On Damage Strain Energy Release Rate Y

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ABSTRACT: In the application of the theory of damage mechanics to solve a wide range of practical engineering problems, a generalized formulation of damage strain energy release rate **Y** is required. This is because the damage strain energy release rate **Y**, which is the thermodynamic conjugate force of damage variable **D**, is used to derive damage evolution equation. For the case of isotropic damage, the damage variable is degenerated to a scalar. Accordingly, the derivation of the damage strain energy release rate is relatively straightforward. However, it is much more involved in the case of anisotropic damage. Currently, no such generalized form of the damage strain energy release rate is available. This paper is intended to present the development of three formulations of damage strain energy release rate based on three different forms of the damage effect tensor **M(D)** for which **D** is a second order symmetric tensor. The realization of the anisotropic damage strain energy rate formulations makes it possible to obtain the solution of practical structural problems where the service loading is often nonproportional.

INTRODUCTION

THE THEORY OF damage mechanics calls for the development of constitutive equations of elasticity and plasticity coupled with damage and the damage evolution equation [1–6]. The latter is used to monitor the progressive material degradation until a macrocrack is initiated. The damage in a material element is caused due to the presence of microcracks/voids which are often represented with a damage variable **D**. Its thermodynamic conjugate **Y** is known as the damage strain energy release rate and is defined as the derivative of strain energy with respect to the damage variable, or $\partial W/\partial \mathbf{D} = \mathbf{Y}$. For the case of isotropic damage, **Y** is a scalar and its derivation is, therefore, straightforward. Even in the case of anisotropic damage, the damage characterization is often achieved for the structures under proportional loading. For the loading condition, the principal coor-

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dinate system of damage coincides with that of stress and the shear stress in the principal coordinate system of damage is zero. However, except for a limited number of cases, most service loading conditions are non-proportional. Under such a loading condition, the principal directions of stress no longer coincide with those of damage. Accordingly, shear stress in the principal coordinate system of damage is non-zero and should be considered. This calls for the derivation of a generalized form of the damage strain energy rate. Unfortunately, such a generalized form is not at present available as highlighted recently by Chaboche that "the damage elastic stiffness tensor and the damage energy release rate are difficult to express in the general case" [7]. It is to remedy the shortcoming in the applicability of the theory of damage mechanics for practical engineering problems that this investigation is aimed.

In the formulation of constitutive equations, the concepts of effective stress and effective strain are introduced. In general, the damage effect tensor **M(D)** is a fourth order tensor according to the definition of effective stress, but a second order damage tensor is often employed in damage analysis [1–6]. The simplification becomes necessary for practical reasons as the elements in a fourth order tensor are difficult if not impossible to measure. In this paper, the representations of three forms of damage effect tensor **M(D)** expressed in terms of second order symmetric damage tensor are first discussed. This is followed by the development of the damage strain energy release rates corresponding to these three forms of **M(D)**.

REPRESENTATIONS OF M(D)

The effective stress is defined as

$$\tilde{\boldsymbol{\sigma}} = \mathbf{M}(\mathbf{D}):\boldsymbol{\sigma} \tag{1}$$

The damage effect tensor **M(D)** is in essence a fourth order tensor. For isotropic damage, the effective stress is reduced to

$$\tilde{\sigma} = \frac{\sigma}{(1 - D)} \tag{2}$$

Actually, the isotropic damage effect tensor may be written as $1/(1 - D)\mathbf{1}$, where **1** is a fourth order unit tensor. When the damage tensor is chosen as a second order symmetric tensor, there are several ways to construct the damage effect tensor **M(D)** [8]. As **M(D)** is a fourth order symmetric tensor, it could be represented by a 6×6 matrix. In the principal coordinate system of damage **D** which is chosen as a second order symmetric tensor for its convenience of application, they may be represented respectively for three different cases as shown below [8]:

Case A

$$[\mathbf{M}_1(\mathbf{D})] = \begin{bmatrix} \frac{1}{(1-D_1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{(1-D_2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{(1-D_3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_2)(1-D_3)}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_1)(1-D_3)}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{(1-D_1)(1-D_2)}} \end{bmatrix} \quad (3)$$

