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AN INVESTIGATION INTO THE STABILITY OF A STEADY STATE VIBRATORY SYSTEM GOVERNED BY THE NONLINEAR DIFFERENTIAL EQUATION $m\ddot{x} + cx\dot{x} + ax = F_o \cos \nu t$

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PREFACE

In my reading of the literature in the field of nonlinear vibrations a single statement caught my attention, a simple sentence stating that the jump phenomenon may be caused by a nonlinear restoring force or a nonlinear damping force. Though I can no longer place the source of the above statement, a continued search of the literature did not reveal any information concerning the jump phenomenon as caused by nonlinear damping.

I made a preliminary investigation of several equations with nonlinear damping terms and only one,

$$m\ddot{x} + cx\dot{x} + ax = F_o \cos \nu t$$

seemed to have the potential to cause the jump phenomenon. It was from this phenomenon that my interest in the stability of nonlinear vibrations stems.

Of the many sources in the literature concerning the stability of dynamic systems, the works of Dr. Chihiro Hayashi particularly caught my attention. Hayashi's book, Forced Oscillations in Non-Linear Systems (1) was of great service in formulating the method of attack for determining the stability of a system with nonlinear damping.

Ironically, though my interest in stability was stimulated by the curious jump phenomenon, the results of this investigation will demonstrate that the instabilities found are not of the jump type.

I am particularly grateful to the National Science Foundation for its grant to me of a Science Faculty Fellowship. Without the aid of the National Science Foundation, I should never have had the opportunity to complete this investigation.

I also wish to acknowledge the generosity of the Aeronautical Engineering Department of the University of Michigan for permission to use the Reac computer. The freedom of access to the Reac that was given me, has truly made the computer an educational tool.

I am sincerely grateful for the cooperation shown me by each of the members of my doctoral committee. In particular, I wish to thank Professor J. Ormondroyd, Chairman, Professor W. Kaplan and Professor L. Rauch for all the conference time and helpful suggestions given me.

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CHAPTER I

STABILITY THEORY

A. Introduction

This investigation has concerned itself with the determination of the stability of the steady-state condition of a forced nonlinear vibratory system which is governed by the differential equation

$$m\ddot{x} + cx\dot{x} + ax = F_0 \cos \nu t. \qquad (1)$$

The nonlinearity of the system is contained in the damping term. The restoring force is deemed to be linear.

A stable steady-state solution of the above system may be defined as a solution which is periodic. An unstable solution is one that is aperiodic and may fall into two classes:

(1) when the maximum amplitude of the vibration tends to increase with time so that in the limit it becomes infinite;

(2) when the periodic solution leaves one stable solution path for another stable solution path, that is the maximum amplitude will suddenly change its value and the solution is momentarily aperiodic.

In order to facilitate the differentiation of the two types of instabilities, they shall be classed as instabilities of the first kind and of the second kind. The definitions of the two types of instabilities are:

(1) when the maximum amplitude of the vibration becomes excessively large so that the amplitude approaches

infinity as time approaches infinity, then the instability shall be considered to be the first kind,

(2) when the maximum amplitude of the vibration leaves one periodic solution path for another but remains finite after arriving at the new periodic solution path, the instability shall be considered to be of the second kind.

An example of the instability of the first kind occurs in an undamped linear system when the forcing function is operating at resonance. The jump phenomenon[2] of the forced Duffing equation is a prime example of the instability of the second kind.

The approach that was taken in this investigation was to impart a small perturbation to the system and observe mathematically if an instability did occur. Applying a small perturbation results in a new differential equation in the variable of the perturbation. It will be demonstrated that the perturbation equation is a Hill type differential equation. The stability of the Hill equation was then studied with reference to the original dynamic system. The results predicted by such a study were checked by computation. A Reeves Electronic Analogue Computer (REAC) C-101, Model 5 and a Reeves Servo Unit S-101, Model 4, manufactured by the Reeves Instrument Corporation, were used for analog studies of the original system.

B. The Hill Equation as a Criterion of Stability

Equation (1) defines a nonlinear system which is activated by a periodic force defined as $F_o \cos \nu t$. Equation (1) may be nondimensionalized by letting $x = \ell v$, where ℓ has the dimensions of x only and v is a pure number. Let

$$\tau = \nu t,$$

where τ is dimensionless. Then

$$\dot{x} = \nu \ell \frac{dv}{d\tau}$$

and

$$\ddot{x} = \nu^2 \ell \frac{d^2 v}{d\tau^2} .$$

Substituting these new parameters into Equation (1), results in a non-dimensional equation;

$$\frac{d^2 v}{d\tau^2} + kv \frac{dv}{d\tau} + \alpha v = B \cos(\tau - \phi), \qquad (2)$$

where

$$k = \frac{c\ell}{m\nu},$$

$$\alpha = \frac{a}{m\nu},$$

$$B = \frac{F_o}{m\ell\nu^2},$$

and the phase angle ϕ has been added as a matter of convenience. The angle ϕ is measured from the solution vector v.

In order to keep the stability theory, about to be developed, in general form, Equation (2) may be generalized to

$$\frac{d^2 v}{d\tau^2} + kf(v) \frac{dv}{d\tau} + g(v) = B \cos(\tau - \phi). \qquad (3)$$

When Equation (3) is specifically related to Equation (2), then

$$f(v) = v,$$

$$g(v) = \alpha v.$$

The advantage of working with the dimensionless equations, Equations (2) or (3), is that the equations are not frequency sensitive as is

Equation (1). The ease with which B, the amplitude of the forcing function, can be changed on the analog computer plus the lack of frequency sensitivity simplifies the laboratory investigation greatly.

Equation (3) may possess both stable and unstable solutions. According to Trefftz[3] if a system which is represented by the differential equation

$$\frac{d^2x}{dt^2} = F(x,\frac{dx}{dt},t), \tag{4}$$

where

$$F(x,\frac{dx}{dt},t+\omega) = F(x,\frac{dx}{dt},t),$$

has a stable solution, the solution will be periodic of least period ω or an integral multiple (different from unity) thereof.

Since the forcing function of Equation (3) is periodic, the steady-state solution to Equation (3) will also be periodic. Let the periodic steady-state solution of Equation (3) be

$$v(\tau) = \Phi(\tau). \tag{5}$$

To determine whether Equation (3) defines a stable system a small perturbation may be imparted to the system. If the amplitude resulting from this perturbation attenuates with time or at the most remains bounded, then the system may be deemed to be stable. Conversely, if the amplitude of the perturbation does not remain bounded then the system is unstable.

Imparting a small perturbation to the system effects Equation (3) as follows:

$$\frac{d^2(v+\xi)}{d\tau^2} + kf(v+\xi)\,\frac{d(v+\xi)}{d\tau} + g(v+\xi) = B\cos(\tau-\phi), \tag{6}$$

where ξ is a small variation from v. The functions $f(v+\xi)$ and $g(v+\xi)$ may be expanded into Taylor series, respectively:

$$f(v+\xi) = f(v) + \xi \frac{df}{dv} + \frac{1}{2} \xi^2 \frac{d^2f}{dv^2} + \ldots.$$

and

$$g(v+\xi) = g(v) + \xi \frac{dg}{dv} + \frac{1}{2} \xi^2 \frac{d^2g}{dv^2} + \ldots.. \qquad (7)$$

By substituting Equations (7) into Equation (3) and neglecting terms in ξ of the second order, since ξ is small, there results

$$\frac{d^2v}{d\tau^2} + \frac{d^2\xi}{d\tau^2} + k[f(v) + \xi \frac{df}{dv}][\frac{dv}{d\tau} + \frac{d\xi}{d\tau}]$$

$$+ g(v) + \xi \frac{dg}{d\tau} = B \cos(\tau-\emptyset). \qquad (8)$$

By expanding Equation (8) and again noting that terms of the second order in ξ may be neglected and also by virtue of Equation (3), the perturbation equation becomes

$$\frac{d^2\xi}{d\tau^2} + kf(v) \frac{d\xi}{d\tau} + [k \frac{df}{d\tau} + \frac{dg}{dv}]\xi = 0 \qquad (9)$$

or

$$\frac{d^2\xi}{d\tau^2} + kF(\tau) \frac{d\xi}{d\tau} + G(\tau)\xi = 0, \qquad (10)$$

where

$$F(\tau) = f(v)$$

and

$$G(\tau) = k \frac{df}{d\tau} + \frac{dg}{dv} .$$

Since $v(\tau)$ has been defined by Equation (4), Equation (9) or (10) is a linear differential equation. However, Equation (10) may be somewhat simplified by assuming a solution for ξ in the form of

$$\xi = \exp[-\frac{k}{2}\int_0^{\tau} F(\tau)d\tau]\eta, \tag{11}$$

where η is a variable still to be determined.

Differentiating Equation (11) with respect to τ the proper number of times and then substituting Equation (11) and its derivatives into Equation (10), yields a simplified linear equation in terms of η. That is,

$$\frac{d^2\eta}{d\tau^2} + [G(\tau) - \frac{k}{2}\frac{dF}{d\tau} - \frac{k^2}{4}F^2(\tau)]\eta = 0. \tag{12}$$

Since, Equation (12) is linear, the solution for η may be taken in the form of

$$\eta = C_1 e^{\mu\tau}\ \Phi_1(\tau) + C_2 e^{-\mu\tau}\ \Phi_2(\tau), \tag{13}$$

where the exponent μ is called the characteristic exponent and determines the stability of Equation (13). Only when μ is imaginary will η be a stable variable. The task then for the moment is to determine the value of μ. When the functions $G(\tau)$, $F(\tau)$, and $F^2(\tau)$ are each expressed as a Fourier series, there will be a relationship between the Fourier coefficients and the characteristic exponent μ.

The primary interest, of course, is not in the stability of η but of ξ. The characteristic exponent of ξ as given by Equation (11) is

$$\mu' = -\frac{k}{2\tau}\int_0^{\tau} F(\tau)d\tau \pm \mu, \tag{14}$$

where μ' is the characteristic exponent of ξ.

However, since $F(\tau)$ is expressed as a Fourier series, only the zero frequency term of the series need be considered. The condition for stable ξ is given by

$$\mathcal{R}e\left(-\frac{ka_o}{4} \pm \mu\right) < 0, \tag{15}$$

where $\frac{a_o}{2}$ is the zero frequency term of the Fourier series representation of $F(\tau)$.

Conversely, if the

$$\mathcal{R}e\left(-\frac{ka_o}{4} \pm \mu\right) > 0, \tag{16}$$

then ξ and consequently v is unstable.

