P_{33} (σ_2) are positive, which is in accordance with most of the experimental results."

These conclusions are erroneous for a number of reasons. First of all, for $n_1 > 0$ and $n_2 > 0$, it is seen from Eqs. (1.70) that, if $\gamma > 0$, σ_1 and σ_2 are both positive while if $\gamma < 0$, they are both negative. The shear rate γ, throughout Leslie's treatment, may be shown to have been taken less than zero. Hence σ_2 was not positive for $n_1 > 0$ and $n_2 > 0$ as Rubin and Elata claim. This further reflects an error in the work of Leslie. In particular, Ericksen[15] has derived the viscometric functions for the fluid model defined by (1.30), and his findings are consistent with (1.70) only if absolute value brackets are placed around the γ's in these equations. The three viscometric functions will then be even functions of the rate of shear, in accordance with the theoretical results of Coleman, Markovitz, and Noll.[7]

As a further source of confusion in their theoretical treatment, Rubin and Elata have, following Leslie, defined the Taylor number in terms of the characteristic parameter μ rather than the actual viscosity $\eta(\gamma)$. It will become apparent from the results of the present study that, unless T is based on $\eta(\gamma)$, the Newtonian and non-Newtonian results may not be properly compared. The two cases considered by Rubin and Elata are listed below along with the values for $T_{c,\eta}$, the critical Taylor number based on η, and $T_{c,N}$, the Newtonian critical value.

$\mu_0 = 1.5$, $\mu_2 = \mu$, $\mu_3 = 0$, $\alpha = -1$, $\epsilon_c \simeq 3.1$, $T_c \simeq 4600$, $T_{c,\eta} \simeq 3500$, $T_{c,N} = 3390$

$\mu_0 = 1.2$, $\mu_2 = \mu_3 = \mu$, $\alpha = -1$, $\epsilon_c \simeq 3.0$, $T_c \simeq 16000$, $T_{c,\eta} \simeq 3700$, $T_{c,N} = 3390$.

One notes that a somewhat less pronounced effect on T_c is obtained by correctly defining the Taylor number.

Giesekus[18] has conducted some "makeshift" experiments on viscoelastic fluids in Couette flow as part of his previously-mentioned theoretical work. For a 4% solution of polyisobutylene in decalin, he found that the onset of instability occurred at a Taylor number which is approximately 3×10^5 times smaller than in the Newtonian case. Furthermore, the critical wave number was about 25% larger than predicted by Taylor's analysis. Giesekus also studied a 1% solution of aluminum napthenate in decalin and obtained a critical Taylor number approximately 900 times lower than for a Newtonian fluid of the same apparent viscosity.

Huppler[24] has recently reported the results of measurements on the three viscometric functions η, σ_1, and σ_2 for solutions of several different polymers at various concentrations. In the present work, it was originally anticipated that these functions might play an important part in determining viscoelastic flow stability. Therefore, two of the polymers studied by Huppler have been chosen for experimental consideration herein. In particular, attention has been given to the Couette stability of high-viscosity grade CMC (carboxymethylcellulose) solutions, ranging in concentration from 0% to 0.55%, and to solutions of Natrosol 250-H HEC (hydroxyethylcellulose), of concentrations between 0% and 1.0%. Furthermore, in order to compare theory with experiment, the stability of a 0.5% CMC solution and a 0.9% HEC solution have, based on Huppler's viscometric functions, been considered in detail analytically.

1.6 OBJECTIVES

The objectives of the present study may now be summarized as follows:

(1) To develop general methods for treating small perturbations on viscometric flows of simple fluids.

(2) To establish, as e.g., Coleman, Markovitz, and Noll[7] have done for viscometric flows, the minimum number of material functions necessary to analytically describe the Taylor-Couette stability of simple fluids with fading memory.

(3) To determine the relationship, if any, between the above material functions and the three viscometric functions η, σ_1 and σ_2.

(4) To list the above material functions for all of the particular fluid models whose stability has been previously analyzed.

(5) To establish which viscosity, in general, is most appropriate for calculating the Taylor number.

(6) To perform stability experiments on polymer solutions whose viscometric functions have been measured experimentally; and

(7) To study analytically the stability of viscoelastic fluids whose viscometric functions are known.

(8) To determine, by a combination of theory and experiment, the validity of the "viscometric hypothesis."

CHAPTER 2

EXPERIMENTAL WORK

2.1 SAMPLES STUDIED

The experimental work of this investigation was undertaken for the purpose of determining the stability behavior of some real viscoelastic fluids in flow between coaxial rotating cylinders. Aqueous solutions of three cellulose-derivative polymers, whose characteristics are summarized in Table 2.1, were chosen for study; the viscometric functions η, σ_1, and σ_2 for the latter two of these have been reported in the literature by Huppler[24] for various particular concentrations.* Since the present theoretical treatment has indicated that η, σ_1, and σ_2 should greatly influence flow stability, detailed numerical calculations have been performed on a 0.5% P-75-XH CMC solution and a 0.9% HEC solution based on Huppler's results. For these concentrations, the current theory and experiment may be compared and, as we shall see later in Chapter 4, one may then evaluate the validity of the "viscometric hypothesis" which postulates that, for sufficiently close approximations to viscometric motions, the flow criteria are determined entirely by η, σ_1, and σ_2.

*Gratitude is owed to Dr. B. Duane Marsh and Professor R. B. Bird of the University of Wisconsin for furnishing samples of these two polymers.

TABLE 2.1

SYSTEMS INVESTIGATED

Polymer	Designation	Producer	Average MW	Viscosity Grade	Conc. Range
CMC*	7MT	Hercules	not known	Medium	0-2.5%
CMC	P-75-XH	Union Carbide	3.5×10^5	High	0-0.55%
HEC**	Natrosol 250-H	Union Carbide	not known	High	0-1.0%

*CMC = Carboxymethylcellulose
**HEC = Hydroxyethylcellulose

2.2 SAMPLE PREPARATION

The volume of sample required for the stability apparatus was approximately two liters. The most concentrated polymer solutions were prepared by first weighing solid polymer on a trip balance, with a least count of 0.1 g. The polymer was then transferred to a gallon container and 2500 cc of distilled water were added; complete solution was accomplished by vigorous and frequent shaking.

Following Lewis,[26] aluminum particles were suspended in each fluid tested in order to facilitate observation of the flow patterns. The particles could not be mixed directly with the polymer solutions, however, since under these circumstances they tended to form "clumps" due to the high viscosity and the accompanying low wetability of the solutions. To avert such difficulties, approximately 0.5 cc bulk of particles was mixed with about one liter of distilled water, which served to wet them quite readily; 500 cc of this suspension was then decanted and combined with the 2500 cc of solution already prepared to yield an intimately dispersed system totaling three liters. Less concentrated solutions were prepared by diluting those of higher concentration.

Of some interest is the effect of the aluminum particles on fluid behavior. Now, Einstein[13] has derived the equation

$$\eta = \eta_o(1+2.5\phi) \tag{2.1}$$

for the viscosity of a dilute suspension of small spheres in a Newtonian fluid. Here η_o refers to the viscosity of the suspending medium while ϕ is the volume fraction of spheres. For volume fractions on the order of magnitude used, Eq. (2.1) predicts a negligible effect on fluid viscosity. Hence, it is fair to assume that the material properties of the solutions studied were not appreciably influenced by the presence of aluminum particles.

Huppler[24] has observed no mechanical degradation in solutions of any of the cellulose derivatives tested. In the present work, degradation due to bacterial growth may also be assumed negligible since the stability behavior of the samples was examined immediately after their preparation.

2.3 STABILITY APPARATUS

The stability apparatus used in this investigation was made available by Professor W. P. Graebel (a member of the Doctoral Committee). It consisted of an inner cylinder of aluminum, having an outside diameter of 5.52 in., contained within a transparent plexiglass outer cylinder, of inside diameter 5.98 in. (Actually, these diameters varied by approximately

± 0.005 in. along the length of the apparatus, and the values quoted here are those measured at the center.) The ratio of the inner cylinder radius to the width of the gap for the system was then $a = r_1/d = 12$.

In order to minimize end effects, the apparatus was originally designed so that its length, approximately 36 in., was relatively large compared to the diameter; the working height of fluid within the apparatus totalled approximately 30 in. (Fig. 2.1).

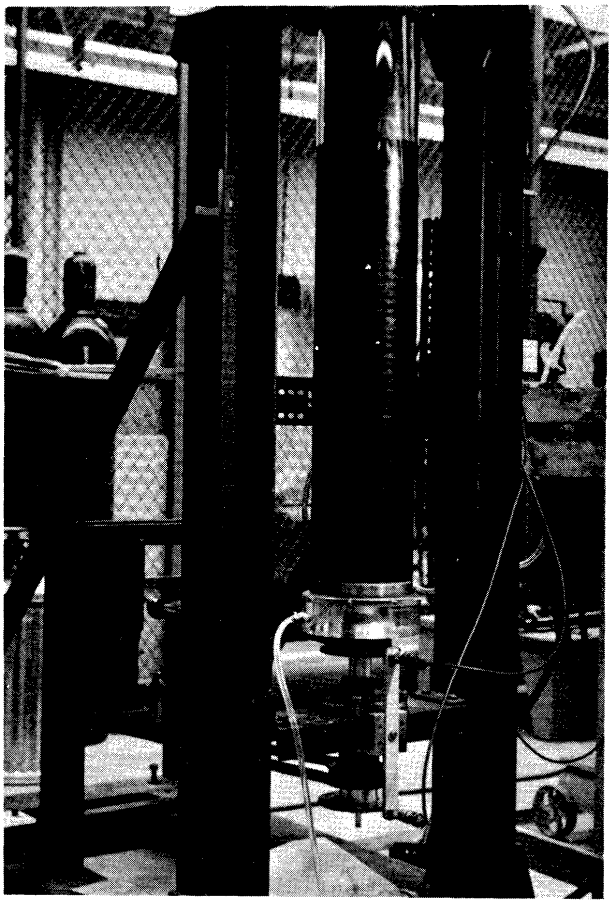


Fig. 2.1. The stability apparatus. The fluid shown is a 2.5% aqueous solution of 7MT CMC. Vortex cells are visible, indicating that the inner cylinder is rotating above the critical speed.

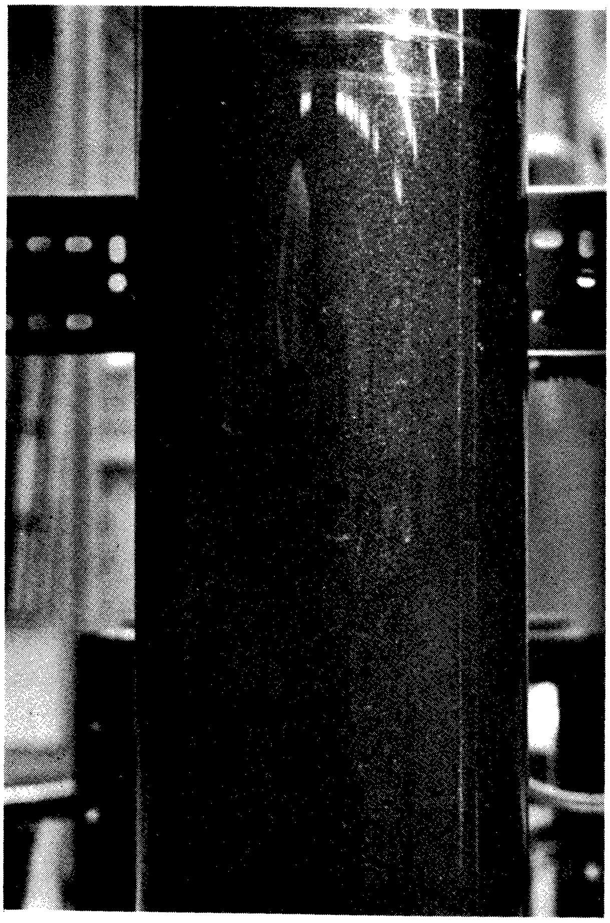
The facility was present for rotating the cylinders independently in either direction by means of two 1/3 hp variable-speed electric motors connected to the cylinder shafts through rubber belts; the maximum attainable speed of rotation was approximately 400 rpm.

In the present work, tests were performed with only the inner cylinder rotating; there were three reasons for this. First of all, the outer cylinder tended to "wobble" when driven, and it was expected that this might influence the steady flow. Secondly, it is frequently desirable to compare stability results at a fixed ratio of the rotational speeds which was automatically assured under these circumstances. Finally, the theoretical findings suggest that similar effects would have been experienced at other speed ratios (at least for cylinders rotating in the same direction).

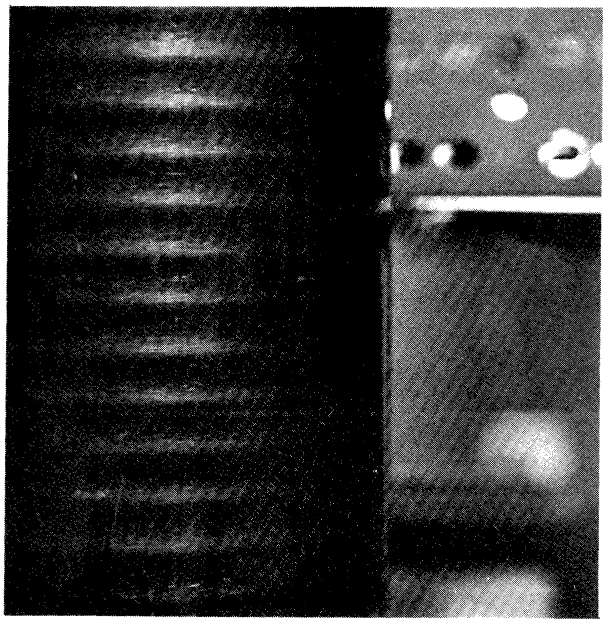
2.4 STABILITY EXPERIMENTS

The rotational speed of the inner cylinder was increased slowly until vortex cells, which appeared as alternate light and dark bands wrapped around the inner cylinder (Fig. 2.2), began forming in certain parts of the system (primarily at the ends). It was next observed that, at a slightly higher rate of rotation, the disturbance completely filled the apparatus. The average of these two limiting speeds was taken as the onset of instability and in no case did this differ by more than 3% from either of the limits.

The rate of rotation of the inner cylinder was determined by counting the number of revolutions and dividing by the corresponding time interval. Revolutions were counted by means of a small protuberance, extending from the top edge of the cylinder, which was allowed to strike a flapper, thus emitting a clearly audible sound at each rotation. The time interval for several counts was measured using a stopwatch.



(a)



(b)

Fig. 2.2. Transition to a vortex pattern. Figure 2.2(a) shows the stability apparatus when the inner cylinder is rotating below the critical speed. Figure 2.2(b) shows the stability apparatus when the critical Taylor number is exceeded. The fluid is a 2.5% aqueous solution of 7MT CMC.

The wavelength of the disturbance could be determined quite accurately in each case by measuring the combined height of several vortex cells at the surface of the transparent outer cylinder. The results of these measurements were reproducible to within 4% of the average value.

After each stability experiment, the polymer solution was drained from the system and its temperature was measured using a standard mercury thermometer. It is estimated that the temperature was known to ±1°F.

Fredrickson[17] has presented the equation

$$T_{max} - T_o = \frac{\eta r_1^2(\Omega_2-\Omega_1)^2}{4k} \tag{2.2}$$

for estimating the temperature rise due to viscous heating in a Couette apparatus. In deriving this relation, it was assumed that the outer cylinder is maintained at the constant temperature T_o and that no heat transfer takes place at the inner cylinder. The quantity k in (2.2) stands for the thermal conductivity of the fluid, while T_{max} refers to the maximum temperature attained in the system. Calculations reveal that, in the present work, the value of $T_{max}-T_o$ was never greater than 1.8°C. As Fredrickson[17] has indicated, a temperature rise of 1 or 2°C is tolerable for most fluids in this apparatus. Therefore, it is reasonable to assume that viscous heating effects were unimportant herein.

2.5 VISCOSITY MEASUREMENTS

In this study, fluid viscosities have been measured using a commercial version of the Weissenberg Rheogoniometer, which is essentially a plate-and-cone viscometer. A detailed description of this instrument may be found elsewhere,[16] and hence we present only the basic features of its design and operation.

The fluid to be tested is sheared between a fixed flat plate and a rotating cone as shown schematically in Fig. 2.3. This flow may be regarded as having a constant rate of shear γ provided that the angle between plate and cone Θ_c is less than 4° (see e.g., Fredrickson[17]); in the present work, Θ_c was 1°37'.

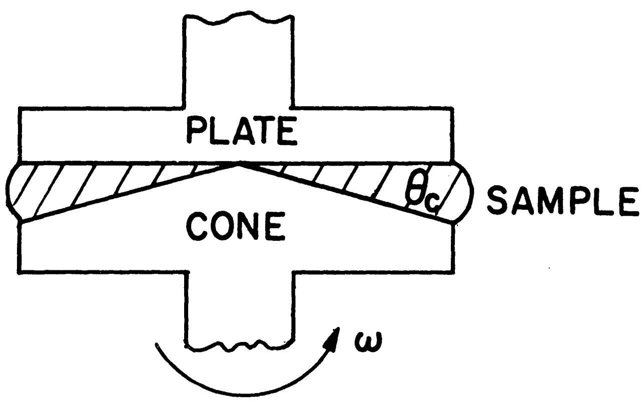


Fig. 2.3. Schematic of plate-and-cone device.

The torque M on the flat plate is related to the apparent viscosity through the equation

$$M = \frac{\pi D^3 \gamma \eta(\gamma)}{12} \tag{2.3}$$

where D is the platen diameter. The shear rate γ, for a small cone angle, is given by

$$\gamma = \omega/\Theta_c \tag{2.4}$$

where γ has units of sec^{-1}, ω is the rotational speed of the lower platen in radians/sec, and Θ_c is in radians. Thus, from measurements of M and ω, one can determine the shear stress-shear rate relation for a given fluid.

A schematic of the entire apparatus is presented in Fig. 2.4. It is seen that the lower conical platen is driven by means of a 1-hp electric motor operating at 1800 rpm; variation of the platen speed is facilitated by a gearbox connected to the output of this motor. Gearbox settings of from 0.0 to 5.9 (in steps of 0.1) correspond to speed reduction ratios of from $1{:}10^{-0.0}$ (1:1) to $1{:}10^{-5.9}$ (1:792,000), respectively. An internal speed reduction of 1:4 is present within the rheogoniometer so that the maximum rotational velocity of the lower platen is 450 rpm; by Eq. (2.4), this corresponds to a maximum shear rate of 1670 sec^{-1} for the present study.

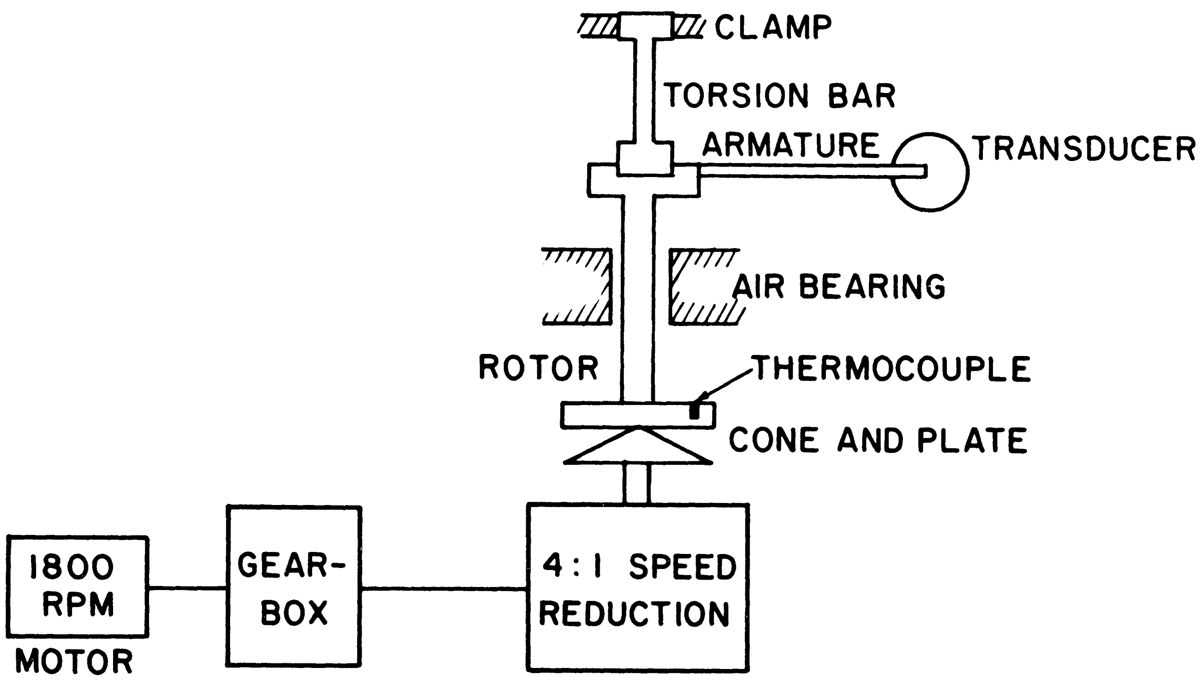


Fig. 2.4. Schematic of rheogoniometer.

The torque on the upper flat plate is determined by the deflection of a torsion bar joined to that platen through a rotor. This rotor passes through an air bearing which serves to minimize frictional effects during testing. The torsion bar armature deflection is measured by means of a torsion-head transducer whose signal is displayed on a meter. For the 7.5 cm diam platens and the 1/8 in. diam torsion bar (torsion bar constant = 3.43×10^{-4} dyne-cm/0.001 in.) used in this work, the shear stress $\gamma\eta(\gamma)$ was related to the above deflection through the equation

$$\gamma\eta(\gamma) = 311\,\Delta \tag{2.5}$$

where Δ is the armature displacement in thousandths of an inch and $\gamma\eta(\gamma)$ is in dynes/cm^2.

In practice, the apex of the lower conical platen is truncated slightly so that the two platens do not actually touch one another; this truncation may be shown to give a negligible contribution towards the total-torque measurement. The upper flat plate is positioned vertically by means of a gap-setting transducer (not shown in Fig. 2.4) which places the hypothetical tip of the cone at the surface of the flat plate.

In the present work, viscosity measurements have been made immediately after each stability experiment by using sample taken directly from the stability apparatus. A wide range of shear rates, which included that occurring at instability, were employed and care was taken to assure that, near the critical shear rate, the temperature differed by less than 0.2°C from its recorded value in the stability test. The method of temperature control, illustrated in Fig. 2.5, was as follows. If the initial platen temperatures (AB), determined by means of a thermocouple embedded in the upper platen, were below that desired, the platens were doused with hot tap water until their temperatures rose a few degrees above this (BC). They were then wiped dry and sample was loaded into the region between them (CD). The system was then allowed to cool slowly (for several minutes) until the temperature dropped to about 0.5°C higher than at instability (DE); at this point, testing was begun. During the entire course of measurements, the temperature was observed to fall an additional 1°C (EF).

Calculations have shown that the several minutes necessary for the platens to cool was sufficient for the system to reach thermal equilibrium. Thus, it is reasonable to assume that the thermocouple gave an accurate representation of the sample temperature.

It was initially observed that air currents generated by the air bearing caused the platens to cool too rapidly during testing; furthermore, these currents gave rise to significant evaporation of the sample from the gap between plate and cone. To protect the system from such effects, the platens were enclosed in a wooden test cell with a plexiglass window. (This apparatus was designed and constructed by Dr. L. F. Carter[4] as part of his doctoral work, and is described in detail in his Ph.D. thesis.)

Fredrickson[17] has presented an equation for calculating the effects of viscous heating in a plate-and-cone device. This is expressible in

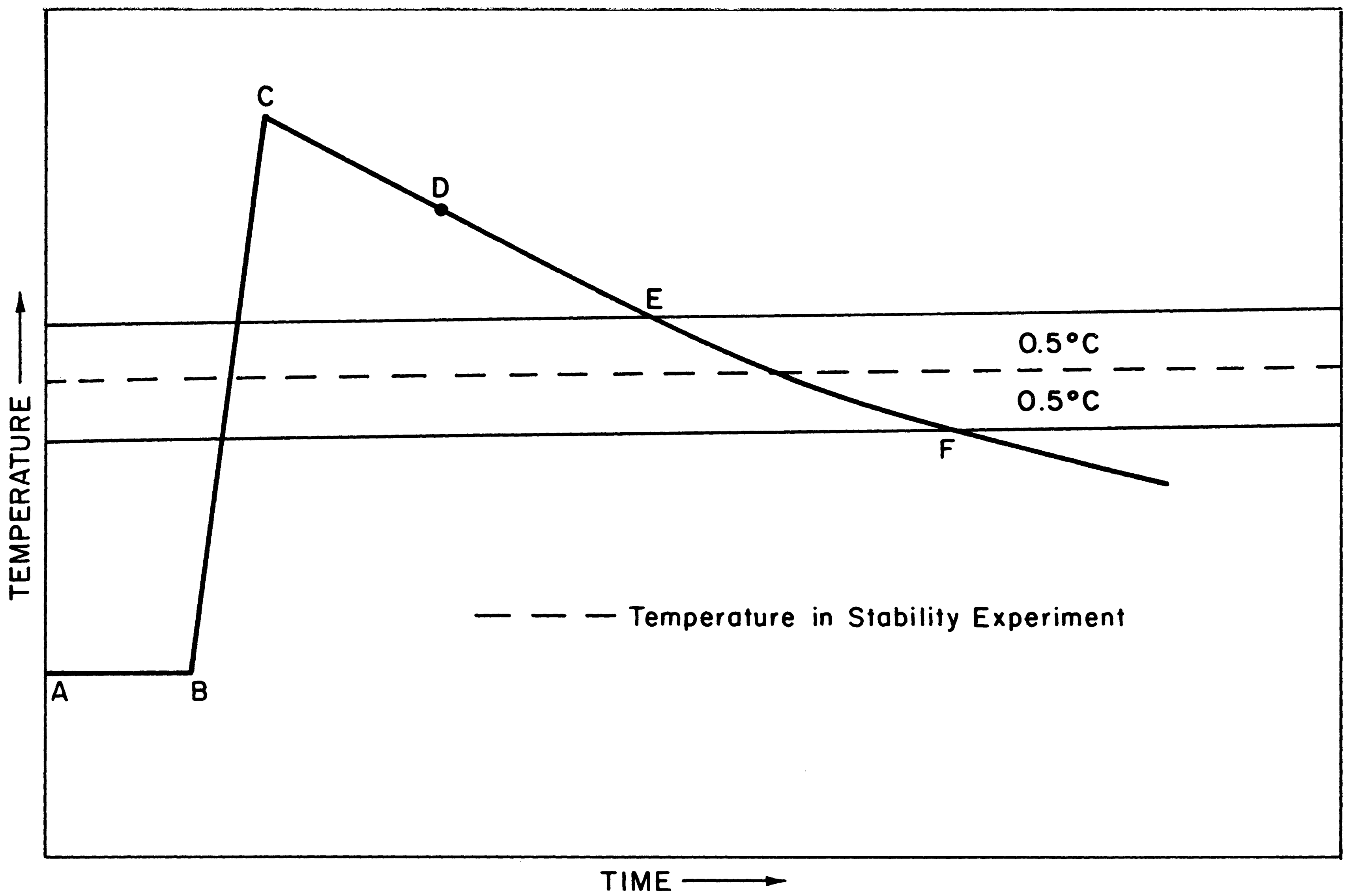


Fig. 2.5. Schematic of the method of temperature control.

the form

$$T_{max} - T_0 = \frac{\eta D^2 \omega^2}{2k} \qquad (2.6)$$

where k is the thermal conductivity of the fluid and T_{max} is the maximum temperature attained in the apparatus. In deriving (2.6), it has been assumed that both platens are kept at the constant temperature T_0 and that no heat transfer takes place at their outer edge. For the present work, the maximum temperature rise due to viscous heating may be shown to have been less than 1.5°C in all cases. Therefore, such effects were not considered significant herein.

2.6 RESULTS AND DISCUSSION

Figures 2.6-2.8 present the results of the viscosity measurements obtained in this study; the viscosity η has been plotted vs. the shear rate γ at fixed values of temperature and composition. It is seen from these graphs that the aqueous solutions of all three polymers tested exhibited shear thinning.

The dashed lines in Figs. 2.7 and 2.8 represent equations which were used in the present theoretical work to approximate the experimental findings of Huppler[24] at the concentrations and temperatures shown. The lines are seen to fall very roughly into appropriate positions with respect to the current experimental results; that is, those solutions which flanked Huppler's in concentration also flanked his in viscosity.

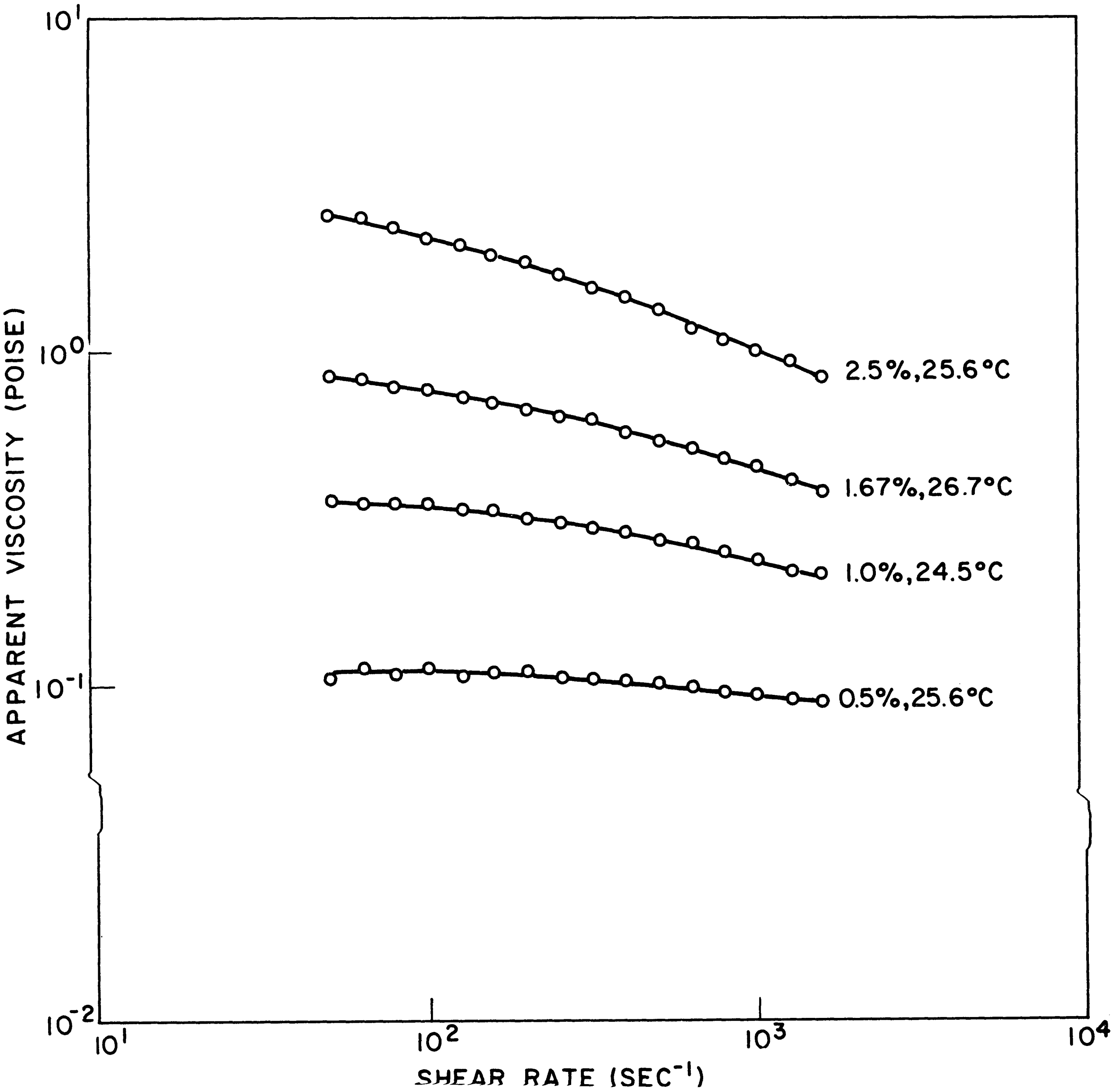


Fig. 2.6. Viscosity vs. shear rate for solutions of 7MT CMC.

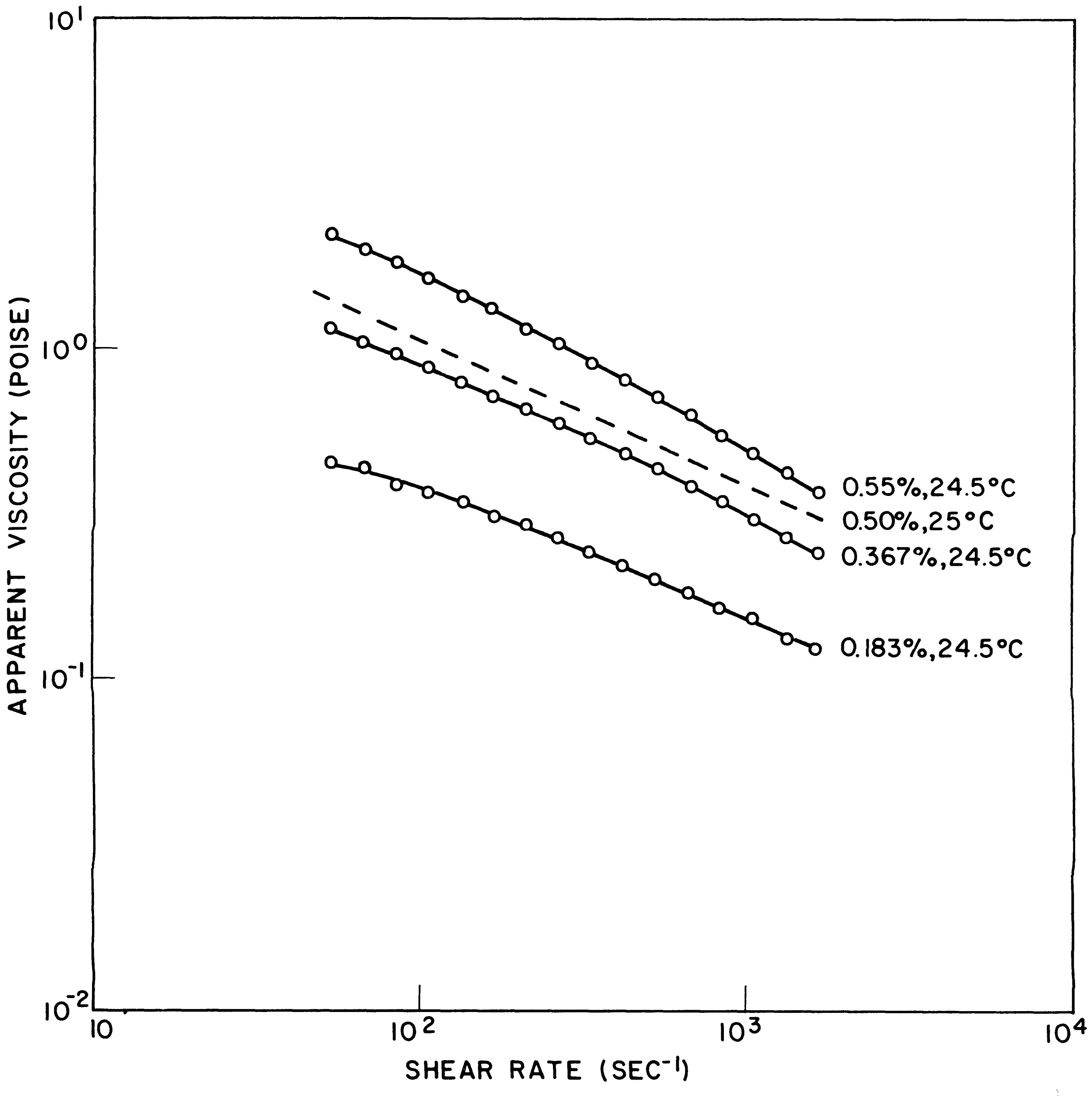


Fig. 2.7. Viscosity vs. shear rate for solutions of P-75-XH CMC.