Case B

$$[\mathbf{M}_2(\mathbf{D})] = \begin{bmatrix} \frac{1}{(1-D_1)} & 0 & 0 & 0 \\ 0 & \frac{1}{(1-D_2)} & 0 & 0 \\ 0 & 0 & \frac{1}{(1-D_3)} & 0 \\ 0 & 0 & 0 & \frac{1}{1-\frac{D_2+D_3}{2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{matrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ \dfrac{1}{1-\dfrac{D_1+D_3}{2}} & 0 \\ 0 & \dfrac{1}{1-\dfrac{D_1+D_2}{2}} \end{matrix} \Bigg] \tag{4}$$

Case C

$$[\mathbf{M}_3(\mathbf{D})] = \begin{bmatrix} \dfrac{1}{(1-D_1)} & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{(1-D_2)} & 0 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{(1-D_3)} & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{2}\left(\dfrac{1}{1-D_2}+\dfrac{1}{1-D_3}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & \dfrac{1}{2}\left(\dfrac{1}{1-D_1}+\dfrac{1}{1-D_3}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & \dfrac{1}{2}\left(\dfrac{1}{1-D_1}+\dfrac{1}{1-D_2}\right) \end{bmatrix} \tag{5}$$

where D_1, D_2 and D_3 are the respective principal values of damage. In the derivation of strain energy release rate **Y**, a general form of **M(D)** in an arbitrary coordinate system in terms of second order symmetric tensor **D** should be obtained first. This is required to deduce the derivative of **M(D)** with respect to **D** for the formulation of the damage strain energy release rate. For the above three forms of **M(D)**, they may be written respectively in tensorial form as

Case A

$$\mathbf{M}_1^2(\mathbf{D}) = \mathbf{N}_1^{-2}(\mathbf{D}) = \mathbf{P}^{-1}(\mathbf{D}) \tag{6}$$

where

$$P_{ijkl} = \frac{1}{2}\left[(\delta_{ik} - D_{ik})(\delta_{jl} - D_{jl}) + (\delta_{il} - D_{il})(\delta_{jk} - D_{jk})\right] \tag{7}$$

Case B

$$\mathbf{M}_2(\mathbf{D}) = (1 - \hat{D})^{-1} \tag{8}$$

where

$$\hat{D}_{ijkl} = \frac{1}{4}(\delta_{ik}D_{jl} + \delta_{il}D_{jk} + \delta_{jk}D_{il} + \delta_{jl}D_{ik}) \tag{9}$$

Case C

$$\mathbf{M}_3(\mathbf{D}) = \hat{\Phi} \tag{10}$$

where

$$\hat{\Phi}_{ijkl} = \frac{1}{4}(\delta_{ik}\Phi_{jl} + \delta_{il}\Phi_{jk} + \delta_{jk}\Phi_{il} + \delta_{jl}\Phi_{ik}) \tag{11}$$

and

$$\Phi = (1 - \mathbf{D})^{-1} \tag{12}$$

From the above representations, it is evident that there are different methods to construct **M(D)** when **D** is a second order symmetric damage tensor.

DERIVATION OF DAMAGE STRAIN ENERGY RELEASE RATE

The damage strain energy release rate **Y** is defined as [3–5],

$$\mathbf{Y} = \boldsymbol{\sigma}:\left[\tilde{\mathbf{E}}^{-1}:\mathbf{M}^{-1}(\mathbf{D}):\frac{\partial \mathbf{M}(\mathbf{D})}{\partial \mathbf{D}}\right]^{s}:\boldsymbol{\sigma} \tag{13}$$

Due to the symmetry of **Y**, $\boldsymbol{\sigma}$ and **D**, Voigt's notation is employed. Then **Y**, $\boldsymbol{\sigma}$ and **D** can be represented in vectorial form as

$$\begin{Bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{Bmatrix} = \begin{Bmatrix} Y_{11} \\ Y_{22} \\ Y_{33} \\ Y_{23} \\ Y_{13} \\ Y_{12} \end{Bmatrix}; \quad \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{Bmatrix} \quad \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \\ D_5 \\ D_6 \end{Bmatrix} = \begin{Bmatrix} D_{11} \\ D_{22} \\ D_{33} \\ 2D_{23} \\ 2D_{13} \\ 2D_{12} \end{Bmatrix} \tag{14}$$

An essential step in the derivation of **Y** is to deduce the derivative of **M(D)** with respect to damage variable **D**. Accordingly, the damage strain energy release rate is derived with Equation (13).