The Fourier series for $F(\tau)$, $F^2(\tau)$ and $G(\tau)$ are given below so that the coefficient notations will be defined.

$$F(\tau) = \frac{a_o}{2} + \sum_{n=1}^{\infty} [a_n \cos n\tau + b_n \sin n\tau]. \tag{17}$$

$$F^2(\tau) = \frac{a'_o}{2} + \sum_{n=1}^{\infty} [a'_n \cos n\tau + b'_n \sin n\tau]. \tag{18}$$

$G(\tau)$ may be expressed by the combination of two series, since

$$G(\tau) = k\frac{df}{d\tau} + \frac{dg}{dv} .$$

Now

$$k\frac{df}{d\tau} = k\frac{dF}{d\tau} = k \sum_{n=1}^{\infty} [n(-a_n \sin n\tau + b_n \cos n\tau)].$$

In general, $g(v)$ may take the form of

$$g(v) = \sum_{\nu=1}^{\infty} \alpha_\nu v^\nu,$$

then

$$\frac{dg}{dv} = \alpha_1 + \sum_{\nu=2}^{\infty} \nu\, \alpha_\nu\, v^{\nu-1}.$$

Consequently, $\frac{dg}{dv}$ may be expressed as a Fourier series in the form of

$$\frac{dg}{dv} = \alpha_1 + \frac{c_o}{2} + \sum_{n=1}^{\infty} [c_n \cos n\tau + d_n \sin n\tau];$$

then

$$G(\tau) = \alpha_1 + \frac{c_o}{2} + \sum_{n=1}^{\infty} [c_n \cos n\tau + d_n \sin n\tau]$$

$$+ k \sum_{n=1}^{\infty} [n(- a_n \sin n\tau + b_n \cos n\tau)]. \qquad (19)$$

For the problem at hand

$$c_o = c_n = d_n = 0, \text{ for all } n,$$

and

$$\alpha_1 = \alpha.$$

Then

$$G(\tau) = \alpha + k \sum_{n=1}^{\infty} [n(- a_n \sin n\tau + b_n \cos n\tau)]. \qquad (20)$$

Substituting the proper series into Equation (12), there results

$$\frac{d^2\eta}{d\tau^2} + [\alpha - \frac{k^2 a'_o}{8} + \sum_{n=1}^{\infty} \left\{(- \frac{kna_n}{2} - \frac{k^2 b'_n}{4}) \sin n\tau\right\}$$

$$+ \sum_{n=1}^{\infty} \left\{(\frac{knb_n}{2} - \frac{k^2 a'_n}{4}) \cos n\tau\right\}]\eta = 0. \qquad (21)$$

Now the extended form of the Hill equation may be written as

$$\frac{d^2\eta}{d\tau^2} + [\Theta_o + 2 \sum_{\nu=1}^{\infty} \left\{\Theta_{ns} \sin 2\nu\tau\right\}$$

$$+ 2 \sum_{\nu=1}^{\infty}\left\{\Theta_{nc} \cos 2\nu\tau\right\}]\eta = 0. \qquad (22)$$

It is a small matter to make the form of Equation (21) conform to that of the Hill equation. Rewriting Equation (21) as

$$\frac{d^2\eta}{d\tau^2} + [\alpha - \frac{k^2a'_o}{8} + 2 \sum_{n=1}^{\infty} \left\{(- \frac{kna_n}{4} - \frac{k^2b'_n}{8})\sin \frac{2n\tau}{2}\right\}$$

$$+ 2 \sum_{n=1}^{\infty}\left\{(\frac{knb_n}{4} - \frac{k^2a'_n}{8})\cos \frac{2n\tau}{2}\right\}]\eta = 0. \qquad (23)$$

Equation (23) is identical in form with Equation (22). The corresponding Hill coefficients are

$$\Theta_o = \alpha - \frac{k^2a'_o}{8},$$

$$\Theta_{ns} = - \frac{kna_n}{4} - \frac{k^2b'_n}{8},$$

$$\Theta_{nc} = \frac{knb_n}{4} - \frac{k^2a'_n}{8},$$

$$n = 1,2,3,\ldots. \qquad (24)$$

Having established that η follows the solution of the Hill equation, the solutions of Whittaker(4) or Ince(5) may be used to establish the value of the characteristic exponent μ. Mention must also be made of the fact that Hayashi(1) includes a discussion of the Hill equation in the appendix to his book. The solutions of Whittaker or Ince, however, involve the use of an infinite matrix and if the Θ's are not known explicitly the task is formidable. An approximate solution to the problem may be made in the manner of Hayashi.[1]

Since η has been established as

$$\eta = C_1 e^{\mu\tau}\Phi_1(\tau) + C_2 e^{-\mu\tau}\Phi_2(\tau), \tag{13}$$

then as a first order approximation a particular solution of the form of

$$\eta = e^{\mu\tau}[C_n \sin \frac{n\tau}{2} + D_n \cos \frac{n\tau}{2}] \tag{25}$$

may be assumed.

Substituting Equation (25) and the necessary derivatives with respect to τ into the differential Equation (23), upon expansion of the terms, there results

$$\begin{aligned}
&[\mu^2 - \frac{n^2}{4} + \Theta_o][C_n \sin \frac{n\tau}{2} + D_n \cos \frac{n\tau}{2}] \\
&+ \sum_{\nu=1}^{\infty} \Theta_{\nu s} C_n \left\{\cos(\frac{n-2\nu}{2})\tau - \cos(\frac{n+2\nu}{2})\tau\right\} \\
&+ \sum_{\nu=1}^{\infty} \Theta_{\nu c} C_n \left\{\sin(\frac{n-2\nu}{2})\tau + \sin(\frac{n+2\nu}{2})\tau\right\} \\
&+ \sum_{\nu=1}^{\infty} \Theta_{\nu s} D_n \left\{\sin(\frac{2\nu-n}{2})\tau + \sin(\frac{2\nu+n}{2})\tau\right\} \\
&+ \sum_{\nu=1}^{\infty} \Theta_{\nu c} D_n \left\{\cos(\frac{2\nu-n}{2})\tau + \cos(\frac{2\nu+n}{2})\tau\right\} \\
&+ n\mu[C_n \cos \frac{n\tau}{2} - D_n \sin \frac{n\tau}{2}] = 0.
\end{aligned} \tag{26}$$

Now the coefficients of $\sin \frac{n\tau}{2}$ and $\cos \frac{n\tau}{2}$ must respectively be equal to zero. Moreover, these coefficients must conform to the above conditions for each n, n=1,2,3,.... Then setting ν=n (note: setting 2ν=n results in $C_{2\nu=n} = D_{2\nu=n} = 0$) yields two equations:

$$[\mu^2 - \frac{n^2}{4} + \theta_o - \theta_{nc}]C_n + [\theta_{ns} - n\mu]D_n = 0,$$

$$[\theta_{ns} + n\mu]C_n + [\mu^2 - \frac{n^2}{4} + \theta_o + \theta_{nc}]D_n = 0. \tag{27}$$

The value of μ can now be determined by setting the determinant of Equations (27) identically equal to zero.

$$\begin{vmatrix} [\mu^2 - \frac{n^2}{4} + \theta_o - \theta_{nc}] & [\theta_{ns} - n\mu] \\ [\theta_{ns} + n\mu] & [\mu^2 - \frac{n^2}{4} + \theta_o + \theta_{nc}] \end{vmatrix} \equiv 0. \tag{28}$$

Whence

$$\mu^4 + 2(\theta_o + \frac{n^2}{4})\mu^2 + (\theta_o - \frac{n^2}{4})^2 - \theta_n^{\ 2} = 0, \tag{29}$$

where

$$\theta_n^{\ 2} = \theta_{ns}^{\ 2} + \theta_{nc}^{\ 2}.$$

Solving for μ^2,

$$\mu^2 = -(\theta_o + \frac{n^2}{4}) \pm \sqrt{\theta_o n^2 + \theta_n^{\ 2}}\ . \tag{30}$$

For the unstable condition of ξ

$$\mathcal{R}e(-\frac{ka_o}{4} \pm \mu) > 0 \tag{16}$$

or

$$\mathcal{R}e(\frac{ka_o}{4} \pm \mu) < 0. \tag{31}$$

It follows then that for unstable ξ

$$\frac{k^2 a_o^{\ 2}}{16} - \mu^2 < 0. \tag{32}$$

Substituting Equation (30) for μ^2 into Equation (32);

$$\frac{k^2 a_o^2}{16} + (\Theta_o + \frac{n^2}{4}) \pm \sqrt{\Theta_o n^2 + \Theta_n^2} < 0 \tag{33}$$

or

$$(\Theta_o - \frac{n^2}{4})^2 + (\Theta_o + \frac{n^2}{4}) \frac{k^2 a_o^2}{8} + \frac{k^4 a_o^4}{256} - \Theta_n^2 < 0. \tag{34}$$

If $(-\tau)$ is substituted for τ in Equation (25), a second particular solution is formed. Carrying out the solution for μ^2 for the second particular solution yields an identical value for μ^2 as given by Equation (30). Therefore, either Equation (33) or (34) may be used as the unstable criterion for v. It must be remembered that these equations represent a first order approximation.

If a_o is equal to zero, as it will be when approximate solutions for v are developed, then Equation (34) simplifies to

$$|\Theta_o - \frac{n^2}{4}| - |\Theta_n^2| < 0, \tag{35}$$

valid as the stability criterion for v when a_o=0.

Throughout this investigation Equation (35) was used as the criterion for the instability of Equation (2).

CHAPTER II

ASSUMED PERIODIC SOLUTIONS

The periodic solution of $v(\tau)$ is given by Equation (17), since $v(\tau) = F(\tau)$. However, a full Fourier series, though exact, is too cumbersome to work with. The task then is to see how rapidly the series converges. This was done by assuming several solutions in Fourier series form.

A. One Term Approximation

Let

$$v = R_1 \cos \tau \ . \tag{36}$$

By substituting v and its derivatives with respect to τ into Equation (2), the amplitude R_1 was found to be

$$R_1 = \frac{B}{\alpha - 1} \ . \tag{37}$$

Equation (37) quite obviously is not satisfactory since it is identical with the undamped linear system; that is, when $k = o$, Equation (2) is reduced to a second order linear equation with no damping. A solution which provides for no nonlinearity is certainly not admissable. It is interesting to note, however, that when instability is predicted, based upon Equation (36), instability does exist. The nature of the variational equation is such that it does contain the damping constant k and the k does reflect the unstable solutions even though Equation (37) is linear. It was found however, that the predictions were less reliable than those predictions which were based upon other assumed solutions.

B. Two Term Approximation

Let

$$v = R_1 \cos \tau + R_2 \cos (2\tau - \beta) \tag{38}$$

where β is a phase angle measured from the R_1 vector. Substituting Equation (38) and its derivatives with respect to τ into Equation (2) and expanding the trigonometric terms, yields:

$$(\alpha-1)R_1 \cos \tau + (\alpha-4)R_2 \cos \beta \cos 2\tau + (\alpha-4)R_2 \sin \beta \sin 2\tau$$

$$+ k[- \frac{R_1^2}{2} \sin 2\tau - \frac{R_1R_2}{2} \cos \beta \{\sin 3\tau - \sin \tau\}$$

$$- \frac{R_1R_2}{2} \sin \beta \{\cos \tau - \cos 3\tau\} - R_1R_2 \cos \beta \{\sin \tau + \sin 3\tau\}$$

$$- R_2^2 \cos^2 \beta \sin 4\tau + 2R_2^2 \cos \beta \sin \beta \cos 4\tau$$

$$+ R_1R_2 \sin \beta \{\cos \tau + \cos 3\tau\} + R_2^2 \sin^2 \beta \sin 4\tau]$$

$$= B \cos \phi \cos \tau + B \sin \phi \sin \tau \; . \tag{39}$$

Setting the coefficients of the $\sin \tau$, $\cos \tau$, $\sin 2\tau$, and $\cos 2\tau$, of the L.H.S. of Equation (39) respectively equal to the corresponding coefficients of the R.H.S. results in four equations.

$$\sin \tau: \quad -\frac{1}{2} kR_1R_2 \cos \beta = B \sin \phi \; .$$

$$\cos \tau: \quad (\alpha-1)R_1 + \frac{1}{2} kR_1R_2 \sin \beta = B \cos \phi \; .$$

$$\sin 2\tau: \quad (\alpha-4)R_2 \sin \beta - \frac{1}{2} kR_1^2 = 0 \; .$$

$$\cos 2\tau: \quad (\alpha-4)R_2 \cos \beta = 0 \; . \tag{40}$$

It is obvious that

$$\cos\beta = 0, \quad \beta = \frac{\pi}{2} .$$

It follows, then, that

$$\sin\phi = 0, \quad \phi = 0 .$$

Subsequently

$$R_2 = \frac{kR_1^2}{2(\alpha-4)} \tag{41}$$

and

$$\frac{k^3R_1^3}{4(\alpha-4)} + (\alpha-1)kR_1 = kB . \tag{42}$$

Equation (42) is the amplitude response equation in terms of R_1 .