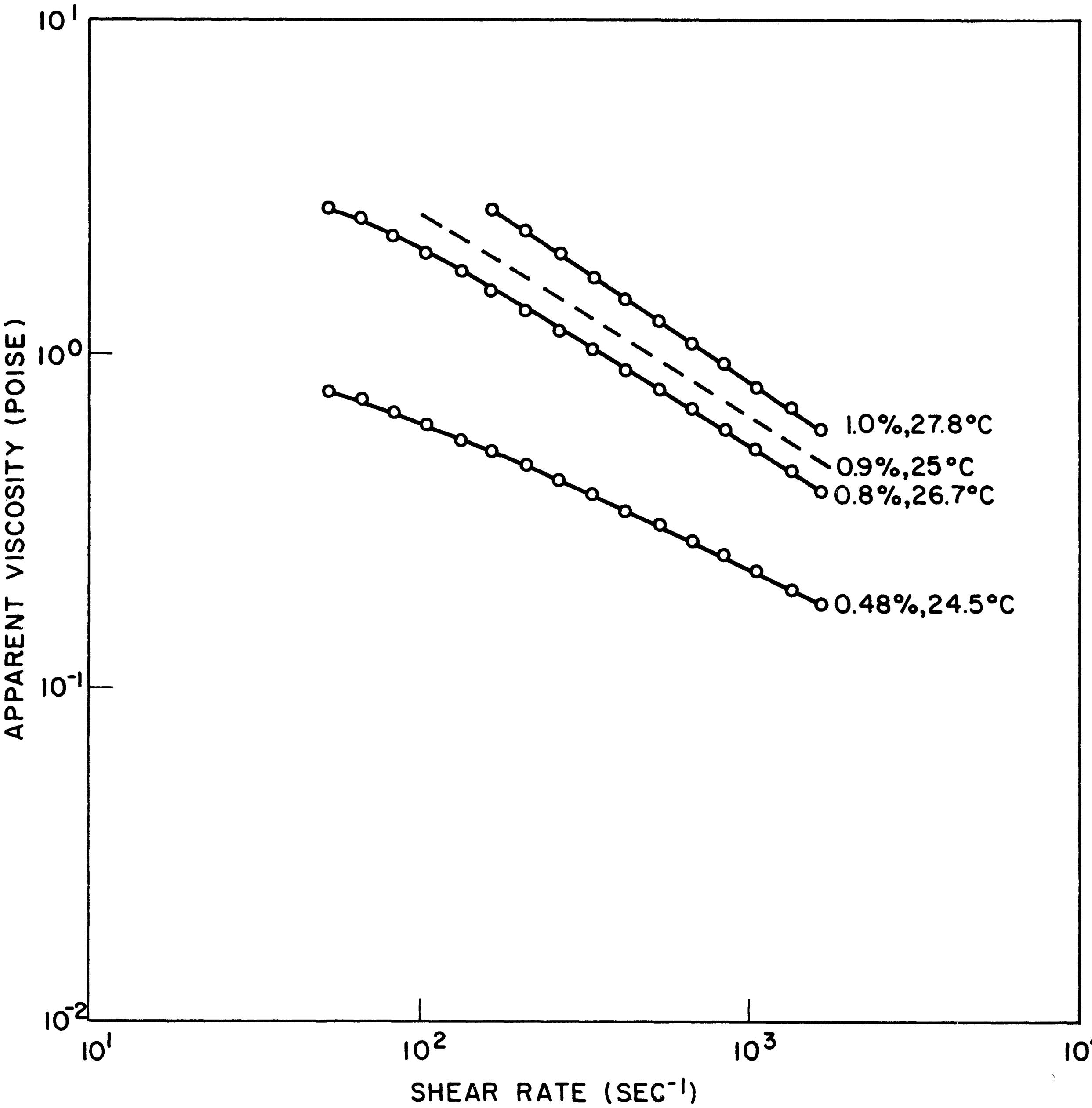


Fig. 2.8. Viscosity vs. shear rate for solutions of Natrosol 250-H.

For each of the investigated fluids, the critical Taylor number as a function of concentration is plotted in Figs. 2.9-2.11; here, T_c is defined by Eq. (1.1), and the apparent viscosity at the critical shear rate $\eta(\gamma)$ has been used in its calculation. The effect of polymer concentration on the critical wave number ϵ_c is shown in Figs. 2.12-2.14.

One can employ Eqs. (1.58) and (1.59) to calculate the theoretical Newtonian value for T_c for the present stability apparatus. It is found that, for $\alpha = -1$ and $a = 12$, $T_{c,N} = 3620$. We also recall that Taylor[39] has shown that $\epsilon_{c,N} \triangleq \pi = 3.14$. These values are indicated as broken lines in Figs. 2.9-2.14.

The only Newtonian fluid considered in the current investigation was pure water which corresponds to the point at zero concentration in the above-mentioned plots. For this case, the critical Taylor number was found to be 4140 which is about 14% higher than predicted by theory. However, this value compares quite favorably with the value, 37% higher than theory, obtained by Merrill et al.,[28] for the comparable situation, and represents fair agreement between theory and experiment.* It is worthwhile noting that the point at zero concentration appears to lie on a smooth curve with the other experimental points and should thus provide a valid basis of comparison for the non-Newtonian results. The critical

*One might argue that the present discrepancy between theory and experiment could be attributed to the fact that, in practice, only finite secondary motions of fluids can be observed whereas, according to the theory, the secondary flow is assumed to be infinitesimal in magnitude. Although such may indeed be the case it should be pointed out that Taylor[39] and Lewis[26] both achieved good confirmation of Taylor's mathematical findings using experimental procedures similar to those employed here.

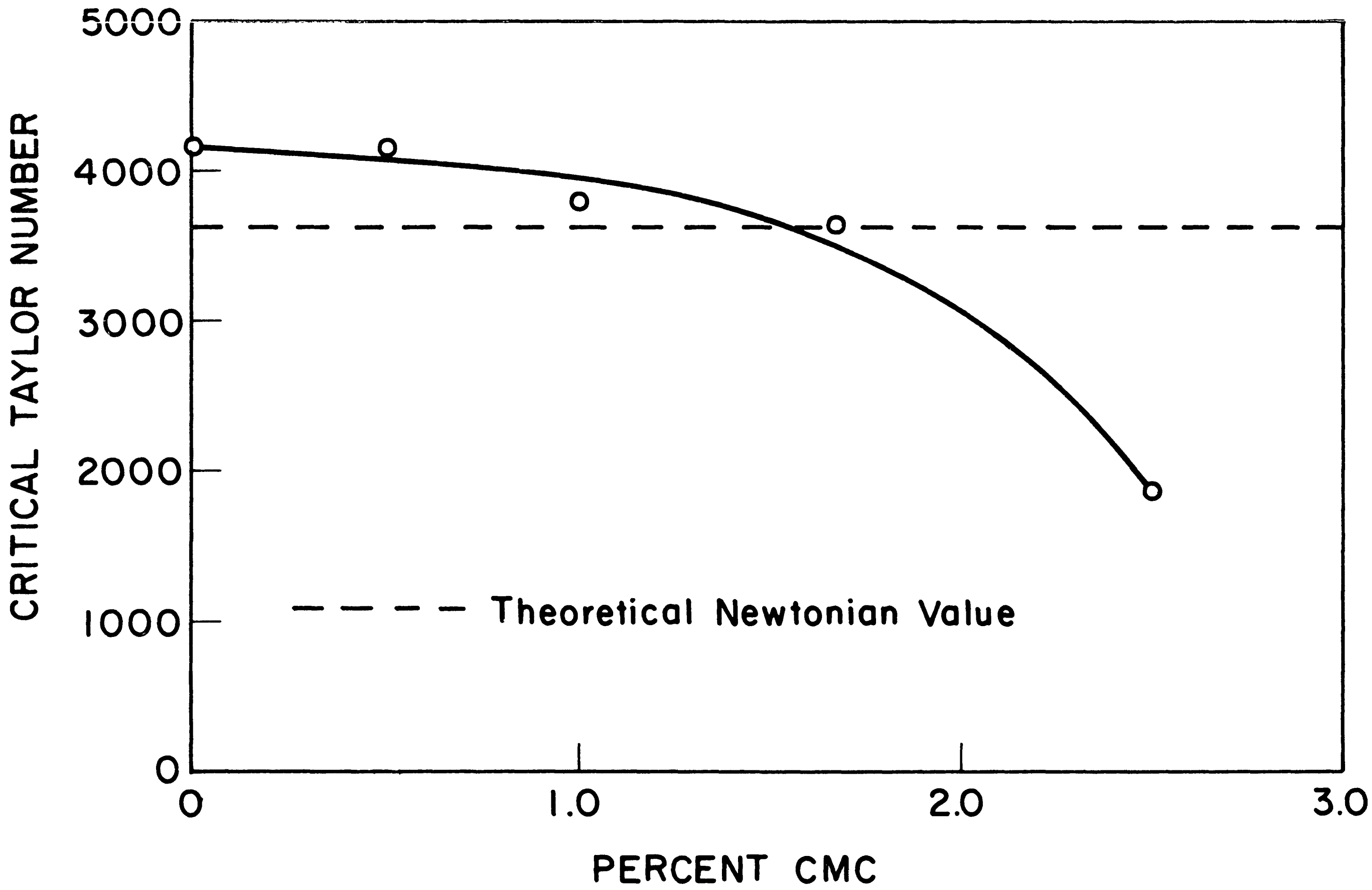


Fig. 2.9. Critical Taylor number as a function of concentration for solutions of 7MT CMC.

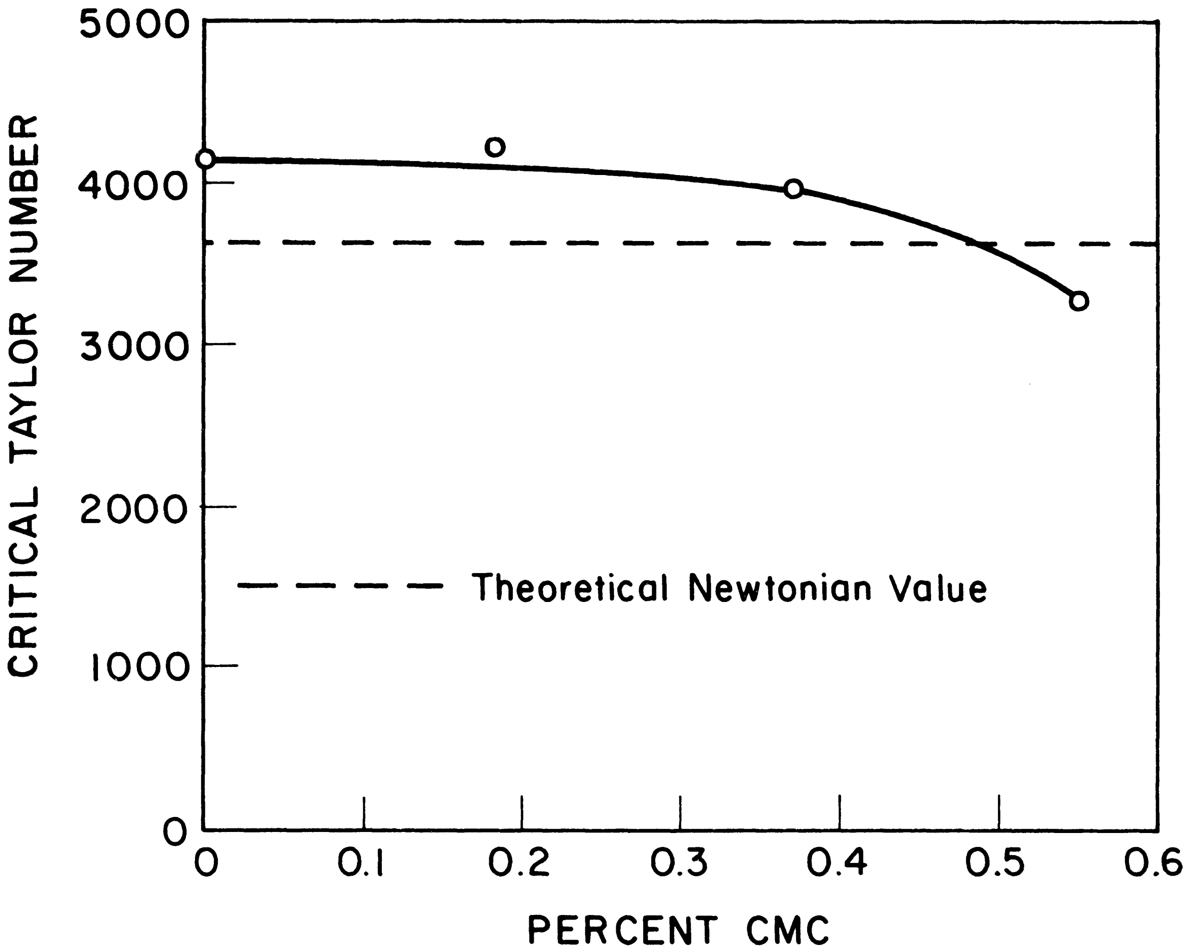


Fig. 2.10. Critical Taylor number as a function of concentration for solutions of P-75-XH CMC.

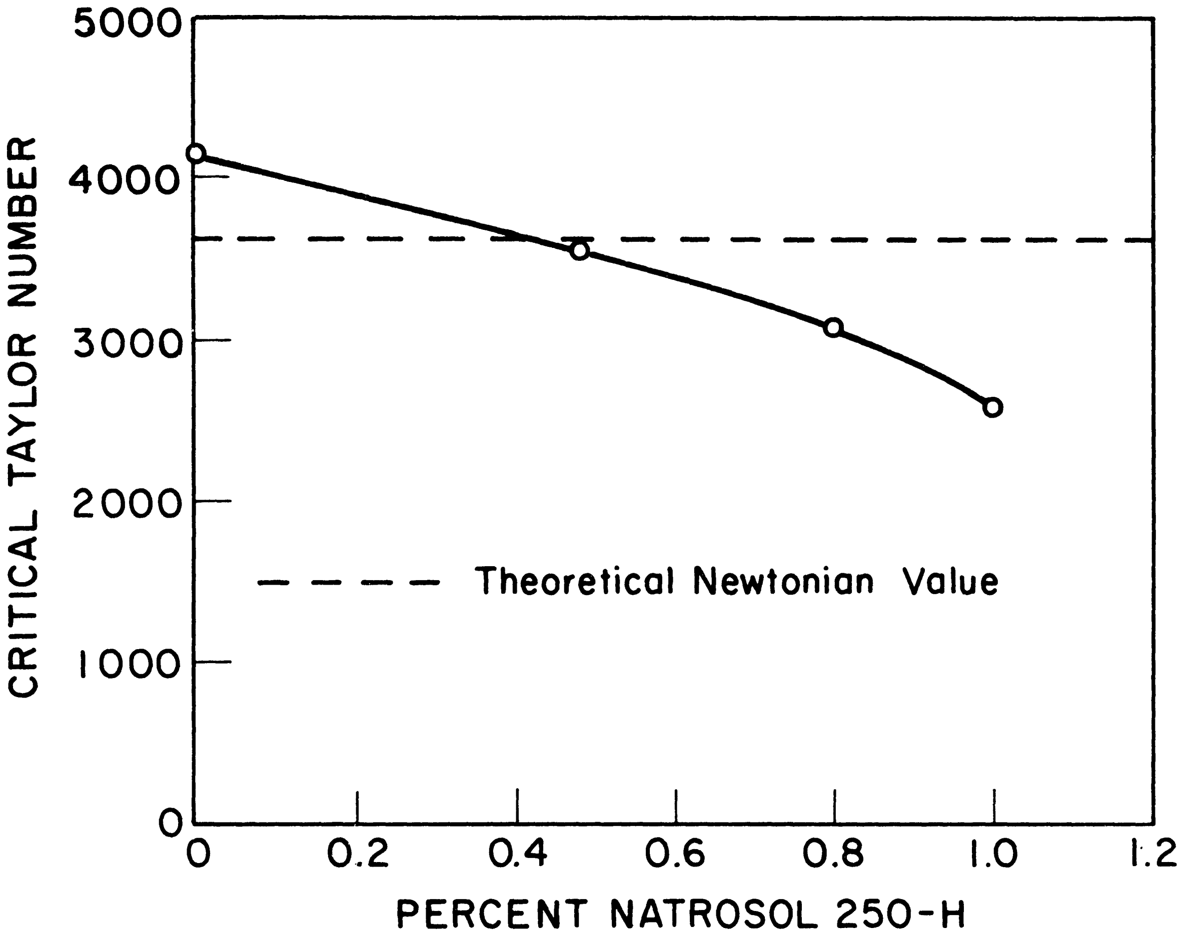


Fig. 2.11. Critical Taylor number as a function of concentration for solutions of Natrosol 250-H.

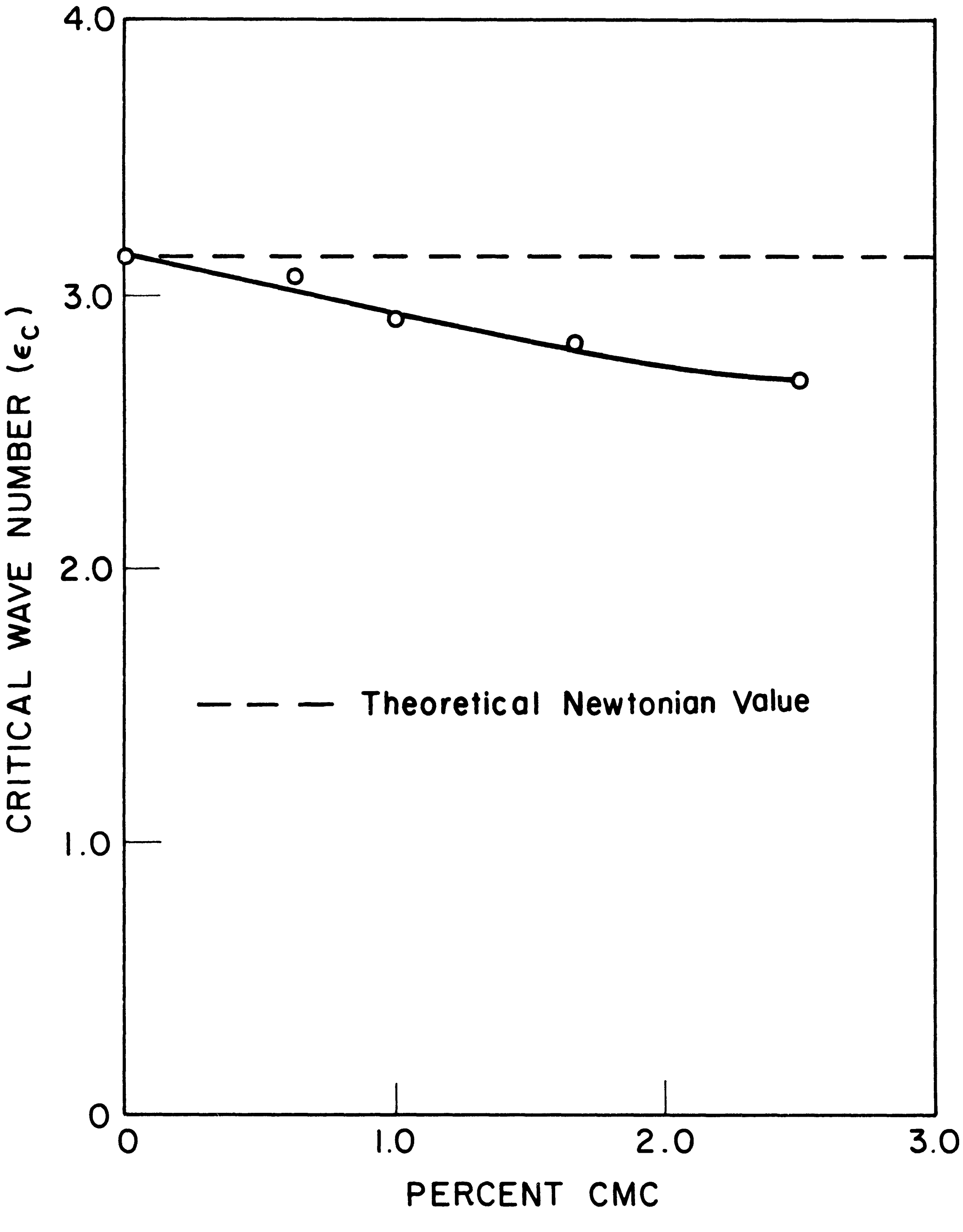


Fig. 2.12. Critical wave number as a function of concentration for solutions of 7MT CMC.

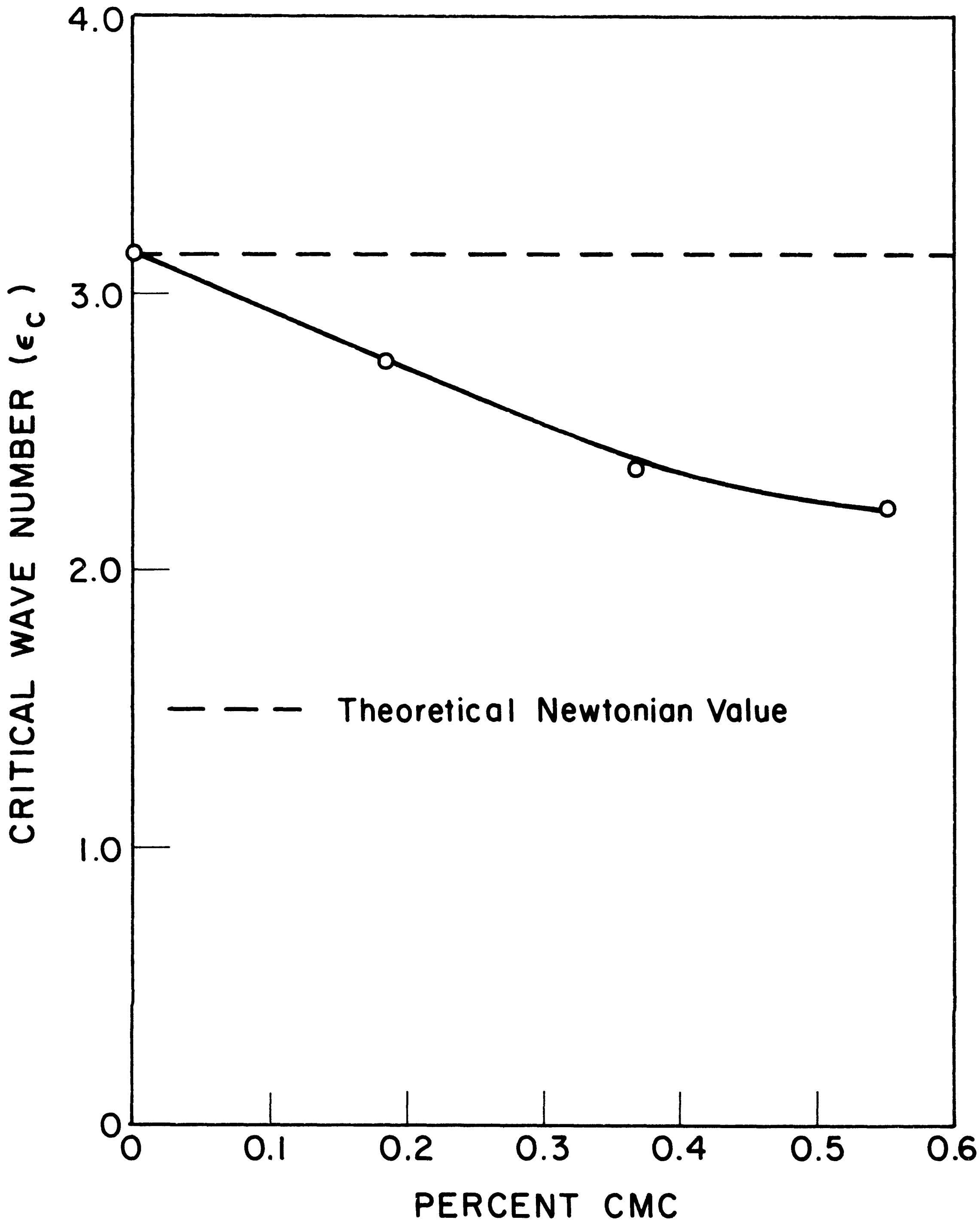


Fig. 2.13. Critical wave number as a function of concentration for solutions of P-75-XH CMC.

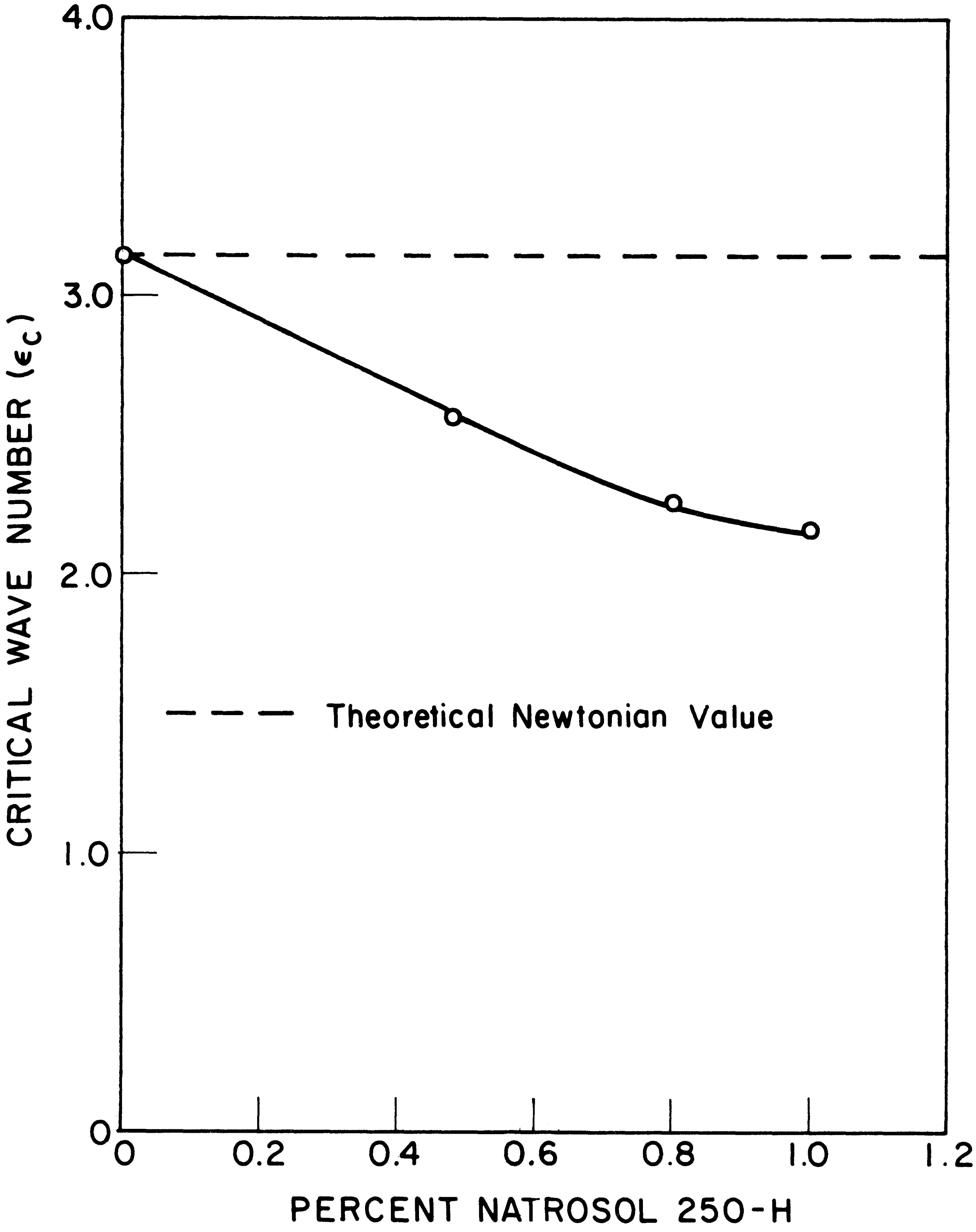


Fig. 2.14. Critical wave number as a function of concentration for solutions of Natrosol 250-H.

wave number obtained herein for the case of pure water was $\epsilon_c = 3.14$.

As can be seen from Figs. 2.9-2.14, the critical Taylor number and critical wave number both decreased with concentration for all of the systems investigated. Thus, the flow was less stable than in the Newtonian case, and the length of the vortex cells was larger. A comparison of these findings with the results of several other stability studies on viscoelastic polymer solutions is presented in Table 2.2. One notes immediately that the only study to report a decrease in the critical wave number is the current one; Rubin and Elata[38] found no change in this quantity for the systems they studied, while Giesekus[18] obtained increases in ϵ_c. The critical Taylor number was observed to decrease in our study, giving qualitative agreement with the findings of Giesekus; however, the magnitude of this decrease was much smaller in the present instance. By way of contrast, Rubin and Elata report increases in T_c, while Merrill et al.,[28] have found, in most cases, no change in this quantity. Solutions of several different polymers have been used in all of the above investigations; thus, it appears that the stability criteria are highly dependent on the systems studied.

In the theoretical portion of the present work, the small-gap approximation has been employed in order to facilitate the mathematical treatment of the Taylor-Couette problem. Unfortunately, the current experimental stability apparatus was not a small-gap system in the strictest sense of the term. This is illustrated by the fact that the theoretical Newtonian value for T_c (corresponding to the present system) was 3620 as

TABLE 2.2

COMPARISON OF EXPERIMENTAL STUDIES

Investigator	$T_c - T_{c,N}$	$\epsilon_c - \epsilon_{c,N}$
Rubin and Elata[38]	Greater than zero	Equal to zero
Giesekus[18]	Much less than zero (almost equal to $-T_{c,N}$)	Greater than zero
Merrill, *et al.*[28]	Zero in most cases Less than zero in some cases	Not determined
This study	Less than zero	Less than zero

compared with 3390 for the small-gap case.[5] A comparison of theory and experiment is still possible however, if one assumes that the fractional reduction in T_c is approximately the same in both the small-gap and large-gap cases. Under such circumstances, one would have $T_c/T_{c,N} \simeq 0.86$ for a 0.5% solution of P-75-XH CMC and $T_c/T_{c,N} \simeq 0.69$ for a 0.9% solution of Natrosol 250-H; the corresponding values for the critical Taylor number for the small-gap case would then be (roughly) 2900 and 2350, respectively.

Before concluding the present discussion of experimental results, one last factor should receive some consideration. In the current theoretical analysis, it has been assumed that the neutral mode of behavior is exhibited by the fluid of interest at the onset of instability; i.e., the disturbance takes the form of a set of cellular vortices, *steady in time*. Now, as we have noted earlier, the experimentally observed vortex pattern (obtained using aluminum particles) was characterized by the appearance of a *steady* array of alternate light and dark bands wrapped around the inner cylinder. Thus, the assumption of neutral stability appears to be valid for the systems studied here.

CHAPTER 3

THEORY OF SMALL PERTURBATIONS ON VISCOMETRIC FLOWS

As was discussed in Section 1.4.5, Coleman, Markovitz, and Noll,[7] have made use of the general constitutive theory for simple fluids to analyze the mechanical behavior of these materials in viscometric flows. It has been shown that, even at very high rates of shear, the state of stress at a material point is determined by three viscometric functions η, σ_1, and σ_2. Unfortunately, most other motions of simple fluids are too complex to be dealt with by general theory, save for those cases where the deformation histories are extremely slow. As a result, rapid flows have, up to the present time, gone largely untreated.

In this chapter, we shall consider a class of motions which are not only amenable to general analysis but which also possess finite rates of strain. These are defined by the presence of a small velocity perturbation superimposed upon some primary viscometric flow. The techniques to be developed here will have obvious application to certain linear stability analyses, such as the current Taylor-Couette problem. However, it is fair to say that there exist many other instances of practical importance for which the methods discussed may find use. Let us briefly cite, as some examples, the following:

(1) Combined steady and oscillatory shear between parallel plates where the amplitude of oscillation is small.

(2) Problems in which the primary viscometric stress distribution is not strictly compatable with the equations of motion so that secondary flows arise. Among these are:

(a) Axial motion in a channel of noncircular cross section.

(b) Flow between a plate and cone.

(c) Torsional flow between rotating flat disks.

It should be noted that Pipkin and Owen[33] have recently derived constitutive relations appropriate for the description of small perturbations on viscometric flows. The present independent development is more general, perhaps simpler to follow, and may be applied more readily than theirs.

3.1 EQUATIONS OF MOTION

The equations of momentum and continuity, governing the flow of any incompressible material, are given respectively by

$$\rho \frac{D\underset{\sim}{v}}{Dt} = \rho\left[\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v}\cdot\underset{\sim}{\nabla}\underset{\sim}{v}\right] = \rho\underset{\sim}{b} + \underset{\sim}{\nabla}\cdot\underset{\sim}{S} \tag{3.1}$$

and

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{v} = 0 \tag{3.2}$$

where $\underset{\sim}{b}$ is the body force/unit mass (assumed here to be due only to gravity) and ρ is the density of the material.