Case A

From Equation (6),

$$\mathbf{M}_1^2(\mathbf{D}):\mathbf{N}_1^2(\mathbf{D}) = \mathbf{M}_1^2(\mathbf{D}):\mathbf{P}(\mathbf{D}) = \mathbf{1} \tag{15}$$

$$\frac{\partial \mathbf{M}_1^2(\mathbf{D})}{\partial \mathbf{D}} = -\mathbf{M}_1^2(\mathbf{D}):\frac{\partial \mathbf{P}(\mathbf{D})}{\partial \mathbf{D}}:\mathbf{M}_1^2(\mathbf{D}) \tag{16}$$

From the above equation, $\partial \mathbf{M}_1(\mathbf{D})/\partial \mathbf{D}$ is obtained as

$$\frac{\partial \mathbf{M}_1(\mathbf{D})}{\partial \mathbf{D}} = \frac{1}{2}\mathbf{M}_1^{-1}(\mathbf{D}):\frac{\partial \mathbf{M}_1^2(\mathbf{D})}{\partial \mathbf{D}} = -\frac{1}{2}\mathbf{M}_1(\mathbf{D}):\frac{\partial \mathbf{P}(\mathbf{D})}{\partial \mathbf{D}}:\mathbf{M}_1^2(\mathbf{D}) \tag{17}$$

Substituting the above into Equation (13), **Y** is obtained as

$$\mathbf{Y} = -\frac{1}{2}\boldsymbol{\sigma}:\left[\tilde{\mathbf{E}}^{-1}:\frac{\partial \mathbf{P}(\mathbf{D})}{\partial \mathbf{D}}:\mathbf{M}_1^2(\mathbf{D})\right]^{s}:\boldsymbol{\sigma} \tag{18}$$

Because **P(D)** is a fourth order symmetric tensor, it may, therefore, be represented by a 6 × 6 matrix with the introduction of Voigt's notation. From Equation (7), we have

$$\begin{bmatrix} (1 - D_{11})^2 & D_{12}^2 & D_{13}^2 \\ D_{21}^2 & (1 - D_{22})^2 & D_{23}^2 \end{bmatrix}$$

$$
[P] = \left|\begin{array}{ccc}
D_{31}^2 & D_{32}^2 & (1 - D_{33})^2 \\
D_{21}D_{31} & -(1 - D_{22})D_{32} & -(1 - D_{33})D_{23} \\
-(1 - D_{11})D_{31} & D_{32}D_{12} & -(1 - D_{33})D_{13} \\
-(1 - D_{11})D_{21} & -(1 - D_{22})D_{12} & D_{13}D_{23}
\end{array}\right.
$$

$$
\begin{array}{cc}
2D_{12}D_{13} & -2(1 - D_{11})D_{13} \\
-2(1 - D_{22})D_{23} & 2D_{23}D_{21} \\
-2(1 - D_{33})D_{32} & -2(1 - D_{33})D_{31} \\
(1 - D_{22})(1 - D_{33}) + D_{23}D_{32} & D_{23}D_{31} - D_{21}(1 - D_{33}) \\
D_{32}D_{13} - D_{12}(1 - D_{33}) & (1 - D_{11})(1 - D_{33}) + D_{13}D_{31} \\
D_{12}D_{23} - D_{13}(1 - D_{22}) & D_{13}D_{21} - D_{23}(1 - D_{11})
\end{array}
$$

$$
\left.\begin{array}{c}
-2(1 - D_{11})D_{12} \\
-2(1 - D_{22})D_{21} \\
2D_{32}D_{31} \\
D_{21}D_{32} - D_{31}(1 - D_{22}) \\
D_{12}D_{31} - D_{32}(1 - D_{11}) \\
(1 - D_{11})(1 - D_{22}) + D_{12}D_{21}
\end{array}\right] \tag{19}
$$

Differentiating it with respect to damage variable **D** gives

$$
\frac{\partial P_{ijkl}}{\partial D_{pq}} = -\frac{1}{2}[(\delta_{ik} - D_{ik})\delta_{jp}\delta_{lq} + (\delta_{jl} - D_{jl})\delta_{ip}\delta_{kq}
+ (\delta_{il} - D_{il})\delta_{jp}\delta_{kq} + (\delta_{jk} - D_{jk})\delta_{ip}\delta_{lq}] \tag{20}
$$