$$v = R_1 \cos\tau + \frac{kR_1^2}{2(\alpha-4)} \sin 2\tau \tag{43}$$

and

$$a_o = 0$$

$$a_1 = R_1 \qquad b_1 = 0$$

$$a_2 = 0 \qquad b_2 = \frac{kR_1^2}{2(\alpha-4)}$$

$$a_n = b_n = 0 \quad \text{for} \quad n = 3, 4, 5, \ldots\ldots .$$

$$F^2(\tau) = \frac{R_1^2}{2} + \frac{k^2R_1^4}{8(\alpha-4)^2} + \frac{R_1^2}{2} \cos 2\tau - \frac{k^2R_1^4}{8(\alpha-4)^2} \cos 4\tau$$

$$+ \frac{kR_1^3}{2(\alpha-4)} \sin\tau + \frac{kR_1^3}{2(\alpha-4)} \sin 3\tau . \tag{44}$$

$$a'_o = R_1^2 + \frac{k^2R_1^2}{4(\alpha-4)^2} ,$$

$$a'_1 = 0 , \qquad b'_1 = \frac{kR_1^3}{2(\alpha-4)} ,$$

$$a'_2 = \frac{R_1^2}{2} , \qquad b'_2 = 0 ,$$

$$a'_3 = 0 , \qquad b'_3 = \frac{kR_1^3}{2(\alpha-4)} ,$$

$$a'_4 = -\frac{k^2R_1^4}{8(\alpha-4)^2} , \qquad b'_4 = 0 ,$$

$$a'_n = b'_n = 0 , \quad \text{for} \quad n = 5, 6, 7, \ldots \quad .$$

The Hill coefficients can now be calculated from Equations (24).

$$\Theta_o = \alpha - \frac{k^2R_1^2}{8} \quad \frac{k^4R_1^4}{32(\alpha-4)^2} ,$$

$$\Theta_{1s} = -\frac{k^3R_1^3}{16(\alpha-4)} ,$$

$$\Theta_{1c} = 0 ,$$

$$\Theta_{2s} = 0 ,$$

$$\Theta_{2c} = \frac{k^2R_1^2}{4(\alpha-4)} - \frac{k^2R_1^2}{16} ,$$

$$\Theta_{3s} = -\frac{k^3R_1^3}{16(\alpha-4)} ,$$

$$\Theta_{3c} = 0 ,$$

$$\Theta_{4s} = 0 ,$$

$$\Theta_{4c} = -\frac{k^4R_1^4}{64(\alpha-4)^2} . \qquad (45)$$

C. Improved Two Term Approximation

Consider for the moment a three term approximation:

$$v = R_1 \cos \tau + R_2 \cos (2\tau - \beta) + R_3 \cos (3\tau - \gamma) , \qquad (46)$$

where β and γ are phase angles measured from the R_1 vector.

Proceeding in the manner outlined in the two term approximation, it is found that

$$\sin \gamma = 0 \quad \text{and} \quad \gamma = 0 ,$$

$$\cos \beta = 0 \quad \text{and} \quad \beta = \frac{\pi}{2} ,$$

$$\sin \phi = 0 \quad \text{and} \quad \phi = 0 .$$

As a result of the above

$$R_2 = \frac{(\alpha-9)kR_1^2}{2(\alpha-4)(\alpha-9) + 3k^2R_1^2} , \qquad (47)$$

$$R_3 = - \frac{3k^2R_1^3}{2[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]} . \qquad (48)$$

The amplitude response relationship is

$$(\alpha-1)kR_1 + \frac{k^2}{2} R_1R_2 - \frac{k^2}{2} R_2R_3 = kB . \qquad (49)$$

Now if the assumption is made that R_3 is small compared to R_2 when α is not in the neighborhood of $\alpha = 9$, then R_3 may be assumed to be negligible. The amplitude response Equation (49) now becomes, upon substituting for R_2 the relationship given by Equation (47),

$$\frac{(\alpha-9)k^3R_1^3}{2[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]} + (\alpha-1)kR_1 = kB . \qquad (50)$$

Then

$$v(\tau) = R_1 \cos \tau + \frac{(\alpha-9)kR_1^2}{2(\alpha-4)(\alpha-9) + 3k^2R_1^2} \sin 2\tau \ . \tag{51}$$

The Hill coefficients are

$$\Theta_o = \alpha - \frac{k^2R_1^2}{8} \quad \frac{(\alpha-9)^2k^4R_1^4}{8[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]^2} \ ,$$

$$\Theta_{1s} = -\frac{(\alpha-9)k^3R_1^3}{8[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]} - \frac{kR_1}{4} \ ,$$

$$\Theta_{1c} = 0 \ ,$$

$$\Theta_{2s} = 0 \ ,$$

$$\Theta_{2c} = \frac{(\alpha-9)k^2R_1^2}{2[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]} - \frac{k^2R_1^2}{16} \ ,$$

$$\Theta_{3s} = -\frac{(\alpha-9)k^3R_1^3}{8[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]} \ ,$$

$$\Theta_{3c} = 0 \ ,$$

$$\Theta_{4s} = 0 \ ,$$

$$\Theta_{4c} = -\frac{(\alpha-9)^2k^4R_1^4}{16[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]^2} \ . \tag{52}$$

D. Three Term Approximation

It is only a small step from the improved two term approximation to the three term approximation. If R_3 is not presumed to be negligible, then the amplitude response equation develops directly from Equation (49), when R_2 and R_3, as defined by Equation (47) and (48) respectively, are substituted. The response equation for the three term approximation is

$$\frac{3(\alpha-9)k^5R_1^5}{4[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]^2} + \frac{(\alpha-9)k^3R_1^3}{2[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]}$$

$$+ (\alpha-1)kR_1 = kB \; . \tag{53}$$

The Hill coefficients are

$$\Theta_o = \alpha - \frac{k^2R_1^2}{8} - \frac{4(\alpha-9)^2k^4R_1^4 + 9k^6R_1^6}{32[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]^2} ,$$

$$\Theta_{1s} = -\frac{kR_1}{4} - \frac{(\alpha-9)k^3R_1^3}{8[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]}$$

$$- \frac{3(\alpha-9)k^5R_1^5}{16[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]^2} ,$$

$$\Theta_{1c} = 0 ,$$

$$\Theta_{2s} = 0 ,$$

$$\Theta_{2c} = \frac{8(\alpha-9)k^2R_1^2 + 3k^4R_1^4}{16[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]} - \frac{k^2R_1^2}{16} ,$$

$$\Theta_{3s} = \frac{(18-\alpha)k^3R_1^3}{8[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]} ,$$

$$\Theta_{3c} = 0 ,$$

$$\Theta_{4s} = 0 ,$$

$$\Theta_{4c} = \frac{3k^4R_1^4}{16[2(\alpha-4)(\alpha-9) + 3k^2R_1^2]} + \frac{(\alpha-9)^2k^4R_1^4}{16[2(\alpha-4)(\alpha-9) + 3kR_1^2]^2} ,$$

$$\Theta_{5s} = \frac{3(\alpha-9)k^5R_1{}^5}{16[2(\alpha-4)(\alpha-9) + 3k^2R_1{}^2]^2} ,$$

$$\Theta_{5c} = 0 ,$$

$$\Theta_{6s} = 0 ,$$

$$\Theta_{6c} = -\frac{9k^6R_1{}^6}{64[2(\alpha-4)(\alpha-9) + 3k^2R_1{}^2]^2} . \tag{54}$$

E. Subharmonic Assumption

It is necessary, for a full discussion, to develop an assumed solution with a subharmonic term of the one half order, that is

$$v = R_{1/2} \cos (\frac{\tau}{2} - \delta) + R_1 \cos \tau , \tag{55}$$

where δ is a phase angle measured from the R_1 vector. If a substitution of

$$\tau' = \frac{\tau}{2} \tag{56}$$

is made, then

$$v = R_{1/2} \cos (\tau' - \delta) + R_1 \cos 2\tau' . \tag{57}$$

Such a substitution makes it possible to use the same stability criterion as defined in Chapter I without changing the n values. Substituting Equation (57) into Equation (2) and equating coefficients as before leads to four equations.

$$\sin \tau': \quad (\alpha-1)R_{1/2} \sin \delta - \frac{1}{2}kR_{1/2}R_1 \cos \delta = 0 . \tag{a}$$

$$\cos \tau': \quad (\alpha-1)R_{1/2} \cos \delta - \frac{1}{2}kR_{1/2}R_1 \sin \delta = 0 . \tag{b}$$

$$\sin 2\tau': \quad -\frac{1}{2} R_{1/2}{}^2 \cos^2 \delta + \frac{1}{2} R_{1/2}{}^2 \sin^2 \delta = B \sin \phi \;. \qquad \text{(c)}$$

$$\cos 2\tau': \quad (\alpha-4)R_1 + kR_{1/2}{}^2 \sin \delta \cos \delta = B \cos \phi \;. \qquad \text{(d)}$$

From Equation (a) and (b)

$$\tan \delta = \frac{kR_1}{2(\alpha-1)} = \frac{2(\alpha-1)}{kR_1} = 1$$

Whence $\beta = \frac{\pi}{4}$ and $kR_1 = 2(\alpha-1)$.

From Equation (c); $\sin \phi = 0$, and $\phi = 0$.

The amplitude response relationship is found from Equation (d).

$$(\alpha-4)kR_1 + \frac{k^2}{2} R_{1/2}{}^2 = kB \;. \qquad (58)$$

Since $kR_1 = 2(\alpha-1)$,

then the amplitude response equation becomes

$$\frac{k}{2} R_{1/2}{}^2 + 2(\alpha-1)(\alpha-4) - kB = 0 \qquad (59)$$

and

$$v = \frac{\sqrt{2}}{2} R_{1/2} \cos \tau' + \frac{\sqrt{2}}{2} R_{1/2} \sin \tau' + \frac{2(\alpha-1)}{k} \cos 2\tau' \;. \qquad (60)$$

The Hill coefficients are

$$\Theta_o = \alpha - \frac{k^2 R_{1/2}^2}{8} - \frac{(\alpha-1)^2}{2} ,$$

$$\Theta_{1s} = - \frac{\sqrt{2}\ (2-\alpha) k R_{1/2}}{8} ,$$

$$\Theta_{1c} = \frac{\sqrt{2}\ (2-\alpha) k R_{1/2}}{8} ,$$

$$\Theta_{2s} = - (\alpha-1) - \frac{k^2 R_{1/2}^2}{16} ,$$

$$\Theta_{2c} = 0 ,$$

$$\Theta_{3s} = - \frac{\sqrt{2}\ (\alpha-1) k R_{1/2}}{8} ,$$

$$\Theta_{3c} = - \frac{\sqrt{2}\ (\alpha-1) k R_{1/2}}{8} ,$$

$$\Theta_{4s} = 0 ,$$

$$\Theta_{4c} = \frac{(\alpha_1-1)^2}{4} . \tag{61}$$

F. Similar System with the Same Stability Criterion

Consider the differential equation

$$\frac{d^2w}{d\tau^2} + k\left(\frac{dw}{d\tau}\right)^2 + \alpha w = B \sin \tau . \tag{62}$$

Let

$$v = \frac{dw}{d\tau} .$$

Then

$$\frac{dv}{d\tau} = \frac{d^2w}{d\tau^2}$$

and

$$w = \int v d\tau + C \ .$$

Substituting Equation (63) into Equation (62) yields

$$\frac{dv}{d\tau} + \frac{kv^2}{2} + \alpha[\int v d\tau + C] = B \sin \tau \ . \tag{63}$$

Differentiating Equation (63) with respect to τ results in

$$\frac{d^2v}{d\tau^2} + kv\frac{dv}{d\tau} + \alpha v = B \cos \tau \ . \tag{64}$$

Equation (64) is identical with Equation (2). Since the integration of v will not change the value of the characteristic exponent μ, the stability criterion for Equation (2) is also valid for Equation (62).