Let us suppose that, within a particular simple fluid, there exists a primary velocity field $\underset{\sim}{v}^{(o)}$ and an associated stress field $\underset{\sim}{S}^{(o)}$ which satisfy (3.1) and (3.2) exactly* and let these fields be disturbed slightly

*The discussion of this section may be easily modified to deal with problems of the type (2) listed above.

by a small perturbation on the motion. Our objective in this section shall be to derive the linearized equations which must be satisfied in order for the new total flow to also be consistent with (3.1) and (3.2). For this purpose, we write

$$\underset{\sim}{v} = \underset{\sim}{v}^{(o)} + \xi \underset{\sim}{v}^{(1)} \tag{3.3}$$

where ξ has a value between 0 and 1 and where $\underset{\sim}{v}^{(1)}$ is some arbitrary, spatially smooth velocity field having the same order of magnitude as $\underset{\sim}{v}^{(o)}$. The new total-stress field $\underset{\sim}{S}$ may then be assumed expressible in the form

$$\underset{\sim}{S} = \underset{\sim}{S}^{(o)} + \xi \underset{\sim}{S}^{(1)} + \underset{\sim}{\mathcal{O}}(\xi) \tag{3.4}$$

where $\underset{\sim}{\mathcal{O}}(\xi)$ is a tensor-valued function with the property that

$$\lim_{\xi \to 0} \frac{\underset{\sim}{\mathcal{O}}(\xi)}{\xi} = \underset{\sim}{0} . \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.1) and (3.2), we obtain

$$\rho \left\{ \frac{D^{(o)}}{Dt} \underset{\sim}{v}^{(o)} + \xi \left[\frac{D^{(o)} \underset{\sim}{v}^{(1)}}{Dt} + \frac{D^{(1)}}{Dt} \underset{\sim}{v}^{(o)} \right] + \xi^2 \frac{D^{(1)}}{Dt} \underset{\sim}{v}^{(1)} \right\}$$

$$= \rho \underset{\sim}{b} + \underset{\sim}{\nabla} \cdot \left[\underset{\sim}{S}^{(o)} + \xi \underset{\sim}{S}^{(1)} + \underset{\sim}{\mathcal{O}}(\xi) \right] \tag{3.6}$$

and

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v}^{(o)} + \xi \underset{\sim}{\nabla} \cdot \underset{\sim}{v}^{(1)} = 0 \tag{3.7}$$

with

$$\frac{D^{(o)}}{Dt} = \frac{\partial}{\partial t} + \underset{\sim}{v}^{(o)} \cdot \underset{\sim}{\nabla} \tag{3.8}$$

and

$$\frac{D^{(1)}}{Dt} = \underset{\sim}{v}^{(1)} \cdot \underset{\sim}{\nabla} \tag{3.9}$$

where, in accordance with Eq. (3.8) and the discussion of the material derivative presented in Section 1.3, it is to be noted that the operator $D^{(o)}/Dt$ defines time-differentiation along a pathline of the primary motion $\underset{\sim}{v}^{(o)}$.

Recalling now that the primary flow satisfies (3.1) and (3.2) identically, one has

$$\rho \left[\frac{D^{(o)}}{Dt} \underset{\sim}{v}^{(1)} + \frac{D^{(1)}}{Dt} \underset{\sim}{v}^{(o)} + \xi \frac{D^{(1)}}{Dt} \underset{\sim}{v}^{(1)} \right] = \underset{\sim}{\nabla} \cdot [\underset{\sim}{S}^{(1)} + \underset{\sim}{\mathcal{O}}(\xi)/\xi] \tag{3.10}$$

and

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v}^{(1)} = 0 \quad . \tag{3.11}$$

Equation (3.11) represents the required perturbation on the equation of continuity. Assuming that $D^{(1)}\underset{\sim}{v}^{(1)}/Dt$ is bounded at all points in the field of flow, we obtain our perturbation on the equation of momentum by taking the limit of (3.10) as $\xi \to 0$. We obtain thus

$$\rho \left[\frac{D^{(o)}}{Dt} \underset{\sim}{v}^{(1)} + \frac{D^{(1)}}{Dt} \underset{\sim}{v}^{(o)} \right] = \underset{\sim}{\nabla} \cdot \underset{\sim}{S}^{(1)} \quad . \tag{3.12}$$

3.2 DERIVATION OF THE STRESS PERTURBATION $\underset{\sim}{S}^{(1)}$

An expression for the stress perturbation $\underset{\sim}{S}^{(1)}$ is obtained by consideration of the constitutive equation for a simple fluid:

$$\underset{\sim}{S} + p\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\mathcal{F}}} [\underset{\sim}{H}_t(t')] \quad . \tag{1.31}$$

We first assume that the deformation history $\underset{\sim}{H}_t(t')$ is analytic in its time arguments and then expand this quantity in a Taylor series about the present time t:

$$\underset{\sim}{H}_t(t') = \sum_{n=0}^{\infty} \frac{(t'-t)^n}{n!} \left[\frac{D^n \underset{\sim}{H}_t(t')}{Dt'^n}\right]_{t'=t} \quad . \tag{3.13}$$

Combining (3.13) with (1.40), we then have

$$\underset{\sim}{H}_t(t') = \sum_{n=0}^{\infty} \frac{(t'-t)^n}{n!} \frac{\mathcal{D}^n \underset{\sim}{E}}{\mathcal{D}t^n} \quad . \tag{3.14}$$

For the velocity field represented by (3.3), Eq. (3.14) gives

$$\underset{\sim}{H}_t(t') = \underset{\sim}{H}_t^{(o)}(t') + \xi \underset{\sim}{H}_t^{(1)}(t') + \xi^2 \underset{\sim}{H}_t^{(2)}(t') + \ldots \tag{3.15}$$

with

$$\underset{\sim}{H}_t^{(o)}(t') = \sum_{n=0}^{\infty} \frac{(t'-t)^n}{n!} \frac{\mathcal{D}^{(o)n}}{\mathcal{D}t^n} \underset{\sim}{E}^{(o)} \tag{3.16}$$

$$\underset{\sim}{H}_t^{(1)}(t') = \sum_{n=0}^{\infty} \frac{(t'-t)^n}{n!} \frac{\mathcal{D}^{(o)n}}{\mathcal{D}t^n} \underset{\sim}{E}^{(1)} + \sum_{n=1}^{\infty} \sum_{m=0}^{n-1} \frac{(t'-t)^n}{n!} \frac{\mathcal{D}^{(o)m}}{\mathcal{D}t^m}$$

$$(x) \frac{\mathcal{D}^{(1)}}{\mathcal{D}t} \frac{\mathcal{D}^{(o)n-m-1}}{\mathcal{D}t^{n-m-1}} \underset{\sim}{E}^{(o)} \tag{3.17}$$

and so forth, where the $\underset{\sim}{H}^{(i)}$ are independent of ξ and where, for any tensor field $\underset{\sim}{A}$,

$$\frac{\mathcal{D}^{(o)}}{\mathcal{D}t} \underset{\sim}{A} = \frac{D^{(o)}}{Dt} \underset{\sim}{A} - \underset{\sim}{\Omega}^{(o)} \cdot \underset{\sim}{A} + \underset{\sim}{A} \cdot \underset{\sim}{\Omega}^{(o)} \tag{3.18}$$

and

$$\frac{\mathcal{D}^{(1)}}{\mathcal{D}t} \underset{\sim}{A} = \frac{D^{(1)}}{Dt} \underset{\sim}{A} - \underset{\sim}{\Omega}^{(1)} \cdot \underset{\sim}{A} + \underset{\sim}{A} \cdot \underset{\sim}{\Omega}^{(1)} \quad . \qquad (3.19)$$

The quantities $\underset{\sim}{E}^{(o)}$ and $\underset{\sim}{\Omega}^{(o)}$ refer, of course, respectively, to the rate-of-strain and vorticity tensors associated with the primary flow $\underset{\sim}{v}^{(o)}$, while $\underset{\sim}{E}^{(1)}$ and $\underset{\sim}{\Omega}^{(1)}$ are obtained from $\underset{\sim}{v}^{(1)}$. It is to be noted that $\underset{\sim}{H}_t^{(o)}(t')$ is the deformation history due to $\underset{\sim}{v}^{(o)}$ alone.

Let us now make the assumption of fading memory, so that the functional $\mathcal{F}$ may be Fréchet differentiated at the primary deformation history $\underset{\sim}{H}_t^{(o)}(t')$:

$$\underset{\sim}{S} + p\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\underset{\sim}{\mathcal{F}}}} [\underset{\sim}{H}_t^{(o)}(t')] + \underset{t'=-\infty}{\overset{t}{\underset{\sim}{\delta\mathcal{F}}}} [\underset{\sim}{H}_t^{(o)}(t'), \underset{\sim}{H}_t(t') - \underset{\sim}{H}_t^{(o)}(t')]$$

$$+ \quad \underset{\sim}{\mathcal{O}}(\|\underset{\sim}{H}_t(t') - \underset{\sim}{H}_t^{(o)}(t')\|) \qquad (3.20)$$

where, as before, $\underset{\sim}{\delta\mathcal{F}}$ is a linear and continuous functional in $[\underset{\sim}{H}_t(t') - \underset{\sim}{H}_t^{(o)}(t')]$ and where

$$\lim_{\|\underset{\sim}{H}_t(t') - \underset{\sim}{H}_t^{(o)}(t')\| \to 0} \left[\frac{\underset{\sim}{\mathcal{O}}(\|\underset{\sim}{H}_t(t') - \underset{\sim}{H}_t^{(o)}(t')\|)}{\|\underset{\sim}{H}_t(t') - \underset{\sim}{H}_t^{(o)}(t')\|} \right] = \underset{\sim}{0} \quad . \qquad (3.21)$$

The norm of $\underset{\sim}{H}_t(t') - \underset{\sim}{H}_t^{(o)}(t')$ is given by

$$\|\underset{\sim}{H}_t(t') - \underset{\sim}{H}_t^{(o)}(t')\| = \left[\int_{-\infty}^{t} \mathrm{tr}[\underset{\sim}{H}_t(t') - \underset{\sim}{H}_t^{(o)}(t')]^2 h^2(t-t')dt' \right]^{1/2} \qquad (3.22)$$

which, from (3.15), becomes

$$\|\underset{\sim}{H}_t(t')-\underset{\sim}{H}_t^{(o)}(t')\| = \xi\left[\int_{-\infty}^{t} tr[\underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{H}_t^{(1)}(t') + \xi(\underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{H}_t^{(2)}(t')+\underset{\sim}{H}_t^{(2)}(t')\cdot\underset{\sim}{H}_t^{(1)}(t')) + \ldots]h^2(t-t')dt'\right]^{1/2}$$

$$= \xi\left[\|\underset{\sim}{H}_t^{(1)}(t')\|^2+2\xi\int_{-\infty}^{t} tr[\underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{H}_t^{(2)}(t')] \,(x)\, h^2(t-t')dt'+\ldots\right]^{1/2} . \quad (3.23)$$

If it is now assumed that the weighting function h(t-t') and the functions $\underset{\sim}{H}_t^{(n)}(t')$ are such that all the integrals appearing in (3.23) are bounded, we have

$$\lim_{\xi\to 0} \frac{\|\underset{\sim}{H}_t(t')-\underset{\sim}{H}_t^{(o)}(t')\|}{\xi} = \|\underset{\sim}{H}_t^{(1)}(t')\| . \quad (3.24)$$

Let us next substitute Eq. (3.15) into (3.20) and recall that $\delta\underset{\sim}{\mathcal{F}}$ is linear in $\underset{\sim}{H}_t(t')-\underset{\sim}{H}_t^{(o)}(t')$. One may then write

$$\underset{\sim}{S} + p\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\mathcal{F}}}[\underset{\sim}{H}_t^{(o)}(t')] + \xi \underset{t'=-\infty}{\overset{t}{\delta\underset{\sim}{\mathcal{F}}}}[\underset{\sim}{H}_t^{(o)}(t'),\underset{\sim}{H}_t^{(1)}(t')]$$
$$+ \sum_{n=2}^{\infty}\xi^n \underset{t'=-\infty}{\overset{t}{\delta\underset{\sim}{\mathcal{F}}}}[\underset{\sim}{H}_t^{(o)}(t'),\underset{\sim}{H}_t^{(n)}(t')]+\underset{\sim}{\mathcal{O}}(\|\underset{\sim}{H}_t(t')-\underset{\sim}{H}_t^{(o)}(t')\|) . \quad (3.25)$$

But, from (3.24), we have that (for n ≥ 2)

$$\lim_{\|\underset{\sim}{H}_t(t')-\underset{\sim}{H}_t^{(o)}(t')\|\to 0}\left[\frac{\xi^n \underset{t'=-\infty}{\overset{t}{\delta\underset{\sim}{\mathcal{F}}}}[\underset{\sim}{H}_t^{(o)}(t'),\underset{\sim}{H}_t^{(n)}(t')]}{\|\underset{\sim}{H}_t(t')-\underset{\sim}{H}_t^{(o)}(t')\|}\right]$$
$$= \lim_{\xi\to 0}\left[\frac{\xi^{n-1} \underset{t'=-\infty}{\overset{t}{\delta\underset{\sim}{\mathcal{F}}}}[\underset{\sim}{H}_t^{(o)}(t'),\underset{\sim}{H}_t^{(n)}(t')]}{\|\underset{\sim}{H}_t^{(1)}(t')\|}\right] = \underset{\sim}{0} \quad (3.26)$$

Therefore, the summation term on the right-hand side of Eq. (3.25) may be grouped together with the term $\mathcal{O}(\|\underset{\sim}{H}_t(t')-\underset{\sim}{H}_t^{(o)}(t')\|$ to yield

$$\underset{\sim}{S} + p\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\underset{\sim}{\mathcal{F}}}} [\underset{\sim}{H}_t^{(o)}(t')] + \epsilon \underset{t'=-\infty}{\overset{t}{\delta\underset{\sim}{\mathcal{F}}}} [\underset{\sim}{H}_t^{(o)}(t'),\underset{\sim}{H}_t^{(1)}(t')] + \underset{\sim}{\mathcal{O}}'(\|\underset{\sim}{H}_t(t')-\underset{\sim}{H}_t^{(o)}(t')\|) \tag{3.27}$$

where $\mathcal{E}'$ also satisfies an equation of the form (3.21). Comparing (3.27) with (3.4), we see finally that $\underset{\sim}{S}^{(1)}$ is expressible as

$$\underset{\sim}{S}^{(1)} + p^{(1)}\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\delta\underset{\sim}{\mathcal{F}}}} [\underset{\sim}{H}_t^{(o)}(t'),\underset{\sim}{H}_t^{(1)}(t')] \tag{3.28}$$

where $p^{(1)}$ is a pressure.

Owing to Eq. (1.44), the relation

$$\underset{t'=-\infty}{\overset{t}{\delta\underset{\sim}{\mathcal{F}}}} [\underset{\sim}{Q}\cdot\underset{\sim}{H}_t^{(o)}(t')\cdot\underset{\sim}{Q}^{\dagger},\underset{\sim}{Q}\cdot\underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{Q}^{\dagger}] = \underset{\sim}{Q}\cdot\left[\underset{t'=-\infty}{\overset{t}{\delta\underset{\sim}{\mathcal{F}}}} [\underset{\sim}{H}_t^{(o)}(t'),\underset{\sim}{H}_t^{(1)}(t')]\right]\cdot\underset{\sim}{Q}^{\dagger} \tag{3.29}$$

must also hold for all orthogonal tensors $\underset{\sim}{Q}$ and for all functions $\underset{\sim}{H}_t^{(1)}(t')$ and $\underset{\sim}{H}_t^{(o)}(t')$ which are consistent with (3.28).

3.3 DERIVATION OF $\underset{\sim}{H}_t^{(1)}(t')$ IN CLOSED FORM

We now demonstrate that, for all viscometric primary flows, Eq. (3.17), our defining relation for $\underset{\sim}{H}_t^{(1)}(t')$, may be written in a convenient closed form.

Let us begin by defining a "primary rotation tensor" $\underset{\sim}{R}_t^{(o)}(t')$ which satisfies the equation

$$\frac{D^{(o)}\underset{\sim}{R}_t^{(o)}(t')}{Dt'} = \underset{\sim}{\Omega}^{(o)}(t')\cdot\underset{\sim}{R}_t^{(o)}(t') \tag{3.30}$$

subject to

$$\underset{\sim}{R}_t^{(o)}(t) = \underset{\sim}{I} \quad . \tag{3.31}$$

As a consequence of (3.30) and (3.31), it is easily verified that, for any tensor field $\underset{\sim}{A}(t)$, the following relation holds:

$$\frac{D^{(o)^n}}{Dt'^n}[\underset{\sim}{R}_t^{(o)\dagger}(t')\cdot\underset{\sim}{A}(t')\cdot\underset{\sim}{R}_t^{(o)}(t')] = \underset{\sim}{R}_t^{(o)\dagger}(t')\cdot\left[\frac{\mathcal{D}^{(o)^n}}{\mathcal{D}t'^n}\underset{\sim}{A}(t')\right]\cdot\underset{\sim}{R}_t^{(o)}(t') \tag{3.32}$$

We now consider the first (single) summation in (3.17). By employing (3.31) and (3.32), we have

$$\sum_{n=0}^{\infty}\frac{(t'-t)^n}{n!}\frac{\mathcal{D}^{(o)^n}}{\mathcal{D}t^n}\underset{\sim}{E}^{(1)} = \sum_{n=0}^{\infty}\frac{(t'-t)^n}{n!}\left[\frac{D^{(o)^n}}{Dt'^n}\right.$$

$$\left.(x)\ [\underset{\sim}{R}_t^{(o)\dagger}(t')\cdot\underset{\sim}{E}^{(1)}(t')\cdot\underset{\sim}{R}_t^{(o)}(t')]\right]_{t'=t} \quad . \tag{3.33}$$

But this expression represents the Taylor series expansion for $\underset{\sim}{R}_t^{(o)\dagger}(t')\cdot\underset{\sim}{E}^{(1)}(t')\cdot\underset{\sim}{R}_t^{(o)}(t')$ about $t'=t$. Therefore, for any primary flow, one may write

$$\underset{\sim}{H}_t^{(1)}(t') = \underset{\sim}{R}_t^{(o)\dagger}(t')\cdot\underset{\sim}{E}^{(1)}(t')\cdot\underset{\sim}{R}_t^{(o)}(t')$$

$$+\sum_{n=1}^{\infty}\sum_{m=0}^{n-1}\frac{(t'-t)^n}{n!}\frac{\mathcal{D}^{(o)^m}}{\mathcal{D}t^m}\frac{\mathcal{D}^{(1)}}{\mathcal{D}t}\frac{\mathcal{D}^{(o)^{n-m-1}}}{\mathcal{D}t^{n-m-1}}\underset{\sim}{E}^{(o)} \tag{3.34}$$

where the quantity $\underset{\sim}{R}_t^{(o)\dagger}(t')\cdot\underset{\sim}{E}^{(1)}(t')\cdot\underset{\sim}{R}_t^{(o)}(t')$ must be evaluated at positions on a pathline of the primary motion since, as has been pointed out earlier in Section 3.1, the operator $D^{(o)}/Dt'$ (appearing in Eqs. (3.30) and (3.33)) defines differentiation along a primary line of flow only.

Now, it has been shown in Section 1.4.5 that the two relations

$$\frac{\mathcal{D}^{(o)^2}}{\mathcal{D}t^2}\,\underset{\sim}{E}^{(o)} + \gamma^{(o)^2}\,\underset{\sim}{E}^{(o)} = \underset{\sim}{0} \tag{3.35}$$

and

$$\frac{D^{(o)}}{Dt}\,\gamma^{(o)} = 0 \tag{3.36}$$

apply to all viscometric primary motions; as will soon be seen, Eq. (3.34) can be simplified further by making use of these. Defining the tensor quantities

$$\underset{\sim}{K} \equiv \frac{\mathcal{D}^{(1)}}{\mathcal{D}t}\,\underset{\sim}{E}^{(o)}(t) \tag{3.37}$$

$$\underset{\sim}{L} \equiv \frac{\mathcal{D}^{(1)}}{\mathcal{D}t}\,\frac{\mathcal{D}^{(o)}}{\mathcal{D}t}\,\underset{\sim}{E}^{(o)}(t) \tag{3.38}$$

$$\underset{\sim}{M} \equiv \underset{\sim}{E}^{(o)}(t)\left[\frac{D^{(1)}\gamma^{(o)^2}}{Dt}\right] \tag{3.39}$$

$$\underset{\sim}{N} \equiv \left[\frac{\mathcal{D}^{(o)}\underset{\sim}{E}^{(o)}(t)}{\mathcal{D}t}\right]\left[\frac{D^{(1)}\gamma^{(o)^2}}{Dt}\right] \tag{3.40}$$

and making use of Eqs. (3.35) and (3.36), one finds that the sums involving m in Eq. (3.34) may be written as

<u>n=1</u>: $\underset{\sim}{K}$

<u>n=2</u>: $\frac{\mathcal{D}^{(o)}}{\mathcal{D}t}\underset{\sim}{K} + \underset{\sim}{L}$

<u>n=3</u>: $\left[\frac{\mathcal{D}^{(o)^2}}{\mathcal{D}t^2} - \gamma^{(o)^2}\right]\underset{\sim}{K} + \frac{\mathcal{D}^{(o)}}{\mathcal{D}t}\underset{\sim}{L} - \underset{\sim}{M}$

n=4: $\left[\frac{\mathcal{D}(o)^3}{\mathcal{D}t^3} - \gamma(o)^2 \frac{\mathcal{D}(o)}{\mathcal{D}t}\right]\underset{\sim}{K} + \left[\frac{\mathcal{D}(o)^2}{\mathcal{D}t^2} - \gamma(o)^2\right]\underset{\sim}{L} - \frac{\mathcal{D}(o)}{\mathcal{D}t}\underset{\sim}{M} - \underset{\sim}{N}$

n=5: $\left[\frac{\mathcal{D}(o)^4}{\mathcal{D}t^4} - \gamma(o)^2 \frac{\mathcal{D}(o)^2}{\mathcal{D}t^2} + \gamma(o)^4\right]\underset{\sim}{K} + \left[\frac{\mathcal{D}(o)^3}{\mathcal{D}t^3} - \gamma(o)^2 \frac{\mathcal{D}(o)}{\mathcal{D}t}\right]\underset{\sim}{L}$

$- \left[\frac{\mathcal{D}(o)^2}{\mathcal{D}t^2} - 2\gamma(o)^2\right]\underset{\sim}{M} - \frac{\mathcal{D}(o)}{\mathcal{D}t}\underset{\sim}{N}$

n=6: $\left[\frac{\mathcal{D}(o)^5}{\mathcal{D}t^5} - \gamma(o)^2 \frac{\mathcal{D}(o)^3}{\mathcal{D}t^3} + \gamma(o)^4 \frac{\mathcal{D}(o)}{\mathcal{D}t}\right]\underset{\sim}{K} + \left[\frac{\mathcal{D}(o)^4}{\mathcal{D}t^4} - \gamma(o)^2 \frac{\mathcal{D}(o)^2}{\mathcal{D}t^2}\right.$

$\left. + \gamma(o)^4\right]\underset{\sim}{L}$

$- \left[\frac{\mathcal{D}(o)^3}{\mathcal{D}t^3} - 2\gamma(o)^2 \frac{\mathcal{D}(o)}{\mathcal{D}t}\right]\underset{\sim}{M} - \left[\frac{\mathcal{D}(o)^2}{\mathcal{D}t^2} - 2\gamma(o)^2\right]\underset{\sim}{N}$

n=7: $\left[\frac{\mathcal{D}(o)^6}{\mathcal{D}t^6} - \gamma(o)^2 \frac{\mathcal{D}(o)^4}{\mathcal{D}t^4} + \gamma(o)^4 \frac{\mathcal{D}(o)^2}{\mathcal{D}t^2} - \gamma(o)^6\right]\underset{\sim}{K}$

$+ \left[\frac{\mathcal{D}(o)^5}{\mathcal{D}t^5} - \gamma(o)^2 \frac{\mathcal{D}(o)^3}{\mathcal{D}t^3} + \gamma(o)^4 \frac{\mathcal{D}(o)}{\mathcal{D}t}\right]\underset{\sim}{L}$

$- \left[\frac{\mathcal{D}(o)^4}{\mathcal{D}t^4} - 2\gamma(o)^2 \frac{\mathcal{D}(o)^2}{\mathcal{D}t^2} + 3\gamma(o)^4\right]\underset{\sim}{M} - \left[\frac{\mathcal{D}(o)^3}{\mathcal{D}t^3} - 2\gamma(o)^2 \frac{\mathcal{D}(o)}{\mathcal{D}t}\right]\underset{\sim}{N}$

and so forth. We turn attention here to only those terms in (3.34) which involve the tensor $\underset{\sim}{K}$. The sum of all such terms $\hat{\hat{\underset{\sim}{K}}}_t(t')$, say, can readily be shown to be represented by

$$\hat{\hat{\underset{\sim}{K}}}_t(t') = \hat{\underset{\sim}{K}}_t(t') - \gamma(o)^2 \int_t^{t'}\int_t^{t_1} \hat{\underset{\sim}{K}}_t(t_2)dt_2dt_1 + \gamma(o)^4 \int_t^{t'}\int_t^{t_1}\int_t^{t_2}\int_t^{t_3} \hat{\underset{\sim}{K}}_t(t_4)dt_4dt_3dt_2dt_1 + \ldots \tag{3.41}$$

where $\hat{\underset{\sim}{K}}_t(t')$ is defined as

$$\hat{\underset{\sim}{K}}_t(t') = \sum_{n=0}^{\infty} \frac{(t'-t)^{n+1}}{(n+1)!} \frac{\mathcal{D}(o)^n}{\mathcal{D}t^n}\underset{\sim}{K} . \tag{3.42}$$

Let us now make the substitution s = t'-t into (3.41) and (3.42). We obtain for them, respectively,

$$\hat{\hat{\underset{\sim}{K}}}_t(t+s) = \hat{\underset{\sim}{K}}_t(t+s) - \gamma(o)^2 \int_0^s \int_0^{s_1} \hat{\underset{\sim}{K}}_t(t+s_2)ds_2ds_1$$

$$+ \gamma(o)^4 \int_0^s \int_0^{s_1} \int_0^{s_2} \int_0^{s_3} \hat{\underset{\sim}{K}}_t(t+s_4)ds_4ds_3ds_2ds_1 + \ldots \qquad (3.43)$$

and

$$\hat{\underset{\sim}{K}}_t(t+s) = \sum_{n=0}^{\infty} \frac{s^{n+1}}{(n+1)!} \frac{\partial(o)^n}{\partial t^n} \underset{\sim}{K} . \qquad (3.44)$$

Next, taking the Laplace transform of $\hat{\hat{\underset{\sim}{K}}}_t(t+s)$ with respect to the new variable s, we have

$$\mathcal{L}[\hat{\hat{\underset{\sim}{K}}}_t(t+s)] = \left[1 - \frac{\gamma(o)^2}{q^2} + \frac{\gamma(o)^4}{q^4} - \ldots\right] \mathcal{L}[\hat{\underset{\sim}{K}}_t(t+s)]$$

$$= \frac{q^2}{q^2+\gamma(o)^2} \mathcal{L}[\hat{\underset{\sim}{K}}_t(t+s)] \qquad (3.45)$$

where q stands for Laplace's operator. The derivative property of the Laplace transform then yields

$$\mathcal{L}\left[\frac{d}{ds} \hat{\underset{\sim}{K}}_t(t+s)\right] = q\mathcal{L}[\hat{\underset{\sim}{K}}_t(t+s)] - \hat{\underset{\sim}{K}}_t(t+s)\Big]_{s=0} . \qquad (3.46)$$

But since, from (3.44), $\hat{\underset{\sim}{K}}_t(t+s)]_{s=0} = \underset{\sim}{0}$, Eq. (3.45) becomes

$$\mathcal{L}[\hat{\hat{\underset{\sim}{K}}}_t(t+s)] = \frac{q}{q^2+\gamma(o)^2} \mathcal{L}\left[\frac{d}{ds} \hat{\underset{\sim}{K}}_t(t+s)\right] . \qquad (3.47)$$

This expression may now be inverted to yield a convolution integral:

$$\hat{\hat{K}}_t(t+s) = \int_0^s \left[\frac{d}{ds_1} \hat{K}_t(t+s_1)\right] \cos \gamma^{(o)}(s-s_1)ds_1 \qquad (3.48)$$

or equivalently

$$\hat{\hat{K}}_t(t') = -\int_{t'}^{t} \left[\frac{d}{dt_1} \hat{K}_t(t_1)\right] \cos \gamma^{(o)}(t'-t_1)dt_1 \qquad (3.49)$$

where

$$\left[\frac{d}{dt'} \hat{K}_t(t')\right] = \sum_{n=0}^{\infty} \frac{(t'-t)^n}{n!} \frac{\partial^{(o)n}}{\partial t^n} K \ . \qquad (3.50)$$

Now, once again employing the "primary rotation tensor" $R_t^{(o)}(t')$ (as has been done in Eq. (3.33)), we obtain from (3.50),

$$\left[\frac{d}{dt'} \hat{K}_t(t')\right] = R_t^{(o)\dagger}(t')\cdot\left[\frac{\mathcal{D}^{(1)}}{\partial t'} E^{(o)}(t')\right]\cdot R_t^{(o)}(t') \ . \qquad (3.51)$$

Finally, then,

$$\hat{\hat{K}}_t(t') = -\int_{t'}^{t} R_t^{(o)\dagger}(t_1)\cdot\left[\frac{\mathcal{D}^{(1)}}{\partial t_1} E^{(o)}(t_1)\right]\cdot R_t^{(o)}(t_1)\cos \gamma^{(o)}(t'-t_1)dt_1. \qquad (3.52)$$

The same type of analysis as presented above may also be applied to the series of the terms in (3.34) involving the tensors L, M, and N. One ultimately arrives at an expression for $H_t^{(1)}(t')$ which holds in all cases where the primary motion $v^{(o)}$ is viscometric. The details of these derivations are left for Appendix A, but the final result is

$$\underset{\sim}{H}_t^{(1)}(t') = \underset{\sim}{R}_t^{(o)\dagger}(t')\cdot\underset{\sim}{E}^{(1)}(t')\cdot\underset{\sim}{R}_t^{(o)}(t') - \int_{t'}^{t}\underset{\sim}{R}_t^{(o)\dagger}(t_1)\cdot\left[\frac{\mathcal{D}^{(1)}}{\mathcal{D}t_1}\underset{\sim}{E}^{(o)}(t_1)\right]$$

$$\cdot\underset{\sim}{R}_t^{(o)}(t')\cos\gamma^{(o)}(t'-t_1)dt_1$$

$$+\frac{1}{\gamma^{(o)}}\int_{t'}^{t}\underset{\sim}{R}_t^{(o)\dagger}(t_1)\cdot\left[\frac{\mathcal{D}^{(1)}}{\mathcal{D}t_1}\frac{\mathcal{D}^{(o)}}{\mathcal{D}t_1}\underset{\sim}{E}^{(o)}(t_1)\right]\cdot\underset{\sim}{R}_t^{(o)}(t_1)\sin\gamma^{(o)}$$

$$\cdot(t_1-t')dt_1$$

$$+\int_{t'}^{t}\underset{\sim}{H}_t^{(o)}(t_1)\left[\frac{D^{(1)}\gamma^{(o)}}{Dt_1}\right][(t_1-t')\sin\gamma^{(o)}(t_1-t')]dt_1$$

$$-\frac{1}{\gamma^{(o)2}}\int_{t'}^{t}\left[\frac{D^{(o)}}{Dt_1}\underset{\sim}{H}_t^{(o)}(t_1)\right]\left[\frac{D^{(1)}\gamma^{(o)}}{Dt_1}\right][\sin\gamma^{(o)}(t_1-t')$$

$$-\gamma^{(o)}(t_1-t')\cos\gamma^{(o)}(t_1-t')]dt_1 \qquad (3.53)$$

which, owing to Eq. (1.54), may also be written in the more compact form

$$\underset{\sim}{H}_t^{(1)}(t') = \underset{\sim}{R}_t^{(o)\dagger}(t')\cdot\underset{\sim}{E}^{(1)}(t')\cdot\underset{\sim}{R}_t^{(o)}(t') - \int_{t'}^{t}\underset{\sim}{R}_t^{(o)\dagger}(t_1)$$

$$\cdot\left[\frac{\mathcal{D}^{(1)}}{\mathcal{D}t_1}\underset{\sim}{H}_{t_1}^{(o)}(t')\right]\cdot\underset{\sim}{R}_t^{(o)}(t_1)dt_1 \quad . \qquad (3.54)$$

3.4 VISCOMETRIC PERTURBATIONS

In the remaining two sections of this chapter, we shall consider a class of motions which will be called "viscometric perturbations." These are flows for which the primary velocity field $\underset{\sim}{v}^{(o)}$ is viscometric and, in addition, the new total motion $\underset{\sim}{v}^{(o)}+\xi\underset{\sim}{v}^{(1)}$ may also be regarded as such. The present section will be devoted to investigation of the kinematics of these motions, while in Section 3.5 we shall examine their rheology.

The following set of equations has been listed in Section 1.4.5 as describing the kinematics of all viscometric flows:

$$\underset{\sim}{H}_t^3(t') = \frac{\gamma^2}{4}\underset{\sim}{H}_t(t') \tag{1.50}$$

$$\frac{D^2}{Dt'^2}\underset{\sim}{H}_t(t') + \gamma^2\underset{\sim}{H}_t(t') = \underset{\sim}{0} \tag{1.52}$$

$$\frac{D}{Dt'}[\underset{\sim}{H}_t(t')]^2 = \underset{\sim}{0} \tag{1.53}$$

and

$$\frac{D\gamma}{Dt} = 0 \quad . \tag{1.49}$$

Substituting (3.3) and (3.15) into these, we have

$$[\underset{\sim}{H}_t^{(o)}(t')]^3 + \xi\Big[[\underset{\sim}{H}_t^{(o)}(t')]^2\cdot\underset{\sim}{H}_t^{(1)}(t') + \underset{\sim}{H}_t^{(o)}(t')\cdot\underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{H}_t^{(o)}(t') + \underset{\sim}{H}_t^{(1)}(t')\cdot[\underset{\sim}{H}_t^{(o)}(t')]^2\Big]+\ldots$$
$$= \frac{\gamma^{(o)2}}{4}\underset{\sim}{H}_t^{(o)}(t') + \xi\left[\frac{2\gamma^{(1)}\gamma^{(o)}\underset{\sim}{H}_t^{(o)}(t')+\gamma^{(o)2}\underset{\sim}{H}_t^{(1)}(t')}{4}\right]+\ldots \tag{3.55}$$

$$\frac{D^{(o)2}}{Dt'^2}\underset{\sim}{H}_t^{(o)}(t') + \gamma^{(o)2}\underset{\sim}{H}_t^{(o)}(t') + \xi\Bigg[\left(\frac{D^{(1)}}{Dt'}\frac{D^{(o)}}{Dt'} + \frac{D^{(o)}}{Dt'}\frac{D^{(1)}}{Dt'}\right)\underset{\sim}{H}_t^{(o)}(t')$$
$$+ \frac{D^{(o)2}}{Dt'^2}\underset{\sim}{H}_t^{(1)}(t') + 2\gamma^{(1)}\gamma^{(o)}\underset{\sim}{H}_t^{(o)}(t') + \gamma^{(o)2}\underset{\sim}{H}_t^{(1)}(t')\Bigg]$$
$$+ \ldots = \underset{\sim}{0} \tag{3.56}$$

$$\frac{D^{(o)}}{Dt'}[\underset{\sim}{H}_t^{(o)}(t')]^2 + \xi\left[\frac{D^{(1)}}{Dt'}[\underset{\sim}{H}_t^{(o)}(t')]^2 + \frac{D^{(o)}}{Dt'}[\underset{\sim}{H}_t^{(o)}(t')\cdot\underset{\sim}{H}_t^{(1)}(t') + \underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{H}_t^{(o)}(t')]\right]+\ldots = \underset{\sim}{0} \tag{3.57}$$

and

$$\frac{D^{(o)}}{Dt}\gamma^{(o)} + \xi\left[\frac{D^{(1)}}{Dt}\gamma^{(o)} + \frac{D^{(o)}}{Dt}\gamma^{(1)}\right] + \ldots = 0 \qquad (3.58)$$

where

$$\gamma^{(1)} = \frac{2}{\gamma^{(o)}}\,\mathrm{tr}[\underset{\sim}{E}^{(o)}\cdot\underset{\sim}{E}^{(1)}] \qquad (3.59)$$

and where, of course, the symbol ... indicates terms of order ξ^2 and higher.

