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FINAL REPORT

AFFINE STRUCTURES IN EUCLIDEAN MANIFOLDS

June 2, 1952, to May 31, 1954

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Project 2051

DETROIT ORDNANCE DISTRICT, DEPARTMENT OF THE ARMY
CONTRACT NO. DA-20-018-ORD-12277

July, 1954

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I. It has been shown that the knot-type of a polygon in Euclidean 3-space E^3 is invariant under orientation-preserving homeomorphisms of E^3 onto itself. That is, if P is a polygon in E^3, and f is an orientation-preserving homeomorphism of E^3 onto E^3, then the knot-types of P and f(P) are the same. In "Affine Structures in 3-manifolds: V. The Triangulation Theorem and Hauptvermutung" (Annals of Mathematics, 56, 96-113 (1952), Theorem 6) it was shown that any semi-locally tamely imbedded simple closed curve J in E^3 can be thrown onto a polygon by an orientation-preserving homeomorphism f of E^3 onto itself. The knot-type of such a curve J can then be defined as that of the polygon f(J); and by the result stated above, the definition is unique, that is, it does not depend on the choice of f. It is clear, also, that the knot-types thus defined for semi-locally tamely imbedded curves are invariant under orientation-preserving homeomorphisms of E^3 onto itself. We have, therefore, a true generalization of the classical theory, the classical combinatorial definitions of knot and knot-type being replaced by topologically invariant definitions.

This result is included in a paper entitled "Affine Structures in 3-Manifolds: VIII. Invariance of the Knot-Types; Local Tame Imbedding" (Annals of Mathematics, 59, 159-170 (1954). (Reprints of this paper have been distributed as a technical report.)

II. In collaboration with O. G. Harrold, the following results have been obtained: We say that a set M in E^3 is locally polyhedral at a point p if p has a closed neighborhood N which intersects M in a polyhedron. K will denote a 2-sphere; I denotes the interior of K, and E denotes the exterior of K, compactified at infinity.

Theorem 1. If K is locally polyhedral except at one point, then both E and I are simply connected, and either the closure of E or the closure of I is a topological 3-cell.

Theorem 2. If K is locally polyhedral except at three points, then either E or I is simply connected.

These results are included in a paper entitled "Almost Locally Polyedral Spheres" (Annals of Mathematics, 57, 575-578 (1953).

III. It has been shown that if M is a compact 3-manifold which is the sum of two open sets, each of which is homeomorphic to Euclidean 3-space E^3, then M is a 3-sphere. This is the solution of a problem originally proposed by J. W. Alexander. It has been published in a paper entitled "Affine Structures in 3-Manifolds: VI. Compact Spaces Covered by Two Euclidean Neighborhoods" (Annals of Mathematics, 58, 107 (1953).

IV. The methods used in obtaining the result stated in I above led also to the following results:

(1) Let D be a tamely imbedded disk, and let e be a tamely imbedded arc, in a triangulated 3-manifold K, such that e pierces D at a point p. Then the union of D and e is tamely imbedded. (Here e pierces D at p if for each sufficiently small open neighborhood U of p, U-D is the sum of two disjoint open sets each of which intersects the component of the intersection of U and e that contains p.)

(2) Let D be a disk, and let e be a linear interval in the triangulated 3-manifold K, such that (a) e pierces D at a point p, and (b) D is locally polyhedral at every point of D-p. Then, for each neighborhood U of p there is a homeomorphism f of K onto itself, such that (a) f is the identity in e and in K-U, (b) f is piecewise linear over every finite polyhedron in K-p, and (c) f(D) is a polyhedron.

(3) Let A be an annulus, let J be a component of the boundary of A, and suppose that A is locally polyhedral at each point of A-J and that J is a polyhedron. Let U be a neighborhood of J. Then there is a homeomorphism f of K onto itself, such that f(A) is a polyhedron and such that f is the identity of J and in the complement of U. Moreover, if P is a polyhedron intersecting A only in J, then f can be chosen so as to be the identity in P.

(4) Let A and B be tamely imbedded 2-manifolds with boundary, whose intersection is a sum of components of their boundaries. Then the union of A and B is tamely imbedded.

(5) Let S^2 be a 2-sphere in 3-space, let I be the closed linear interval from 0 to 1, and let U be the interior of S^2. Suppose that there is a homeomorphism f, throwing the Cartesian product S^2xI into the closure of U, such that for each point p of S^2, f(px0) is p. Then the closure of U is a 3-cell. (Of course it does not follow that S^2 is tamely imbedded.)

(6) Let L be a linear graph in the triangulated 3-manifold K, and suppose that L is locally tamely imbedded, in the sense that every point of L has a neighborhood in L which is tamely imbedded in K. Then L is tamely imbedded in K.

Some of these results were anticipated by R. H. Bing and published separately by him. They have also been published in a paper entitled "Affine Structures in 3-Manifolds: VII. Disks Which Are Pierced by Intervals" (Annals of Mathematics, 58, 403-308 (1953) and in part VIII of the same series, cited above.

V. A number of results have been obtained on simplicial homeomorphisms of the 3-sphere. Among these are the following:

(1) Let f be a simplicial homeomorphism of the 3-sphere onto itself, preserving orientation. Let the period of f be n. Then f has period n at each nonfixed point of f.

(2) Let f be as in (1), and suppose that the fixed-point set F of f is a single simple closed polygon P. Then there is a tubular neighborhood T of P, such that (a) f, restricted to T, is homeomorphic to a latitudinal rotation of a solid torus onto itself, and (b) if N is the closure of the complement of T, and B is the boundary of N, then N has a decomposition into polyhedral 3-manifolds with boundary N_1, N_2, $\cdots$, N_n, such that the sets N_i are permuted by f, and such that each set N_i intersects B in a set homeomorphic to a plane annulus.

These results (and the methods used in obtaining them) represent first steps in an attempt to verify a conjecture of P. A. Smith to the effect that if P is as in (2), then P must be unknotted. There appears to be some ground for hoping that Smith's conjecture is related (methodologically, at least) to the Dehn Lemma and to the Poincare Conjecture.

VI. A very large part of the effort expended on this project was devoted to attempts to solve certain very difficult problems, notably the Dehn Lemma and the Poincare Conjecture. These efforts have not led to any results worth reporting.