CHAPTER III

THEORY PREDICTIONS AND TEST RESULTS

For convenience in discussion the theory predictions and comparison of these predictions with the Reac test results, the range of α is separated into three zones:

Zone 1, $1 < \alpha \leq 4$;

Zone 2, $0 < \alpha \leq 1$;

Zone 3, $\alpha > 4$.

Recalling that α was defined by Equation (2) to be

$$\alpha = \frac{a}{m\nu^2} ,$$

then for $\alpha = 1$ the system is at resonance with respect to Equation (1) if the linear vibration theory definition of resonance is applied. For $\alpha < 1$, the system is above resonance and for $\alpha > 1$ the system is below resonance. Therefore zones 1 and 3 are subresonant regions and zone 2 is at or above resonance. Zone 1 may be considered to be of moderate subresonant frequencies. Zone 3 is well below resonance and covers the range of low frequency input. Zone 2 covers a wide range of high frequencies from resonance to infinite frequency in the limit. The use of the linear system definition of resonance seems to be justified since tests on the Reac revealed that the natural frequency of the autonomous system was very nearly equal to $\sqrt{\alpha}$.

A. Zone 1, $1 < \alpha \leq 4$

At the outset of this discussion it is best to point out that during all of the tests on the analogue computer only instabilities of the first kind were found. This phenomenon will be discussed in greater detail further on in this section. As a result of the observation of only instabilities of the first kind, the first points of instability were calculated for the various approximate solutions of $v(\tau)$ reported in Chapter II.

Since a_o is always zero in the approximate solutions, the criterion for instability is given by Equation (35). The question which immediately arises is which value of n is applicable for the first point of instability. The choice of n may be set forth as a rule:

> the applicable value of n for the first point of instability is found when $(\alpha - \frac{n^2}{4})$ has its smallest positive value.

The value

$$\alpha - \frac{n^2}{4} = 0$$

is ruled out as a first point of instability since only at $kR_1 = 0$ will the Hill criterion be on the border line of stability. The border line of stability is represented by the equation

$$\left|\Theta_o - \frac{n^2}{4}\right| - \left|\Theta_n\right| = 0. \qquad (65)$$

To determine the first point of instability, Equation (65) was satisfied by selecting a proper kR_1 by the trial and error method. After Equation (61) was solved, kB was determined from the amplitude response equation.

Table I is a zone 1 table of values for the first points of kR_1 and kB for the various approximate solutions. The right hand column reports the values of kB as determined by the tests on the analog computer.

TABLE I

COMPARISON OF FIRST POINTS OF INSTABILITY, ZONE 1

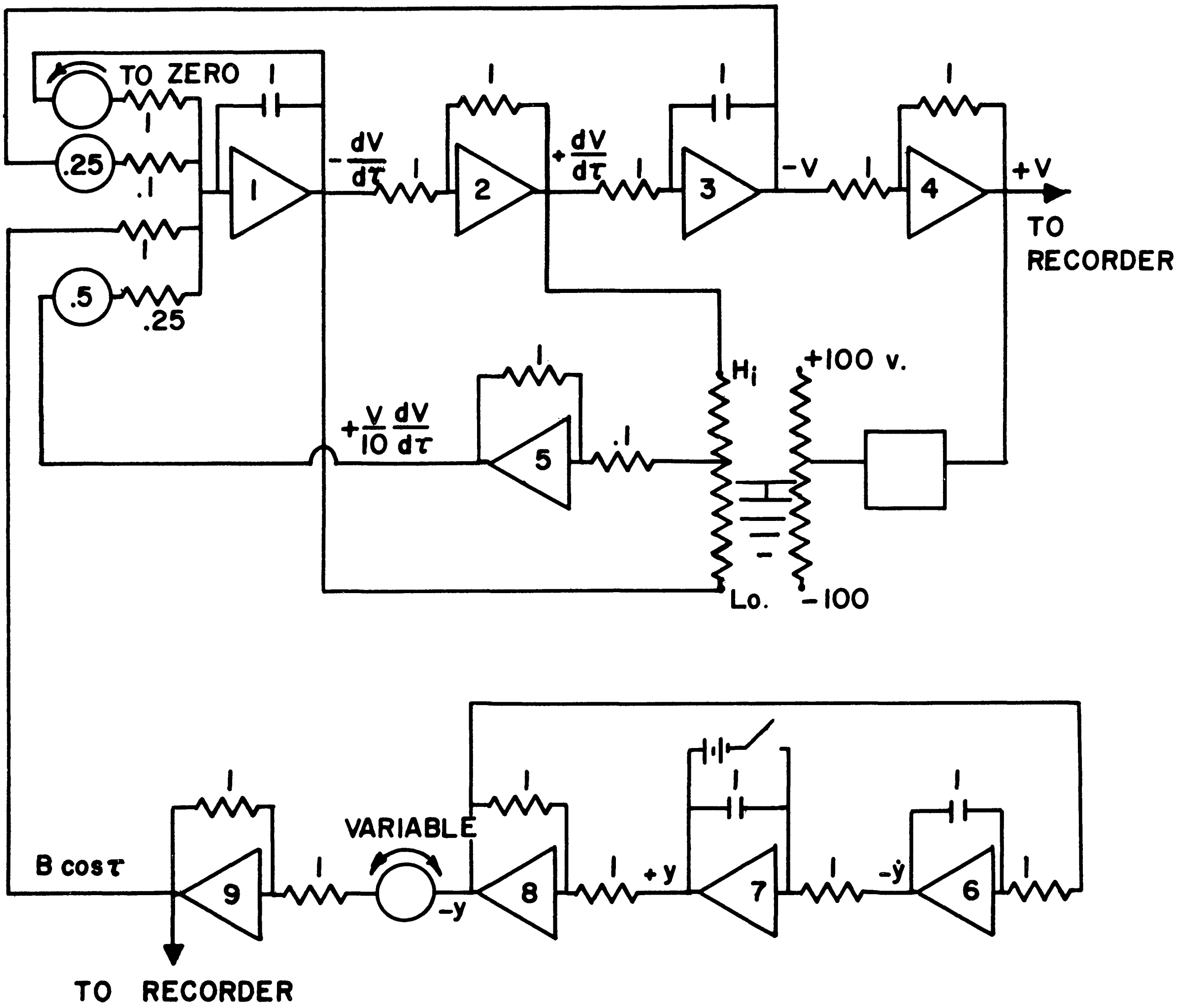
		Two Terms		Improved Two Terms		Three Terms		Analog Computer
α	n	kR_1	kB	kR_1	kB	kR_1	kB	kB
1.5	2	1.30	0.42	1.33	0.46	1.35	0.45	0.36
2.0	2	1.72	1.08	1.84	1.27	1.90	1.20	1.06
2.5	3	1.18	1.50	1.15	1.55	1.05	1.40	1.35
3.0	3	1.17	1.94	1.85	2.85	1.55	2.41	2.30
3.5	3	1.00	2.00	2.26	4.13	1.75	3.06	2.96
4.0	3	***	***	2.62	5.68	1.66	2.91	2.74

*** Theory fails because of pole at $\alpha = 4$.

Before discussing the results of Table I, the technique and circuits which were used on the Reac need to be presented. An investigation of the energy dissipated over a cycle reveals that the net energy loss is zero. This is due to the even order of the damping term or, in relation to Equation (3), to the odd order of f(v). Therefore, the transient will persist over an extremely long period of time unless additional damping is introduced into the analog computer. The technique used on the Reac was to introduce a linear damping term of $b\frac{dv}{d\tau}$ into the system. A value of b=0.10 was arbitrarily chosen. After the transient was damped out the value of b was slowly reduced to zero. With every increment of B the process of damping out the transient was repeated. At the point of instability, even though the transient was effectively damped out, there was a considerable time lapse before the system became unstable. Therefore, it is thought likely that some transient was present at the time the instability occurred. The presence of some transient may account in part for the fact that test instabilities were lower than those predicted by theory.

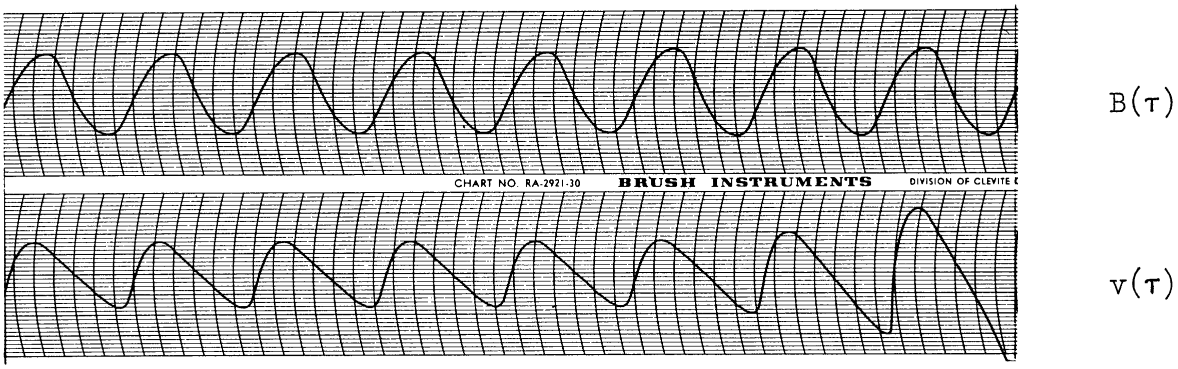
The schematic diagram for the Reac circuit is given in Figure 1. The diagram shows that B cos τ was generated by the computer itself, rather than using an outside source. This preference lies in two aspects; (1) the D.C. bias which was evident at low amplitude output of the sine generator, (2) the phase relationship of the driving function to the system as given by Equation (2). By using a forcing function generated by the computer, the amplitude phase could be guaranteed. Since Equation (2) is not frequency sensitive, all tests were made with the forcing frequency equal to one radian per second. This simplification required only that B, the amplitude of the forcing function, be varied to determine the point of instability.