Recalling now that the primary flow $\underset{\sim}{v}^{(o)}$ is viscometric, we may drop those terms involving only primary quantities (those with no coefficient involving ξ) from (3.55)-(3.58). By dividing the resulting expressions by ξ and taking the limit as ξ approaches zero, one then obtains

$$[\underset{\sim}{H}_t^{(o)}(t')]^2\cdot\underset{\sim}{H}_t^{(1)}(t')+\underset{\sim}{H}_t^{(o)}(t')\cdot\underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{H}_t^{(o)}(t')+\underset{\sim}{H}_t^{(1)}(t')\cdot[\underset{\sim}{H}_t^{(o)}(t')]^2$$

$$= \frac{[2\gamma^{(1)}\gamma^{(o)}\underset{\sim}{H}_t^{(o)}(t')+\gamma^{(o)^2}\underset{\sim}{H}_t^{(1)}(t')]}{4} \qquad (3.60)$$

$$\left[\frac{D^{(o)^2}}{Dt'^2} + \gamma^{(o)^2}\right]\underset{\sim}{H}_t^{(1)}(t') + 2\gamma^{(1)}\gamma^{(o)}\underset{\sim}{H}_t^{(o)}(t')$$

$$+ \left(\frac{D^{(1)}}{Dt'}\frac{D^{(o)}}{Dt'} + \frac{D^{(o)}}{Dt'}\frac{D^{(1)}}{Dt'}\right)\underset{\sim}{H}_t^{(o)}(t') = \underset{\sim}{0} \qquad (3.61)$$

$$\frac{D^{(1)}}{Dt'}[\underset{\sim}{H}_t^{(o)}(t')]^2 - \frac{D^{(o)}}{Dt'}[\underset{\sim}{H}_t^{(o)}(t')\cdot\underset{\sim}{H}_t^{(1)}(t')+\underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{H}_t^{(o)}(t')] = \underset{\sim}{0} \qquad (3.62)$$

$$\frac{D^{(1)}\gamma^{(o)}}{Dt} + \frac{D^{(o)}\gamma^{(1)}}{Dt} = 0 \;. \qquad (3.63)$$

Thus, a viscometric perturbation will be defined mathematically as a flow of the form $\underset{\sim}{v} = \underset{\sim}{v}^{(o)}+\xi\underset{\sim}{v}^{(1)}$ for which

(1) $\underset{\sim}{H}_t^{(o)}(t')$ satisfies (1.49)-(1.53),

(2) $\xi \to 0$, and

(3) Eqs. (3.60)-(3.63) are satisfied by $\underset{\sim}{H}_t^{(1)}(t')$.

3.5 RHEOLOGY OF VISCOMETRIC PERTURBATIONS

We have proven in Section 3.2 that the stress perturbation $\underset{\sim}{S}^{(1)}$, due to a small perturbation on any primary flow, will be given by

$$\underset{\sim}{S}^{(1)} + p^{(1)}\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\underset{\sim}{\delta \mathcal{F}}}} [\underset{\sim}{H}_t^{(o)}(t'),\underset{\sim}{H}_t^{(1)}(t')] \tag{3.28}$$

where $\underset{\sim}{\delta\mathcal{F}}$ is a linear functional in $\underset{\sim}{H}_t^{(1)}(t')$. Let us now investigate the particularly simple form which this functional assumes whenever the velocity field $\underset{\sim}{v}^{(o)}+\xi\underset{\sim}{v}^{(1)}$ corresponds to a viscometric perturbation.

For all viscometric flows, it was shown in Section 1.4.5 that the stress tensor $\underset{\sim}{S}$ may be written as

$$\underset{\sim}{S} + p\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\underset{\sim}{\mathcal{F}}}} [\underset{\sim}{H}_t(t')] = 2\eta\underset{\sim}{H}_t(t')]_{t'=t} + \frac{2(\sigma_1+\sigma_2)}{\gamma^2}\underset{\sim}{H}_t^2(t')]_{t'=t}$$

$$+ \frac{(\sigma_2-\sigma_1)}{\gamma^2}\left[\frac{D}{Dt'}\underset{\sim}{H}_t(t')\right]_{t'=t} . \tag{1.55}$$

After substituting Eq. (3.15) into this expression, subtracting those portions of the resulting equation due to $\underset{\sim}{v}^{(o)}$ alone, dividing by ξ, and taking the limit as $\xi \to 0$, one finally obtains

$$\underset{\sim}{S}^{(1)} + p^{(1)}\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\delta \mathcal{F}}} [\underset{\sim}{H}_t^{(o)}(t'),\underset{\sim}{H}_t^{(1)}(t')] = 2\eta\underset{\sim}{H}_t^{(1)}(t')]_{t'=t}$$

$$+ \frac{2(\sigma_1+\sigma_2)}{\gamma^{(o)2}} [\underset{\sim}{H}_t^{(o)}(t')\cdot\underset{\sim}{H}_t^{(1)}(t')+\underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{H}_t^{(o)}(t')]_{t'=t}$$

$$+ \frac{(\sigma_2-\sigma_1)}{\gamma^{(o)2}} \left[\frac{D^{(o)}}{Dt'} \underset{\sim}{H}_t^{(1)}(t')\right]_{t'=t}$$

$$+ \frac{(\sigma_2-\sigma_1)}{\gamma^{(o)2}} \left[\frac{D^{(1)}}{Dt'} \underset{\sim}{H}_t^{(o)}(t')\right]_{t'=t} + 2\gamma^{(1)}\eta'\underset{\sim}{H}_t^{(o)}(t')]_{t'=t}$$

$$+ \frac{2\gamma^{(1)}}{\gamma^{(o)3}} [\gamma^{(o)}(\sigma_1'+\sigma_2') - 2(\sigma_1+\sigma_2)][\underset{\sim}{H}_t^{(o)}(t')]^2_{t'=t}$$

$$+ \frac{\gamma^{(1)}}{\gamma^{(o)3}} [\gamma^{(o)}(\sigma_2'-\sigma_1')-2(\sigma_2-\sigma_1)] \left[\frac{D^{(o)}}{Dt'} \underset{\sim}{H}_t^{(o)}(t')\right]_{t'=t} \quad (3.64)$$

where primes denote differentiation with respect to γ and where η, σ_1, σ_2, η', σ_1', and σ_2' are all evaluated at $\gamma = \gamma^{(o)}$. Hence, we see that, whenever $\underset{\sim}{v}^{(o)}+\xi\underset{\sim}{v}^{(1)}$ is representative of a viscometric perturbation, the stress perturbation $\underset{\sim}{S}^{(1)}$ may be obtained simply by a direct substitution into Eq. (3.64).

Let us now consider viscometric perturbations for which the derivatives $D^{(1)}/Dt$ of all primary quantities are zero. In such instances, Eq. (3.60) remains unchanged, but Eqs. (3.61)-(3.63) become

$$\left[\frac{D^{(o)2}}{Dt'^2} + \gamma^{(o)2}\right]\underset{\sim}{H}_t^{(1)}(t') + 2\gamma^{(o)}\gamma^{(1)}\underset{\sim}{H}_t^{(o)}(t') = \underset{\sim}{0} \quad (3.65)$$

$$\frac{D^{(o)}}{Dt'} [\underset{\sim}{H}_t^{(o)}(t')\cdot\underset{\sim}{H}_t^{(1)}(t') + \underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{H}_t^{(o)}(t')] = \underset{\sim}{0} \quad (3.66)$$

and

$$\frac{D^{(o)}\gamma^{(1)}}{Dt} = 0 \quad (3.67)$$

Since terms involving $D^{(1)}/Dt$ no longer appear in (3.65)-(3.67), the corresponding stress perturbation is found from (3.64) by dropping the term

$$\frac{(\sigma_2-\sigma_1)}{\gamma^{(o)2}}\left[\frac{D^{(1)}}{Dt'}\,\underset{\sim}{H}_t^{(o)}(t')\right]_{t'=t} \quad .$$

We then have

$$\underset{\sim}{S}^{(1)} + p^{(1)}\underset{\sim}{I} = \underset{t'=-\infty}{\overset{t}{\underset{\sim}{\delta \mathcal{F}}}}\,[\underset{\sim}{H}_t^{(o)}(t'),\underset{\sim}{H}_t^{(1)}(t')] = 2\eta\underset{\sim}{H}_t^{(1)}(t')]_{t'=t}$$
$$+ \frac{2(\sigma_1+\sigma_2)}{\gamma^{(o)2}}\,[\underset{\sim}{H}_t^{(o)}(t')\cdot\underset{\sim}{H}_t^{(1)}(t') + \underset{\sim}{H}_t^{(1)}(t')\cdot\underset{\sim}{H}_t^{(o)}(t')]_{t'=t}$$
$$+ \frac{(\sigma_2-\sigma_1)}{\gamma^{(o)2}}\left[\frac{D^{(o)}}{Dt'}\,\underset{\sim}{H}_t^{(1)}(t')\right]_{t'=t} + \frac{2\gamma^{(1)}}{\gamma^{(o)3}}\,[\gamma^{(o)}(\sigma_1'+\sigma_2')$$
$$- 2(\sigma_1+\sigma_2)][\underset{\sim}{H}_t^{(o)}(t')]^2_{t'=t}$$
$$+ \frac{\gamma^{(1)}}{\gamma^{(o)3}}\,[\gamma^{(o)}(\sigma_2'-\sigma_1')-2(\sigma_2-\sigma_1)]\left[\frac{D^{(o)}}{Dt'}\,\underset{\sim}{H}_t^{(o)}(t')\right]_{t'=t}$$
$$+ 2\gamma^{(1)}\eta'\underset{\sim}{H}_t^{(o)}(t')]_{t'=t} \quad . \tag{3.68}$$

Thus, (3.68) will yield the appropriate stress perturbation whenever $\underset{\sim}{v}^{(o)}$ is viscometric and, in addition, Eqs. (3.60) and (3.65)-(3.67) hold for the perturbed flow.

We next suppose that there exists a perturbation on a viscometric deformation history which may be written in the following form:

$$\underset{\sim}{H}_t^{(1)}(t') = [\underset{\sim}{H}_t^{(1)}(t')]_v + [\underset{\sim}{H}_t^{(1)}(t')]_{nv} \tag{3.69}$$

where $[\underset{\sim}{H}_t^{(1)}(t')]_v$ satisfies (3.60)-(3.63) while $[\underset{\sim}{H}_t^{(1)}(t')]_{nv}$ does not; accordingly, the subscripts v and nv will refer here to "viscometric" and "nonviscometric" respectively. Since the functional $\delta \mathfrak{F}$ of Eq. (3.28) is linear in $H_t^{(1)}(t')$, the quantities $[H_t^{(1)}(t')]_v$ and $[H_t^{(1)}(t')]_{nv}$ may be treated separately, in so far as their respective contributions to the total stress perturbation are concerned. It therefore follows immediately that the stress perturbation due to $[H_t^{(1)}(t')]_v$ is obtained by substitution of $[H_t^{(1)}(t')]_v$ for $H_t^{(1)}(t')$ in (3.64); similarly, if $[H_t^{(1)}(t')]_v$ satisfies Eqs. (3.65)-(3.67) and (3.60) (but not (3.61)-(3.63)), then the stress perturbation due to $[H_t^{(1)}(t')]_v$ is obtained from (3.68) on replacing $H_t^{(1)}(t')$ by $[H_t^{(1)}(t')]_v$.

CHAPTER 4

TAYLOR-COUETTE STABILITY

4.1 THE PRIMARY FLOW

We now apply the general theory developed in Chapter 3 to treat the current Taylor-Couette stability problem. It is supposed that a simple fluid is contained between two long coaxial cylinders of radii r_1 and r_2 which are rotating at angular velocities Ω_1 and Ω_2, respectively. Cylindrical polar coordinates (r,Θ,z) are employed, with the axis of the cylinders lying along the z-axis. The undisturbed Couette motion consists of a velocity field $\underset{\sim}{v}^{(o)}$ with physical components given by

$$v_r^{(o)} = 0, \quad v_\Theta^{(o)} = v^{(o)}(r), \quad v_z^{(o)} = 0 \quad . \tag{4.1}$$

Now, as we have indicated earlier, tangential motion between concentric rotating cylinders represents a viscometric flow; thus, from Eq. (1.54), we have

$$\underset{\sim}{H}_t^{(o)}(t') = \underset{\sim}{E}^{(o)} \cos \gamma^{(o)}(t-t') - \frac{1}{\gamma^{(o)}} \frac{\mathcal{D}^{(o)}\underset{\sim}{E}^{(o)}}{\mathcal{D}t} \sin \gamma^{(o)}(t-t') \tag{4.2}$$

where, in the present instance, one may readily show that

$$\gamma^{(o)} = r\,\frac{d(v^{(o)}/r)}{dr} \tag{4.3}$$

$$\underset{\sim}{E}^{(o)} \Longleftrightarrow \frac{\gamma^{(o)}}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.4}$$

and

$$\frac{\mathcal{D}^{(o)}\underset{\sim}{E}^{(o)}}{\mathcal{D}t} \Longleftrightarrow \frac{\gamma^{(o)^2}}{2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad . \qquad (4.5)$$

The corresponding stress distribution is given by

$$\begin{aligned} S_{r\theta}^{(o)} &= \gamma^{(o)}\eta(\gamma^{(o)}) \\ S_{\theta\theta}^{(o)}-S_{zz}^{(o)} &= \sigma_1(\gamma^{(o)}) \\ S_{rr}^{(o)}-S_{zz}^{(o)} &= \sigma_2(\gamma^{(o)}) \\ S_{\theta z}^{(o)} = S_{rz}^{(o)} &= 0 \quad . \end{aligned} \qquad (4.6)$$

One can theoretically obtain an equation for the velocity of the primary flow $v^{(o)}(r)$ by substituting the above stress components into the equations of motion and applying the appropriate boundary conditions. In particular, we have

$$S_{r\theta}^{(o)} = \gamma^{(o)}\eta = C/r^2 \qquad (4.7)$$

$$\begin{aligned} v^{(o)}/r &= \Omega_1 \text{ at } r = r_1 \\ v^{(o)}/r &= \Omega_2 \text{ at } r = r_2 \end{aligned} \qquad (4.8)$$

where C is a constant. It is seen from Eq. (4.7) that, in general, the shear rate $\gamma^{(o)}$ is a function of radial position within the gap between the cylinders. Therefore, since the viscosity function $\eta(\gamma)$ may vary with shear rate, there exists no general solution to (4.7) and (4.8). However, the solution to these equations can be approximated if the gap width $d = r_2 - r_1$ is regarded as small compared to either r_1 or r_2. Under such circumstances, C/r^2 will remain relatively constant as r varies between r_1 and r_2, and one finally obtains

$$\frac{v^{(o)}}{r} \doteq \Omega_1 + \frac{(r-r_1)}{d}(\Omega_2 - \Omega_1) \tag{4.9}$$

$$\gamma^{(o)} \doteq (\Omega_2 - \Omega_1) r_1 / d \tag{4.10}$$

and

$$\frac{dv^{(o)}}{dr} \doteq \gamma^{(o)} \quad . \tag{4.11}$$

4.2 THE STRESS PERTURBATION $\underset{\sim}{S}^{(1)}$

Let us now assume, following Taylor,[39] that the primary Couette motion $\underset{\sim}{v}^{(o)}$ is disturbed slightly by a steady, rotationally symmetric velocity perturbation $\xi \underset{\sim}{v}^{(1)}$, for which $\xi \to 0$. We write accordingly

$$v_r^{(1)} = u, \quad v_\Theta^{(1)} = v, \quad v_z^{(1)} = w \tag{4.12}$$

where u, v, and w are functions of r and z only. In this section our objective will be to determine the stress perturbation $\underset{\sim}{S}^{(1)}$ due to the altered motion.

One can first calculate the deformation history perturbation $\underset{\sim}{H}_t^{(1)}(t')$ by substituting (4.1) and (4.12) into Eq. (3.54). The result is expressible in the form

$$\underset{\sim}{H}_t^{(1)}(t') = [\underset{\sim}{H}_t^{(1)}(t')]_v + [\underset{\sim}{H}_t^{(1)}(t')]_{nv,1} + [\underset{\sim}{H}_t^{(1)}(t')]_{nv,2} + [\underset{\sim}{H}_t^{(1)}(t')]_{nv,3}$$
$$+ [\underset{\sim}{H}_t^{(1)}(t')]_{nv,4} \tag{4.13}$$

where, in cylindrical coordinates,

$$[\underset{\sim}{H}_t^{(1)}(t')]_v \Longleftrightarrow \frac{1}{2}\begin{bmatrix} r\,\frac{\partial(u/r)}{\partial r} & r\,\frac{\partial(v/r)}{\partial r} & \frac{\partial w}{\partial r} \\ r\,\frac{\partial(v/r)}{\partial r} & -r\,\frac{\partial(u/r)}{\partial r} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial r} & \frac{\partial v}{\partial z} & 0 \end{bmatrix} \cos\gamma^{(o)}(t-t')$$

$$- \frac{1}{2}\begin{bmatrix} r\frac{\partial(v/r)}{\partial r} & -r\,\frac{\partial(u/r)}{\partial r} & \frac{\partial v}{\partial z} \\ -r\frac{\partial(u/r)}{\partial r} & -r\,\frac{\partial(v/r)}{\partial r} & -\frac{\partial w}{\partial r} \\ \frac{\partial v}{\partial z} & -\frac{\partial w}{\partial r} & 0 \end{bmatrix} \sin\gamma^{(o)}(t-t') + \frac{r}{2}\,\frac{\partial(v/r)}{\partial r}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\gamma^{(o)}(t-t')\cos\gamma^{(o)}(t-t')$$

$$- \frac{r}{2}\,\frac{\partial(v/r)}{\partial r}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\gamma^{(o)}(t-t')\sin\gamma^{(o)}(t-t') \qquad (4.14)$$

$$[\underset{\sim}{H}_t^{(1)}(t')]_{nv,1} \Longleftrightarrow \frac{1}{2r}\,\frac{\partial(ur)}{\partial r}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (4.15)$$

$$[\underset{\sim}{H}_t^{(1)}(t')]_{nv,2} \Longleftrightarrow \frac{\partial w}{\partial z}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{4.16}$$

$$[\underset{\sim}{H}_t^{(1)}(t')]_{nv,3} \Longleftrightarrow \frac{1}{2}\frac{\partial u}{\partial z}\left[\begin{array}{c:c:c} 0 & 0 & 2\cos\frac{\gamma^{(o)}}{2}(t-t')-\cos\gamma^{(o)}(t-t') \\ 0 & 0 & 2\sin\frac{\gamma^{(o)}}{2}(t-t')-\sin\gamma^{(o)}(t-t') \\ 2\cos\frac{\gamma^{(o)}}{2}(t-t')-\cos\gamma^{(o)}(t-t') & 2\sin\frac{\gamma^{(o)}}{2}(t-t')-\sin\gamma^{(o)}(t-t') & 0 \end{array}\right] \tag{4.17}$$

$$[\underset{\sim}{H}_t^{(1)}(t')]_{nv,4} \Longleftrightarrow \frac{u}{4}(t-t')^2\,\gamma(o)\,\frac{d\gamma^{(o)}}{dr}\begin{bmatrix} \cos\gamma^{(o)}(t-t') & \sin\gamma^{(o)}(t-t') & 0 \\ \sin\gamma^{(o)}(t-t') & -\cos\gamma^{(o)}(t-t') & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-\frac{u}{2}(t-t')\frac{d\gamma^{(o)}}{dr}\begin{bmatrix} -\sin\gamma^{(o)}(t-t') & \cos\gamma^{(o)}(t-t') & 0 \\ \cos\gamma^{(o)}(t-t') & \sin\gamma^{(o)}(t-t') & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{4.18}$$

The subscripts v and nv are used here, as in Section 3.5, to denote "viscometric" and "nonviscometric" respectively; that is, one can easily verify that $[\underset{\sim}{H}_t^{(1)}(t')]_v$ satisfies Eqs. (3.65)-(3.67) and (3.60) with

$$\gamma^{(1)} = r\,\frac{\partial(v/r)}{\partial r} \quad . \tag{4.19}$$

Therefore, based on the discussion of Section 3.5, the stress perturbation $\underset{\sim}{S}_v^{(1)}$, due to $[\underset{\sim}{H}_t^{(1)}(t')]_v$, follows immediately from (3.68):

$$\begin{aligned}
[S_{rr}^{(1)}-S_{zz}^{(1)}]_v &= r\,\frac{\partial(u/r)}{\partial r}\,\eta + r\,\frac{\partial(v/r)}{\partial r}\,\sigma_2' \\
[S_{\theta\theta}^{(1)}-S_{zz}^{(1)}]_v &= -r\,\frac{\partial(u/r)}{\partial r}\,\eta + r\,\frac{\partial(v/r)}{\partial r}\,\sigma_1' \\
[S_{\theta r}^{(1)}]_v &= r\,\frac{\partial(u/r)}{\partial r}\,\frac{(\sigma_1-\sigma_2)}{2\gamma^{(o)}} + r\,\frac{\partial(v/r)}{\partial r}\,(\eta+\eta'\gamma^{(o)}) \\
[S_{rz}^{(1)}]_v &= \frac{\partial w}{\partial r}\,\eta + \frac{\partial v}{\partial z}\,\frac{\sigma_2}{\gamma^{(o)}} \\
[S_{z\theta}^{(1)}]_v &= \frac{\partial w}{\partial r}\,\frac{\sigma_1}{\gamma^{(o)}} + \frac{\partial v}{\partial z}\,\eta \quad .
\end{aligned} \tag{4.20}$$

We are next faced with the problem of determining the remainder of the stress perturbation arising from the "nonviscometric" terms in $\underset{\sim}{H}_t^{(1)}(t')$. As will presently be shown, it will be necessary to define several new material functions of the primary shear rate $\gamma^{(o)}$ (other than the three viscometric functions η, σ_1, and σ_2) in order to account for these stresses.

Let us begin by considering the contribution towards $\underset{\sim}{S}^{(1)}$ due only to $[\underset{\sim}{H}_t^{(1)}(t')]_{nv,1}$. Making use of Eq. (3.28), one notes that the appropriate stress perturbation $\underset{\sim}{S}_{nv,1}^{(1)}$ will be given by

$$[\underset{\sim}{S}^{(1)}+p^{(1)}\underset{\sim}{I}]_{nv,1} = \underset{t'=-\infty}{\overset{t}{\delta\mathcal{F}}} [\underset{\sim}{H}_t^{(o)}(t'), \underset{\sim}{H}_t^{(1)}(t')_{nv,1}] \qquad (4.21)$$

where it is recalled that $\delta\mathcal{F}$ is linear in $[\underset{\sim}{H}_t^{(1)}(t')]_{nv,1}$ and satisfies the relation

$$\underset{\sim}{Q} \cdot \underset{t'=-\infty}{\overset{t}{\delta\mathcal{F}}} [\underset{\sim}{H}_t^{(o)}(t'), [\underset{\sim}{H}_t^{(1)}(t')]_{nv,1}] \cdot \underset{\sim}{Q}^{\dagger} = \underset{t'=-\infty}{\overset{t}{\delta\mathcal{F}}} [\underset{\sim}{Q} \cdot \underset{\sim}{H}_t^{(o)}(t') \cdot \underset{\sim}{Q}^{\dagger},$$

$$\underset{\sim}{Q} \cdot [\underset{\sim}{H}_t^{(1)}(t')_{nv,1}] \cdot \underset{\sim}{Q}^{\dagger}] \qquad (4.22)$$

for any orthogonal tensor $\underset{\sim}{Q}$. We now choose $\underset{\sim}{Q}$ in (4.22) to be the orthogonal tensor whose matrix is

$$\underset{\sim}{Q} \Longleftrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} . \qquad (4.23)$$

Substituting (4.23) into (4.22) and employing Eqs. (4.2), (4.4), (4.5), and (4.15), one finds that

$$\underset{\sim}{Q} \cdot \underset{t'=-\infty}{\overset{t}{\delta\mathcal{F}}} [\underset{\sim}{H}_t^{(o)}(t'), [\underset{\sim}{H}_t^{(1)}(t')]_{nv,1}] \cdot \underset{\sim}{Q}^{\dagger} = \underset{t'=-\infty}{\overset{t}{\delta\mathcal{F}}} [\underset{\sim}{H}_t^{(o)}(t'), [\underset{\sim}{H}_t^{(1)}(t')]_{nv,1}]. \qquad (4.24)$$

Hence, the stress perturbation $\underset{\sim}{S}^{(1)}_{nv,1}$ due to the history $[H_t^{(1)}(t')]_{nv,1}$ must have the property that

$$\underset{\sim}{Q} \cdot \underset{\sim}{S}^{(1)}_{nv,1} \cdot \underset{\sim}{Q}^{\dagger} = \underset{\sim}{S}^{(1)}_{nv,1} . \qquad (4.25)$$

The matrix of $\underset{\sim}{Q} \cdot \underset{\sim}{S}^{(1)}_{nv,1} \cdot \underset{\sim}{Q}^{\dagger}$ is readily calculated to be

$$\underset{\sim}{Q} \cdot \underset{\sim}{S}^{(1)}_{nv,1} \cdot \underset{\sim}{Q}^{\dagger} \Longleftrightarrow \begin{bmatrix} S^{(1)}_{rr} & S^{(1)}_{r\theta} & -S^{(1)}_{rz} \\ S^{(1)}_{r\theta} & S^{(1)}_{\theta\theta} & -S^{(1)}_{\theta z} \\ -S^{(1)}_{rz} & -S^{(1)}_{\theta z} & S^{(1)}_{zz} \end{bmatrix}_{nv,1} \quad . \qquad (4.26)$$

It follows then from (4.25) and (4.26) that

$$[S^{(1)}_{rz}]_{nv,1} = [S^{(1)}_{\theta z}]_{nv,1} = 0 \quad . \qquad (4.27)$$

That is, $\underset{\sim}{S}^{(1)}_{nv,1}$ must have the form

$$\underset{\sim}{S}^{(1)}_{nv,1} \Longleftrightarrow \begin{bmatrix} S^{(1)}_{rr} & S^{(1)}_{r\theta} & 0 \\ S^{(1)}_{r\theta} & S^{(1)}_{\theta\theta} & 0 \\ 0 & 0 & S^{(1)}_{zz} \end{bmatrix}_{nv,1} \qquad (4.28)$$

Now, since the functional $\underset{\sim}{\delta\mathfrak{z}}$ is linear in $\underset{\sim}{H}^{(1)}_{t}(t')$, it is seen from (4.15) that the components of $\underset{\sim}{S}^{(1)}_{nv,1}$ will be directly proportional to $\partial(ur)/r\partial r$; moreover, the "constants" of proportionality will be material functions of the rate of shear $\gamma^{(o)}$. Hence one can write

$$\begin{aligned} [S^{(1)}_{r\theta}]_{nv,1} &= \frac{1}{r}\frac{\partial(ur)}{\partial r}\frac{\sigma_3}{\gamma^{(o)}} \\ [S^{(1)}_{\theta\theta}-S^{(1)}_{zz}]_{nv,1} &= \frac{1}{r}\frac{\partial(ur)}{\partial r}\eta_1 \qquad (4.29) \\ [S^{(1)}_{rr}-S^{(1)}_{zz}]_{nv,1} &= \frac{1}{r}\frac{\partial(ur)}{\partial r}\eta_2 \quad . \end{aligned}$$

Our purpose in defining the new material functions η_1, η_2, and σ_3 in this manner shall become apparent later. Suffice it to say at this point that, by again applying (4.22), we can show that all three will be even functions of the shear rate.

A treatment similar to that presented above may also be employed to determine the stress perturbations $\underset{\sim}{S}^{(1)}_{nv,2}$, $\underset{\sim}{S}^{(1)}_{nv,3}$, and $\underset{\sim}{S}^{(1)}_{nv,4}$ arising from $[\underset{\sim}{H}_t^{(1)}(t')]_{nv,2}$, $[\underset{\sim}{H}_t^{(1)}(t')]_{nv,3}$, and $[\underset{\sim}{H}_t^{(1)}(t')]_{nv,4}$, respectively. As a final result, we obtain the total stress perturbation $\underset{\sim}{S}^{(1)}$:

$$
\begin{aligned}
S^{(1)}_{rr} + p^{(1)} &= r\frac{\partial(u/r)}{\partial r}\eta + r\frac{\partial(v/r)}{\partial r}\sigma_2' + \frac{1}{r}\frac{\partial(ur)}{\partial r}\eta_2 + u\frac{d\gamma^{(o)}}{dr}\frac{\lambda_1}{\gamma^{(o)}} \\
S^{(1)}_{\theta\theta} + p^{(1)} &= -r\frac{\partial(u/r)}{\partial r}\eta + r\frac{\partial(v/r)}{\partial r}\sigma_1' + \frac{1}{r}\frac{\partial(ur)}{\partial r}\eta_1 + u\frac{d\gamma^{(o)}}{dr}\frac{\lambda_2}{\gamma^{(o)}} \\
S^{(1)}_{zz} + p^{(1)} &= 2\eta_3\frac{\partial w}{\partial z} \\
S^{(1)}_{r\theta} &= r\frac{\partial(u/r)}{\partial r}\frac{(\sigma_1-\sigma_2)}{2\gamma^{(o)}} + r\frac{\partial(v/r)}{\partial r}(\eta+\eta'\gamma^{(o)}) \\
&\quad + \frac{1}{r}\frac{\partial(ur)}{\partial r}\frac{\sigma_3}{\gamma^{(o)}} + u\frac{d\gamma^{(o)}}{dr}\lambda_3 \\
S^{(1)}_{rz} &= \frac{\partial w}{\partial r}\eta + \frac{\partial v}{\partial z}\frac{\sigma_2}{\gamma^{(o)}} + \frac{\partial u}{\partial z}\eta_4 \\
S^{(1)}_{\theta z} &= \frac{\partial w}{\partial r}\frac{\sigma_1}{\gamma^{(o)}} + \frac{\partial v}{\partial z}\eta + \frac{\partial u}{\partial z}\frac{\sigma_4}{\gamma^{(o)}}
\end{aligned}
\tag{4.30}
$$

where the new material functions η_1, η_2, η_3, η_4, σ_3, σ_4, λ_1, λ_2, and λ_3 are all even functions of γ evaluated at $\gamma = \gamma^{(o)}$.