The above expression is a sixth order tensor. Theoretically, **Y** may be obtained by substituting it into Equation (18), but this is a complex way to derive **Y**. The procedure employed instead in this investigation is to first derive $\partial\mathbf{P}/\partial D_{11}$, $\partial\mathbf{P}/\partial D_{22}$, $\partial\mathbf{P}/\partial D_{33}$, $\partial\mathbf{P}/\partial D_{23}$, . . . , respectively, which are fourth order tensors. These are evaluated in the principal coordinate system of damage, and the damage strain energy release rate is obtained from Equation (18),

$$
Y_1 = \frac{1}{E}\left(\frac{\sigma_1^2}{V_{11}^3} - \frac{\nu\sigma_2\sigma_1}{V_{11}^2V_{22}} - \frac{\nu\sigma_3\sigma_1}{V_{11}^2V_{33}} + \frac{(1+\nu)\sigma_5^2}{V_{31}^2V_{11}} + \frac{(1+\nu)\sigma_6^2}{V_{12}^2V_{11}}\right) \tag{21}
$$

$$
Y_2 = \frac{1}{E}\left(-\frac{\nu\sigma_2\sigma_1}{V_{11}V_{22}^2} + \frac{\sigma_2^2}{V_{22}^3} - \frac{\nu\sigma_3\sigma_1}{V_{22}^2V_{33}} + \frac{(1+\nu)\sigma_4^2}{V_{23}^2V_{22}} + \frac{(1+\nu)\sigma_6^2}{V_{12}^2V_{22}}\right) \tag{22}
$$

$$Y_3 = \frac{1}{E}\left(-\frac{\nu\sigma_1\sigma_3}{V_{11}V_{33}^2} - \frac{\nu\sigma_2\sigma_3}{V_{22}V_{33}^2} + \frac{\sigma_3^2}{V_{33}^3} + \frac{(1+\nu)\sigma_5^2}{V_{23}^2V_{33}} + \frac{(1+\nu)\sigma_5^2}{V_{31}^2V_{33}}\right) \tag{23}$$

$$Y_4 = \frac{1}{2E}\left(\frac{-2\nu\sigma_1\sigma_4}{V_{11}V_{22}V_{33}} + \frac{(1-\nu)\sigma_2\sigma_4}{V_{23}^2V_{22}} + \frac{(1-\nu)\sigma_3\sigma_4}{V_{23}^2V_{33}}\right.$$
$$\left. + \frac{2(1+\nu)\sigma_5\sigma_6}{V_{11}V_{22}V_{33}} + \frac{(1+\nu)\sigma_2\sigma_4}{V_{23}^2V_{22}} + \frac{(1+\nu)\sigma_3\sigma_4}{V_{23}^2V_{33}}\right) \tag{24}$$

$$Y_5 = \frac{1}{2E}\left(\frac{(1-\nu)\sigma_1\sigma_5}{V_{11}^2V_{33}} + \frac{-2\nu\sigma_2\sigma_5}{V_{11}V_{22}V_{33}} + \frac{(1-\nu)\sigma_3\sigma_5}{V_{11}V_{33}^2}\right.$$
$$\left. + \frac{2(1+\nu)\sigma_4\sigma_6}{V_{11}V_{22}V_{33}} + \frac{(1+\nu)\sigma_1\sigma_5}{V_{31}^2V_{11}} + \frac{(1+\nu)\sigma_3\sigma_5}{V_{31}^2V_{33}}\right) \tag{25}$$

$$Y_6 = \frac{1}{2E}\left(\frac{(1-\nu)\sigma_1\sigma_6}{V_{11}^2V_{22}} + \frac{(1-\nu)\sigma_2\sigma_6}{V_{11}V_{22}^2} + \frac{-2\nu\sigma_3\sigma_6}{V_{11}V_{22}V_{33}}\right.$$
$$\left. + \frac{2(1+\nu)\sigma_4\sigma_5}{V_{11}V_{22}V_{33}} + \frac{(1+\nu)\sigma_1\sigma_6}{V_{12}^2V_{11}} + \frac{(1+\nu)\sigma_2\sigma_6}{V_{12}^2V_{22}}\right) \tag{26}$$

where

$$\begin{aligned}
V_{11} &= 1 - D_{11} \\
V_{22} &= 1 - D_{22} \\
V_{33} &= 1 - D_{33} \\
V_{23} &= \sqrt{(1 - D_{22})(1 - D_{33})} \\
V_{31} &= \sqrt{(1 - D_{11})(1 - D_{33})} \\
V_{12} &= \sqrt{(1 - D_{22})(1 - D_{11})}
\end{aligned}$$