A comparison of the results as reported in Table I, indicates that for $\alpha = 1.50$ and $\alpha = 2.00$, the two term approximation appears to predict closer results than do the other approximations. A study of the tape from a Brush recorder shows that for the low range of α in zone 1 the fundamental harmonic dominates. Figure 2a is a photograph of a Brush recorder tape for $\alpha = 2.00$, k = 0.20. It is quite evident from this tape that the fundamental is the strongest of the harmonics. There are, however, overtones of the higher harmonics, as evidenced by

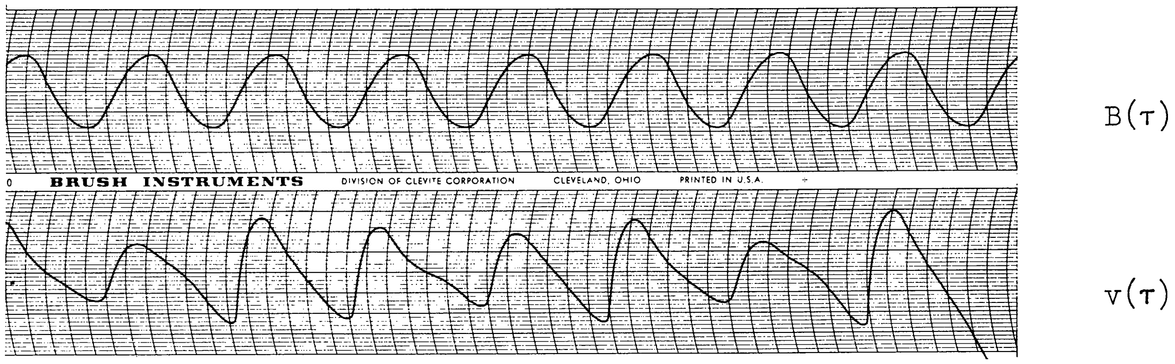


$$\frac{d^2v}{d\tau^2} + 0.20\ v\ \frac{dv}{d\tau} + 2.50v = B\cos(\tau - \phi)$$

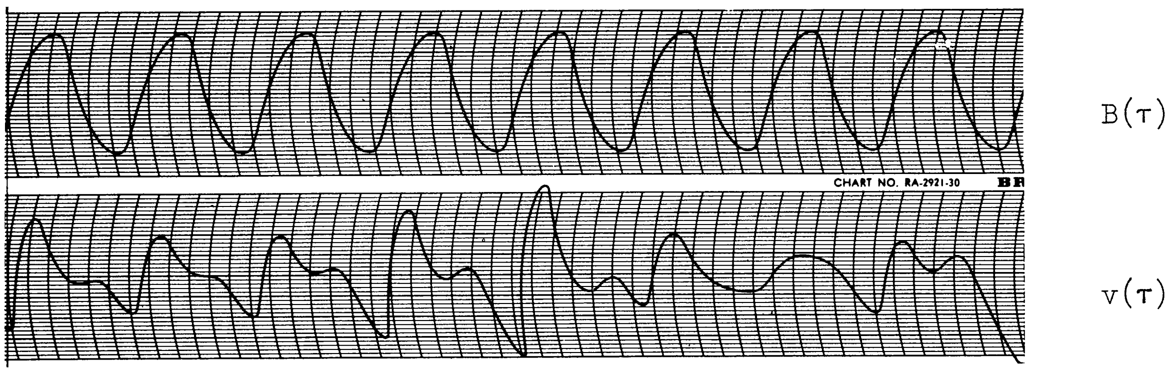
Figure 1. Reac Circuit Diagram



a. $\alpha=2.0$



b. $\alpha=2.5$



c. $\alpha=4.0$

Figure 2. Brush Recorder Tapes, Zone 1

the distortion of the fundamental. For the balance of zone 1 when $2.5 \leq \alpha \leq 4$, the three term approximation predicts the instability more closely than does the other two approximations. Though the improved two term approximation predicts the instabilities fairly closely, at no point in the entire zone 1 does it predict the instability closer than does the three term approximation. Though the two term approximation is better than the improved two term approximation for $\alpha = 2.5$, the two term approximation is not as good for $\alpha = 3.0$ and $\alpha = 3.5$. At $\alpha = 4$ the two term approximation breaks down completely in regard to making a prediction since Θ_0 and Θ_3 have infinite values.

All of the above seem to indicate that for $\alpha = 2.25$ the influence of the second harmonic and probably the third comes more strongly into being. Figure 2b is a tape for $\alpha = 2.5$. The influence of the second harmonic, though not marked, can be seen. Figure 2c is a tape recording for $\alpha = 4$. The strong influence of the second harmonic is now in evidence.

The conclusion that this investigator has drawn from the above evidence is that the three term approximation is more representative of the entire zone 1 and should be used as the assumed solution for $v(\tau)$. The temptation to break zone 1 into two regions is great; one region $1.00 < \alpha \leq 2.25$, the other $2.25 < \alpha \leq 4.00$. However, if R_3 is negligible, another conclusion may be drawn that the assumed solution should have a fourth term for the entire zone. The logic of this statement rests in the fact that the fourth harmonic combines with the second harmonic to form a component of the second harmonic.

When the assumed solution for $v(\tau)$ was taken to be a four term approximation and when the like coefficients were equated, the resulting simultaneous algebraic equations were found to be so nonlinear that it was impossible to find an explicit relationship between the amplitude of each of the higher harmonics and the fundamental. Such explicit relationships probably can be determined by using a digital computer.

In correlating the first points of instability as given by the three term approximation with the Reac test results, one must bear in mind that two previous assumptions have been made in arriving at the predicted points of instability. Also, it is the investigator's understanding that the computer will develop an instability slightly before the mathematical prediction for any system. Coupling the afore-mentioned with the possibility of the presence of some transient, the investigator feels that for zone 1 the three term approximate solution for $v(\tau)$ yields good predictable results for the instabilities in the zone.

As was reported earlier the instabilities found were always of the first kind. This was a surprising condition. Figure 3 represents the amplitude response relationship when $\alpha = 2.5$, which is typical for zone 1. All three amplitudes are plotted against B, based upon the three term approximation. At kB = 2.08, the slope of each of the amplitudes is infinite. One would therefore anticipate that an instability would occur at the point of infinite slope. Moreover, one would anticipate an instability of the second kind. This anticipation was not borne out in the tests. The point of instability on the Reac

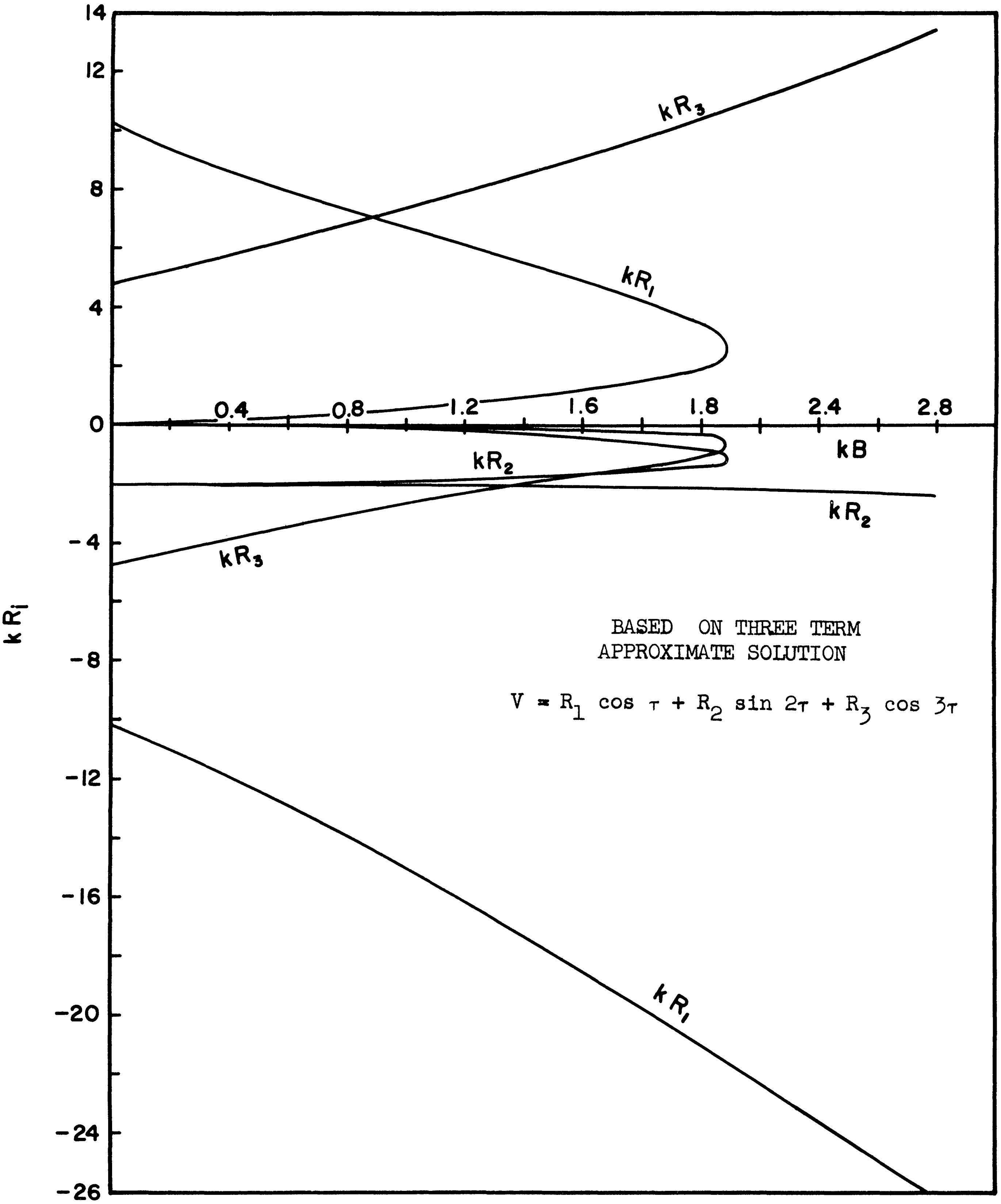


Figure 3. Amplitude Response Relationship, α = 2.5.

occurred at $kB = 1.35$. This instability agrees quite closely with the prediction of $n = 3$ for the three term approximation. When the point of instability for $n = 2$ was calculated it was found to be $kR_1 = 2.31$ and $kB = 2.08$. This new point of instability coincides with the point of infinite slope.

Now the mechanism of instability which develops at a point of infinite slope is quite understandable. As kB increases the amplitude must jump to another solution path. The characteristic exponent μ' assumes a real value momentarily and reverts to an imaginary value during the jump process. It is not locial to apply the above mechnaism to the instability as predicted when $n = 3$. Even though there is a second stable path available, the reason for leaving the original stable path is not clear. An examination of the assumed solution for the Hill equation shows that the build up of the perturbation for $n = 3$ is due to a three halves order harmonic. This fact also sheds little light on the cause of the instability occurring when it does.

In order to be certain that another stable solution path did not exist, a new test on the Reac was devised. By replacing τ by $(-\tau)$ in Equation (2) the stable solution should become unstable and the unstable solution which consists of that part of the curves in Figure 3 which have a negative slope, should become stable. The new equation when τ is replaced by $(-\tau)$ is

$$\frac{d^2v}{d\tau^2} - kv\frac{dv}{d\tau} + \alpha v = B \cos(\tau + \phi). \qquad (66)$$

The new circuit for the computer was readily converted from the circuit in Figure 1. All that needed to be done was to interchange the Hi and

Lo inputs to the potentiometer of the servo unit. The unstable system was tested and the instability points were found to be identical with the original stable system. As a further check, a stable test was made at a low kB value. With the settings of the stable circuit unchanged, the interchange of the Hi and Lo connections was made on the theory that there would be a small amplitude for the stable circuit and a large amplitude for the unstable circuit. Such theory was not borne out. The amplitudes were identical. The only noticeable difference was in the phase angle between the forcing function and the amplitude response.