4.3 THE DISTURBANCE EQUATIONS

Appropriate expressions for the perturbations on the equations of momentum and continuity for this problem are found by substituting (4.1) and (4.12) into Eqs. (3.11) and (3.12), respectively, and employing cylindrical polar coordinates. We obtain

(continuity) $$\frac{1}{r}\frac{\partial(ur)}{\partial r} + \frac{\partial w}{\partial z} = 0 \tag{4.31}$$

(momentum)

$$-2\rho v[v^{(o)}/r] = \frac{\partial S_{rr}^{(1)}}{\partial r} + \frac{\partial S_{rz}^{(1)}}{\partial z} + \frac{S_{rr}^{(1)} - S_{\theta\theta}^{(1)}}{r}$$

$$\rho u \left[\frac{\partial v^{(o)}}{\partial r} + \frac{v^{(o)}}{r}\right] = \frac{1}{r^2}\frac{\partial(r^2 S_{r\theta}^{(1)})}{\partial r} + \frac{\partial S_{\theta z}^{(1)}}{\partial z} \qquad (4.32)$$

$$0 = \frac{1}{r}\frac{\partial(r S_{rz}^{(1)})}{\partial r} + \frac{\partial S_{zz}^{(1)}}{\partial z} \quad .$$

Now, in order to facilitate the succeeding analysis, we make the usual simplifying assumption that the annular gap between the cylinders is small compared to their radii. The primary shear rate $\gamma^{(o)}$, which may then be presumed constant, is given by Eq. (4.10), and, in addition, the components of the stress perturbation reduce to

$$S_{rr}^{(1)} + p^{(1)} = \frac{\partial u}{\partial r}[\eta + \eta_2] + \frac{\partial v}{\partial r}\sigma_2'$$

$$S_{\theta\theta}^{(1)} + p^{(1)} = \frac{\partial u}{\partial r}[\eta_1 - \eta] + \frac{\partial v}{\partial r}\sigma_1'$$

$$S_{zz}^{(1)} + p^{(1)} = 2\eta_3 \frac{\partial w}{\partial z}$$

$$S_{r\theta}^{(1)} = \frac{\partial u}{\partial r}\left[\frac{\sigma_3 + (\sigma_1 - \sigma_2)/2}{\gamma^{(o)}}\right] + \frac{\partial v}{\partial r}[\eta + \eta' \gamma^{(o)}] \qquad (4.33)$$

$$S_{rz}^{(1)} = \frac{\partial w}{\partial r}\eta + \frac{\partial v}{\partial z}\frac{\sigma_2}{\gamma^{(o)}} + \frac{\partial u}{\partial r}\eta_4$$

$$S_{\theta z}^{(1)} = \frac{\partial w}{\partial r}\frac{\sigma_3}{\gamma^{(o)}} + \frac{\partial v}{\partial z}\eta + \frac{\partial u}{\partial z}\frac{\sigma_4}{\gamma^{(o)}}$$

where terms involving $d\gamma^{(o)}/dr$ have been dropped from (4.30). Furthermore, for the small-gap case, the equations of continuity and momentum become

$$\frac{\partial u}{\partial r} + \frac{\partial w}{\partial z} = 0 \tag{4.34}$$

$$-2\rho v\left[\Omega_1 + (\Omega_2-\Omega_1)\frac{(r-r_1)}{d}\right] = \frac{\partial S_{rr}^{(1)}}{\partial r} + \frac{\partial S_{rz}^{(1)}}{\partial z} - \frac{S_{\theta\theta}^{(1)}}{r_1}$$

$$\rho u(\Omega_2-\Omega_1)\frac{r_1}{d} = \frac{\partial S_{r\theta}^{(1)}}{\partial r} + \frac{\partial S_{\theta z}^{(1)}}{\partial z} \tag{4.35}$$

$$0 = \frac{\partial S_{rz}^{(1)}}{\partial r} + \frac{\partial S_{zz}^{(1)}}{\partial z} \quad .$$

Substituting (4.33) into (4.35) and making use of (4.34), we then have

$$-2\rho v\left[\Omega_1+(\Omega_2-\Omega_1)\frac{(r-r_1)}{d}\right] = -\frac{\partial p^{(1)}}{\partial r} + \eta_2\frac{\partial^2 u}{\partial r^2} + \eta_4\frac{\partial^2 u}{\partial z^2} - \sigma_2'\frac{\partial^2 v}{\partial r^2} + \frac{\sigma_2}{\gamma(o)}\frac{\partial^2 v}{\partial z^2} - \frac{(\eta_1-\eta)}{r_1}\frac{\partial u}{\partial r} - \frac{\sigma_1'}{r_1}\frac{\partial v}{\partial r}$$

$$\rho u(\Omega_2-\Omega_1)\frac{r_1}{d} = \left[\frac{\sigma_3-(\sigma_1+\sigma_2)/2}{\gamma(o)}\right]\frac{\partial^2 u}{\partial^2 r} + \frac{\sigma_4}{\gamma(o)}\frac{\partial^2 u}{\partial z^2} + [\eta+\eta'\gamma(o)]\frac{\partial^2 v}{\partial r^2} + \eta\frac{\partial^2 v}{\partial z^2} \tag{4.36}$$

$$0 = -\frac{\partial p^{(1)}}{\partial z} + \eta\frac{\partial^2 w}{\partial r^2} + (2\eta_3-\eta_4)\frac{\partial^2 w}{\partial z^2} + \frac{\sigma_2}{\gamma(o)}\frac{\partial^2 v}{\partial z\partial r} \quad .$$

To reduce these equations to dimensionless form, we define the following quantities:

$$\begin{aligned}
X &= \frac{r-r_1}{r_2-r_1} = \frac{r-r_1}{d} & u^* &= u/\Omega_1 d \\
Z &= z/d & v^* &= v/\Omega_1 d \\
\alpha &= (\Omega_2-\Omega_1)/\Omega_1 & w^* &= w/\Omega_1 d \\
a &= r_1/d & R &= \rho d^2\Omega_1/\eta(\gamma^{(o)}) \\
p^* &= p^{(1)}/\Omega_1\eta(\gamma^{(o)}) & T &= -2\alpha a R^2
\end{aligned} \tag{4.37}$$

The Reynolds number R and the Taylor number T are specified here in terms of the apparent viscosity $\eta(\gamma^{(o)})$ rather than by some other "characteristic" point on the viscosity curve for the fluid. Any other definition of these quantities would serve only to introduce an additional and superfluous parameter into the analysis. Thus, we have established that <u>the most appropriate and meaningful definitions of R and T are based on the apparent viscosity evaluated at the gap shear rate.</u>

From (4.34), (4.36), and (4.37) one now obtains

$$\frac{\partial u^*}{\partial X} + \frac{\partial w^*}{\partial Z} = 0 \tag{4.38}$$

$$-2v^*(1+\alpha X)R = -\frac{\partial p^*}{\partial X} + \eta_2\frac{\partial^2 u^*}{\partial X^2} + \eta_4\frac{\partial u^{*2}}{\partial Z^2} + \frac{\sigma_2'}{\eta}\frac{\partial^2 v^*}{\partial X^2} + \frac{\sigma_2}{\eta\gamma(o)}\frac{\partial^2 v^*}{\partial Z^2}$$

$$- \frac{(\eta_1-\eta)}{\eta a}\frac{\partial u^*}{\partial X} + \frac{\sigma_1'}{\eta a}\frac{\partial v^*}{\partial X}$$

$$-\frac{T}{2R}u^* = \left[1 + \frac{\eta'\gamma^{(o)}}{\eta}\right]\frac{\partial^2 v^*}{\partial X^2} + \frac{\partial^2 v^*}{\partial Z^2} + \left[\frac{\sigma_3-(\sigma_1+\sigma_2)/2}{\eta\gamma(o)}\right]\frac{\partial^2 u^*}{\partial X^2} + \frac{\sigma_4}{\eta\gamma(o)}\frac{\partial^2 u^*}{\partial Z^2}$$

$$0 = -\frac{\partial p^*}{\partial Z} + \frac{\partial^2 w^*}{\partial X^2} + \frac{(2\eta_3-\eta_4)}{\eta}\frac{\partial^2 w^*}{\partial Z^2} + \frac{\sigma_2}{\eta\gamma(o)}\frac{\partial^2 v^*}{\partial X\partial Z} \; . \tag{4.39}$$

Next, resolving the disturbance into its normal modes, we make the usual assumption that the disturbance is spatially periodic in the z-direction and write

$$u^* = \epsilon\psi(X)\sin\epsilon Z$$

$$v^* = -\epsilon V(X)\sin\epsilon Z \tag{4.40}$$

$$w^* = \frac{d\psi}{dX}\cos\epsilon Z$$

$$p^* = \epsilon P(X)\sin\epsilon Z \tag{4.41}$$

where ϵ is the dimensionless wave number of the disturbance. One can readily verify that (4.40) satisfies Eq. (4.38), the equation of continuity, identically. Substitution of (4.40) and (4.41) into (4.39) then yields

$$2V(1+\alpha X)R + \frac{\sigma_2'}{\eta} D^2V - \frac{\sigma_2}{\eta\gamma^{(o)}} \epsilon^2 V = -DP + \frac{\eta_2}{\eta} D^2\psi - \frac{\eta_4}{\eta} \epsilon^2\psi - \frac{(\eta_1-\eta)}{\eta a} D\psi - \frac{\sigma_1'}{\eta a} DV \qquad (4.42)$$

$$\left[1 + \frac{\eta'\gamma^{(o)}}{\eta}\right] D^2V - \epsilon^2 V = \frac{T}{2R}\psi + \frac{[\sigma_3-(\sigma_1+\sigma_2)/2]}{\eta\gamma^{(o)}} D^2\psi - \frac{\sigma_4}{\eta\gamma^{(o)}} \epsilon^2\psi \qquad (4.43)$$

and

$$0 = -\epsilon^2 P + D^3\psi - \frac{(2\eta_3-\eta_4)}{\eta} \epsilon^2 D\psi - \frac{\sigma_2}{\eta\gamma^{(o)}} \epsilon^2 DV \qquad (4.44)$$

where

$$D \equiv d/dX .$$

We can now eliminate **P** between Eqs. (4.42) and (4.44); the final disturbance equations are then given by

$$[\beta_1 D^2-\epsilon^2]V = \left[\frac{T}{2R} + \beta_2 D^2 - \beta_3\epsilon^2\right] \psi \qquad (4.45)$$

and

$$\left[D^4-\beta_4\epsilon^2 D^2 + \frac{2\beta_9}{a} \epsilon^2 D+\beta_5\epsilon^4\right]\psi = \epsilon^2 \left[-2R(1+\alpha X)+\beta_6 D^2+\beta_7\epsilon^2 + \frac{\beta_8}{a} D\right] V \qquad (4.46)$$

where the β's are <u>rheological parameters</u> defined as*

*For convenience, we henceforth drop the superscript on the primary shear rate $\gamma^{(o)}$ with the understanding that γ and $\gamma^{(o)}$ are now one and the same.

$$\beta_1 = \frac{d\ \ell n[\eta\gamma]}{d\ \ell n\ \gamma} \tag{4.47}$$

$$\beta_2 = \left[\frac{\sigma_3-(\sigma_1+\sigma_2)/2}{\eta\gamma}\right] \tag{4.48}$$

$$\beta_3 = \sigma_4/\eta\gamma \tag{4.49}$$

$$\beta_4 = (2\eta_3+\eta_2-\eta_4)/\eta \tag{4.50}$$

$$\beta_5 = \eta_4/\eta \tag{4.51}$$

$$\beta_6 = \frac{\sigma_2}{\eta\gamma}\left[1 - \frac{d\ \ell n\sigma_2}{d\ \ell n\ \gamma}\right] \tag{4.52}$$

$$\beta_7 = \sigma_2/\eta\gamma \tag{4.53}$$

$$\beta_8 = \frac{\sigma_1}{\eta\gamma}\frac{d\ \ell n\sigma_1}{d\ \ell n\ \gamma} \tag{4.54}$$

and

$$\beta_9 = (\eta_1-\eta)/2\eta \quad . \tag{4.55}$$

As can be seen from these equations, there are eight material functions, depending on the shear rate γ, which are necessary to define the Taylor-Couette stability of a simple fluid in the narrow-gap limit. Three of these are the viscometric functions η, σ_1, and σ_2, but the other five β_2, β_3, β_4, β_5, and β_9 are entirely new. One notes, from (4.48) and (4.49), that β_2 and β_3 are odd functions of γ while, from (4.50), (4.51), and (4.55), we see that β_4, β_5, and β_9 are even functions of γ.

Equations (4.45) and (4.46) represent two ordinary, homogeneous, differential equations in two unknowns, ψ and V. Together with the homogeneous boundary conditions,

$$\psi = D\psi = V = 0 \text{ at } X = 0,1 \tag{4.56}$$

they determine a characteristic-value or eigenvalue problem for the Taylor number T, as function $T(\epsilon)$ of the dimensionless wave number ϵ (for fixed α and a); the smallest value, T_c, of $T(\epsilon)$ determines the critical conditions T_c, ϵ_c at which instability sets in.

4.4 LOW-SHEAR-RATE APPROXIMATION

There are presently no known methods, either theoretical or experimental, for determining values for the five new material functions β_2 to β_5 and β_9 at finite rates of shear. However, as we shall see below, the variation of these quantities in the limit as $\gamma \to 0$ can be predicted, to terms of first order in γ, by making use of simple-fluid theory.

We recall from Section 1.5.1 that the fluid model considered by Datta,[11]

$$\underset{\sim}{S} + p\underset{\sim}{I} = \alpha_1 \underset{\sim}{A}_1 + \alpha_2 \underset{\sim}{A}_1^2 + \alpha_3 \underset{\sim}{A}_2 \tag{1.66}$$

has been shown to describe the limiting behavior of simple fluids with fading memory at extremely low rates of deformation. Here

$$\begin{aligned} \alpha_1 &= \lim_{\gamma \to 0} \eta(\gamma) \\ \alpha_2 &= \lim_{\gamma \to 0} \sigma_1(\gamma)/\gamma^2 \\ \alpha_3 &= \lim_{\gamma \to 0} (\sigma_2 - \sigma_1)/2\gamma^2 \quad . \end{aligned} \tag{1.67}$$

One finds from Datta's disturbance equations that

$$\beta_2 = \beta_3 = (\alpha_2+\alpha_3)\gamma/\alpha_1$$
$$\beta_4 = 2,\ \beta_5 = 1,\ \beta_9 = 0 \quad . \tag{4.57}$$

Therefore, in the limit as $\gamma \to 0$, we have

$$\beta_2 = \beta_3 = (\sigma_1+\sigma_2)/2\eta\gamma$$
$$\beta_4 = 2,\ \beta_5 = 1,\ \beta_9 = 0 \quad . \tag{4.58}$$

An obvious suggestion which one might put forth at this point would be to approximate the material functions β_2 to β_5 and β_9 by Eqs. (4.58), even at finite rates of shear. If such a procedure were valid, then the solution to the disturbance equations would be governed entirely by the viscometric functions η, σ_1, and σ_2; hence, knowledge of these three functions would suffice to determine the flow stability of a particular fluid _a priori_. One may now conclude the following: If at the point of instability, the approximation (4.58) were reasonable, _absolutely nothing new concerning the fluid of interest_ (beyond η, σ_1, and σ_2) _would be learned from a stability experiment_. Conversely, if the approximation were _not_ valid, then new information _would_ be obtained. In fact, any difference between the results of a stability experiment and the solution to the disturbance equations with the approximation (4.58) would presumably be attributable to deviations from (4.58).

The validity of the "viscometric hypothesis" is likewise contingent upon whether Eqs. (4.58) represent a reasonable approximation. One will recall that this hypothesis requires that Eq. (1.57) describe motions, such as the present one, which are close to viscometric flows; it is readily verified that Eqs. (4.58) are precisely what would have resulted if Eq. (1.57) had been chosen as the original rheological model for the stability analysis.

In certain cases, a stability experiment can be used to advantage in determining which of two fluid models is more appropriate for representing the constitutive behavior of a particular viscoelastic fluid. The primary requirement for these models, of course, should be that they both adequately describe the experimental data on η, σ_1, and σ_2; in this way, the material functions β_1 and β_6 to β_8 would be the same for both. If the models then differed sufficiently in their predictions of β_2 to β_5 and β_9, the corresponding stability analyses would yield different results. Based on a comparison of the theoretical findings with experiment, a choice of the best model could then be made.

4.5 PREVIOUS ANALYSES

As has been pointed out earlier, all previous theoretical treatments of the small-gap Taylor-Couette stability problem have involved consideration of the behavior of various idealized simple-fluid models; these treatments will henceforth represent special cases of the present, general analysis. In order to illustrate more clearly the relationship between past studies and the current work Table 4.1 has been prepared. This table presents a listing of the eight material functions η, σ_1, σ_2, β_2, β_3, β_4, β_5, and β_9 for each of the previous investigations.

TABLE 4.1

MATERIAL FUNCTIONS IN PREVIOUS ANALYSES*

Investigator	Model	η	σ_1	σ_2	β_2	β_3	β_4	β_5	β_9
Chandrasekhar[5]	Newtonian Eq. (1.19)	η_o	0	0	0	0	2	1	0
Graebel[21]	Reiner-Rivlin Eq. (1.20)	α_1	$\alpha_2\gamma^2$	$\alpha_2\gamma^2$	$(\sigma_1+\sigma_2)/2\eta\gamma$	$(\sigma_1+\sigma_2)/2\eta\gamma$	2	1	0
Graebel[22]	Bingham Plastic Eq. (1.62)	$\mu\left[1+\frac{\vartheta}{\mu\|\gamma\|}\right]$	0	0	0	0	2	1	0
Thomas & Walters[40]	Walters B' Eq. (1.24)	η_o	$2K_o\gamma^2$	0	$(\sigma_1+\sigma_2)/2\eta\gamma$	$(\sigma_1+\sigma_2)/2\eta\gamma$	2	1	0
Chan Man Fong[6]	Walters A' Eq. (1.23)	η_o	0	$-2K_o\gamma^2$	$(\sigma_1+\sigma_2)/2\eta\gamma$	$(\sigma_1+\sigma_2)/2\eta\gamma$	$2+3S_o\gamma^2/\eta_o$	$1+3S_o\gamma^2/\eta_o$	0
Datta[11]	Fluid of Second Grade Eq. (1.66)	α_1	$\alpha_2\gamma^2$	$(\alpha_2+2\alpha_3)\gamma^2$	$(\sigma_1+\sigma_2)/2\eta\gamma$	$(\sigma_1+\sigma_2)/2\eta\gamma$	2	1	0
Davies[12]	Modified Walters B' Eq. (1.72)	$\eta_o+\mu_oK_o\gamma^2$	$2K_o\gamma^2$	0	$(\sigma_1+\sigma_2)/2\eta\gamma$ $-3\mu_oS_o\gamma^4/\eta_o$	$(\sigma_1+\sigma_2)/2\eta\gamma$	2	1	$3S_o\gamma^2/\eta_o$**
Thomas & Walters[41]	Walters B' Eq. (1.24)	η_o	$2K_o\gamma^2$	0	$(\sigma_1+\sigma_2)/2\eta\gamma$	$(\sigma_1+\sigma_2)/2\eta\gamma$	2	1	$3S_o\gamma^2/\eta_o$
Leslie[25]	Ericksen Anisotropic Eq. (1.30)	$\mu+\mu_3$ $+\mu_2n_1^2n_2^2$	$n_1n_2(\mu_2n_2^2$ $+2\mu_3)\gamma$	$n_1n_2(\mu_2n_1^2$ $+2\mu_3)\gamma$	0	$n_1(1-$ $\mu/\eta)/n_2$	$(4\mu+n_1^2(1-$ $2n_1^2)\mu_2)/2\eta$	$(\mu+n_1^4\mu_2$ $+2n_1^2\mu_3)/\eta$	0

*It is interesting to note that, due to the simplicity of many of the models considered, the approximations shown in Eqs. (4.58) are found to occur quite frequently in this table.

$$**S_o = \int_0^\infty t^2\phi(t)dt$$

One study not included in Table 4.1 is that of Giesekus,[18] who attempted to generalize the theoretical analysis of Datta[11] for finite rates of shear. He substituted

$$\alpha_1 = \eta$$
$$\alpha_2 = \sigma_1/\gamma^2 \qquad (4.59)$$
$$\alpha_2 + 2\alpha_3 = \sigma_2/\gamma^2$$

into Datta's disturbance equations and obtained

$$\begin{aligned} \beta_1 &= 1 & \beta_6 &= -\sigma_2/\eta\gamma \\ \beta_2 = \beta_3 &= (\sigma_1+\sigma_2)/2\eta\gamma & \beta_7 &= \sigma_2/\eta\gamma \\ \beta_4 &= 2 & \beta_8 &= (2\sigma_1+\sigma_2)/\eta\gamma \\ \beta_5 &= 1 & \beta_9 &= 0 \end{aligned} \qquad (4.60)$$

By comparing these equations with Eqs. (4.47)-(4.55), one notes that most of the expressions in (4.60) are clearly incorrect, except of course at $\gamma = 0$.

4.6 SPECIAL CASES

In this section, we briefly discuss two special cases. The first is that of negligible inertia where $T \to 0$ and the other is plane Couette flow where $a \to \infty$.

4.6.1 Negligible Inertia

There are numerous ways in which the disturbance equations for the current study could have been written. However, in the form we have presented them (Eqs. (4.45) and (4.46)), the terms due to inertial effects (those containing the Taylor number T and the Reynolds number R) have been completely separated from the "rheological terms" (those involving the β's).

Giesekus[18] was the first investigator to consider the possibility that, for a particular viscoelastic fluid, instability might occur at an extremely small Taylor number, where inertial effects are negligible. Under such circumstances, Eqs. (4.45) and (4.46) become

$$\begin{aligned} [\beta_1 D^2 - \epsilon^2]V &= [\beta_2 D^2 - \beta_3 \epsilon^2]\psi \\ \left[D^4 - \beta_4 \epsilon^2 D^2 + \frac{2\beta_9}{a}\epsilon^2 D + \beta_5 \epsilon^4\right]\psi &= \epsilon^2\left[\beta_6 D^2 + \beta_7 \epsilon^2 + \frac{\beta_8 D}{a}\right]V \quad . \end{aligned} \tag{4.61}$$

Apparently, a modified Taylor stability criterion could no longer be regarded as applicable to these new disturbance equations since they do not even contain the Taylor number. Instead, an entirely new criterion results, in which the rheological coefficients determine the point of instability.

4.6.2 Plane Couette Flow (Simple Shear Between Parallel Planes)

The disturbance equations which describe this situation are obtained by allowing $a \to \infty$ while γ is held fixed in Eqs. (4.45) and (4.46). We then have

$$\begin{aligned} [\beta_1 D^2 - \epsilon^2]V &= [-R' + \beta_2 D^2 - \beta_3 \epsilon^2]\psi \\ [D^4 - \beta_4 \epsilon^2 + \beta_5 \epsilon^4]\psi &= \epsilon^2[\beta_6 D^2 + \beta_7 \epsilon^2]V \end{aligned} \tag{4.62}$$

where

$$R' = \rho d^2\gamma/\eta \quad . \tag{4.63}$$

For the case of a Newtonian fluid, these equations yield only the trivial solution, $\psi = V = 0$; however, for certain idealized viscoelastic fluids, Giesekus[18] has shown that solutions other than the trivial one are possible.

4.7 SOLVING THE DISTURBANCE EQUATIONS

In the present work, three methods have been used to solve the characteristic-value problem defined by Eqs. (4.45), (4.46), and (4.56). The first and second of these were analytic techniques, and were employed only to obtain approximate expressions for the Taylor number as a function of the dimensionless wave number; the third method, a numerical technique, was used to obtain more precise results.

4.7.1 First Analytic Method

The first analytic method, which can be attributed to Chandrasekhar,[5] is illustrated by considering the solution of the simplified equations

$$[\beta_1 D^2 - \epsilon^2]V = \frac{T}{2R}\psi + \beta_2[D^2-\epsilon^2]\psi \tag{4.64}$$

$$[D^2-\epsilon^2]^2\psi = \epsilon^2\left[-2R(1+\alpha X)+\beta_6 D^2+\beta_7\epsilon^2 + \frac{\beta_8}{a}D\right]V \tag{4.65}$$

which result when the approximations shown in Eqs. (4.58) are made.

We begin by writing

$$V = \sum_{m=1}^{\infty} A_m \sin m\pi X \tag{4.66}$$

thus satisfying automatically the boundary conditions on V. Substituting (4.66) into (4.65) and solving for ψ, one then obtains

$$\psi = -2R\epsilon^2 \sum_{m=1}^{\infty} \frac{A_m}{[(m\pi)^2+\epsilon^2]} \Bigg[C_m \cosh \epsilon X + D_m \sinh \epsilon X + E_m X \cosh \epsilon X$$

$$+ F_m X \sinh \epsilon X + \left[1+\alpha X + \frac{\beta_6(m\pi)^2-\beta_7\epsilon^2}{2R}\right] \sin m\pi X + I_m \cos m\pi X\Bigg] \tag{4.67}$$

where

$$I_m = m\pi \left[\frac{4\alpha}{(m\pi)^2+\epsilon^2} - \frac{\beta_8}{2Ra}\right] \tag{4.68}$$

and where C_m, D_m, E_m, and F_m are determined from the boundary conditions on ψ:

$$\psi = D\psi = 0 \text{ at } X = 0,1 \ . \tag{4.56}$$

Equations (4.66) and (4.67) are next substituted into (4.64) to yield

$$\sum_{m=1}^{\infty} \frac{B_m[\beta_1(m\pi)^2+\epsilon^2][(m\pi)^2+\epsilon^2]^2}{T\epsilon^2} = \sum_{m=1}^{\infty} B_m \Bigg[\left(C_m + \frac{4\beta_2 R}{T}\epsilon F_m\right)\cosh \epsilon X$$

$$+ \left(D_m + \frac{4\beta_2 R}{T}\epsilon E_m\right) \sinh \epsilon X + E_m X \cosh \epsilon X + F_m X \sinh \epsilon X$$

$$+ \left(1 - \frac{2\beta_2 R}{T}\right)\left(1 + \alpha X + \frac{\beta_6(m\pi)^2-\beta_7\epsilon^2}{2R}\right) \sin m\pi X + J_m \cos m\pi X\Bigg] \tag{4.69}$$

where

$$B_m = A_m/[(m\pi)^2+\epsilon^2]^2$$

and (4.70)

$$J_m = \left[1 - \frac{2\beta_2 R}{T}[(m\pi)^2+\epsilon^2]\right] + 4\alpha m\pi \frac{\beta_2 R}{T} \ .$$

Multiplying this equation by sin $n\pi X$ and integrating between $X = 0$ and $X = 1$, we then obtain an expression of the form

$$\sum_{m=1}^{\infty} B_m K_{mn} = 0 \quad (n=1,2,\ldots) \tag{4.71}$$

which represents an infinite system of linear homogeneous equations in the variables B_m. The condition under which this system has a nontrivial solution is that the determinant of the coefficients K_{mn} should vanish. Since in the present treatment, we are seeking only an approximate solution for T as a function of ϵ, we set only the first element K_{11} of the K-matrix equal to zero. The first approximation, after considerable reduction, is then given by

$$T = \frac{[\pi^2+\epsilon^2]^2[\beta_1\pi^2+\epsilon^2]}{\epsilon^2[1+\alpha/2]\left[1 - \frac{8\pi^2\epsilon}{[\pi^2+\epsilon^2]^2}\frac{(1+\cosh\epsilon)}{(\epsilon+\sinh\epsilon)}\right]\left[1 - \left(\frac{2R}{T}\right)\beta_2(\pi^2+\epsilon^2)\right]\left[1 + \frac{\beta_6\pi^2-\beta_7\epsilon^2}{2R(1+\alpha/2)}\right]} \tag{4.72}$$

4.7.2 Second Analytic Method

The second analytic technique employed was in many respects, quite similar to the first. It was essentially a <u>Galerkin</u> method (see e.g., Crandall[10]), and has been used by Walowit[43] in treating the Newtonian stability problem in the case of a radial temperature gradient. Consider here the complete disturbance equations:

$$[\beta_1 D^2-\epsilon^2]V = \left[\frac{T}{2R} + \beta_2 D^2 - \beta_3\epsilon^2\right]\psi \tag{4.45}$$

$$\left[D^4-\beta_4\epsilon^2D^2 + \frac{2\beta_9}{a}\epsilon^2D+\beta_5\epsilon^4\right]\psi = \epsilon^2\left[-2R(1+\alpha X)+\beta_6D^2+\beta_7\epsilon^2 + \frac{\beta_8 D}{a}\right]V \quad (4.46)$$

Let us write

$$V = -\sum_{m=1}^{\infty} A_m V_m$$

and (4.73)

$$\psi = \sum_{m=1}^{\infty} B_m \psi_m$$

where

$$V_m = X(1-X)X^{m-1}$$

and (4.74)

$$\psi_m = [X(1-X)]^2 X^{m-1}$$

One notes that all boundary conditions on both ψ and V are thus satisfied. Substituting (4.73) and (4.74) into (4.45) and (4.46), we then have

$$\sum_{m=1}^{\infty}\left[A_m(\beta_1D^2-\epsilon^2)V_m + B_m\left[\frac{T}{2R} + \beta_2D^2 - \beta_3\epsilon^2\right]\psi_m\right] = 0$$

(4.75)

$$\sum_{m=1}^{\infty}\left[B_m\left[D^4-\beta_4\epsilon^2D^2+\frac{2\beta_9}{a}\epsilon^2D+\beta_5\epsilon^4\right]\psi_m+A_m\epsilon^2\left[-2R(1+\alpha X)+\beta_6D^2+\beta_7\epsilon^2 + \frac{\beta_8D}{a}\right]V_m\right] = 0$$

These equations are next multiplied by ψ_n and V_n, respectively and integrated between X = 0 and X = 1 to yield expressions of the form

$$\sum_{m=1}^{\infty} [C_{mn}A_m + D_{mn}B_m] = 0 \quad (n=1,2,\ldots)$$

$$\sum_{m=1}^{\infty} [E_{mn}A_m + F_{mn}B_m] = 0 \quad (n=1,2,\ldots) \quad . \tag{4.76}$$

Equations (4.76) represent a doubly infinite set of linear homogeneous equations in the variables A_m and B_m. By setting the determinant of the system equal to zero, we obtain the required characteristic equation for T. As a first approximation, we have*

$$T = \frac{28}{27} \frac{[504+12\beta_4\epsilon^2+\beta_5\epsilon^4][10\beta_1+\epsilon^2]}{\epsilon^2[1+\alpha/2]\left[1 - \frac{2R}{T}\left(\frac{28}{3}\beta_2+\beta_3\epsilon^2\right)\right]\left[1 + \frac{\frac{28}{3}\beta_6-\beta_7\epsilon^2}{2R(1+\alpha/2)}\right]} \tag{4.77}$$

This same method may also be used to solve Eqs. (4.61) for the case of negligible inertia and Eqs. (4.62) for the case of plane Couette flow. One ultimately finds that the first approximations to the characteristic equations for these situations are given respectively by

$$\frac{28}{27} \frac{[504+12\beta_4\epsilon^2+\beta_5\epsilon^4][10\beta_1+\epsilon^2]}{\epsilon^2\left[\beta_7\epsilon^2 - \frac{28}{3}\beta_6\right]\left[\beta_3\epsilon^2 + \frac{28}{3}\beta_2\right]} = 1 \tag{4.78}$$

and

$$\mathcal{R}' = \frac{28}{27} \frac{[504+12\beta_4\epsilon^2+\beta_5\epsilon^4][10\beta_1+\epsilon^2]}{\epsilon^2\left[\beta_7\epsilon^2 - \frac{28}{3}\beta_6\right]} - \left\{\frac{28}{3}\beta_2+\beta_3\epsilon^2\right\} \quad . \tag{4.79}$$

4.7.3 Numerical Method

The numerical method used in solving the disturbance equations was

*The similarities between Eqs. (4.77) and (4.72) should be noted.

apparently first successfully applied by Beard, Davies, and Walters.[1] It consists of replacing the current boundary-value problem with a related initial-value problem.

In any eigenvalue analysis, there is an arbitrary amplitude associated with the eigenfunctions. This indeterminancy can be removed in the present instance simply by imposing the additional initial condition

$$D^2\psi = 1 \text{ at } X = 0 \quad . \tag{4.80}$$

We next assume values for $D^3\psi$ and DV at X = 0:

$$D^3\psi]_{X=0} = G_1, \quad DV]_{X=0} = G_2 \tag{4.81}$$

where G_1 and G_2 represent first approximations. An initial-value problem is now defined by Eqs. (4.45) and (4.46) with boundary conditions

$$\psi = D\psi = V = 0, \quad D^2\psi = 1, \quad D^3\psi = G_1, \quad DV = G_2 \tag{4.82}$$

at X=0. Once the coefficients in (4.45) and (4.46) are specified, one can integrate the system of equations numerically by using a standard Runge-Kutta technique (see e.g., Carnahan, Luther, and Wilkes[3]) from X = 0 to X = 1. In the present work, the method of obtaining these coefficients was as follows:

(1) Making use of Eq. (4.10), rewrite the definition of the Taylor number T in the form

$$T = -2\alpha a \left[\frac{\rho d^2 \Omega_1}{\eta}\right]^2 = -\frac{2}{\alpha a}\left[\frac{\gamma T}{E\ell}\right]^2 \left[\frac{\eta_0}{\eta}\right]^2 \tag{4.83}$$

where τ is an appropriately chosen characteristic time for the fluid, η_o is the zero-shear viscosity, and $E\ell$ is the elasticity number, a dimensionless constant:

$$E\ell = \frac{\eta_o \tau}{\rho d^2} \quad . \tag{4.84}$$

(2) Define the modified Taylor number T_m and the modified Reynolds number R_m by

$$T_m = T\left(\frac{\eta}{\eta_o}\right)^2 = -\frac{2}{\alpha a}\left[\frac{\gamma\tau}{E\ell}\right]^2 = -2\alpha a R_m^2 \quad . \tag{4.85}$$

(3) Express the fluid viscosity η in the form

$$\eta = \eta_o/\beta_o\ (\gamma\tau) \tag{4.86}$$

where β_o defines a new dimensionless function.

(4) Express the material coefficients $\beta_i(\gamma)$ (i=1,...,9) in the form

$$\beta_i = \beta_i(\gamma\tau) \quad . \tag{4.87}$$

(5) Assume a value for the modified Taylor number T_m:

$$T_m = G_3 \quad . \tag{4.88}$$

(6) Calculate the corresponding modified Reynolds number R_m:

$$R_m = \frac{[\gamma\tau]}{[\alpha a][E\ell]} = \sqrt{G_3/-2\alpha a} \quad . \tag{4.89}$$

(7) Calculate the dimensionless shear rate γ_T

$$\gamma_T \;=\; [\alpha a][E\ell]R_m \;. \tag{4.90}$$

(8) Evaluate the material functions $\beta_i(\gamma_T)$ $(i=0,\ldots,9)$ at the dimensionless shear rate.

(9) Calculate the actual Taylor number and actual Reynolds number from the equations

$$\begin{aligned} T &= T_m\beta_o^2 \\ R &= R_m\beta_o \;. \end{aligned} \tag{4.91}$$

The solutions for ψ, $D\psi$, and V obtained by integrating the disturbance equations from X = 0 to X = 1 will depend, presumably in a continuous fashion, on the values chosen for the three quantities G_1, G_2, and G_3. Of course, for arbitrary choices, the boundary conditions (4.56) will not necessarily be satisfied at X = 1. Therefore, one must determine appropriate corrections to G_1, G_2, and G_3 in order that these conditions may be met. Such corrections are obtained in the present work using a Newton-Raphson technique;[3] in particular, we write, to terms of the first order,

$$\begin{bmatrix} \dfrac{\partial\psi}{\partial G_1} & \dfrac{\partial\psi}{\partial G_2} & \dfrac{\partial\psi}{\partial G_3} \\[2ex] \dfrac{\partial(D\psi)}{\partial G_1} & \dfrac{\partial(D\psi)}{\partial G_2} & \dfrac{\partial(D\psi)}{\partial G_3} \\[2ex] \dfrac{\partial V}{\partial G_1} & \dfrac{\partial V}{\partial G_2} & \dfrac{\partial V}{\partial G_3} \end{bmatrix}_{X=1} \begin{bmatrix} \delta G_1 \\[2ex] \delta G_2 \\[2ex] \delta G_3 \end{bmatrix} = - \begin{bmatrix} \psi \\[2ex] D\psi \\[2ex] V \end{bmatrix}_{X=1} \tag{4.92}$$

where δG_i (i=1,2,3) represent the required changes in the G_i. Iteration on G_1, G_2, and G_3 is repeated until satisfactory convergence of ψ, $D\psi$, and V at X = 1 has been achieved.

The nine partial derivatives in Eq. (4.92) have been evaluated herein by a finite-difference process. That is, each of the quantities G_1, G_2, and G_3 was individually given a small perturbation, and the corresponding differences in ψ, $D\psi$, and V at X = 1 were noted.

By making use of the methods outlined above, one can determine the Taylor number for any value of the dimensionless wave number ϵ. The smallest T over all ϵ represents the critical Taylor number T_c at which vortex cells first appear in the flow. In this study, T_c has been found numerically by use of the "golden-section" minimum searching technique (see e.g., Wilde[46]).

Numerical calculations have been performed on the IBM 7090 digital computer located at The University of Michigan Computing Center. A short description of the programs and subroutines employed is presented in Appendix D.

4.8 QUALITATIVE DISCUSSION

Let us now examine the qualitative effects of the various rheological coefficients β_1 to β_9 on the critical parameters T_c and ϵ_c. For this purpose, we consider Eq. (4.77), our approximate solution to the complete disturbance equations.

One notes immediately that the variables β_8 and β_9 are absent from (4.77). Therefore, to a first approximation, β_8 and β_9 have a relatively small influence on the stability criteria. This fact has been verified numerically; in one numerical case, β_8 was varied from -2.4 to +2.4 while, in another case, β_9 was varied from 0 to 2; in both instances, it was found that T_c and ϵ_c changed by less than 0.05%. Hence, we see that, for moderate values of β_8 and β_9, the critical parameters T_c and ϵ_c are determined primarily by six material functions (η, σ_2, and β_2 to β_5) of the shear rate rather than by eight. The corresponding disturbance equations for this situation are given by

$$[\beta_1 D^2 - \epsilon^2]V = \left[\frac{T}{2R} + \beta_2 D^2 - \beta_3 \epsilon^2\right]\psi \tag{4.93}$$

$$[D^4 - \beta_4 \epsilon^2 D^2 + \beta_5 \epsilon^4]\psi = \epsilon^2[-2R(1+\alpha X) + \beta_6 D^2 + \beta_7 \epsilon^2]V \quad .$$

In studying the effects of the other material coefficients β_1 to β_7 on the stability criteria T_c and ϵ_c, we begin by noting that Eq. (4.77) may be written in the general form

$$T = f[\beta_i(T), T, R(T), \epsilon] \quad (i=1,\ldots,7) \tag{4.94}$$

for fixed values of the parameters α and a. Taking the derivative of T with respect to ϵ, we have

$$\frac{dT}{d\epsilon} = \frac{\partial f}{\partial \beta_i}\frac{d\beta_i}{dT}\frac{dT}{d\epsilon} + \frac{\partial f}{\partial T}\frac{dT}{d\epsilon} + \frac{\partial f}{\partial R}\frac{dR}{dT}\frac{dT}{d\epsilon} + \frac{\partial f}{\partial \epsilon} \tag{4.95}$$

or equivalently

$$\frac{dT}{d\epsilon} = \frac{[\partial f/\partial \epsilon]}{\left[1 - \left[\frac{\partial f}{\partial \beta_i}\frac{d\beta_i}{dT} + \frac{\partial f}{\partial T} + \frac{\partial f}{\partial R}\frac{dR}{dT}\right]\right]} \quad . \tag{4.96}$$

The critical Taylor number T_c and the critical wave number ϵ_c are found by setting $dT/d\epsilon$ equal to zero. From (4.96), one obtains

$$\frac{\partial f}{\partial \epsilon}\,[\beta_i(T_c), T_c, R_c(T_c), \epsilon_c] = 0 \tag{4.97}$$

while from (4.94),

$$T_c = f[\beta_i(T_c), T_c, R_c(T_c), \epsilon_c] \tag{4.98}$$

where

$$T_c = -2\alpha a R_c^2 \ . \tag{4.99}$$

Equations (4.97) and (4.98) represent two simultaneous equations in the two unknowns T_c and ϵ_c.