which are the diagonal terms of **M(D)** in the principal coordinate system of **D** as shown in Equation (3). It can be readily observed that Y_4, Y_5 and Y_6 are not zero even in the principal coordinate system of damage, when the principal coordinate system of damage is not coincident with that of stress. Although the above expressions of $Y_{1\text{-}6}$ are in the principal coordinate system of damage, its value in an arbitrary coordinate system could be easily obtained through the tensor transformation rule.

Case B

From the Equation (8),

$$\frac{\partial \mathbf{M}_2(\mathbf{D})}{\partial \mathbf{D}} = \frac{\partial(1 - \hat{\boldsymbol{D}})^{-1}}{\partial \mathbf{D}} = (1 - \hat{\boldsymbol{D}})^{-1}:\frac{\partial \hat{\boldsymbol{D}}}{\partial \mathbf{D}}:(1 - \hat{\boldsymbol{D}})^{-1} = \mathbf{M}_2(\mathbf{D}):\frac{\partial \hat{\boldsymbol{D}}}{\partial \mathbf{D}}:\mathbf{M}_2(\mathbf{D}) \tag{27}$$

As $\hat{\boldsymbol{D}}$ is a symmetric fourth order tensor, it can be represented by a 6×6 matrix with Voigt's notation.

$$[\hat{D}] = \begin{bmatrix} D_{11} & 0 & 0 & 0 & D_{13} & D_{12} \\ 0 & D_{22} & 0 & D_{23} & 0 & D_{12} \\ 0 & 0 & D_{33} & D_{23} & D_{31} & 0 \\ 0 & D_{32}/2 & D_{23}/2 & (D_{33} + D_{22})/2 & D_{21}/2 & D_{31}/2 \\ D_{31}/2 & 0 & D_{13}/2 & D_{13}/2 & (D_{11} + D_{33})/2 & D_{32}/2 \\ D_{21}/2 & D_{12}/2 & 0 & D_{13}/2 & D_{23}/2 & (D_{11} + D_{22})/2 \end{bmatrix} \tag{28}$$

Differentiating $\hat{\boldsymbol{D}}$ with respect to $\mathbf{D}$, we obtain

$$\frac{\partial \hat{D}_{ijkl}}{\partial D_{uv}} = \frac{1}{4}(\delta_{ik}\delta_{ju}\delta_{lv} + \delta_{il}\delta_{ju}\delta_{kv} + \delta_{jk}\delta_{iu}\delta_{lv} + \delta_{jl}\delta_{iu}\delta_{iv}) \tag{29}$$

For Case B, the derivatives of $\hat{\boldsymbol{D}}$ with respect to D_{11}, D_{22}, D_{33}, D_{23}, . . . , are first evaluated, which are also fourth order tensors. The damage strain energy release rate **Y** in the principal coordinate system of damage is obtained by first substituting the resulting derivatives in the principal coordinate system of damage in Equation (27). **Y** is then obtained with Equation (13)

$$Y_1 = \frac{1}{E}\left(\frac{\sigma_1^2}{W_{11}^3} - \frac{\nu\sigma_1\sigma_2}{W_{11}^2 W_{22}} - \frac{\nu\sigma_1\sigma_3}{W_{11}^2 W_{33}} + \frac{(1+\nu)\sigma_5^2}{W_{31}^3} + \frac{(1+\nu)\sigma_6^2}{W_{12}^3}\right) \tag{30}$$

$$Y_2 = \frac{1}{E}\left(-\frac{\nu\sigma_1\sigma_2}{W_{22}^2 W_{11}} + \frac{\sigma_2^2}{W_{22}^3} - \frac{\nu\sigma_2\sigma_3}{W_{22}^2 W_{33}} + \frac{(1+\nu)\sigma_4^2}{W_{23}^3} + \frac{(1+\nu)\sigma_6^2}{W_{12}^3}\right) \tag{31}$$

$$Y_3 = \frac{1}{E}\left(-\frac{\nu\sigma_1\sigma_3}{W_{33}^2 W_{11}} - \frac{\nu\sigma_2\sigma_3}{W_{33}^2 W_{22}} + \frac{\sigma_3^2}{W_{33}^3} + \frac{(1+\nu)\sigma_4^2}{W_{23}^3} + \frac{(1+\nu)\sigma_5^2}{W_{31}^3}\right) \tag{32}$$