Since Equation (66) has a negative sign before the damping term, the computer results seemed to be somewhat confusing at first glance. However, if Equation (2) is transformed by letting

$$v = \frac{dw}{d\tau} ,$$

then Equation (2) becomes

$$\frac{d^2w}{d\tau^2} + \frac{k}{2}\left(\frac{dw}{d\tau}\right)^2 + \alpha w = B \sin(\tau-\phi) + A, \tag{67}$$

where A is a constant of integration. If $(-\tau)$ is substituted into Equation (67) for τ, then

$$\frac{d^2w}{d\tau^2} + \frac{k}{2}\left(\frac{dw}{d\tau}\right)^2 + \alpha w = - B \sin(\tau+\phi) + A. \tag{68}$$

Equations (67) and (68) have identical L.H.S. and vary only in the R.H.S. by a phase shift. It appears that for both $(+\tau)$ and $(-\tau)$ there are stable asymptotic solutions, that is, periodic solutions which are identical in form except for the change in phase. Therefore, there is an unstable solution for $(+\tau)$ that is only a phase shift away from the stable solution. Likewise for $(-\tau)$ there is an unstable solution which is also a phase shift away from the stable solution. The investigator was not able to develop a technique which could accomplish the phase shift necessary to find either of the unstable solutions.

The question of whether there is another stable solution path is still unresolved. Attempts to get the vibration on another solution path by means of overdamping and suddenly releasing the damping were of no avail. Attempts to find another solution path by starting the Reac well beyond the point of infinite slope also proved futile. The lack of success in finding another stable solution path does not necessarily infer that such a solution path does not exist. The question still remains unresolved and should be subjected to further investigation.

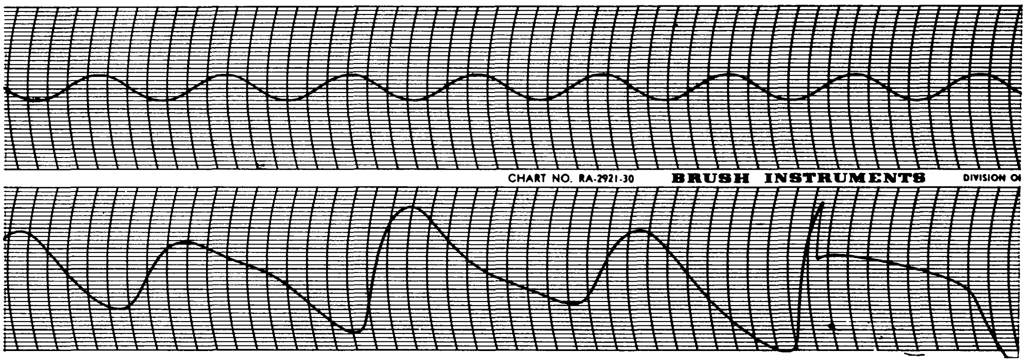
There is another important matter which must be discussed before concluding this section. This matter concerns the values of α which are slightly greater than $\alpha = \frac{n^2}{4}$. Consider the case $\alpha = 2.26$. The appropriate n for the stability criterion for the first point of instability in this case is n=3. The predicted first point is kB = 0.25. If the instability for n=2 is calculated, it is found to be kB = 1.64. The Reac instability point was found to be kB = 1.52. It appears that the first unstable point is governed by n=2. At first it was thought that an instability of the second kind must exist in the neighborhood of kB = 0.25. However, when a search was made for this jump it was not evident. The search covered not only $\alpha = 2.26$, but also other values of α between $\alpha = 2.26$ and $\alpha = 2.50$. A second look at the instability prediction based on n=3 revealed that for $0.25 \leq kB \leq \infty$ the system was unstable. Therefore, if the predictions were correct, only an instability of the first kind could exist at kB = 0.25.

There appears to be no apparent reason why a prediction based upon n=3 fails at $\alpha = 2.26$ but exists at $\alpha = 2.5$. Accepting the facts as found by Reac test, then there must exist a transition zone somewhere between $2.26 < \alpha < 2.50$ wherein the prediction for instability by n=3 comes into being. Using the Reac an exploration for this zone was undertaken. Until $\alpha = 2.40$, n=2 seemed to govern the first observed point of instability. Between $\alpha = 2.40$ and $\alpha = 2.50$ there appears to be a falling off from the prediction for n=2. The transition zone appears to be in the range of $2.40 < \alpha < 2.50$. No clear line of demarcation was observed.

B. Zone 2, $0 < \alpha \leq 1$

The first points of instability were calculated for various values of α in zone 2 using the three term approximation. For all values of α in zone 2, n=1 is valid. When the calculated points were checked on the Reac, it was found that the Reac did not duplicate its own points of instability. A test was then made for $\alpha = 0$. At this value of α, the system was found to be unstable for a very low amplitude of B. Furthermore, it took a very long time for the Reac to saturate its amplifiers. At $\alpha = 0$ the system is nonvibratory, but $v(\tau)$ increased with time for any small value of B. The time it took the amplifiers to saturate was about ten minutes. Upon the completion of this test, new tests were run for α equal to 0.25, 0.50, 0.75, and 1.00. In all cases the response was vibratory, but given three or more minutes the system became unstable. The values of B used for these tests were about one quarter of a volt. It was necessary to have some voltage input so that the autonomous equation was not being examined. In all cases of α examined other than

$\alpha = 0$, the wave form was found to be a subharmonic of the one-half order. Figure 4 is a photograph of the tape for $\alpha = 0.50$, $k = 0.10$, $B = 0.15$. The subharmonic of the one-half order is clearly evident in this picture. The cusp recorded on the tape is a change in the attenuator of the amplifier of the Brush recorder just before the system became unstable.



B(τ)

v(τ)

α=0.50

Figure 4. Brush Recorder Tape, Zone 2

With the evidence of the one-half harmonic in hand, a solution for $v(\tau)$ was assumed containing the one-half harmonic. The solution and the Hill coefficients have been reported in Section E, Chapter II. Upon examining the Hill coefficients, it was found that for n=2, the stability criterion as given by Equation (35) reports the system to be unstable for $0 < \alpha \leq 1$, for $kR_1 = 0$. That is

$$|\Theta_o - 1| - |\Theta_2| < 0, \tag{69}$$

for $kR_1 = 0$. Equation (69) with its Θ's evaluated becomes

$$|\alpha\text{-}1 - \frac{(\alpha\text{-}1)^2}{2}| - |\alpha\text{-}1| < 0. \tag{70}$$

Therefore, there is agreement between the stability predictions and the Reac tests for zone 2.

Professor Rauch suggested that a probable explanation for the relatively large period of time necessary for the system to saturate the amplifiers of the Reac might be found by examining the phase plane solution for the autonomous part of Equation (2). Such a study was undertaken.

The autonomous part of Equation (67) was used for the phase plane analysis rather than the one for Equation (2). Therefore, the autonomous equation being examined is

$$\frac{d^2w}{d\tau^2} + \frac{k}{2}\left(\frac{dw}{d\tau}\right)^2 + \alpha w = 0. \tag{71}$$

Let

$$z = \frac{dw}{d\tau}, \tag{72}$$

then Equation (71) reduces to

$$\frac{dz}{d\tau} = -\frac{k}{2} z^2 - \alpha w \tag{73}$$

By dividing Equation (73) by Equation (72), τ may be eliminated, that is

$$z\frac{dz}{dw} = -\frac{k}{2} z^2 - \alpha w. \tag{74}$$

The isoclines for zero slope occur when

$$\frac{dz}{dw} = 0 \tag{75}$$

and are

$$w = -\frac{k}{2\alpha} z^2. \tag{76}$$

The isoclines for zero slope are a family of parabolas, the vertices of which are all at the origin of the phase plane. There is a parabola for each value of $\frac{k}{2\alpha}$. Figure 5 represents a family of isoclines when k is

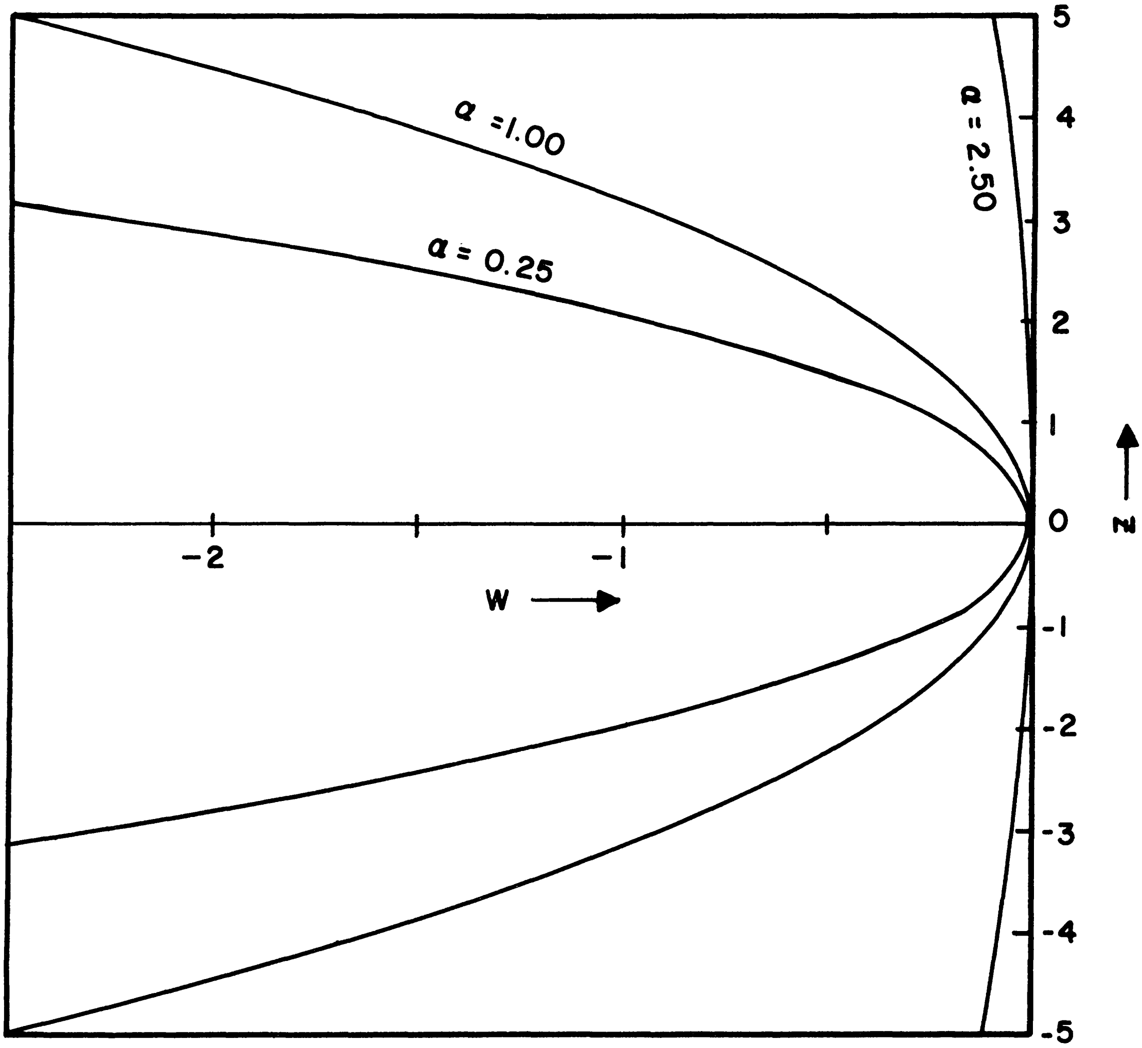


Figure 5. Zero Isoclines, k = 0.20.

held constant and α ia allowed to vary. It can be readily seen that as α approaches infinity in the limit, the parabola approaches the z axis. Also as α approaches zero in the limit the parabola tends to coalesce with the negative w axis. By virtue of Equation (72), it can be shown that everywhere on the w axis the slopes are vertical.