Let us now allow $\beta_i(T_c)$ to vary independently of T_c* and write

$$\frac{d}{d\beta_i(T_c)}\frac{\partial f}{\partial \epsilon}[\beta_i(T_c), T_c, R_c, \epsilon_c]$$
$$= \left[\frac{\partial^2 f}{\partial\epsilon\partial\beta_i} + \frac{\partial^2 f}{\partial\epsilon\partial T}\frac{dT_c}{d\beta_i(T_c)} + \frac{\partial^2 f}{\partial\epsilon\partial R}\frac{dR_c}{dT_c}\frac{dT_c}{d\beta_i(T_c)} + \frac{\partial^2 f}{\partial\epsilon^2}\frac{d\epsilon_c}{d\beta_i(T_c)}\right]_{T_c,\epsilon_c} = 0 \tag{4.100}$$

and

$$\frac{dT_c}{d\beta_i(T_c)} = \left[\frac{\partial f}{\partial\beta_i} + \frac{\partial f}{\partial T}\frac{dT_c}{d\beta_i(T_c)} + \frac{\partial f}{\partial R}\frac{dR_c}{dT_c}\frac{dT_c}{d\beta_i(T_c)} + \frac{\partial f}{\partial\epsilon}\frac{d\epsilon_c}{d\beta_i(T_c)}\right]_{T_c,\epsilon_c} \tag{4.101}$$

*This can be accomplished by varying the material parameters.

Substituting (4.97) and (4.99) into (4.101) and solving for $dT_c/d\beta_i(T_c)$, we have

$$\frac{dT_c}{d\beta_i(T_c)} = \left. \frac{[\partial f/\partial \beta_i]}{\left[1 - \left[\frac{\partial f}{\partial T} + \frac{\partial f}{\partial R}\frac{R_c}{2T_c}\right]\right]} \right]_{T_c,\epsilon_c} \tag{4.102}$$

Moreover, from (4.100),

$$\frac{d\epsilon_c}{d\beta_i(T_c)} = - \left. \frac{\left[\frac{\partial^2 f}{\partial\epsilon\partial\beta_i} + \left(\frac{\partial^2 f}{\partial\epsilon\partial T} + \frac{\partial^2 f}{\partial\epsilon\partial R}\frac{R_c}{2T_c}\right)\frac{dT_c}{d\beta_i(T_c)}\right]}{[\partial^2 f/\partial\epsilon^2]} \right]_{T_c,\epsilon_c} \tag{4.103}$$

The terms

$$\left[\frac{\partial f}{\partial T} + \frac{\partial f}{\partial R}\frac{R_c}{2T_c}\right]_{T_c,\epsilon_c} \quad \text{and} \quad \left[\frac{\partial^2 f}{\partial\epsilon\partial T} + \frac{\partial^2 f}{\partial\epsilon\partial R}\frac{R_c}{2T_c}\right]_{T_c,\epsilon_c}$$

in these equations can be evaluated using Eq. (4.77). One obtains

$$\left[\frac{\partial f}{\partial T} + \frac{\partial f}{\partial R}\frac{R_c}{2T_c}\right]_{T_c,\epsilon_c} = \frac{1}{2}\left[\frac{\dfrac{(\frac{28}{3}\beta_6 - \beta_7\epsilon^2)}{2R_c(1+\alpha/2)}}{1 + \dfrac{(\frac{28}{3}\beta_6 - \beta_7\epsilon^2)}{2R_c(1+\alpha/2)}} - \frac{\dfrac{2R_c}{T_c}(\frac{28}{3}\beta_2 + \beta_3\epsilon^2)}{1 - \dfrac{2R_c}{T_c}(\frac{28}{3}\beta_2 + \beta_3\epsilon^2)}\right]_{T_c,\epsilon_c} \tag{4.104}$$

and

$$\left[\frac{\partial^2 f}{\partial\epsilon\partial T} + \frac{\partial^2 f}{\partial\epsilon\partial R}\frac{R_c}{2T_c}\right]_{T_c,\epsilon_c} = -\left[\frac{\dfrac{2R_c}{T_c}\beta_3\epsilon_c}{1 - \dfrac{2R_c}{T_c}(\frac{28}{3}\beta_2 + \beta_3\epsilon^2)} + \frac{\dfrac{\beta_7\epsilon_c}{2R_c(1+\alpha/2)}}{1 + \dfrac{(\frac{28}{3}\beta_6 - \beta_7\epsilon^2)}{2R_c(1+\alpha/2)}}\right]_{T_c,\epsilon_c} \tag{4.105}$$

Now, in the present work, the expression given by (4.104) was, in all instances, small in comparison with unity. Thus, it appears from (4.102) that the sign of $dT_c/d\beta_i(T_c)$ was determined entirely by the sign of $[\partial f/\partial \beta_i]_{T_c,\epsilon_c}$. This quantity has been calculated for each of the cases i=1,...,7 and the final results are presented in Table 4.2.

TABLE 4.2

EFFECT ON T_c OF A SMALL CHANGE IN $\beta_i(T_c)$

Parameter	$dT_c/d\beta_i(T_c)$
β_1	Greater than zero
β_2	Greater than zero
β_3	Greater than zero
β_4	Greater than zero
β_5	Greater than zero
β_6	Less than zero
β_7	Greater than zero

In investigating the influence on ϵ_c of a small change in $\beta_i(T_c)$, we consider here (for simplicity) only deviations relative to the Newtonian case, $\beta_1 = 1$, $\beta_2 = \beta_3 = \beta_6 = \beta_7 = 0$, $\beta_4 = 2$, $\beta_5 = 1$. Under these circumstances, from (4.103) and (4.105), one has

$$\frac{d\epsilon_c}{d\beta_i(T_c)} = - \left.\frac{[\partial^2 f/\partial\epsilon\partial\beta_i]}{[\partial^2 f/\epsilon^2]}\right]_{T_c,\epsilon_c} . \qquad (4.106)$$

Recalling now that at the minimum point of a function, the second derivative is positive, it is seen that

$$\left.\frac{\partial^2 f}{\partial\epsilon^2}\right]_{T_c,\epsilon_c} > 0 \quad . \qquad (4.107)$$

Therefore, for $[\partial^2 f/\partial\epsilon\partial\beta_i]_{T_c,\epsilon_c} > 0$, we have that $d\epsilon_c/d\beta_i(T_c) < 0$, and vice versa. Table 4.3 presents the results of a study on the sign of $[\partial^2 f/\partial\epsilon\partial\beta_i]_{T_c,\epsilon_c}$ for each of the cases $i = 1,\ldots,7$.

TABLE 4.3

EFFECT ON ϵ_c OF A SMALL CHANGE IN $\beta_i(T_c)$*

Parameter	$d\epsilon_c/d\beta_i(T_c)$
β_1	Greater than zero
β_2	Zero
β_3	Less than zero
β_4	Less than zero
β_5	Less than zero
β_6	Zero
β_7	Less than zero

*Relative to Newtonian case.

4.9 PRESENTATION AND DISCUSSION OF NUMERICAL RESULTS

In this section, the numerical results of the current investigation are presented. Consideration is given first to the Newtonian case, where the values obtained here for the critical parameters T_c and ϵ_c are compared with those found by other authors. Next, we examine the stability behavior of a fluid defined by the general constitutive model $\underset{\sim}{S} + p\underset{\sim}{I} = 2\eta(\gamma)\underset{\sim}{E}$. Thirdly, we employ the approximations presented in Eqs. (4.58) to solve the disturbance equations for the case of two fluids whose viscometric functions η, σ_1, and σ_2 have been determined experimentally by Huppler;[24] the current experimental and numerical results are then compared and certain definite conclusions are reached concerning the validity of the "viscometric hypothesis." Finally, we consider numerically the effect on T_c and ϵ_c of varying the parameters β_2, β_3, β_4, and β_5.

4.9.1 The Newtonian Case

The disturbance equations describing this situation are given by

$$\begin{aligned} [D^2-\epsilon^2]V &= \frac{T}{2R}\psi \\ [D^2-\epsilon^2]^2\psi &= -2\epsilon^2R(1+\alpha X)V \end{aligned} \qquad (4.108)$$

which may be expressed in the form

$$\begin{aligned} [D^2-\epsilon^2]V &= \psi \\ [D^2-\epsilon^2]^2\psi &= -T\epsilon^2(1+\alpha X)V \end{aligned} \qquad (4.109)$$

if the variable ψ is suitably redefined. It is observed from equations (4.109) that the Reynolds number $R = \sqrt{T/(-2\alpha a)}$ (or equivalently the radius-to-gap ratio $a = r_1/d$) does not occur explicitly in the small-gap Newtonian stability problem. Thus, for this case, the stability criteria T_c and ϵ_c are functions only of the parameter α.

In order to illustrate the accuracy of the present numerical procedure, Table 4.4, which contains a comparison of the current findings with those of several other authors (for the Newtonian case), has been prepared. One notes the excellent agreement between the present and past results.

TABLE 4.4

VALUES OF T_c AND ϵ_c FOR NEWTONIAN CASE

Investigator	Critical Parameters	$\alpha = -0.5$	$\alpha = -1.0$	$\alpha = -1.5$	$\alpha = -2.0$
Thomas and Walters[40]	T_c	2285	3404	6431	--
	ϵ_c	3.12	3.12	3.20	--
Chan Man Fong[6]	T_c	2290	3414	--	--
	ϵ_c	3.10	3.10	--	--
Beard, Davies, and Walters[1]	T_c	2275	--	--	--
	ϵ_c	--	--	--	--
Chandrasekhar[5]	T_c	2275	3390	6417	18677
	ϵ_c	3.12	3.12	3.20	4.00
Harris and Reid[23]	T_c	--	3390	6414	18663
	ϵ_c	--	3.13	3.20	4.00
This study	T_c	2275	3390	6414	18669
	ϵ_c	3.12	3.13	3.20	4.00

4.9.2 Case of a Fluid Defined by $\underset{\sim}{S}+p\underset{\sim}{I} = 2\eta(\gamma)\underset{\sim}{E}$

A fluid obeying the general constitutive law

$$\underset{\sim}{S} + p\underset{\sim}{I} = 2\eta(\gamma)\underset{\sim}{E} \qquad (4.110)$$

exhibits neither normal stresses nor elasticity; nonetheless, this model provides a worthwhile basis for comparison with real viscoelastic behavior since it possesses a viscosity which depends on shear rate, a property common to all viscoelastic fluids. The usual Bingham plastic model, the Ostwald-de Waele "power-law" fluid, the Reiner-Philippoff fluid, the Eyring fluid, and the Ellis fluid are all special cases of

(4.110). Hence, the need for any future stability treatments using these or other such simple models should be obviated by the present analysis.

By redefining the variable ψ, it is possible to express the disturbance equations for (4.110) in the form

$$\begin{aligned} [\beta_1 D^2 - \epsilon^2]V &= \psi \\ [D^2-\epsilon^2]^2\psi &= -T\epsilon^2[1+\alpha X]V \quad . \end{aligned} \tag{4.111}$$

We see then that the stability criteria for this model will be governed by the two variables α and β_1.

Owing to the limited amount of computer time available in the present work, numerical calculations have been restricted to the case of $\alpha = -1$. However, suitable approximations for T_c and ϵ_c for other values of α can be determined using Eq. (4.77). In particular, it is found that, for $-1 < \alpha < 0$

$$\frac{T_c(\alpha,\beta_1)}{T_c(-1,\beta_1)} \doteq \frac{T_c(\alpha,1)}{T_c(-1,1)}$$

and (4.112)

$$\epsilon_c(\alpha,\beta_1) \doteq \epsilon_c(-1,\beta_1)$$

where $T_c(\alpha,1)$ and $T_c(-1,1)$ refer to the critical values associated with the Newtonian case.

Table 4.5 presents the results of the numerical calculations performed herein for the model defined by (4.110), and Figs. 4.1 and 4.2

show plots of T_c and ϵ_c vs. β_1 (for $\alpha = -1$). It will be observed that both T_c and ϵ_c increase with increasing β_1, in qualitative agreement with the predictions of Tables 4.2 and 4.3.

TABLE 4.5

NUMERICAL RESULTS FOR A FLUID DEFINED BY (4.110); $\alpha = -1$

β_1	T_c	ϵ_c
2.00	5009	3.50
1.00	3390	3.13
0.99	3373	3.12
0.98	3356	3.12
0.97	3338	3.11
0.96	3321	3.10
0.95	3304	3.10
0.90	3217	3.07
0.80	3040	3.01
0.60	2672	2.85
0.40	2276	2.74
0.20	1822	2.30
0.10	1542	2.00

From the present experimental findings, the values of β_1 occurring at instability have been calculated for each of the polymer solutions investigated, and these values have then been used in conjunction with Figs. 4.1 and 4.2 in an attempt to predict the experimentally observed stability behavior. Table 4.6, which contains a comparison between theory and experiment, presents the corresponding fractional reductions in T_c and ϵ_c. One notes that the theoretical results based on Eq. (4.110) are in qualitative agreement with experiment, in that the critical parameters both decreased with polymer concentration. However, the predicted decrease in ϵ_c was, in all cases, somewhat smaller than observed.

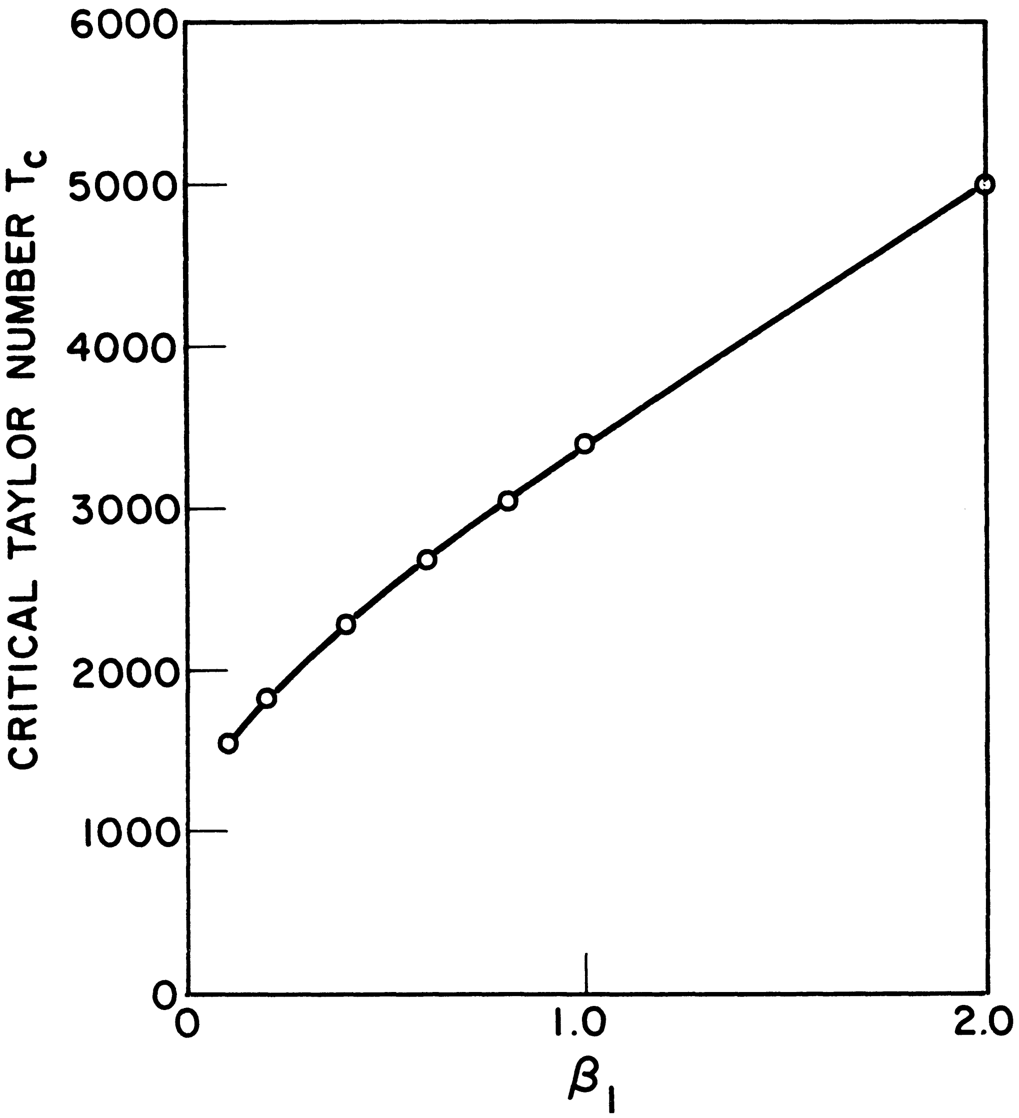


Fig. 4.1. Plot of T_c against β_1 for a fluid defined by (4.110); α = -1.

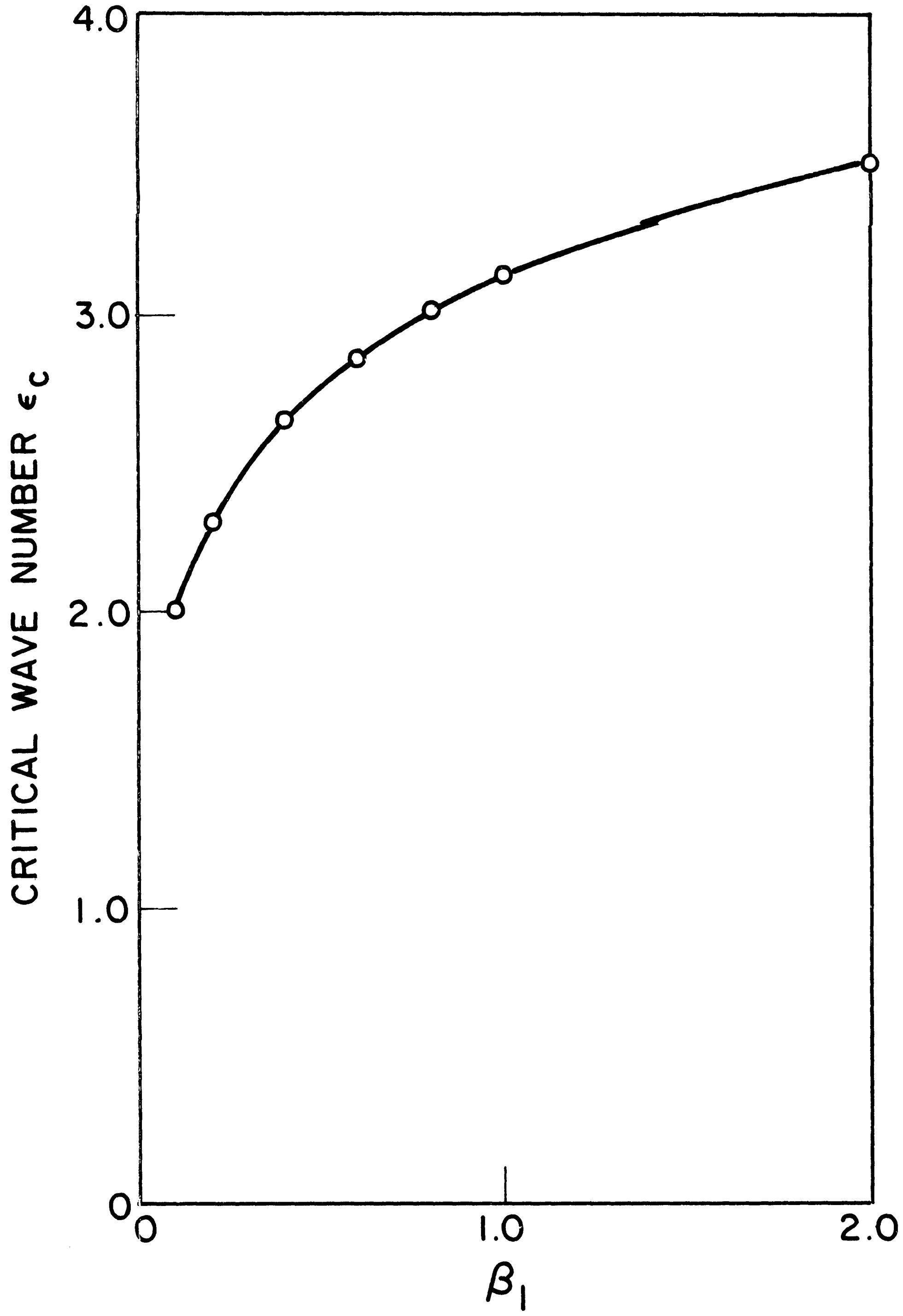


Fig. 4.2. Plot of ϵ_c against β_1 for a fluid defined by (4.110); α = -1.

TABLE 4.6

FRACTIONAL REDUCTIONS IN T_c AND ϵ_c

Substance	Conc. (Wt.%)	Experimental		Theoretical*	
		$T_c/T_{c,N}$	$\epsilon_c/\epsilon_{c,N}$	$T_c/T_{c,N}$	$\epsilon_c/\epsilon_{c,N}$
7MT CMC	0.5	1.0	0.97	1.0	1.0
	1.0	0.92	0.93	0.94	0.97
	1.67	0.88	0.90	0.89	0.95
	2.5	0.45	0.85	0.80	0.91
P-75-XH CMC	0.183	1.01	0.88	0.82	0.92
	0.367	0.95	0.75	0.80	0.91
	0.55	0.79	0.71	0.70	0.89
Natrosol 250-H HEC	0.48	0.86	0.82	0.80	0.91
	0.80	0.74	0.72	0.67	0.85
	1.0	0.63	0.69	0.64	0.82

*Based on Eq. (4.110) as a constitutive model.

4.9.3 Approximate Analysis

Metzner, White, and Denn[29] have suggested that the equation

$$\underset{\sim}{S} + p\underset{\sim}{I} = \eta(\gamma)\underset{\sim}{A}_1 + \frac{\sigma_1}{\gamma^2}\underset{\sim}{A}_1^2 + \frac{(\sigma_2-\sigma_1)}{2\gamma^2}\underset{\sim}{A}_2 \qquad (1.57)$$

or equivalently

$$\underset{\sim}{S} + p\underset{\sim}{I} = 2\eta(\gamma)\underset{\sim}{E} + \frac{2(\sigma_1+\sigma_2)}{\gamma^2}\underset{\sim}{E} + \frac{\sigma_2-\sigma_1}{\gamma^2}\frac{\mathcal{D}\underset{\sim}{E}}{\mathcal{D}t} \qquad (1.56)$$

might be useful in describing motions, such as the current one, which are extremely close approximations to viscometric flows (viscometric hypothesis).* If this were the case, then, as has been indicated earlier in Section 4.4, the "unknown" material functions β_2 to β_5 and β_9 would be given by the low-shear-rate approximation (4.58), and, in addition, the Taylor-Couette stability behavior of a fluid would be governed entirely

*The reader will recall that (1.57) is applicable in the strictest sense only to purely viscometric flows.

by η, σ_1, and σ_2. In order to explore these possibilities fully, the following course of action has been taken. First of all, the viscometric data of Huppler[24] on solutions of 0.5% P-75-XH CMC and 0.9% Natrosol 250-H HEC have been represented (in the power-law region) by empirical expressions of the form

$$\begin{aligned} \eta &= \eta_o / |\gamma\tau|^{p_o} \\ \sigma_1 &= c_1 \eta_o |\gamma| / |\gamma\tau|^{p_1} \\ \sigma_2 &= c_2 \eta_o |\gamma| / |\gamma\tau|^{p_2} \end{aligned} \tag{4.113}$$

where τ is a characteristic time,* η_o is the zero-shear viscosity, and c_1, c_2, p_o, p_1, and p_2 are dimensionless material constants; the particular values employed for these quantities are shown in Table 4.7.

TABLE 4.7

MATERIAL CONSTANTS USED TO APPROXIMATE HUPPLER'S[24] DATA

Substance	τ(sec)	η_o(poise)	c_1	c_2	p_o	p_1	p_2
P-75-XH CMC	0.76	7.42	0.275	0.0812	0.446	0.13	0.19
Natrosol 250-H	0.148	13.7	1.99	0.127	0.61	0.423	0.375

Next, based on Eqs. (4.58), (4.113), and Table 4.7, the disturbance equations for the two fluids of interest have been solved for the case

*In the present work, the characteristic time τ has been taken to be the reciprocal of the shear rate occurring at the point of intersection of the zero-shear viscosity line with the power law line in Fig. 1.2. It is found then that for $|\gamma\tau| < 1$, $\eta \simeq \eta_o$ while for $|\gamma\tau| > 1$, η is approximated by the power law.

where $\alpha = -1$ and $a = 12$.* The numerical results obtained are presented in Table 4.8 along with the predicted small-gap values of T_c and ϵ_c, determined from the stability tests.

TABLE 4.8

COMPARISON OF APPROXIMATE THEORY WITH EXPERIMENT

Substance	Theoretical		Experimental	
	T_c	ϵ_c	T_c	ϵ_c
P-75-XH CMC	1771	3.73	2900	2.2
Natrosol 250-H	1001	5.25	2350	2.2

As can be seen, a considerably greater reduction in stability is found theoretically than was actually observed. Even more noteworthy is the fact that the critical wave number ϵ_c decreased from its Newtonian value of 3.13 in the experiments while, in the calculations, this quantity was greater then $\epsilon_{c,N}$. One could argue that such discrepencies in results might be attributed to small deviations of Eqs. (4.113) and Table 4.7 from the exact viscometric functions. However, calculations have shown this not to be the case. For example, it was necessary to actually reverse the sign on σ_2 before the experimental and numerical findings could be brought into even rough agreement.** Thus, we have

*It is recalled that values of $a = 12$ and $\alpha = -1$ correspond to the current stability apparatus and experiments.

**Dr. B. Duane Marsh[27] has indicated in a private communication that Huppler's results for σ_2 have been checked by an independent technique, and that the data from the two methods agree to within 10% for HEC solutions, giving in each case the same sign.

demonstrated that Eqs. (4.58) do not represent a reasonable approximation for β_2 to β_5 and β_9 at finite rates of shear and, furthermore, that the "viscometric hypothesis" is unacceptable in the present instance.

4.9.4 Effect of Varying β_2, β_3, β_4, and β_5

In order to illustrate the fact that agreement between the present theoretical development and experiment is possible for certain values of the parameters β_2 to β_5,* these quantities have been allowed to vary, while η, σ_1, σ_2 (and thus β_1, β_6, β_7, and β_8) have retained their functional dependence as given in the previous section. The disturbance equations for the two fluids of interest were then solved numerically for each of the cases listed in Tables 4.9 and 4.10. From the first en-

TABLE 4.9

EFFECT OF VARYING β_2 to β_5 FOR P-75-XH CMC

β_2	β_3	β_4	β_5	T_c	ϵ_c
0	0	2	1	2067	3.47
-1	0	2	1	1863	3.50
1	0	2	1	2290	3.44
0	-1	2	1	1798	3.95
0	1	2	1	2301	3.15
0	2	2	1	2511	2.92
-2	1	2	1	1882	3.22
2	-1	2	1	2223	3.86
0.5	0.5	2	1	2301	3.28
-0.5	-0.5	2	1	1838	3.70
-1	1	2	1	2082	3.19
-3	3	2	1	2032	2.85
0	0	2	2	2349	3.00
0	0	2	0.5	1851	4.03
0	0	3	1	2407	3.26
0	0	1	1	1706	3.71

*In Section 4.8, we have shown that the dependence of T_c and ϵ_c on β_9 is very weak.

TABLE 4.10

EFFECT OF VARYING β_2 TO β_5 FOR NATROSOL 250-H

β_2	β_3	β_4	β_5	T_c	ϵ_c
0	0	2	1	1925	3.36
1	0	2	1	2133	3.31
-1	0	2	1	1738	3.40
0	-1	2	1	1683	3.88
0	1	2	1	2133	3.03
0	2	2	1	2314	2.79
1	-0.5	2	1	2007	3.54
2	-1	2	1	2080	3.80
-2	1	2	1	1748	3.11
0.5	0.5	2	1	2138	3.15
-0.5	-0.5	2	1	1718	3.60
-1	1	2	1	1932	3.06
-3	3	2	1	1869	2.72
0	0	3	1	2228	3.12
0	0	1	1	1600	3.64
0	0	2	2	2158	2.89
0	0	2	0.5	1743	3.93

tries in these tables, it is apparent that replacement of β_2 to β_5 by their respective Newtonian values, of 0,0,2, and 1, yields considerable improvement over the theoretical results shown in Table 4.8.

Figures 4.3 and 4.4 indicate graphically the effect on T_c and ϵ_c caused by varying the parameters β_2 and β_3, with β_4 and β_5 being held fixed at $\beta_4 = 2$, $\beta_5 = 1$. Similarly, Figs. 4.5 and 4.6 show the dependence of the stability criteria on β_4 and β_5, with $\beta_2 = \beta_3 = 0$. In these figures, the experimentally predicted small-gap values of T_c and ϵ_c are indicated using the symbol ⊗, and the numbers in parenthesis represent points in the original β_2-β_3 or β_4-β_5 planes which have been mapped into

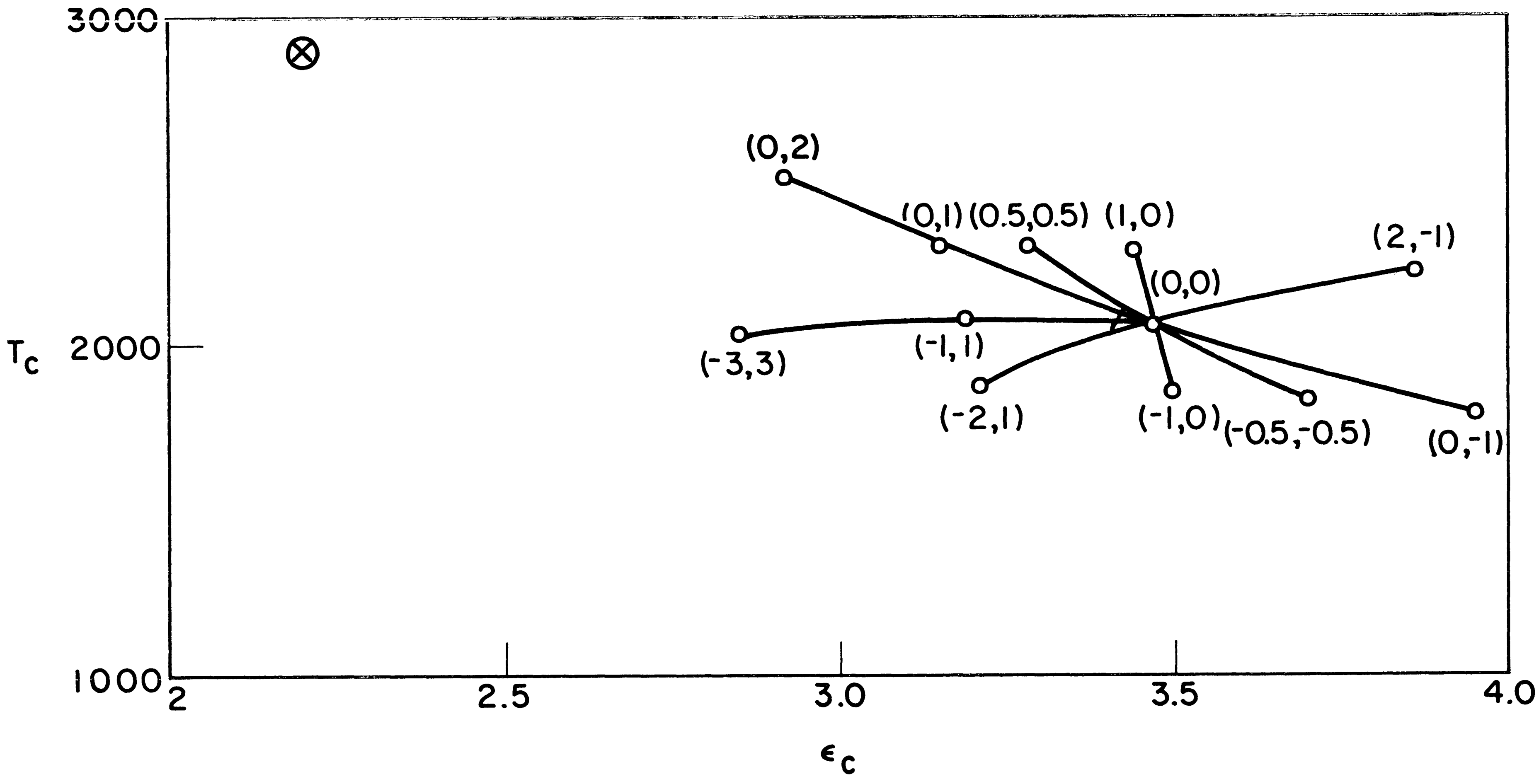


Fig. 4.3. Mapping of the β_2-β_3 plane into the T_c-ϵ_c plane for a 0.5% solution of P-75-XH CMC; $\alpha = -1$, $a = 12$, $\beta_4 = 2$, $\beta_5 = 1$, $\beta_9 = 0$.

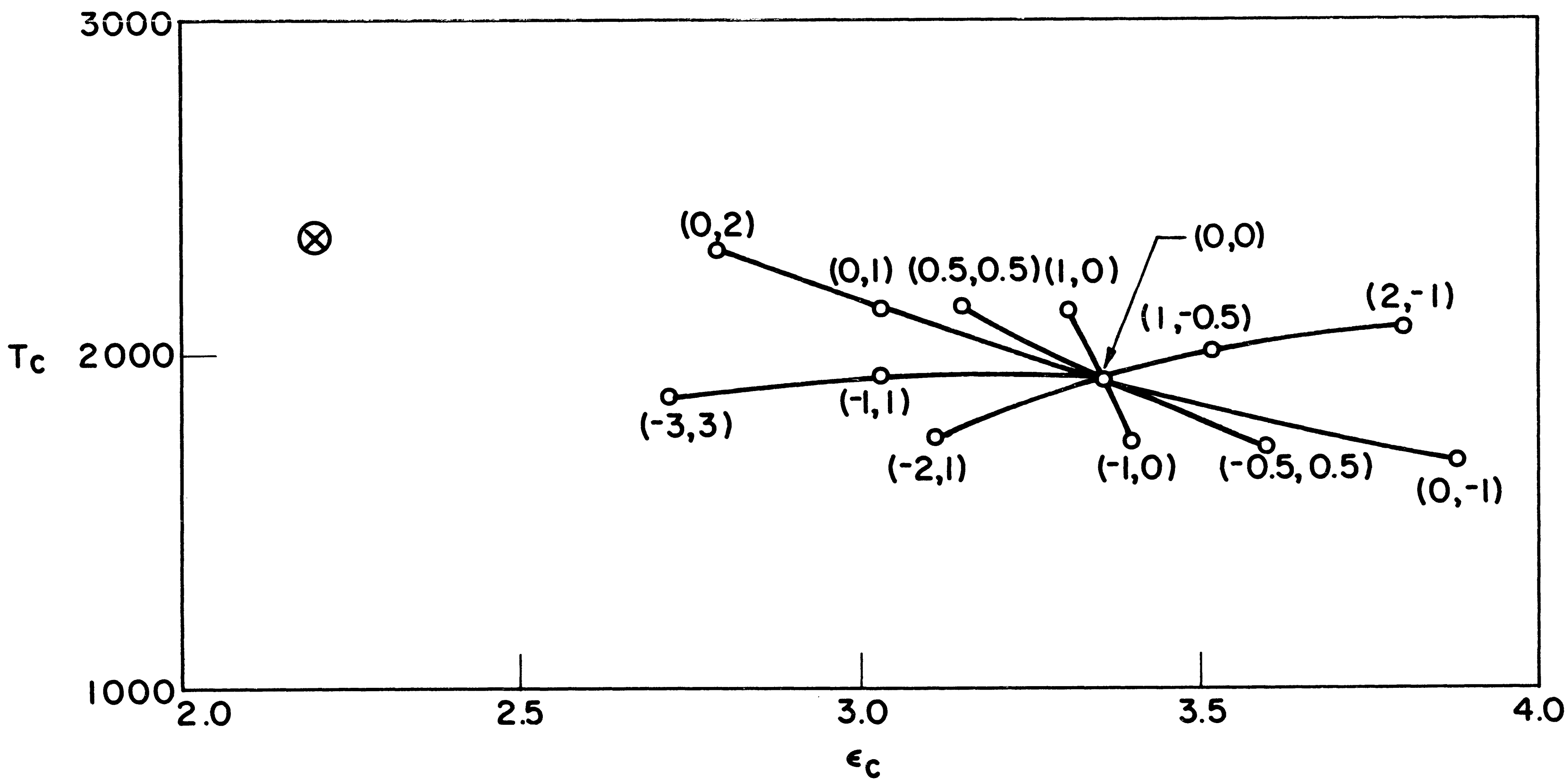


Fig. 4.4. Mapping of the β_2-β_3 plane into the T_c-ϵ_c plane for a 0.9% solution of Natrosol 250-H; $\alpha = -1$, $a = 12$, $\beta_4 = 2$, $\beta_5 = 1$, $\beta_9 = 0$.