$$Y_4 = \frac{1}{2E}\left[\left(-\frac{\nu}{W_{11}W_{22}W_{23}} - \frac{\nu}{W_{11}W_{33}W_{23}}\right)\sigma_1\sigma_4 + \left(\frac{1}{W_{22}^2 W_{23}} - \frac{\nu}{W_{22}W_{33}W_{23}}\right)\sigma_2\sigma_4\right.$$

$$+ \left(\frac{1}{W_{33}^2 W_{23}} - \frac{\nu}{W_{22}W_{33}W_{23}}\right)\sigma_3\sigma_4 + \frac{(1+\nu)\sigma_2\sigma_4}{W_{23}^2 W_{22}} + \frac{(1+\nu)\sigma_3\sigma_4}{W_{23}^2 W_{33}}$$

$$+ \frac{(1+\nu)\sigma_5\sigma_6}{W_{31}^2 W_{12}} + \frac{(1+\nu)\sigma_5\sigma_6}{W_{12}^2 W_{31}}\Bigg] \tag{33}$$

$$Y_5 = \frac{1}{2E}\Bigg[\left(\frac{1}{W_{11}^2 W_{31}} - \frac{\nu}{W_{11}W_{33}W_{31}}\right)\sigma_1\sigma_5 + \left(-\frac{\nu}{W_{11}W_{22}W_{31}} - \frac{\nu}{W_{22}W_{33}W_{31}}\right)\sigma_2\sigma_5$$

$$+ \left(\frac{1}{W_{33}^2 W_{31}} - \frac{\nu}{W_{11}W_{33}W_{31}}\right)\sigma_3\sigma_5 + \frac{(1+\nu)\sigma_6\sigma_4}{W_{23}^2 W_{12}} + \frac{(1+\nu)\sigma_6\sigma_4}{W_{12}^2 W_{23}}$$

$$+ \frac{(1+\nu)\sigma_5\sigma_1}{W_{31}^2 W_{11}} + \frac{(1+\nu)\sigma_5\sigma_3}{W_{31}^2 W_{33}}\Bigg] \tag{34}$$

$$Y_6 = \frac{1}{2E}\Bigg[\left(\frac{1}{W_{11}^2 W_{12}} - \frac{\nu}{W_{11}W_{22}W_{12}}\right)\sigma_1\sigma_6 + \left(\frac{1}{W_{22}^2 W_{12}} - \frac{\nu}{W_{22}W_{11}W_{12}}\right)\sigma_2\sigma_6$$

$$+ \left(-\frac{\nu}{W_{11}W_{33}W_{23}} - \frac{\nu}{W_{22}W_{33}W_{12}}\right)\sigma_3\sigma_6 + \frac{(1+\nu)\sigma_5\sigma_4}{W_{23}^2 W_{31}} + \frac{(1+\nu)\sigma_5\sigma_4}{W_{31}^2 W_{23}}$$

$$+ \frac{(1+\nu)\sigma_1\sigma_6}{W_{12}^2 W_{11}} + \frac{(1+\nu)\sigma_1\sigma_6}{W_{12}^2 W_{22}}\Bigg] \tag{35}$$

where

$$\begin{aligned}
W_{11} &= 1 - D_{11}\\
W_{22} &= 1 - D_{22}\\
W_{33} &= 1 - D_{33}\\
W_{23} &= 1 - (D_2 + D_3)/2\\
W_{31} &= 1 - (D_1 + D_3)/2\\
W_{12} &= 1 - (D_2 + D_1)/2
\end{aligned}$$

which are the diagonal terms of **M(D)** in the principal coordinate system of **D** as shown in Equation (4). The above formulations have been incorporated in a finite element program and employed in a failure analysis [6].