In order that the type of singularity be determined, let

$$u = z^2, \tag{77}$$

then

$$\frac{du}{dw} = 2z \frac{dz}{dw} .$$

Substituting Equation (77) into Equation (74), there results

$$\frac{du}{dw} + ku = - 2\alpha w. \tag{78}$$

Equation (78) is a first order ordinary differential equation, the solution of which consists of two parts, the complementary and particular solutions. It may be readily shown that the complete solution to Equation (78) is

$$u = \frac{2\alpha}{k^2}[\frac{Ck^2}{2\alpha} e^{-kw} + 1] - \frac{2\alpha}{k} w \tag{79}$$

or

$$z^2 = \frac{2\alpha}{k^2}[\frac{Ck^2}{2\alpha} e^{-kw} + 1] - \frac{2\alpha}{k} w. \tag{80}$$

A phase plane solution has been set forth for an equation of the type of Equation (71) by Professor Rauch.[6] It is shown therein that the singularity is a center at the origin of the phase plane. A separatrix is generated when C=0. The system exhibits stable periodic oscillations whenever $\frac{-2\alpha}{k^2} < C < 0$ and has an unstable solution path when $C \geq 0$. Figure 6 is a reproduction of a phase plane solution taken from Professor Rauch's[6] notes. To correlate the symbolism in Figure 6 to the symbols used herein:

$$w = x,$$

$$y = z,$$

$$k = 2h,$$

and

$$\alpha = 1.$$

It is important to note that the vertex of the separatrix is independent of α. For a fixed k only the latus rectum of the parabola representing the separatrix is affected. For increasing α the latus rectum increases, that is, the parabola is opening wider and in the limit approaches a vertical line. The above may be interpreted to mean that if the initial conditions for starting the solution of the autonomous equation are an initial displacement equal to $\frac{1}{k}$ and an initial velocity of zero, the solution will always be unstable whatever the value of α. On the other hand, if the initial displacement is less than $\frac{1}{k}$, then for an unstable solution along the separatrix an initial velocity is also needed: the larger the value of α, the larger the initial velocity need be.

Consider the phase plane equation for the entire system as given by Equation (67):

$$z \frac{dz}{dw} = - \frac{k}{2} z^2 - \alpha w + B \sin(\tau - \phi) + A. \qquad (81)$$

If B is very small and the term $B \sin(\tau-\phi)$ is for the moment neglected, the complete solution for Equation (81) is

$$z^2 = \frac{2}{k}[(A + \frac{\alpha}{k}) \left\{ \frac{Ck^2 \, e^{-kw}}{2(kA+\alpha)} + 1 \right\} - \alpha w]. \qquad (82)$$

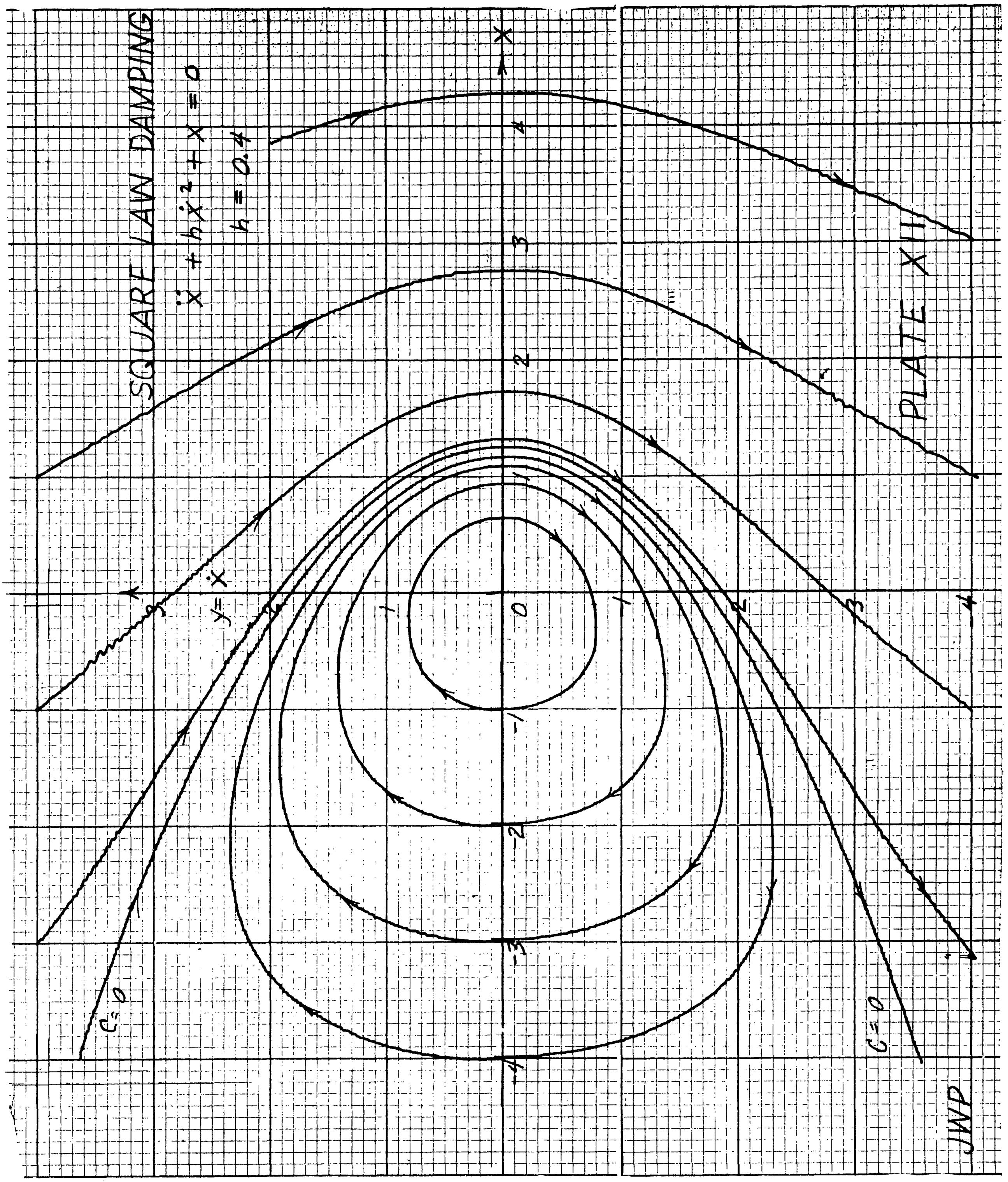


Figure 6. Phase Plane Diagram

The vertex of the separatrix now occurs at

$$w = \frac{1}{k} + \frac{A}{\alpha}, \quad z = 0. \tag{83}$$

It can be readily seen that the vertex of the separatrix is shifted by the amount $\frac{A}{\alpha}$ from the separatrix of the autonomous equation. This shift is to the right or to the left depending upon whether A is positive or negative.

The purpose of the foregoing discussion is merely to show that the nature of the separatrix for Equation (81), when B is small, is quite similar to the separatrix for the autonomous system. Referring to Figure 6, if a small initial displacement is given to the system, the oscillation will follow one of the periodic solution paths. Now if instead of a small initial displacement, a periodic forcing function $B \sin(\tau - \phi)$, where B is very small, is given to the system, there are two possibilities as to what the solution path might be: (1) a periodic solution about the singular point as a center; (2) an aperiodic solution about the singular point as an unstable focus. If the system remains in dynamic equilibrium when the forcing function is applied, then the solution path will be periodic. If, however, the system does not remain in equilibrium due to excess energy being introduced into the system by the periodic forcing function, the energy build-up will set the solution on the unstable focus path. The energy build-up may be likened to the energy build-up of an undamped linear vibratory system being excited at resonant frequency. It appears, then, from the evidence submitted in the first part of this section that the solution paths for all α in zone 2 follow unstable spiral solution paths with respect to the integral curves and, hence, the time element is necessary before the maximum amplitude of the vibration becomes excessively large. However, after due consideration, such a generalization cannot be made. At this point, it is safe to say that aperiodic solutions do exist in zone 2. There still is the possible existence of periodic solutions in this zone.

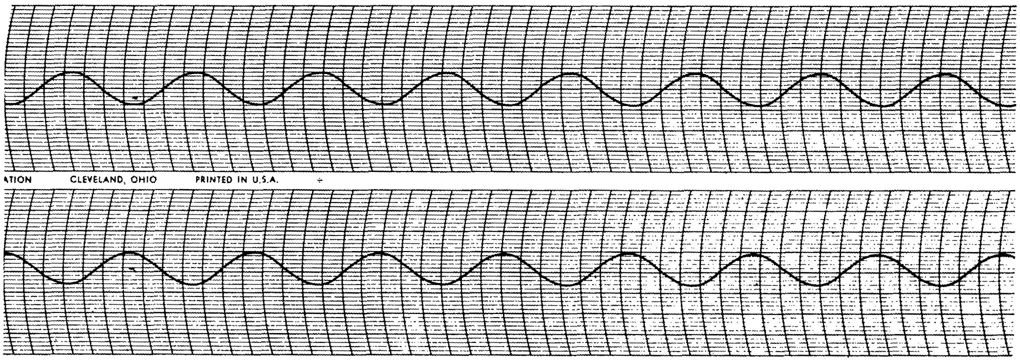
Since the existence of aperiodic solutions in zone 2 has been established and since the phase plane diagrams are quite similar for all α when B is small, the argument for aperiodic solutions may be extended for zones 1 and 3 as well. It may, therefore, be concluded that aperiodic solutions may exist in all zones of α. Conversely, since it has been established that stable periodic solutions exist in zone 1, then periodic solutions may exist in zone 2. Therefore, with the aforementioned possibilities in mind, new solutions were effected on the Reac.

It is important to recall that for the prior solutions, found by the Reac, the transients were controlled by effectively damping them out by linear damping. A different technique was used for the new series of Reac investigation; namely, no linear damping was introduced. The transient and the steady-state solutions, if such a differentiation can be made, coalesced into a single solution path with respect to the phase plane. In respect to zone 2, it was found that when B was small nearly all solutions were periodic. Only a few aperiodic solutions were found. The aperiodic solutions attained extremely large maximum amplitudes in a few minutes. Thus, verification was found that the two types of solutions, periodic and aperiodic, do exist in zone 2.

There was an important difference between the wave forms of the solutions using the different techniques of linear damping or no linear damping. As shown in Figure 4, when the transient was damped out the wave form was predominately of the one-half subharmonic. For the undamped system, the wave form was predominately of the fundamental. A photograph of a recorder tape for the periodic solution of $\alpha = 0.50$, $k = 0.20$, is shown in Figure 7a. The tape represents the periodic solution at approximately six minutes after the solution started. The impressed voltage is $B = 0.20$, the maximum amplitude of response is approximately 0.38 volts. The difference in wave form and in type of solution must be due to the difference in solution techniques. The addition and removal of linear damping allows a subharmonic of the one-half order to form and it has been demonstrated that a solution containing the one-half harmonic is unstable. The difference in the two techniques is essentially the difference of starting two solutions with different initial conditions. It is well-know that the solution to a nonlinear differential equation is dependent upon the initial conditions of the solution.

While searching for a periodic solution for $\alpha = 1.00$, the interesting phenomenon of a beat was noticed. That a beat should occur for $\alpha = 1.00$ when B is small is not surprising, since the natural frequency of the system at that value of α is nearly equal to one. Figure 7b is a tape showing the beat phenomenon of the periodic solution for $\alpha = 1.00$, $k = 0.20$. The beat period is seen to be quite large.