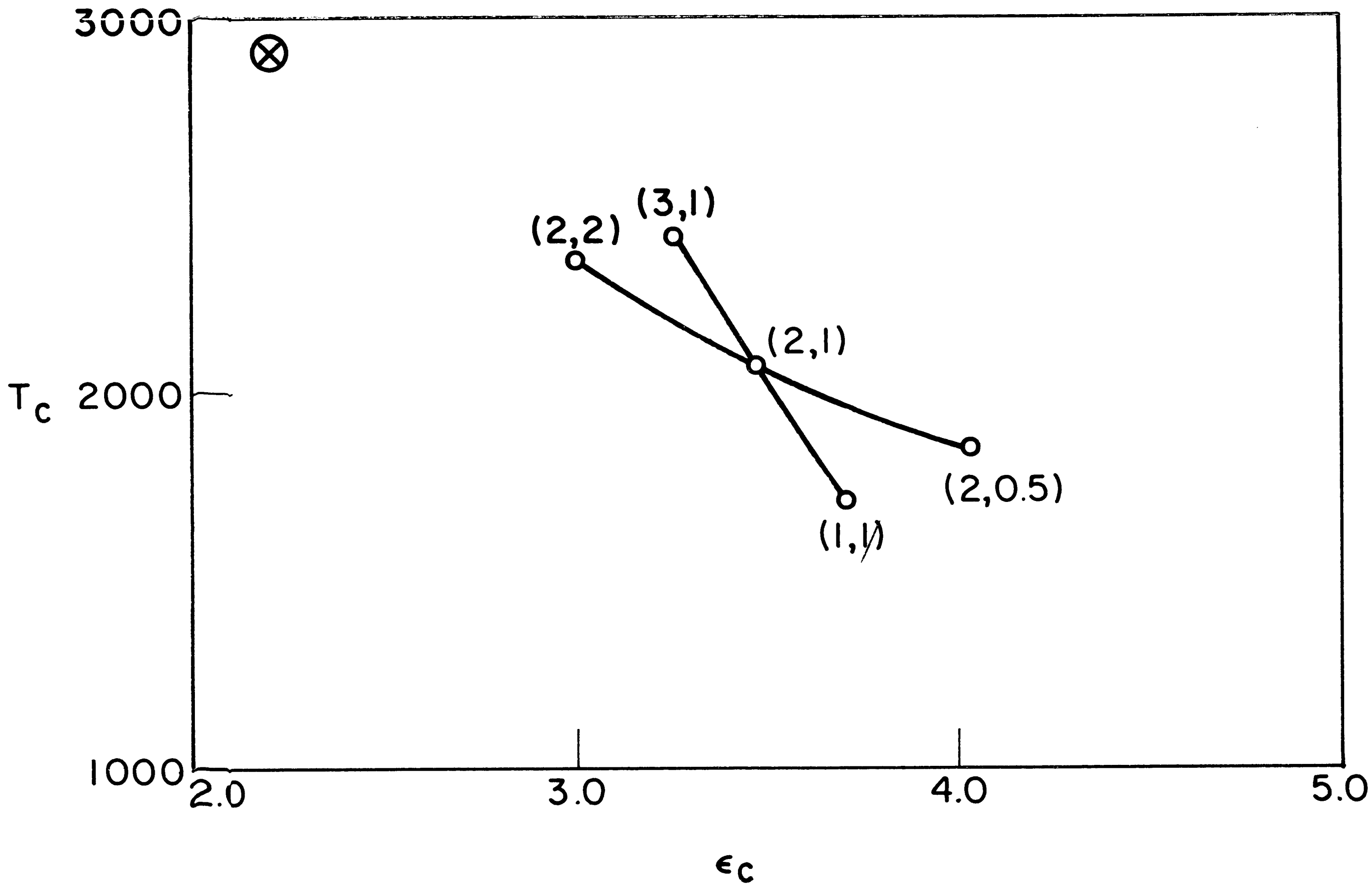


Fig. 4.5. Mapping of the β_4-β_5 plane into the T_c-ϵ_c plane for a 0.5% solution of P-75-XH CMC; $\alpha = -1$, a = 12, $\beta_2 = \beta_3 = \beta_5 = 0$.

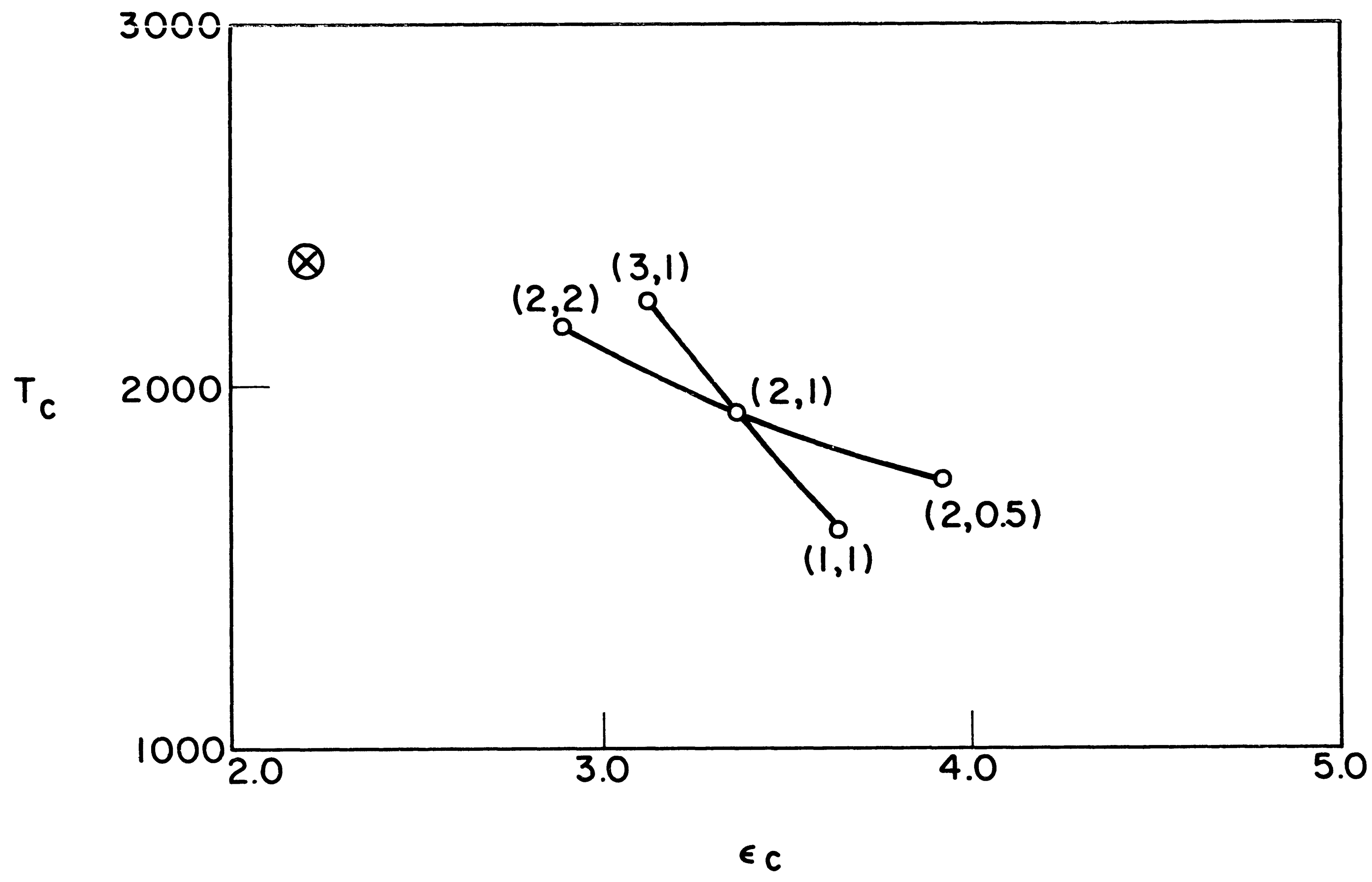


Fig. 4.6. Mapping of the β_4-β_5 plane into the T_c-ϵ_c plane for a 0.9% solution of Natrosol 250-H; $\alpha = -1$, $a = 12$, $\beta_2 = \beta_3 = \beta_9 = 0$.

the T_c-ϵ_c plane. Although no special attempt was made to achieve agreement between theory and experiment in any of the cases shown, it is obvious that certain appropriately chosen (but non-unique) combinations of the parameters β_2 to β_5 could readily accomplish this.

CHAPTER 5

SUMMARY AND CONCLUSIONS

The stability of viscoelastic flow in the narrow channel between two long concentric rotating cylinders has been considered both theoretically and experimentally.

In the mathematical analysis, the general Coleman-and-Noll[9] simple fluid with fading memory is employed as the rheological model. Since all previous treatments of the small-gap stability problem have dealt with special cases of this model, the earlier works are shown to represent special cases of the current investigation. General techniques are presented for treating small perturbations on viscometric flows of simple fluids, and these techniques are subsequently applied in analyzing the pertinent stability behavior, under the usual assumption of neutral, axisymmetric disturbances. It is found that there are exactly eight material functions of the undisturbed rate of shear which are necessary to define the problem. Three of these are the familiar viscometric functions η, σ_1, and σ_2, but the other five are entirely new, and their forms can be deduced, at present, only in the limit of vanishing shear rates. Furthermore, numerical calculations reveal that, in practice, only six of the functions should actually have a substantial influence on flow stability.

Another important result of the theoretical investigation concerns the definition of one of the stability criteria, the Taylor number. It is shown that this quantity should be calculated using the apparent fluid viscosity, as evaluated at the shearing conditions in the apparatus.

Stability tests and viscosity measurements are reported for aqueous solutions of cellulose-derivative polymers whose viscometric functions have been presented in the literature, for certain concentrations. In all instances, it is found experimentally that the flow is less stable than in the Newtonian case, and that the wavelength of the disturbance is greater. By comparing these findings with the results of numerical stability calculations (based on empirical expressions for the viscometric functions of the fluids investigated), one concludes that certain well-known, approximate fluid models do not suffice to describe the observed stability behavior, and, in particular, that the "viscometric hypothesis" suggested by Metzner, White and Denn[29] is unacceptable in the present instance. Additional calculations, however, reveal that the five newly-defined material functions of the present work can be appropriately (but not uniquely) chosen to give agreement between theory and experiment.

At present, the measurement of the three viscometric functions η, σ_1, and σ_2 constitutes an important part of the laboratory testing of viscoelastic fluids. This is mainly because a knowledge of these functions enables one to predict the stress distributions associated with an entire class of fluid motions, namely, viscometric flows. By way of contrast, the five new material functions of the current analysis may very possibly be peculiar only to the Taylor-Couette experiment. Therefore, since a stability test does not suffice to determine the variation of the new material functions in a unique manner, it appears that there is but one important rheological application for this test. That is, it represents one additional means for experimentally testing simplified rheological models to determine whether they are adequate for describing more complicated flows of real viscoelastic fluids; the current theoretical development provides a basis for interpreting the results of such a test.

Although the present work eliminates the need for further theoretical studies on the Taylor-Couette stability of simple fluids (at least for the small-gap, neutral-stability case), it would be interesting, as further theoretical work, to determine whether the new material functions found here are common to other stability problems.

APPENDIX A

$\underset{\sim}{H}_t^{(1)}(t')$ IN CLOSED FORM

In this appendix, we present the remaining details of the derivation of $\underset{\sim}{H}_t^{(1)}(t')$ in closed form, which was begun in Section 3.3.

Let us start out by defining $\hat{\hat{\underset{\sim}{L}}}_t(t')$, $\hat{\hat{\underset{\sim}{M}}}_t(t')$, and $\hat{\hat{\underset{\sim}{N}}}_t(t')$ to be the summations of all terms in (3.34) involving the tensors $\underset{\sim}{L}$, $\underset{\sim}{M}$, and $\underset{\sim}{N}$ respectively. It is readily shown that

$$\hat{\hat{\underset{\sim}{L}}}_t(t') = \hat{\underset{\sim}{L}}_t(t') - \gamma(o)^2 \int_t^{t'}\int_t^{t_1} \hat{\underset{\sim}{L}}_t(t_2)dt_2dt_1 + \gamma(o)^4 \int_t^{t'}\int_t^{t_1}\int_t^{t_2}\int_t^{t_3} \hat{\underset{\sim}{L}}_t(t_4)dt_4dt_3dt_2dt_1 + \ldots \quad \text{(A-1)}$$

$$\hat{\hat{\underset{\sim}{M}}}_t(t') = -\left[\hat{\underset{\sim}{M}}_t(t') - 2\gamma(o)^2 \int_t^{t'}\int_t^{t_1} \hat{\underset{\sim}{M}}_t(t_2)dt_2dt_1 + 3\gamma(o)^4 \int_t^{t'}\int_t^{t_1}\int_t^{t_2}\int_t^{t_3} \hat{\underset{\sim}{M}}_t(t_4)dt_4dt_3dt_2dt_1 + \ldots\right] \quad \text{(A-2)}$$

$$\hat{\hat{\underset{\sim}{N}}}_t(t') = -\left[\hat{\underset{\sim}{N}}_t(t') - 2\gamma(o)^2 \int_t^{t'}\int_t^{t_1} \hat{\underset{\sim}{N}}_t(t_2)dt_2dt_1 + 3\gamma(o)^4 \int_t^{t'}\int_t^{t_1}\int_t^{t_2}\int_t^{t_3} \hat{\underset{\sim}{N}}_t(t_4)dt_4dt_3dt_2dt_1 \ldots\right] \quad \text{(A-3)}$$

where

$$\hat{\underset{\sim}{L}}_t(t') = \sum_{n=0}^{\infty} \frac{(t'-t)^{n+2}}{(n+2)!} \frac{\mathcal{D}(o)^n}{\mathcal{D}t^n} \underset{\sim}{L}(t) \quad \text{(A-4)}$$

$$\hat{\underset{\sim}{M}}_t(t') = \sum_{n=0}^{\infty} \frac{(t'-t)^{n+3}}{(n+3)!} \frac{\mathcal{D}^{(o)n}}{\mathcal{D}t^n} \underset{\sim}{M}(t) \tag{A-5}$$

$$\hat{\underset{\sim}{N}}_t(t') = \sum_{n=0}^{\infty} \frac{(t'-t)^{n+4}}{(n+4)!} \frac{\mathcal{D}^{(o)n}}{\mathcal{D}t^n} \underset{\sim}{N}(t) \ . \tag{A-6}$$

Substituting s = t'-t into (A-1)-(A-3), we then have

$$\hat{\hat{\underset{\sim}{L}}}_t(t+s) = \hat{\underset{\sim}{L}}_t(t+s) - \gamma^{(o)2}\int_0^s\int_0^{s_1} \hat{\underset{\sim}{L}}_t(t+s_2)ds_2ds_1$$

$$+ \gamma^{(o)4}\int_0^s\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} \hat{\underset{\sim}{L}}_t(t+s_4)ds_4ds_3ds_2ds_1+\ldots \tag{A-7}$$

$$\hat{\hat{\underset{\sim}{M}}}_t(t+s) = -\Bigg[\hat{\underset{\sim}{M}}_t(t+s)-2\gamma^{(o)2}\int_0^s\int_0^{s_1} \hat{\underset{\sim}{M}}_t(t+s_2)ds_2ds_1$$

$$+ 3\gamma^{(o)4}\int_0^s\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} \hat{\underset{\sim}{M}}_t(t+s_4)ds_4ds_3ds_2ds_1+\ldots\Bigg] \tag{A-8}$$

$$\hat{\hat{\underset{\sim}{N}}}_t(t+s) = -\Bigg[\hat{\underset{\sim}{N}}_t(t+s) - 2\gamma^{(o)2}\int_0^s\int_0^{s_1} \hat{\underset{\sim}{N}}_t(t+s_2)ds_2ds_1$$

$$+ 3\gamma^{(o)4}\int_0^s\int_0^{s_1}\int_0^{s_2}\int_0^{s_3} \hat{\underset{\sim}{N}}_t(t+s_4)ds_4ds_3ds_2ds_1+\ldots\Bigg] \tag{A-9}$$

Next, taking the Laplace transforms of Eqs. (A-7)-(A-9) with respect to the new variable s, one finds that

$$\mathcal{L}[\hat{\hat{\underset{\sim}{L}}}_t(t+s)] = \frac{1}{q^2+\gamma^{(o)2}} \mathcal{L}\left[\frac{d^2}{ds^2}\hat{\underset{\sim}{L}}_t(t+s)\right] \tag{A-10}$$

$$\mathcal{L}[\hat{\hat{\underset{\sim}{M}}}_t(t+s)] = \frac{-q}{[q^2+\gamma^{(o)2}]^2} \mathcal{L}\left[\frac{d^3}{ds^3}\hat{\underset{\sim}{M}}_t(t+s)\right] \tag{A-11}$$

$$\mathcal{L}[\hat{\hat{N}}_t(t+s)] = -\frac{1}{[q^2+\gamma^{(o)2}]^2}\,\mathcal{L}\left[\frac{d^4}{ds^4}\,\hat{N}_t(t+s)\right] \qquad \text{(A-12)}$$

where q denotes Laplace's operator. These expressions may be inverted to yield convolution integrals:

$$\hat{\hat{L}}_t(t+s) = \int_o^s \left[\frac{d^2}{ds_1^2}\,\hat{L}_t(t+s_1)\right] \frac{\sin\gamma^{(o)}(s-s_1)}{\gamma^{(o)}}\,ds_1 \qquad \text{(A-13)}$$

$$\hat{\hat{M}}_t(t+s) = -\int_o^s \left[\frac{d^3}{ds_1^3}\,\hat{M}_t(t+s_1)\right] \frac{(s-s_1)\sin\gamma^{(o)}(s-s_1)ds_1}{2\gamma^{(o)}} \qquad \text{(A-14)}$$

$$\hat{\hat{N}}_t(t+s) = -\int_o^s \left[\frac{d^4}{ds_1^4}\,\hat{N}_t(t+s_1)\right] \frac{[\sin\gamma^{(o)}(s-s_1)-\gamma^{(o)}(s-s_1)\cos\gamma^{(o)}(s-s_1)]ds_1}{2\gamma^{(o)3}} \qquad \text{(A-15)}$$

or equivalently

$$\hat{\hat{L}}_t(t') = \int_{t'}^t \left[\frac{d^2}{dt_1^2}\,\hat{L}_t(t_1)\right] \frac{\sin\gamma^{(o)}(t_1-t')dt_1}{\gamma^{(o)}} \qquad \text{(A-16)}$$

$$\hat{\hat{M}}_t(t') = \int_{t'}^t \left[\frac{d^3}{dt_1^3}\,\hat{M}_t(t_1)\right] \frac{(t_1-t')\sin\gamma^{(o)}(t_1-t')dt_1}{2\gamma^{(o)}} \qquad \text{(A-17)}$$

$$\hat{\hat{N}}_t(t') = -\int_{t'}^t \left[\frac{d^4}{dt_1^4}\,\hat{N}_t(t_1)\right] \frac{[\sin\gamma^{(o)}(t_1-t')-\gamma^{(o)}(t_1-t')\cos\gamma^{(o)}(t_1-t')]dt_1}{2\gamma^{(o)3}} \qquad \text{(A-18)}$$

Now, by again making use of the primary rotation tensor $R_t^{(o)}(t')$ (as has been done with Eq. (3.33)), it is possible to show, from (A-4)-(A-6), that

$$\frac{d^2}{dt'^2}[\hat{\underset{\sim}{L}}_t(t')] = \underset{\sim}{R}_t^{(o)\dagger}(t') \cdot \left[\frac{\mathcal{D}^{(1)}}{\mathcal{D}t'}\frac{\mathcal{D}^{(o)}}{\mathcal{D}t'}\underset{\sim}{E}^{(o)}(t')\right] \cdot \underset{\sim}{R}_t^{(o)}(t') \qquad \text{(A-19)}$$

$$\frac{d^3}{dt'^3}[\hat{\underset{\sim}{M}}_t(t')] = [\underset{\sim}{R}_t^{(o)\dagger}(t') \cdot \underset{\sim}{E}^{(o)}(t') \cdot \underset{\sim}{R}_t^{(o)}(t')] \frac{D^{(1)}\gamma^{(o)2}}{Dt'}$$

$$= \underset{\sim}{H}_t^{(o)}(t') \frac{D^{(1)}\gamma^{(o)2}}{Dt'} \qquad \text{(A-20)}$$

$$\frac{d^4}{dt'^4}[\hat{\underset{\sim}{N}}_t(t')] = \left[\underset{\sim}{R}_t^{(o)\dagger}(t') \cdot \frac{\mathcal{D}^{(o)}\underset{\sim}{E}^{(o)}(t')}{\mathcal{D}t'} \cdot \underset{\sim}{R}_t^{(o)}(t')\right] \frac{D^{(1)}\gamma^{(o)2}}{Dt'}$$

$$= \frac{D^{(o)}\underset{\sim}{H}_t^{(o)}(t')}{Dt'} \frac{D^{(1)}\gamma^{(o)2}}{Dt'} . \qquad \text{(A-21)}$$

Therefore, substituting (A-19)-(A-21) into (A-16)-(A-18) we have finally that

$$\hat{\hat{\underset{\sim}{L}}}_t(t') = \frac{1}{\gamma^{(o)}} \int_{t'}^{t} \underset{\sim}{R}_t^{(o)\dagger}(t_1) \cdot \left[\frac{\mathcal{D}^{(1)}}{\mathcal{D}t_1}\frac{\mathcal{D}^{(o)}}{\mathcal{D}t_1}\underset{\sim}{E}^{(o)}(t_1)\right] \cdot \underset{\sim}{R}_t^{(o)}(t_1)$$

$$(x) \sin \gamma^{(o)}(t_1 - t')dt_1 \qquad \text{(A-22)}$$

$$\hat{\hat{\underset{\sim}{M}}}_t(t') = \int_{t'}^{t} \underset{\sim}{H}_t^{(o)}(t_1) \frac{D^{(1)}\gamma^{(o)}}{Dt_1} (t_1 - t')\sin \gamma^{(o)}(t_1 - t')dt_1 \qquad \text{(A-23)}$$

$$\hat{\hat{\underset{\sim}{N}}}_t(t') = - \frac{1}{\gamma^{(o)2}} \int_{t'}^{t} \frac{D^{(o)}\underset{\sim}{H}_t^{(o)}(t_1)}{Dt_1} \frac{D^{(1)}\gamma^{(o)}}{Dt_1} [\sin \gamma^{(o)}(t_1 - t')$$

$$- \gamma^{(o)}(t_1 - t')\cos \gamma^{(o)}(t_1 - t')]dt_1 \qquad \text{(A-24)}$$

APPENDIX B

EXPERIMENTAL DATA

The experimental data of the current investigation are reported in this section. First the results of the stability tests are presented, and then the viscosity data are given. Units are as follows: concentration in weight percent, time in seconds, angular velocity in revolutions per minute, height of vortex cells in cm., armature displacement in 0.001 in., shear rate in sec^{-1}, shear stress in $dyne/cm^2$, and viscosity in poise.

Stability Data

1. Pure Water

Temperature = 79°F, revolutions = 3, time = 54.9±0.3, angular velocity = 3.28±0.02, height/20 vortex cells = 11.7±0.1, height/cell = 0.585±0.005, ϵ_c = 3.14±0.03, T_c = 4140

2. 7MT CMC

Concentration	0.5	1.0	1.67	2.5
Temperature (°F)	78	76	80	78
Revolutions	40	100	100	100
Time	57.7±0.6	51.0±0.6	27.0±0.3	17.2±0.5
Angular velocity	41.6±0.4	117.5±1.1	222±3	348±10
Shear rate	52.3±0.5	148±2	279±3	438±13
Critical viscosity	0.111	0.33	0.64	1.41
T_c	4140	3790	3630	1880
Height/20 cells	12.0±0.2	12.6±0.2	13.0±0.1	13.7±0.1
Height/cell	0.600±0.010	0.630±0.010	0.650±0.005	0.685±0.005
ϵ_c	3.04±0.05	2.91±0.05	2.82±0.03	2.68±0.02

3. P-75-XH CMC

Concentration	0.183	0.367	0.55
Temperature (°F)	76	76	76
Revolutions	100	100	100
Time	48.5±1.0	27.8±0.5	20.5±0.3
Angular velocity	123.8±2.6	216±4	293±4
Shear rate	155.5±3.2	271±5	368±5
Critical viscosity	0.327	0.590	0.880
T_c	4200	3950	3270
Height/20 cells	13.3±0.5	15.5±0.5	16.5±0.5
Height/cell	0.665±0.025	0.775±0.025	0.825±0.025
ϵ_c	2.76±0.10	2.36±0.08	2.22±0.07

4. Natrosol 250-H HEC

Concentration	0.48	0.80	1.00
Temperature (°F)	76	80	82
Revolutions	100	200	200
Time	36.8±0.5	39.2±0.5	30.9±0.5
Angular velocity	163±2	306±4	388±6
Shear rate	205±3	384±5	488±8
Critical viscosity	0.470	0.950	1.32
T_c	3550	3080	2590
Height/20 cells	14.3±0.5	16.3±0.5	17.0±0.5
Height/cell	0.715±0.025	0.815±0.025	0.850±0.025
ϵ_c	2.56±0.09	2.25±0.07	2.16±0.06

Viscosity Data

1. Pure Water

Viscosity of water at 79°F[32] = 0.00872

2. 7MT CMC

0.5% Solution (25.6°C)

Gearbox	Armature Displacement	Shear Rate	Shear Stress	Viscosity
1.5	0.018	52.8	5.59	0.104
1.4	0.024	66.4	7.45	0.112
1.3	0.029	83.6	9.00	0.108
1.2	0.038	105.3	11.8	0.112
1.1	0.046	132.9	14.3	0.107
1.0	0.059	167.0	18.2	0.109
0.9	0.074	210.0	23.0	0.110
0.8	0.090	264.0	27.9	0.106
0.7	0.112	334.0	34.8	0.104
0.6	0.138	420.0	42.8	0.102
0.5	0.172	528.0	53.4	0.101
0.4	0.211	664.0	65.5	0.986
0.3	0.256	836.0	79.5	0.950
0.2	0.314	1053.0	97.5	0.929
0.1	0.387	1329.0	120.0	0.903
0.0	0.480	1670.0	149.0	0.892

1.0% Solution (24.5°C)

Gearbox	Armature Displacement	Shear Rate	Shear Stress	Viscosity
1.5	0.061	52.8	18.9	0.358
1.4	0.075	66.4	23.2	0.350
1.3	0.093	83.6	28.8	0.344
1.2	0.117	105.3	36.3	0.346
1.1	0.144	132.9	44.7	0.336
1.0	0.176	167.0	55.6	0.333
0.9	0.214	210.0	66.5	0.316
0.8	0.260	264.0	80.6	0.306
0.7	0.318	334.0	98.6	0.295
0.6	0.378	420.0	117.0	0.285
0.5	0.462	528.0	143.0	0.271
0.4	0.560	664.0	174.0	0.262
0.3	0.670	836.0	208.0	0.249
0.2	0.800	1053.0	248.0	0.236
0.1	0.930	1329.0	288.0	0.216
0.0	1.14	1670.0	354.0	0.212

1.67% Solution (26.7°C)

Gearbox	Armature Displacement	Shear Rate	Shear Stress	Viscosity
1.5	0.143	52.8	44.4	0.840
1.4	0.176	66.4	54.6	0.823
1.3	0.212	83.6	65.8	0.786
1.2	0.258	105.3	80.0	0.762
1.1	0.312	132.9	96.8	0.728
1.0	0.377	167.0	117.0	0.700
0.9	0.455	210.0	141.0	0.672
0.8	0.540	264.0	168.0	0.636
0.7	0.655	334.0	203.0	0.608
0.6	0.775	420.0	240.0	0.571
0.5	0.910	528.0	282.0	0.535
0.4	1.08	664.0	335.0	0.505
0.3	1.28	836.0	397.0	0.475
0.2	1.52	1053.0	471.0	0.448
0.1	1.77	1329.0	549.0	0.413
0.0	2.05	1670.0	636.0	0.381

2.5% Solution (25.6°C)

Gearbox	Armature Displacement	Shear Rate	Shear Stress	Viscosity
1.5	0.435	52.8	135	2.56
1.4	0.535	66.4	166	2.52
1.3	0.630	83.6	195	2.34
1.2	0.735	105.3	228	2.17
1.1	0.885	132.9	275	2.07
1.0	1.04	167.0	322	1.93
0.9	1.23	210.0	382	1.82
0.8	1.44	264.0	447	1.69
0.7	1.68	334.0	521	1.56
0.6	1.93	420.0	599	1.46
0.5	2.22	528.0	689	1.31
0.4	2.52	664.0	781	1.18
0.3	2.90	836.0	900	1.08
0.2	3.40	1053.0	1055	1.00
0.1	3.95	1329.0	1225	0.920
0.0	4.45	1670.0	1370	0.820

3. P-75-XH CMC

0.183% Solution (24.5°C)

Gearbox	Armature Displacement	Shear Rate	Shear Stress	Viscosity
1.5	0.077	52.8	23.9	0.453
1.4	0.090	66.4	28.9	0.435
1.3	0.106	83.6	32.8	0.392
1.2	0.126	105.3	39.1	0.372
1.1	0.147	132.9	45.6	0.343
1.0	0.171	167.0	53.0	0.318
0.9	0.199	210.0	61.7	0.294
0.8	0.229	264.0	71.0	0.269
0.7	0.260	334.0	80.6	0.242
0.6	0.300	420.0	93.0	0.222
0.5	0.342	528.0	106.0	0.201
0.4	0.390	664.0	121.0	0.183
0.3	0.448	836.0	139.0	0.166
0.2	0.515	1053.0	160.0	0.153
0.1	0.585	1329.0	182.0	0.137
0.0	0.678	1670.0	210.0	0.126

0.367% Solution (24.5°C)

Gearbox	Armature Displacement	Shear Rate	Shear Stress	Viscosity
1.5	0.194	52.8	60.2	1.14
1.4	0.225	66.4	69.8	1.05
1.3	0.260	83.6	80.6	0.964
1.2	0.298	105.3	92.4	0.880
1.1	0.342	132.9	106.1	0.797
1.0	0.387	167.0	120.0	0.718
0.9	0.445	210.0	138.1	0.657
0.8	0.510	264.0	158.1	0.599
0.7	0.580	334.0	180.1	0.539
0.6	0.658	420.0	204.0	0.486
0.5	0.740	528.0	230.0	0.436
0.4	0.825	664.0	256.0	0.386
0.3	0.925	836.0	287.0	0.344
0.2	1.03	1053.0	320.0	0.305
0.1	1.16	1329.0	360.0	0.270
0.0	1.31	1670.0	406.0	0.243

0.55% Solution (24.5°C)

Gearbox	Armature Displacement	Shear Rate	Shear Stress	Viscosity
1.5	0.375	52.8	116.3	2.20
1.4	0.425	66.4	132.0	1.99
1.3	0.490	83.6	152.0	1.82
1.2	0.558	105.3	173.0	1.65
1.1	0.630	132.9	195.5	1.47
1.0	0.710	167.0	220.0	1.32
0.9	0.790	210.0	245.0	1.17
0.8	0.885	264.0	274.0	1.04
0.7	0.982	334.0	305.0	0.913
0.6	1.090	420.0	338.0	0.805
0.5	1.22	528.0	378.0	0.716
0.4	1.36	664.0	422.0	0.637
0.3	1.50	836.0	465.0	0.556
0.2	1.67	1053.0	518.0	0.493
0.1	1.84	1329.0	571.0	0.429
0.0	2.02	1670.0	626.0	0.375

4. Natrosol 250-H HEC

0.48% Solution (24.5°C)

Gearbox	Armature Displacement	Shear Rate	Shear Stress	Viscosity
1.5	0.132	52.8	40.9	0.776
1.4	0.156	66.4	48.3	0.729
1.3	0.181	83.6	56.1	0.671
1.2	0.208	105.3	64.5	0.614
1.1	0.239	132.9	74.1	0.557
1.0	0.275	167.0	85.3	0.511
0.9	0.315	210.0	97.6	0.465
0.8	0.358	264.0	111.0	0.420
0.7	0.402	334.0	125.0	0.374
0.6	0.460	420.0	143.0	0.340
0.5	0.520	528.0	161.0	0.305
0.4	0.590	664.0	183.0	0.276
0.3	0.670	836.0	208.0	0.249
0.2	0.755	1053.0	234.0	0.223
0.1	0.840	1329.0	260.0	0.196
0.0	0.955	1670.0	296.0	0.177

0.8% Solution (26.7°C)

Gearbox	Armature Displacement	Shear Rate	Shear Stress	Viscosity
1.5	0.465	52.8	144	2.73
1.4	0.538	66.4	167	2.52
1.3	0.600	83.6	186	2.22
1.2	0.675	105.3	210	2.00
1.1	0.745	132.9	231	1.74
1.0	0.822	167.0	255	1.53
0.9	0.910	210.0	282	1.34
0.8	1.00	264.0	310	1.18
0.7	1.11	334.0	344	1.03
0.6	1.21	420.0	376	0.896
0.5	1.34	528.0	416	0.787
0.4	1.47	664.0	456	0.687
0.3	1.61	836.0	500	0.598
0.2	1.76	1053.0	546	0.520
0.1	1.92	1329.0	595	0.447
0.0	2.10	1670.0	651	0.390

1.0% Solution (27.8°C)

Gearbox	Armature Displacement	Shear Rate	Shear Stress	Viscosity
1.0	1.44	167	447	2.68
0.9	1.56	210	484	2.30
0.8	1.69	264	525	1.99
0.7	1.83	334	568	1.70
0.6	1.96	420	608	1.45
0.5	2.12	528	658	1.25
0.4	2.31	664	717	1.08
0.3	2.51	836	779	0.931
0.2	2.70	1053	838	0.797
0.1	2.95	1329	915	0.688
0.0	3.18	1670	986	0.590

APPENDIX C

CRITICAL EIGENFUNCTIONS

The critical eigenfunctions ψ_c and V_c are defined herein as the eigenfunctions associated with the critical parameters T_c and ϵ_c. For any particular simple fluid, ψ_c and V_c can be determined using the present numerical techniques, by integrating the disturbance equations from $X = 0$ to $X = 1$ for $T = T_c$ and $\epsilon = \epsilon_c$. As an example, Figs. C.1 and C.2 show plots of ψ_c and V_c vs. X corresponding to the Newtonian case with $\alpha = -1$.