Case C

From the Equation (10),

$$\frac{\partial \mathbf{M}_3(\mathbf{D})}{\partial \mathbf{D}} = \frac{\partial \hat{\Phi}}{\partial \mathbf{D}} = \frac{\partial \hat{\Phi}}{\partial \Phi}\frac{\partial \hat{\Phi}}{\partial \mathbf{D}} \tag{36}$$

and with Equation (12), we have

$$\frac{\partial \Phi_{ij}}{\partial D_{uv}} = \Phi_{iu}\Phi_{iv} \tag{37}$$

As $\hat{\Phi}$ is a fourth order symmetric tensor, it can be represented by 6×6 matrix as follows:

$$[\hat{\Phi}] = \begin{bmatrix} \Phi_{11} & 0 & 0 & 0 & \Phi_{13} & \Phi_{12} \\ 0 & \Phi_{22} & 0 & \Phi_{23} & 0 & \Phi_{12} \\ 0 & 0 & \Phi_{33} & \Phi_{23} & \Phi_{31} & 0 \\ 0 & \Phi_{32}/2 & \Phi_{23}/2 & (\Phi_{33}+\Phi_{22})/2 & \Phi_{21}/2 & \Phi_{31}/2 \\ \Phi_{31}/2 & 0 & \Phi_{13}/2 & \Phi_{12}/2 & (\Phi_{11}+\Phi_{33})/2 & \Phi_{32}/2 \\ \Phi_{21}/2 & \Phi_{12}/2 & 0 & \Phi_{13}/2 & \Phi_{23}/2 & (\Phi_{11}+\Phi_{22})/2 \end{bmatrix} \tag{38}$$

Differentiating $\hat{\Phi}$ with respect to Φ,

$$\frac{\partial \hat{\Phi}_{ijkl}}{\partial \Phi_{uv}} = \frac{1}{4}(\delta_{ik}\delta_{ju}\delta_{lv} + \delta_{il}\delta_{ju}\delta_{kv} + \delta_{jk}\delta_{iu}\delta_{lv} + \delta_{jl}\delta_{iu}\delta_{iv}) \tag{39}$$

The damage strain energy release rate can be similarly obtained as those employed in Cases A and B by first obtaining the derivative of the damage effect tensor to damage tensor. Y is then derived with Equation (13) as

$$Y_1 = \frac{1}{E}\left(\frac{\sigma_1^2}{U_{11}^3} - \frac{\nu\sigma_1\sigma_2}{U_{11}^2U_{22}} - \frac{\nu\sigma_1\sigma_3}{U_{11}^2U_{33}} + \frac{(1+\nu)\sigma_5^2}{U_{11}^2U_{31}} + \frac{(1+\nu)\sigma_6^2}{U_{12}U_{11}^2}\right) \tag{40}$$

$$Y_2 = \frac{1}{E}\left(-\frac{\nu\sigma_1\sigma_2}{U_{22}^2U_{11}} + \frac{\sigma_2^2}{U_{22}^3} - \frac{\nu\sigma_2\sigma_3}{U_{22}^2U_{33}} + \frac{(1+\nu)\sigma_4^2}{U_{22}^2U_{23}} + \frac{(1+\nu)\sigma_6^2}{U_{12}U_{22}^2}\right) \tag{41}$$

$$Y_3 = \frac{1}{E}\left(-\frac{\nu\sigma_1\sigma_3}{U_{33}^2U_{11}} - \frac{\nu\sigma_1\sigma_2}{U_{33}^2U_{22}} + \frac{\sigma_3^2}{U_{33}^3} + \frac{(1+\nu)\sigma_4^2}{U_{33}^2U_{23}} + \frac{(1+\nu)\sigma_5^2}{U_{31}U_{33}^2}\right) \tag{42}$$

$$Y_4 = \frac{1}{2E}\left(\frac{-2\nu\sigma_1\sigma_4}{U_{11}U_{23}^2} + \frac{(1-\nu)\sigma_2\sigma_4}{U_{22}U_{23}^2} + \frac{(1-\nu)\sigma_3\sigma_4}{U_{33}U_{23}^2} + \frac{(1+\nu)\sigma_2\sigma_4}{U_{23}^3}\right.$$
$$\left. + \frac{(1+\nu)\sigma_3\sigma_4}{U_{23}^3} + \frac{(1+\nu)\sigma_5\sigma_6}{U_{31}U_{23}^2} + \frac{(1+\nu)\sigma_5\sigma_6}{U_{12}U_{23}^2}\right) \tag{43}$$

$$Y_5 = \frac{1}{2E}\left(\frac{(1-\nu)\sigma_1\sigma_5}{U_{11}U_{31}^2} - \frac{2\nu\sigma_2\sigma_5}{U_{22}U_{31}^2} + \frac{(1-\nu)\sigma_3\sigma_5}{U_{33}U_{31}^2} + \frac{(1+\nu)\sigma_4\sigma_6}{U_{23}U_{31}^2}\right.$$