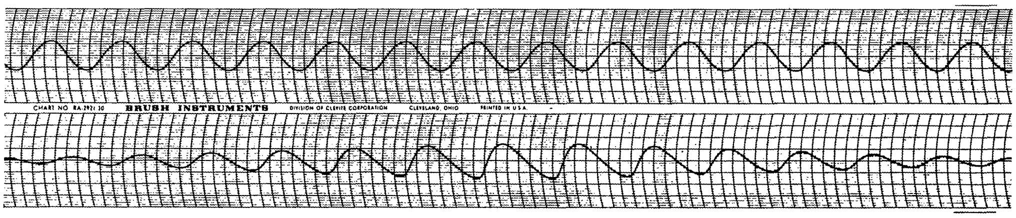
Having determined that periodic solutions do occur in zone 2, the next task was to determine the first point of instability.



B(τ)

v(τ)

a. α=0.50



B(τ)

v(τ)

b. α=1.00

Figure 7. Brush Recorder Tapes, No Damping, Zone 2

Table II shows the comparison between the predicted and the actual first point of instability for $\alpha = 0.50$ and 1.00, based upon the three term approximate periodic solution.

TABLE II

COMPARISON OF FIRST POINTS OF INSTABILITY, ZONE 2

		Three Terms		Analog Computer
α	n	kR_1	kB	kB
0.50	1	-1.75	0.40	0.38
1.00	1	-1.75	0.41	0.40

Following the line of reason that if a periodic solution exists in zone 2, an aperiodic solution may exist in zones 1 and 3, a search was made for such a solution for various values of $\alpha > 1$. After many attempts to find an aperiodic solution in zone 1, one was found at $\alpha = 2.5$, $k = 0.2$, $B = 1.00$. It required a time period of 28 minutes before the amplifiers of the Reac saturated. It was quite apparent at the start of this solution that the solution was aperiodic since the maximum amplitude of vibration was constantly increasing, though slowly. The increase of time necessary for the system to saturate the amplifiers when compared to $\alpha = 1.00$ may be explained in part by examining the zero isoclines of Figure 5. Since the latus rectum of the zero isoclines is larger for $\alpha = 2.5$ than $\alpha = 1.00$, the periodic solution paths for $\alpha = 2.5$ will be less flat than those for $\alpha = 1.00$. Therefore, it probably takes the unstable focus solution for $\alpha = 2.5$ a longer time to reach the separatrix, assuming of course, the two separatrices are in the same neighborhood of the phase plane. An intensive search for an aperiodic solution for other values of α in zones 1 and 3 was made without success. The lack of success, however, is no indication that aperiodic solutions do not exist; it merely points out the difficulties in setting the initial conditions for the aperiodic solutions.

C. Zone 3, $\alpha > 4$

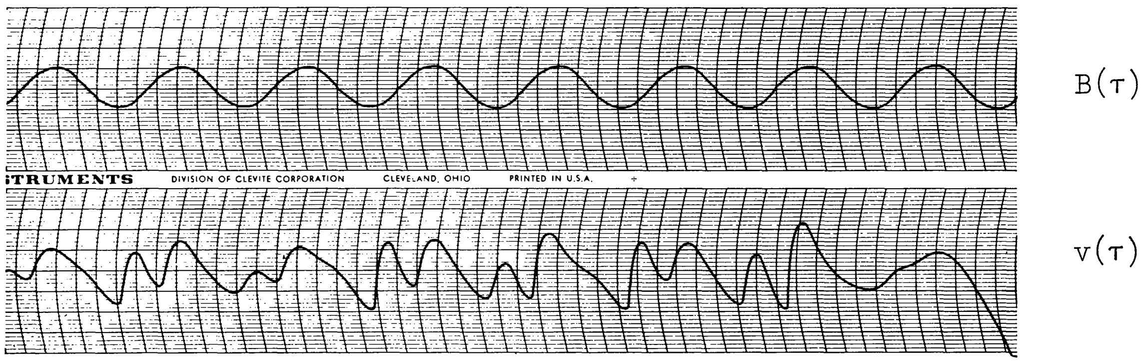
Predictions based upon the three term approximation seemed to give the best results for zone 3. The predicted results, however, are not in as close agreement as were the results for zone 1 and zone 2. Table III records the results for three α's. Though the agreement for $\alpha = 5$ and $\alpha = 7$ are fair, the theory seems to break down for $\alpha = 9$. This disagreement in theory and in test is understandable in view of the discussion for the need of a four-term approximation contained in Section A of this chapter. The difficulties of assuming a four-term approximation were also discussed in Section A.

TABLE III

COMPARISON OF FIRST POINTS OF INSTABILITY, ZONE 3

		Three Terms		Analog Computer
α	n	kR_1	kB	kB
5	4	1.28	5.40	4.56
7	5	1.55	9.49	10.00
9	5	4.20	33.56	17.20

Strong evidence of the second harmonic was apparent in zone 3 tests. Figure 8 is a photograph of a recording tape for $\alpha = 7$, $k = 0.40$. There is clear evidence of the higher harmonics predominating.



$\alpha = 7.0$

Figure 8. Brush Recorder Tape, Zone 3

The difficulties encountered in zone 3 were anticipated. In reading the literature concerning nonlinear vibrations many references were made concerning the higher harmonics which were found at low frequencies when tests were made. It seems appropriate to quote a paragraph from a paper by Dana Young[7] concerning the test findings of the Duffing equation. The form of the Duffing equation simulated on an analog computer by Mr. Young was:

$$\ddot{x}_1 + \alpha\dot{x}_1 + x_1 + \beta_1 x_1^3 = \sin \omega t \ . \qquad (83)$$

To quote from Mr. Young:

> "It is found that for small nonlinearities the analog computer results check quite closely with the calculated values from the approximate solution. For large values of β_1, a number of differences develop, some of which had not previously been expected. At low frequencies of the forcing function the periodic solution contained many higher harmonics of large amplitude. For certain values of the parameters the solution is actually unsymmetrical. In the study of the stability

> it is shown that there are many narrow regions of instability within which the solution develops irregularities."

Though the problem discussed by Mr. Young and the problem concerned with in this investigation are of different natures, the above quoted paragraph seems to be relevant to both problems.

D. Consolidated Results

Figure 9 is a graph showing the comparison of the predicted and test results for the periodic solutions for all three zones based upon the three term approximation. The values for $\alpha = 9$ have been omitted; first, because of the difficulty of including this result on the graph without a drastic scale change and second, because of the great disparity between the predicted and the test results.

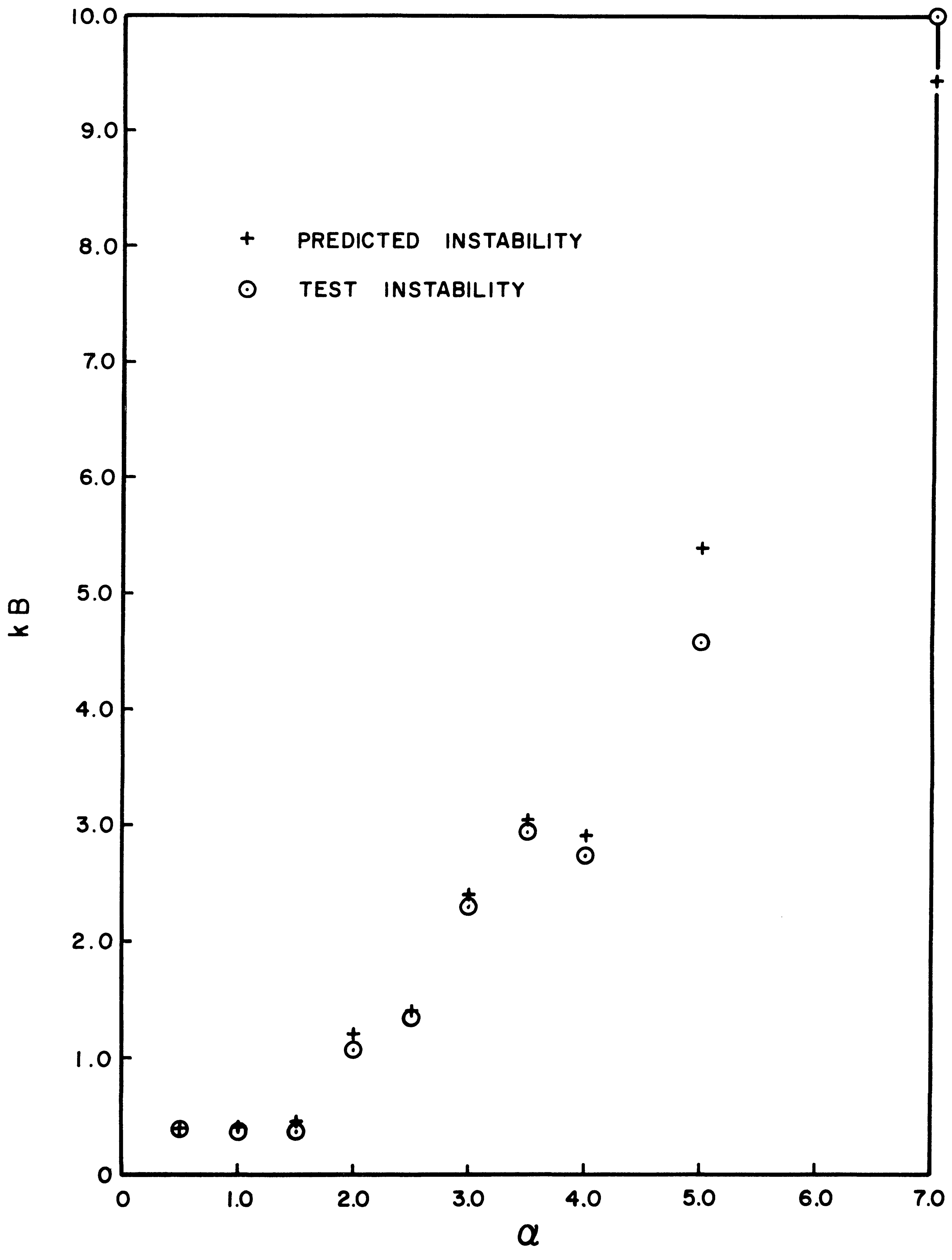


Figure 9. Consolidated Results.

CHAPTER IV

SUMMARY

The prediction of the first points of instability of the steady-state vibratory system governed by the equation

$$m\ddot{x} + cx\dot{x} + ax = F_o \cos \nu t \qquad (1)$$

has been made within the realm of two assumptions: (1) an assumed solution for the Hill equation; (2) an assumed periodic solution for the vibration. Three assumed periodic solutions for $v(\tau)$ were presented. It was found that the three term approximate solution gave good results over a wider range of α than did the other two solutions. The need for a four term approximate periodic solution has been demonstrated. This need is greater as α assumes a large value or ν becomes small. At large values of α the predictions of the first points of instability become unreliable. Whether the unreliability at large α is due to the approximation made for the Hill equation or to the assumed periodic solution for $v(\tau)$ requires further study.

There is a region of α when investigated by the stability theory that does not respond to the expected n value. This region is just beyond $\alpha = \frac{n^2}{4}$, when n has the value defined by the rule for choice of proper n. A study of the perturbation equation, as given by Equation (12) may contribute to the understanding of what is happening in this boundary area.

The instabilities found by Reac test were always of the first kind, that is the amplitudes became extremely large so that in the limit they did not remain finite. No instabilities of the second kind, jump phenomenon, were discerned.

It was also demonstrated that stable periodic solutions and unstable aperiodic solutions exist for the system defined by Equation (1). Whether a solution will be periodic or aperiodic depends on the initial conditions of the solution.

The mechanisms of the perturbation equation and the Hill equation are fascinating ones. A great deal about the stabilities might be learned from a general study of these equations. Certainly a study of the perturbation equation simulated on an analogue computer for several nonlinear systems should reveal a good deal of information.

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