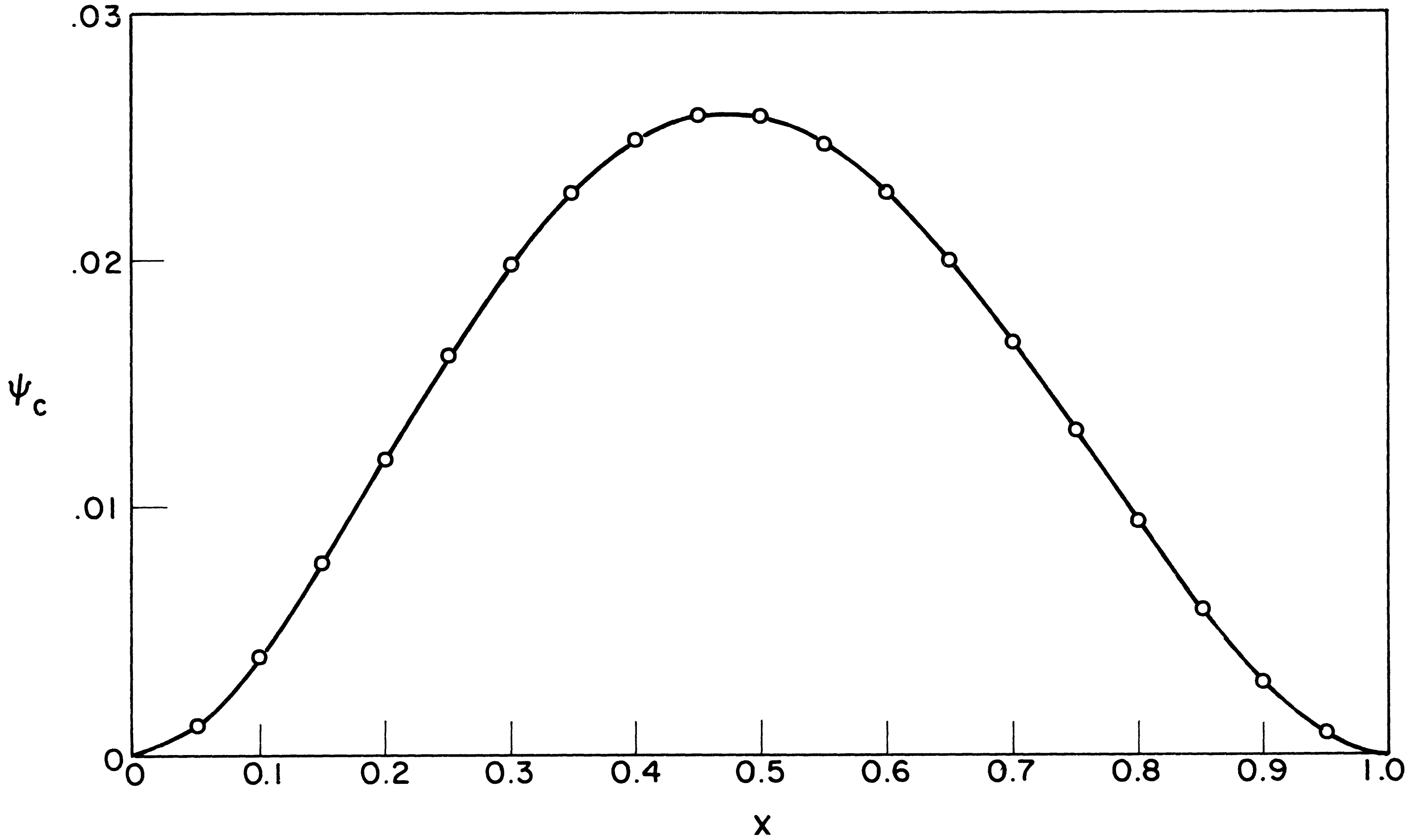


Fig. C.1. Critical eigenfunction ψ_c vs. X for Newtonian case; α = -1.

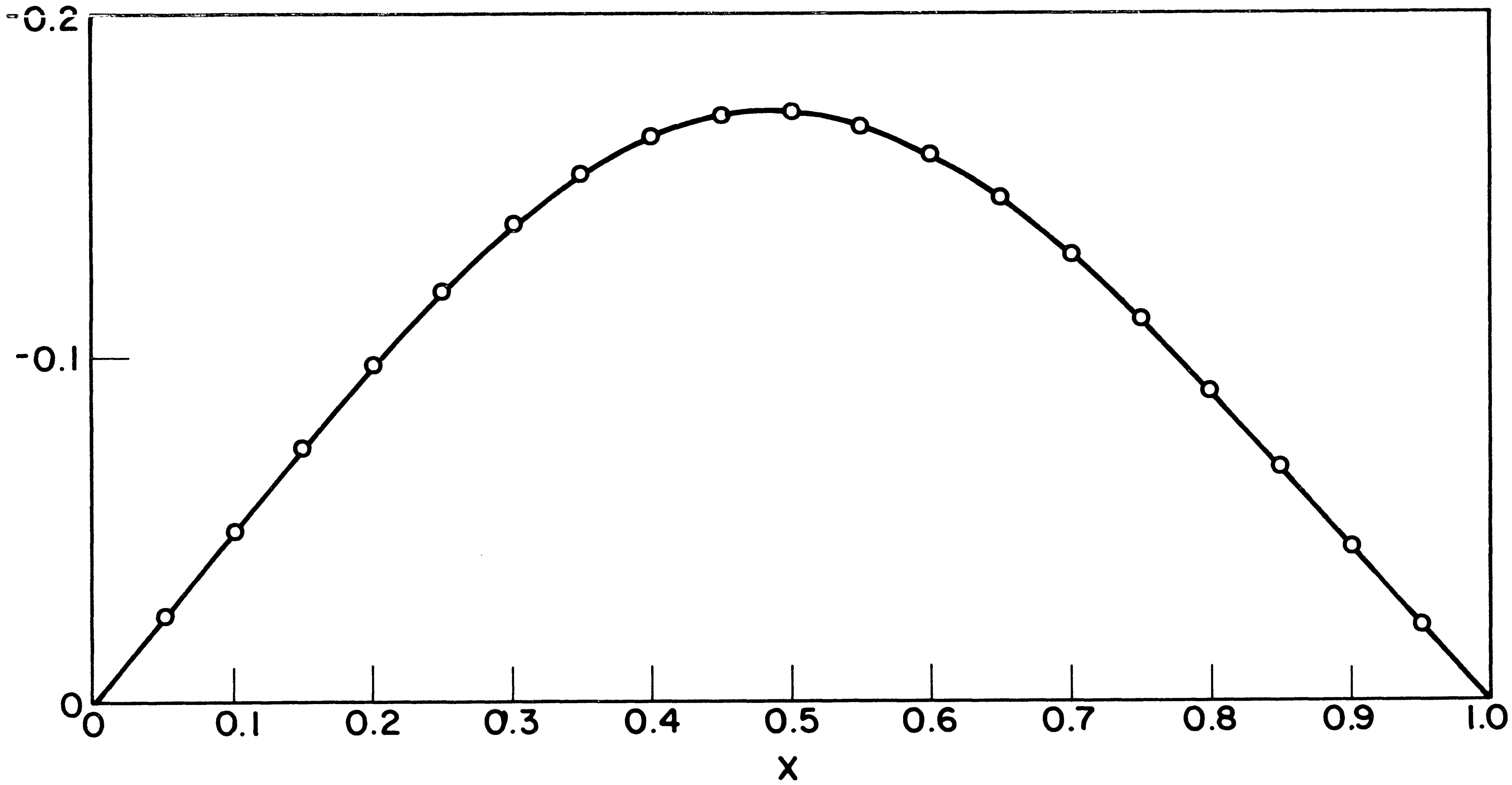


Fig. C.2. Critical eigenfunction V_c vs. X for Newtonian case; α = -1.

APPENDIX D

COMPUTER ANALYSIS

A main program and three subroutines (BETA., SOLVE., and INVERT.) were used in this work to (a) numerically solve the characteristic value problem arising as a consequence of the current Taylor-Couette stability analysis, and (b) determine the critical parameters T_c and ϵ_c. The basic techniques employed have been presented in Section 4.7.3. All programs were written in the "MAD" computer language, and the time required to determine each eigenvalue was approximately 2 sec.

The following is a summary of the primary duties of the main program:

(1) To direct the reading of data and the prinout of results.

(2) To initiate and implement the "golden section" minimum searching procedure.

(3) To call upon the subroutine BETA. to calculate the rheological coefficients β_0 through β_9, for any prespecified dependence of these quantities on the dimensionless shear rate $\gamma\tau$.

(4) To call upon the subroutine SOLVE. to integrate the disturbance equations from X = 0 to X = 1, using a Runge-Kutta technique.*

(5) To determine the matrix of partial derivatives appearing in Eq. (4.92).

(6) To call upon the subroutine INVERT.** to solve Eq. (4.92).

---

*The computer library Runge-Kutta subroutine was employed in SOLVE.
**INVERT. is a modified version of the program appearing on pp. 390-391 of Applied Numerical Methods.[3]

(7) To iterate on the quantities G_1, G_2, and G_3 until satisfactory convergence is achieved.

A description of the important symbols employed is given below.

Program Symbol	Definition
A	$a = r_1/d$
A(I,J)	Elements of the matrix of partial derivatives appearing in Eq. (4.92). A vector consisting of ψ, $D\psi$, and V at X = 1 is appended as an additional column.
ALPHA	$\alpha = (\Omega_2-\Omega_1)/\Omega_1$
B	αa
BET(I)	β_i (i = 0,...,9)
COEF(I)	c_i (i = 1,2,) in Eqs. (4.113)
CRIT	ϵ_c
DEL(I)	δG_i (i = 1,2,3) in Eq. (4.92)
DELX	ΔX, step size in Runge-Kutta integration
EL	$E\ell$ = elasticity number = $\tau\eta_0/pd^2$
EPS	ϵ
GESNU(I)	GUESS(I) * RATIO(I)
GUESS(I)	G_i (i = 1,2,3)
LEFT	Left boundary in the golden section interval (LEFT < ϵ_c < RIGHT)
MAX	Total number of applications of the golden section test
NU	Maximum allowable fractional change in G_i during final iteration

Program Symbol	Definition
POWER(I)	p_i (i = 0,1,2) in Eqs. (4.113)
PSI(I)	ψ_i (i = 1,...,6); $\psi_1 = \psi$, $\psi_2 = D\psi$, $\psi_3 = V$, $\psi_4 = D^2\psi$, $\psi_5 = D^3\psi$, $\psi_6 = DV$
RATIO(I)	(1.0-RATIO(I)) represents the fractional change in G_i used in numerically determining the nine partial derivatives in Eq. (4.92)
RIGHT	Right boundary of the golden section interval (LEFT < ϵ_c < RIGHT)
R	Modified Reynolds number, R_m
SOLN	Taylor number (eigenvalue)
TAUGAM	$\lvert\gamma_T\rvert$
TAYLOR	T_c
TEST	Boolean variable which specifies whether or not the critical eigenfunctions ψ_c and V_c should be printed

The MAD program listings and a typical computer output are presented in the pages which follow.

```
$COMPILE MAD,PRINT OBJECT,DUMP,EXECUTE                                          036933   03/24/67     10:37 36.1 AM

MAD (06 JAN 1967 VERSION) PROGRAM LISTING ... ... ...


                THIS PROGRAM SOLVES THE EIGENVALUE PROBLEM WHICH ARISES AS
                A RESULT OF ANALYSIS OF THE STABILITY OF A VISCOELASTIC
                FLUID CONTAINED BETWEEN CONCENTRIC ROTATING CYLINDERS.  THE
                EIGENVALUE PROBLEM IS HEREIN BROKEN UP INTO A SET OF 6 FIRST
                ORDER DIFFERENTIAL EQUATIONS IN 6 UNKNOWNS (PSI(1)...PSI(6))
                SUBJECT TO THE BOUNDARY CONDITIONS (AT X=0 AND X=1),
                PSI(1)...PSI(3)=0.  THE COEFFICIENTS IN THE DIFFERENTIAL
                EQUATIONS ARE ALL FUNCTIONS OF THE TAYLOR NUMBER (GUESS(3)),
                THE DIMENSIONLESS WAVE NUMBER (EPS), AND GEOMETRIC AND
                RHEOLOGICAL PARAMETERS.  SINCE, IN AN EIGENVALUE PROBLEM, THE
                SOLUTION IS SPECIFIED UP TO AN ARBITRARY CONSTANT MULTIPLIER,
                PSI(4) IS TAKEN TO BE EQUAL TO 1.0 AT X=0.  FOR ANY ASSUMED
                VALUES TO THE TAYLOR NUMBER AND MISSING INITIAL CONDITIONS
                (AT X=0, PSI(5)=GUESS(1) AND PSI(6)=GUESS(2)), THE EQUATIONS
                ARE INTEGRATED BETWEEN X=0 AND X=1 USING A RUNGE-KUTTA
                FOUR-POINT INTEGRATION SCHEME.  OF COURSE, FOR AN ARBITRATY
                GUESS, THE VALUES OF PSI(1)...PSI(3) AT X=1 WOULD NOT LIKELY
                TURN OUT TO BE ZERO, BUT THE INITIAL GUESS IS SUBSEQUENTLY
                CORRECTED USING A NEWTON-RAPHSON TECHNIQUE.  DERIVITIVES FOR
                THIS TECHNIQUE ARE HEREIN ESTIMATED BY A FINITE DIFFERENCE
                APPROXIMATION.  ITERATION CONTINUES UNTIL THE MAGNITUDES OF
                THE CORRECTIONS TO THE GUESSES BECOME SMALL COMPARED TO THE
                GUESSES THEMSELVES.

                THE CRITICAL TAYLOR NUMBER (I.E. THE MINIMUM TAYLOR NUMBER
                OVER ALL DISTURBANCE WAVE NUMBERS) IS LOCATED USING THE
                GOLDEN SECTION SEARCHING TECHNIQUE.  THE PROGRAM EXAMINES
                THE REGION BETWEEN EPS=LEFT AND EPS=RIGHT FOR A MINIMUM AS
                FOLLOWS.  AFTER EVALUATING THE TAYLOR NUMBER AT THE .382 AND
                .618 POINTS OF THIS INTERVAL, IT ELIMINATES ONE PORTION
                (EITHER BETWEEN 0 AND .382 OR BETWEEN .618 AND 1) AS NOT
                CONTAINING THE MINIMUM.  LEFT OR RIGHT IS THEN ADJUSTED SO
                THAT THE INTERVAL IS SHRUNK TO .618 OF ITS ORIGINAL SIZE.
                THIS PROCEDURE IS REPEATED COUNT TIMES.


                ....DIMENSIONING AND MODE DECLARATION....

                 DIMENSION PSI(6),GUESS(4),GESNU(4),DEL(3),RATIO(4),A(4*4),        *001
               1 BET(9),EPS(2),SOLN(2),COEF(9),POWER(9)                            *001
                 INTEGER I,N,INDEX,COUNT,K,MAX                                      *002
                 BOOLEAN TEST                                                       *003
                 REFERENCES ON                                                      *004


                ....PRINTOUT....

START            PRINT COMMENT $1SOLUTION PERFORMED FOR FOLLOWING VALUES OF PA    *005
               1 RAMETERS$                                                          *005
                 READ AND PRINT DATA                                                *006
                 B=ALPHA*A                                                          *007
                 PRINT COMMENT $0THE TAYLOR NUMBER AS A FUNCTION OF THE DIMENS    *008
               1 IONLESS WAVE NUMBER IS GIVEN BY$                                   *008
```

```
          ....GOLDEN SECTION....

           INDEX=0                                                     *009
           COUNT=0                                                     *010
           EPS(1)=LEFT+.382*(RIGHT-LEFT)                               *011
           EPS(2)=LEFT+.618*(RIGHT-LEFT)                               *012
GOLDEN     COUNT=COUNT+1                                               *013
           THROUGH END, FOR VALUES OF K=1,2                            *014
           WHENEVER K.E.INDEX, TRANSFER TO END                         *015          01

          ....INITIALIZATION....

BEGIN      THROUGH NEXT, FOR I=1,1,I.G.4                               *016          01
           GESNU(I)=GUESS(I)                                           *017          02
NEXT       A(I,4)=0.                                                   *018          02
           THROUGH SETUP, FOR VALUES OF N=4,1,2,3                      *019          01
           GESNU(N)=GUESS(N)*RATIO(N)                                  *020          02
           WHENEVER N.GE.3                                             *021          02
           GESNU(2)=GUESS(2)                                           *022    01    02
           R=SQRT.(GESNU(3)/(-2.*B))                                   *023    01    02
           EXECUTE BETA.(BET,R,B,EL,COEF,POWER)                        *024    01    02
           END OF CONDITIONAL                                          *025    01    02
           WHENEVER N.E.2, GESNU(1)=GUESS(1)                           *026          02
           THROUGH INIT, FOR I=1,1,I.G.3                               *027          02
INIT       PSI(I)=0.                                                   *028          03
           PSI(4)=1.                                                   *029          02
           PSI(5)=GESNU(1)                                             *030          02
           PSI(6)=GESNU(2)                                             *031          02

          ....INTEGRATION....

           EXECUTE SOLVE.(EPS(K),ALPHA,BET,GESNU(3),R,PSI,DELX,0,A)    *032          02

          ....SETUP OF MATRIX OF COEFFICIENTS....

           THROUGH SETUP, FOR I=1,1,I.G.3                              *033          02
SETUP      A(I,N)=(PSI(I)+A(I,4))/ (GUESS(N)*(RATIO(N)-1.))            *034          03

          ....MATRIX INVERSION....

           EXECUTE INVERT.(A,DEL,3)                                    *035          01

          ....ITERATION ON GUESSES....

           THROUGH CHANGE, FOR N=1,1,N.G.3                             *036          01
CHANGE     GUESS(N)=GUESS(N)+DEL(N)                                    *037          02
           THROUGH VEST, FOR I=1,1,I.G.3                               *038          01
VEST       WHENEVER .ABS.(DEL(I)).G..ABS.(NU*GUESS(I)),TRANSFER TO BEGIN  *039       02

          ....GOLDEN SECTION....

           R=SQRT.(GUESS(3)/(-2.*B))                                   *040          01
```

```
          EXECUTE BETA.(BET,R,B,EL,COEF,POWER)                          *041          01
          SOLN(K)=GUESS(3)*BET(0).P.2                                   *042          01
END       PRINT RESULTS EPS(K), SOLN(K)                                 *043          01

         ....CHECK ON WHETHER INTERVAL HAS BEEN DIVIDED SUFFICIENTLY..

          WHENEVER COUNT.E.MAX, TRANSFER TO SKIP                        *044

         ....INTERVAL SHRINKING....

          WHENEVER SOLN(2).L.SOLN(1)                                    *045
          LEFT=EPS(1)                                                   *046      01
          EPS(1)=EPS(2)                                                 *047      01
          SOLN(1)=SOLN(2)                                               *048      01
          EPS(2)=LEFT+.618*(RIGHT-LEFT)                                 *049      01
          INDEX=1                                                       *050      01
          OTHERWISE                                                     *051      01
          RIGHT=EPS(2)                                                  *052      01
          EPS(2)=EPS(1)                                                 *053      01
          SOLN(2)=SOLN(1)                                               *054      01
          EPS(1)=LEFT+.382*(RIGHT-LEFT)                                 *055      01
          INDEX=2                                                       *056      01
          END OF CONDITIONAL                                            *057      01
          TRANSFER TO GOLDEN                                            *058

         ....FINAL RESULTS....

SKIP      WHENEVER SOLN(2).L.SOLN(1)                                    *059
          CRIT=EPS(2)                                                   *060      01
          TAYLOR=SOLN(2)                                                *061      01
          OTHERWISE                                                     *062      01
          CRIT=EPS(1)                                                   *063      01
          TAYLOR=SOLN(1)                                                *064      01
          END OF CONDITIONAL                                            *065      01
          PRINT COMMENT $0THE CRITICAL TAYLOR NUMBER IS        $        *066
          PRINT RESULTS TAYLOR                                          *067
          PRINT COMMENT$0FOR A CRITICAL DIMENSIONLESS WAVE NUMBER OF$   *068
          PRINT RESULTS CRIT                                            *069
          PRINT COMMENT $0THE RHEOLOGICAL PARAMETERS ARE$               *070
          PRINT RESULTS BET(0)...BET(9)                                 *071
          WHENEVER TEST                                                 *072
          THROUGH FINAL, FOR I=1,1,I.G.3                                *073      01
FINAL     PSI(I)=0.                                                     *074      01      01
          PSI(4)=1.                                                     *075      01
          PSI(5)=GUESS(1)                                               *076      01
          PSI(6)=GUESS(2)                                               *077      01
          WHENEVER INDEX.E.1                                            *078      01
          INDEX=2                                                       *079      02
          OTHERWISE                                                     *080      02
          INDEX=1                                                       *081      02
          END OF CONDITIONAL                                            *082      02
          PRINT COMMENT$0THE CRITICAL EIGENFUNCTIONS ARE$               *083      01
          EXECUTE SOLVE.(EPS(INDEX),ALPHA,BET,GUESS(3),R,PSI,DELX,1,A)  *084      01
          END OF CONDITIONAL                                            *085      01
          PRINT RESULTS GUESS(1)...GUESS(3)                             *086
          TRANSFER TO START                                             *087
          END OF PROGRAM                                                *088
```

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TABLE OF REFERENCES

**** SYMBOL TABLE REFERENCES

      ALPHA    007,032,084
      A        007,018,032,034,035,084
      BEGIN    016,039
      BET      024,032,041,042,071,084
      B        007,023,024,040,041
      CHANGE   036,037
      COEF     024,041
      COUNT    010,013,044
      CRIT     060,063,069
      DEL      035,037,039
      DELX     032,084
      EL       024,041
      END      014,015,043
      EPS      011,012,032,043,046,047,049,052,053,055,060,063,084
      FINAL    073,074
      GESNU    017,020,022,023,026,030,031,032
      GOLDEN   013,058
      GUESS    017,020,022,026,034,037,039,040,042,076,077,084,086
      INDEX    009,015,050,056,078,079,081,084
      INIT     027,028
      I        016,017,018,027,028,033,034,038,039,073,074
      K        014,015,032,042,043
      LEFT     011,012,046,049,055
      MAX      044
      NEXT     016,018
      N        019,020,021,026,034,036,037
      NU       039
      POWER    024,041
      PSI      028,029,030,031,032,034,074,075,076,077,084
      RATIO    020,034
      RIGHT    011,012,049,052,055
      R        023,024,032,040,041,084
      SETUP    019,033,034
      SKIP     044,059
      SOLN     042,043,045,048,054,059,061,064
      START    005,087
      TAYLOR   061,064,067
      TEST     072
      VEST     038,039

**** FUNCTION REFERENCES

      BETA     024,041
      INVERT   035
      SOLVE    032,084
      SQRT     023,040

THE FOLLOWING NAMES HAVE OCCURRED ONLY ONCE IN THIS PROGRAM.
COMPILATION WILL CONTINUE.

      MAX            *044
      NU             *039
      TEST           *072
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    $COMPILE MAD,PRINT OBJECT,DUMP,EXECUTE                                               036933   03/24/67     10:37 48.9 AM

MAD (06 JAN 1967 VERSION) PROGRAM LISTING ... ... ...

               EXTERNAL FUNCTION(EPS,ALPHA,BET,T,R,PSI,DELX,BOOL,A)                             *001
               ENTRY TO SOLVE.                                                                  *002

              THE SUBROUTINE SOLVE. INTEGRATES THE DIFFERENTIAL EQUATIONS
              ASSOCIATED WITH THE EIGENVALUE PROBLEM UNDER CONSIDERATION
              (IN THE INTERVAL 0.L.X.L.1) SUBJECT TO ANY PRESCRIBED INITIAL
              CONDITIONS AND TAYLOR NUMBER.  A RUNGE-KUTTA FOUR POINT
              INTEGRATION SCHEME IS EMPLCYED.

              ....DIMENSIONING....

               DIMENSION       Q(6),F(6)                                                        *003
               BOOLEAN BOOL                                                                     *004

              ....INITIALIZATION....

               EPSQ=EPS*EPS                                                                     *005
               X=0.                                                                             *006
               EXECUTE SETRKD.(6,PSI(1),F(1),Q,X,DELX)                                          *007
               WHENEVER BOOL                                                                    *008
               PRINT COMMENT$0                               X                                  *009        01
             1    PSI               PSI'                V$                                      *009
               END OF CONDITIONAL                                                               *010        01
    WRITE      WHENEVER BOOL, PRINT FORMAT RESULT,X,PSI(1)...PSI(3)                             *011

              ....CHECK ON WHETHER INTEGRATION IS COMPLETE....

               WHENEVER X.G..99, TRANSFER TO FINISH                                             *012

              ....INTEGRATION....

    CALC       S=RKDEQ.(0)                                                                      *013
               WHENEVER S.L.1.5                                                                 *014
               F(1)=PSI(2)                                                                      *015        01
               F(2)=PSI(4)                                                                      *016        01
               F(3)=PSI(6)                                                                      *017        01
               F(4)=PSI(5)                                                                      *018        01
               F(6)=(EPSQ*PSI(3)+T*BET(0)*PSI(1)/(2.*R)+BET(2)*PSI(4)                           *019        01
             1 -BET(3)*EPSQ*PSI(1))/BET(1)                                                      *019
               F(5)=EPSQ*(BET(4)*PSI(4)-EPSQ*BET(5)*PSI(1)                                      *020        01
             1 -2.*R*(1.+ALPHA*X)*PSI(3)*BET(0)+BET(6)*F(6)                                     *020
             2 +BET(7)*EPSQ*PSI(3)+(BET(8)*PSI(6)-2.*BET(9)*PSI(2))/A)                          *020
               TRANSFER TO CALC                                                                 *021        01
               END OF CONDITIONAL                                                               *022        01
               TRANSFER TO WRITE                                                                *023
               VECTOR VALUES RESULT=$1H ,F36.2,3F18.6*$                                         *024
    FINISH     FUNCTION RETURN                                                                  *025
               END OF FUNCTION                                                                  *026
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$COMPILE MAD,PRINT OBJECT,DUMP,EXECUTE                    036933   03/24/67    10:37 55.7 AM

MAD (06 JAN 1967 VERSION) PROGRAM LISTING ... ... ...

        EXTERNAL FUNCTION(BET,R,B,EL,COEF,POWER)                        *001
        ENTRY TO BETA.                                                  *002

       THE SUBROUTINE BETA. CALCULATES THE RHEOLOGICAL PARAMETERS
       BET(0)...BET(9) WHICH APPEAR IN THE EIGENVALUE PROBLEM UNDER
       CONSIDERATION.

       ....FREQUENTLY APPEARING QUANTITIES....

        TAUGAM=-B*R*EL                                                  *003
        SIG1=-COEF(1)*TAUGAM.P.(POWER(0)-POWER(1))                      *004
        SIG2=-COEF(2)*TAUGAM.P.(POWER(0)-POWER(2))                      *005

       ....CALCULATION OF BET(0)...BET(9)....

        BET(0)=TAUGAM.P.POWER(0)                                        *006
        BET(1)=1.-POWER(0)                                              *007
        BET(2)=(SIG1+SIG2)/2.                                           *008
        BET(3)=(SIG1+SIG2)/2.                                           *009
        BET(4)=2.                                                       *010
        BET(5)=1.                                                       *011
        BET(6)=POWER(2)*SIG2                                            *012
        BET(7)=SIG2                                                     *013
        BET(8)=(1.-POWER(1))*SIG1                                       *014
        BET(9)=0.                                                       *015

        FUNCTION RETURN                                                 *016
        END OF FUNCTION                                                 *017
```

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$COMPILE MAD,PRINT OBJECT,DUMP,EXECUTE                                    999107   06/22/66     10 18 36.0 AM

MAD (29 APR 1966 VERSION) PROGRAM LISTING ... ... ...

              EXTERNAL FUNCTION (A,DEL,M)                                   *001
              ENTRY TO INVERT.                                              *002

             THE SUBROUTINE INVERT. IS AN ADAPTATION OF THE PROGRAM
             APPEARING ON PP.390-391 OF 'APPLIED NUMERICAL METHODS' BY
             CARNAHAN, WILKES, AND LUTHER.  IT SOLVES M SIMULTANEOUS
             LINEAR EQUATIONS USING THE GAUSS-JORDAN REDUCTION SCHEME WITH
             THE MAXIMUM PIVOT CRITERION.  THE MXM MATRIX OF COEFFICIENTS
             WITH THE VECTOR OF CONSTANTS APPENDED AS THE M+1 TH COLUMN IS
             STORED IN THE A ARRAY.

              NORMAL MODE IS INTEGER                                        *003
              FLOATING POINT BIGA, AJCK, A, DEL                             *004
              DIMENSION IR(10), JC(10)                                      *005
              MP1=M+1                                                       *006
              THROUGH L1, FOR K=1,1,K.G.M                                   *007
              BIGA=0.                                                       *008         01
              THROUGH L2, FOR I=1,1,I.G.M                                   *009         01
              THROUGH L2, FOR J=1,1,J.G.M                                   *010         02
              THROUGH L3, FOR I1=1,1,I1.E.K                                 *011         03
              THROUGH L3, FOR J1=1,1,J1.E.K                                 *012         04
L3            WHENEVER I.E.IR(I1).OR.J.E.JC(J1), TRANSFER TO L2             *013         05
              WHENEVER .ABS.A(I,J) .G. BIGA                                 *014         03
              BIGA=.ABS.A(I,J)                                              *015    01   03
              IR(K)=I                                                       *016    01   03
              JC(K)=J                                                       *017    01   03
L2            END OF CONDITIONAL                                            *018    01   03
              BIGA=A(IR(K),JC(K))                                           *019         01
              THROUGH L4, FOR J=1,1,J.G.MP1                                 *020         01
L4            A(IR(K),J)=A(IR(K),J)/BIGA                                    *021         02
              THROUGH L1, FOR I=1,1,I.G.M                                   *022         01
              WHENEVER I.NE.IR(K)                                           *023         02
              AJCK=A(I,JC(K))                                               *024    01   02
              THROUGH L6, FOR J=1,1,J.G.MP1                                 *025    01   02
L6            A(I,J)=A(I,J)-AJCK*A(IR(K),J)                                 *026    01   03
L1            END OF CONDITIONAL                                            *027    01   02
              THROUGH L7, FOR I=1,1,I.G.M                                   *028
L7            DEL(JC(I))=A(IR(I),MP1)                                       *029         01
              FUNCTION RETURN                                               *030
              END OF FUNCTION                                               *031
```

```
SOLUTION PERFORMED FOR FOLLOWING VALUES OF PARAMETERS

    NU=1E-3,DELX=.05,RATIO(1)=1.05,1.01,1.01,2.,MAX=13,TEST=0B,

    ALPHA=-1.0,A=12.0,RIGHT=4.0,LEFT=3.0,COEF(1)= 0.00,0.00

    BET(2)=C.0,BET(3)=0.0,BET(4)=2.,BET(5)=1.,BET(9)=0.

    POWER(0)=0.0,.423,.375,EL=5.90,GUESS(1)=-7.0.,-.5, 3E3,-1.,TEST=1B*
THE TAYLOR NUMBER AS A FUNCTION OF THE DIMENSIONLESS WAVE NUMBER IS GIVEN BY
        EPS(1) =     3.382000,     SOLN(1) =   3420.783661
        EPS(2) =     3.618000,     SCLN(2) =   3499.300537
        EPS(1) =     3.236076,     SCLN(1) =   3395.898743
        EPS(2) =     3.382000,     SOLN(2) =   3420.783661
        EPS(1) =     3.145924,     SOLN(1) =   3390.279144
        EPS(2) =     3.236076,     SOLN(2) =   3395.898743
        EPS(1) =     3.090181,     SCLN(1) =   3390.762146
        EPS(2) =     3.145924,     SCLN(2) =   3390.279144
        EPS(1) =     3.145924,     SOLN(1) =   3390.279144
        EPS(2) =     3.180344,     SOLN(2) =   3391.511444
        EPS(1) =     3.124623,     SOLN(1) =   3390.099457
        EPS(2) =     3.145924,     SOLN(2) =   3390.279144
        EPS(1) =     3.111475,     SOLN(1) =   3390.212036
        EPS(2) =     3.124623,     SOLN(2) =   3390.099457
        EPS(1) =     3.124623,     SOLN(1) =   3390.099457
        EPS(2) =     3.132764,     SOLN(2) =   3390.116333
        EPS(1) =     3.119607,     SOLN(1) =   3390.122253
        EPS(2) =     3.124623,     SCLN(2) =   3390.099457
        EPS(1) =     3.124623,     SOLN(1) =   3390.099457
        EPS(2) =     3.127738,     SOLN(2) =   3390.098358
        EPS(1) =     3.127738,     SOLN(1) =   3390.098358
        EPS(2) =     3.129654,     SCLN(1) =   3390.102264
```

```
EPS(1) =     3.126545,      SOLN(1) =   3390.097748

EPS(2) =     3.127738,      SOLN(2) =   3390.098358

EPS(1) =     3.125813,      SOLN(1) =   3390.098083

EPS(2) =     3.126545,      SOLN(2) =   3390.097748

THE CRITICAL TAYLOR NUMBER IS

           TAYLOR =   3390.097748

FOR A CRITICAL DIMENSIONLESS WAVE NUMBER OF

           CRIT =     3.126545

THE RHEOLOGICAL PARAMETERS ARE

                    BET(0)...BET(9)

 1.000000E+00   1.000000E+00    .000000E+00    .000000E+00   2.000000E+00   1.000000E+00  -.000000E+00  -.000000E+00
 -.000000E+00    .000000E+00

THE CRITICAL EIGENFUNCTIONS ARE
```

X	PSI	PSI'	V
.00	.000000	.000000	.000000
.05	.001109	.041665	-.024597
.10	.003915	.068225	-.049351
.15	.007718	.081941	-.073886
.20	.011931	.084979	-.097460
.25	.016073	.079422	-.119160
.30	.019764	.067270	-.138048
.35	.022722	.050421	-.153277
.40	.024757	.030657	-.164167
.45	.025766	.009610	-.170261
.50	.025722	-.011255	-.171349
.55	.024666	-.030655	-.167471
.60	.022699	-.047497	-.158908
.65	.019974	-.060866	-.146146
.70	.016683	-.070002	-.129843
.75	.013055	-.074259	-.110768
.80	.009348	-.073054	-.089742
.85	.005851	-.065796	-.067567
.90	.002881	-.051810	-.044939
.95	.000795	-.030251	-.022357
1.00	.000000	.000000	.000000

```
              GUESS(1)...GUESS(3)

-6.988391E+00   -4.906935E-01    3.390098E+03
```

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