$$+ \frac{(1 + \nu)\sigma_1\sigma_5}{U_{31}^3} + \frac{(1 + \nu)\sigma_3\sigma_5}{U_{31}^3} + \frac{(1 + \nu)\sigma_4\sigma_6}{U_{12}U_{31}^2}\Bigg) \tag{44}$$

$$Y_6 = \frac{1}{2E}\Bigg(\frac{(1 - \nu)\sigma_1\sigma_6}{U_{11}U_{12}^2} + \frac{(1 - \nu)\sigma_2\sigma_6}{U_{22}U_{12}^2} - \frac{2\nu\sigma_3\sigma_6}{U_{33}U_{12}^2} + \frac{(1 + \nu)\sigma_5\sigma_4}{U_{23}U_{12}^2}$$

$$+ \frac{(1 + \nu)\sigma_1\sigma_6}{U_{12}^3} + \frac{(1 + \nu)\sigma_2\sigma_6}{U_{12}^3} + \frac{(1 + \nu)\sigma_4\sigma_5}{U_{31}U_{12}^2}\Bigg) \tag{45}$$

where

$$U_{11} = 1 - D_{11}$$
$$U_{22} = 1 - D_{22}$$
$$U_{33} = 1 - D_{33}$$
$$U_{23} = 2(1 - D_2)(1 - D_3)/(2 - D_2 - D_3)$$
$$U_{31} = 2(1 - D_1)(1 - D_3)/(2 - D_1 - D_3)$$
$$U_{12} = 2(1 - D_2)(1 - D_1)/(2 - D_2 - D_1)$$

which are the diagonal terms of **M(D)** in the principal coordinate system of **D** as shown in Equation (5).

DISCUSSION

The three different forms of **M(D)** have been formulated based on a second order symmetric tensor of **D**. As shown in Equations (3)–(5), the first three diagonal terms are the same for the three cases considered, indicating that the damage effects on the normal stress are equivalent. The other three diagonal terms relating to shear stress are, however, different for the cases.

When the principal coordinate system of damage is coincident with that of stress, then $\sigma_4 = \sigma_5 = \sigma_6 = 0$, Y_1, Y_2, and Y_3 in the principal coordinate system of damage are easily obtained. It is evident that the formulations of Y_1, Y_2, and Y_3 are the same for all three cases. These formulations are also identical in form to that of **Y** derived for the proportional loading case where the principal coordinate system of stress σ and damage coincide. For the case of proportional loading, the principal directions of $\boldsymbol{\sigma}$ do not change, and Y_4, Y_5 and Y_6 are zero. This leads to the elements of $\dot{D}_4$, $\dot{D}_5$, and $\dot{D}_6$ being zero. In this way, the principal directions of damage **D** will not change and always coincide with that of stress during the loading process.

When the principal coordinate system of damage **D** does not coincide with that of stress during the non-proportional loading process, σ_4, σ_5, and σ_6 in the principal coordinate system of damage are non-zero. Y_4, Y_5, and Y_6 have correspondingly real numbers as well as that of the damage rate $\dot{D}_4$, $\dot{D}_5$, and $\dot{D}_6$. This is because the principal direction of damage rotates constantly during the loading process. Consequently, **Y**'s representation in general case should be derived first in order to make possible the non-proportional loading analysis, a loading condition that is often observed in real-life structures under service loading.

CONCLUSIONS

In this paper, the damage strain energy release rates are derived for three forms of **M(D)** based on the second order symmetric tensor of **D**. The relation between **M(D)**, which is a fourth order symmetric tensor, and the second order symmetric damage tensor **D** is elucidated. The generalized tensor representations of **M(D)** in terms of **D** developed in the paper are important in extending the present applicability of the damage mechanics to solve a wide range of engineering problems. This is compared with the matrix representation in the principal coordinate system of damage **D** as shown in Equations (3)–(5) which are only special cases of the generalized representation and cannot be used to derive the generalized formulations of **Y**. The formulations of **Y** obtained are necessary to conduct stress analysis for structures under non-proportional loading which is hitherto lacking, when **Y** is chosen to be the controlling thermodynamics force in the damage evolution equation.

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